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SERAP ŞAHİNKAYA
A THESIS SUBMITTED for THE DEGREE of
DOCTOR of PHILOSOPHY
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ÜYE
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ÜYE : Prof. Dr. Vasfi ELDEM

ÜYE : Doç.Dr. Ünsal TEKİR

ÜYE : Yrd.Doç.Dr. Uğur KAPLAN

ÜYE : Yrd.Doç.Dr. Aslı GÜÇLÜKAN İLHAN

**ONAY**

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# SUMMARY

Local cohomology was introduced by Grothendieck in the early 1960s, in part to answer a conjecture of Pierre Samuel about when certain types of commutative rings are unique factorization domains. Local cohomology has become an indepensable tool and is the subject in lots of researchs. In the first chapter, we summarize the recent developments about local cohomology modules. In the second chapter of the thesis, we give some basic notions and definitions related to this topic. In the third chapter, after recalling some results about $\mathfrak{a}$-weakly cofinite and weakly Laskerian modules we provide a partial answer to the following question: '' When is the set of associated primes of the *n*-th local cohomology module of the module $M$ with respect to the ideal $\mathfrak{a}$, $H_{\mathfrak{a}}^{n}(M)$ finite? '' Also, we try to answer when the *n*-th generalized local cohomology module of the pair $(M,N)$ with respect to the ideal $\mathfrak{a}$, $H_{\mathfrak{a}}^{i}(M,N)$, *is* $\mathfrak{a}$-weakly cofinite. In the last chapter, over a Gorenstein local ring $R$ of finite Krull dimension, we discuss the case in which the socle of $H_{\mathfrak{a}}^{n}(M)$ is weakly Laskerian.

**Key Words: Local cohomology modules, Generalized local cohomology modules, (weakly) Laskerian modules, weakly cofinite modules.**

# ÖZET

Yerel kohomoloji kavramı ilk olarak Grothendieck tarafından 1960 yılında Pierre Samuel'in belirli tipteki değişmeli halkaların tek türlü çarpanlarına ayrılabilir bölge olması ile ilgili varsayımını kısmen de olsa cevaplamak için tanımlanmıştır. Yerel kohomoloji o zamandan beri bir çok araştırmanın konusu ve vazgeçilmez bir aracı olmuştur. Birinci bölümde yerel kohomoloji modülleri ile ilgili son gelişmeleri özetledik. Tezin ikinci bölümünde konumuzla alakalı temel tanım ve kavramlar verildi. Üçüncü bölümde ilk olarak, $\mathfrak{a}$-zayıf eşsonlu modüller ve zayıf Laskerian modüllerle ilgili sonuçları hatırlattık. Daha sonra $M$ modülün $\mathfrak{a}$ idealine göre, $n$-inci yerel kohomoloji modülü olan $H_{\mathfrak{a}}^{n}(M)$ nin ilişkili asallarının kümesinin ne zaman sonlu olacağı ile ilgili soruyu kısman cevapladık. Ayrıca $(M, N)$ çiftinin $\mathfrak{a}$ idealine göre, $n$-inci genelleştirilmiş yerel kohomloji modülü olan $H_{\mathfrak{a}}^{i}(M, N)$ nin ne zaman $\mathfrak{a}$-zayıf eşonlu olduğu ile ilgili soruyu cevaplamaya çalıştık. Son bölümde sonlu Krull boyutlu Gorenstein yerel halkası olan $R$ üzerinde $H_{\mathfrak{a}}^{n}(M)$ nin temelinin ne zaman zayıf Laskerian olduğu sorusunu tartıştık.

**Anahtar Kelimeler: Yerel kohomoloji modüller, genelleştirilmiş yerel kohomoloji modüller, (zayıf) Laskerian modüller, zayıf eşsonlu modüller.**

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# TABLE OF CONTENTS

# LIST OF SYMBOLS and ABBREVIATONS

| **Abbreviatons and Acronyms** | | **Explanations** |
|---|---|---|
| $\boldsymbol{R}$ | : | Commutative ring |
| $\boldsymbol{\mathfrak{a}}$ | : | Ideal of $\boldsymbol{R}$ |
| $\mathbb{N}_{\mathbf{0}}$ | : | The set of non-negative integers |
| $\mathbb{N}$ | : | The set of positive integers |
| $\mathbb{Z}$ | : | The set of integers |
| $\mathbf{V}(\boldsymbol{\mathfrak{a}})$ | : | The set of prime ideals of **R** containing $\boldsymbol{\mathfrak{a}}$ |
| $\boldsymbol{Ann}(\boldsymbol{x})$ | : | The annhilator of an element $\boldsymbol{x}$ |
| $\boldsymbol{\Gamma}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{M})$ | : | An $\boldsymbol{\mathfrak{a}}$-torsion functor |
| $\boldsymbol{H}^{\boldsymbol{n}}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{M})$ | : | The *n*-th local cohomology of $\boldsymbol{M}$ with respect to the ideal $\boldsymbol{\mathfrak{a}}$ |
| $\boldsymbol{H}^{\boldsymbol{n}}_{\boldsymbol{\mathfrak{a}}}(\boldsymbol{N},\boldsymbol{M})$ | : | The *n*-th generalized local cohomology module of the pair $(\boldsymbol{N},\boldsymbol{M})$ with respect to the ideal $\boldsymbol{\mathfrak{a}}$ |
| Spec($\boldsymbol{R}$) | : | The set of prime ideals of the ring $\boldsymbol{R}$ |
| $\mathbf{Hom}_{\boldsymbol{R}}(\boldsymbol{M},\boldsymbol{N})$ | : | The class of $\boldsymbol{R}$-module homomorphisms from $\boldsymbol{M}$ to $\boldsymbol{N}$ |
| $\boldsymbol{\Gamma}_{\boldsymbol{\mathfrak{a}}}$ | : | $\boldsymbol{\mathfrak{a}}$-torsion functor |
| $\boldsymbol{Ker}(\boldsymbol{f})$ | : | The kernel of the homomorphism $\boldsymbol{f}$ |
| $\boldsymbol{Im}(\boldsymbol{f})$ | : | The image of the homomorphism $\boldsymbol{f}$ |
| $\boldsymbol{J}(\boldsymbol{R})$ | : | Jacobson radical of $\boldsymbol{R}$ |
| $\boldsymbol{Rad}(\boldsymbol{M})$ | : | Radical of the module $\boldsymbol{M}$ |
| $\boldsymbol{M}_{\boldsymbol{\mathfrak{p}}}$ | : | Localization *of* $\boldsymbol{M}$ *at* ideal $\boldsymbol{\mathfrak{p}}$. |
| $\mathbf{Ass}_{\boldsymbol{R}}\boldsymbol{M}$ | : | The set of all associated primes of the module $\boldsymbol{M}$ |
| $\mathbf{Supp}_{\boldsymbol{R}}\boldsymbol{M}$ | : | Support of the module $\boldsymbol{M}$ |
| $\sqrt{\boldsymbol{I}}$ | : | The radical of the ideal $\boldsymbol{I}$ |

# 1. INTRODUCTION

Local cohomology was introduced by Grothendieck in the early 1960s in order to answer a conjecture which is formed by Pierre Samuel. The conjecture was about when certain types of commutative rings are unique factorization domains. Since then local cohomology has become an indepensable tool and also is the subject of many research. Local cohomology modules can be used to measure the depth of a module on an ideal and as a way to test the Cohen –Macaulay and Gorenstein properties. Moreover, the cohomology of coherent sheaves on projective varieties can be recovered from graded components of local cohomology modules, providing useful insights into theorems about projective varieties that were originally proved in other ways.

For each $i \in \mathbb{N}_0$, the $i$th local cohomology module of $N$ with respect to an ideal $\mathfrak{a}$ is defined as

$$H_{\mathfrak{a}}^{i}(N) = \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}_{R}^{i}(R/\mathfrak{a}^{n}, N) \tag{1.1}$$

It is well-known that for a Noetherian ring $R$, an ideal $\mathfrak{a}$ of $R$ and a finitely generated $R$-module $M$, the local cohomology modules $H_{\mathfrak{a}}^{j}(M)$ are not always finitely generated. On the other hand, if $R$ is local and $\mathfrak{m}$ is a maximal ideal of $R$, then $H_{\mathfrak{m}}^{j}(M)$ are Artinian modules. It is an easy consequence of Matlis duality [Matlis, 1958] and the basic work of Grothendieck [Grothendieck, 1966] that this property is equivalent to $\mathrm{Supp}_{\mathrm{R}}(H_{\mathfrak{m}}^{j}(M)) \subseteq \{\mathfrak{m}\}$ and $Hom_R(R/\mathfrak{m}, H_{\mathfrak{m}}^{j}(M))$ is finitely generated. In view of these facts, Grothendieck [Grothendieck, 1968] conjectured the following:

Let $R$ be a commutative Noetherian local ring with an ideal $\mathfrak{a}$ and $M$ any finitely generated module, then module $Hom_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{j}(M))$ is finitely generated for all $j$.

Hartshorne [Hartshorne, 1970] showed that this is false in general. However, he defined an $R$-module $M$ to be an $\mathfrak{a}$-cofinite if $\mathrm{Supp}_{\mathrm{R}}(\mathrm{M}) \subseteq \mathrm{V}(\mathfrak{a})$, where $\mathrm{V}(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$, and $\mathrm{Ext}_{\mathrm{R}}^{\mathrm{i}}(\mathrm{R}/\mathfrak{a}, \mathrm{M})$ is finitely generated for all $\mathrm{i} \in \mathbb{N}_0$. Later he refined Grothendieck's conjecture and asked when the local cohomology modules $H_{\mathfrak{a}}^{j}(M)$'s of a finitely generated module are $\mathfrak{a}$-cofinite. In general, the answer is negative, even if $R$ is a regular local ring. Let $R = k[[x; y; u; v]]$ be the formal power series ring in four variables over a field $k$, $\mathfrak{m}$ the maximal ideal of $R$, $\mathfrak{a} = (x, u)R$ and $M = R/(xy - uv)$. Hartshorne showed that $Hom_R(R/\mathfrak{m}, H_{\mathfrak{a}}^{2}(M))$ is not finitely generated, and hence $Hom_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{2}(M))$ can not be finitely generated see, [Hartshorne, 1970]. In the positive direction, he proved for a finitely generated $R$-module $M$ and complete regular local ring $(R, \mathfrak{m})$, if either $\mathfrak{a}$ is a non zero principal or $\mathfrak{a}$ is prime ideal with $dim\, R/\mathfrak{a} = 1$, then the $H_{\mathfrak{a}}^{i}(M)$'s are $\mathfrak{a}$-cofinite. Hartshorne's work provides motivation for some more recent results on this question. Huneke and Koh, extended Hartshorne's second case. Theorem 4.1 of [Huneke and Koh, 1991] shows that if $R$ is a complete Gorenstein domain, $\mathfrak{a}$ is an ideal of $R$ with $dim\, R/\mathfrak{a} = 1$ and $N$ is finitely generated with support in $\mathrm{V}(\mathfrak{a})$, then $\mathrm{Ext}_{\mathrm{R}}^{\mathrm{i}}\left(\mathrm{N}, H_{\mathfrak{a}}^{j}(M)\right)$ is finitely generated for all $i$ and $j$ and for all finitely generated modules $M$.

This result was further was extended by Delfino (see Theorem 3 of [Delfino, 1994]). Delfino was able to remove the Gorenstein condition on $R$ as long as $R$ satisfies one of the three conditions (here $K$ denotes a coefficient ring of $R$, $q$ the uniformizing parameter of $K$):

i) $K$ is a field;
ii) $K$ is not a field and $q \in \mathfrak{a}$;
iii) $K$ is not a field and $q$ is not an element of any minimal prime of $R/\mathfrak{a}$.

Delfino and Marley eliminated the complete domain hypothesis entirely. They prove in [Delfino and Marley, 1997] that if $M$ is a finitely generated module over a commutative Noetherian ring $R$ and $\mathfrak{a}$ is an ideal such that $dimR/\mathfrak{a} = 1$, then the local cohomology modules $H_{\mathfrak{a}}^{j}(M)$ are $\mathfrak{a}$-cofinite for all $j$.

Hartshorne's work also provided motivation for study in the special case when $M = R$. Theorem 2.3 of [Huneke and Koh, 1991] shows that if $R$ is a regular local ring of characteristic $p > 0$, $\mathfrak{a}$ is any ideal of $R$, $j > bight(\mathfrak{a})$, where $bight(\mathfrak{a}) = \max\{ht\, \mathfrak{p} : \mathfrak{p}\ is\ a\ minimal\ prime\ ideal\ of\ \mathfrak{a}\}$ and $Hom_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{j}(R))$ is finitely generated, then $H_{\mathfrak{a}}^{j}(R) = 0$.

The relation between $\mathfrak{m}$-cofiniteness and the Artinian condition is given by the following remark. It is an easy consequence of Matlis duality [Matlis, 1958] and the basic work of Grothendieck [Grothendieck, 1966].

*Remark 1.1: Let $(R, \mathfrak{m}, k)$ be a complete local Noetherian ring and let $M$ be an $R$-module. Then the following properties are equivalent:*

*i) $M$ is Artinian;*
*ii) $M$ is isomorphic to a submodule of a finite direct sum of copies of the injective hull $E$ of the residue field $k$ of $R$;*
*iii) There is a finitely generated $R$-module $N$ such that $Hom_R(N,E) = M$, and $Hom_R(M,E) = N$;*
*iv) $Supp(M) = \{\mathfrak{m}\}$ and $Hom_R(k,M)$ is finitely generated;*
*v) $M$ is $\mathfrak{m}$-cofinite.*

In [Belshof et al., 1996], the writers obtained similar results as above, but for a larger class of modules. First they focused to the case that $R$ is a complete Gorenstein domain and $\mathfrak{a}$ is an ideal of $R$ such that $dimR/\mathfrak{a} = 1$. Let $E = E_R(k)$ be the injective hull of the residue field and let $(-)^v$ denote the functor $Hom(-,E)$. Recall that a module is *Matlis reflexive* if $M \cong (M^v)^v$. The class of Matlis reflexive modules over a complete ring includes all finitely generated and Artinian modules. They show that if $M$ and $N$ are Matlis reflexive with $Supp(N) \subseteq V(\mathfrak{a})$, then $Ext_R^i(N, H_{\mathfrak{a}}^j(M))$ is Matlis reflexive for all $i$ and $j$. In fact, $Ext_R^i(N, H_{\mathfrak{a}}^j(M))$ is finitely generated for all $i$ and positive $j$. Also they gave an example to show that $Ext_R^i(N, H_{\mathfrak{a}}^0(M))$ need not be finitely generated. However, it turns out that this is enough to show that all the Bass numbers of $H_{\mathfrak{a}}^j(M)$ are finite. Later, they then remove the domain assumption on $R$ and study the case that $M = R$ and $N$ is finitely generated. In this situation they proved that $Ext_R^i(N, H_{\mathfrak{a}}^j(R))$ is Matlis reflexive. As a corollary, they showed that $H_{\mathfrak{a}}^j(R)$ has finite Bass numbers for every $j$.

Since for an $\mathfrak{a}$-cofinite module $N$ we have $Ass_R N = Ass_R(Hom_R(R/\mathfrak{a}, N))$, it turns out that $Ass_R N$ is finite. Huneke [Huneke, 1992] raised the following question: If $M$ is a finitely generated $R$-module, then the set of associated primes of $H_{\mathfrak{a}}^j(R)$ is finite for all ideals $\mathfrak{a}$ of $R$ and all $j \geq 0$. Singh [Singh, 2000] gave a counterexample to this conjecture. However, it is known that this conjecture is true in many situations. For example, Brodmann and Lashgari [Brodmann and Lashgari, 2000] showed that if for a finitely generated $R$-module $M$ and an integer $t$, the local cohomology modules $H_{\mathfrak{a}}^{0}(M), H_{\mathfrak{a}}^{1}(M), \cdots, H_{\mathfrak{a}}^{t-1}(M)$ are all finitely generated, then $Ass_R(H_{\mathfrak{a}}^{t}(M))$ is (see Theorem 2.2) finite. In the case that the ring $R$ is regular and contains a field of prime characteristic $p > 0$, Huneke and Sharp showed that the set of associated primes of $H_{\mathfrak{a}}^{n}(M)$ is finite [Huneke and Sharp, 1993]. If $R$ is regular local ring containing a field of characteristic zero, Lyubeznik showed that $H_{\mathfrak{a}}^{n}(M)$ has only finitely many associated prime ideals, see [Lyubeznik, 1993], [Lyubeznik, 1997], [Lyubeznik, 2000]. Recently, Lyubeznik has proved this result for unramified regular local rings of mixed characteristic [Lyubeznik, 2000]. Singh gave a counter-example to the question of Huneke [Singh, 2000]. He proved that if $M$ is finitely generated $R$-module, then the set of associated primes of $H_{\mathfrak{a}}^{n}(M)$ is infinite for all ideals $\mathfrak{a}$ of $R$ and all $i \geq 0$. He constructed an example of a hypersurface $R$ such that the local cohomology module $H_{\mathfrak{a}}^{3}(R)$ has $p$-torsion elements for every prime integer $p$, and consequently has infinitely many associated primes.

An $R$-module $M$ is called weakly Laskerian iff $Ass_R(M/N)$ is finite for all submodules $N$. Let $M$ be a weakly Laskerian module. In [Divaani-Aazar and Mafi, 2004 ], the writers proved that the set of associated primes of the first non-$\mathfrak{a}$-cofinite local cohomology module of $M$ with respect to $\mathfrak{a}$ is finite which clearly implies the result mentioned above due to Brodmann and Lashgari. Also regarding the Artinianness of local cohomology, it is shown in [Divaani-Aazar and Esmkhani, 2005] that in many cases weakly Laskerian modules behave similar to finitely generated modules.

In [Divaani-Aazar and Mafi, 2006], the writers proved that the local cohomology modules of a weakly Laskerian module are weakly cofinite in several cases. As a result, they deduce that the sets of associated primes of the local cohomology modules of a weakly Laskerian module are finite in these cases. For instance, it was shown that if $\mathfrak{a}$ is a principal ideal of $R$ and $M$ is a weakly Laskerian $R$-module, then $Ass_R(H_{\mathfrak{a}}^{i}(M))$ is finite for all $i \geq 0$. Also, they proved that if $R$ is a local ring and $M$ is a weakly Laskerian $R$-module, then $Ass_R(H_{\mathfrak{a}}^{i}(M))$ is finite for all $i \geq 0$ in two cases:

i) $dimR \leq 3$,

ii) $dim\, R/\mathfrak{a} \leq 1$.

The following generalization of local cohomology theory is given by J. Herzog [Herzog, 1974]. Let $R$ be a local with unique maximal ideal $\mathfrak{m}$. For each $i \geq 0$,

$$H^i_{\mathfrak{m}}(,): \mathcal{C}(R) \times \mathcal{C}(R) \to \mathcal{C}(R) \tag{1.2}$$

is the functor defined by $H^i_{\mathfrak{m}}(M, N) = \lim_{\alpha} \mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(M/\mathfrak{m}^{\alpha}M, N)$ (where $\mathcal{C}(R)$ denotes the category of all $R$-modules and $R$-homomorphisms). In [Bijan-Zadeh, 1980], this definition is generalized to the case of an arbitrary commutative Noetherian ring $R$ and an arbitrary ideal $\mathfrak{a}$ of $R$. There are not many results concerning the vanishing of this generalized local cohomology modules. It is shown that if $M, N$ are non-zero finite $R$-modules, then $grade_N(M/\mathfrak{a}M)$, which is the length of any maximal $N$-sequence contained in $(0:_R M/\mathfrak{a}M)$, see 5.5 in [Bijan-Zadeh, 1980], is the smallest integer $i$ such that $H^i_{\mathfrak{a}}(M, N)$ is non-zero. Also, if $M$ is a finite $R$-module of finite projective dimension $p.dim_R(M) < \infty$ and $N$ an $R$-module of finite Krull dimension, then for any $i > dim + Np.dim_R(M)$ the generalized local cohomology module $H^i_{\mathfrak{a}}(M, N)$ is zero, see 5.1 in [Bijan-Zadeh, 1980].

The finiteness property of generalized local cohomology modules is not well understood either. Recently, in [Yassemi, 2001] it is shown that for any nonzero principal ideal $\mathfrak{a}$ of $R$, the $R$-module $H^i_{\mathfrak{a}}(M, N)$ is an $\mathfrak{a}$ -cofinite module for all $j \geq 0$, whenever $M$ and $N$ are finitely generated $R$-module and $R$ is a Gorenstein ring. Also writers asked the following question: Assume $M$ and $N$ are finitely generated $R$-modules and $\mathfrak{a}$ is an ideal of $R$ with dimension one or $\mathfrak{a}$ is principal. Is the module $H^i_{\mathfrak{a}}(M, N)$ $\mathfrak{a}$ -cofinite for all $j \geq 0$?

Asadollahi et al., showed that $Hom_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M, N))$ is finitely generated, whenever $H^i_{\mathfrak{a}}(M, N)$ is the first non-finitely generated generalized local cohomology module of finitely generated modules $M$ and $N$ over a Noetherian ring $R$ with respect to an ideal $\mathfrak{a} \subseteq R$, see [Asadollahi et al., 2002]. Hence it has only finitely many associated primes. This generalizes the main results of [Brodman and Lashgari, 2000] and [Khashyarmanesh and Salarian, 1999]. They proved that

$Hom_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M,N))$ is a finitely generated $R$-module for all $i \geq 0$, whenever $M$ and $N$ are finite $R$-modules and $\mathfrak{a}$ is a non-zero principal ideal, which gives a positive answer to the Grothendieck's conjecture for principal ideals in the context of this generalization of local cohomology theory. They showed that, if $M$ is projective then $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all principal ideals $\mathfrak{a}$ and all $i \geq 0$, therefore they obtained the main result of [Kawasaki, 1998].

The local cohomology modules are not finitely generated in general. So the following questions summarize the natural problems concerning local cohomology theory, see [Huneke, 1992]:

i) When is the set of associated primes of $H^n_{\mathfrak{a}}(M)$ finite ?
ii) When is $H^n_{\mathfrak{a}}(N)$ $\mathfrak{a}$-cofinite?
iii) When is the socle of $H^n_{\mathfrak{a}}(N)$ finitely generated?

Since the concept of weakly Laskerian and weakly cofinite modules are natural generalizations of the concepts of finitely generated and cofinite modules, respectively, we have the following natural questions:

iv) When is $H^n_{\mathfrak{a}}(N)$ $\mathfrak{a}$-weakly cofinite?
v) When is the socle of $H^n_{\mathfrak{a}}(N)$ weakly Laskerain?

On the other hand, a generalization of the local cohomology functor has been given by Herzog in [Herzog, 1974] (see also [Suzuki, 1978]). For each $i \in \mathbb{N}_0$, the $i$-th generalized local cohomology module of the pair $(N, M)$ with respect to an ideal $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(M,N) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(M/\mathfrak{a}^n M, N) \tag{1.3}$$

Clearly, $H^i_{\mathfrak{a}}(R,N) \cong H^i_{\mathfrak{a}}(N)$ for all $i \in \mathbb{N}_0$.

In this thesis, we provide a partial answers to the questiones 1, 4 and 5. Moreover, we show that there is a certain connection between finiteness property of $H^n_{\mathfrak{a}}(N)$ and $H^n_{\mathfrak{a}}(M,N)$. (See Theorem 3.4 and Corollary 3.5)

# 2. BASIC NOTIONS AND DEFINITIONS

In this section, we will give some basic definitions and notions. We refer to the books [Enochs and Jenda, 2000], [Anderson and Fuller, 2011], [Broadmann and Sharp, 2000] and lectures notes from the web site [Web 1, 2006] for the definitions, theorems, propositions and lemmas which do not have references in this section.

*Definition 2.1: A unitary ring is a set R which satisfy the following properties for all* $a, b, c \in R$

i) $(a + b) + c = a + (b + c)$ *(+ is associative)*
ii) $0 + a = a$ *(0 is the zero element)*
iii) $a + b = b + a$ *(+ is commutative)*
iv) $a + (-a) = (-a) + a = 0$ *(inverse of element )*
v) $(a.b).c = a.(b.c)$ *(multiplication is associative)*
vi) $a.(b + c) = (a.b) + (a.c)$ *(left distributivity)*
vii) $(b + c).a = (b.a) + (c.a)$ *(right distributivity)*
viii) There exists a unit element $1 \in R$ such that $a.1 = 1.a = a$

*A ring R is called* commutative *if it is commutative with respect to multiplication.*

*Examples 2.2:*

*i) The ring of integeres* $\mathbb{Z}$ *under the usual operations of addition and multiplication is a commutative ring with identity.*
*ii) The quotient group* $\mathbb{Z}/n\mathbb{Z}$ *is a commutative ring with identity under the operations of addition and multiplication of residue classes.*

*Definition 2.3: For an arbitrary ring* $R$*, a subset I is called a* right ideal *of* $R$ *if it satisfies the following conditions:*

*i*) $(I,+)$ *is a subgroup of* $(R,+)$

*ii)* $x.r \in I$, *for all* $x$ *in* $I$ *and all* $r$ *in* $R$.

*Similarly, one can define a* left ideal *of* $R$ *as follows :*

*i*) $(I,+)$ *is a subgroup of* $(R,+)$

*ii)* $r.x \in I$, *for all* $x$ *in* $I$ *and all* $r$ *in* $R$.

*If* $R$ *is commutative then every right ideal is also left ideal and vice verca.*

*Examples 2.4:*

*i*) $n\mathbb{Z}$ *is an ideal of* $\mathbb{Z}$, *for any* $n \in \mathbb{Z}$, *and these are the only ideals of* $\mathbb{Z}$. *(since in particular these are the only subgroups of* $\mathbb{Z}$*.)*

*ii) The subring 0 is the trivial ideal.*

*Definition 2.5: Let* $R$ *be a ring and* $(M,+)$ *a commutative group. If for* $m \in M$ *and* $r \in R$ *the function* $M \times R \longrightarrow M$, *defined by* $mr$, *satisfies the following conditions for all* $x, y \in M$ *and* $a, b \in R$, *then* $M$ *is called a* left $R$-module *and denoted by* $M_R$.

i) $(x+y)a = xa + ya$

ii) $x(a+b) = xa + xb$

iii) $x(ab) = (xa)b$

*Examples 2.6:*

*i) Let* $R$ *be an arbitrary ring. Then* $M = R$ *is a left* $R$*-module, where the action is given by left multiplication . Similarly,* $R$ *is right* $R$*-module over itself.*

*ii) Let* $R = F$ *be a field. Every vector space over* $F$ *is an* $F$*-module.*

*Definition 2.7: An* $R$*-module* $M$ *is called* finitely generated *if and only if there exist* $a_1, a_2, \cdots, a_n$ *in* $M$ *such that for all* $x \in M$, *there exist* $r_1, r_2, \cdots, r_n$ *in* $R$ *such that* $x = r_1 a_1 + r_2 a_2 + \cdots + r_n a_n$ .

*Examples 2.8:*

*i) Let* $R = \mathbb{Z}$ *and let* $M$ *be an* $R$*-module, that is an abelian group. If* $a \in M$, *then* $\mathbb{Z}a$ *is the cyclic subgroup of* $M$, *generated by* $a$. *In particular, it is finitely generated.*
*ii) Let* $R$ *be a ring by definition and let* $M$ *be the left* $R$*-module* $R$. *Since* $R = R1$, $R$ *is finitely generated.*

*Definition 2.9: Let* $N$ *be a submodule of the* $R$*-modul*e $M$. *Since,* $N$ *is an additive subgroup of* $M$, *the quotient group* $M/N$ *is well-defined. Via the operation* $r(x + N) = rx + N$, $M/N$, *becomes an* $R$*-modul*e *which is called the* quotient module of $M$.

*Definition 2.10: If a module has no nonzero proper submodule then it is called* simple module.

*Definition 2.11: If a module* $M$ *is the direct sum of simple submodules then it is called* semisimple module.

*Definition 2.12: Let* $N \leq M$ *then* $N$ *is called an* essential submodule *of* $M$ *if for every submodule* $H$ *of* $M$, $H \cap N = 0$ *implies* $H = 0$.

*Definition 2.13: The s*ocle of a module $M$ *over a ring* $R$ *is the sum of the nonzero minimal submodules of* $M$. *In set notation;*

$$Soc(M) = \sum\{N \mid N\ is\ a\ simple\ submodule\ of\ M\}. \tag{2.1}$$

$$Soc(M) = \bigcap\{E \mid E\ is\ an\ essential\ submodule\ of\ M\}. \tag{2.2}$$

For a semisimple module $M$, we have $Soc(M) = M$.

*Definition 2.14: A ring* $R$ *is called* Noetherian *if one of the following equivalent conditions are satisfied:*

*i) Every ascending chain of ideals in* $R$ *like*

$$I_1 \subset I_2 \subset \cdots I_n \subset \cdots \tag{2.3}$$

*is eventually stationary. That is, there exists an integer* $n$ *such that*

$$I_n = I_{n+1} = \cdots \tag{2.4}$$

*ii) Every ideal of R has a finite set of generators.*

*Examples 2.15:*

*i) Every Principal Ideal Domain, such as the integers, is Noetherian since every ideal is generated by a single element.*
*ii) Any field including fields of rational numbers, real numbers and complex numbers is Noetherian.*
*iii) A Dedekind domain is Noetherian since every ideal is generated by at most two elements.*
*iv) The ring of polynomials in finitely many variables is Noetherian.*

Dually one can define Artinian ring as following:

*Definition 2.16: A ring R is called* Artinian *if every descending chain of ideals in R like*

$$I_1 \supset I_2 \supset \cdots I_n \supset \cdots \tag{2.5}$$

*is eventually stationary. That is, there exists an integer* $n$ *such that*

$$I_n = I_{n+1} = \cdots \tag{2.6}$$

*Examples 2.17:*

*i) Let* $n > 1$ *be an integer. Since the ring* $R = \mathbb{Z}/n\mathbb{Z}$ *is finite, it is Artinian.*
*ii) Suppose* $f$ *is a nonzero polynomial in* $k[x]$, *where* $k$ *is a field. Then the quotient ring* $R = k[x]/(f(x))$ *is Artinian.*

Although the descending chain condition is a dual notion of the ascending chain condition in rings, in fact it is a stronger condition. Clearly, as a consequence of Akizuki-Hopkins-Levitzki theorem, a left (right) Artinian ring is automatically a left (right) Noetherian ring.

*Definition 2.18: A module* $M$ *is called* Noetherian (Artinian) *if every ascending (descending) chain of submodules of* $M$ *terminates.*

Unlike the situation in the rings, there exist Artinian modules which are not Noetherian modules. For example, consider the $p$-primary component of $\mathbb{Q}/\mathbb{Z}$, regarded as $\mathbb{Z}$-module. The chain

$$\langle 1/p \rangle \subset \langle 1/p^2 \rangle \subset \langle 1/p^3 \rangle \subset \cdots \tag{2.7}$$

does not terminate, so $\mathbb{Z}(p^\infty)$ is not Noetherian. Nevertheless, every descending chain of proper submodules ends: Such chain has the following form:

$$\langle 1/n_1 \rangle \supseteq \langle 1/n_2 \rangle \supseteq \langle 1/n_3 \rangle \supseteq \cdots \tag{2.8}$$

for some integers $n_1,\ n_2, n_3, \cdots$ and the inclusion of $\langle 1/n_{i+1} \rangle \subseteq \langle 1/n_i \rangle$ implies that $n_{i+1}$ must divide $n_i$. So $n_1,\ n_2, n_3, \cdots$ is a decreasing chain of positive integers. So that the sequence terminates, which makes $\mathbb{Z}(p^\infty)$ Artinian.

*Definition 2.19: A* category $\mathcal{C}$ *consists of :*

*i) a class Ob* $\mathcal{C}$, *whose elements are called the objects of* $\mathcal{C}$*;*

*ii) for each pair* $(A, B)$ *of objects of* $\mathcal{C}$*, a set* $Hom_{\mathcal{C}}(A, B)$*, whose elements are called morphisms of* $A$ *into* $B$*;*
*iii) for each triple* $(A, B, C)$ *of objects of* $\mathcal{C}$*, a mapping*

$$\circ : Hom_{\mathcal{C}}(B, C) \times Hom_{\mathcal{C}}(A, B) \longrightarrow Hom_{\mathcal{C}}(A, C) \qquad (2.9)$$

*is called composition.*

Let us fix some notations. If $f \epsilon\ \mathrm{Hom}_{\mathcal{C}}(A, B)$ then we write $f: A \longrightarrow B$ . The composition of morphisms $f: A \longrightarrow B$ and $g: B \longrightarrow C$ will be denoted by $gf$ or $g \circ f$ where $\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$.

The followings conditions are the axioms of category:

i) If $A, B, C$ and $D$ are objects of $\mathcal{C}$ and $f: A \longrightarrow B$ , $g: B \longrightarrow C$ and $h: C \longrightarrow D$ are morphisms, then $(hg)f = h(gf)$ (associativity of composition).
ii) For every $A \in \mathrm{Ob}\, \mathcal{C}$ , there exists an element of $\mathrm{Hom}_{\mathcal{C}}(A, A)$, which we will denote it by $1_A$, such that $f \circ 1_A = f$ and $1_A \circ g = g$ for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$.

*Examples 2.20:*

*i) Set is the category of all sets. For any two sets* $A$ *and* $B$, $Hom(A, B)$ *is the set of all functions from* $A$ *to* $B$*. Composition of morphisms is the familiar composition of functions. The identity in* $Hom(A, A)$ *is the map* $1_A(a) = a$*, for all* $a \in A$*.*
*ii) Grp is the category of all groups, where morphisms are group homomorphisms.*

*Definition 2.21: Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. A* functor (*or a* covariant functor*)* $F: \mathcal{C} \longrightarrow \mathcal{D}$ *assigns to every object* $C \in Ob\mathcal{C}$ *an object* $F(C) \in Ob\mathcal{D}$ *and to every morphism* $f: C \longrightarrow C'$ *in* $\mathcal{C}$ *a morphism* $F(f): F(C) \longrightarrow F(C')$ *in* $\mathcal{D}$ *such that the following axioms are satisfied:*

*i)* $F(g \circ f) = F(g) \circ F(f)$ *for every* $f: C \longrightarrow C'$ *and* $g: C' \longrightarrow C''$, *ii)* $F(1_C) = 1_{F(C)}$ *for every* $C \in Ob\mathcal{C}$.

Fix an $R$- module$M_R$. Then there exists a covariant functor

$$\mathrm{Hom}(M_R, -): \mathrm{Mod} - R \longrightarrow Ab \tag{2.10}$$

given by $\mathrm{Hom}(M_R, -)(\mathrm{N}) = \mathrm{Hom}(M_R, \mathrm{N})$ and $\mathrm{Hom}(M_R, -)(\mathrm{f}) = \mathrm{f}_*$.

*Definition 2.22: Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. A* contravariant functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ *assigns to every* $C \in Ob\mathcal{C}$ *an object* $F(C) \in Ob\mathcal{D}$*, and to every morphism* $f: C \longrightarrow C'$ *a morphism* $F(f): F(C') \longrightarrow F(C)$ *such that the following axioms are satisfied:*

*i)* $F(g \circ f) = F(f) \circ F(g)$ *for every* $f: C \longrightarrow C'$ *and* $g: C' \longrightarrow C''$,
*ii)* $F(1_C) = 1_{F(C)}$ *for every* $C \in Ob\mathcal{C}$.

Fix an $R$- module $N_R$ . Then we know that there exists a contravariant functor

$$\mathrm{Hom}(-, N_R): \mathrm{Mod} - R \longrightarrow Ab. \tag{2.11}$$

given by $\mathrm{Hom}(-, N_R)(\mathrm{M}) = \mathrm{Hom}(M, N_R)$ and $\mathrm{Hom}(-, N_R)(\mathrm{f}) = \mathrm{f}_*$.

It is well-known that Hom functors are "left exact", that is, for a fixed module $M_R$ , if $0 \longrightarrow N'_R \longrightarrow N_R \longrightarrow N''_R$ is exact, then so is $0 \longrightarrow \mathrm{Hom}(M_R, N'_R) \longrightarrow \mathrm{Hom}(M_R, N_R) \longrightarrow \mathrm{Hom}(M_R, N''_R)$ and for a fixed module $N_R$, if $M'_R \longrightarrow M_R \longrightarrow M''_R \longrightarrow 0$ is exact, then so is $0 \longrightarrow \mathrm{Hom}(M''_R, N_R) \longrightarrow \mathrm{Hom}(M_R, N_R) \longrightarrow \mathrm{Hom}(M'_R, N_R)$ .

*Proposition 2.23: The following conditions are equivalent for an* $R$*-module* $E_R$*:*

*i) The functor* $\mathrm{Hom}(-, E_R): \mathrm{Mod} - R \longrightarrow Ab$ *is exact, that is for every exact sequence* $0 \longrightarrow M'_R \longrightarrow M_R \longrightarrow M''_R \longrightarrow 0$ *of right R-modules, the sequence* $0 \longrightarrow \mathrm{Hom}(M''_R, E_R) \longrightarrow \mathrm{Hom}(M_R, E_R) \longrightarrow \mathrm{Hom}(M'_R, E_R) \longrightarrow 0$ *of abelian groups is exact.*

*ii) For every monomorphism* $M'_R \longrightarrow M_R$ *of right* $R$*-modules,* $\mathrm{Hom}(M_R, E_R) \longrightarrow \mathrm{Hom}(M'_R, E_R)$ *is an epimorphism of abelian groups.*

*iii) The functor* $\mathrm{Hom}(-, E_R)\colon \mathrm{Mod} - R \longrightarrow Ab$ *is exact, that is for every exact sequence* $0 \longrightarrow M'_R \longrightarrow M_R \longrightarrow M''_R \longrightarrow 0$ *of right* $R$*-modules, the sequence* $0 \longrightarrow \mathrm{Hom}(M''_R, E_R) \longrightarrow \mathrm{Hom}(M_R, E_R) \longrightarrow \mathrm{Hom}(M'_R, E_R) \longrightarrow 0$ *of abelian groups is exact.*

*iv) For every submodule* $M'_R$ *of a right* $R$*-module* $M_R$*, every morphism* $M'_R \longrightarrow E_R$ *can be extended to a morphism* $M_R \longrightarrow E_R$.

*v) For every monomorphism* $f\colon {M_R}' \longrightarrow M_R$ *and every homomorphism* $g\colon M'_R \longrightarrow E_R$*, there exists a morphism* $h\colon M_R \longrightarrow E_R$ *with* $h \circ f = g$.

*Definition 2.24: A module* $E_R$ *is* injective *if it satisfies one of the equivalent conditions of Proposition 2.23.*

*Examples 2.25:*

*i) The set of rational numbers* $\mathbb{Q}$ *is an injective* $\mathbb{Z}$*-module.*

*ii) The quotient module* $\mathbb{Q}/\mathbb{Z}$ *of the injective* $\mathbb{Z}$*-module* $\mathbb{Q}$ *is an injective* $\mathbb{Z}$*-module.*

*iii)* $\mathbb{Z}$ *is not injective* $\mathbb{Z}$*-module. This follows from the fact that the exact sequence* $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ *corresponding the multiplication by 2 does not split.*

*Definition 2.26: An exact sequence of the form*

$$0 \longrightarrow M_R \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots \qquad (2.12)$$

*where the* $E_i$*'s are injective modules is called an* injective resolution *of* $M$.

*Definition 2.27: An* $R$*-module* $M$ *is said to have* injective dimension at most *n, denoted* inj dim$\leq n$*, if there is an injective resolution* $0 \longrightarrow M_R \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots E^n \longrightarrow 0$.

*Definition 2.28: Let* $N$ *be a submodule of the* $R$*-module* $M$. $M$ *is called the* injective hull of $N$ *if* $M$ *is an injective* $R$*-module which is also an essential extension of* $N$.

The injective hull or injective envelope of a module is the smallest injective module containing it and the largest essential extension of it. Every module $M$ has an injective hull. The injective hull of $M$ is unique up to isomorphisms. Nevertheless the isomorphism is not necessarily unique.

*Examples 2.29:*

*i) The injective hull of an injective module is itself.*
*ii) The injective hull of an integral domain is its field of fractions.*
*iii) The injective hull of a cyclic p-group (as* $\mathbb{Z}$*-module) is a Prüfer group.*

*Definition 2.30: Let R be a ring. A two-sided ideal I of R is called* maximal *if* $I \neq R$ *and the only ideal strictly containing* $I$ *is* $R$.

*Definition 2.31: A commutative ring R is called* local *if it has a unique maximal ideal.*

*Examples 2.32:*

*i) All fields (and skew fields) are local rings, because {0} is the only maximal ideal in such rings.*
*ii) A nonzero ring in which every element is either a unit or nilpotent is a local ring.*
*iii) Discrete valuation rings are important class of local rings. These rings are local principal ideal domains that are not fields.*
*iv) Every ring of formal power series over a field (even in several variables) is local. For these rings the maximal ideal consists of those power series without constant term.*
*v) More generally, given for a commutative ring R and a prime ideal* $\mathfrak{p}$ *of R, the localization of R at* $\mathfrak{p}$ *is local; in this localization the maximal ideal is the ideal which is generated by* $\mathfrak{p}$.

*Definition 2.33: The* characteristic *of a ring R, which is denoted by char(R), is defined to be the smallest number of times one must use the ring's multiplicative* *identity element 1 in a sum to get the additive identity element 0. For the ring in which this sum never reaches the additive identity is said to have characteristic zero.*

The characteristic of any field is either 0 or a prime number. A field of nonzero characteristic is called a field of finite characteristic or a field of positive characteristic.

*Definition 2.34: An M-regular sequence is a sequence* $r_1, r_2, \cdots, r_d$ *in R such that* $r_i$ *is a nonzero divisor on* $M/(r_1,, \cdots, r_{i-1})M$ *for* $i = 1 \cdots, d$.

*Definition 2.35: Let R be a Noetherian ring,* $\mathfrak{a}$ *an ideal in* $R$*, and* $M$ *a finitely generated* $R$*-module. The* depth *of* $\mathfrak{a}$ *on* $M$ *is the supremum of the lengths of all* $M$*-regular sequences of elements of* $\mathfrak{a}$ *I.*

*Examples 2.36:*

*i) For a prime number* $p$*, the local ring* $\mathbb{Z}_{(p)}$ *is the subring of the rational numbers which consists of fractions whose denominator is not a multiple of* $p$*. The element* $p$ *is a non-zero-divisor in* $\mathbb{Z}_{(p)}$*, and the quotient ring of* $\mathbb{Z}_{(p)}$ *which is generated by the ideal* $p$ *is the field* $\mathbb{Z}/(p)$*. So,* $p$ *can not be extended to a longer regular sequence in the maximal ideal (p), in fact the local ring* $\mathbb{Z}_{(p)}$ *has depth 1.*

*ii) For any field* $k$ *, the elements* $x_1, x_2, \cdots, x_n$ *in the polynomial ring* $A = k[x_1, x_2, \cdots, x_n]$ *form a regular sequence. It follows that the localization R of A at the maximal ideal* $\mathfrak{m} = (x_1, x_2, \cdots, x_n)$ *has depth at least* $n$*. In fact, R has depth equal to* $n$*; that is, there is no regular sequence in the maximal ideal of length greater than* $n$*.*

*Definition 2.37: A ring R is called* Gorenstein *if R is both left and right noetherian and it has finite self-injective dimension on both the left and the right.*

*Definition 2.38: An* $R$*-module* $M$ *is called* torsion-free *if* $rx = 0$ *for* $r \in R$ *and* $x \in M$ *implies* $r = 0$ *or* $x = 0$*.* $M$ *is said to be a* torsion module *if for each* $x \in M$*, there is an* $0 \neq r \in R$ *with* $rx = 0$*.*

*Definition 2.39: $M$ be an $R$-module, set $\Gamma_{\mathfrak{a}}(M) = \bigcup_{n\in\mathbb{N}}(0:_M \mathfrak{a}^n)$. The set of elements of $M$ which are annihilated by some powers of $\mathfrak{a}$ is called $\mathfrak{a}$-torsion functor, which we show it by $\Gamma_{\mathfrak{a}}(M) = \bigcup_{n\in\mathbb{N}}(0:_M \mathfrak{a}^n)$. $\Gamma_{\mathfrak{a}}$ is a covariant $R$ -linear functor, from $\mathcal{C}(R)$ to itself.*

*Definition 2.40: Let $M$ be an $R$-module. We shall say that $M$ is* $\mathfrak{a}$-torsion-free *if $\Gamma_{\mathfrak{a}}(M) = 0$ and that $M$ is* $\mathfrak{a}$ -torsion *if $\Gamma_{\mathfrak{a}}(M) = M$.*

We now present some important results which is about $\mathfrak{a}$ –torsion modules and $\mathfrak{a}$ -torsion-free modules [Brodman and Sharp 1998]. But before this we recall the Prime Avoidance Lemma. The idea of the Prime Avoidance Lemma is that if we can avoid the $\mathfrak{p}_j$ individually, in other words, for each $j$ , if we can find an element in $\mathfrak{q}$ but not in $\mathfrak{p}_j$ , then we can avoid all the $\mathfrak{p}_j$ simultaneously, that is, we can find a single element in $\mathfrak{p}$ that is in none of the $\mathfrak{p}_j$ . We will state and prove the contrapositive.

*Theorem 2.41: (Prime Avoidance Lemma) Suppose that $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n, \mathfrak{q}$ are ideals of a ring $R$, and suppose that $\mathfrak{q} \subset \bigcup_{j=1}^{n}\mathfrak{p}_j$. If at most 2 of the $\mathfrak{p}_j$'s are not prime, then $\mathfrak{q}$ is contained in one of the $\mathfrak{p}_j$'s .*

The following Lemmas and Remarks can be found from [Brodman and Sharp 1998].

*Lemma 2.42: Let $M$ be an $R$-module.*

*i) If $\mathfrak{a}$ contains a non-zero divisor on $M$, then $M$ is $\mathfrak{a}$-torsion-free, that* is $\Gamma_{\mathfrak{a}}(M) = 0$.
*ii) Assume now that $M$ is finitely generated. Then $M$ is $\mathfrak{a}$-torsion-free if and only if $\mathfrak{a}$ contains a non-zero divisor on $M$.*

*Lemma 2.43: For an $R$-module $M$, the module $M/\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion-free.*

*Remarks 2.44:*

*i) If $M$ is an $\mathfrak{a}$-torsion $R$-module, that is $\Gamma_{\mathfrak{a}}(M) = M$, then all submodules of $M$ and all $R$-homomorphic images of $M$ are also $\mathfrak{a}$-torsion.*
*ii) For each $R$-module $L$ and each $i \in \mathbb{N}_0$, the i-th local cohomology module $H_{\mathfrak{a}}^{i}(L)$ is an $\mathfrak{a}$-torsion $R$-module.*

The Artin–Rees lemma is a basic result about modules over a Noetherian ring, along with results such as the Hilbert basis theorem. The Artin–Rees lemma was proved in the 1950s by the mathematicians Emil Artin and David Rees in the independent works.

*Lemma 2.45: (Artin Rees Lemma) Let I be an ideal in a Noetherian ring R; let M be a finitely generated R-module and let N a submodule of M. Then there exists an integer k ≥ 1 so that, for n ≥ k,*

$$I^n M \cap N = I^{n-k}\Big((I^k M) \cap N\Big) \tag{2.13}$$

*Proposition 2.46: Let $I$ be an injective $R$-module. Then $\Gamma_{\mathfrak{a}}(I)$ is also an injective $R$-module.*

The following Corollaries are due to [Brodman and Sharp 1998].

*Corollary 2.47: Let $I$ be an injective $R$-module. Then the canonical exact sequence*

$$0 \to \Gamma_{\mathfrak{a}}(I) \to I \to I/\Gamma_{\mathfrak{a}}(I) \to 0 \tag{2.14}$$

*splits.*

*Corollary 2.48: Let $M$ be an $\mathfrak{a}$-torsion $R$-module. Then there exists an injective resolution of $M$ in which each term is an $\mathfrak{a}$-torsion $R$-module.*

*Corollary 2.49:*

*i) Let $M$ be an $\mathfrak{a}$-torsion $R$-module. Then $H_{\mathfrak{a}}^{i}(M) = 0$ for all $i > 0$.*

*ii) For each $R$-module $N$, we have $H_{\mathfrak{a}}^{i}\big(\Gamma_{\mathfrak{a}}(N)\big) = 0$ for all $i > 0$.*

*iii) For each $R$-module $N$, the natural epimorphism $\pi: N \to N/\Gamma_{\mathfrak{a}}(N)$ induces isomorphisms*

$$H_{\mathfrak{a}}^{i}(\pi): H_{\mathfrak{a}}^{i}(N) \overset{\cong}{\to} H_{\mathfrak{a}}^{i}\big(\Gamma_{\mathfrak{a}}(N)\big) \ for\ all\ i > 0\,. \tag{2.15}$$

*Proposition 2.50: We present some properties of local cohomology modules which can be found in [Brodman and Sharp 1998]. Let $M$ be an arbitrary $R$-module. Then:*

*i) To calculate $H_{\mathfrak{a}}^{i}(M)$, one proceeds as follows. Take an injective resolution*

$$\boldsymbol{I^d}: 0 \xrightarrow{d^{-1}} E^0 \overset{d^0}{\to} E^1 \longrightarrow \cdots E^i \overset{d^i}{\to} E^{i+1} \xrightarrow{d^{i+1}} \cdots \tag{2.16}$$

*of $M$, so that there is an $R$-homomorphism $\alpha: M \to E^0$ such that the sequence*

$$0 \xrightarrow{d^{-1}} M_R \overset{\alpha}{\to} E^0 \overset{d^1}{\to} E^1 \longrightarrow \cdots E^i \overset{d^i}{\to} E^{i+1} \xrightarrow{d^{i+1}} \cdots \tag{2.17}$$

*is exact . Apply the functor $\Gamma_{\mathfrak{a}}$ to the complex $\boldsymbol{I^d}$ to obtain*

$$0 \xrightarrow{\Gamma_{\mathfrak{a}}(d^{-1})} \Gamma_{\mathfrak{a}}(E^0) \longrightarrow \cdots \longrightarrow \Gamma_{\mathfrak{a}}\big(E^i\big) \xrightarrow{\Gamma_{\mathfrak{a}}(d^i)} \Gamma_{\mathfrak{a}}\big(E^{i+1}\big) \longrightarrow \cdots \tag{2.18}$$

*Take the i-th cohomology module of this complex; the result,*

$$Ker(\Gamma_{\mathfrak{a}}(d^i))/Im(\Gamma_{\mathfrak{a}}(d^{i-1})), \tag{2.19}$$

*which, by a standard fact of homological algebra, is independent (up to $R$-isomorphism) of the choice of injective resolution $\boldsymbol{I^d}$ of $M$, is $H_{\mathfrak{a}}^{i}(M)$.*

*ii) Since $\Gamma_{\mathfrak{a}}$ is covariant and $R$-linear, it is automatic that each local cohomology functor $H_{\mathfrak{a}}^{i}$ $(i \in \mathbb{N}_0)$ is again covariant and $R$-linear.*
*iii) Since $\Gamma_{\mathfrak{a}}$ is left exact , $H_{\mathfrak{a}}^{0}$ is naturally eguivalent to $\Gamma_{\mathfrak{a}}$. Thus, loosely, we can use this natural equivalence to identify these two functors.*

Now we are ready to give the main definitions of this section. They are from [Brodman and Sharp 1998].

*Definition 2.51: For $i \in \mathbb{N}_0$, the $i$th right derived functor of $\Gamma_{\mathfrak{a}}$ is denoted by $H_{\mathfrak{a}}^{i}$ and will be referred to as the* i-th local cohomology functor *with respect to* $\mathfrak{a}$. *It is defined as following :*

$$H_{\mathfrak{a}}^{i}(M) = \varinjlim_{\mathrm{m} \in \mathbb{N}} \mathrm{Ext}_{\mathrm{R}}^{\mathrm{i}}(R/\mathfrak{a}^{m}, M) \qquad (2.20)$$

*Definition 2.52: For each $n \in \mathbb{N}_0$, the $i$th* generalized local cohomology module *of the pair $(N, M)$ with respect to an ideal* $\mathfrak{a}$ *is defined as*

$$H_{\mathfrak{a}}^{n}(N, M) = \varinjlim_{\mathrm{m} \in \mathbb{N}} \mathrm{Ext}_{\mathrm{R}}^{\mathrm{n}}(N/\mathfrak{a}^{m}N, M). \qquad (2.21)$$

We will denote the zero module with one element by $0$. We now consider *sequences of modules,* where by a sequence of modules

$$\dots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \dots \qquad (2.22)$$

we mean a family of modules $M_i$ indexed by integer numbers and a set of module morphisms $f_i \colon M_i \longrightarrow M_{i+1}$. Sequences can be either finite or infinite on one side or both sides.

*Definition 2.53: A sequence of modules is called a* $0$-sequence (or a complex of modules, or a chain complex of modules) *if $f_i(M_i) \subseteq ker\, f_{i+1}$ for every index $i$. Equivalently, if $f_{i+1}f_i = 0$ for every $i$.*

*Definition 2.54: A sequence is* exact in $M_i$ *if* $f_{i-1}(M_{i-1}) = \ker f_i$. *A sequence is* exact *if it is exact in* $M_i$ *for every* $i$.

*Definition 2.55: Let* $R$ *be a ring and* $\mathfrak{p}$ *be an ideal of* $R$ *such that* $\mathfrak{p} \neq 0$. *If for all* $a, b \in R$, $ab \in \mathfrak{p}$ *implies either* $a \in \mathfrak{p}$ *or* $b \in \mathfrak{p}$, *then* $\mathfrak{p}$ *is called a prime ideal of* $R$. *All of the prime (resp. maximal) ideals of the ring* $R$ *is denoted by* $\operatorname{Spec}(R)$ $(\operatorname{MaxSpec}(R))$.

*Examples 2.56:*

*i) A positive integer* $n$ *is a prime number if and only if the ideal* $n\mathbb{Z}$ *is a prime ideal in* $\mathbb{Z}$.
*ii) If* $R$ *denotes the ring* $\mathbb{C}[X, Y]$ *of polynomials in two variables with complex coefficients, then the ideal generated by the polynomial* $Y^2 - X^3 - X - 1$ *is a prime ideal.*
*iii) In the ring* $\mathbb{Z}[X]$ *of all polynomials with integer coefficients, the ideal generated by* $2$ *and* $X$ *is a prime ideal. It consists of all those polynomials whose constant coefficient is even.*
*iv) In the ring* $\mathbb{Z}$ *of integers the maximal ideals are the principal ideals generated by a prime number.*
*v) All nonzero prime ideals are maximal in a principal ideal domain.*

*Definition 2.57: Let* $R$ *be a ring and* $I$ *be an ideal of* $R$. *A prime ideal* $\mathfrak{p}$ *containing* $I$ *is said to be minimal over* $I$ *if there are no prime ideal properly contained in* $\mathfrak{p}$ *which also contain* $I$.

*Definition 2.58: Let* $R$ *be a ring and* $I$ *be an ideal of* $R$. $I$ *is said to be a* primary ideal *if whenever* $ab \in I$, *then either* $a \in I$ *or* $b^n \in I$ *for some* $n > 0$. *It is obviuous that every prime ideal is a primary ideal.*

*Definition 2.59: Let* $P$ *be a primary ideal, then a primary ideal* $I$ *is called* P-primary *if* $P = \sqrt{I}$. *Here* $\sqrt{I}$ *is the radical of the ideal* $I$ *and defined by* $\sqrt{I} = \{r \in R : \exists\, n \in \mathbb{N}, r^n \in I\}$.

*Example 2.60: If P is a maximal prime ideal, then any ideal containing a power of P is P-primary. Not all P-primary ideals need be powers of P; for example the ideal* $(x,y^2)$ *is P-primary for the ideal P=*$(x,y)$ *in the ring* $k[x,y]$*, but is not a power of* $P$.

*Definition 2.61: Let* $M$ *be an* $R$*-module. A prime ideal* $\mathfrak{p}$ *is said to be* associated prime ideal *of* $M$ *if* $\mathfrak{p} = Ann(x)$ *for some* $x \in M$*. It is equivalent to say that* $M$ *containing a cyclic submodule, say* $Rx$*, which is isomorphic to* $R/\mathfrak{p}$*. The set of associated prime ideals is denoted by* $Ass(M)$.

In commutative algebra associated primes are linked to the Lasker-Noether primary decomposition of ideals in commutative Noetherian rings. Specifically, if an ideal $J$ is decomposed as a finite intersection of primary ideals, the radicals of these primary ideals are prime ideals, and this set of prime ideals coincides with $Ass_R(R/J)$. It is possible, even for a commutative local ring, that the set of associated primes of a finitely generated module is empty. However, in any ring satisfying the ascending chain condition on ideals (for example, any right or left Noetherian ring) every nonzero module has at least one associated prime.

*Definition 2.62: An element* $r$ *of a ring* $R$ *is said to be:*

*i) A right zero divisor if* $r \neq 0$ *and there exists* $s \in R$ *such that* $s \neq 0$ *and* $sr = 0$*;*
*ii) A left zero divisor if* $r \neq 0$ *and there exists* $s \in R$ *such that* $s \neq 0$ *and* $rs = 0$;
*iii) A zero divisor if it is either a right zero-divisor or a left zero-divisor.*

*Definition 2.63: A submodule* $N$ *of an* $R$*-module* $M$ *is said to be a* primary submodule if $N \neq M$ *and* $xy \in N$ and $x \notin N$ *implies* $y^n M \subset N$ *for some* $n > 0$*. It is clear that* $N$ *is primary submodule of* $M$ *if and only if every zero-divisor* $r$ *of* $M/N$ *is nilpotent for* $M/N$*, that is,* $r^n(M/N) = 0$ *for some* $n > 0$*, or equivalently* $r \in \sqrt{Ann(M/N)}$.

*Proposition 2.64: The prime ideal $P$ is associated to $M$ if and only if there is an injective $R$-module homomorphism from $R/P$ to $M$. Therefore if $N$ is a submodule of $M$, then $Ass(N) \subseteq Ass(M)$.*

*Proposition 2.65: For any prime ideal $\mathfrak{p}$, $Ass(R/\mathfrak{p}) = \{\mathfrak{p}\}$. That is, the annihilator of any nonzero element of $R/\mathfrak{p}$ is $\mathfrak{p}$.*

The next result gives us considerable information about the elements that belong to associated primes.

*Theorem 2.66: Let $z(M)$ be the set of zero-divisors of $M$. Then*

$$\cup\{\mathfrak{p} : \mathfrak{p} \in Ass(M)\} \subseteq z(M), \tag{2.23}$$

*with equality if $R$ is Noetherian.*

*Proposition 2.67: If $N$ is a submodule of $M$, then $Ass(M) \subseteq Ass(N) \cup Ass(M/N)$.*

We now establish the connection between associated primes and primary decomposition, and show that under wide conditions, there are only finitely many associated primes.

*Definition 2.68: A* primary decomposition *of the submodule $N$ of $M$ is given by $N = \bigcap_{i=1}^{r} N_i$, where the $N_i$ are $P_i$-primary submodules. The decomposition is* reduced *if the $P_i$ are distinct and $N$ cannot be expressed as the intersection of a proper subcollection of the $N_i$.*

We can always extract a reduced primary decomposition from an unreduced one, by discarding those $N_i$'s that contain $\bigcap_{j\neq i} N_j$ and intersecting those $N_i$'s that are $P$-primary for the same $P$. The following result justifies this process.

*Definition 2.69: The proper submodule $N$ of $M$ is* irreducible *if $N$ cannot be expressed as $N_1 \cap N_2$ with $N$ properly contained in the submodules $N_i$, $i = 1, 2$.*

*Proposition 2.70: If $N$ is an irreducible submodule of the Noetherian module $M$, then $N$ is primary.*

*Theorem 2.71: If $N$ is a proper submodule of the Noetherian module $M$, then $N$ has a primary decomposition, hence a reduced primary decomposition.*

*Theorem 2.72: Let $M$ be a nonzero finitely generated module over the Noetherian ring $R$, so that by Theorem 2.71, every proper submodule of $M$ has a reduced primary decomposition. In particular, the zero module can be expressed as $\bigcap_{i=1}^{r} N_i$, where $N_i$ is $P_i$-primary. Then $Ass(M) = \{P_1, \cdots, P_r\}$, a finite set.*

*Corollary 2.73: Let $N$ be a submodule of $M$ (finitely generated over the Noetherian ring $R$). Then $N$ is $P$-primary iff $Ass(M/N) = \{P\}$.*

In [Divaani-Aazar and Mafi, 2004], the writers defined weakly Laskerian modules as following:

*Definition 2.74: An $R$-module $M$ is said to be* Laskerian *if any submodule of $M$, is an intersection of a finite number primary submodules, and* weakly Laskerian *if the set of associated primes of any quotient module of $M$ is finite.*

Every Laskerian module is weakly Laskerian. Clearly, the class of weakly Laskerian $R$-modules includes all finitely generated, Artinian and linearly compact $R$-modules.

*Definition 2.75: Let $S$ be a* multiplicative subset *of $R$, that is, $1 \in S$ and $S$ is closed under multiplication. Then the* localization *of $R$ with respect to $S$, denoted by $S^{-1}R$, is the set of all equivalence classes $(a, s)$ with $a \in R$ and $s \in S$ under equivalence relation $(a, s) \sim (b, t)$ if there is an $s' \in S$ such that $(at - bs)s' = 0$. The equivalence class $(a, s)$ is denoted by $a/s$.*

*Definition 2.76: Let* $S \subset R$ *be a multiplicative set and* $M$ *be an* $R$*-module. Then the* localization *of* $M$ *with respect to* $S$*, denoted* $S^{-1}M$ *is defined as for* $S^{-1}R$*.* $S^{-1}M$ *is an abelian group under addition and is an* $S^{-1}R$*-module via* $(a/s)(x/t) = ax/st$.

*Definition 2.77: Let* $\mathfrak{p}$ *be a prime ideal of* $R$*. Then* $S = R - \mathfrak{p}$ *is a multiplicative subset of* $R$*. In this case* $S^{-1}M$ *is denoted by* $M_{\mathfrak{p}}$ *and it is called the* localization *of* $M$ *at* $\mathfrak{p}$.

*Examples 2.78:*

*i) Given a commutative ring* $R$*, we can consider the multiplicative set* $S$ *of non-zero divisors (i.e. elements* $a$ *of* $R$ *such that multiplication by* $a$ *is an injection from* $R$ *into itself.) The ring* $S^{-1}R$ *is called the total quotient ring of* $R$*.* $S$ *is the largest multiplicative set such that the canonical mapping from* $R$ *to* $S^{-1}R$ *is injective. When* $R$ *is an integral domain, this is none other than the fraction field of* $R$*.*
*ii) The ring* $\mathbb{Z}/n\mathbb{Z}$ *where* $n$ *is composite is not an integral domain. When* $n$ *is a prime power it is a finite local ring, and its elements are either units or nilpotent. This implies it can be localized only to a zero ring. But when* $n$ *can be factorised as* $ab$ *with* $a$ *and* $b$ *coprime and greater than 1, then* $\mathbb{Z}/n\mathbb{Z}$ *is by the Chinese remainder theorem isomorphic to* $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$*. If we take* $S$ *to consist only of* $(1,0)$ *and* $1 = (1,1)$*, then the corresponding localization is* $\mathbb{Z}/a\mathbb{Z}$*.*

*Definition 2.79: The* support *of an* $R$*-module* $M$*, denoted* $Supp(M)$*, is the set of all prime ideals* $\mathfrak{p}$ *of* $R$ *such that* $M_{\mathfrak{p}} \neq 0$.

If $M$ is a finitely generated $R$-module, then $Supp(M)$, is the set of all prime ideals containing the annihilator of $M$. In particular, it is closed.

*Proposition 2.80:* $Supp\ R/I = V(I)$.

Supports and annihilators are connected by the following basic result.

*Theorem 2.81: If* $M$ *is a finitely generated* $R$*-module, then* $SuppM = V(Ann(M))$.

And now we connect associated primes and annihilators.

*Proposition 2.82: If $M$ is a finitely generated module over the Noetherian ring $R$, then*

$$\bigcap_{P \in Ass(M)} P = \sqrt{Ann(M)} \tag{2.24}$$

*Proposition 2.83: Let $M$ be a finitely generated module over the Noetherian ring $R$, and let $P$ be any prime ideal of $R$. The following conditions are equivalent:*

*i) $P \in SuppM;$*
*ii) $P \supseteq Q$ for some $Q \in Ass(M);$*
*iii) $P \supseteq Ann(M).$*

*Theorem 2.84: Let $M$ be a finitely generated module over the Noetherian ring $R$. Then $Ass(M) \subseteq SuppM$ and the minimal elements of $Ass(M)$ and $SuppM$ are the same.*

*Definition 2.85: Let $I$ be a* directed set, *that is, $I$ is a partially ordered set such that for any $i,j \in I$ there is a $k \in I$ with $i,j \leq k$. Let $\{M_i\}_{i \in I}$ be a family of $R$-modules and suppose for each pairs $i,j \in I$ with $i \leq j$ there is an $R$-homomorphism $f_{ji}: M_i \to M_j$ such that*

*i) $f_{ii} = id_{M_i}$ for any $i \in I$*
*ii) If $i \leq j \leq k$, then $f_{kj}f_{ji} = f_{ki}$.*

*Then we say that the $R$-modules $M_i$ together with the homomorphisms $f_{ji}$ form a* direct (or inductive) system *which is denoted $\big((M_i),\ (f_{ji})\big)$.*

*Definition 2.86: The* direct (inductive) limit *of a direct system $\big((M_i),\ (f_{ji})\big)$ of $R$-modules is an $R$-module $M$ with $R$-homomorphism $g_i: M_i \to M$ for $i \in I$ with*

$g_j f_{ji} = g_i$ *whenever* $i \leq j$ *and such that if* $(N, \{h_i\})$ *is another such family, then there is unique* $R$*-homomorphism* $f: M \to N$ *such that* $f g_i = h_i$ *for all* $i \in I$. *The direct limit is unique up to isomorphism and it is denoted by* $\lim\limits_{\rightarrow} \mathrm{M_i}$.

*Examples 2.87:*

*i) A collection of subsets* $M_i$ *of a set* $M$ *can be partially ordered by inclusion. If the collection is directed, its direct limit is the union* $\bigcup M_i$

*ii) Let* $p$ *be a prime number. Consider the direct system composed of the groups* $\mathbb{Z}/p^n\mathbb{Z}$ *and the homomorphisms* $\mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^{n+1}\mathbb{Z}$ *which are induced by multiplication by* $p$. *The direct limit of this system consists of all the roots of unity of order some power of* $p$, *and is called the Prüfer group* $\mathbb{Z}(p^\infty)$.

Hartshorne defined an $R$-module $M$ to be $\mathfrak{a}$-cofinite as following [Hartshorne, 1970].

*Definition 2.88: An* $R$*-module* $M$ *is said to be* $\mathfrak{a}$*-cofinite if* $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$, *where* $V(\mathfrak{a})$ *denotes the set of prime ideals of* $R$ *containing* $\mathfrak{a}$, *and* $Ext_R^i(R/\mathfrak{a}, M)$ *is finitely generated for all* $i \in \mathbb{N}_0$.

Following [Divaani-Aazar and Mafi, 2006] , Divaani-Aazar and Mafi defined the $\mathfrak{a}$- weakly cofinite module.

*Definition 2.89: An* $R$*-module* $M$ *said to be* $\mathfrak{a}$*- weakly cofinite if* $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ *and* $Ext_R^i(R/\mathfrak{a}, M)$ *is weakly Laskerian for all* $i \in \mathbb{N}_0$.

It is well-known that every $\mathfrak{a}$-cofinite module is $\mathfrak{a}$- weakly cofinite.

*Definition 2.90: An* injective resolution *of a module* $M_R$ *is a complex* $\mathbb{E}$ *such that* $E_i = 0$ *for every* $i < 0$ , $E_i$ *is injective for every* $i \in \mathbb{Z}$ *and there exists a monomorphism* $\varepsilon : M \longrightarrow E_0$ *such that the complex*

$$0 \longrightarrow M_R \xrightarrow{\varepsilon} E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \tag{2.25}$$

*is an exact sequence.*

*Lemma 2.91: For every module $M_R$ there is an exact sequence*

$$0 \longrightarrow M_R \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \tag{2.26}$$

*with the $E_i$'s injective modules.*

*Definition 2.92: Given an injective coresolution of $M$,*

$$0 \xrightarrow{d^{-1}} M_R \xrightarrow{d_0} E^0 \xrightarrow{d^1} E^1 \longrightarrow \cdots E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} \cdots \tag{2.27}$$

*The module $Imd_i$ is called the* i-th cosyzygy *of $M$.*

# 3. GENERALIZED LOCAL COHOMOLOGY AND WEAKLY LASKERIAN MODULES

Throught this section we assume that $R$ is a commutative Noetherian ring with non-zero identity.

An $R$-module $M$ is said to be Laskerian if any submodule of $M$ is an intersection of a finite number of primary submodules. Obviously, any Noetherian module is Laskerian. An $R$-module $M$ is said to be weakly Laskerian if the set of associated primes of any quotient module of $M$ is finite [Divani-Aazar and Mafi, 2004].

*Example 3.1:*

*i) Any Laskerian module is weakly Laskerian. In particular, any Noetherian module is weakly Laskerian.*

*ii) It is known that the set of associated primes of an Artinian module is a finite set consisting of maximal ideals. Hence any Artinian module is weakly Laskerian*

*iii) Recall that a module M is said to have finite Goldie dimension if M does not contain any infinite direct sum of non zero submodules, or equivalently, the injective envelope $E(M)$ of $M$, decomposes as a finite direct sum of indecomposable injective submodules. Because any $R$-module $C$,* we *have* $\mathrm{Ass}_R(C) = \mathrm{Ass}_R(E(C))$, *it turns out that any module with finite Goldie dimension has only finitely many associated prime ideals. This yields that a module whose all quotients have finite Goldie dimension is weakly Laskerian.*

*iv) Let E be the minimal injective cogenerator of R and M an R- module. If for an $R$-module $M$, the natural map from M to $Hom_R(Hom_R(M, E), E)$, is an isomorphism, then M is said to be Matlis reflexive. An $R$- module M is Matlis reflexive if and only if M has a finitely generated submodule S such that $M/S$ is Artinian and $R/Ann_R M$, is a complete semi-local ring. Also, any quotient of an $R$- module M has finite Goldie dimension if and only if M has a finitely generated submodule S such that $M/S$ Artinian. Thus, by (iii), any Matlis reflexive module is weakly Laskerian.*

*v) An $R$- module M is said to be linearly compact if each system of congruences $x \equiv x_i(M_i)$, indexed by a set I and where the $M_i$ are submodules of $M$, has a solution $x$, whenever it has a solution for every finite subsystem. It is known that the* *category of linearly compact $R$- modules form a Serre subcategory of the category of all $R$- modules. In particular, every quotinet of a linearly compact module is also linealy compact. On the other hand, a linearly compact module $M$ has finite Goldie dimension. Thus, if $M$ is a linearly compact module, then any quotient of $M$ has finite Goldie dimension, and so $M$ is weakly Laskerian.*

Before stating one of the main theorem of this section we will give the following useful lemma which was proved in [Divani-Aazar and Mafi, 2004] and [Divani-Aazar and Mafi, 2006].

*Lemma 3.2:*

*i) The set of associated primes of an $\mathfrak{a}$-weakly cofinite module is finite.*
*ii) Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of $R$-modules. Then $M$ is weakly Laskerian if and only if $L$ and $N$ are weakly Laskerian. Hence if $0 \longrightarrow L \longrightarrow M \longrightarrow N$ is an exact sequence such that both end terms are weakly Laskerian $R$-modules, then $M$ is also weakly Laskerian*
*iii) Let $N$ be a finitely generated $R$-module and $M$ be a weakly Laskerian $R$-module. Then $Ext_R^i(N, M)$ is weakly Laskerian for all $i \in \mathbb{N}_0$.*
*iv) If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence and two of modules in the sequence are $\mathfrak{a}$-weakly cofinite, then so is the third.*

*Proof 3.2:*

*i) Firstly we prove that the set of associated primes is finite for $\mathfrak{a}$-cofinite module. Then, we can deduce that the set of associated primes of an $\mathfrak{a}$-weakly cofinite module is finite since the set of associated primes of weakly Laskerian module is finite. Since $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$, it follows that $\operatorname{Ass}_R M = \operatorname{Ass}_R(0:_M \mathfrak{a})$. As $M$ is $\mathfrak{a}$-cofinite the module $(0:_M \mathfrak{a}) \cong Ext_R^0(R/\mathfrak{a}, M)$ is finitely generated. Thus $\operatorname{Ass}_R M$ is finite.*
*ii) It is clear.*
*iii) Since $R$ is a Noetherian ring and $N$ is finitely generated, it follows that $N$ posseses a free resolution*

$$F.:\cdots \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0, \tag{3.1}$$

*consisting of finitely generated free modules. If* $F_i = \bigoplus^n R$ *for some integer* $n$*, then* $Ext_R^i(N,M) = H^i(Hom_R(F.,M))$ *is a subquotient of* $\bigoplus^n M$. *Therefore, it folows from (ii) that* $Ext_R^i(N,M)$ *is weakly Laskerian for all* $i \in \mathbb{N}_0$.

*iv) It is enough to prove this for* $\mathfrak{a}$*-cofinite modules due to the statement (ii) Consider the long exact sequence induced by the exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0, \tag{3.2}$$

*Then it follows easily from the fact that if, in an exact sequence, two of the modules in the exact sequence are Noetherian, then the third one is also Noetherian.*

The following lemma was proved by [Asadollahi et al., 2002] and it will be used for the proof of the one of the main theorem of this section.

*Lemma 3.3: Let N be an* $\mathfrak{a}$*-torsion* $R$*-module. Then* $H_{\mathfrak{a}}^i(M,N)$ *is a finitely generated* $R$*-module for all* $i \geq 0$.

*Proof 3.3: It is clear that, for each* $i$, $H_{\mathfrak{a}}^i(R,N) = H_{\mathfrak{a}}^i(N)$ *is a finitely generated* $R$*-module, and it follows easily (using the additivity of the generalized local cohomology functor) that* $H_{\mathfrak{a}}^i(F,N)$ *is finitely generated whenever* $F$ *is a finitely generated free* $R$*-module.*

*Suppose* $M$ *is an arbitrary finitely generated* $R$*-module.* $M$ *can be included in an exact sequence*

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0, \tag{3.3}$$

*of finite* $R$*-modules in which* $F$ *is free. Since* $N$ *is an* $\mathfrak{a}$*-torsion* $R$*-module, by Corollary 2.28, there exists an injective resoluiton* $E^\circ$ *of* $N$ *in which each term is an* $\mathfrak{a}$*-torsion* $R$*-module. Since all the terms of* $E^\circ$ *are injective, the above sequence induces an exact sequence*

$$0 \longrightarrow Hom_R(M, E^\circ) \longrightarrow Hom_R(F, E^\circ) \longrightarrow Hom_R(L, E^\circ) \longrightarrow 0, \qquad (3.4)$$

*of complexes. Note that, for a finitely generated* $R$*-module* $S$*, and an* $\mathfrak{a}$*-torsion* $R$*-module* $T$*, we have* $\Gamma_{\mathfrak{a}}\big(Hom_R(S,T)\big) = Hom_R(S,T)$*. Hence in the view of Corollary 2.46, we obtain the following long exact sequence of generalized local cohomology modules, which induces from the above exact sequence of complexes by applying the functor,* $\Gamma_{\mathfrak{a}}$ *on it*

$$\begin{aligned}
&0 \to H^0_{\mathfrak{a}}(M,N) \to H^0_{\mathfrak{a}}(F,N) \to H^0_{\mathfrak{a}}(L,N)\\
&\to H^1_{\mathfrak{a}}(M,N) \to H^1_{\mathfrak{a}}(F,N) \to H^1_{\mathfrak{a}}(L,N)\\
&\to \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.5)\\
&\to H^i_{\mathfrak{a}}(M,N) \to H^{\mathrm{i}}_{\mathfrak{a}}(F,N) \to H^{\mathrm{i}}_{\mathfrak{a}}(L,N)\\
&\to H^{i+1}_{\mathfrak{a}}(M,N) \to \cdots
\end{aligned}$$

*We have already noted that* $H^i_{\mathfrak{a}}(F,N)$ *is finitely generated for all* $i \geq 0$*. Hence* $H^1_{\mathfrak{a}}(M,N)$ *is finitely generated. But* $M$ *was an arbitrary finitely generated* $R$*-module, so,* $H^1_{\mathfrak{a}}(L,N)$ *is finitely generated. The above long exact sequence shows that* $H^2_{\mathfrak{a}}(M,N)$ *is finitely generated. Now use induction and so we are done.*

In the following theorem, by using the ideas of the proof of Theorem 1.2 which was proved in [Asadollahi et al., 2002], we provide a partial answer to when the set of associated primes of $H^n_{\mathfrak{a}}(M)$ is finite.

*Theorem 3.4: Let* $r$ *be a non-negative integer and* $M$ *be a finitely generated* $R$*-module. If* $N$ *and* $H^i_{\mathfrak{a}}(N,M)$ *are weakly Laskerain modules for all* $i < r$*, then* $Hom_R(R/\mathfrak{a}, H^r_{\mathfrak{a}}(M,N))$ *is also weakly Laskerian. In particular,* $Ass_R(H^r_{\mathfrak{a}}(M,N))$ *is finite.*

*Proof 3.4: We use induction on* $r$*. If* $r = 0$*,* $H^0_{\mathfrak{a}}(M,N) \cong Hom_R(M, \Gamma_{\mathfrak{a}}(N))$ *is weakly Laskerian. Now let* $r \in \mathbb{N}$*. From the exact sequence*

$$0 \to \Gamma_{\mathfrak{a}}(N) \to N \to N/\Gamma_{\mathfrak{a}}(N) \to 0\,, \tag{3.6}$$

*we obtain an exact sequence*

$$H_{\mathfrak{a}}^{i}(M,\Gamma_{\mathfrak{a}}(N)) \xrightarrow{f_i} H_{\mathfrak{a}}^{i}(M,N) \xrightarrow{g_i} H_{\mathfrak{a}}^{i}(M,N/\Gamma_{\mathfrak{a}}(N)) \to H_{\mathfrak{a}}^{i+1}(M,\Gamma_{\mathfrak{a}}(N) \tag{3.7}$$

*Moreover by Lemma 3.2 and Lemma 3.1,*

$$H_{\mathfrak{a}}^{i}\big(M,\Gamma_{\mathfrak{a}}(N)\big) \cong Ext_{R}^{i}\big(M,\Gamma_{\mathfrak{a}}(N)\big) \tag{3.8}$$

*is weakly Laskerian for all* $i \in \mathbb{N}$. *Therefore,* $H_{\mathfrak{a}}^{i}(M,N/\Gamma_{\mathfrak{a}}(N))$ *is weakly Laskerian for all* $i < r-1$. *Also we have two exact sequences*

$$0 \to Hom_R(R/\mathfrak{a},Imf_r) \to Hom_R(R/\mathfrak{a},H_{\mathfrak{a}}^{r}(M,N)) \to Hom_R(R/\mathfrak{a},Img_r) \tag{3.9}$$

*and*

$$Hom_R(R/\mathfrak{a},Img_r) \to Hom_R(R/\mathfrak{a},H_{\mathfrak{a}}^{r}(M,N/\Gamma_{\mathfrak{a}}(N))) \tag{3.10}$$

*In view of (3.2),* $Imf_r$ *is weakly Laskerian. Also, it follows from the exact sequence (3.1) that* $H_{\mathfrak{a}}^{i}(M,N/\Gamma_{\mathfrak{a}}(N))$ *is weakly Laskerian. Hence by using the exact sequences (3.3) and (3.4), it is enough for us to prove that* $Hom_R(R/\mathfrak{a},H_{\mathfrak{a}}^{r}(M,N/\Gamma_{\mathfrak{a}}(N)))$ *is weakly Laskerian. Thus, we can assume that* $\Gamma_{\mathfrak{a}}(N) = 0$, *so there is an element* $x \in \mathfrak{a}$ *which is a nonzero divisor on* $N$. *The exact sequence*

$$0 \to N \xrightarrow{x} N \to N/xN \to 0, \tag{3.11}$$

*induces an exact sequence*

$$H_{\mathfrak{a}}^{r-1}(M,N) \xrightarrow{k} H_{\mathfrak{a}}^{r-1}(M,N/xN) \xrightarrow{h} H_{\mathfrak{a}}^{r}(M,N) \xrightarrow{x} H_{\mathfrak{a}}^{r}(M,N). \tag{3.12}$$

*By assumption, we can see that* $Imk$ *is weakly Laskerian. Hence* $H_{\mathfrak{a}}^{i}(M, N/xN)$ *is weakly Laskerian for all* $i < r - 1$ *and by induction hypothesis,*

$$Hom_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{r-1}(M, N/xN)) \tag{3.13}$$

*is also weakly Laskerian. Now consider the exact sequence*

$$0 \to Imk \to H_{\mathfrak{a}}^{r-1}(M, N/xN) \to Imh \to 0. \tag{3.14}$$

*Since* $Hom_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{r-1}(M, N/xN))$ *is weakly Laskerian, by applying the functor,* $Hom_R(R/\mathfrak{a}, -)$, *we see that* $Hom_R(R/\mathfrak{a}, Imh)$ *is also weakly Laskerian. Finally, we consider the exact sequence*

$$0 \to Imh \to H_{\mathfrak{a}}^{r}(M, N) \overset{x}{\to} H_{\mathfrak{a}}^{r}(M, N). \tag{3.15}$$

*Since* $x \in \mathfrak{a}$ *and* $Hom_R(R/\mathfrak{a}, Imh)$ *is weakly Laskerian,* $0:_{Hom_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{r}(M,N)} x$ *is weakly Laskerian, and the result is clear.*

The following corollary is a consequence of the previous theorem. Also it is in the [Divani-Aazar and Mafi, 2004].

*Corollary 3.5: Let* $r$ *be a non negative integer. Assume that* $N$ *is weakly Laskerian and* $H_{\mathfrak{a}}^{i}(N)$ *is weakly Laskerian for all* $i < r$. *Then* $Ass_R(H_{\mathfrak{a}}^{r}(N))$ *is finite.*

The following theorem is another main result in this section.

*Theorem 3.6: Let* $r$ *be a non negative integer. Assume that* $N$ *is finitely generated* $R$*-module such that* $Ext_R^{j}(M, H_{\mathfrak{a}}^{i}(N))$ *is weakly Laskerian for all* $i, j \in \mathbb{N}_0$ *with* $i \leq r$. *Then* $H_{\mathfrak{a}}^{i}(M, N)$ *is* $\mathfrak{a}$*-weakly cofinite for all* $i \leq r$.

*Proof 3.6: We use induction on* $r$. *If* $r = 0$, *then*

$$Ext_R^j(R/\mathfrak{a}, H_\mathfrak{a}^0(M,N)) \cong Ext_R^j\left(R/\mathfrak{a}, Hom_R\left(M, \Gamma_\mathfrak{a}(N)\right)\right) \tag{3.16}$$

*is weakly Laskerian for all* $j \in \mathbb{N}_0$. *Now suppose that* $r \in \mathbb{N}$ *and the case* $r-1$ *is settled. From the short exact sequence*

$$0 \to \Gamma_\mathfrak{a}(N) \to N \to N/\Gamma_\mathfrak{a}(N) \to 0, \tag{3.17}$$

*we obtain the long exact sequence*

$$\cdots \to H_\mathfrak{a}^r(M, \Gamma_\mathfrak{a}(N)) \xrightarrow{f} H_\mathfrak{a}^r(M,N) \xrightarrow{g} H_\mathfrak{a}^r(M, N/\Gamma_\mathfrak{a}(N)) \xrightarrow{h} H_\mathfrak{a}^{r+1}(M, \Gamma_\mathfrak{a}(N) \to \cdots \tag{3.18}$$

*so for all for all* $j \in \mathbb{N}_0$, *we have two exact sequences*

$$Ext_R^j(R/\mathfrak{a}, Imf) \to Ext_R^j(R/\mathfrak{a}, H_\mathfrak{a}^r(M,N)) \to Ext_R^j(R/\mathfrak{a}, Img) \tag{3.19}$$

*and*

$$Ext_R^{j-1}(R/\mathfrak{a}, Imh) \to Ext_R^j(R/\mathfrak{a}, Img) \to Ext_R^j\left(R/\mathfrak{a}, H_\mathfrak{a}^r(M, N/\Gamma_\mathfrak{a}(N))\right). \tag{3.20}$$

*Therefore, it is enough to prove that* $Ext_R^j\left(R/\mathfrak{a}, H_\mathfrak{a}^r(M, N/\Gamma_\mathfrak{a}(N))\right)$ *is weakly Laskerian for all* $j \in \mathbb{N}_0$. *Moreover, in view of Lemma 3.2,*

$$H_\mathfrak{a}^i(M, N/\Gamma_\mathfrak{a}(N)) \cong Ext_R^j\left(M, \Gamma_\mathfrak{a}(N)\right) \tag{3.21}$$

*for all* $j \in \mathbb{N}_0$. *Hence by using the above long exact sequence in conjection with the fact that* $H_\mathfrak{a}^i(N/\Gamma_\mathfrak{a}(N)) \cong H_\mathfrak{a}^i(N)$ *for all* $i \in \mathbb{N}_0$, *we can assume that* $\Gamma_\mathfrak{a}(N) = 0$. *Let* $E := E(N)$ *be an injective hull of* $N$ *and* $P := E/N$. *Then it is easy to see that* $\Gamma_\mathfrak{a}(E) = 0$. *Hence,* $H_\mathfrak{a}^0(M,E) \cong Hom_R\left(M, \Gamma_\mathfrak{a}(E)\right) = 0$. *Also, since* $E$ *is injective, we can obtain*

$$H_\mathfrak{a}^i(M,E) = \varinjlim_{n \in \mathbb{N}} Ext_R^i\,(M/\mathfrak{a}^n M, E) = 0 \tag{3.22}$$

*for all* $i \in \mathbb{N}$*. Therefore, by using the exact sequence* $0 \to N \to E \to P \to 0$ *and the long exact sequences of (generalized) local cohomology modules derived from it,* $H_{\mathfrak{a}}^{i}(M,P) \cong H_{\mathfrak{a}}^{i+!}(M,N)$ *and* $H_{\mathfrak{a}}^{i}(P) \cong H_{\mathfrak{a}}^{i+1}(N)$ *for all* $i \in \mathbb{N}_0$*. Now by induction hyothesis,* $H_{\mathfrak{a}}^{r-1}(M,P)$ *is* $\mathfrak{a}$*-weakly cofinite, hence* $H_{\mathfrak{a}}^{r}(M,N)$ *is also* $\mathfrak{a}$*-weakly cofinite.*

The following corollary establishes a connection between question (4), in the section of introduction , for ordinary local cohomology modules and (5) in the context of generalized local cohomology modules.

*Corollary 3.7: Let* $r$ *be a non negative integer. Assume that* $H_{\mathfrak{a}}^{i}(N)$ *is weakly Laskerian for all* $i \leq r$ *. Then, for every finitely generated* $R$*-module* $M$*,* $H_{\mathfrak{a}}^{i}(M,N)$ *is* $\mathfrak{a}$*-weakly cofinite for all* $i \leq r$*.*

# 4. SOCLES OF LOCAL COHOMOLOGY MODULES

In [Trlifaj and Koşan, 2011], Trlifaj and Koşan studied the finiteness of socles of local cohomology modules over a Gorenstein local ring. In this section, by using their methods, we provide a positive answer to the question v), in the section (1). To do this, let

$$E^{*}: 0 \longrightarrow N_R \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots \tag{4.1}$$

Be a (fixed) minimal injective coresolutin of $N$. Suppose that $\Omega_i(N)$ is the cosyzygy of $N$, in $E^{*}$. Moreover, we set $E^0 \coloneqq \Omega_0(N)(= N)$. By slight modifications of the proof of Lemma 1 in [Trlifaj and Koşan, 2011], we establish the following lemma:

*Lemma 4.1: Let* $s \geq 0$, $\mathfrak{a} \subseteq \mathfrak{b}$ *be ideals of commutative Noetherian ring* $R$, *and let* $M, N$ *be* $R$*-modules (not necessarily weakly Laskerian). For* $s = 0$, *assume that* $Hom_R(R/\mathfrak{b}, Hom_R(M,N))$ *is weakly Laskerian. For* $s \geq 1$, *assume that* $Ext_R^s(M,N) = 0$, *and that* $Ext_R^1(R/\mathfrak{b}, Hom_R(M, E^{s-1}))$ *and* $Ext_R^2(R/\mathfrak{b}, H_{\mathfrak{a}}^0(M, \Omega_{s-1}(N)))$ *are weakly Laskerian* $R$ *-modules. Then* $Hom_R(R/\mathfrak{b}, H_{\mathfrak{a}}^s(M,N))$ *is a weakly Laskerian* $R$*-module. Moreover, if we assume that* $F$ *is a finitely generated* $R$ *-module such that* $Supp_R(F) \subseteq V(\mathfrak{b})$ *then* $Hom_R(F, H_{\mathfrak{a}}^s(M,N))$ *is a weakly Laskerian* $R$*-module.*

*Proof 4.1: We will prove this by induction on* $s$.

*i) If* $s = 0$, *then by assumtion,* $Hom_R(R/\mathfrak{b}, Hom_R(M,N))$ *is weakly Laskerian. Since for each* $n \in \mathbb{N}$, *there is a canonical inclusion* $Hom_R(M/\mathfrak{b}^n M, N) \subseteq Hom_R(M,N)$, *also* $H_{\mathfrak{a}}^0(M,N) \subseteq Hom_R(M,N)$. *So* $Hom_R(R/\mathfrak{b}, H_{\mathfrak{a}}^0(M,N))$ *is weakly Laskerian.*

*ii) If* $s = 1$, *then,* $Ext_R^1(R/\mathfrak{b}, Hom_R(M,N))$ *and* $Ext_R^2(R/\mathfrak{b}, H_{\mathfrak{a}}^0(M,N))$ *are weakly Laskerian* $R$*-modules by assumption. We will prove that* $Hom_R(R/\mathfrak{b}, H_{\mathfrak{a}}^1(M,N))$ *is a weakly Laskerian* $R$*-module. Consider the exact sequence*

$$0 \to H_{\mathfrak{a}}^0(M,N) \subseteq Hom_R(M,N) \to Hom_R(M,N)/H_{\mathfrak{a}}^0(M,N) \to 0. \tag{4.2}$$

*Then, we have the long exact sequence*

$$\cdots \to Ext_R^1(R/\mathfrak{b}, Hom_R(M,N)) \to Ext_R^1(R/\mathfrak{b}, Hom_R(M,N)/H_{\mathfrak{a}}^0(M,N) \to Ext_R^2(R/\mathfrak{b}, H_{\mathfrak{a}}^0(M,N) \to \cdots, \tag{4.3}$$

*So* $Ext_R^1(R/\mathfrak{b}, Hom_R(M,N)/H_{\mathfrak{a}}^0(M,N))$ *is weakly Laskerian. Note that*

$$0 \to Hom_R(M,N)/H_{\mathfrak{a}}^0(M,N) \to D_{\mathfrak{a}}(M,N) \to H_{\mathfrak{a}}^1(M,N) \to 0 \tag{4.4}$$

*is exact where*

$$D_{\mathfrak{a}}(M,-) \cong \lim_{n\in\mathbb{N}} Hom_R\,(\mathfrak{a}^n M,-) \tag{4.5}$$

*is the generalized ideal transform functor (see Theorem 2.2.4 in [Brodmann and Sharp, 1998]). Then the sequence*

$$\cdots \to Hom_R(R/\mathfrak{b}, D_{\mathfrak{a}}(M,N)) \to Hom_R(R/\mathfrak{b}, H_{\mathfrak{a}}^1(M,N)) \to Ext_R^1(R/\mathfrak{b}, Hom_R(M,N)/H_{\mathfrak{a}}^0(M,N)) \to \cdots, \tag{4.6}$$

*is exact. Since* $\mathfrak{a} \subseteq \mathfrak{b}$, $D_{\mathfrak{a}}(M,N)$ *contains no non zero elements annihilated by* $\mathfrak{b}$, *so the module* $Hom_R(R/\mathfrak{b}, D_{\mathfrak{a}}(M,N))$ *is zero. We have already prove that* $Ext_R^1(R/\mathfrak{b}, Hom_R(M,N)/H_{\mathfrak{a}}^0(M,N))$ *is weakly Laskerian, hence the same is true of the module* $Hom_R(R/\mathfrak{b}, H_{\mathfrak{a}}^1(M,N))$.

*Now suppose that* $s > 1$, *and assume that the claim is true for* $s-1$. *We will prove it for* $s$. *Since*

$$H_{\mathfrak{a}}^s(M,N) \cong \lim_{n\in\mathbb{N}} Ext_R^{s-1}\left(M/\mathfrak{a}^n M, \Omega_1(N)\right) = H_{\mathfrak{a}}^{s-1}\left(M, \Omega_1(N)\right), \tag{4.7}$$

*it suffices to prove that* $Hom_R(R/\mathfrak{b}, H_{\mathfrak{a}}^{s-1}\left(M, \Omega_1(N)\right))$ *is weakly Laskerian. For this purpose, we only have to verify the assumptions of our lemma for* $s-1$ *and the pair* $(M, \Omega_1(N))$.

*The first assumption for* $(M,\Omega_1(N))$ *says that* $Ext_R^{s-1}\big(M,\Omega_1(N)\big) = 0$. *This holds since* $Ext_R^{s-1}\big(M,\Omega_1(N)\big) \cong Ext_R^{s}(M,N) = 0$ . *The second assumption for* $(M,\Omega_1(N))$ *says the module* $Ext_R^1(R/\mathfrak{b}, Hom_R(M,E^{1+s-2}))$ *is weakly Laskerian; but this is just the second assumption for* $(M,N)$. *Also the third assumption for* $(M,\Omega_1(N))$ *and* $(M,N)$ *coincide because* $\Omega_{s-2}(N)\Omega_1(N) = \Omega_{s-1}(N)$.

*For the moreover part, we first recall a classical result of Gruson saying that* $F$ *contains a finite chain of submodules* $0 \subseteq F_0 \subseteq \cdots \subseteq F_n = F$ *such that* $F_i/F_{i-1}$ *is a homomorphic image of a finite direct sum of copies of* $R/\mathfrak{b}$, *for each* $0 < i \leq n$. *So there is an epimorphism* $(R/\mathfrak{b})^{m_i} \to F_i/F_{i-1}$ , *for some* $m_i > 0$ . *Then* $Hom_R(F_i/F_{i-1}, H_{\mathfrak{a}}^s(M,N)) \subseteq (Hom_R(R/\mathfrak{b}, H_{\mathfrak{a}}^s(M,N)))^{m_i}$ *and the latter module is weakly Laskerian by the first part. By induction on* $n$ , *we obtain that* $Hom_R(F, H_{\mathfrak{a}}^s(M,N))$ *is weakly Laskerian.*

Now let $(R,\mathfrak{m})$ be a $k$-dimensional Gorenstein local ring and $\mathfrak{a}$ be an ideal of $R$ . Suppose that

$$0 \longrightarrow R \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \to G^k \to 0 \tag{4.8}$$

is the minimal injective coresolutin of $R$. Also for each $i$, let $Q_i$ be the set of prime ideals in $V(\mathfrak{a})$ of height $i$ and $K_i := \oplus_{\mathfrak{p}_i \in Q_i} E(R/\mathfrak{p}_i)$, where for a prime ideal $\mathfrak{p}$ of $R$, $E(R/\mathfrak{p})$ is the injective hull of the residue field $R/\mathfrak{p}$. Put $S_i := \Omega_i(R) + K_i$.

By using a method similar to the one in the proof of Lemma 2 of [Trlifaj and Koşan, 2011], in conjuction with Lemma 4.1, one can obtain the following theorem:

*Theorem 4.2: Let* $R$ *be a Gorenstein local ring of Krull dimension* $k$ *and* $\mathfrak{a}$ *be an ideal of* $R$. *Then the following conditions are equivalent:*

*i) The socle of* $G_i/S_i$ *is weakly Laskerian for each* $1 \leq i < k$.
*ii) The socle of* $H_{\mathfrak{a}}^n(R)$ *is weakly Laskerian for each* $n \in \mathbb{N}_0$.

*Proof 4.2 : Clearly, both (i) and (ii) hold for* $\mathfrak{a} = R$, *so we may assume that* $\mathfrak{a} \subseteq \mathfrak{m}$. *Assume (i). In order to prove (ii), we have to only verify the assumptions of previous Lemma, in the given setting, that is, for* $s = n$, $M = N = R$, $\mathfrak{b} = \mathfrak{m}$, $E^i = G_{i-1}$, *for*

*$i \geq 1$ (where $G_i = 0$ for $i \geq k+1$ ) and $F = R/\mathfrak{m}$ , where $\mathfrak{m}$ denotes the maximal ideal of $R$.*

*First, $Hom_R(R/\mathfrak{m}, Hom_R(R,R)) \cong, Hom_R(R/\mathfrak{m}, R) \cong Soc(R)$ is weakly Laskerian. Clearly $Ext_R^n(R,R) = 0$, for $n \geq 1$ since $R$ is projective. For $n = 1$, we see that $Ext_R^n(R/\mathfrak{m}, R)$ is weakly Laskerian because $R/\mathfrak{m}$ and $R$ are such. If $n \geq 2$, then*

$$Ext_R^1(R/\mathfrak{m}, Hom_R(R, G_{n-2})) = Ext_R^1(R/\mathfrak{m}, G_{n-2}) = 0, \tag{4.9}$$

*because $G_{n-2}$ is injective. It remains to prove that $Ext_R^1(R/\mathfrak{m}, H_\mathfrak{a}^0(\Omega_{n-1}(R))\,)$ is weakly Laskerian for each $n \geq 1$. We recall that for a module, $M$, $H_\mathfrak{a}^0(M) \cong \Gamma_\mathfrak{a}(M)$ where*

$$\Gamma_\mathfrak{a}(M) = \{x \in M \mid \exists\, k \geq 0 \colon \mathfrak{a}^k x = 0\}. \tag{4.10}$$

*In particular, $\Gamma_\mathfrak{a}(E(R/p)) = E(R/p)$, when $p \in V(\mathfrak{a})$ and $\Gamma_\mathfrak{a}(E(R/p)) = 0$, when $p \notin V(\mathfrak{a})$. For $n = 1$, we have $H_\mathfrak{a}^0(\Omega_0(R)) = H_\mathfrak{a}^0(R) \subseteq Hom_R(R,R) \cong R$, hence $H_\mathfrak{a}^0(\Omega_0(R))$ is weakly Laskerian, and so is $Ext_R^2(R/\mathfrak{m}, H_\mathfrak{a}^0(\Omega_0(R))\,)$. If $n \geq k$ , then $\Gamma_\mathfrak{a}(\Omega_{n-1}(R)) = \Omega_{n-1}(R)$ is injective, so we have $Ext_R^2(R/\mathfrak{m}, H_\mathfrak{a}^0(\Omega_{n-1}(R))\,) = 0$. Assume $1 < n \leq k$. Then $\Omega_{n-1}(R)$ is an essential submodule of $G_{n-1}$. So $\Gamma_\mathfrak{a}(\Omega_{n-1}(R)) = \Omega_{n-1}(R) \cap K_{n-1}$ and $K_{n-1}$ is the injective envelope of $\Gamma_\mathfrak{a}(\Omega_{n-1}(R))$. We have*

$$K_{n-1}/\Gamma_\mathfrak{a}(\Omega_{n-1}(R)) \cong S_{n-1}/\Omega_{n-1}(R) \subseteq G_{n-1}/\Omega_{n-1}(R) = \Omega_n(R) \subseteq G_n \tag{4.11}$$

*so the exact sequence*

$$0 \to \Gamma_\mathfrak{a}(\Omega_{n-1}(R)) \to K_{n-1} \to K_{n-1}/\Gamma_\mathfrak{a}(\Omega_{n-1}(R)) \to 0 \tag{4.12}$$

*yields*

$$\begin{aligned} Ext^2_R(R/\mathfrak{m}, H^0_{\mathfrak{a}}(\Omega_{n-1}(R))\,) &= Ext^2_R\left(R/\mathfrak{m}, \Gamma_{\mathfrak{a}}\big(\Omega_{n-1}(R)\big)\right) \\ &\cong Ext^2_R(R/\mathfrak{m}, S_{n-1}/\Omega_{n-1}(R)\,) \end{aligned} \tag{4.13}$$

*First consider the case of* $n = k$. *Then* $G_n = E(R/\mathfrak{m}) = \Omega_n(R)$ *is an injective module containing* $S_{n-1}/\Omega_{n-1}(R)$*, so in order to prove that* $Ext^2_R(R/\mathfrak{m}, H^0_{\mathfrak{a}}(\Omega_{n-1}(R))\,)$ *is weakly Laskerian, it is enough to show that* $Hom_R(R/\mathfrak{m}, G_n/S_{n-1}/\Omega_{n-1}(R))$ *is such. But this is clear because the modulo* $G_n$*, and hence* $G_n/S_{n-1}/\Omega_{n-1}(R)$ *is Artinian.*

*Now assume* $1 < n < k$. *In order to prove that* $Ext^1_R(R/\mathfrak{m}, S_{n-1}/\Omega_{n-1}(R)\,)$ *is weakly Laskerian, it suffices to show that* $Hom_R(R/\mathfrak{m}, G_n/S_{n-1}/\Omega_{n-1}(R))$ *is such. We have the exact sequence*

$$\begin{aligned} 0 \to G_{n-1}/\Omega_{n-1}(R)/S_{n-1}/\Omega_{n-1}(R) \subseteq G_n/S_{n-1}/\Omega_{n-1}(R) \to\ G_n/\Omega_n(R) \\ \to 0 \end{aligned} \tag{4.14}$$

*where* $G_n/G_{n-1}/\Omega_{n-1}(R) =\ G_n/\Omega_n(R) = \Omega_{n+1}(R) \subseteq G_{n+1}$. *Note that if* $n < k-1$, *then* $Hom_R(R/\mathfrak{m}, G_{n+1}) = 0$ *and if* $n = k-1$, *then* $Hom_R(R/\mathfrak{m}, G_{n+1}) = Hom_R\big(R/\mathfrak{m}, E(R/\mathfrak{m})\big) \cong Soc_R(E(R/\mathfrak{m})) \cong R/\mathfrak{m}$. *So in either case* $Hom_R(R/\mathfrak{m}, G_n/S_{n-1}/\Omega_{n-1}(R))$ *is weakly Laskerian. It remains to show that* $Hom_R(R/\mathfrak{m},\ G_n/S_{n-1})$ *is weakly Laskerian. But this is exactly our assumption in i). This finishes the proof of ii). Assume ii). First note that* $Hom_R(R/\mathfrak{m}, H^1_{\mathfrak{a}}(\Omega_i(R))$ *is weakly Laskerian for each* $i \geq 1$, *because* $H^n_{\mathfrak{a}}(N) \cong H^{n-1}_{\mathfrak{a}}(\Omega_1(N))$ *for each module* $N$, *and each* $n > 1$.

*We claim that* $Ext^1_R(R/\mathfrak{m}, D_{\mathfrak{a}}(\Omega_i(R))\,) = 0$ *for all* $1 \leq i < k$ *where* $D_{\mathfrak{a}}(-)$ *is the classical ideal transform functor. Since* $D_{\mathfrak{a}}(-)$ *is left exact, we have the exact sequence*

$$0 \to D_{\mathfrak{a}}\big(\Omega_i(R)\big) \to D_{\mathfrak{a}}(G_i) \to T_i \to 0, \tag{4.15}$$

*where* $T_i \subseteq D_{\mathfrak{a}}(G_i/\Omega_i(R)) = D_{\mathfrak{a}}\big(\Omega_{i+1}(R)\big)$.

*Notice that* $Hom_R(R/\mathfrak{m}, T_i) \subseteq Hom_R\left(R/\mathfrak{m}, D_{\mathfrak{a}}\left(\Omega_{i+1}(R)\right)\right) = 0$ *because* $D_{\mathfrak{a}}\left(\Omega_{i+1}(R)\right)$ *contains no elements annihilated by* $\mathfrak{m}(\supseteq \mathfrak{a})$. *On the other hand, by injectivity of* $G_i$ *yield the exact sequence*

$$0 \to G_i/\Gamma_{\mathfrak{a}}(G_i) \to D_{\mathfrak{a}}(G_i) \to H_{\mathfrak{a}}^1(G_i) = 0, \tag{4.16}$$

*where* $G_i/\Gamma_{\mathfrak{a}}(G_i)$ *is injective, so* $Ext_R^1(R/\mathfrak{m}, D_{\mathfrak{a}}(\Omega_i(R))) = 0$. *Then our cliam follows from the exactness of the sequence*

$$\begin{aligned} 0 \to Hom_R(R/\mathfrak{m}, T_i) &\to Ext_R^1(R/\mathfrak{m}, D_{\mathfrak{a}}(\Omega_i(R))) \\ &\to Ext_R^1(R/\mathfrak{m}, D_{\mathfrak{a}}(G_i)) = 0 \end{aligned} \tag{4.17}$$

*We have the exact sequence*

$$\begin{aligned} &Hom_R\left(R/\mathfrak{m}, H_{\mathfrak{a}}^1(\Omega_i(R))\right) \to Ext_R^1(R/\mathfrak{m}, \Omega_i(R)/\Gamma_{\mathfrak{a}}(\Omega_i(R))) \\ &\to Ext_R^1(R/\mathfrak{m}, D_{\mathfrak{a}}(\Omega_i(R))) = 0. \end{aligned} \tag{4.18}$$

*This shows that the module* $Ext_R^1(R/\mathfrak{m}, \Omega_i(R)/\Gamma_{\mathfrak{a}}(\Omega_i(R)))$ *is weakly Laskerian.*

*Finally,* $\Omega_i(R)/\Gamma_{\mathfrak{a}}(\Omega_i(R)) \cong S_i/K \subseteq G_i/K_i \cong K_i{}'$. *Since* $i < k$ , $Hom_R(R\mathfrak{m}, K_i{}') = 0$ , *so* $Hom_R(R/\mathfrak{m}, G_i/S_i) \cong Ext_R^1(R/\mathfrak{m}, S_i/K_i)$ *is weakly Laskerian and (i) holds.*

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# BIOGRAPHY

Serap ŞAHİNKAYA, was born in 01.10.1986 in İstanbul-Kartal. She completed primary and secondary education in İstanbul. She graduated from Kocaeli University, Department of Mathematics, in 2008. She got her master degree from Gebze Institute of Technology in February, 2011. She has been working as a research asistant at Department of Mathematics of Gebze Institute of Technology since July 2009.