FUSION OF W STATES USING OPTICAL QUANTUM GATES

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ABSTRACT

FUSION OF W STATES USING OPTICAL QUANTUM GATES

In this work, after talking about historical developments leading to ideas of information processing based on quantum dynamics, we will be constructing basic mathematical expressions for qubits and quantum gates, the Toffoli gate and the cNOT gate. To do this, we will mention the superposition of polarization states of photons and their state matrices defining our quantum system since these matrices purely depend on the qubits. Later, we will mention one of the basic theorems in quantum information processing, the no-cloning theorem that states that no operation can be defined cloning some quantum state to steal information. We will also point out the fact that quantum systems must be isolated from environment by showing an example. In the later section, we will talk about the reversibility property of quantum information processing by examining quantum Toffoli and cNOT gates. By showing famous examples, the Deutsch algorithm and the Grover search algorithm, we will adress the fact that quantum information processing can be more advantageous than classical algorithms. Lastly, we will focus on the construction of the optical networks to fuse photonic W states, that is the main subject of this work. For this purpose, we will examine the previous works that use optical setups, and then propose two methods fusing two W states. The first proposal consists of two cascaded Toffoli gates and the basic fusion gate whereas the second proposal consists of a Toffoli and a cNOT gates together with the basic fusion gate. We will use some theoretical and experimental results to show that our setups are more realizable with current photonics technology. We will also talk about why creating large-scale W states is important by pointing out their unique superpositions.

ÖZET

OPTİK KUANTUM KAPILARI KULLANARAK W DURUMLARINI BİRLEŞTİRME

Bu çalışmada, kuantum dinamiklerini temel alan bilgi işleme fikirlerine yol açan tarihsel gelişmelerden bahsettikten sonra kubit ve kuantum kapıları(Toffoli kapısı ve cNOT kapısı) için temel matematiksel ifadeler oluşturacağız. Bunun için fotonların polarizasyon durumlarının süperpozesi ve bunların sadece kubitlere bağlı olduğu için kuantum sistemlerini tanımlayan durum matrislerinden bahsedeceğiz. Kuantum bilgi işleme konusundaki temel teoremlerden olan ve bilgi çalmak için herhangi bir kuantum durumunu klonlayan bir operatörün olmadığını gösteren no-cloning teoremi hakkında konuşacağız. Aynı zamanda, bir örnekle, kuantum sistemlerinin çevreden izole edilmesi gerektiğinden bahsedeceğiz. Sonraki bölümde, kuantum Toffoli ve cNOT kapılarını inceleyerek kuantum bilgi işleminin geri döndürülebilirliği hakkında konuşacağız. Deutsch algoritması ve Grover arama algoritması gibi ünlü örnekleri göstererek, kuantum bilgi işleminin klasik bilgi işlemlerinden daha avantajlı olduğunu vurgulayacağız. Son olarak, fotonik W durumlarını birleştirmek için kullanılan optik ağların kurulumuna odaklanacağız ki bu çalışmanın ana konusu. Bu amaçla, optik ağlar kullanan önceki çalışmaları inceleyeceğiz, ve ardından iki W durumunu birleştiren iki yöntem öne süreceğiz. İlk önermede iki tane sıralı Toffoli kapısı ve temel füzyon kapısı varken ikinci önermede bir Toffoli, bir cNOT ve temel füzyon kapısı bulunur. Tasarımlarımızın bugünkü fotonik teknolojiyle daha anlaşılabilir olduğunu göstermek için bazı teorik ve deneysel verileri kullanacağız. W durumlarının kendilerine has süperpozelerini vurgulayarak büyük ölçekli W durumları oluşturmanın neden önemli olduğundan bahsedeceğiz.

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LIST OF SYMBOLS

$a(\theta \rightarrow \alpha)$	Probability amplitude for a photon polarized at θ to pass through an analyzer oriented along α
G	Grover search operation
k_B	Boltzmann constant
k_o	The number of operations needed to find a specific bit on the database of Grover search algorithm
$p(\theta \rightarrow \alpha)$	Probability for a photon polarized at θ to pass through an analyzer oriented along α
T	Absolute temperature
$U(t, t_0)$	A unitary operator that describes a quantum evolution in a time interval between t_0 and t
$\|W_i\rangle$	A W state of i qubits
$< x >$	The expectation value of some variable x
Γ	The factor that determines the relation between the probability of decoherence and interaction time
δ_{ij}	Kronecker delta
ρ_A	State matrix of some qubit A
σ_i	Pauli spin matrices
τ_D	The time needed for a qubit to be incoherent with a unit probability
$\otimes$	Tensor product
$\uparrow$	the NAND gate's operation
$\oplus$	mod 2 addition

LIST OF ACRONYMS/ABBREVIATIONS

D1	Detector 1
D2	Detector 2
HWP	Half-wave plate
PBS	Polarizing beam splitter
SLOCC	Stochastic local operations and classical communications
T1	The first Toffoli gate
T2	The second Toffoli gate

1. INTRODUCTION

1.1. Historical Background

Quantum information and quantum computation are promising fields of research, and give scientists and engineers hope for new discoveries of the dynamics hidden from our minds and eyes. Simply we can say that quantum information and its computation is about information processing tasks based on quantum mechanical laws.

To look at the history of quantum information, we shall begin with quantum mechanics. The rules of quantum mechanics are counter-intuitive, and even experts find it hard to understand since we are used to think classically. We should note that understanding of quantum mechanical information processing may lead us to understand the underlying facts of quantum physics. It is also likely that this will lead us to develop mathematical tools to sharpen our intuitions about quantum mechanics.

In the 1980s, one of the earliest discoveries was made by realizing that cloning, which is easy in classical algorithms, is impossible in quantum mechanics. This phenomenon is explained by the no-cloning theorem [1, 2].

Another contribution to quantum information dating to the 1970s is obtaining control over single quantum systems. Obtaining control over single quantum systems was a new method when it was achieved and it allows us to take a look at the insights of nature in a better way as they happened in the historical scenes of science. While constructing our quantum algorithms, we will have complete control over our qubits and their spaces in which operations take place.

It can also be argued that quantum information and quantum computation are necessities in today's world because large amounts of data are to be kept and transmitted, and also because of the fact that computers dealing with large amount of data should be fabricated functioning in smaller sizes where quantum effects are not

negligible.

Quantum computers offer more speedy performance than classical computers. The Grover Search Algorithm [3] and the Deutsch Algorithm [4] are good examples of why quantum computers waste less time and use less number of operations. We will be discussing these algorithms in detail to show explicitly why quantum information theory is very promising as far as performance is concerned.

Another challenge which must be met while constructing classical or quantum algorithms is the noise effect. This is an important test for an algorithm to pass and to prove its efficiency. The development of quantum error correcting codes is one of the steps ahead [5, 6].

To solve problems efficiently, a new kind of algorithm was developed by David Deutsch in 1985. Deutsch attempted to define a quantum device that is capable of doing computational tasks and simulating any arbitrary physical system. The laws of physics are ultimately quantum mechanical so Deutsch considered computing devices based on quantum mechanics. The device proposed by Deutch is called the Universal Quantum Computer and it is not clear whether Deutsch's notion of Universal Quantum Computer can be capable of simulating any arbitrary physical system.

What Deutsch tried to answer is "Could we construct new algorithms that can work faster and more efficent than classical ones?". He gave an example of an algorithm whose power might exceed those of classical computers. This step taken by Deutsch was improved by Peter Shor in 1994 [7]. His demonstration shows that the problem of finding a prime factor of an integer and the discrete logarithm problem can be solved more efficiently in quantum computers. Shor's results are the powerful indication that quantum computers can be more powerful than classical ones. Further powerful evidence of this fact came in 1995 when Lov Grover showed that a searching task can be performed faster in quantum computers [3]. This search algorithm is called The Grover Search Algorithm and will be discussed later.

We do not know at what other tasks quantum computers can be faster than classical ones. But we know that we are used to classical algorithms and anything that comes from our minds is most likely to be classical ideas. Our way of thinking is rooted in classical physics and this is one of the challenges we must face. Another challenge that is to be met is that we should construct our algorithms in a more efficient way than that of classical algorithms. These two problems make quantum computation much more challenging.

We now look at the historical background of information theory contributing to quantum information theory. In the 1940s, while computer science was exploding, another branch was developing: communication and information theory. In 1948, Claude Shannon published a pair of remarkable papers explaining the foundations of information [8].

> "Shannon was interested in two key questions related to the communication of information over a communications channel. First, what resources are required to send information over a communications channel? For example, telephone companies need to know how much information they can reliably transmit over a given telephone cable. Second, can information be transmitted in such a way that it is protected against noise in the communications channel [9]?"

Shannon answered these questions by proving two fundamental theorems of information theory. The first one is Shannon's noiseless channel coding theorem which quantifies the physical resources needed to store output from an information source. Shannon's second fundamental theorem is noisy channel coding theorem which tells us how much information we can transmit in a reliable way through the noisy communication channels. In other words, Shannon found an upper limit of information to transmit reliably but he did not show how to reach that limit. Since then researchers have constructed error-correcting codes to get close to that limit.

In the developments of quantum information theory, same research has been done and an analogue to Shannon's noiseless coding theorem has been provided [10]. A theory of quantum error-correction codes has been made which allows quantum computers to compute effectively in the presence of noise, and also allows communication to take

place reliably.

In 1992 Charles Bennett and Stephen Wiesner showed it is possible to send classical bits through quantum channels [11]. They constructed such channels being able to transmit two classical bits by sending only one qubit which is a quantum bit. This fact is called superdense coding.

Another striking fact about the nature of quantum is quantum entanglement. In the last section, we will be focussing on photonic W states whose entanglement is interesting since these quantum states' robustness is very high. This is important because noise effects from environment causes incoherence giving us false results.

> "Entanglement is a uniquely quantum mechanical resource that plays a key role in many of the most interesting applications of quantum computation and quantum information; entanglement is iron to the classical world's bronze age. In recent years there has been a tremendous effort trying to better understand the properties of entanglement considered as a fundamental resource of Nature, of comparable importance to energy, information, entropy, or any other fundamental resource. Although there is as yet no complete theory of entanglement, some progress has been made in understanding this strange property of quantum mechanics. It is hoped by many researchers that further study of the properties of entanglement will yield insights that facilitate the development of new applications in quantum computation and quantum information [9]."

1.2. Mathematical Formulation: The Qubit

Our first example of a qubit will be the polarization of light. The polarization of light was demostrated for the first time by the Chevalier Malus in 1809 [12]. He observed the light of sun reflected by the glass of window through a crystal of spar. He observed that when the crystal was rotated, one of the two images of sun disappeared. This led to the scientific discovery of the polarization of light, which means light ray is decomposed into two rays perpendicular to each other. In the twentieth century, Ideas on transmitting information by light developed and the fact that light is polarized raised the questions on whether information transmitting can be applied to light rays or not. The answer is yes! But we should not forget that the polarization of light is a very basic identification of a qubit.

The photon polarization can be used to transmit information, for example, by an optical fiber. When the photon is polarized along x axis, we can name it the bit 0. When the photon is polarized along y axis, we can name it the bit 1. Let us call two people exchanging information Alice and Bob. For example, Alice sends Bob a series of photons polarized as yyxxxyxy. These polarizations are determined by Alice's polarizer. By using analyzer, Bob decodes the message and gets a series of bits as 11000101. As long as we see, it may seem to us that there is no difference between classical information processes and this process. However, the crucial question is, what if a photon is polarized at 45°? We can define the photons polarized at some angle as a linear superposition of a photon polarized along x axis and a photon polarized along y axis. The polarization of photon is a basic and good example of a qubit because polarized light is much richer than an ordinary bit defined in classical algorithms. This is because of the fact that photon as a linear superposition can take all values between 0 and 1, and contains an infinite amount of information. This statement can be opposed by the argument that the measurement of a qubit can give only 1 and 0 after testing it by an analyzer. The question whether or not this property can be used to process information faster is important and we shall see that it is possible by the Deutsch algorithm and the Grover Search algorithm.

We said that photons can be defined as linear superpositions and to express them mathematically, we should introduce a two-dimensional vector space H. This is called the two dimensional Hilbert Space in which all polarization properties of light rays can be defined. Let us choose our orthonormal basis vectors as $|x\rangle$ and $|y\rangle$ corresponding to polarizations along x axis and y axis. Any polarization can be decomposed in these bases:

$$|\Phi\rangle = \lambda\,|x\rangle + \mu\,|y\rangle\,. \tag{1.1}$$

Dirac notation is used to define vectors of H. While constructing our quantum algorithms, we will be using these superpositionally defined waves for specific purposes. In the equation above, λ and μ can be complex or real numbers. The space H is

therefore a complex vector space containing real vector spaces.

Let us take two vectors, $|\Phi\rangle$ given by Equation 1.1 and $|\Psi\rangle$ given by

$$|\Psi\rangle = \nu\,|x\rangle + \sigma\,|y\rangle\,. \tag{1.2}$$

The scalar product of these vectors is expressed as $\langle\Psi\,|\,\Phi\rangle$, and

$$\langle\Psi\,|\,\Phi\rangle = \nu^*\lambda + \sigma^*\mu = \langle\Phi\,|\,\Psi\rangle^*\,, \tag{1.3}$$

where c^* is the complex conjugate of c. By using this kind of product, we can find the norm of $|\Phi\rangle$:

$$\langle\Phi\,|\,\Phi\rangle = \|\Phi\|^2 = |\lambda|^2 + |\mu|^2. \tag{1.4}$$

The vectors $|x\rangle$ and $|y\rangle$ are orthogonal and they have unit form:

$$\langle x\,|\,x\rangle = \langle y\,|\,y\rangle = 1, \langle x\,|\,y\rangle = 0. \tag{1.5}$$

We know that every wave(complex vector) defined in Hilbert Space obeys the normalization rule as follows:

$$\|\Phi\|^2 = |\lambda|^2 + |\mu|^2 = 1. \tag{1.6}$$

Polarization states are represented mathematically by normalized vectors in Hilbert space denoted by H.

Recall that we defined a light wave polarized at 45° in the beginning of this section. This light wave is defined as

$$|45^\circ\rangle = \cos 45^\circ\,|x\rangle + \sin 45^\circ\,|y\rangle\,. \tag{1.7}$$

If we would like to generalize this formula, we can write as such

$$|\theta\rangle = \cos\theta\,|x\rangle + \sin\theta\,|y\rangle\,. \tag{1.8}$$

The vector $|\theta\rangle$ is the mathematical description of a wave linearly polarized at angle θ.

Until now we talked about photons polarized at some angle and their mathematical description. As mentioned in the beginning, this is the very basic example of a qubit. That is to say, I showed all these mathematical descriptions to familiarize with notations and how a qubit will be defined in the next sections. However, we should not forget photon polarizaton is just one example of a qubit.

Let us simply talk about probability amplitudes and probability calculations which will be used in the same sense while setting algorithms. The probability amplitude for a photon polarized at θ to pass through an analyzer oriented along α is as follows:

$$a(\theta \to \alpha) = \cos(\theta - \alpha) = \langle \alpha \,|\, \theta \rangle \tag{1.9}$$

We already know that probability measurements are made by taking square of amplitudes,

$$p(\theta \to \alpha) = \cos^2(\theta - \alpha) = |\langle \alpha \,|\, \theta \rangle|^2. \tag{1.10}$$

These probablity calculations are based on polarizer-analyzer systems and polarization of a photon. Bob may use an analyzer to receive bits sent by Alice, however this kind of calculation means destruction of waves.

One crucial thing should be noted here. If the probability of passing the analyzer is 0 or 1, we can exactly know the polarization of the photon sent. For example, if Alice sends N photons polarized along x axis to Bob where N is a large number, and Bob has an analyzer oriented along x axis, we deduce from that the probability of passing the

test is 1 and all the photons pass the test. This gives us the information about which angle photons are polarized at. When the analyzer is oriented along y axis, probability becomes 0, so no photon can pass the test. This also informs us the polarization angle which is 0°. On the other hand, unless the probability is 0 or 1, there is no way of knowing unambiguously the angle at which photons are prepared. This is the fundamental fact of measuring in quantum physics. There is difference in measurements between classical physics and quantum physics. In the classical algorithms, what is to test is exactly the same before and after measurement. In quantum algorithms, it is not possible to determine the state before the measurement after testing it. We could get a photon polarized along y axis or x axis, both of these may happen, so it is incorrect to think that state is one of these before the measurement.

To summarize, our example of a qubit which is the polarization of a photon, is a good start to understand the basic principle of qubit. This will be generalized in later sections such that more complex information processing is expressed mathematically.

1.3. The Basics of Quantum Mechanics

To generalize the results obtained in the case of photon polarization, we will examine the basics of quantum mechanics.

First of all, state vectors belonging to Hilbert Space are represented by Dirac notation using bras and kets. A ket denoted by $|\Psi\rangle$ is generally defined on an infinite dimensional Hilbert Space. For the purpose of quantum information theory, we will define a finite dimensional Hilbert Space for our algorithms and we will consider normalized state vectors on this space, $\|\Psi\|^2 = 1$. $|\Psi\rangle$ is called the state vector of a quantum system constructed by our quantum algorithmic mechanisms.

Secondly, if $|\Psi\rangle$ and $|\Phi\rangle$ are two physical states, the probability amplitude $a(\Phi \to \Psi)$ of finding $|\Phi\rangle$ in $|\Psi\rangle$ is given by $\langle\Psi \,|\, \Phi\rangle$:

$$a(\Phi \to \Psi) = \langle\Psi \,|\, \Phi\rangle\,, \tag{1.11}$$

and the probability of passing the $|\Psi\rangle$ test is

$$p(\Phi \to \Psi) = |a(\Phi \to \Psi)|^2 = |\langle \Psi \,|\, \Phi \rangle|^2. \tag{1.12}$$

$|\Phi\rangle$ is the prepared quantum system and $|\Psi\rangle$ is the test(analyzer).

When the state is analyzed, it can be said that it was projected onto $|\Psi\rangle$. Let P_Ψ be the projector operating on $|\Phi\rangle$ as below:

$$|P_\Psi \Phi\rangle \equiv P_\Psi |\Phi\rangle = |\Psi\rangle \langle \Psi \,|\, \Phi \rangle \,, \tag{1.13}$$

as we see, projecting some quantum state onto some subspace or a vector can be represented mathematically.

Let us simply describe our polarizations mathematically by the projectors P_x and P_y which are

$$P_x = |x\rangle \langle x| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_y = |y\rangle \langle y| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.14}$$

The identity operator can be written as

$$I = P_x + P_y = |x\rangle \langle x| + |y\rangle \langle y| \,. \tag{1.15}$$

This is a special case of the *completeness relation* on a two dimensional Hilbert space H. We can generalize it by defining a Hilbert Space of N dimensions:

$$\sum_{i=1}^{N} |i\rangle \langle i| = I, \langle i \,|\, j \rangle = \delta_{ij}. \tag{1.16}$$

The projector P_x commutes with P_y:

$$[P_x, P_y] = P_x P_y - P_y P_x, \tag{1.17}$$

where we express [A,B] as the commutator of two operators A and B. In preceding section, we defined the state $|\theta\rangle$ given by Equation 1.8. There is another state $|\theta_\perp\rangle$:

$$|\theta_\perp\rangle = -\sin\theta\,|x\rangle + \cos\theta\,|y\rangle\,. \tag{1.18}$$

$|\theta\rangle$ and $|\theta_\perp\rangle$ form an orthogonal basis as $|x\rangle$ and $|y\rangle$ do. We can define projectors onto the states $|\theta\rangle$ and $|\theta_\perp\rangle$ as:

$$P_\theta = |\theta\rangle\langle\theta| = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}, \tag{1.19}$$

$$P_{\theta_\perp} = |\theta_\perp\rangle\langle\theta_\perp| = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}. \tag{1.20}$$

These projectors do no commute with P_x and P_y, as can be verified by the calculation below:

$$[P_x, P_\theta] = \begin{pmatrix} 0 & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & 0 \end{pmatrix}. \tag{1.21}$$

Since P_x and P_y do not commute, they cannot be simultaneously measured. We say that $|x\rangle$ and $|\theta\rangle$ are incompatible. P_x, P_y, P_θ, $P_{\theta_\perp}$ represent the observables of physical properties mathematically.

In the general case, a Hilbert space with dimension N, denoted by $H^{(N)}$, contains an orthonormal basis $|n\rangle$, $n = 1, ..., N$, which will be associated to N compatible tests $|n\rangle$. If the quantum system is in the state $|\Phi\rangle$, the probability of passing the test $|n\rangle$ is $p_n = |\langle n\,|\,\Phi\rangle|^2$ and $\sum_n p_n = 1$.

It will be useful to mention expectation value calculation in quantum mechanics. For example, let us say that the probability of measuring 0 is p_0 and the probability of

measuring 1 is p_1, and we shall call this observable s. Thus the expectation value can be found as below:

$$< s >= 1 \times p_1 + 0 \times p_0 = p_1. \tag{1.22}$$

The probability of passing the test $|\Psi\rangle$ for the test $|\Phi\rangle$ can be shown as

$$p(\Phi \to \Psi) = |\langle \Psi \mid \Phi \rangle|^2 = \langle \Phi \mid \Psi \rangle \langle \Psi \mid \Phi \rangle = \langle \Phi | \left(|\Psi\rangle \langle \Psi| \right) |\Phi\rangle \,, \tag{1.23}$$

then it can be expressed such that

$$|\langle \Psi \rangle \, \Phi|^2 = \langle \Phi | \, P_\Psi \, |\Phi\rangle =< P_\Psi >_\Phi \,. \tag{1.24}$$

Equation 1.24 is called the expectation value of the operator P_Ψ in the state $|\Phi\rangle$.

The fact that we can generalize this fact permits us to construct the physical properties of a quantum system using projectors. To give an example, in the case of photon polarization, let us introduce an operator denoted by M. This operator gives +1(eigenvalue) when a photon is polarized along x axes and gives -1 when a photon is polarized along y axes. As we said before, this operator defines an observable which is a physical property. In the following equation, we can write this operator as

$$M = P_x - P_y, \tag{1.25}$$

satisfying

$$M \, |x\rangle = + \, |x\rangle \,, M \, |y\rangle = - \, |y\rangle \,. \tag{1.26}$$

The expectation value of M is

$$< M >= 1 \times p(M = 1) + (-1) \times p(M = -1). \tag{1.27}$$

If the photon is in the polarization state $|\theta\rangle$, then the expectation value of M is

$$< M >= \langle\theta| M |\theta\rangle = \langle\theta| P_x |\theta\rangle - \langle\theta| P_y |\theta\rangle = \cos^2\theta - \sin^2\theta = \cos 2\theta. \tag{1.28}$$

The operator M is a Hermitian operator ($M = M^\dagger$), and physical properties of a quantum system are generally represented by Hermitian operators defining *observables*.

Let us summarize what we did so far in this section. The physical properties are defined mathematically by Hermitian operators two of which are momentum and position operators. These operators obviously correspond to momentum and position of a particle. Let us introduce an observable shown by M. This operator is defined in the basis $|n\rangle$ where $n = 1, ..., N$ and gives an eigenvalue a_n for a state $|n\rangle$,

$$M |n\rangle = a_n |n\rangle . \tag{1.29}$$

In our case, the eigenvalues are nondegenerate and M and I can be written as

$$M = \sum_{i=1}^{N} a_n |n\rangle \langle n| , I = \sum_{n=1}^{N} |n\rangle \langle n| , \tag{1.30}$$

where N is the dimension of the Hilbert space of states. If one measures the state a_n of M, then the state is $|n\rangle$ afterwards: this is the state vector collapse. It means that, as mentioned before, the state changes after measurement. By using the M operator expressed above we can calculate the expectation value:

$$\langle\Phi| M |\Phi\rangle = \sum_{i=1}^{N} a_n \langle\Phi | n\rangle \langle n | \Phi\rangle =< M >_\Phi . \tag{1.31}$$

This expectation value is calculated after large number S of states are measured by the operator M,

$$< M >_\Phi = \lim_{S\to+\infty} \frac{1}{S}(M_1 + M_2 + ... + M_S), \tag{1.32}$$

where M_i is the measurement of i^{th} state. This nondegenerate case can be easily generalized to the degenerate case.

All these basics of quantum mechanics are to be used while constructing our algorithms and defining our operations on qubits mathematically. We mainly follow the introductory method from Le Bellac's book for the sections, *Mathematical Formulation: The Qubit* and *The Basics of Quantum Mechanics* [13].

2. QUANTUM CORRELATIONS

We will see that going from one qubit to two qubits leads to important results, because it introduces correlations between two qubits in the quantum mechanical system. Going from one-qubit systems to 2-qubit or n-qubit systems makes us able to configure special structures of information processing and see why quantum computing is remarkably rich.

2.1. Two-qubit States

We will construct two-qubit states by using the tensor product which will be shown by an elementary example. In our example, let us define a two dimensional vector space H_A whose basis vectors are $\cos x$ and $\sin x$. There are functions in this space and the general expression for them is

$$f_A(x) = \lambda_A \cos x + \mu_A \sin x, \tag{2.1}$$

and let H_B be another two dimensional vector space and the functions defined in this space are expressed as follows:

$$f_B(y) = \lambda_B \cos y + \mu_B \sin y. \tag{2.2}$$

Function of two variables can be constructed by tensor product of f_A and f_B,

$$f_A(x)f_B(y) = \lambda_A\lambda_B \cos x \cos y + \mu_A\lambda_B \sin x \cos y + \lambda_A\mu_B \cos x \sin y + \mu_A\mu_B \sin x \sin y. \tag{2.3}$$

The bases of the tensor product space are $\cos x \cos y, \cos x \sin y, \sin x \cos y$, $\sin x \sin y$. Any function can be decomposed on these bases and is denoted by $g(x, y)$. Equation

2.3 is not the general form of tensor product functions. If we write the general form as

$$f_A(x)f_B(y) = \alpha \cos x \cos y + \beta \sin x \cos y + \gamma \cos x \sin y + \delta \sin x \sin y, \qquad (2.4)$$

we see that there is a condition that $\alpha\delta = \beta\gamma$, and this makes the equation specific form, not general.

We shall continue with this method to construct our two-qubit states mathematically. In the same sense of constructing tensor product of functions, let us represent our two-qubit states by firstly introducing two dimensional vector spaces H_A and H_B containing qubit A and qubit B. We should not confuse these vector spaces with the spaces introduced for functions. Our new spaces are Hilbert spaces in which quantum states(two-qubit states) are defined. Let us define these states in the general form:

$$|\varphi_A\rangle = \lambda_A |0_A\rangle + \mu_A |1_A\rangle , \qquad (2.5)$$

$$|\varphi_B\rangle = \lambda_B |0_B\rangle + \mu_B |1_B\rangle . \qquad (2.6)$$

We know that these states are normalized. As you can see, these states are one-qubit states. By taking the tensor product of these states as in the case of tensor product of functions, we get

$$\begin{aligned} |\varphi_A\rangle \otimes |\varphi_B\rangle &= |\varphi_A \otimes \varphi_B\rangle \\ &= \lambda_A\lambda_B |0_A \otimes 0_B\rangle + \lambda_A\mu_B |0_A \otimes 1_B\rangle + \mu_A\lambda_B |1_A \otimes 0_B\rangle + \mu_A\mu_B |1_A \otimes 1_B\rangle . \end{aligned} \qquad (2.7)$$

$H_A \otimes H_B$ has been constructed as the tensor product of H_A and H_B. We have to note that $|\varphi_A \otimes \varphi_B\rangle$ is also normalized as every state defined in Hilbert space.

Recall that the tensor product of f_A and f_B is not the general form of functions. This result is also valid for Equation 2.7. $|\varphi_A \otimes \varphi_B\rangle$ is not the generalized formula of the states defined in the space $H_A \otimes H_B$. If we write Equation 2.7 in the form

$$|\varphi_A \otimes \varphi_B\rangle = \alpha_{00} |0_A \otimes 0_B\rangle + \alpha_{01} |0_A \otimes 1_B\rangle + \alpha_{10} |1_A \otimes 0_B\rangle + \alpha_{11} |1_A \otimes 1_B\rangle , \quad (2.8)$$

we can easily see that there is a condition that limits the subset of the states in the form Equation 2.8: $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$. Not every state is necessarily in this form. Note that Equation 2.8 is the general form of states without the condition that $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$. To be more explanatory, let us look at an example of a state:

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|0_A \otimes 1_B\rangle + |1_A \otimes 0_B\rangle). \quad (2.9)$$

Here $\alpha_{00} = \alpha_{11} = 0$ and $\alpha_{10} = \alpha_{01} = \frac{1}{\sqrt{2}}$, so $\alpha_{00}\alpha_{11} \neq \alpha_{01}\alpha_{10}$. We also define the tensor product $M_A \otimes M_B$ of two operators which correspond to observables in the vector space H_A and H_B respectively,

$$[M_A \otimes M_B]_{i_A p_B; j_A q_B} = [M_A]_{i_A j_A} [M_B]_{p_B q_B}. \quad (2.10)$$

As an example, we can give the tensor product of two 2×2 matrices which are

$$M_A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, M_B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

The matrix $M_A \otimes M_B$ is a 4×4 matrix as below:

$$M_A \otimes M_B = \begin{pmatrix} aM_B & bM_B \\ cM_B & dM_B \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta & b\alpha & b\beta \\ a\gamma & a\delta & b\gamma & b\delta \\ c\alpha & c\beta & d\alpha & d\beta \\ c\gamma & c\delta & d\gamma & d\delta \end{pmatrix}. \quad (2.11)$$

Until now we have introduced two-qubit states and their mathematical expression defined in the vector space, that is the tensor product of two spaces. These are the basic matheatical tools which allow us to construct quantum parallelism and quantum algorithms whose entries are the qubits, each of which is defined in its own space.

A two-qubit state which is not formed as $|\varphi_A \otimes \varphi_B\rangle$ is called an entangled state. The property of an entangled state is that the qubit A cannot be in a definite state. For example, in Equation 2.7 qubit A is the definite state $|\varphi_A\rangle$ so Equation 2.7 cannot be an entangled state. To give a more explanatory example, let M be a physical property of the qubit A. To express this operator in the space $H_A \otimes H_B$, we need to represent it by $M \otimes I$. We calculate the expectation value of M in the state $|\Phi\rangle$ as

$$\begin{aligned} M_{ave} &= \langle\Phi| M |\Phi\rangle \\ &= \frac{1}{2}(\langle 0_A \otimes 1_B| + \langle 1_A \otimes 0_B|)(|MO_A \otimes 1_B\rangle + |M1_A \otimes 0_B\rangle) \\ &= \frac{1}{2}(\langle 0_A| M |0_A\rangle + \langle 1_A| M |1_A\rangle), \end{aligned} \tag{2.12}$$

where we have used

$$\langle 0_B | 0_B\rangle = \langle 1_B | 1_B\rangle = 1, \langle 0_B | 1_B\rangle = \langle 1_B | 0_B\rangle = 0.$$

It can be shown that there is no state

$$|\varphi_A\rangle = \lambda |0_A\rangle + \mu |1_A\rangle \tag{2.13}$$

such that

$$\langle\Phi| M |\Phi\rangle = \langle\varphi_A| M |\varphi_A\rangle .$$

By computing the equation above, we obtain

$$\langle\varphi_A|\, M\, |\varphi_A\rangle = |\lambda|^2 \langle 0_A|\, M\, |0_A\rangle + \lambda^*\mu \langle 0_A|\, M\, |1_A\rangle + \lambda\mu^* \langle 1_A|\, M\, |0_A\rangle + |\mu|^2 \langle 1_A|\, M\, |1_A\rangle . \tag{2.14}$$

It is obvious that $\langle\Phi|\, M\, |\Phi\rangle \neq \langle\varphi_A|\, M\, |\varphi_A\rangle$. Even though we give the value $\frac{1}{\sqrt{2}}$ to λ and μ, the terms with $\lambda^*\mu$ and $\lambda\mu^*$ do not vanish. We deduce from this result that qubit A defined in the state $|\Phi\rangle$ is an incoherent mixture of $|0_A\rangle$ and $|1_A\rangle$, but not a linear superposition of these two states as in the state $|\varphi_A\rangle$.

2.2. The quantum no-cloning theorem

For further use, we should not contradict the quantum no-cloning theorem [1, 2] while constructing our quantum algorithms. Not coming into conflict with this theorem is important. The importance of this theorem is that, by this theorem, the impossibbility of someone stealing information without being detected is guaranteed. This is one of the fundamental properties of quantum information processing attracting many researchers. While constructing our circiuts for the fusion of W-state networks, we will be counting on this property providing us safe networks. Let us demonstrate the theorem by introducing a universal unitary operator whose action is as below:

$$U\, |X_1 \otimes \varphi\rangle = |X_1 \otimes X_1\rangle . \tag{2.15}$$

Here, X_1 is the original qubit we wish to clone and φ is a blank page on which we wish to put the copy of X_1. The reason why we called our operator universal is that its action does not depend on X_1, and it must clone any qubit that can be defined on the Hilbert space. Note that the one who wants to do copying does not know the original state and that is why one wants to duplicate it. Now, let us introduce another copying

action:

$$U\left|X_2 \otimes \varphi\right\rangle = \left|X_2 \otimes X_2\right\rangle. \tag{2.16}$$

Let us now calculate the scalar product of $|X_1 \otimes X_1\rangle$ and $|X_2 \otimes X_2\rangle$ in two different ways:

$$X = \langle X_1 \otimes X_1 \,|\, X_2 \otimes X_2\rangle = (\langle X_1 \,|\, X_2\rangle)^2, \tag{2.17}$$

$$X = \langle X_1 \otimes \varphi|\, U^\dagger U \,|X_2 \otimes \varphi\rangle = \langle X_1 \,|\, X_2\rangle. \tag{2.18}$$

For $X_1 = X_2$, the last two results become 1. If they are orthogonal to each other, both results become 0. What if X_2 is not a basis state vector and it is a superposition of two orthogonal states, that are X_1 and X_3? We can write X_2 as

$$|X_2\rangle = a\,|X_1\rangle + b\,|X_3\rangle, \tag{2.19}$$

by taking the scalar product of the qubits X_2 and X_1, we get

$$\langle X_1 \,|\, X_2\rangle = a. \tag{2.20}$$

Let us again look at Equation 2.17 and Equation 2.18 after we put the result above,

$$X = \langle X_1 \otimes X_1 \,|\, X_2 \otimes X_2\rangle = a^2, \tag{2.21}$$

$$X = \langle X_1 \otimes \varphi|\, U^\dagger U \,|X_2 \otimes \varphi\rangle = a. \tag{2.22}$$

As we see, results are different unless a is equal to 1 or 0. We can conclude that if a method duplicates the qubit, it can duplicate other qubits that are orthogonal, but can

not duplicate a qubit which is not orthogonal. In the case a qubit is a superposition of two or more orthogonal states, one can not copy that qubit.

2.3. Decoherence

Let us look at the entangled state containing qubit A and qubit B below,

$$|\Psi\rangle = \lambda\,|0_A \otimes 0_B\rangle + \mu\,|1_A \otimes 1_B\rangle \tag{2.23}$$

and from this we can find the state matrix of qubit A by first calculating $|\Psi\rangle\langle\Psi|$,

$$\begin{aligned} |\Psi\rangle\langle\Psi| = {} & |\lambda|^2\,|0_A \otimes 0_B\rangle\langle 0_A \otimes 0_B| + \lambda\mu^*\,|0_A \otimes 0_B\rangle\langle 1_A \otimes 1_B| \\ & + \lambda^*\mu\,|1_A \otimes 1_B\rangle\langle 0_A \otimes 0_B| + |\mu|^2\,|1_A \otimes 1_B\rangle\langle 1_A \otimes 1_B|\,, \end{aligned} \tag{2.24}$$

and then the state matrix of qubit A is

$$Tr_B(|\Psi\rangle\langle\Psi|) = \langle 0_B\,|\,\Psi\rangle\langle\Psi\,|\,0_B\rangle + \langle 1_B\,|\,\Psi\rangle\langle\Psi\,|\,1_B\rangle = |\lambda|^2\,|0_A\rangle\langle 0_A| + |\mu|^2\,|1_A\rangle\langle 1_A|\,, \tag{2.25}$$

which can be written in the matrix form as

$$\rho_A = Tr_B(|\Psi\rangle\langle\Psi|) = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & |\mu|^2 \end{pmatrix}. \tag{2.26}$$

Note that λ and μ are complex numbers and information on the phases of these complex numbers seems to have been lost, and we see that the matrix above is a diagonal matrix in the basis $|0_A\rangle$, $|1_A\rangle$. Generalization of the preceding argument is the following theorem:

> "If a pair of states of the system of interest becomes correlated with mutually orthogonal states of another system, then all the phase coherence between the orthogonal states of the first system is lost [13]."

The fact, that all the phase coherence between the orthogonal states of the first system is lost, means *decoherence*. Often state matrices of states are nondiagonal meaning coherences, however here we have a matrix that is diagonal. To remark one thing, let us look at our state matrix in different basis. Let our basis be $|+_A\rangle$ and $|-_A\rangle$ which are in the form

$$|\pm_A\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle \pm |1_A\rangle).$$

Here we can write $|0_A\rangle$, $|1_A\rangle$, $|0_B\rangle$, $|1_B\rangle$ in terms of $|\pm_A\rangle$ and $|\pm_B\rangle$. They are as follows:

$$\begin{aligned}
|0_A\rangle &= \frac{1}{\sqrt{2}}(|+_A\rangle + |-_A\rangle), \\
|1_A\rangle &= \frac{1}{\sqrt{2}}(|+_A\rangle - |-_A\rangle) \\
|0_B\rangle &= \frac{1}{\sqrt{2}}(|+_B\rangle + |-_B\rangle), \\
|1_B\rangle &= \frac{1}{\sqrt{2}}(|+_B\rangle - |-_B\rangle).
\end{aligned}$$

After showing our qubit states in terms of new basis, we put them in Equation 2.23 and get our $|\Psi\rangle$ in the new form as

$$\begin{aligned}
|\Psi\rangle = \frac{\lambda}{2}(|+_A \otimes +_B\rangle + |+_A \otimes -_B\rangle + |-_A \otimes +_B\rangle + |-_A \otimes -_B\rangle) \\
+\frac{\mu}{2}(|+_A \otimes +_B\rangle - |+_A \otimes -_B\rangle - |-_A \otimes +_B\rangle + |-_A \otimes -_B\rangle).
\end{aligned} \tag{2.27}$$

After long calculations we reach our density(state) matrix $|\Psi\rangle\langle\Psi|$ and by taking its trace in qubit B, finally we find our density matrix of qubit A which was found in different basis previously,

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & |\lambda|^2 - |\mu|^2 \\ |\lambda|^2 - |\mu|^2 & 1 \end{pmatrix}. \tag{2.28}$$

So we expressed the same density matrix in different basis, by looking at its elements we see that phase information is lost. But, actually, it is locally lost. What does "locally

lost" mean? It means that if we measure the physical properties of qubit A only, this shows us that qubit A is in a state that is incoherent and affected by environment. Note that qubit B is rooted in the environment. On the other hand, joint properties of qubit A and qubit B are still dependent on phases of λ and μ. In this case, the qubits become entangled and are not isolated. Any state matrix of any qubit in this kind of systems is diagonal or almost diagonal. To compute perfectly, the qubits must be isolated and phase informations must be kept in state matrices. This is necessary for our fusion process since it fails if we get some other output as a resultant state.

What we deduce from all this is that if we want to have control over the operations of a quantum computer, it is crucial that the computer is immune to decoherence. In other words, the computer must be completely isolated from the environment.

The characteristic time which gives the time interval of the decay of coherence, and it is called the *decoherence time*. The decoherence time is inversely proportional to some positive power of the size of the system. If we have more qubits, the decoherence time gets smaller meaning that states become incoherent more easily. The following example is an explanatory one, and makes us understand the nature of decoherence better. Let us suppose interaction time is Δt and during this time interval interaction of a qubit with the environment has the following effect:

$$|0\rangle \rightarrow |0\rangle\,, |1\rangle \rightarrow -\,|1\rangle$$

with a probability $p = \Gamma\Delta t \ll 1$. As you can see, to make p equal to 1, we need Δt to be $\tau_D = \frac{1}{\Gamma}$. Here the states of environment are not written explicitly, but the important thing is that $|0\rangle$ and $|1\rangle$ do not become entangled with the states of environment. For example, if the qubit is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{2.29}$$

the state after interaction turns into

$$|\psi^{'}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{2.30}$$

with probability p. The state's relation here is broken and we get a different kind of relation after the effect of environment. This phenomenon is called *phase flip*. Let us look at the n-qubit state,

$$|\Psi_n\rangle = \frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle). \tag{2.31}$$

Note that the first and the second term of the equation are the tensor products of n qubits of 0 and n qubits of 1 respectively. To save some space, I omitted the symbol $\otimes$. After the interaction with environment, the state $|\Psi_n\rangle$ will be transformed into

$$|\Psi_n^{'}\rangle = \frac{1}{\sqrt{2}}(|00...0\rangle - |11...1\rangle). \tag{2.32}$$

We assumed that all of the qubits interact with environment without being affected by other qubits, and note that we get the above result due to the phase flip of only one qubit. The probability of phase flipping in Δt is

$$p_n = n\Gamma\Delta t. \tag{2.33}$$

See that our new probability formula is the old one multiplied by n. From this, we conclude that probability of phase flipping, in other words probability of change in the state will be higher by the factor of n. Therefore it is more likely for a state to interact with environment when there are more qubits. Why do we multiply p by n when there are n-qubit states? We said every qubit interacts with environment independently and when we increse the number of qubits, there will be more qubits to interact with the same probability. We should not forget that, in our case only qubit 1 interacts with environment. It is sufficient that only one of the qubits interacts with environment to cause phase flip as expressed in Equation 2.32. When there are n qubits of 1, the probability is multiplied by n because of the fact that any one of them can interact with

the probability p. Recall that we talked about decoherence time τ_D. What happens to the decoherence time here? To make p_n equal to 1, we need Δt to be $1/n\Gamma$. As we can see, the decoherence time is shorter and it is more likely for a state to be incoherent.

To give a much more explicit example, let us describe a model which is called the *phase damping channel*. In this model the state of the qubit does not change whereas the state of environment changes and causes decoherence. If environment, which is initially in the state $|0_E\rangle$, changes into $|1_E\rangle$ ($|2_E\rangle$) if the qubit is in the state $|0_A\rangle$ ($|1_A\rangle$). These interactions are shown as below:

$$|0_A 0_E\rangle \to \sqrt{1-p}\,|0_A 0_E\rangle + \sqrt{p}\,|0_A 1_E\rangle = |0_A\rangle \otimes \Big(\sqrt{1-p}\,|0_E\rangle + \sqrt{p}\,|1_E\rangle\Big),$$
$$|1_A 0_E\rangle \to \sqrt{1-p}\,|1_A 0_E\rangle + \sqrt{p}\,|1_A 2_E\rangle = |1_A\rangle \otimes \Big(\sqrt{1-p}\,|0_E\rangle + \sqrt{p}\,|2_E\rangle\Big).$$

We can give an example of the above interactions as photon scattering. A photon is scattered by a single qubit and the final state of photon depends on the state of the qubit. Let us also look at entanglement which was mentioned previously. In our model single qubits of 0 and 1 do not become entangled but, if we construct any linear combination of the states $|0_A\rangle$ and $|1_A\rangle$, it becomes entangled. The generalized initial state of qubit and environment is in the following equation:

$$|\Phi\rangle = \Big(\lambda\,|0_A\rangle + \mu\,|1_A\rangle\Big) \otimes |0_E\rangle\,, \tag{2.34}$$

so the density matrix of qubit A is

$$\rho_A = \begin{pmatrix} |\lambda|^2 & \lambda\mu^* \\ \lambda^*\mu & |\mu|^2 \end{pmatrix} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}. \tag{2.35}$$

Interactions defined between qubit A and environment can be represented by a unitary operator defined in the Hilbert space $H_A \otimes H_E$. Let us represent the action of this

operator on our state:

$$\begin{aligned} U|\Phi\rangle &= \lambda U|0_A 0_E\rangle + \mu U|1_A 0_E\rangle \\ &= \lambda\sqrt{1-p}\,|0_A 0_E\rangle + \lambda\sqrt{p}\,|0_A 1_E\rangle + \mu\sqrt{1-p}\,|1_A 0_E\rangle + \mu\sqrt{p}\,|1_A 2_E\rangle\,. \end{aligned} \tag{2.36}$$

This is our new state after interaction with environment. Our new density matrix of qubit A:

$$\begin{aligned} \rho_A^{'} &= Tr_E[U|\Phi\rangle\langle\Phi|U^\dagger] \\ &= |\lambda|^2|0_A\rangle\langle 0_A| + |\mu|^2|1_A\rangle\langle 1_A| + \lambda\mu^*(1-p)|0_A\rangle\langle 1_A| + \lambda^*\mu(1-p)|1_A\rangle\langle 0_A| \end{aligned} \tag{2.37}$$

or

$$\rho_A^{'} = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}. \tag{2.38}$$

After n interactions we find

$$\rho_A^{'} = \begin{pmatrix} \rho_{00} & (1-p)^n\rho_{01} \\ (1-p)^n\rho_{10} & \rho_{11} \end{pmatrix}. \tag{2.39}$$

If interactions between environment and qubits keep going on continuously for a time t for $n \to \infty$, we take $p = \Gamma t$ such that $t = n\Delta t$ and our density matrix becomes

$$\lim_{n\to\infty} \rho_A^{'} = \begin{pmatrix} \rho_{00} & \rho_{01}e^{-\Gamma t} \\ \rho_{10}e^{-\Gamma t} & \rho_{11} \end{pmatrix}. \tag{2.40}$$

Observe that

$$\rho_{01}(t) = \rho_{01}(1 - \Gamma\Delta t)^{t/\Delta t} \to \rho_{01}e^{-\Gamma t} \tag{2.41}$$

when $\Delta t \to 0$. It can easily be seen that our density matrix of qubit A decay into a diagonal matrix meaning the qubit A becomes incoherent mixture of qubit 0 and qubit 1:

$$\rho(t) \to \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}, t \to \infty. \tag{2.42}$$

What can we conclude from these explicitly shown calculations? We mentioned that phase informations are lost in a density matrix when a qubit interacts with environment and becomes incoherent.

There are some observations made about the decoherence phenomenon in brain processes. Some illuminating results have been reached for the neural firings providing information transmission throughout the neural systems [15, 16]. Calculations by Tegmark showed that timescale of quantum phenomena is so small such that it can be thought of as noise.

Specifically, Tegmark estimated that the decoherence time of quantum effects while transmitting information along neurons is 10^{-20} sec at brain temperature (~ 310 K). We know that human brain is warm and wet and that makes decoherence time much shorter making impossible to build our quantum algorithms into the human neural system. Recall that our decoherence time gets shorter by the factor of n when we increase qubit number to n. In contrast, the time scale of the fastest neuron firing is in the order of 10^{-3} seconds. In conclusion, neuron firings are not long enough so that quantum effects behave as noise in information processing. For a neural microtubule, which is a tiny protein structure within a neuron and thought to be isolated for quantum coherence and computation [17–19], Tegmark obtained the decoherence time

10^{-13} sec. This emprical result led us to the fact that microtubules are not isolated enough and quantum computation can not be implemented in that protein structure. Additionally, arguing that microtubules offer isolation for quantum coherences lack any emprical support [20]. The effects of quantum-level phenomenon are ruled out by the large timescale discrepancies described [15, 16]. The results Tegmark obtained are well summarized in the paper by LITT and others [20].

Let us summarize what we did so far in this section. We described decoherence mathematically by defining the state matrix of a qubit and constructed a space where environment and qubits are defined. Then we introduced an explicit example of decoherence by specifically defining interactions between our qubits and the qubits from environment with our mathematical tools. We also showed that in an infinite timescale every qubit becomes incoherent in our example mathematically. Lastly, to make decoherence phenomenon understandable in the physical reality, we looked at some results on decoherence in the human brain process. Decoherence is a crucial concept such that only without it we will be able to operate in quantum algorithms. That is why we emphasised quantum decoherence strongly.

You can look at the good instructions of two-qubit states, the quantum no-cloning theorem and decoherence in the Le Bellac's textbook [13].

3. QUANTUM COMPUTATION

3.1. General Remarks

To represent N numbers in classical bits, we need n bits such that $2^n \geq N$. For example, if we want to represent eight numbers, we need three bits. Let our numbers be 0, 1, ..., 7 and they can be represented in the system of base two,

$$0 = (000), 1 = (001), 2 = (010), 3 = (011),$$
$$4 = (100), 5 = (101), 6 = (110), 7 = (111).$$

As in the case of classical bits, we can store eight numbers by using three bits but this time our bits are qubits. Their configuration is in the following:

$$0 = |000\rangle, 1 = |001\rangle, 2 = |010\rangle, 3 = |011\rangle,$$
$$4 = |100\rangle, 5 = |101\rangle, 6 = |110\rangle, 7 = |111\rangle.$$

See that we have omitted tensor product symbol $\otimes$ to save space. For example, $|000\rangle$ is abbreviated notation for $|0_A \otimes 0_B \otimes 0_C\rangle$ where the qubits A, B and C are the states defined in their own Hilbert spaces that are H_A, H_B and H_C. Three-qubit states are defined in the space which is the tensor product of the three spaces, and denoted by $H_A \otimes H_B \otimes H_C$. It is easy to generalize three-qubit states to n-qubit states defined in $H^{\otimes n}$ in which there are 2^n states denoted by $|x\rangle$ where

$$0 \leq x \leq 2^n - 1. \tag{3.1}$$

The importance of $|x\rangle$ is that they are our computational basis on which we construct our quantum algorithms and define operations affecting qubits for a specific purpose. By the way, we can use the spins of electrons as the physical support of qubits as we

can also use the polarization of light as mentioned before. We know that the spin states are defined in matrices as below:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.2}$$

When we write, for example, $|000\rangle$, it is simply the tensor product of these three states,

$$|000\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ : \\ 0 \end{pmatrix}. \tag{3.3}$$

The result is a 8×1 matrix. Let us consider a state which is a superposition of eight states in three qubits. When we measure the state, we get only one of the eight states and not all of the eight states. One can ask what the advantage of quantum algorithms is, therefore we need to go further to exploit its possibilities. We will see them while constructing our quantum algorithms. In this section, let us just show the schematic depiction of the principle for a quantum computer's function.

Calculation and operation are performed on quantum algorithms as shown in figure 3.1 where n qubits are prepared in the state $|0\rangle$ at time $t = t_0$. The initial states prepared belong to a Hilbert space of 2^n dimensions, $H^{\otimes n}$. The initial states undergo quantum evolution described by a unitary operator $U(t, t_0)$ that acts in $H^{\otimes n}$. This operation is done for a desired purpose, for example, to determine whether a function is balanced or not. As mentioned in the section of decoherence we have to avoid any interaction with environment, otherwise states that are to be calculated become incoherent giving different results. For the phenomenon of decoherence to occur, it is necessary that isolation of any quantum algorithm is not good. Once the operations acting on qubits have been done, a measurement is made at time t in order to obtain the result. We have to note that operations are done between t_0 and t and our unitary operator takes our states from t_0 to t by evaluating them through the process. The

time interval between t_0 and t must be short enough such that any interaction with environment can be avoided. We do measurements after the qubits are operated by $U(t, t_0)$, and to avoid the destruction of states we can not do any measurement while states are being evaluated by our algorithm. States are only measured at the entrance and the exit of the box shown in Figure 3.1. It is also important that these operations are reversible and the inverse of U can be shown as

$$U^{-1}(t, t_0) = U(t_0, t). \tag{3.4}$$

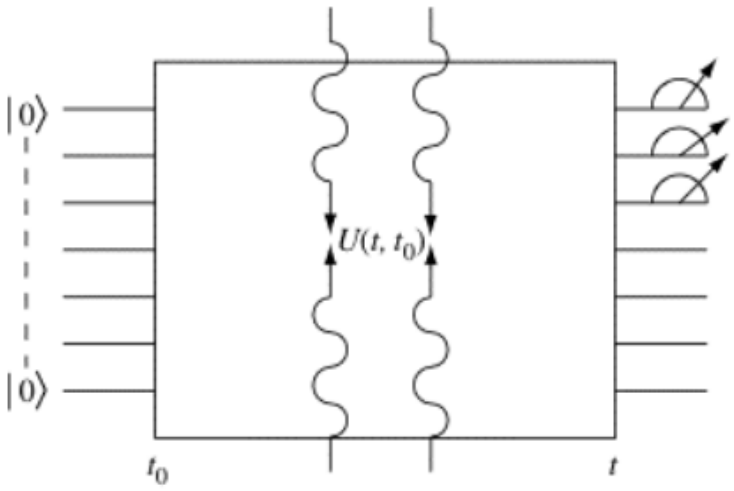


Figure 3.1. Schematic depiction of a quantum calculation. The arrows represent the operations manipulating qubits and diagrams are read from left to right. This figure is taken from [13].

3.2. Reversibility of Quantum Operations

The process of manipulating the initial qubits at $t = t_0$ to turn them into the final states at time t must be necessarily reversible. This is not the case for classical algorithms whose operations are not reversible. This is because of the fact that these operations correspond to mechanisms transforming 2 bits to 1 bit. This makes reversible calculation impossible due to energy dissipation and increase in entropy. Our

example gate is the NAND gate denoted by $\uparrow$,

$$x \uparrow y = 1 \oplus xy, \tag{3.5}$$

where $\oplus$ is mod 2 addition, and transformations are as below:

$$(00) \to 1, (01) \to 1, (10) \to 1, (11) \to 0. \tag{3.6}$$

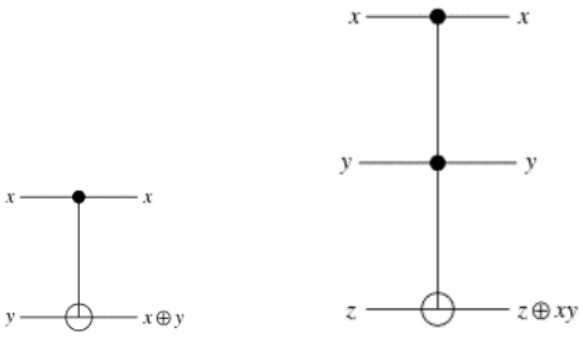
As it is seen, the knowledge of the final state does not allow us to reconstruct the initial state. We should note that the NAND gate and the COPY operation are the tools needed to construct any logic circuit. The interesting question is whether all the operations performed on a classical computer are irreversible or not. The question was raised by Landauer and Bennett who wondered if it were possible to do operations without energy dissipation [21–23]. All these operations are information processings and in spite of its abstract nature, information is carried by some physical support. Otherwise, what we do remains as mathematics, not physics! Bennett obtained a satisfactory solution of the paradox of the Maxwell demon [21]. According to Landauer, the loss of a bit in a NAND operation costs a thermodynamical entropy of $k_B ln2$ per bit, where k_B is the Boltzmann constant [22]. This leads to the energy dissipation of an energy $\Delta E = k_B T ln2$ into environment. T is the absolute temperature of computer. This problem remains theoretical for now due to high energy consumption of electricity. For a PC, the energy dissipated per erased bit is $\Delta E \sim 500 k_B T$. However, in the future, this will hopefully be a more practical problem thanks to improvements in the efficiency of the usage of computers.

What we try to do here is constructing the same operations, that can be classically performed in an irreversible way, on our quantum algorithms such that they are reversible. It is possible to do the COPY operation and the NAND operation on quantum algorithms aiming at the same tasks. To do the COPY operation, we should not come into conflict with the no-cloning theorem. Let us introduce our new gates equivalent to the quantum version of the COPY and the NAND operations respectively: the *control-NOT* gate and the *Toffoli* gate (Figure 3.2a and 3.2b). Let (x,y)

be the bits entering cNOT gate and x is called the control bit whereas y is called the target bit. The operation is as follows:

$$cNOT : |x, y\rangle \rightarrow |x, x \oplus y\rangle \, . \tag{3.7}$$

Recall that $\oplus$ is mod 2 addition. In this case, y gets only the values of 1 and 0. If y is 0, then the operation copies the bit x meaning this is the quantum version of the COPY operation. If y is 1, the second bit coming out of operation is certainly different from x. For example, if x and y are both 1, $x \oplus y = 0$. The cNOT operation is a permutation of the basis vectors. Transformations of the cNOT gate are



(a) The cNOT gate (b) The Toffoli gate

Figure 3.2. Black points represent control bits and circles represent target bits. These figures are taken from [13].

$$|00\rangle \rightarrow |00\rangle \, , \quad |01\rangle \rightarrow |01\rangle \, , \quad |10\rangle \rightarrow |11\rangle \, , \quad |11\rangle \rightarrow |10\rangle \, . \tag{3.8}$$

This operation is reversible due to one-to-one correspondence. We should also analyze the Toffoli gate which has three entering bits two of which are control bits and other one is the target bit. Let (x,y) be our control bits and the transformation of the Toffoli gate is

$$|x, y, z\rangle \rightarrow |x, y, z \oplus xy\rangle \, . \tag{3.9}$$

We already mentioned the NAND gate and its transformation,

$$x \uparrow y = 1 \oplus xy.$$

When z is equal to 1, the Toffoli gate acts like the NAND gate. In this case, the difference between them is that the Toffoli operation is reversible. Let us show below that it is also one-to-one operation meaning the operation is reversible, we will do it by giving all possible values to x, y and z. All possible transformations are as below:

$$|0,0,0\rangle \to |0,0,0\rangle\,, \quad |0,0,1\rangle \to |0,0,1\rangle\,, \quad |0,1,0\rangle \to |0,1,0\rangle\,, \quad |1,0,0\rangle \to |1,0,0\rangle\,,$$
$$|1,0,1\rangle \to |1,0,1\rangle\,, \quad |0,1,1\rangle \to |0,1,1\rangle\,, \quad |1,1,0\rangle \to |1,1,1\rangle\,, \quad |1,1,1\rangle \to |1,1,0\rangle\,.$$

In conclusion, the cNOT gate and the Toffoli gate act like reversible COPY and NAND operations respectively. By using these useful tools, we will be constructing our quantum algorithms for some purposes that are to be discussed later.

3.3. Quantum Gates

The responsible operator for quantum evolution is a unitary matrix operating on qubits. For a n-qubit system, we need to express these operators in 2^n dimensional Hilbert space $H^{\otimes n}$. Any operator defined in a Hilbert space must be unitary and our gates, that are operators, must be $2^n \times 2^n$ unitary matrices. Here, I would like to mention a theorem which allows us to limit our cases to one-qubit and two-qubit systems. This theorem says that we can decompose any unitary transformation on $H^{\otimes n}$ into a product of cNOT gates and unitary transformations on one qubit [13].

It should be noted that operations must be generalized to two-qubit systems in order to obtain generalized unitary transformations. The theorem mentioned above guarantees that this is suffient for obtaining generalized transformations. This theorem is called the existence theorem, however it is not always necessary to use this theorem explicitly to solve a given problem. We already looked at cNOT gate transformations

and, now let me write the gate as a matrix for a 4 dimensional Hilbert space,

$$cNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix}. \tag{3.10}$$

Let us check by looking at transformation of each qubit. Transformations are already given in Equation 3.8. To show explicitly, let us write each qubit in matrix form and then calculate their matrix multiplication by cNOT gate:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.11}$$

by doing the other calculations in the same way we find

$$|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3.12}$$

now let us multiply $|00\rangle$ with the cNOT operator,

$$cNOT\,|00\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.13}$$

It can be easily seen that $cNOT\,|01\rangle = |01\rangle$, and the other two qubits' transformations

are as follows:

$$cNOT\,|10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3.14}$$

$$cNOT\,|10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{3.15}$$

So it is checked that our matrix representation of cNOT gate is correct. If we would like to generalize cNOT gate, we replace σ_x by a unitary operator U in matrix representation and it is called cU gate:

$$cU = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}. \tag{3.16}$$

As in the case of cNOT operation, qubit is left unchanged if control bit is 0. Otherwise, target bit is transformed as such $|y\rangle \rightarrow U\,|y\rangle$. To construct a matrix providing above conditions generally, we define U as

$$U = C\sigma_x B\sigma_x A \tag{3.17}$$

where

$$CBA = I. \tag{3.18}$$

cU and cNOT gates are sufficient tools to construct the Toffoli gate where $U =$

$\sqrt{\sigma_x}$ which is

$$\sqrt{\sigma_x} = \frac{1}{1+i}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{3.19}$$

that is not possible to define mathematically on classical algorithms. We can see the schematic depictions of cU gate and the Toffoli gate in figure 3.3. Let us show the calculation of the lower circuit of figure 3.3 by showing each cU and cNOT gate explic-

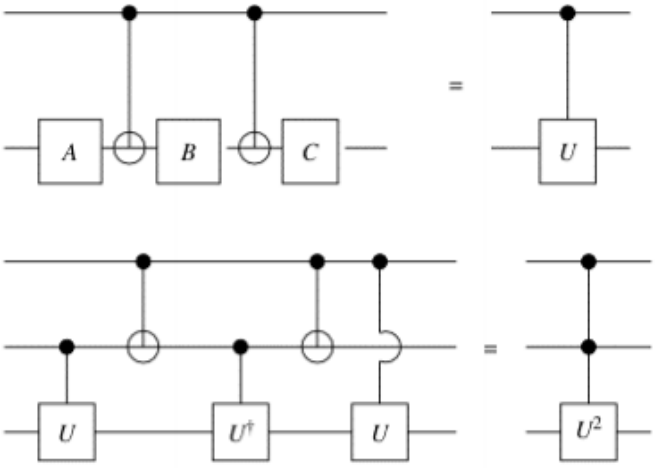


Figure 3.3. The upper circuit is the cU gate and the lower one is the Toffoli gate. These figures are taken from [13].

itly and then doing matrix multiplications just before we show qubits' transformations. Finally, we will calculate the transformations of all of the eight 3-qubit states defined in 8 dimensional Hilbert space. First we write all matrices together to define whole process,

$$c_1 U_3 c_1 NOT_2 c_2 U_3^\dagger c_1 NOT_2 c_2 U_3 \left|\Psi\right\rangle = \left|\Psi_5\right\rangle \tag{3.20}$$

where $\left|\Psi\right\rangle$ is the most general state to be transformed,

$$\left|\Psi\right\rangle = a\left|000\right\rangle + b\left|001\right\rangle + c\left|010\right\rangle + d\left|011\right\rangle + e\left|100\right\rangle + f\left|101\right\rangle + g\left|110\right\rangle + h\left|111\right\rangle. \tag{3.21}$$

Recall that each state is a basis state vector corresponding to a dimension in our Hilbert

space. We mentioned that defined U matrix in our cU gate must be a unitary matrix

$$U = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \tag{3.22}$$

and its product with its hermitian conjugate

$$U^{\dagger}U = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.23}$$

providing the below conditions that

$$|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1, \quad \alpha^*\gamma + \beta^*\delta = \gamma^*\alpha + \delta^*\beta = 0. \tag{3.24}$$

In this case, our cNOT gate is a 4×4 matrix in which σ_x is replaced with U,

$$cNOT = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}. \tag{3.25}$$

Let us construct our cU and cNOT gates, but we should not forget that all operations

are to be done in 8 dimensional Hilbert space $H^{\otimes 3}$:

$$c_2U_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & U \end{pmatrix}, \tag{3.26}$$

$$c_1NOT_2 = \begin{pmatrix} I & 0 \\ 0 & \sigma_x \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix}, \tag{3.27}$$

$$c_2U_3^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & U^\dagger \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & U^\dagger & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & U^\dagger \end{pmatrix}. \tag{3.28}$$

Let us give our intermediate result which is the state after four of the operators are implemented,

$$c_1NOT_2c_2U_3^\dagger c_1NOT_2c_2U_3 \left|\Psi\right\rangle = \left|\Psi_4\right\rangle \tag{3.29}$$

which can be represented in matrix form as

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & U^\dagger & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & U^\dagger \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & U \end{pmatrix} \left|\Psi\right\rangle = \left|\Psi_4\right\rangle. \tag{3.30}$$

It should be noted here that our gates are 8×8 matrices where I and U are 2×2

matrices. The multiplication of these matrices simplify the equation above:

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & U^\dagger & 0 \\ 0 & 0 & 0 & U \end{pmatrix} |\Psi\rangle = |\Psi_4\rangle \tag{3.31}$$

and

$$\begin{aligned} |\Psi_4\rangle = a\,|000\rangle + b\,|001\rangle + c\,|010\rangle + d\,|011\rangle + (\alpha^* e + \beta^* f)\,|100\rangle \\ +(\gamma^* e + \delta^* f)\,|101\rangle + (\alpha g + \gamma h)\,|110\rangle + (\beta g + \delta h)\,|111\rangle\,. \end{aligned}$$

To find our final state we implement c_1U_3 on $|\Psi_4\rangle$,

$$\begin{aligned} c_1U_3\,|\Psi_4\rangle = ac_1U_3\,|000\rangle + bc_1U_3\,|001\rangle + cc_1U_3\,|010\rangle + dc_1U_3\,|011\rangle + (\alpha^* e + \beta^* f)c_1U_3\,|100\rangle \\ +(\gamma^* e + \delta^* f)c_1U_3\,|101\rangle + (\alpha g + \gamma h)c_1U_3\,|110\rangle + (\beta g + \delta h)c_1U_3\,|111\rangle\,. \end{aligned}$$

The first four terms are not affected by the operator just because the control bits are 0 in those terms. The other four terms are operated as below:

$$\begin{aligned} c_1U_3 \quad \Big((\alpha^* e + \beta^* f)\,|100\rangle + (\gamma^* e + \delta^* f)\,|101\rangle + (\alpha g + \gamma h)\,|110\rangle + (\beta g + \delta h)\,|111\rangle\Big) \\ + unaffected \;\; terms = \Big(\alpha(\alpha^* e + \beta^* f) + \gamma(\gamma^* e + \delta^* f)\Big)\,|100\rangle + \Big(\beta(\alpha^* e \\ + \beta^* f) + \delta(\gamma^* e + \delta^* f)\Big)\,|101\rangle + \Big(\alpha(\alpha g + \gamma h) + \gamma(\beta g + \delta h)\Big)\,|110\rangle \\ + \Big(\beta(\alpha g + \gamma h) + \delta(\beta g + \delta h)\Big)\,|111\rangle + unaffected \;\; terms \end{aligned}$$

which results in

$$\begin{aligned}|\Psi_{final}\rangle = &\left(|\alpha|^2 e + \alpha\beta^* f + |\gamma|^2 e + \gamma\delta^* f\right)|100\rangle \\ &+ \left(\beta\alpha^* e + |\beta|^2 f + \delta\gamma^* e + |\delta|^2 f\right)|101\rangle + \left(\alpha^2 g + \alpha\gamma h + \gamma\beta g + \gamma\delta h\right)|110\rangle \\ &+ \left(\beta\alpha g + \beta\gamma h + \delta\beta g + \delta^2 h\right)|111\rangle + unaffected \;\; terms.\end{aligned}$$

We already know that $|\alpha|^2 + |\gamma|^2 = |\beta|^2 + |\delta|^2 = 1$ and also $\alpha\beta^* + \gamma\delta^* = \alpha\gamma^* + \beta\delta^* = 0$, so the above equation simplifies to

$$\begin{aligned}|\Psi_{final}\rangle = e\,|100\rangle + f\,|101\rangle + \Big(g(\alpha^2 + \gamma\beta) + h(\alpha\gamma + \gamma\delta)\Big)|110\rangle \\ + \Big(g(\beta\alpha + \delta\beta) + h(\beta\gamma + \delta^2)\Big)|111\rangle + other \;\; terms.\end{aligned} \tag{3.32}$$

We can also calculate U^2,

$$U^2 = \begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix}\begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix} = \begin{pmatrix}\alpha^2 + \gamma\beta & \alpha\gamma + \gamma\delta \\ \beta\alpha + \delta\beta & \beta\gamma + \delta^2\end{pmatrix}. \tag{3.33}$$

It can be easily seen that our final state can be expressed with the elements of U^2:

$$\begin{aligned}|\Psi_{final}\rangle = a\,|000\rangle + b\,|001\rangle + c\,|010\rangle + d\,|011\rangle + e\,|100\rangle + f\,|101\rangle \\ + (gU_{11}^2 + hU_{12}^2)\,|110\rangle + (gU_{21}^2 + hU_{22}^2)\,|111\rangle\,.\end{aligned} \tag{3.34}$$

Now look at the lower circuit in figure 3.3 and see that there are two control bits and one target bit. We already know the transformation of the Toffoli gate which is

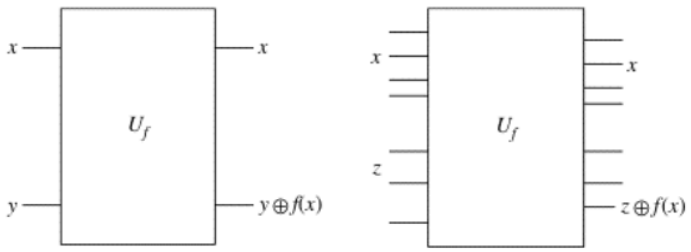
$$|x, y, z\rangle \rightarrow |x, y, z \oplus xy\rangle\,,$$

and the condition that both of the control bits must be one must be provided to have changed states. Our final state tells us our circuit is precisely working as the Toffoli gate. This is a good example to show how we use cU and cNOT gates to construct our quantum algorithms serving similar purposes as the classical ones do, but in a much more effective and faster way. These two gates are the most basic tools to construct algorithms, specifically for our case in which we fuse W states.

Let us now begin to state the basic ideas of *quantum parallelism* by introducing the operation U_f (Figure 3.4a) doing transformations as below:

$$(x, y) \to (x, y \oplus f(x)), \tag{3.35}$$

where f(x) is a function of only x. The above transformation can be generalized to more qubits but now we will look at the simplest case. This simplest case contains two-qubit states as you can see. There are also generalized systems containing $n + m$



(a) U_f for two qubits (b) U_f for the generalized case

Figure 3.4. The schematic depictions of U_f. These figures are taken from [13].

qubits as you can see in Figure 3.4b. For $y = 0$, transformation is simply

$$(x, 0) \to (x, f(x)). \tag{3.36}$$

Here, we are using two qubits so that our transformation U_f can be a unitary matrix. Otherwise, using the transformation $x \to f(x)$ would be sufficient but it is not always true that this is one-to-one correspondence. In other words, by using two qubits we get a one-to-one transformation. Recall that one-to-one correspondence means it is a reversible operation which is suitable for quantum algorithm. If we implement U_f on

a state twice, we get the initial state meaning $U_f^2 = I$:

$$(x, y) \to (x, y \oplus f(x)) \to (x, y \oplus f(x) \oplus f(x)) \tag{3.37}$$

where $\oplus$ is mod 2 addition and $f(x) \oplus f(x) = 0$. Finally we get the initial state that is (x, y). If $U_f^2 = I$, U_f is a permutation of the four basis vectors so U_f is a unitary matrix. We know that U_f is a unitary matrix, and from this we can deduce U_f is a hermitian matrix as well,

$$U_f U_f = U_f^\dagger U_f = I. \tag{3.38}$$

Let us start using ket notation and express U_f as a quantum operator,

$$U_f \left|x \otimes y\right\rangle = \left|x \otimes y \oplus f(x)\right\rangle, \quad U_f \left|x \otimes 0\right\rangle = \left|x \otimes f(x)\right\rangle. \tag{3.39}$$

Let us introduce the Hadamard gate that is important for our algorithmic calculations:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{3.40}$$

if we apply the Hadamard gate on the states $|0\rangle$ and $|1\rangle$, we get

$$H \left|0\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle), \quad H \left|1\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle). \tag{3.41}$$

These operations will be used as the parts of quantum algorithmic circuits. Let us give an example as an exercise. In a two-qubit system, let both of the qubits be $|0\rangle$. If we apply a Hadamard gate only on the first qubit, we get

$$(H \otimes I) \left|0 \otimes 0\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle) \otimes \left|0\right\rangle = \frac{1}{\sqrt{2}}(\left|0 \otimes 0\right\rangle + \left|1 \otimes 0\right\rangle) \tag{3.42}$$

where the applied operator is denoted by $H \otimes I$ in a two-qubit system. Physically, a Hadamard gate can be implemented by using a half-wave plate(HWP) that is an optical device that alters the polarization state of a light wave travelling through it. This phenomenon is well presented in the sections on the reversibility of quantum operations and the quantum gates in Le Bellac's textbook [13].

4. QUANTUM ALGORITHMS

In this section, we will discuss two quantum algorithms, the Deutsch Algorithm and the Grover Search Algorithm, to show the importance of quantum information processing. We will see, by using less operations than classical algorithms, we can accomplish some tasks.

4.1. The Deutsch Algorithm

The remarkable step forward in quantum information processing was taken by David Deutsch in 1985 [4]. The Deutsch algorithm is a very basic but an explanatory algorithm to understand the effectiveness of quantum information processing where we use less operations to gain information on if a function f(x) is constant or balanced. Before getting into this, let us again look at Equation 3.42 and apply U_f on that state,

$$U_f \left| H0 \otimes 0 \right\rangle = \frac{1}{\sqrt{2}}(\left| 0 \otimes f(0) \right\rangle + \left| 1 \otimes f(1) \right\rangle). \tag{4.1}$$

As we can see, information on f(0) and f(1) is carried simultaneously by the state. However, this superposition does not give any advantage over a classical algorithm if we would like to construct a table of f(x) for each value on x. As mentioned in the beginning of this section, we can have an advantageous situation if our purpose is to find if f(x) is constant or balanced. By the way, "balanced" means that, in a certain interval of a variable, a function of this variable gets 0 and 1 for equal number of times in base 2. We will explain this more explicitly in the succeeding paragraphs. Our variable x can be defined in base 10 as well as f(x), but in our algorithmic calculations function will be transformed to value in base 2. For example, in 8-dim Hilbert space we define 8 basis state vectors one of which is $\left| 101 \right\rangle$ corresponding 5 in base 10, but f(5) will be considered in base 2. Therefore, for a n-qubit system,

$$x \in [0, 2^n - 1], \quad f(x) \in [0, 1], \tag{4.2}$$

there are two cases in which $f(0) = f(1)$ or $f(0) \neq f(1)$. For the simplest case, we take $n = 1$, hence $x \in (0, 1)$. The case where $n = 1$ is called the Deutsch Algorithm. If $f(0) = f(1)$, it means that f(x) is constant, otherwise f(x) is a balanced function taking two values 0 and 1. Observe that f(x) takes one 0 and one 1. We should notice that we do not know the values of f(0) and f(1). What we try to do here is exploiting information on whether f(x) is constant or balanced.

Before doing calculations, let us consider the classical case where we want to decide if f(x) is constant or balanced. In this case, it is obvious that we need to calculate f(0) and f(1) seperately, which means we do two operations. On the contrary, we need only one operation in quantum algorithms. We can compare this with checking the two sides of a coin to see if the sides are same(two heads or two tails) or different(one head and one tail). These are the equivalent problems and the remarkable thing is that we do not need to look at the two sides of a coin to realize it has same or different sides.

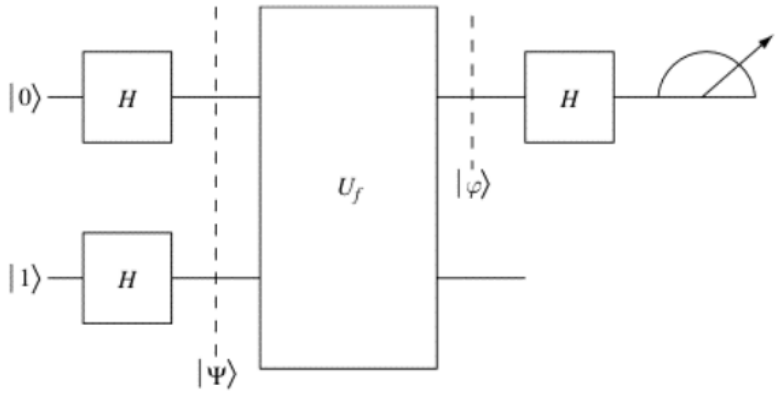


Figure 4.1. The Deutsch Algorithm. This figure is taken from [13].

When we look at figure 4.1, we see there are two input qubits $|0\rangle$ and $|1\rangle$ prepared initially. Some sources call $|0\rangle$ as an input register and $|1\rangle$ as an output register but I prefer calling both of them as input registers due to the fact that process is going from left to right and the final state is attained after the final Hadamard gate is applied. We have initial states on which Hadamard gates are applied giving us $|\Psi\rangle$,

$$|\Psi\rangle = (H|0\rangle) \otimes (H|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{4.3}$$

which can be shown as

$$|\Psi\rangle = \frac{1}{2}\Big(\sum_{x=0}^{1}|x\rangle\Big) \otimes (|0\rangle - |1\rangle). \tag{4.4}$$

Let us do tensor product and expand $|\Psi\rangle$:

$$|\Psi\rangle = \frac{1}{2}(|0\otimes 0\rangle - |0\otimes 1\rangle + |1\otimes 0\rangle - |1\otimes 1\rangle). \tag{4.5}$$

After we get $|\Psi\rangle$, we apply U_f on it as depicted schematically in a box in Figure 4.1,

$$U_f\,|\Psi\rangle = \frac{1}{2}\Big(\,|0\otimes f(0)\rangle - |0\otimes(1\oplus f(0))\rangle + |1\otimes f(1)\rangle - |1\otimes(1\oplus f(1))\rangle\,\Big). \tag{4.6}$$

Notice that we have not decided what $f(x)$ is yet and we only wrote the general transformation formula of U_f that was mentioned earlier. We can talk about two cases $f(x)$ is either constant or not. Let us first look at the case our function is constant which means it can take only two values 0 and 1 in base 2. For $f(x) = 0$, the equation above remains unchanged:

$$U_f\,|\Psi\rangle = |\Psi\rangle\,. \tag{4.7}$$

If $f(x) = 1$, then the eigenvalue of this eigenfunction is -1,

$$U_f\,|\Psi\rangle = -\,|\Psi\rangle\,. \tag{4.8}$$

We can express the last two equations as

$$U_f\,|\Psi\rangle = \pm\,|\Psi\rangle = \pm\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{4.9}$$

for $f(x) = 0$ or 1. We can express the same equation in a general form,

$$U_f\,|\Psi\rangle = \frac{1}{\sqrt{2}}((-1)^{f(0)}\,|0\rangle + (-1)^{f(1)}\,|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{4.10}$$

Before implementing the last Hadamard gate on this state, let us reach the same general form for the case $f(x)$ is not constant.

What if $f(x)$ is not constant? In base 2, we know a function can take only two values 0 and 1 as mentioned before. For the case $f(x)$ is not constant, there are two possibilities:

$$f(0) = 0, \quad f(1) = 1 \quad or \quad f(0) = 1, \quad f(1) = 0.$$

Let us first consider the case $f(0) = 0$ and $f(1) = 1$, in this case Equation 4.6 becomes

$$U_f \left|\Psi\right\rangle = \frac{1}{2}\Big(\left|0 \otimes 0\right\rangle - \left|0 \otimes 1\right\rangle + \left|1 \otimes 1\right\rangle - \left|1 \otimes 0\right\rangle \Big). \tag{4.11}$$

This can be written as

$$\begin{aligned} U_f \left|\Psi\right\rangle &= \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle) \otimes \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle) \\ &= \frac{1}{\sqrt{2}}((-1)^{f(0)} \left|0\right\rangle + (-1)^{f(1)} \left|1\right\rangle) \otimes \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle). \end{aligned} \tag{4.12}$$

For the case $f(0) = 1$ and $f(1) = 0$,

$$\begin{aligned} U_f \left|\Psi\right\rangle &= \frac{1}{2}\Big(\left|0 \otimes 1\right\rangle - \left|0 \otimes 0\right\rangle - \left|1 \otimes 1\right\rangle + \left|1 \otimes 0\right\rangle \Big) \\ &= \frac{1}{\sqrt{2}}(-\left|0\right\rangle + \left|1\right\rangle) \otimes \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle) \\ &= \frac{1}{\sqrt{2}}((-1)^{f(0)} \left|0\right\rangle + (-1)^{f(1)} \left|1\right\rangle) \otimes \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle). \end{aligned} \tag{4.13}$$

So we conclude that we finally arrive at the same general form for all the cases in which our function $f(x)$ is defined differently. Let us look at the final state in all the cases after applying the final Hadamard gate to the state $\left|\varphi\right\rangle$ which is

$$\left|\varphi\right\rangle = \frac{1}{\sqrt{2}}((-1)^{f(0)} \left|0\right\rangle + (-1)^{f(1)} \left|1\right\rangle). \tag{4.14}$$

Notice that we have omitted $\frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle)$ which means we apply the final Hadamard

gate only in the first qubit's Hilbert space. It can be seen in the schematic depiction of Deutsch algorithm. By applying the final gate we get

$$\begin{aligned} H|\varphi\rangle &= \frac{1}{\sqrt{2}}((-1)^{f(0)}H|0\rangle + (-1)^{f(1)}H|1\rangle) \\ &= \frac{1}{2}\Big((-1)^{f(0)}(|0\rangle + |1\rangle) + (-1)^{f(1)}(|0\rangle - |1\rangle)\Big) \\ &= \frac{1}{2}\Big(((-1)^{f(0)} + (-1)^{f(1)})|0\rangle + ((-1)^{f(0)} - (-1)^{f(1)})|1\rangle\Big). \end{aligned} \tag{4.15}$$

Let us look what happens for all of the four cases:

$$\begin{aligned} f(0) = f(1) = 0 \Rightarrow \quad & H|\varphi\rangle = |0\rangle, \\ f(0) = f(1) = 1 \Rightarrow \quad & H|\varphi\rangle = -|0\rangle, \\ f(0) = 0, f(1) = 1 \Rightarrow \quad & H|\varphi\rangle = |1\rangle, \\ f(0) = 1, f(1) = 0 \Rightarrow \quad & H|\varphi\rangle = -|1\rangle. \end{aligned}$$

The calculation of the final state gives us information on whether $f(x)$ is constant or balanced. We should know that, at first, we do not know the final state. We send one of the basis state vectors to see what final state is, for example, if we send $|0\rangle$ and then get

$$|\langle 0 \,|\, \Psi_{final}\rangle|^2 = 0, \tag{4.16}$$

$|\Psi_{final}\rangle = |1\rangle$ or $-|1\rangle$ meaning $f(x)$ is not constant but balanced. If the probability amplitude is 1, then $f(x)$ is constant. Notice that our final calculation does not give a clue about values the function takes for all x values. For example, if $|\langle 0 \,|\, \Psi_{final}\rangle|^2 = 1$, $|\Psi_{final}\rangle = |0\rangle$ or $-|0\rangle$ meaning we are not sure which constant value the function takes.

Up to now, we saw how Deutsch algorithm works and how it serves the same purpose in less operations compared to classical algorithms. Here are the few things to be noted:

1) The Deutsch Algorithm is the simplest case where we use two-qubit states, and the generalized case is called The Deutsch-Jozsa algorithm [24] that will not be mentioned

in this work.

2) On the circuit of Deutsch algorithm, we see two lines defining Hilbert spaces for each qubit we use as registers and we express the whole state as the tensor product of these two states. By using quantum parallelism, without interfering in the inside mechanism of Deutsch algorithm we gain information owing to calculations made at the end of circuit.

4.2. The Grover Search Algorithm

In this section, we will discuss the Grover search algorithm [3] providing us a much faster search engine than classical engines. This search operation will be denoted by G that is the composition of suboperators $H^{\otimes n}$, X and O as shown below:

$$G = H^{\otimes n} X H^{\otimes n} O \tag{4.17}$$

where $H^{\otimes n}$ is the Hadamard gate applied on n-qubit system, and X and O are to be defined in the following:

$$O\left|x\right\rangle = (-1)^{\delta_{xy}}\left|x\right\rangle, \tag{4.18}$$

$$X\left|x\right\rangle = -(-1)^{\delta_{x0}}\left|x\right\rangle = (2\left|0\right\rangle\left\langle 0\right| - I)\left|x\right\rangle. \tag{4.19}$$

where $|x\rangle$ and $|y\rangle$ denote basis states. In Equation 4.18, y is a specific n-qubit state, and we store our data base using n qubits giving us 2^n numbers shown as some variable x,

$$x \in [0, 2^n - 1]. \tag{4.20}$$

Let us evaluate our G:

$$G = H^{\otimes n}(2\left|0\right\rangle\left\langle 0\right| - I)H^{\otimes n}O = (H^{\otimes n}2\left|0\right\rangle\left\langle 0\right|H^{\otimes n} - H^{\otimes n2})O = (2\left|\Psi\right\rangle\left\langle\Psi\right| - I)O \tag{4.21}$$

where I is the identity operator and $H^{\otimes n}|0\rangle = \frac{1}{2^{n/2}}\sum_{x=0}^{2^n-1}|x\rangle = |\Psi\rangle$. Not to be confused, it should be noted that $|0\rangle$ is a n-qubit state. Each qubit of this state is in the zero state.

As mentioned before, we have a data base containing n qubits. This data base is actually a quantum state having 2^n computational bases, and that is $|\Psi\rangle$. Each basis corresponds to a number x whose interval is in Equation 4.20. Let us suppose that we would like to find some specific number on this data base, say $|y\rangle$. Now let us rewrite $|\Psi\rangle$,

$$\begin{aligned} |\Psi\rangle &= \sqrt{\frac{1}{2^n}}\sum_{x\neq y}|x\rangle + \sqrt{\frac{1}{2^n}}|y\rangle \\ &= \sqrt{1-\frac{1}{2^n}}\sqrt{\frac{1}{2^n-1}}\sum_{x\neq y}|x\rangle + \sqrt{\frac{1}{2^n}}|y\rangle \\ &= \sqrt{1-\frac{1}{2^n}}|\alpha\rangle + \sqrt{\frac{1}{2^n}}|y\rangle \end{aligned} \tag{4.22}$$

where $|\alpha\rangle = \sqrt{\frac{1}{2^n-1}}\sum_{x\neq y}|x\rangle$. Since $|\Psi\rangle$ is normalized, we can interchange the factors with cosine and sinus of some angle:

$$|\Psi\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|y\rangle\,. \tag{4.23}$$

Let us apply the operator O on this state:

$$\begin{aligned} O|\Psi\rangle &= (-1)^{\delta_{xy}}\cos\frac{\theta}{2}\sqrt{\frac{1}{2^n-1}}\sum_{x\neq y}|x\rangle + (-1)^{\delta_{xy}}\sin\frac{\theta}{2}|y\rangle \\ &= \cos\frac{\theta}{2}|\alpha\rangle - \sin\frac{\theta}{2}|y\rangle\,, \end{aligned} \tag{4.24}$$

this result leads us to the conclusion that operator O is similar to the rotation operator in a two-dimensional plane. If we think of a two-dimensional Hilbert space whose bases are $|\alpha\rangle$ and $|y\rangle$, the operator O corresponds to a reflector with respect to $|\alpha\rangle$. What

happens if we apply the operator $(2\left|\Psi\right\rangle\left\langle\Psi\right| - I)$, that is a part of the operator G, on the state $\lambda\left|\Psi\right\rangle + \mu\left|\Phi\right\rangle$? Assume $\left\langle\Psi \mid \Phi\right\rangle = 0$ and let us look at what happens:

$$\begin{aligned}(2\left|\Psi\right\rangle\left\langle\Psi\right| - I)(\lambda\left|\Psi\right\rangle + \mu\left|\Phi\right\rangle) &= 2\lambda\left|\Psi\right\rangle - \lambda\left|\Psi\right\rangle - \mu\left|\Phi\right\rangle \\ &= \lambda\left|\Psi\right\rangle - \mu\left|\Phi\right\rangle .\end{aligned} \tag{4.25}$$

This is the same operation with O. Remember that our operator G is defined as follows:

$$G = (2\left|\Psi\right\rangle\left\langle\Psi\right| - I)O \tag{4.26}$$

meaning G is an operator rotating with the angle θ. This is explicitly shown in the following:

$$\begin{aligned}G\left|\Psi\right\rangle &= (2\left|\Psi\right\rangle\left\langle\Psi\right| - I)O\left|\Psi\right\rangle \\ &= (2\left|\Psi\right\rangle\left\langle\Psi\right| - I)(\cos\frac{\theta}{2}\left|\alpha\right\rangle - \sin\frac{\theta}{2}\left|y\right\rangle) \\ &= (2\left|\Psi\right\rangle\left\langle\Psi\right| - I)(\cos\frac{\theta}{2}\left|\alpha\right\rangle + \sin\frac{\theta}{2}\left|y\right\rangle - 2\sin\frac{\theta}{2}\left|y\right\rangle) \\ &= (2\left|\Psi\right\rangle\left\langle\Psi\right| - I)(\left|\Psi\right\rangle - 2\sin\frac{\theta}{2}\left|y\right\rangle) \\ &= \left|\Psi\right\rangle - 4\sin\frac{\theta}{2}\left|\Psi\right\rangle\left\langle\Psi \mid y\right\rangle + 2\sin\frac{\theta}{2}\left|y\right\rangle \\ &= \left|\Psi\right\rangle - 4\sin^2\frac{\theta}{2}\left|\Psi\right\rangle + 2\sin\frac{\theta}{2}\left|y\right\rangle \\ &= \cos\frac{\theta}{2}\left|\alpha\right\rangle + \sin\frac{\theta}{2}\left|y\right\rangle - 4\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\left|\alpha\right\rangle - 4\sin^3\frac{\theta}{2}\left|y\right\rangle + 2\sin\frac{\theta}{2}\left|y\right\rangle .\end{aligned} \tag{4.27}$$

By doing straightforward calculations we get

$$G\left|\Psi\right\rangle = \cos\frac{3\theta}{2}\left|\alpha\right\rangle + \sin\frac{3\theta}{2}\left|y\right\rangle \tag{4.28}$$

meaning G is the same operation with the rotating operator in two-dimensional plane. After k iterations of G, our resultant state becomes

$$G^k\left|\Psi\right\rangle = \cos\frac{(2k+1)\theta}{2}\left|\alpha\right\rangle + \sin\frac{(2k+1)\theta}{2}\left|y\right\rangle . \tag{4.29}$$

Recall that we would like to find $|y\rangle$ on our data base, therefore we need to find some value of k to this. We need k_o such that

$$\cos\frac{(2k+1)\theta}{2} = 0. \tag{4.30}$$

Let us find k_o:

$$\cos k\theta\cos\frac{\theta}{2} - \sin k\theta\sin\frac{\theta}{2} = \sqrt{1-\frac{1}{2^n}}\cos k\theta - \sqrt{\frac{1}{2^n}}\sin k\theta = 0. \tag{4.31}$$

We then can find that $\tan k\theta = \sqrt{2^n-1}$ and $\cos k\theta = 1/\sqrt{2^n}$. From this, We deduce that

$$k_o = \frac{1}{\theta}\arccos 1/\sqrt{2^n}. \tag{4.32}$$

For n is very large number, $\sin\frac{\theta}{2} = \sqrt{\frac{1}{2^n}} \simeq \frac{\theta}{2}$ giving us

$$k_o = \frac{\sqrt{2^n}}{2}\arccos 1/\sqrt{2^n} \simeq \frac{\sqrt{2^n}}{2}\frac{\pi}{2} = \frac{\pi\sqrt{2^n}}{4}. \tag{4.33}$$

As you can see, the number of operations needed to get $|y\rangle$ is proportional to $\sqrt{2^n}$. We know that 2^n is the number of all the bits stored, and via classical algorithms we need, on average, $2^n/2$ operations to find what we need. In conclusion, The Grover search algorithm provides us an engine finding some specific information in less operations and also wasting less time. As n increases to a large number, the number of operations is excessively reduced.

5. THE FUSION OF W-STATE NETWORKS

Until now we discussed historical and mathematical background of quantum information processing, and we mentioned the quantum no-cloning theorem and decoherence phenomenon. Then, we talked about Toffoli and cNOT gates that are to be used in our fusion networks. Also we showed the mathematical representations of these gates and the quantum states. In the previous section, we argued why quantum information processing is a promising field of research by showing famous algorithms, The Deutsch algorithm and The Grover search algorithm as good examples. We have not done anything new yet. The main theme of this work is fusion process of W states, and we will now construct two quantum circuits fusing two W states which is a specific type of quantum states whose robustness is very high. As mentioned in decoherence section, robustness is important since noise effects may disturb our states causing decoherence. It should also be noted that we will be using polarization property of photons as the physical reality on which we encode information.

Before constructing our circuits, let us look at W states and discuss how its robustness is high. The general form of a W state of n qubits is as below:

$$|W_n\rangle = \frac{1}{\sqrt{n}}[|(n-1)_H\rangle_a|1_V\rangle_1 + \sqrt{n-1}|W_{(n-1)}\rangle_a|1_H\rangle_1] \tag{5.1}$$

where H is for horizontal polarization and V is for vertical polarization. The three-qubit and four-qubit examples of W states are in the following:

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|HHV\rangle + |HVH\rangle + |VHH\rangle),$$
$$|W_4\rangle = \frac{1}{\sqrt{4}}(|HHHV\rangle + |HHVH\rangle + |HVHH\rangle + |VHHH\rangle).$$

A W state's uniqueness comes from the fact that only one qubit in each component is vertically polarized meaning it has an interesting entanglement property. Even one of the qubits is lost during interactions with environment, the remaining state of two

qubits is still entangled, and it makes this state more resistant. It is significant to create multipartite entangled states for specific quantum information processings [9]. The W state, that is one of the non-biseparable classes, cannot be converted to other classes of entangled states such as GHZ, Dicke and cluster states under stochastic local operations and classical communications (SLOCC) [25]. It should be noted that some quantum information tasks require specific entangled states [26–30], that is why we need to focus on creating larger scale entangled states one of which is the W state, the main subject of this work. By using photonics, successful theoretical and experimental proposals were published on GHZ [31–34] and cluster states [35]. On the other hand, it is much more challenging to create large scale W states due to their sophisticated structures. A lot of effort has been put into this endeavor, and there have been both theoretical and experimental achievements [36–42]. There are also theoretical proposals to fuse multiple W states [43, 44]. Among these, Ozdemir *et al.*'s proposal is the milestone surpassing the success limit of previous ones and creating arbitrarily large scale W states [42]. We will now discuss this paper together with Bugu *et al.*'s proposal serving the same purpose with higher success probability than Ozdemir *et al.* [45]. The reason why we will discuss them together is that Bugu *et al.* succeded in increasing the success probability by enhancing Ozdemir *et al.*'s network.

In the fusion process constructed by Ozdemir *et al.*, there are two parties, Alice and Bob, having n- and m-partite entangled W states respectively. Their aim is to fuse these two states by using a fusion gate (FG). The Symbolic depiction of the fusion process is in Figure 5.1. Their multipartite entangled states are defined mathematically as follows:

$$|W_n\rangle_A = \frac{1}{\sqrt{n}}(|(n-1)_H\rangle_a|1_V\rangle_1 + \sqrt{n-1}|W_{(n-1)}\rangle_a|1_H\rangle_1) \tag{5.2}$$

$$|W_m\rangle_B = \frac{1}{\sqrt{m}}(|(m-1)_H\rangle_b|1_V\rangle_2 + \sqrt{m-1}|W_{(m-1)}\rangle_b|1_H\rangle_2) \tag{5.3}$$

where the photons in mode 1 (2) are sent to FG and other photons are kept intact at their site of Alice and Bob. Here, H photons are horizontal polarized whereas V photons are vertical polarized. When tripartite W states are used, our expressions can

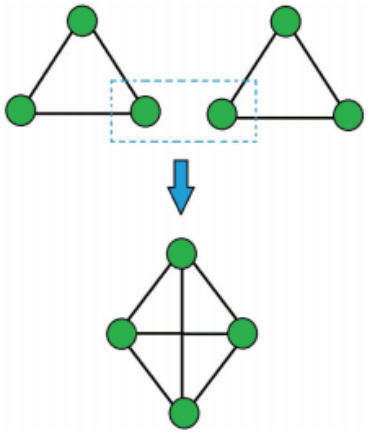


Figure 5.1. The fusion process of Ozdemir *et al.*'s proposal. One photon from each party is sent to FG to create a larger scale W state. Here, two W states of three qubits are fused. This figure is taken from the paper by Bugu *et al.* [45].

be written as

$$|W_3\rangle_{A,B} = \frac{1}{\sqrt{3}}[|2_H\rangle_{a,b}|1_V\rangle_{1,2} + \sqrt{2}|W_2\rangle_{a,b}|1_H\rangle_{1,2}], \tag{5.4}$$

where $|W_2\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)$. To be more explicit, let us write our tripartite entangled states as

$$|W_3\rangle_{A,B} = \frac{1}{\sqrt{3}}[|HHV\rangle_{A,B} + |HVH\rangle_{A,B} + |VHH\rangle_{A,B}]. \tag{5.5}$$

One photon from each party goes into FG in Figure 5.2 and the polarization of one of the photons is rotated by $\pi/2$. In the FG, photons are then mixed on a polarizing beam splitter (PBS) such that their polarizations determine the trajectory they will follow. The states in mode 3 and 4 are measured in the basis $(|D\rangle, |\bar{D}\rangle)$ where $|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ and $|\bar{D}\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$. After the PBS, there are four possible outcomes: (i) $|1_H\rangle_1|1_H\rangle_2 \rightarrow |0\rangle_3|1_H1_V\rangle_4$ meaning only detector 2 (D2) clicks. In this case, tripartite states lose their H polarized photons implying the remaining photons are $|W_2\rangle$ states that can be recycled to restart the fusion process. (ii)$|1_V\rangle_1|1_V\rangle_2 \rightarrow |1_H1_V\rangle_3|0\rangle_4$, if both photons entering FG are V polarized, then only detector 1 (D1) clicks which means fusion fails and all W states are destroyed. (iii)$|1_H\rangle_1|1_V\rangle_2 \rightarrow |1_H\rangle_3|1_H\rangle_4$, when the photon in mode 1 is H polarized and the photon in mode 2 is V polarized, both of

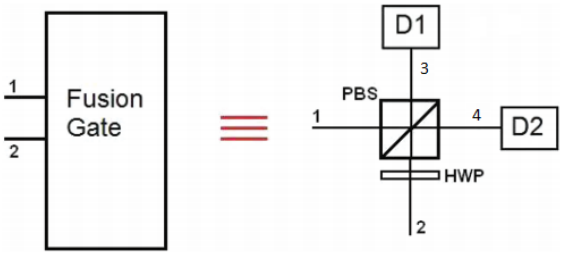


Figure 5.2. Two photons are sent from two W states to the FG. The figure is taken from Bugu *et al.*'s paper [45].

the photons entering the PBS are H polarized for the photon in mode 2 is rotated by the half-wave plate (HWP) and becomes H polarized. Each detector receives one H polarized photon and a coincidence is observed meaning that W states are fused succesfully. (iv) $|1_V\rangle_1|1_H\rangle_2 \rightarrow |1_V\rangle_3|1_V\rangle_4$, when the photon in mode 1 is V polarized and the photon in mode 2 is H polarized, both of the photons entering PBS are V polarized for the photon in mode 2 is rotated by HWP and becomes V polarized. Each detector receives one V polarized photon and, in this case, a coincidence is observed as well meaning W states are fused succesfully. The detection system (Figure 5.3) can not discriminate the cases (iii) and (iv) in which both detectors click. When a H polarized photon enters the detector, it is transformed into the state $\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$. If the input photon is V polarized, the transformed state is $\frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$. In any case, only one of the subdetectors click meaning we cannot know if the initial state is $|H\rangle$ or $|V\rangle$. It just tells us that a photon is detected or not. Note that $|0\rangle$ is for vacuum states meaning no spatial mode is occupied by a photon. When n- and m-partite entangled states are fused instead of two tripartite entangled states, we get (n+m-2)-qubit W states. The probabilities of four possible outcomes of the fusion process of n- and m-partite entangled W states are given in Table 5.1. Total probability of a successful fusion process is

$$P_s = \frac{n+m-2}{nm}. \tag{5.6}$$

This strategy of Ozdemir *et al.* [42] has the flaw that there is the risk of destruction of the whole network causing more expensive information processing. Bugu *et al.* [45] introduced a Fredkin gate and using an ancillary photon they have improved the

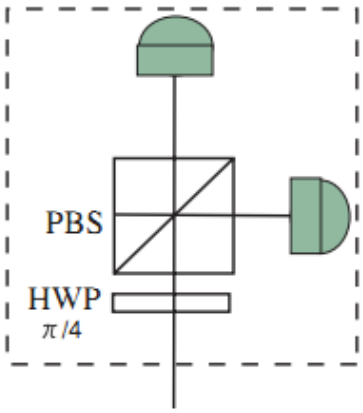


Figure 5.3. This is the more detailed structure of D1 and D2 used on our setup. As you can see, each detector is composed of two subdetectors, PBS and a quarter-wave plate.

Table 5.1. The probabilities of four possible outcomes.

Polarizations of Photons	Probability	Result
H,H	$\frac{(n-1)(m-1)}{nm}$	recycle
H,V	$\frac{(n-1)}{nm}$	success
V,H	$\frac{(m-1)}{nm}$	success
V,V	$\frac{1}{nm}$	failure

probability. In addition to this, they have also got rid of the probability of failure case in which both of the photons entering FG gate are V polarized. A good side of this strategy is that the ancillary photon is added to the resultant W state, which contains (n+m-1) photons, meaning it has one more qubit than the resultant W state in the previous network. The Fredkin gate, which is also called a controlled-SWAP gate, is a universal gate that can be applied to any logical operation by using an ancillary photon. It is a three-qubit gate which swaps target bits when the control bit is V polarized. The ancillary photon added to the system is H polarized, therefore our four possible inputs of three qubits entering the Fredkin gate are $|HHH\rangle$, $|HVH\rangle$, $|VHH\rangle$ and $|VVH\rangle$. The first qubit in mode 1 is the control bit on which swapping operation depends, and the third qubit is the ancillary photon. The transformations of the Fredkin gate are

as follows: $|HHH\rangle \rightarrow |HHH\rangle$, $|HVH\rangle \rightarrow |HVH\rangle$, $|VHH\rangle \rightarrow |VHH\rangle$ and $|VVH\rangle \rightarrow |VHV\rangle$. Swapping occurs only when two V polarized photons enter the Fredkin gate causing the change of polarization of one photon sent to the FG. When the photon in mode 1 is V polarized and the photon in mode 2 is H polarized, both detectors click, and a coincidence is observed meaning our fusion is successful. Hence, by adding the Fredkin gate to our algorithm and using an ancillary photon, we see that even if the photons coming from Alice and Bob are V polarized, $|VH\rangle$ is the input of the FG. Failure is turned into succesful fusion while other possible results remain the same with the ones found by Ozdemir and others. Only one of the four states, that is $|HHH\rangle$, does not lead to success, but the remaining W states can be saved for further use as mentioned before. Thanks to the Fredkin gate, our total probability of success is increased by $\frac{1}{nm}$ which is the probability of the case in which we turned failure into success,

$$P_s = \frac{n+m-2}{nm} + \frac{1}{nm} = \frac{n+m-1}{nm}. \tag{5.7}$$

The schematic depiction of the fusion obtained by using the Fredkin gate and FG can be seen in Figure 5.4. Here, we should note that three important developments have been achieved. First, the probability of success increased as seen above. Second, this algorithm proposed by Bugu *et al.* prevents W states from being destroyed lowering the cost of the network. Thereby, it makes this algorithm more optimum. Third, our ancillary photon is added to the resultant W state making it larger by one qubit than the previously acquired W state in the setup by Ozdemir and others. The comparison of the possible results between the two algorithms can be seen in Table 5.2. Until now various algorithmic methods have been considered to create larger scale W states and to decrease the cost of fusion. Bugu *et al.*'s strategy, with the idea of using a Fredkin gate and an ancillary photon, is the most optimum among all the algorithms. Now we introduce two Toffoli gates instead of a Fredkin gate serving exactly the same purpose which is swapping two qubits when the control bit is V polarized [46]. As in the case of the previous setup, an ancillary photon will be added to the resultant W state. The photons in mode a (b) in Equation 5.2 (5.3) are kept intact while the photons in mode 1 (2) are sent to fusion. As mentioned before, there are four possible inputs including

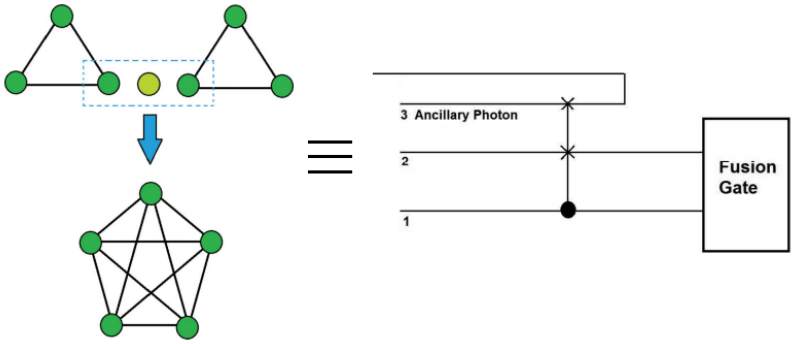


Figure 5.4. The schematic depiction of fusion obtained by Bugu *et al.* [45] and the corresponding network.

Table 5.2. The comparison of results from two algorithms (F: Fredkin gate, FG: Fusion gate).

Input	Probability	Result in FG	Result in FG and F
H,H	$\frac{(n-1)(m-1)}{nm}$	recycle	recycle
H,V	$\frac{(n-1)}{nm}$	success	success
V,H	$\frac{(m-1)}{nm}$	success	success
V,V	$\frac{1}{nm}$	failure	success

the ancillary photon: $|HHH\rangle$, $|HVH\rangle$, $|VHH\rangle$ and $|VVH\rangle$. These three-qubit states are transformed by two Toffoli gates whose schematic depictions are shown in Figure 5.5. Two of the three qubits are the control bits shown as black dots in Figure 5.5 and the third one is the target bit. The action of a Toffoli gate is such that it changes

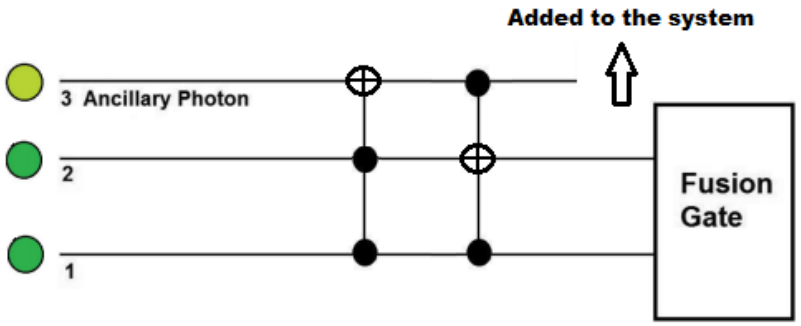


Figure 5.5. The schematic depiction of the fusion gate and two Toffoli gates acting on an ancillary photon and two photons coming from W states.

the polarizaton of the target bit when the contol bits are V polarized, which means its action is similar to a cNOT gate except a cNOT gate acts on one control bit and one target bit. As seen in Figure 5.5, for the first Toffoli gate (T1), the photons in mode 1 and 2 are the control bits whereas our ancillary photon is the target bit, and for the second Toffoli gate (T2), the photons in mode 2 and 3 are the control bits whereas the photon in mode 1 is the target bit. The transformations of this two-Toffoli system are as follows:

$$|HHH\rangle \rightarrow_1 |HHH\rangle \rightarrow_2 |HHH\rangle\,, |HVH\rangle \rightarrow_1 |HVH\rangle \rightarrow_2 |HVH\rangle\,,$$
$$|VHH\rangle \rightarrow_1 |VHH\rangle \rightarrow_2 |VHH\rangle\,, |VVH\rangle \rightarrow_1 |VVV\rangle \rightarrow_2 |HVV\rangle$$

where $\rightarrow_1$ and $\rightarrow_2$ denote T1 and T2 respectively. By comparison, we could easily see that the photons entering the FG are the same with the ones in Bugu *et al.*'s proposal, and it means that the probability of fusion is the same which is given in Equation 5.7. The probability of recycling is found the same as well with the result in Bugu *et al.*'s strategy.

It might be argued that using two Toffoli gates is more costly than using only a Fredkin gate. It is true that we use more resources when two Toffoli gates are applied to our 3-qubit system due to the fact that two 3-qubit gates are implemented whereas only one 3-qubit gate is used in Bugu *et al.*'s proposal. On the other hand, we should also take into account the applicability of our algorithm if we would like to determine whether it is worth using or not. For this, we can get help from measurement based one-way quantum computation on graph states. Measurement based one-way quantum computation is the revolutionary theoretical proposal of universal computation [47]. To the best of our knowledge, only Toffoli gates can be implemented by using graph states [48], and it is not yet known if graph states can be used to form a Fredkin gate. The Fredkin gate can be implemented by using linear optics with the success probability in the order of 10^{-3} [49, 50]. However, the Toffoli gate can be constructed by using weighted graph states of 6, 7 and 8 qubits giving the success probabilities of 1/4, 1/2 and 1, respectively [48]. Even though we use the graph states of 6 qubits,

which gives the lowest probability of a successful application of the Toffoli gate, the whole circuit works with the probability of 1/16 that is much higher than the success probability of the linear optical Fredkin gate.

In this section, we have discussed the basic optical fusion gate of Ozdemir *et al.* and Bugu *et al.*'s proposal increasing the total success probability as well as the size of the resultant W state. Then, we have seen that interchanging the Fredkin gate in Bugu *et al.*'s proposal with two cascaded Toffoli gates increases the probability of successful application of the circuit fusing two W states. Now, we propose a similar interchange as done before except we integrate a cNOT gate instead of the second Toffoli gate as can be seen in Figure 5.6 [51]. The Toffoli gate does operation on qubits only when the photons in mode 1 and 2 are V polarized. This causes a change in the polarization state of the ancillary photon. Our cNOT gate does act if and only if the ancillary photon is V polarized. This means that both gates act only when the inputs in the spatial mode 1 and 2 are V polarized resulting in the same outcomes as in the case of Bugu *et al.*'s strategy. All possible transformations are as below:

$$|HHH\rangle \rightarrow_1 |HHH\rangle \rightarrow_2 |HHH\rangle\,, \quad |HVH\rangle \rightarrow_1 |HVH\rangle \rightarrow_2 |HVH\rangle\,,$$
$$|VHH\rangle \rightarrow_1 |VHH\rangle \rightarrow_2 |VHH\rangle\,, \quad |VVH\rangle \rightarrow_1 |VVV\rangle \rightarrow_2 |VHV\rangle\,,$$

where $\rightarrow_1$ denotes the transformation of the first Toffoli gate whereas $\rightarrow_2$ denotes our cNOT gate's transformation. We succeed in getting the same probability of successful

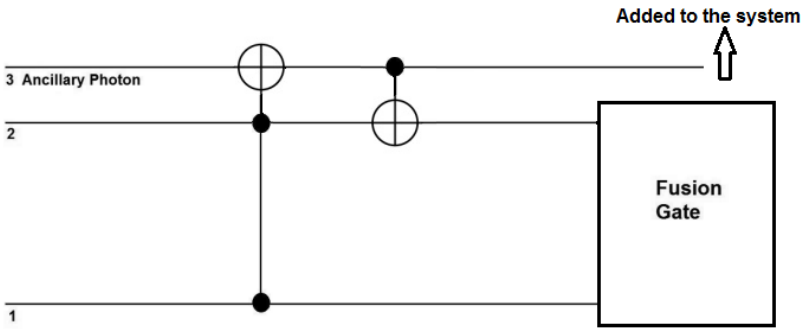


Figure 5.6. The circuit composed of the basic fusion gate, a Toffoli gate and a cNOT gate.

fusion which is (n + m - 1)/nm. There are two possible ways to implement this setup: The linear optical Toffoli [52] and the linear optical cNOT gates [53] are constructed, and a hybrid circuit composed of a Toffoli gate implemented by using weighted graph states as mentioned before [48] and the linear optical cNOT gate working with a probability of almost unity [53].

In the first method, both Toffoli and cNOT gates are implemented by using linear optical tools. Toffoli gate can be used by using linear optics with the probability of 1/32 [52]. Also, the linear optical cNOT gate is shown to be working with almost unit probability [53]. Therefore, our circuit is much more realizable with current optics technology.

In the second method, we propose a hybrid circuit since a Toffoli gate is theoretically realized via one-way quantum computation as in the case of two-Toffoli system whereas our cNOT gate is implemented with the same linear optical way as in the first method. Together with the linear optical cNOT gate, the second setup is more suitable since we want our fusion to be successful to reduce cost. We already mentioned that Toffoli gates can be implemented with unit probability by using graph states of 8 qubits [48]. By this method, we can implement our circuit in a nearly deterministic way.

6. CONCLUSION

In this work, we presented historical developments leading to revolutionary method of information processing. Later, we gave the basic details on the formulation of the qubit and its difference from a classical bit. To do this, we talked about the superposition of polarization states of a photon. Then, we mentioned the basics of quantum mechanics talking about different bases of polarization states and their state matrices that define the quantum system. Also, we talked about the basic projecting calculations and their probabilities. Previously, we talked about the polarization states of a photon which is only one qubit. After this, we also talked about two-qubit states and their state matrices by calculating the tensor products of two systems. Later, we adressed the fact that no operation can be defined cloning some quantum state to steal information. This is important since we want to keep the information we are transmitting secret! We showed how environmental effects cause dechorence in quantum states changing their state matrices. We did this in two-qubit system to show an example of how quantum states become incoherent. This is not something we want because we need to know our states do not become entangled with environmental states to make our algorithms work properly. In the next section, we talked about three-qubit quantum states and quantum NAND and COPY operations. We pointed out the advantage of these operations since they are reversible. It was shown that the quantum NAND gate and the quantum COPY operation correspond cNOT and Toffoli gates respectively. Later on, we talked about the cNOT gate and the Toffoli gate in detail by giving their mathematical expressions and showing their operations mathematically. In the following, we also talked about quantum parallelism and the Hadamard gate.

Since we want to show how quantum information processing is faster than classical algorithms doing some basic tasks in less operations. For this purpose, we showed how the Deutsch Algorithm and the Grover Search Algorithm work [3, 4]. In the Deutsch Algorithm, we saw that it is theoretically possible to know if two sides of a coin are the same or different without the need of looking at both sides. In the Grover Search Algorithm, we saw that, in a N-qubit system, the number of operations needed to find

some specific qubit is proportional to $\sqrt{N}$ whereas we need $\frac{N}{2}$ operations in classical algorithms.

After we talked about the advantages of quantum information processing, we focussed on the fusion of W states that is the main subject of the original work in this thesis. This is an important process since the robustness of W states is high due to their sophisticated structures. Even if one of the qubits is destroyed, the remaining state is still a W state of smaller size. These properties make these states more useful to use as inputs since we do not want our input states to become incoherent. We mentioned the papers on the fusion of W states by Ozdemir *et al.* [42] and Bugu *et al.* [45]. In the Ozdemir *et al.*'s proposal, the basic fusion gate was used to fuse two W states. In the Bugu *et al.*'s proposal, a Fredkin gate was integrated to the basic fusion gate increasing the size of the resultant W state and the probability of fusion. Then, we proposed two algorithms doing exactly the same fusion with the same probability (not considering the probability of successful implementation of the gates). We interchanged the Fredkin gate with two cascaded Toffoli gates. By this method, we made the circuit more realizable with current photonics technology since Toffoli gates can be implemented via one-way quantum computation. Then, we suggested integrating a Toffoli gate and a cNOT gate to the basic fusion gate instead of a Fredkin gate or two Toffoli gates. By doing so, we made our circuit more realizable via linear optics, giving a higher probability of successful application of the circuit. Also, we mentioned the hybrid circuit consisting of a linear optical cNOT gate and a Toffoli gate implemented via one-way quantum computation.

To conclude, we constructed the circuits fusing two W states by using more realizable methods with current photonics technology. For this purpose, we used experimental and theoretical results to support our proposals.

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