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GRADUATE SCHOOL of ENGINNERING and SCIENCES

QUALITATIVE ANALYSIS of DYNAMICAL SYSTEMS on TIME SCALES with INITIAL TIME DIFFERENCE

BÜLENT OĞUR

A THESIS SUBMITTED for THE DEGREE of
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SUMMARY

In this thesis, dynamic systems on time scales is studied. The basic definitions and theorems about the time scales are given. Dynamic inequalities and comparison methods are developed on arbitrary time scales with initial time difference. Stability theory of dynamical systems on time scales that have initial time difference is developed with the help of Lyapunov-like functions. Boundedness criteria of dynamic system on time scales is established with initial time difference by employing Lyapunov-like functions. The behavior of solutions of perturbed dynamic system with respect to original unperturbed dynamic system that have initial time difference are investigated on arbitrary time scales. Notions of stability, asymptotic stability and instability with initial time difference are introduced. Sufficient conditions of stability properties are given with the help of Lyapunov like functions.

Key Words: Time Scales; Stability; Initial Time Difference (ITD); Lyapunov Functions; Boundedness; Comporison Method.

ÖZET

Bu tezde zaman skalasında dinamik sistemler çalışıldı. Zaman skalası hakkında temel tanım ve teoremler verildi. Başlangıç zaman farklı dinamik eşitsizlikler ve karşılaştırma metodu keyfi zaman skalasında geliştirildi. Liapunov (Lyapunov) fonksiyonları yardımıyla başlangıç zaman farkı olan dinamik sistemler için keyfi zaman skalasında kararlılık teorisi geliştirildi. Lyapunov fonksiyonları ile başlangıç zaman farkı olan dinamik sistemler için keyfi zaman skalasında sınırlılık kriteri verildi. Keyfi zaman skalasında başlangıç zaman farkı olan pertörb dinamik çözümleri orijinal pertörb olmayan dinamik sistemin çözümlerine göre davranışı incelendi. Başlangıç zaman farklı kararlılık, asimptotik kararlılık ve kararsızlık kavramları tanıtıldı. Liapunov fonksiyonları yardımıyla kararlılık özellikleri için yeterli koşullar verildi.

Anahtar Kelimeler: Zaman Skalası; Kararlılık; Başlangıç Zaman Farkı (ITD); Lyapunov Fonksiyonu; Sınırlılık; Karşılaştırma Metodu.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
Δ	:	Operator of Delta
rd	:	Right dense
ld	:	Left dense
ITD	:	Initial Time Difference
$\mathbb{T}$	:	Time scale
GYTE	:	Gebze Yüksek Teknoloji Enstitüsü
rs	:	Right scattered
ls	:	Left scattered
$\mathbb{C}_h$	:	Hilger Complex Plane
$\mathbb{R}_h$	:	Hilger Real Axis

1. INTRODUCTION

Differential and difference equations are among the most important mathematical tools used in producing models of physical and biological sciences and engineering. Difference equations are also appears in the study of discretization methods for differential equations. Several results of differential equations do just translate themselves into analogous results in difference equations. This naturally raises the question whether it is possible to unify the theory of differential and difference equations into a single set up. The answer is yes and we now have necessary calculus and the fundamental existence theory for dynamic systems on times scales [1] - [3], [11], [14]. A brief of results of this theory is given in Chapter 2. From a modeling point of view, it is perhaps more realistic to model a phenomenon by a dynamic system which incorporates both continuous and discrete times, namely, time as an arbitrary closed set of real called time scales. Many phenomenons in physical and biological sciences and engineering are interpreted in terms of dynamic equations and their solutions. As a result, stability theory plays a central role in systems theory and engineering. There are different kinds of stability problems that arise in the study of dynamical systems on time scale. Asymptotic stability, exponentially asymptotic stability, eventual stability, Lipschitz stability, practical stability, stability of conditionally invariant sets are examples of stability problems that studied in the literature [4] - [6], [9], [10], [12] - [14].

The investigation of initial value problems with a perturbation in the space variable is well known when the perturbation is restricted to the space variable with initial time unchanged [4] - [6], [9], [10], [12] - [14]. However, in real world applications, it is possible to make an error in the initial time in addition to the initial position. This results in a problem of measuring the difference of two solutions which differ in initial time and initial position. So far, several studies have been made on this problem to explore the stability, boundedness, etc., criteria for differential systems relative to initial time difference [17], [19] - [22], [24], [26], [27]. In Chapter 3, we investigate stability and boundedness criteria for dynamic systems on time scales relative to initial time difference by using comparison method. In Subsections, we present definitions and necessary background material. We discuss and compare the differences between classical notion of stability and the notion of initial time difference (ITD) stability. We obtain new dynamic inequalities and comparison results on time scales relative to initial time difference. In the framework of these results we obtain several ITD-stability and ITD-boundedness properties from the stability and boundedness properties of the comparison dynamic system. Finally, as an application of these new results we give examples.

In real world applications, it is necessary to consider a particular dynamic system with a perturbation term. In addition to perturbing the given particular dynamic system, it is possible to make an error in initial times as well as in initial positions. Therefore, one needs to investigate the qualitative and quantitative properties of a given particular dynamic system under these perturbations. A principal technique employed in stability theory is investigating stability properties of a particular dynamic system under small perturbations. This technique is employed in many ways [5], [9], [13] - [15]. Another techniques are also used in [5], [7] - [9], [13] - [15]. In [21], [22], [26], [27] the authors considered the case that perturbed dynamic system and original unperturbed dynamic system which have different initial time. In Chapter 4, we consider the problem of determining the behavior of solutions of a perturbed dynamic equation with respect to those of original unperturbed dynamic system that have initial time difference (ITD). We consider this problem on arbitrary time scales, nonempty closed subset of real numbers, and therefore we obtained a general result that can be applied discrete and continuous cases simultaneously. We begin with a preliminary section which includes the basic concepts and definitions. Then we give the obtained novel results.

2. TIME SCALES FUNDAMENTALS

In this chapter we give the main definitions and characteristics of the calculus on time scales initiated by Aulbach and Hilger [1], [11] which comprise those features of the differential and difference calculus as they are relevant for the development of a qualitative theory of dynamical systems. We note that the contents of such a development of some higher ranging calculus is quite extensive. So we suffice only with giving the essentials necessary for the further aims of this work and refer to [2], [3], [14] for more details.

We begin this chapter with basic definitions of time scales and give the special type of Induction Principle that is used as a main tool in the arguments. Next, we continue by paying special attention to concepts of differentiability of a function on time scales. After that, we introduce the integrability of a function on time scales. In this section we give the concepts such as continuity, right dense (rd)- continuity, regularity of a function which posses important roles in the analysis of discrete and continuous scales in a unified manner. As a result of these concepts, we end this chapter by considering exponential functions on time scales, structure of dynamical systems on time scales and some dynamic inequalities that will be employed throughout the thesis.

2.1. Basic Definitions of Time Scales

A time scale is an arbitrary nonempty closed subset of real numbers. It is denoted by $\mathbb{T}$. Thus $\mathbb{R}$, $\mathbb{Z}$, i.e., the real numbers, the integers and the closed interval $[0,1]$ are examples of time scales. $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$, i.e., the rational numbers, the irrational numbers and the open interval $(0,1)$ are not time scales. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology and, for our future purposes, unbounded from above with $t_0 \geq 0$ as a minimal element.

Because of the fact that a time scale is not necessarily connected, we define the forward and backward jump operators on $\mathbb{T}$ as follows.

Definition 2.1: Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = inf\{s \in \mathbb{T}, s > t\} \tag{2.1}$$

while the backward jump operator $\rho: \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = sup\{s \in \mathbb{T}, s < t\} \tag{2.2}$$

A point $t \in \mathbb{T}$ is called right dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left dense if $\rho(t) = t$, left scattered if $\rho(t) < t$, dense if $t = \sigma(t) = \rho(t)$, isolated if $\sigma(t) > t > \rho(t)$.

Finally, the graininess function $\mu: \mathbb{T} \to \mathbb{R}_{+}$, which measure the gap between a point t and its right neighbor, is defined by

$$\mu(t) = \sigma(t) - t. \tag{2.3}$$

We need below the set $\mathbb{T}^{\kappa}$ is derived from the time scale $\mathbb{T}$ as follows : If a time scale $\mathbb{T}$ has a maximal element which is also left-scattered then it is called a degenerate point. $\mathbb{T}^{\kappa}$ represents the set of all non-degenerate points of $\mathbb{T}$. This set cuts off an eventually existing isolated maximum of $\mathbb{T}$. In summary,

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if } \sup\mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup\mathbb{T} = \infty \end{cases} \tag{2.4}$$

Example 2.1*: Let us briefly consider the two examples $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.*

- *If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$*

$$\sigma(t) = inf\{s \in \mathbb{T}, s > t\} = inf(t, \infty) = t \tag{2.5}$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be

$$\mu(t) \equiv 0 \;\; for\ all \quad t \in \mathbb{R}; \tag{2.6}$$

- *If* $\mathbb{T} = \mathbb{Z}$, *then we have for any* $t \in \mathbb{Z}$

$$\sigma(t) = inf\{s \in \mathbb{T}, s > t\} = inf\{t+1, t+2, \dots\} = t+1 \tag{2.7}$$

and similarly $\rho(t) = t - 1$. *Hence every point* $t \in \mathbb{Z}$ *is isolated. The graininess function* μ *is the case is*

$$\mu(t) \equiv 1 \quad for\ all \quad t \in \mathbb{Z}. \tag{2.8}$$

A basic tool which will be employed in the proofs later is contained in the following theorem as an induction principle.

Theorem 2.1: Let $t_0 \in \mathbb{Z}$ *and assume that* $\{A(t): t \in [t_0, \infty)\}$ *is a family of statements satisfying:*

i) The statement $A(t_0)$ *is true.*
ii) If $t \in [t_0, \infty)$ *is right-scattered and* $A(t)$ *is true, then* $A(\sigma(t))$ *is also true.*
iii) If $t \in [t_0, \infty)$ *is right-dense and* $A(t)$ *is true, then there is a neighborhood* U *of* t *such that* $A(s)$ *is true for all* $s \in U \cap (t, \infty)$.
iv) If $t \in (t_0, \infty)$ *is left-dense and* $A(s)$ *is true for all* $s \in [t_0, t)$, *then* $A(t)$ *is true.*

Then $A(t)$ *is true for all* $t \in [t_0, \infty)$.

Proof 2.1: See [2], [3].□

2.2. Differentiation of Functions on Time Scales

In this section we consider a function $u: \mathbb{T} \to \mathbb{R}$ and define the so-called *delta (or Hilger) derivative* of u at a point $t \in \mathbb{T}^{\kappa}$

Definition 2.2: Assume that $u: \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^{\kappa}$*. Then we define delta derivative* $u^{\Delta}(t)$ *at* t *to be the number (provided it exists) with the property that given any* $\varepsilon > 0$*, there is a neighborhood* N *of* t *such that*

$$|[u(\sigma(t)) - u(s)] - u^{\Delta}(t)(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \forall s \in N \tag{2.9}$$

Alternatively one can define,

$$u^{\Delta}(t) := \lim_{s \to t, s \neq \sigma(t)} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s}. \tag{2.10}$$

Theorem 2.2: Assume $u: \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^{\kappa}$*. Then we have the following:*

i) If u *is differentiable at* t*, then* u *is continuous at* t*.*

ii) If u *is continuous at* t *and* t *is right-scattered, then* u *is differentiable at* t *with*

$$u^{\Delta}(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)}. \tag{2.11}$$

iii) If t *is right-dense, then* u *is differentiable at* t *iff the limit*

$$\lim_{s \to t} \frac{u(t) - u(s)}{t - s} \tag{2.12}$$

exists as a finite number. In this case

$$u^{\Delta}(t) = \lim_{s \to t} \frac{u(t) - u(s)}{t - s}. \tag{2.13}$$

iv) If u *is differentiable at* t*, then*

$$u(\sigma(t)) = u(t) + \mu(t)u^{\Delta}(t). \tag{2.14}$$

Proof 2.2: See [2], [3].□

Example 2.2: We consider the two cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

- *If $\mathbb{T} = \mathbb{R}$, then Theorem 2.2 (iii) yields that $u: \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff*

$$u^{'}(t) = \lim_{s \to t} \frac{u(t) - u(s)}{t - s} \tag{2.15}$$

exists, i.e., iff u is differentiable (in the ordinary sense) at t. In this case we then have

$$u^{\Delta}(t) = \lim_{s \to t} \frac{u(t) - u(s)}{t - s} = u^{'}(t) \tag{2.16}$$

by Theorem 2.1 (iii).

- *If $\mathbb{T} = \mathbb{Z}$, then Theorem 2.1 (ii) yields that that $u: \mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ with*

$$u^{\Delta}(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)} = \frac{u(t+1) - u(t)}{1} = u(t+1) - u(t) = \Delta u(t) \tag{2.17}$$

where Δ is the usual forward difference operator defined by the last equation above.

Theorem 2.3: Assume $u, w: \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$. Then:

i) The sum $u + w: \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}^{\kappa}$ with

$$(u + w)^{\Delta}(t) = u^{\Delta}(t) + w^{\Delta}(t); \tag{2.18}$$

ii) For any constant $\alpha \in \mathbb{R}$, the function $\alpha u: \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}^{\kappa}$ with

$$(\alpha u)^{\Delta}(t) = \alpha u^{\Delta}(t); \tag{2.19}$$

iii) The product $uw: \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t \in \mathbb{T}^\kappa$ *with*

$$(uw)^\Delta(t) = u^\Delta(t)w(t) + u\big(\sigma(t)\big)w^\Delta(t) = u(t)w^\Delta(t) + u^\Delta(t)w(\sigma(t)); \qquad (2.20)$$

iv) If $u(t)u\big(\sigma(t)\big) \neq 0$, *then* $\frac{1}{u}$ *is differentiable at* $t \in \mathbb{T}^\kappa$ *with*

$$\left(\frac{1}{u}\right)^\Delta(t) = \frac{-u^\Delta(t)}{u(t)u\big(\sigma(t)\big)}; \qquad (2.21)$$

v) If $w(t)w\big(\sigma(t)\big) \neq 0$, *then* $\frac{u}{w}$ *is differentiable at* $t \in \mathbb{T}^\kappa$ *with*

$$\left(\frac{u}{w}\right)^\Delta(t) = \frac{u^\Delta(t)w(t) - u(t)w^\Delta(t)}{w(t)w\big(\sigma(t)\big)}. \qquad (2.22)$$

Proof 2.3: See [2], [3].□

In the course of this thesis, we frequently employ the generalized delta derivatives corresponding to Dini derivatives.

Definition 2.3: For each $t \in \mathbb{T}$, *let* N *be a neighborhood of* t. *Then we define the upper right Dini derivative* $D^+u^\Delta(t)$ *by the condition: for a given* $\varepsilon > 0$, *there exists a right neighborhood* $N_\varepsilon \subset N$ *of* t *such that*

$$\frac{u(\sigma(t)) - u(s)}{\sigma(t) - s} < D^+u^\Delta(t) + \varepsilon \ \ for \ \ s \in N_\varepsilon, \qquad (2.23)$$

in case $t \in \mathbb{T}$ *is right-scattered and* $u(t)$ *is continuous at* t *we have*

$$D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)} \qquad (2.24)$$

where $\mu(t) = \sigma(t) - t$. *Alternatively one can define,*

$$D^{+}u^{\Delta}(t) := \lim_{s\to t, s\neq\sigma(t)} sup \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s}. \tag{2.25}$$

2.3. Integration of Functions on Time Scales

Now, we define a class of functions which are integrable on arbitrary time scales. To achieve this, we first introduce the following two concepts.

Definition 2.4: A function $u: \mathbb{T} \to \mathbb{R}$ *is called regulated provided its right-sided limits exist (finite) at all right-dense points in* $\mathbb{T}$ *and its left-sided limits exist (finite) at all left dense points in* $\mathbb{T}$.

Definition 2.5: A function $u: \mathbb{T} \to \mathbb{R}$ *is called rd-continuous provided it is continuous at right-dense points in* $\mathbb{T}$ *and its left-sided limits exist (finite) at left-dense points in* $\mathbb{T}$. *The set of rd-continuous functions* $u: \mathbb{T} \to \mathbb{R}$ *will be denoted by* $C_{rd}(\mathbb{T}, \mathbb{R})$.

The following theorem gives some result about rd-continuous functions and regulated functions.

Theorem 2.4: Assume $u: \mathbb{T} \to \mathbb{R}$

i) If u *is continuous, then* u *is rd-continuous.*
ii) If u *is rd-continuous, then* u *is regulated.*
iii) The jump operator σ *is rd-continuous.*
iv) If u *is regulated or rd-continuous, then so is* $u\big(\sigma(t)\big)$.
v) Assume u *is continuous. If* $w: \mathbb{T} \to \mathbb{R}$ *is regulated or rd-continuous, then* $u \circ w$ *has that property too.*

Definition 2.6: A continuous function $u: \mathbb{T} \to \mathbb{R}$ *is pre-differentiable with (region of differentiation)* D, *provided* $D \subset \mathbb{T}^{\kappa}$, $\mathbb{T}^{\kappa} \backslash D$ *is countable and contains no right-scattered elements of* $\mathbb{T}$, *and* u *is differentiable at each* $t \in D$.

Theorem 2.5: Let u be regulated. Then there exists a function U which is pre-differentiable with region of differentiation D such that

$$U^{\Delta}(t) = u(t) \tag{2.26}$$

holds for all $t \in D$.

Definition 2.7: Assume $u: \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function as in Theorem 2.5 is called a pre-antiderivative of u.

We define the indefinite integral of a regulated function u by

$$\int u(t)\Delta t = U(t) + C \tag{2.27}$$

where C is an arbitrary constant and U is a pre-antiderivative of u.

We define the Cauchy integral by

$$\int_r^s u(t)\Delta t = U(s) - U(r) \quad for\ all\ r, s \in \mathbb{T} \tag{2.28}$$

A function $U: \mathbb{T} \to \mathbb{R}$ is called an anti-derivative of u: $\mathbb{T} \to \mathbb{R}$ provided

$$U^{\Delta}(t) = u(t) \ \ holds\ for\ all\ t \in \mathbb{T}^{\kappa}. \tag{2.29}$$

Definition 2.8: The mapping $f: \mathbb{T}^{\kappa} \times \mathbb{R} \to \mathbb{R}$ is called rd-continuous if it

i) is continuous at each (t, x) with right-dense or maximal t
and
ii) the limits $f(t^-, x) = \lim_{\substack{(s,y)\to(t,x) \\ s<t}} f(s, y)$ and $\lim_{y\to x} f(t, y)$ exist at each (t, x) with left-dense t.

2.4. The Exponential Function

In this section we will study linear first order dynamic equations and initial value problems for them. In fact, we will construct the solution of the initial value problem

$$y^{\Delta} = p(t)y \qquad y(t_0) = y_0 \tag{2.30}$$

explicitly, and we will call this solution the exponential function associated with the given time scale. Many of the results in this section can be found in [2], [3] and for more details we refer to them. Before defining the exponential function on time scales, we must introduce some terminology. So, we now introduce what we call the Hilger complex plane.

Definition 2.9: For $h > 0$ *we define the Hilger complex numbers, the Hilger real axis, the Hilger alternating axis, and the Hilger imaginary circle as*

- $\mathbb{C}_h := \left\{z \in \mathbb{C}: z \neq \frac{1}{h}\right\}$,
- $\mathbb{R}_h := \left\{z \in \mathbb{C}_h: z \in \mathbb{R} \ and\ z > -\frac{1}{h}\right\}$,
- $\mathbb{A}_h := \left\{z \in \mathbb{C}_h: z \in \mathbb{R} \ and\ z < -\frac{1}{h}\right\}$,
- $\mathbb{I}_h := \left\{z \in \mathbb{C}_h: \left|z + \frac{1}{h}\right| = \frac{1}{h}\right\}$,

respectively. For $h = 0$, *let* $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{I}_0 := i\,\mathbb{R}$, *and* $\mathbb{A}_0 := \emptyset$.

For $h > 0$, let Z_h be the strip $\mathbb{Z}_h := z \in \mathbb{C}$: $-\frac{\pi}{h} < Im(z) < \frac{\pi}{h}$ and for $h = 0$ let $\mathbb{Z}_h := \mathbb{C}$.

Definition 2.10: For $h > 0$, we define the cylinder transformation $\xi_h: \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \frac{1}{h} Log(1 + zh), \tag{2.31}$$

where Log is the principal logarithm function. For $h = 0$, *we define* $\xi_0(z) = z$ *for all* $z \in \mathbb{C}$.

We call ξ_h the cylinder transformation because when $\mathrm{h} > 0$ we can view $\mathbb{Z}_h$ as a cylinder if we glue the bordering lines $\mathrm{Im(z)} = -\frac{1}{\mathrm{h}}$ and $\mathrm{Im(z)} = \frac{1}{\mathrm{h}}$ of $\mathbb{Z}_h$ together to form a cylinder.

Definition 2.11: We say that a function $p: \mathbb{T} \to \mathbb{R}$ *is regressive provided*

$$1 + \mu(t)p(t) \neq 0 \;\; for\ all\ t \in \mathbb{T}^{\kappa} \tag{2.32}$$

holds. The set of all regressive and rd-continuous functions $p: \mathbb{T} \to \mathbb{R}$ *will be denoted by* $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.12: The set $\mathcal{R}^+$ *of all positively regressive elements of* $\mathcal{R}$ *is defined by*

- $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) := \{p \in \mathcal{R}: 1 + \mu(t)p(t) > 0 \; for\ all\ t \in \mathbb{T}\}$.

We now ready to define the generalized exponential function $e_p(t, s)$ on time scales.

Definition 2.13: If $p \in \mathcal{R}$, *then we define the exponential function by*

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big)\Delta\tau\right) for\ s,t \in \mathbb{T} \tag{2.33}$$

where the cylinder transformation $\xi_h(z)$ *is introduced in Definition 2.10.*

In the following theorem, we list some properties of the generalized exponential function.

Theorem 2.6: If $p, q \in \mathcal{R}$*, then*

i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$*;*

iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$*;*

iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*

v) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$*;*

vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$

Proof 2.6: See [2], [3].□

2.4.1. Structure of Dynamical Systems on Time Scales

In this subsection, we introduce the structure of a dynamical system on time scales.

Definition 2.14: The first order linear dynamic equation

$$y^{\Delta} = p(t)y(t) \tag{2.34}$$

is called regressive if $p \in \mathcal{R}$.

Theorem 2.7: Suppose (2.34) is regressive. Let $t_0 \in \mathbb{T}$ *and* $y_0 \in \mathbb{R}$*. Then the unique solution of the initial value problem*

$$y^{\Delta} = p(t)y(t) \qquad y(t_0) = y_0 \tag{2.35}$$

is given by

$$y(t) = e_p(t,t_0)y_0 \tag{2.36}$$

Definition 2.15: If $p \in \mathcal{R}$ and $f: \mathbb{T} \to \mathbb{R}$ *is rd-continuous, then the dynamic equation*

$$y^{\Delta} = p(t)y(t) + f(t) \tag{2.37}$$

is called regressive.

Theorem 2.8: Suppose (2.37) is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. Then the unique solution of the initial value problem

$$y^{\Delta} = p(t)y(t) + f(t) \qquad y(t_0) = y_0 \tag{2.38}$$

is given by

$$y(t) = y_0 e_p(t, t_0) + \int_{t_0}^{t} e_p\big(t, \sigma(\tau)\big) f(\tau)\Delta\tau \tag{2.39}$$

For more details about the dynamical systems on time scales, we refer to [2], [3], [14].

2.5. Dynamic Inequalities and Comparison Results

In this section, we consider the fundamental dynamic inequalities that are needed for our future purposes. We restrict ourselves to scalar dynamic equations.

In the following theorem, we give a result relative to a system of strict dynamic inequalities.

Theorem 2.9: Let $\mathbb{T}$ be a time scale. Let $v, w: \mathbb{T} \to \mathbb{R}$ be mappings that are differentiable at each $t \in \mathbb{T}^{\kappa}$ *satisfying*

$$v^{\Delta}(t) \le g\big(t, v(t)\big), \qquad for\ t \in \mathbb{T} \tag{2.40}$$

$$w^{\Delta}(t) > g\big(t, w(t)\big), \qquad for\ t \in \mathbb{T} \tag{2.41}$$

where $g \in C_{rd}[\mathbb{T} \times \mathbb{R}, \mathbb{R}]$ *and* $g(t,x)\mu(t)$ *is nondecreasing in* x *for each* $t \in \mathbb{T}$. *Then* $v(t) < w(t)$, *for all* $t \in \mathbb{T}$ *whenever* $v(t_0) < w(t_0)$.

Proof 2.9: See [14].□

Corollary 2.1: If in Theorem 2.9 the inequalities (2.40) *and* (2.41) *is replaced by* $v^{\Delta}(t) < g\big(t, v(t)\big)$, $w^{\Delta}(t) \geq g\big(t, w(t)\big)$, *the conclusion* $v(t) < w(t)$, *for* $t \in \mathbb{T}$ *is still valid.*

In the next theorem, we give a result similar to that of Theorem 2.9 but in this case we consider non-strict dynamic inequalities.

Theorem 2.10: Assume the hypotheses of Theorem 2.9 with (2.40) *and* (2.41) *replaced by*

$$v^{\Delta}(t) \leq g\big(t, v(t)\big), \qquad for\ t \in \mathbb{T} \tag{2.42}$$

$$w^{\Delta}(t) \geq g\big(t, w(t)\big), \qquad for\ t \in \mathbb{T} \tag{2.43}$$

Suppose further that for $x \geq y$ *and* $t \in \mathbb{T}$ *is right dense*

$$g(t,x) - g(t,y) \leq L(x - y), \qquad L > 0 \tag{2.44}$$

Then $v(t) \leq w(t)$, *for all* $t \in \mathbb{T}$ *whenever* $v(t_0) \leq w(t_0)$.

Proof 2.10: See [14].□

Definition 2.16: Let $r(t)$ *be a solution of the scalar dynamic equation* $u^{\Delta} = g(t, u(t))$ $u(t_0) = u_0$ *on* $[t_0, t_0 + a)$. *Then* $r(t)$ *is said to be a maximal solution if, for every solution of* $u^{\Delta} = g(t, u(t))$ $u(t_0) = u_0$ *existing on* $[t_0, t_0 + a)$, *the inequality*

$$u(t) \leq r(t), \qquad t \in [t_0, t_0 + a) \tag{2.45}$$

holds. A minimal solution $\rho(t)$ *may be defined similarly by reversing the inequality* (2.45).

An important technique in the theory of differential equations is concerned with estimating a function satisfying a differential inequality by the extremal solutions, of the corresponding differential equation. One of the results that is widely used is the following comparison theorem:

Theorem 2.11: Let $g \in C_{rd}[\mathbb{T} \times \mathbb{R}, \mathbb{R}]$ *and* $g(t,x)\mu(t)$ *is nondecreasing in* x *for each* $t \in \mathbb{T}$. *Let* $m: [t_0, t_0 + a)_{\mathbb{T}} \to \mathbb{R}$ *be a mapping that is differentiable for each* $t \in [t_0, t_0 + a)_{\mathbb{T}}$ *satisfying*

$$m^{\Delta}(t) \leq g\big(t, m(t)\big), \qquad for\ t \in [t_0, t_0 + a)_{\mathbb{T}}. \tag{2.46}$$

Then, $m(t_0) \leq u_0$ *implies that* $m(t) \leq r(t),\ t \in [t_0, t_0 + a)_{\mathbb{T}}$, *where* $r(t)$ *is the maximal solution of the scalar dynamic equation* $u^{\Delta} = g(t, u(t)) \quad u(t_0) = u_0$ *existing on* $[t_0, t_0 + a)_{\mathbb{T}}$.

Proof 2.11: See [14].□

We now give Gronwall inequality on time scales.

Theorem 2.12: Let $y, f \in C_{rd}$ *and* $p \in \mathcal{R}^+, p \geq 0$. *Then*

$$y(t) \leq f(t) + \int_{t_0}^{t} y(\tau)p(\tau)\,\Delta\tau \quad for\ all\ t \in \mathbb{T} \tag{2.47}$$

implies

$$y(t) \leq f(t) + \int_{t_0}^{t} e_p\big(t, \sigma(\tau)\big) f(\tau)p(\tau)\,\Delta\tau \quad for\ all\ t \in \mathbb{T} \tag{2.48}$$

Proof 2.12: See [2], [3].

Corollary 2.2: Let $y \in C_{rd}$, $p \in \mathcal{R}^+$, $p \geq 0$ *and* $\alpha \in \mathbb{R}$. *Then*

$$y(t) \leq \alpha + \int_{t_0}^{t} y(\tau)p(\tau)\,\Delta\tau \quad for\ all\ t \in \mathbb{T} \tag{2.49}$$

implies

$$y(t) \leq \alpha e_p(t, t_0) \quad for\ all\ t \in \mathbb{T} \tag{2.50}$$

Proof 2.2: See [2], [3].□

3. STABILITY and BOUNDEDNESS of DYNAMIC EQUATIONS on TIME SCALES with INITIAL TIME DIFFERENCE

3.1. Introduction

The qualitative behavior of differential-difference systems has been explored extensively by employing Lyapunov-like functions and corresponding inequalities [13], [15], [16]. The investigation of initial value problems with a perturbation in the space variable is well known when the perturbation is restricted to the space variable with initial time unchanged [9], [10], [13], [15], [16]. However, in real world applications, it is possible to make an error in the initial time in addition to the initial position. This results in a problem of measuring the difference of two solutions which differ in initial time and initial position. So far, several studies have been made on this problem to explore the stability, boundedness, etc., criteria for differential systems relative to initial time difference [13], [20] - [22], [24], [26], [27].

In this chapter, we investigate stability and boundedness criteria for dynamic systems on time scales relative to initial time difference by using comparison method. In Section 2, we present basic definitions and necessary background material. In Section 3, we discuss and compare the differences between classical notion of stability and the notion of initial time difference (ITD) stability. In Section 4, we obtain new dynamic inequalities and comparison results on time scales relative to initial time difference. In the framework of of these results, we obtain several ITD-stability and ITD-boundedness properties from the stability and boundedness properties of the comparison dynamic system. Moreover, as an application of these new results, we give an example.

3.2. Basic Definitions and Concepts

Consider the dynamic systems

$$x^{\Delta} = f(t,x), \qquad x(t_0) = x_0, \quad for\ t \geq t_0 \geq 0, \qquad t, t_0 \in \mathbb{T}, \tag{3.1}$$

$$y^{\Delta} = f(t,y), \qquad y(\tau_0) = y_0, \quad for\ t \geq \tau_0 \geq 0, \qquad t, \tau_0 \in \mathbb{T} \tag{3.2}$$

where $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$. Suppose that the function f is smooth enough to guarantee existence, uniqueness and right dense (rd)-continuous depence of solutions $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, \tau_0, y_0)$ (3.1), (3.2); respectively. Assume that $x(t) = x(t, t_0, x_0)$ is the solution that we shall study stability and boundedness criteria with respect to it. Set $\eta = \tau_0 - t_0 > 0$ and denote $S(\rho) = \{x \in R^{\mathrm{n}}: \|x\| < \rho\}$ and $S^c(\rho) = \{x \in R^{\mathrm{n}}: \|x\| \geq \rho\}$ for some $\rho > 0$.

Before we can establish our comparison theorem, stability and boundedness criteria for dynamic systems we need to introduce the following definitions.

Definition 3.1: The solution $x(t) = x(t, t_0, x_0)$ *of (3.1) is said to be*

S1) equistable with ITD, if given $\varepsilon > 0$ and $t_0 \in \mathbb{T}$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ and $\tilde{\delta} = \tilde{\delta}(\varepsilon, t_0) > 0$ such that $\|y(t+\eta, \tau_0, y_0) - x(t, t_0, x_0)\| < \varepsilon$ for $t \geq t_0$ whenever $\|y_0 - x_0\| < \delta$ and $|\eta| < \tilde{\delta}$;

S2) uniformly stable with ITD, *if (S1) holds with* δ *and* $\tilde{\delta}$ *independent of* $t_0 \in \mathbb{T}$*;*

S3) attractive with ITD, if for each $\varepsilon > 0$ and $t_0 \in \mathbb{T}$, there exist $\delta_0 = \delta_0(t_0) > 0$, $\tilde{\delta}_0 = \tilde{\delta}_0(t_0) > 0$ and a $T = T(\varepsilon, t_0) > 0$ such that if $\|y_0 - x_0\| < \delta_0$ and $|\eta| < \delta_0$ implies that $\|y(t+\eta, \tau_0, y_0) - x(t, t_0, x_0)\| < \varepsilon$ for $t \geq t_0 + T$;

S4) uniformly attractive with ITD, *if* δ_0 *and* T *in (S3) are independent of* $t_0 \in \mathbb{T}$*;*

S5) asymptotically stable with ITD, *if (S1) and (S3) hold simultaneously;*

S6) uniformly asymptotically stable with ITD, if (S_2) and (S_4) hold simultaneously.

Definition 3.2: The dynamic system (3.1) is said to be

B1) equibounded with ITD, *if given* $\alpha > 0$ *and* $t_0 \in \mathbb{T}$, *there exists* $\tilde{\delta} = \tilde{\delta}(\alpha, t_0) > 0$ *and* $\beta = \beta(\alpha, t_0) > 0$ *such that* $\|y(t + \eta, \tau_0, y_0) - x(t, t_0, x_0)\| < \beta$ *for* $t \geq t_0$ *whenever* $\|y_0 - x_0\| < \alpha$ *and* $|\eta| < \tilde{\delta}$*;*

B2) uniformly bounded with ITD, *if (B1) holds with* β *and* $\tilde{\delta}$ *independent of* $t_0 \in \mathbb{T}$*;*

B3) quasi-ultimately equibounded with ITD, *if for each* $\alpha > 0$ *and* $t_0 \in \mathbb{T}$, *there exists* $\tilde{\delta}_0 = \tilde{\delta}_0(t_0) > 0$ *and* $\beta_0 = \beta_0(t_0) > 0$ *and a* $T = T(\alpha, t_0) > 0$ *such that* $\|y(t + \eta, \tau_0, y_0) - x(t, t_0, x_0)\| < \beta_0$ *for* $t \geq t_0 + T$ *whenever* $\|y_0 - x_0\| < \alpha$ *and* $|\eta| < \tilde{\delta}_0$*;*

B4) quasi-uniformly ultimately equibounded with ITD, *if* β_0 *and* T *in (B3) are independent of* $t_0 \in \mathbb{T}$*;*

B5) ultimately equibounded with ITD, *if (B1) and (B1) hold simultaneously;*

B6) uniformly ultimately equibounded with ITD, *if (B2) and (B4) hold simultaneously.*

We now introduce a certain class of function which we need in the sequel.

Definition 3.3: A function $\varphi(r)$ *is said to belong to the class* $\mathcal{K}$ *if* $\in C_{rd}[\mathbb{T}, \mathbb{R}_+]$, $\varphi(0) = 0$, $\varphi(r) \to \infty$ *as* $r \to \infty$ *and* $\varphi(r)$ *is strictly monotone increasing in* r*.*

Definition 3.4: A real-valued function $V(t,x) \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$ *with* $V(t,0) \equiv 0$ *is said to be* positive definite (negative definite)*if there exists a function* $\varphi(r) \in \mathcal{K}$ *such that the relations holds for* $(t,x) \in \mathbb{T} \times S(\rho)$ *respectively;*

- $V(t,x) \geq \varphi(\|x\|)$
- $V(t,x) \leq -\varphi(\|x\|)$

Definition 3.5: A real-valued function $V(t,x) \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$ *is said to be* decrescent *if a function* $\varphi(r) \in \mathcal{K}$ *exists such that*

- $V(t,x) \geq \varphi(\|x\|), \quad for \quad (t,x) \in \mathbb{T} \times S(\rho).$

Definition 3.6: For a real real-valued function $V(t,x) \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$ *we define the* Dini-derivatives along the trajectories of (3.1) *as follows: for a given* $\varepsilon > 0$, *there exists a neighborhood* $\mathcal{N}(\varepsilon)$ *of* $t \in \mathbb{T}$ *such that*

$$\frac{V\left(\sigma(t), x\big(\sigma(t)\big)\right) - V\left(s, x\big(\sigma(t)\big) - (\sigma(t) - s) f\big(t, x(t)\big)\right)}{\sigma(t) - s} < D^+V^{\Delta}(t,x) + \varepsilon, \qquad s \epsilon\, \mathcal{N}(\varepsilon), \qquad s > t \tag{3.3}$$

for $(t,x) \in \mathbb{T} \times S(\rho)$ *in case* $t \in \mathbb{T}$ *is right-scattered and* $V(t, x(t))$ *is continuous at* $t \in \mathbb{T}$, *we have*

$$D^+V^{\Delta}(t,x) = \frac{V\left(\sigma(t), x\big(\sigma(t)\big)\right) - V(t, x(t))}{\mu(t)} \tag{3.4}$$

where $\mu(t) = \sigma(t) - t$. *Alternatively one can define;*

$$D^+V^{\Delta}(t,x) := \lim_{s \to t, s \neq \sigma(t)} sup \frac{V\left(\sigma(t), x\big(\sigma(t)\big)\right) - V\big(s, x(t)\big)}{\sigma(t) - s} \tag{3.5}$$

Definition 3.7: For a real real-valued function $V(t,x) \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$ *we define the generalized derivatives (*Dini-like derivatives) as follows there exists a neighborhood $\mathcal{N}(\varepsilon)$ of $t \in \mathbb{T}$ such that

$$\frac{1}{\mu(t,s)}\Big[V\left(\sigma(t), \tilde{y}\big(\sigma(t)\big) - x\big(\sigma(t)\big)\right) - V\Big(s, \tilde{y}\big(\sigma(t)\big) - x\big(\sigma(t)\big) - \mu(t,s)\big\{f\big(t+\eta, \tilde{y}(t)\big) - f\big(t, x(t)\big)\big\}\Big)\Big] < D^+V^{\Delta}(t, \tilde{y} - x) + \varepsilon, \qquad s \epsilon\, \mathcal{N}(\varepsilon), \qquad s > t \tag{3.6}$$

where $\mu(t,s) = \sigma(t) - s, \eta > 0$ *and* $\tilde{y}(t, t_0, y_0) = y(t + \eta, \tau_0, y_0)$.

3.3. Classical and Initial Time Difference (ITD) Notions of Stability on Time Scales

3.3.1. Classical Notion of Stability

Consider the followings IVPs on time scales $\mathbb{T}$,

$$x^{\Delta} = f(t,x), \qquad x(t_0) = x_0 \quad for\ \ t \geq t_0,\ \ t \in \mathbb{T} \tag{3.7}$$

$$x^{\Delta} = f(t,x), \qquad x(t_0) = y_0 \quad for\ \ t \geq t_0,\ \ t \in \mathbb{T} \tag{3.8}$$

where $f \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}^n]$ *and* $S(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho < \infty\}$.

Assume that $f(t,0) = 0$ for all $t \in \mathbb{T}$ so that $x \equiv 0$ is a null solution of (3.7) through $(t_0, 0)$. Now, we can state the well known definitions concerning the stability of the null solution.

Definition 3.8: The null solution $x \equiv 0$ *of* (3.7) *is said to be* stable *if and only if for each* $\varepsilon > 0$ *and for all* $t_0 \in \mathbb{T}$, *there exists a positive function* $\delta = \delta(\varepsilon, t_0)$ *that is rd-continuous in* $t_0 \in \mathbb{T}$ *for each* ε *such that*

$$\|x_0\| < \delta \quad implies \quad \|x(t, t_0, x_0)\| < \varepsilon, \qquad for\ \ t \geq t_0. \tag{3.9}$$

If δ *is independent of* t_0, *then the null solution* $x \equiv 0$ *of* (3.7) *is said to be* uniformly stable. *with respect to the solution* $x(t, t_0, x_0)$.

Definition 3.9: The solution $x(t, t_0, y_0)$ *of* (3.8) *is said to be* stable *with respect to the solution* $x(t, t_0, x_0)$ *of* (3.7) *for* $t \geq t_0$ *if and only if given any* $\varepsilon > 0$ *and* $t_0 \in \mathbb{T}$ *there exists a positive function* $\delta = \delta(\varepsilon, t_0) > 0$ *that is rd-continuous in* t_0 *for each* $\varepsilon > 0$ *such that*

$$\|y_0 - x_0\| < \delta \quad implies \quad \|x(t, t_0, y_0) - x(t, t_0, x_0)\| < \varepsilon \quad for \quad t \geq t_0. \tag{3.10}$$

If δ *is independent of* t_0, *then the solution of the system is* uniformly stable *with respect to the solution* $x(t, t_0, x_0)$.

We remark that for the purpose of studying the classical stability of a given solution $x(t, t_0, y_0)$ of the system (3.8), it is convenient to make a change of variable. Let $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ be the unique solutions of (3.7) and (3.8); respectively. Setting

$$z(t, t_0, y_0 - x_0) = x(t, t_0, y_0) - x(t, t_0, x_0) \text{ for } t \geq t_0 \text{ and } t_0 \in \mathbb{T} \tag{3.11}$$

Then

$$z^{\Delta}(t, t_0, y_0 - x_0) = x^{\Delta}(t, t_0, y_0) - x^{\Delta}(t, t_0, x_0) \tag{3.12}$$

$$\begin{gathered} z^{\Delta}(t, t_0, y_0 - x_0) = f(t, z(t, t_0, y_0 - x_0) + x(t, t_0, x_0)) - f(t, x(t, t_0, x_0)) \\ z^{\Delta}(t, t_0, y_0 - x_0) = \tilde{f}(t, z(t, t_0, y_0 - x_0)) \end{gathered} \tag{3.13}$$

It is easy to observe that $z(t, t_0, y_0 - x_0) \equiv 0$ is a solution of the transformed system if $y_0 - x_0 = 0$ which implies that $\tilde{f}(t, 0) = 0$. Since $z^{\Delta}(t) = 0$ and $z(t, t_0, 0) = 0$ is the null solution, the solution $x(t, t_0, 0) \equiv 0$ of (3.7) corresponds to the identically null solution of $z^{\Delta} = \tilde{f}(t, z)$ where $\tilde{f}(t, z(t, t_0, y_0 - x_0)) = f(t, z(t, t_0, y_0 - x_0) + x(t, t_0, x_0)) - f(t, x(t, t_0, x_0))$. Hence, we can always assume, without any loss of generality, that $x(t, t_0, x_0) \equiv 0$ is the null solution of the given system (3.7) and we can limit our study of stability to that of the null solution. However, it is impossible to do the same transformation for (ITD) stability.

3.3.2. New Notion of (ITD) Stability

Let $x(t, t_0, x_0)$ be a solution of (3.1) and $\tilde{x}(t, t_0, y_0) = x(t + \eta, \tau_0, y_0)$ where $x(t, \tau_0, y_0)$ is any solution of the system (3.2) for $t \geq \tau_0 \geq 0$. Let us make a transformation similar to that in (3.11). Set

$$z(t, t_0, y_0 - x_0) = x(t, t_0, x_0) - x(t + \eta, \tau_0, y_0) \text{ for } t \geq t_0 \text{ and } t_0 \in \mathbb{T} \tag{3.14}$$

Then

$$
\begin{gathered}
z^{\Delta}(t,t_0,y_0-x_0)=x^{\Delta}(t,t_0,x_0)-x^{\Delta}(t+\eta,\tau_0,y_0)\\
z^{\Delta}(t,t_0,y_0-x_0)=\tilde{f}(\eta;t,z(t,t_0,y_0-x_0))
\end{gathered}
\tag{3.15}
$$

One can observe that even if $y_0=x_0$, $z(t,t_0,0)$ is not zero and is not the null solution of the transformed system and $x(t+\eta,\tau_0,y_0)$, where $x(t,\tau_0,y_0)$ is any solution of the system (3.2) for $t\geq\tau_0\geq 0$, does not correspond to the identically zero solution of $z^{\Delta}(t)=\tilde{f}(\eta;t,z)$. Therefore, we do not use the stability properties of the null solution in order to find (ITD) stability properties using this approach.

3.4. Main Results

3.4.1. Dynamic Inequalities and Comparison Results

3.4.1.1. Dynamic Inequalities

We start with the fundamental result in the theory of differential inequalities parallel to the well known results [13], [14].

Theorem 3.1: Assume that

i) $\alpha,\beta\in C_{rd}[\mathbb{T},\mathbb{R}]$, $f\in C_{rd}[\mathbb{T}\times\mathbb{R},\mathbb{R}]$ *and* α,β *are* Δ*-differentiable for each* $t\in\mathbb{T}$ *and satisfy*

$$\alpha^{\Delta}\leq f(t,\alpha),\qquad \alpha(t_0)\leq x_0,\qquad t_0\geq 0 \tag{3.16}$$

$$\beta^{\Delta}> f(t,\beta),\qquad \beta(\tau_0)> x_0,\qquad \tau_0\geq 0 \tag{3.17}$$

ii) $f(t,x)\mu(t)$ *is non-decreasing in* $x\in\mathbb{R}$ *for each* $t\in\mathbb{T}$*;*
iii) $\tau_0>t_0$ *and* $f(t,x)$ *is non-decreasing in* $t\in\mathbb{T}$ *for each* $x\in\mathbb{R}$.

Then

- $\alpha(t)<\beta(t+\eta)$, $t\geq t_0$
- $\alpha(t-\eta)<\beta(t)$, $t\geq\tau_0$, *where* $\eta=\tau_0-t_0>0$.

Proof 3.1: We now prove this cases seperately.

• *For convenience let* $\beta_0(t) = \beta(t+\eta)$ *so that* $\beta_0(t_0) = \beta(t_0+\eta) = \beta(\tau_0) > x_0 \geq \alpha(t_0)$. *Also* $\beta_0^{\Delta}(t) = \beta^{\Delta}(t+\eta) > f(t+\eta, \beta_0(t))$, $t \geq t_0$. *We apply the induction principle to the statement:* $\{A(t): \alpha(t) < \beta(t+\eta), t \in T, t \geq t_0\}$

– *Let* $t = t_0$. *Since* $\alpha(t_0) < \beta_0(t_0) = \beta(t_0+\eta)$, *it follows that* $A(t_0)$ *is true.*

– *Let* t *be right-scattered and* $A(t)$ *is true. We shall show that* $A(\sigma(t))$ *is true. Using the definition of the derivative for right-scattered point and by (i), (iii), we have*

$$\begin{aligned}\alpha(\sigma(t)) - \beta_0(\sigma(t)) &= (\alpha^{\Delta}(t) - \beta_0^{\Delta}(t))\mu(t) + (\alpha(t) - \beta_0(t))\\ &< (f(t,\alpha(t)) - f(t+\eta, \beta_0(t)))\mu(t) + (\alpha(t) - \beta_0(t))\\ &< (f(t,\alpha(t)) - f(t, \beta_0(t)))\mu(t) + (\alpha(t) - \beta_0(t))\end{aligned} \tag{3.18}$$

Then since $A(t)$ *is true, by assumptions (ii) it follows that* $\alpha(\sigma(t)) - \beta_0(\sigma(t)) < 0$. *Hence* $A(\sigma(t))$ *is true.*

– *Let* t *be right-dense and* U *be a neighborhood of* t. *Assume that* $A(t)$ *is true. We need to show that* $A(s)$ *is true for* $s > t, s \in U$, *i.e,* $\alpha(s) < \beta_0(s), s > t, s \in U$. *If this is not true, then there exists a point* $s_0 > t, s_0 \in U$ *such that*

$$\alpha(s_0) = \beta_0(s_0) \text{ and } \alpha(s) < \beta_0(s) \quad t \leq s < s_0 \tag{3.19}$$

Hence, since the point in U *are dense, for sufficiently small* $h > 0$,

$$\alpha(s_0 - h) - \alpha(s_0) < \beta_0(s_0 - h) - \beta_0(s_0) \tag{3.20}$$

which implies that

$$\alpha^{\Delta}(s_0) \geq \beta_0^{\Delta}(s_0) \tag{3.21}$$

it follows that $f(s_0, \alpha(s_0)) \geq \alpha^{\Delta}(s_0) \geq \beta_0^{\Delta}(s_0) > f(s_0 + \eta, \beta_0(s_0))$. *Then by (iii) and* (3.19) *we obtain that*

$$f(s_0, \alpha(s_0)) > f(s_0 + \eta, \beta_0(s_0)) > f(s_0, \beta_0(s_0)) = f(s_0, \alpha(s_0)) \qquad (3.22)$$

which is a contradiction. So this proves that $\alpha(s) < \beta_0(s)$ *for* $s > t$, $s \in U$, *and therefore* $A(s)$ *is true.*

– *Let* t *be left-dense and* $A(s)$ *is true for* $s < t$. *We need to show that* $A(t)$ *is true. Since* $\alpha(t)$ *and* $\beta_0(t)$ *are* Δ*-differentiable, they are continuous. Hence it follows that*

$$\alpha(t) = \lim_{s \to t^-} \alpha(s) \leq \lim_{s \to t^-} \beta_0(s) = \beta_0(t) \qquad (3.23)$$

It remains to show that $\alpha(t) = \beta_0(t)$ *is not possible. Assume to the contrary that* $(t) = \beta_0(t)$. *Then by (i), (iii)*

$$\begin{aligned} \alpha^{\Delta}(t) - \beta_0^{\Delta}(t) &< f(t, \alpha(t)) - f(t + \eta, \beta_0(t)) \\ &< f(t, \alpha(t)) - f(t, \beta_0(t)) = 0 \end{aligned} \qquad (3.24)$$

On the other hand since $A(s)$ *is true for* $s < t$ *and* $\alpha(t) = \beta_0(t)$ *we have* $\alpha(s) - \alpha(t) < \beta_0(s) - \beta_0(t)$. *Hence it follows that* $\alpha^{\Delta}(t) \geq \beta_0^{\Delta}(t)$. *But this contradict with* (3.24).

Therefore we obtain that $A(t)$ *is true. Thus by induction principle, we conclude that* $\alpha(t) < \beta(t + \eta), t \in \mathbb{T}, t \geq t_0$.

• *For convenience let* $\alpha_0(t) = \alpha(t - \eta)$ *so that* $\alpha_0(\tau_0) = \alpha(\tau_0 - \eta) = \alpha(t_0) \leq x_0 < \beta(\tau_0)$. *Also* $\alpha_0^{\Delta}(t) = \alpha^{\Delta}(t - \eta) \leq f(t - \eta, \alpha_0(t)) \quad t \geq \tau_0$. *We apply the induction principle to the statement:* $\{A(t): \alpha(t - \eta) < \beta(t), t \in \mathbb{T}, t \geq \tau_0\}$.

– *Let* $t = t_0$. *Since* $\alpha_0(\tau_0) = \alpha(t_0) < \beta(\tau_0)$, *it follows that* $A(\tau_0)$ *is true.*

– *Let* t *be right-scattered and* $A(t)$ *is true. We shall show that* $A(\sigma(t))$ *is true.*

Using the definition of the derivative for right-scattered point and by (i), (iii), we have

$$\begin{aligned}\alpha_0(\sigma(t)) - \beta(\sigma(t)) &= (\alpha_0^{\Delta}(t) - \beta^{\Delta}(t))\mu(t) + (\alpha_0(t) - \beta(t)) \\ &< (f(t-\eta, \alpha_0(t)) - f(t, \beta(t)))\mu(t) + (\alpha_0(t) - \beta(t)) \\ &< (f(t, \alpha_0(t)) - f(t, \beta(t)))\mu(t) + (\alpha_0(t) - \beta(t))\end{aligned} \tag{3.25}$$

Then since $A(t)$ *is true, by assumptions (ii) it follows that* $\alpha_0(\sigma(t)) - \beta(\sigma(t)) < 0$. *Hence* $A(\sigma(t))$ *is true.*

– Let t *be right-dense and* U *be a neighborhood of* t. *Assume that* $A(t)$ *is true. We need to show that* $A(s)$ *is true for* $s \geq t,\ s \in U$, *i.e,* $\alpha_0(t) < \beta(t), s \geq t,\ s \in U$. *If this is not true there exists a point* $s_0 > t,\ s_0 \in U$ *such that*

$$\alpha_0(s_0) = \beta(s_0) \text{ and } \alpha_0(s) < \beta(s) \ \ t \leq s < s_0 \tag{3.26}$$

Hence, since the point in U *are dense, for sufficiently* $h > 0$,

$$\alpha_0(s_0 - h) - \alpha_0(s_0) < \beta(s_0 - h) - \beta(s_0) \tag{3.27}$$

which implies that

$$\alpha_0^{\Delta}(s_0) \geq \beta^{\Delta}(s_0) \tag{3.28}$$

it follows that $f(s_0 - \eta, \alpha_0(s_0)) \geq \alpha_0^{\Delta}(s_0) \geq \beta^{\Delta}(s_0) > f(s_0, \beta(s_0))$.
Then by (iii) and (3.26) *we obtain that* $f(s_0, \alpha_0(s_0)) \geq f(s_0 - \eta, \alpha_0(s_0)) > f(s_0, \beta(s_0)) = f(s_0, \alpha_0(s_0))$, *which is a contradiction. So this proves that* $\alpha_0(s) < \beta(s)$ *for* $s \geq t, s \in U$, *and therefore* $A(s)$ *is true.*

– Let t *be left-dense and* $A(s)$ *is true for* $s < t$. *We need to show that* $A(t)$ *is true. Since* $\alpha(t)$ *and* $\beta(t)$ *are* Δ*-differentiable, they are continuous. Hence it follows that*

$$\alpha_0(t) = \lim_{s\to t^-} \alpha_0(s) \leq \lim_{s\to t^-} \beta(s) = \beta(t) \tag{3.29}$$

It remains to show that $\alpha_0(t) = \beta(t)$ *is not possible. Assume to the contrary that* $_0(t) = \beta(t)$ *. Then by (i), (iii)*

$$\begin{aligned} \alpha_0^\Delta(t) - \beta^\Delta(t) &< f(t-\eta, \alpha_0(t)) - f(t, \beta(t)) \\ &< f(t, \alpha(t)) - f(t, \beta_0(t)) = 0 \end{aligned} \tag{3.30}$$

On the other hand since $A(s)$ *is true and* $\alpha_0(t) = \beta(t)$ *we have* $\alpha_0(s) - \alpha_0(t) < \beta(s) - \beta(t)$ *which implies that* $\alpha_0^\Delta(t) \geq \beta^\Delta(t)$*. But this contradict with* (3.30)*. Therefore we obtain that* $A(t)$ *is true. Thus by induction principle, we conclude that* $\alpha(t-\eta) < \beta(t), \quad t \in \mathbb{T}, \quad t \geq \tau_0$.□

Theorem 3.2: Assume that

i) $\alpha, \beta \in C_{rd}[\mathbb{T}, \mathbb{R}]$, $f \in C_{rd}[\mathbb{T} \times \mathbb{R}, \mathbb{R}]$ *and* α, β *are* Δ*-differentiable for each* $t \in \mathbb{T}$ *and satisfy*

$$\begin{aligned} \alpha^\Delta &\leq f(t, \alpha), \qquad \alpha(t_0) \leq x_0, \qquad t_0 \geq 0 \\ \beta^\Delta &\geq f(t, \beta), \qquad \beta(\tau_0) \geq x_0, \qquad \tau_0 \geq 0 \end{aligned} \tag{3.31}$$

ii) $f(t,x) - f(t,y) \leq L(x-y), x \geq y,\ L > 0$ *and* $t \in \mathbb{T}$ *;*
iii) $f(t,x)\mu(t)$ *is non-decreasing in* $x \in \mathbb{R}$ *for each* $t \in \mathbb{T}$*;*
iv) $\tau_0 > t_0$ *and* $f(t,x)$ *is non-decreasing in* $t \in \mathbb{T}$ *for each* $x \in \mathbb{R}$.

Then

- $\alpha(t) \leq \beta(t+\eta), t \geq t_0$
- $\alpha(t-\eta) \leq \beta(t),\ t \geq \tau_0$, *where* $\eta = \tau_0 - t_0 > 0$.

Proof 3.2:

• *For convenience let* $\beta_0(t) = \beta(t+\eta)$ *so that* $\beta_0(t_0) = \beta(t_0+\eta) = \beta(\tau_0) \geq x_0 \geq \alpha(t_0)$. *Also* $\beta_0^{\Delta}(t) = \beta^{\Delta}(t+\eta) \geq f(t+\eta, \beta_0(t))$, $t \geq t_0$. *We apply the induction principle to the statement:* $\{A(t): \alpha(t) \leq \beta(t+\eta), t \in \mathbb{T}, t \geq t_0\}$.

– *Let* $t = t_0$. *Since* $\alpha(t_0) \leq \beta_0(t_0) = \beta(t_0+\eta)$, *it follows that* $A(t_0)$ *is true.*

– *Let* t *be right-scattered and* $A(t)$ *is true. We shall show that* $A(\sigma(t))$ *is true. Using the definition of the derivative for right-scattered point and by (i), (iv), we have*

$$\begin{aligned} \alpha(\sigma(t)) - \beta_0(\sigma(t)) &= (\alpha^{\Delta}(t) - \beta_0^{\Delta}(t))\mu(t) + (\alpha(t) - \beta_0(t)) \\ &\leq (f(t, \alpha(t)) - f(t+\eta, \beta_0(t)))\mu(t) + (\alpha(t) - \beta_0(t)) \\ &\leq (f(t, \alpha(t)) - f(t, \beta_0(t)))\mu(t) + (\alpha(t) - \beta_0(t)) \end{aligned} \tag{3.32}$$

Then since $A(t)$ *is true, by assumptions (iii) it follows that* $\alpha(\sigma(t)) - \beta_0(\sigma(t)) \leq 0$. *Hence* $A(\sigma(t))$ *is true.*

– *Let* t *be right-dense and* U *be a neighborhood of* t. *Assume that* $A(t)$ *is true. We need to show that* $A(s)$ *is true for* $s > t$, $s \in U$. *Set* $\tilde{\beta}_0(s) = \beta_0(s) + \varepsilon Q(s)$, *for* $s \geq t, s \in U$ *and* $\varepsilon > 0$ *sufficiently small, where* $Q(s) > 0$ *satisfies*

$$Q^{\Delta}(s) > LQ(s), \quad Q(t) > 0. \tag{3.33}$$

Then clearly

$$\tilde{\beta}_0(s) > \beta_0(s) \tag{3.34}$$

We shall show that $\alpha(s) < \beta_0(s)$ *for* $s > t, s \in U$. *Then by (i), (ii) and (iv), we have*

$$
\begin{aligned}
\tilde{\beta}_0^{\Delta}(s) &= \beta_0^{\Delta}(s) + \varepsilon Q^{\Delta}(s) \\
&> f(s, \beta_0(s)) + \varepsilon L Q(s) \\
&\geq f(s, \tilde{\beta}_0(s)) - L(\tilde{\beta}_0(s) - \beta_0(s)) + \varepsilon L Q(s) \\
&= f(s, \tilde{\beta}_0(s))
\end{aligned}
\tag{3.35}
$$

Also $\tilde{\beta}_0(t) > \beta_0(t) \geq \alpha(t)$. *Hence by Theorem 3.1, we obtain*

$$\alpha(s) < \tilde{\beta}_0(s) \quad s > t, s \in U \tag{3.36}$$

Since $\varepsilon > 0$ *is arbitrary letting* $\varepsilon \to 0^{+}$ *we obtain*

$$\alpha(s) \leq \beta_0(s) \quad s > t, s \in U. \tag{3.37}$$

This proves that $A(s)$ *is true for* $s > t, s \in U$.

– Let t *be left-dense and* $A(s)$ *is true for* $s < t$. *We need to show that* $A(t)$ *is true. Since* $\alpha(t)$ *and* $\beta(t)$ *are* Δ*-differentiable, they are continuous. Hence it follows that*

$$\alpha(t) = \lim_{s \to t^{-}} \alpha(s) \leq \lim_{s \to t^{-}} \beta(s + \eta) = \beta(t + \eta) \tag{3.38}$$

proving A(t) is true.
Thus by induction principle, we conclude that $\alpha(t) \leq \beta(t + \eta), t \in \mathbb{T}, t \geq t_0$

• *For convenience let* $\alpha_0(t) = \alpha(t - \eta)$ *so that* $\alpha_0(\tau_0) = \alpha(\tau_0 - \eta) = \alpha(t_0) \leq x_0 \leq \beta(\tau_0)$. *Also* $\alpha_0^{\Delta}(t) = \alpha^{\Delta}(t - \eta) \leq f(t - \eta, \alpha_0(t)), t \geq \tau_0$. *We apply the induction principle to the statement:* $\{A(t): \alpha(t - \eta) \leq \beta(t), t \in \mathbb{T}, t \geq \tau_0\}$.

– Let $t = t_0$. *Since* $\alpha_0(\tau_0) = \alpha(\tau_0 - \eta) = \alpha(t_0) \leq \beta(\tau_0)$, *it follows that* $A(\tau_0)$ *is true.*

– Let t be right-scattered and $A(t)$ is true. We shall show that $A(\sigma(t))$ is true. Using the definition of the derivative for right-scattered point and by (i), (iv), we have

$$\begin{aligned}\alpha_0(\sigma(t)) - \beta(\sigma(t)) &= (\alpha_0^\Delta(t) - \beta^\Delta(t))\mu(t) + (\alpha_0(t) - \beta(t)) \\ &\leq (f(t-\eta, \alpha_0(t)) - f(t, \beta(t)))\mu(t) + (\alpha_0(t) - \beta(t)) \\ &\leq (f(t, \alpha_0(t)) - f(t, \beta(t)))\mu(t) + (\alpha_0(t) - \beta(t))\end{aligned} \tag{3.39}$$

Then since $A(t)$ is true, by assumption (iii) it follows that $\alpha_0(\sigma(t)) - \beta(\sigma(t)) \leq 0$. Hence $A(\sigma(t))$ is true.

– Let t be right-dense and U be a neighborhood of t. Assume that $A(t)$ is true. We need to show that $A(s)$ is true for $s \geq t, s \in U$. Set $\tilde{\alpha}_0(s) = \alpha_0(s) - \varepsilon Q(s)$, for $s \geq t, s \in U$ and $\varepsilon > 0$ sufficiently small, where $Q(s) > 0$ satisfies

$$Q^\Delta(s) > LQ(s), \quad Q(t) > 0 \tag{3.40}$$

Then clearly

$$\tilde{\alpha}_0(s) < \alpha_0(s). \tag{3.41}$$

We shall show that $\tilde{\alpha}_0(s) < \beta(s)$ for $s > t, s \in U$. Then by (i), (ii) and (iv), we have

$$\begin{aligned}\tilde{\alpha}_0^\Delta(s) &= \alpha_0^\Delta(s) - \varepsilon Q^\Delta(s) \\ &< f(s, \alpha_0(s)) - \varepsilon LQ(s) \\ &\leq f(s, \tilde{\alpha}_0(s)) + L(\alpha_0(s) - \tilde{\alpha}_0(s)) - \varepsilon LQ(s)\end{aligned} \tag{3.42}$$

$$= f(s, \tilde{\alpha}_0(s)) \tag{3.43}$$

Also $\tilde{\alpha}_0(t) < \alpha_0(t) \leq \beta(t)$. Hence by Theorem 3.1, we obtain

$$\tilde{\alpha}_0(s) < \beta(s) \text{ for } s > t, s \in U \tag{3.44}$$

Since $\varepsilon > 0$ is arbitrary letting $\varepsilon \to 0^+$ we obtain

$$\alpha_0(s) \leq \beta(s) \quad for\ s > t, s \in U. \tag{3.45}$$

This proves that $A(s)$ is true for $s > t, s \in U$.

– Let t be left-dense and $A(s)$ is true for $s < t$. We need to show that $A(t)$ is true. Since $\alpha(t)$ and $\beta(t)$ are Δ-differentiable, they are continuous. Hence it follows that

$$\alpha(t-\eta) = \lim_{s \to t^-} \alpha(s-\eta) \leq \lim_{s \to t^-} \beta(s) = \beta(t) \tag{3.46}$$

proving A(t) is true.

Thus by induction principle, we conclude that $\alpha(t-\eta) \leq \beta(t), t \in \mathbb{T}, t \geq t_0$.□

Remark 3.1: It is possible to weaken the hypotheses in above Theorem 3.1 and Theorem 3.2. The conclusions of the theorems remain true if the functions $\alpha(t)$ and $\beta(t)$ are only assumed to be rd-continuous. In this case the first condition Theorem 3.1 and Theorem 3.2 are replaced by

$$\begin{aligned} D^+\alpha^\Delta &\leq f(t,\alpha), \qquad \alpha(t_0) \leq x_0, \qquad t_0 \geq 0 \\ D^+\beta^\Delta &> f(t,\beta), \qquad \beta(\tau_0) > x_0, \qquad \tau_0 \geq 0 \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} D^+\alpha^\Delta &\leq f(t,\alpha), \qquad \alpha(t_0) \leq x_0, \qquad t_0 \geq 0 \\ D^+\beta^\Delta &\geq f(t,\beta), \qquad \beta(\tau_0) > x_0, \qquad \tau_0 \geq 0 \end{aligned} \tag{3.48}$$

respectively. Here D^+ is the upper right hand Dini derivative defined as in Definition 2.4.

Remark 3.2: The conclusion of the Theorem 3.1 remains true if the first condition is replaced by

$$\begin{aligned} &\alpha^{\Delta} < f(t,\alpha), \qquad \alpha(t_0) \le x_0, \qquad t_0 \ge 0 \\ &\beta^{\Delta} \ge f(t,\beta), \qquad \beta(\tau_0) > x_0, \qquad \tau_0 \ge 0 \end{aligned} \tag{3.49}$$

3.4.1.2. Comparison Results

The most commonly used technique in the theory of dynamic equations is concerned with estimating a function satisfying a dynamic inequality by the extremal solutions of the related dynamic equation. The following theorem gives such a estimate with initial time difference on time scales.

Theorem 3.3: Assume that

i) $m \in C_{rd}[\mathbb{T},\mathbb{R}_+]$, $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+,\mathbb{R}]$ *and*

$$D^{+}m^{\Delta}(t) \le g(t,m(t)), \qquad m(t_0) \le w_0 \quad t_0 \ge 0 \tag{3.50}$$

ii) the maximal solution $r(t) = r(t,\tau_0,w_0)$ *of* $w^{\Delta} = g(t,w), w(\tau_0) = w_0 \ge 0$ $\tau_0 \ge 0$ *exists for* $t \ge \tau_0$;
iii) $\tau_0 > t_0$ *and* $g(t,w)$ *is non-decreasing in* $t \in \mathbb{T}$ *for each* $w \in \mathbb{R}_+$;
iv) $g(t,w)\mu(t)$ *is non-decreasing in* $w \in \mathbb{R}_+$ *for each* $t \in \mathbb{T}$.

Then

- $m(t) \le r(t+\eta),\ t \ge t_0$;
- $m(t-\eta) \le r(t),\ t \ge \tau_0$.

Proof 3.3: It is well known that [14] *if* $w(t,\varepsilon)$ *is any solution of*

$$w^{\Delta} = g(t,w) + \varepsilon, \qquad w(\tau_0) = w_0 + \varepsilon \tag{3.51}$$

for $\varepsilon > 0$ *is sufficiently small, then* $\lim_{\varepsilon\to 0} w(t,\varepsilon) = r(t,\tau_0,w_0)$ *on every compact interval* $[\tau_0,\tau_0+\xi] \cap \mathbb{T}$, $\xi > 0$. *Hence setting* $w_0(t,\varepsilon) = w(t+\eta,\varepsilon)$ *we have*

$$w_0(t_0, \varepsilon) = w(t_0 + \eta, \varepsilon) = w(\tau_0, \varepsilon) = w_0 + \varepsilon > w_0 \geq m(t_0) \tag{3.52}$$

and also

$$w_0^{\Delta}(t,\varepsilon) = w^{\Delta}(t + \eta, \varepsilon) = g(t + \eta, w_0(t,\varepsilon)) + \varepsilon > g(t + \eta, w_0(t, \varepsilon)) \tag{3.53}$$

then since $\eta > 0$ *by (iii)*

$$w_0^{\Delta}(t,\varepsilon) > g(t, w_0(t,\varepsilon)). \tag{3.54}$$

On the other hand by (i) we get

$$D^{+}m^{\Delta}(t) \leq g\big(t, m(t)\big) \quad and \quad m(t_0) < w_0(t_0,\varepsilon). \tag{3.55}$$

Therefore by Theorem 3.1, we obtain

$$m(t) < w_0(t,\varepsilon) = w(t + \eta, \varepsilon). \tag{3.56}$$

Since $\varepsilon > 0$ *is arbitrary, we obtain by letting* $\varepsilon \to 0$ *that* $m(t) \leq r(t + \eta)$ $t \geq t_0$. *This proves (a). For the proof of (b) set* $m_0(t) = m(t - \eta)$ *so that*

$$m_0(\tau_0) = m(\tau_0 - \eta) = m(t_0) \leq w_0 < w_0 + \varepsilon = w(\tau_0, \varepsilon) \tag{3.57}$$

and also by (i) and (iii)

$$D^{+}m_0^{\Delta}(t) = D^{+}m^{\Delta}(t - \eta) \leq g(t - \eta, m_0(t)) \leq g(t, m_0(t)). \tag{3.58}$$

On the other hand we have $w_0^{\Delta}(t,\varepsilon) > g(t, w_0(t, \varepsilon))$. *Therefore by Theorem 3.1, we have* $m(t - \eta) < w(t,\varepsilon)$. *Since* $\varepsilon > 0$ *is arbitrary, we obtain by letting* $\varepsilon \to 0$ *that* $m(t - \eta) \leq r(t)$ $t \geq \tau_0$. *This proves (b).*□

In the following theorem, we obtain a comparison result in terms of Lyapunov-like functions with ITD.

Theorem 3.4: Assume that

i) $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $V(t,x)$ *is locally Lipschitzian in* $x \in \mathbb{R}^n$ *for each* $t \in \mathbb{T}$, $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}]$ *and*

$$D^+V^{\Delta}(t, y(t+\eta) - x(t)) \leq g(t, V(t, y(t+\eta) - x(t))) \tag{3.59}$$

ii) the maximal solution $r(t) = r(t, \tau_0, u_0)$ *of* $u^{\Delta} = g(t,u), u(\tau_0) = u_0 \geq 0$, *exists for* $t \geq \tau_0 \geq 0$;
iii) $\tau_0 > t_0$ *and* $g(t,u)$ *is non-decreasing in* $t \in \mathbb{T}$ *for each* $u \in \mathbb{R}_+$;
iv) $g(t,u)\mu(t)$ *is non-decreasing in* $u \in \mathbb{R}_+$ *for each* $t \in \mathbb{T}$.

Then

$$V(t_0, y_0 - x_0) \leq u_0 \tag{3.60}$$

implies that

$$V(t, y(t+\eta, \tau_0, y_0) - x(t, t_0, x_0)) \leq r(t+\eta, \tau_0, u_0) \;\; for \;\; t \geq t_0 \tag{3.61}$$

Proof 3.4: Define $m(t) = V(t, y(t+\eta, \tau_0, y_0) - x(t, t_0, x_0))$ *so that*

$$m(t_0) = V(t_0, y_0 - x_0) \leq u_0. \tag{3.62}$$

Further set $r_0(t) = r(t+\eta, \tau_0, u_0)$ *so that* $r_0(t_0) = r(\tau_0) = u_0$ *and also* $r_0^{\Delta}(t) = r^{\Delta}(t+\eta) = g(t+\eta, r_0(t))$. *We apply the induction principle to the statement:* $\{A(t): V(t, y(t+\eta, \tau_0, y_0) - x(t, t_0, x_0)) \leq r(t+\eta, \tau_0, u_0), t \in \mathbb{T}, t \geq t_0\}$.

– *Let* $t = t_0$. *Since* $m(t_0) = V(t_0, y_0 - x_0) \leq u_0 = r(t_0+\eta, \tau_0, u_0)$, *it follows that* $A(t_0)$ *is true.*

– *Let* t *be right-scattered and* $A(t)$ *is true. We shall show that* $A(\sigma(t))$ *is true. Using the definition of the derivative for right-scattered point and by (iii), we have*

$$
\begin{aligned}
m(\sigma(t)) - r_0(\sigma(t)) &= (D^+ m^\Delta(t) - r_0^\Delta(t))\mu(t) + (m(t) - r_0(t)) \\
&\leq (g(t, m(t)) - g(t + \eta, r_0(t)))\mu(t) + (m(t) - r_0(t)) \\
&\leq (g(t, m(t)) - g(t, r_0(t)))\mu(t) + (m(t) - r_0(t))
\end{aligned} \tag{3.63}
$$

Then since $A(t)$ is true, by assumption (iv) it follows that

$$
m(\sigma(t)) - r_0(\sigma(t)) \leq 0 \tag{3.64}
$$

In view of the fact that

$$
\begin{aligned}
&\frac{m(\sigma(t)) - m(t)}{\mu(t)} \\
&= \frac{V\left(\sigma(t), y(\sigma(t) + \eta) - x(\sigma(t))\right) - V(t, y(t + \eta) - x(t))}{\mu(t)}
\end{aligned} \tag{3.65}
$$

we see that $A(\sigma(t))$ is true.

– *Let t be right-dense and U be a neighborhood of t. Assume that $A(t)$ is true. We need to show that $A(s)$ is true for $s > t, s \in U$. Let $z(t, t_0, y_0 - x_0) = y(t + \eta, \tau_0, y_0) - x(t, t_0, x_0)$ so that*

$$
\begin{aligned}
z^\Delta = \tilde{f}(t, z; \eta) &= [f(t + \eta, y(t + \eta)) - f(t, x(t))], \\
z(t_0) &= y_0 - x_0
\end{aligned} \tag{3.66}
$$

$$
\begin{aligned}
&\frac{m(s+h) - m(s)}{h} \\
&= \frac{V(s+h, z(s+h)) - V(s+h, z(s) + h\tilde{f}(s, z(s)))}{h} \\
&+ \frac{V(s+h, z(s) + h\tilde{f}(s, z(s))) - V(s, z(s))}{h} \\
&= \frac{V(s+h, z(s) + h\tilde{f}(s, z(s)) + h\varepsilon(h)) - V(s+h, z(s) + h\tilde{f}(s, z(s)))}{h} \\
&+ \frac{V(s+h, z(s) + h\tilde{f}(s, z(s))) - V(s, z(s))}{h}
\end{aligned} \tag{3.67}
$$

Since V is locally Lipschitzian in x and $L > 0$ is the Lipschitz constant and ε is error term, we have

$$\begin{aligned} &D^{+}m^{\Delta}(s) \\ &\leq \lim_{h\to 0^{+}} L\|\varepsilon(h)\| \\ &+ \lim_{h\to 0^{+}, s+h\in\mathbb{T}} sup \frac{V(s+h, z(s)+hf(s,z(s))) - V(s,z(s))}{h} \\ &= D^{+}V^{\Delta}(s, y(s+\eta) - x(s)) \leq g(s, m(s)) \end{aligned} \tag{3.68}$$

Since $A(t)$ is true, by Theorem 3.3, we obtain that

$$\begin{aligned} m(s) = V(s, y(s+\eta, \tau_0, y_0) - x(s, t_0, x_0)) \leq r(s+\eta, \tau_0, u_0), \\ for \quad s \geq t, s \in U \end{aligned} \tag{3.69}$$

– Let t be left-dense and $A(s)$ is true for $s < t$. We need to show that $A(t)$ is true. This follows by rd-continuity of $V(t,z)$ and $r(t)$.
Thus by induction principle, we conclude that

$$V(t, y(t+\eta, \tau_0, y_0) - x(t, t_0, x_0)) \leq r(t+\eta, \tau_0, u_0) \quad t \in \mathbb{T} \quad t \geq t_0. \tag{3.70}$$

3.4.2. Stability and Boundedness Criteria

A very general comparison principle is obtained under much less limiting assumption by using the notion of Lyapunov function together with theory of dynamic inequalities. In this setup, one can see Lyapunov function as a transformation which reduces the study of stability, boundedness properties relative to ITD of a given complicated dynamic system to the study of stability, boundedness properties of a relatively simpler scalar dynamic equation.

Let us consider the following scalar dynamic equation

$$u^{\Delta} = g(t,u), \qquad u(\tau_0) = u_0 \geq 0 \ \ for \quad t \in \mathbb{T}, \qquad t \geq \tau_0 \tag{3.71}$$

where $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_{+}, \mathbb{R}]$.

3.4.2.1. Stability Criteria

Theorem 3.5: Assume that

i) $V \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, $V(t,x)$ *is locally Lipschitzian in* $x \in \mathbb{R}^n$ *for each* $t \in \mathbb{T}$, $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}]$ *and*

$$D^+V^\Delta(t, y(t+\eta) - x(t)) \leq g(t, V(t, y(t+\eta) - x(t))) \tag{3.72}$$

ii) the maximal solution $r(t) = r(t, \tau_0, u_0)$ *of* (3.71) *exists for* $t \geq \tau_0$*;*
iii) there exists $a, b \in \mathcal{K}$ *such that* $b(\|x\|) \leq V(t,x) \leq a(\|x\|)$ *for* $(t,x) \in \mathbb{T} \times S(\rho)$*;*
iv) $g(t,u)\mu(t)$ *is non-decreasing in* $u \in \mathbb{R}_+$ *for each* $t \in \mathbb{T}$*;*
v) $\tau_0 > t_0$ *and* $g(t,u)$ *is non-decreasing in* $t \in \mathbb{T}$ *for each* $u \in \mathbb{R}_+$ *and* $g(t,0) = 0$.

Then the stability properties of the null solution of (3.71) *imply the corresponding initial time difference stability properties of the solution* $x(t, t_0, x_0)$.

Proof 3.5: Firstly, let the null solution of the (3.71) *is equistable. Let* $0 < \varepsilon < \rho$ *and* $\tau_0 \in \mathbb{T}$ *be given. Then by definition of equistability given* $b(\varepsilon) > 0$, $\tau_0 \in \mathbb{T}$, $\exists \delta_1 = \delta_1(\varepsilon, \tau_0) > 0$ *such that*

$$u(t) < b(\varepsilon) \quad provided\ that \quad u_0 < \delta_1, \qquad t \geq \tau_0, \qquad t \in \mathbb{T} \tag{3.73}$$

where $u(t, \tau_0, u_0)$ *is any solution of the* (3.71). *Choose* $\delta_2 = \delta_2(\varepsilon, \tau_0) > 0$ *as* $0 < a(\delta_2) < \delta_1$. *Obviously;* $\lim_{(\tau_0, y_0) \to (t_0, x_0)} \|y(t+\eta, \tau_0, y_0) - x(t, t_0, x_0)\| = 0$. *Then given* $\varepsilon > 0$ *and* $\tau_0 \in \mathbb{T}$, *there exist* $\tilde{\delta} = \tilde{\delta}(\varepsilon, \tau_0) > 0$ *and* $\delta_3 = \delta_3(\varepsilon, \tau_0) > 0$ *such that*

$$\|y(t+\eta) - x(t)\| < \varepsilon \quad if \quad \|y_0 - x_0\| < \delta_3 \ \ and \ \ |\eta| < \tilde{\delta} \\ for \quad t_0 \leq t \leq \tau_0 \tag{3.74}$$

Let $u_0 = a(\|y_0 - x_0\|)$ *and choose* $\delta = min(\delta_2, \delta_3)$. *Then we claim that*

$$\|y(t+\eta)-x(t)\| < \varepsilon \quad if \quad \|y_0 - x_0\| < \delta_3 \quad and \quad |\eta| < \tilde{\delta} \quad for \quad t \geq t_0 \tag{3.75}$$

If it is not true, becasuse of (3.74)*, there exist a* $t_1 > \tau_0$ *and a solution* $y(t,\tau_0,y_0)$ *of* (3.2) *with* $\|y_0 - x_0\| < \delta$ *and* $|\eta| < \tilde{\delta}$ *such that*

$$\|y(t_1+\eta)-x(t_1)\| \geq \varepsilon, \qquad \|y(t+\eta)-x(t)\| < \varepsilon, \qquad for \quad t_0 \leq t < t_1 \tag{3.76}$$

Moreover since $\|y_0 - x_0\| < \delta$*, by (iii) we have*

$$V(t_0, y_0 - x_0) \leq a(\|y_0 - x_0\|) < a(\delta) < \delta_1 \tag{3.77}$$

Hence, by (i), (ii), (3.76) *and Theorem 3.4, we obtain the following estimate*

$$V(t, y(t+\eta)-x(t)) \leq r(t+\eta, \tau_0, u_0), \qquad t_0 \leq t < t_1 \tag{3.78}$$

Consequently, the relations (3.73), (3.76), (3.78) *and (iii) leads to the contradiction*

$$\begin{aligned} b(\varepsilon) \leq b(\|y(t_1+\eta)-x(t_1)\|) \leq V(t_1, y(t_1+\eta)-x(t_1)) \\ \leq r(t_1+\eta, \tau_0, u_0) < b(\varepsilon) \end{aligned} \tag{3.79}$$

which proves that the solution $x(t, t_0, x_0)$ *of* (3.1) *is equistable with ITD. Secondly, let the null solution of the* (3.71) *is uniformly equistable, then* δ_1 *is independent of* τ_0*, it follows that* δ *is independent of* τ_0*, thus by the same procedure above we obtain uniform stability with ITD of* $x(t, t_0, x_0)$ *of* (3.1)*. Thirdly, let the null solution of the* (3.71) *is asymptotically equistable. It follows that* $x(t, t_0, x_0)$ *of* (3.1) *is equistable with ITD. Consequently, it can be chosen that* $\varepsilon = \rho$ *and* $\delta_0 = \delta_0(\varepsilon, \tau_0) > 0$, $\tilde{\delta}_0 = \tilde{\delta}_0(\varepsilon, \tau_0) > 0$ *such that*

$$\|y(t+\eta)-x(t)\| < \rho \quad if \quad \|y_0 - x_0\| < \delta_0 \quad and \quad |\eta| < \delta_0, \qquad t \geq t_0 \tag{3.80}$$

To prove attractivity, we let $0 < \varepsilon < \rho$*. Since* $u \equiv 0$ *of* (3.71) *is asymptotically equistable, given* $b(\varepsilon) > 0$, $\tau_0 \in \mathbb{T}$ *there exist* $\delta_1 = \delta_1(\varepsilon, \tau_0) > 0$ *and* $T = T(\varepsilon, \tau_0) > 0$ *such that*

$$u(t) < b(\varepsilon) \quad provided\ that \quad u_0 < \delta_1, \qquad t \geq \tau_0 + T \tag{3.81}$$

Choose $u_0 = a(\|y_0 - x_0\|)$ *and* $\delta_2 = \delta_2(\varepsilon, \tau_0) > 0$ *as* $0 < a(\delta_2) < \delta_1$. *Since* $\lim_{(\tau_0, y_0) \to (t_0, x_0)} \|y(t + \eta, \tau_0, y_0) - x(t, t_0, x_0)\| = 0$, *it follows that given* $\varepsilon > 0$ *and* $\tau_0 \in \mathbb{T}$, *there exist* $\tilde{\delta}_1 = \tilde{\delta}_1(\varepsilon, \tau_0) > 0$ *and* $\delta_3 = \delta_3(\varepsilon, \tau_0) > 0$ *such that*

$$\begin{gathered} \|y(t + \eta) - x(t)\| < \varepsilon \quad if \quad \|y_0 - x_0\| < \delta_3 \quad and \quad |\eta| < \tilde{\delta}_1 \\ for \quad t_0 \leq t \leq \tau_0 \end{gathered} \tag{3.82}$$

Let $\delta = min(\delta_0, \delta_2, \delta_3)$ *and* $\tilde{\delta} = min(\tilde{\delta}_0, \tilde{\delta}_1)$, *and by (i), (ii),* (3.80), *Theorem 3.4, we have the estimate*

$$V(t, y(t + \eta) - x(t)) \leq r(t + \eta, \tau_0, u_0), \qquad t \geq t_0, \qquad t \in \mathbb{T} \tag{3.83}$$

Now, suppose that there exists a sequence $\{t_k\} \in \mathbb{T}, t_k \to \infty$ *as* $k \to \infty, t_k > \tau_0 + T$ *and a solution* $y(t, \tau_0, y_0)$ *of* (3.2) *with* $\|y_0 - x_0\| < \delta$ *and* $|\eta| < \tilde{\delta}$ *such that*

$$\|y(t_k + \eta) - x(t_k)\| \geq \varepsilon \tag{3.84}$$

The relations (iii), (3.81), (3.83), (3.84), *leads to the contradiction,*

$$\begin{gathered} b(\varepsilon) \leq b(\|y(t_k + \eta) - x(t_k)\|) \leq V(t_k, y(t_k + \eta) - x(t_k)) \\ \leq r(t_k + \eta, \tau_0, u_0) < b(\varepsilon) \end{gathered} \tag{3.85}$$

which proves that the solution $x(t, t_0, x_0)$ *of* (3.1) *is asymptotically equistable with ITD. Finally, assume that* $u \equiv 0$ *of* (3.71) *is uniformly asymptotically equistable. Therefore* δ *and* T *are independent of* τ_0. *It follows that* $x(t, t_0, x_0)$ *of* (3.1) *is uniformly asymptotically equistable with ITD.*□

3.4.2.2. Boundedness Criteria

In this section we shall discuss boundedness criteria of dynamic systems by employing comparison method. For boundedness properties of solutions, we do not have to need the existence of null solutions.

Theorem 3.6: Assume that

i) $V \in C_{rd}[\mathbb{T} \times S^c(\rho), \mathbb{R}_+]$, $V(t,x)$ *is locally Lipschitzian in* $x \in \mathbb{R}^n$ *for each* $t \in \mathbb{T}$, $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}]$ *and*

$$D^+V^{\Delta}(t, y(t+\eta) - x(t)) \le g(t, V(t, y(t+\eta) - x(t))) \tag{3.86}$$

ii) the maximal solution $r(t) = r(t, \tau_0, u_0)$ *of* (3.71) *exists for* $t \ge \tau_0$*;*
iii) there exists $a, b \in \mathcal{K}$ *such that* $b(\|x\|) \le V(t,x) \le a(\|x\|)$ *for* $(t,x) \in \mathbb{T} \times S(\rho)$*;*
iv) $g(t,u)\mu(t)$ *is non-decreasing in* $u \in \mathbb{R}_+$ *for each* $t \in \mathbb{T}$*;*
v) $\tau_0 > t_0$ *and* $g(t,u)$ *is non-decreasing in* $t \in \mathbb{T}$ *for each* $u \in \mathbb{R}_+$.

Then boundedness property of (3.71) *imply the corresponding boundedness property with ITD of* (3.1)

Proof 3.6: Assume that the dynamic equation (3.71) *is equibounded. Let* $\alpha > \rho$ *and* $\tau_0 \in \mathbb{T}$ *be given and let* $\|y_0 - x_0\| \le \alpha$. *Define* $\alpha_1 = a(\alpha) > 0$. *From the equiboundedness of* (3.71), *it follows that, given* $\alpha_1 > 0$ *there exist* $\beta_1 = \beta_1(\alpha_1, \tau_0) > 0$ *such that*

$$u(t, \tau_0, u_0) < \frac{\beta_1}{2} \quad \text{provided that} \quad u_0 \le \alpha_1, \qquad t \ge \tau_0 \tag{3.87}$$

Obviously; $\lim_{\eta \to 0^+} |u(t+\eta) - u(t)| = 0$. *Therefore, given* $\frac{\beta_1}{2} > 0$ *there exists a* $\tilde{\delta} = \tilde{\delta}(\alpha, \tau_0) > 0$ *such that*

$$|u(t+\eta)-u(t)|<\frac{\beta_1}{2} \qquad provided\ that \qquad |\eta|<\delta, \qquad t\geq\tau_0 \tag{3.88}$$

Since $b(r)\to\infty$ *as* $r\to\infty$, *there exist* $\beta=\beta(\alpha,\tau_0)>0$ *such that*

$$b(\beta)\geq\beta_1 \tag{3.89}$$

Now we claim that

$$\|y(t+\eta)-x(t)\|<\beta \quad if \quad \|y_0-x_0\|\leq\alpha \quad and \quad |\eta|<\delta \quad t\geq t_0 \tag{3.90}$$

If this were false, there would exist some solutions $y(t,\tau_0,y_0)$ *of* (3.2) *with* $\|y_0-x_0\|\leq\alpha$ *and* $|\eta|<\delta$ *and* $t_1>t_2>\tau_0$ *such that*

$$\|y(t_2+\eta)-x(t_2)\|=\alpha \quad \|y(t_1+\eta)-x(t_1)\|=\beta \tag{3.91}$$

and

$$\rho<\alpha\leq\|y(t+\eta)-x(t)\|\leq\beta, \qquad for \qquad t_2\leq t\leq t_1 \tag{3.92}$$

Set $z(t,t_0,y_0-x_0)=y(t+\eta,\tau_0,y_0)-x(t,t_0,x_0)$ *so that* $z(t)$ *is solution of*

$$z^\Delta=\tilde{f}(t,z;\eta) \tag{3.93}$$

where $\tilde{f}(t,z;\eta)=f(t,z(t,t_0,y_0-x_0)+x(t,t_0,x_0))-f(t,x(t,t_0,x_0))$. *Choose* $u_0=a(\|z(t_2)\|)$, *where* $z(t_2)=z(t_2,t_0,y_0-x_0)$ *so that by (iii)*

$$V(t_2,z(t_2))\leq a(\|z(t_2)\|)=a(\alpha)=\alpha_1 \tag{3.94}$$

Hence the condition (i) and Theorem 3.4 shows that, because of (3.91) and (3.92)

$$V(t,\tilde{z}(t,t_2,z(t_2)))\leq r(t+\eta,t_2,u_0), \qquad for \quad t_2\leq t\leq t_1 \tag{3.95}$$

where $\tilde{z}(t, t_2, z(t_2))$ *is any solutions of* $z^{\Delta} = \tilde{f}(t, z; \eta)$ *through* $(t_2, z(t_2))$. *Thus the estimate* (3.95) *is true for* $y(t + \eta, \tau_0, y_0) - x(t, t_0, x_0)$ *on the interval* $[t_2, t_1] \cap \mathbb{T}$. *We therefore obtain, from the foregoing considerations, using* (3.87), (3.88), (3.89), (3.91), (3.92), (3.95) *and condition (iii)*

$$\begin{aligned} b(\beta) \le V(t_1, y(t_1 + \eta) - x(t_1)) \le r(t_1 + \eta, t_2, u_0) < r(t_1, t_2, u_0) + \frac{\beta_1}{2} \\ < \beta_1 \le b(\beta) \end{aligned} \tag{3.96}$$

This contradiction proves (B1). For the case that the scalar system is uniformly equibounded, β_1 *is independent of* τ_0. *Therefore in a similar way, we obtain the uniformly equiboundedness with ITD of* (3.1). *Now assume that the dynamic equation* (3.71) *is ultimately equibounded. This implies that the dynamic system* (3.1) *is equibounded. Hence given* $\rho > 0$, $t_0 \in \mathbb{T}$, *there exist* $B > 0$ *and* $\tilde{\delta}_0 > 0$ *such that*

$$\|y(t + \eta) - x(t)\| < B \quad if \quad \|y_0 - x_0\| \le \rho \quad and \quad |\eta| < \delta \quad t \ge t_0 \tag{3.97}$$

Let now $\alpha > \rho$ *and* $\tau_0 \in \mathbb{T}$ *be given, and let* $\rho \le \|y_0 - x_0\| \le \alpha$. *Define* $\alpha_1 = a(\alpha) > 0$. *By the definition of quasi-equi-ultimately boundedness, it follows that, given* $\alpha_1 > 0$, $\tau_0 \in \mathbb{T}$, *there exists positive numbers* N_1 *and* $T = T(\alpha, \tau_0)$ *such that*

$$u(t, \tau_0, u_0) < \frac{N_1}{2} \quad if \quad u_0 \le \alpha_1 \quad t \ge \tau_0 + T \tag{3.98}$$

Obviously $\lim_{\eta \to 0^+} |u(t + \eta) - u(t)| = 0$. *Therefore, given* $\frac{N_1}{2} > 0$ *there exists a* $\tilde{\delta}_1 = \tilde{\delta}_1(\alpha, \tau_0) > 0$ *such that*

$$|u(t + \eta) - u(t)| < \frac{N_1}{2} \quad provided\ that \quad |\eta| < \delta_1, \qquad t \ge \tau_0 \tag{3.99}$$

Let $N^* = max(N, B)$, *where* N *is chosen so as to satisfy the inequality*

$$b(N) \ge N_1 \tag{3.100}$$

and $\tilde{\delta} = min(\tilde{\delta}_0, \tilde{\delta}_1)$. *Clearly,* $N^* > \rho$, *and the choice of* N *is possible since* $b(u) \to \infty$ *as* $u \to \infty$. *We claim that, with this* N^*, $\tilde{\delta}$ *and* T *the definition (B3) holds. Suppose that this is false. Since the solutions* $y(t,\tau_0,y_0)$, $x(t,t_0,x_0)$ *with* $\|y_0 - x_0\| \le \rho$ *satisfy* $\|y(t+\eta) - x(t)\| < N^*$, *it is enough to consider only solutions* $y(t,\tau_0,y_0)$ *with* $\rho \le \|y_0 - x_0\| \le \alpha$. *Set* $u_0 = a(\|y_0 - x_0\|)$, *then assumption (i) and Theorem 3.4 yields, for such solutions, because of* (3.97), *the inequality*

$$V(t, y(t+\eta) - x(t)) \le r(t+\eta, \tau_0, u_0), \qquad t \ge t_0 \tag{3.101}$$

Let there exists a sequence $\{t_k\}$, $t_k > \tau_0$, $t_k > \tau_0 + T$, $t_k \to \infty$ *as* $k \to \infty$ *and some solutions* $y(t,\tau_0,y_0)$ *of* (3.2) *with* $\rho \le \|y_0 - x_0\| \le \alpha$ *and* $|\eta| < \delta$ *such that* $\|y(t_k+\eta) - x(t_k)\| \ge N^*$. *Then we get the following inequality, using* (3.98)-(3.101)

$$\begin{aligned} b(N^*) \le V(t_k, y(t_k+\eta) - x(t_k)) &\le r(t_k+\eta, \tau_0, u_0) \\ &< r(t_k, \tau_0, u_0) + \frac{N_1}{2} < N_1 \le b(N) \end{aligned} \tag{3.102}$$

whence we have $N^* < N$. *This is absurd in view of the definition of* N^*, *since* $N^* \ge N$. *Thus the system* (3.1) *is equi-ultimately bounded with ITD. Next let the dynamic equation* (3.71) *be uniformly ultimately equibounded. Then* N_1 *is independent of* τ_0. *Hence, in a similar way, we obtain the uniformly ultimately equiboundedness with ITD of* (3.1). *This completes the proof.*□

3.4.3. Applications

In this section we give an example that applies the results of the preceding section.

Example 3.1: Consider the scalar dynamic system

$$x^{\Delta} = -x + h(t), \qquad x(t_0) = x_0, \qquad for \quad t \ge t_0, \qquad t \in \mathbb{T} \tag{3.103}$$

where $h(t)$ *is a non-increasing function. Let* $V(x) = x^2$ *and* $b(r) = \frac{r^2}{2}$, $a(r) = 2r^2$, *so that*

$$b(x) \leq V(x) \leq a(x) \tag{3.104}$$

Then

$$\begin{aligned} D^+V^\Delta(t, \tilde{y} - x) &= 2(\tilde{y} - x)(\tilde{y}^\Delta - x^\Delta) + \mu(t)(\tilde{y}^\Delta - x^\Delta)^2 \\ &= 2(\tilde{y} - x)\big(-(\tilde{y} - x) + h(t+\eta) - h(t)\big) \\ &+ \mu(t)\big(-(\tilde{y} - x) + h(t+\eta) - h(t)\big)^2 \\ &\leq (\mu(t) - 2)(\tilde{y} - x)\text{^}2 \quad since \quad h(t) \quad is\ decreasing \\ &= (\mu(t) - 2)V \end{aligned} \tag{3.105}$$

where $\tilde{y} = \tilde{y}(t) = y(t+\eta, \tau_0, y_0)$, *where* $y(t, \tau_0, y_0)$ *is solution of* (3.103) *through* (τ_0, y_0), *and* $x = x(t) = x(t, t_0, x_0)$ *is solution of* (3.103) *through* (t_0, x_0). *Then the corresponding scalar comparison equation is*

$$u^\Delta = (\mu(t) - 2)u, \qquad u(\tau_0) = u_0, \qquad t \geq \tau_0 \tag{3.106}$$

If $1 + \mu(t)(\mu(t) - 2) \neq 0, t \in \mathbb{T}^\kappa$, *i.e, the function* $\mu(t) - 2$ *is regressive, then the solution of comparison equation is*

$$u(t, \tau_0, u_0) = u_0 e_{\mu(t)-2}(t, \tau_0) = u_0 exp\left(\int_{\tau_0}^t \frac{Log((\mu(s) - 1)^2}{\mu(s)} \Delta s\right) \tag{3.107}$$

where $e_{\mu(t)-2}(t, \tau_0)$ *is the generalized exponential function on time scales, Log is the principal logarithm function.*

Now it is easy to see that:

1. *The null solution of* (3.106) *is stable if*

$$\int_{\tau_0}^t \frac{Log((\mu(s) - 1)^2}{\mu(s)} \Delta s \leq N(\tau_0), \qquad t \geq \tau_0 \tag{3.108}$$

where $N(\tau_0)$ *is finite for any* $\tau_0 \geq 0$. *It follows by Theorem* (3.5) *that the solution* $x(t, t_0, x_0)$ *of* (3.103) *is stable with ITD.*

2. The null solution of (3.106) *is uniformly stable if*

$$\int_{\tau_0}^{t} \frac{Log((\mu(s)-1)^2}{\mu(s)} \Delta s \leq N, \qquad t \geq \tau_0 \tag{3.109}$$

with N *is constant. It follows by Theorem* (3.5) *that the solution* $x(t, t_0, x_0)$ *of* (3.103) *is uniformly stable with ITD.*

3. The null solution of (3.106) *is asymptotically stable if*

$$\lim_{t\to\infty} \int_{\tau_0}^{t} \frac{Log((\mu(s)-1)^2}{\mu(s)} \Delta s = -\infty \tag{3.110}$$

It follows by Theorem (3.5) *that the solution* $x(t, t_0, x_0)$ *of* (3.103) *is asymptotically stable with ITD.*□

4. STABILITY OF PERTURBED DYNAMIC SYSTEM ON TIME SCALES WITH INITIAL TIME DIFFERENCE

4.1. Introduction

In real world applications, it is necessary to consider a particular dynamic system with a perturbation term. In addition to perturbing the given particular dynamic system, it is possible to make an error in initial times as well as in initial positions. Therefore, one need to investigate the qualitative and quantitative properties of a given particular dynamic system under these perturbations.

A principal technique employed in stability theory is investigating stability properties of a particular dynamic system under small perturbations. This technique is employed in many ways [5], [13]. Another techniques are also used in [5], [7], [8], [13]. In [21], [22], [26], [27], the authors considered the case that perturbed dynamic system and original unperturbed dynamic system which have different initial time.

In the present paper, we consider the problem of determining the behavior of solutions of a perturbed dynamic equation with respect to those of original unperturbed dynamic system that have initial time difference (ITD). We consider this problem on arbitrary time scales, nonempty closed subset of real numbers, and therefore we obtained a general result that can be applied discrete and continuous cases simultaneously

We begin with a preliminary section which includes the basic concepts and definitions. After that, we give the obtained novel results.

4.2. Basic Definitions and Concepts

We will consider the dynamic system

$$x^{\Delta} = f(t,x), \qquad x(t_0) = x_0 \tag{4.1}$$

where $f \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}^{\mathrm{n}}]$, $S(\rho) = \{x \in \mathbb{R}^{\mathrm{n}}: \|x\| \leq \rho\}$. Here $\|x\|$ denotes any n-dimensional norm of the vector x.

In addition to dynamical system (4.1), we also consider the associated perturbed dynamical system

$$y^{\Delta} = f(t, y) + R(t, y), \qquad y(\tau_0) = y_0 \tag{4.2}$$

where $R \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}^{\mathrm{n}}]$ is called the perturbation term.

We assume that $f, R \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}^{\mathrm{n}}]$ are smooth enough to guarantee the existence, uniqueness and rd-continuous dependence of solutions of (4.1) and (4.2).

We now give the definitions of stability which identify the possible behavior for solution of perturbed dynamic system (4.2) which we will need later.

Definition 4.1: The solutions of the perturbed dynamic system (4.2) *are said to be stable with respect to unperturbed dynamic system* (4.1) *with initial time difference if, given* $\varepsilon > 0$ *and* $\tau_0 \in \mathbb{T}$, *there exist* $\delta(\varepsilon, \tau_0) > 0$ *and* $\tilde{\delta}(\varepsilon, \tau_0) > 0$ *such that* $\|y_0 - x_0\| < \delta$ *and* $|\tau_0 - t_0| < \tilde{\delta}$ *implies*

$$\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| < \varepsilon \tag{4.3}$$

for $t \geq \tau_0$, *for every solution* $y(t, \tau_0, y_0)$ *of the perturbed dynamic system* (4.2), *where* $\eta = \tau_0 - t_0 > 0$.

Definition 4.2: The solutions of the perturbed dynamic system (4.2) *are said to be asymptotically stable with respect to unperturbed dynamic system* (4.1) *with initial time difference if, they are stable with respect to equation* (4.1) *with initial time difference and if, given* $\varepsilon > 0$ *and* $\tau_0 \in \mathbb{T}$, *there exist* $\delta_0(\tau_0) > 0$, $\tilde{\delta}_0(\tau_0) > 0$ *and* $T = T(\varepsilon, \tau_0) > 0$ *such that* $\|y_0 - x_0\| < \delta_0$ *and* $|\tau_0 - t_0| < \tilde{\delta}_0$ *implies*

$$\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| < \varepsilon \tag{4.4}$$

for $t \geq \tau_0 + T$, *for every solution* $y(t, \tau_0, y_0)$ *of the perturbed system* (4.2).

Definition 4.3: The solutions of the perturbed dynamic system (4.2) *are said to be unstable with respect to unperturbed dynamic system* (4.1) *with initial time difference if, they are not stable with respect to unperturbed dynamic system* (4.1) *with initial time difference.*

Definition (4.1) and Definition (4.2) are equivalent to the statement that all solutions of the perturbed dynamic system (4.2) which start sufficiently close to the initial conditions of the unperturbed solution respectively remain close to it or eventually approach it. Definition (4.3) requires that for each solution of the unperturbed equation (4.1), a solution of the perturbed equation (4.2) can be found which starts arbitrarily close to the unperturbed solution and which eventually diverges from it.

We stress that all of the above definitions are independent of the behavior of the solutions of the unperturbed dynamic system. Indeed, we particularly show that the equilibria of the original dynamic equations may be stable, asymptotically stable or even unstable. We illustrate this situations as the following on different time scales.

Example 4.1: Let $\mathbb{T} = \mathbb{Z}$. *Consider the dynamic equation*

$$x^{\Delta} = \Delta\, x = c, \qquad x(t_0) = x_0 \tag{4.5}$$

where c *is any constant, whose solution is given by* $x(t, t_0, x_0) = x_0 + c(t - t_0)$ *which is unstable. In addition, consider the perturbed equation*

$$\Delta\, y = c + g(t+1), \qquad y(\tau_0) = y_0 \tag{4.6}$$

where $\{g(t)\}$ *is any sequence for which* $\sum^{\infty} g(t) = 0$. *The corresponding solution is then given by* $y(t, \tau_0, y_0) = y_0 + c(t - \tau_0) + \sum_{\tau_0}^{t} g(k)$. *Then the difference is*

$$y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0) = y_0 - x_0 + \sum_{k=\tau_0}^{t} g(k) \tag{4.7}$$

which can be made arbitrarily small. Therefore the solution of the perturbed equation is stable with respect to unperturbed equation with initial time difference. □

Example 4.2: Let $\mathbb{T} = \mathbb{R}$*. Consider the dynamic equation*

$$x^{\Delta} = x' = -ax, \qquad x(t_0) = x_0 \tag{4.8}$$

where $a > 0$*, whose asymptotically stable solution is given by*

$$x(t, t_0, x_0) = x_0 \exp(-a(t - t_0)). \tag{4.9}$$

Further, consider the associated perturbed equation

$$y' = -(a + b)y, \qquad y(\tau_0) = y_0 \tag{4.10}$$

whose solution is $y(t, \tau_0, y_0) = y_0 \exp(-(a + b)(t - t_0))$*. As a consequence,*

$$\begin{aligned} y(t, \tau_0, y_0) &- x(t - \eta, t_0, x_0) \\ &= \exp(-a(t - t_0))[y_0 \exp(-b(t - t_0)) - x_0]. \end{aligned} \tag{4.11}$$

- *If* $b > 0$*, this difference approaches* 0 *as* $t \to \infty$ *and thus the perturbed solutions are asymptotically stable with respect to unperturbed equation with ITD.*
- *If* $b < 0$*, then the perturbed solutions are unstable with respect to unperturbed equation with ITD.*

□

Considering the dynamical systems (4.1) and (4.2), for $V(t, x) \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, we define the generalized derivative with respect to the systems (4.1) and (4.2).

Definition 4.4: Let $V \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, $V(t,u)$ *is locally Lipschitzian in* u.

$$D^+V^{\Delta}(t,u,\eta) := \lim_{s \to t, s \neq \sigma(t)} sup \frac{V(t+\mu(t), u(t) + \mu(t)\tilde{f}(t,u(t);\eta)) - V(s,u(t))}{\sigma(t) - s} \tag{4.12}$$

for $(t,u,\eta) \in \mathbb{T} \times S(\rho) \times \mathbb{R}$, *where* $\mu(t) = \sigma(t) - t$, $u(t,\tau_0,u_0) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$ *for* $t \geq \tau_0$, *and* $\tilde{f}(t,u(t);\eta) = f(t,u(t) + x(t-\eta)) + R(t,u(t) + x(t-\eta)) - f(t-\eta, x(t-\eta))$.

4.3. Main Results

We now present several theorems which supply sufficient conditions for the above types of behavior to hold in terms of the existence of rd-continuous real scalar, Lyapunov-type, functions $V(t,x)$. We first give the comparison result in terms of Lyapunov-like functions which we employ frequently later.

Theorem 4.1: Assume that

i) $V \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, $V(t,u)$ *is locally Lipschitzian in* u *and*

$$D^+V^{\Delta}(t,u,\eta) \leq g(t, V(t,u), |\eta|), \qquad for \quad (t,u,\eta) \in \mathbb{T} \times S(\rho) \times \mathbb{R} \tag{4.13}$$

where $u = u(t) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$, $\eta = \tau_0 - t_0 > 0$, $g \in C_{rd}[\mathbb{T} \times \mathbb{R}^2, \mathbb{R}_+]$;

ii) the maximal solution $r(t,\tau_0,w_0,|\eta|)$ *of* $w^{\Delta} = g(t,w,|\eta|)$, $w(\tau_0) = w_0 \geq 0$ *exists for* $t \geq \tau_0 \geq 0$;

iii) $g(t,w,|\eta|)\mu(t)$ *is non-decreasing in* $w \in \mathbb{R}$ *for each* $\eta \in \mathbb{R}$ *and* $t \in \mathbb{T}$.

Then $V(\tau_0, y_0 - x_0) \leq w_0$ *implies*

$$V(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) \leq r(t,\tau_0,w_0,|\eta|), \qquad t \geq \tau_0 \tag{4.14}$$

Proof 4.1: Set $u(t) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$ for $t \geq \tau_0$ so that $u(\tau_0) = y_0 - x_0$ and

$$\begin{aligned} u^{\Delta}(t) &= f(t,y(t,\tau_0,y_0)) + R(t,y(t,\tau_0,y_0)) - f(t-\eta,x(t-\eta,t_0,x_0)) \\ &= f(t,u(t) + x(t-\eta,t_0,x_0)) + R(t,u(t) + x(t-\eta,t_0,x_0)) - f(t \\ &\qquad - \eta,x(t-\eta,t_0,x_0)) = \tilde{f}(t,u(t);\eta), \qquad for\ t \geq \tau_0. \end{aligned} \tag{4.15}$$

We apply induction principle to the statement: $\{A(t)\colon V(t,u(t,\tau_0,u_0)) \leq r(t,\tau_0,w_0,|\eta|),\ \ t \geq \tau_0,\ \ t,\tau_0 \in \mathbb{T}\}$.

- *Let* $t = \tau_0$. *Since* $V(\tau_0,y_0 - x_0) \leq w_0$, *it follows that* $A(\tau_0)$ *is true.*

- *Let* t *be right-scattered and A(t) is true. We shall show that* $A(\sigma(t))$ *is true. Set* $m(t) = V(t,u(t))$. *Then using the definition of the derivative for right-scattered point, we have the inequality*

$$\begin{aligned} m(\sigma(t)) - r(\sigma(t)) &= (D^{+}m^{\Delta}(t) - r^{\Delta}(t))\mu(t) + (m(t) - r(t)) \\ &\leq (g(t,m(t)) - g(t,r(t)))\mu(t) + (m(t) - r(t)). \end{aligned} \tag{4.16}$$

Then since $A(t)$ *is true, by assumption (iii) it follows that*

$$m(\sigma(t)) - r(\sigma(t)) \leq 0. \tag{4.17}$$

In view of the fact that

$$\frac{m(\sigma(t)) - m(t)}{\mu(t)} = \frac{V(\sigma(t),u(\sigma(t))) - V(t,u(t))}{\mu(t)} \tag{4.18}$$

we see that $A(\sigma(t))$ *is true.*

- *Let* t *be right-dense and* U *be a neighborhood of* t. *Assume that* $A(t)$ *is true. We need to show that* $A(s)$ *is true for* $s > t$, $s \in U$.

$$
\begin{aligned}
&m(s+h)-m(s)\\
&\qquad = V(s+h,u(s+h)) - V(s+h,u(s)+h\tilde{f}(s,u(s);\eta))\\
&\qquad + V(s+h,u(s)+h\tilde{f}(s,u(s);\eta)) - V(s,u(s))\\
&\qquad = V(s+h,u(s)+h\tilde{f}(s,u(s);\eta)+h\varepsilon(h)) - V(s\\
&\qquad + h,u(s)+h\tilde{f}(s,u(s);\eta)) + V(s+h,u(s)\\
&\qquad + h\tilde{f}(s,u(s);\eta)) - V(s,u(s))
\end{aligned}
\tag{4.19}
$$

Since V is locally Lipschitzian in u and $L > 0$ is the Lipschitz constant and ε is the error term, we have

$$
\begin{aligned}
&D^{+}m^{\Delta}(s)\\
&\leq \lim_{h\to 0^{+}} L\|\varepsilon(h)\|\\
&+ \lim_{h\to 0^{+}, s+h\in\mathbb{T}} sup \frac{V(s+h),u(s)+h\tilde{f}(s,u(s);\eta)) - V(s,u(s))}{h}\\
&= D^{+}V^{\Delta}(s,u(s)) \leq g(s,m(s))
\end{aligned}
\tag{4.20}
$$

Since $A(t)$ is true, by Theorem 2.11, we obtain that

$$
m(s) = V(s,u(s,\tau_0,u_0)) \leq r(s,\tau_0,w_0,|\eta|), \qquad for \;\; s \geq t, \qquad s \in U \tag{4.21}
$$

• Let t be left-dense and $A(s)$ is true for $s < t$. We need to show that $A(t)$ is true. This follows by rd-continuity of $V(t,u)$ and $r(t)$.
Thus by induction principle, we conclude that

$$
V(t,u(t,\tau_0,u_0)) \leq r(t,\tau_0,w_0,|\eta|), \qquad t \in \mathbb{T}, \qquad t \geq \tau_0. \tag{4.22}
$$

□

Remark 4.1: If the inequality (i) is reversed and $V(\tau_0, y_0 - x_0) \geq w_0$, then we have to replace the conclusion by $V(t,u(t,\tau_0,u_0)) \geq r_{*}(t,\tau_0,w_0,|\eta|)$,$t \in \mathbb{T}$, $t \geq \tau_0$, where $r_{*}(t,\tau_0,w_0,|\eta|)$ is the minimal solution of comparison equation.

The following theorem satisfies the requirement that both the function $V(t, y(t) - x(t-\eta))$ remains well defined and that the difference $y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)$ of the solution of the perturbed dynamic system (4.2) with shifted solution of the original unperturbed dynamic system (4.1) remains on $S(\rho)$.

Theorem 4.2: Assume that

i) $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$ *is Lipschitzian in time and space such that*

$$\begin{aligned} &\|f(t, u(t, \tau_0, u_0) + x(t-\eta, t_0, x_0)) - f(t-\eta, x(t-\eta, t_0, x_0))\| \\ &\quad \le L(t)\|u(t)\| + N(t)|\eta| \end{aligned} \tag{4.23}$$

where $u(t, \tau_0, u_0) = y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)$ *for* $t \ge \tau_0$, $u_0 = y_0 - x_0$ *and* $\eta = \tau_0 - t_0 > 0$*;*

ii) The perturbation term $R(t, y)$ *satisfies* $\|R(t, y)\| \le a\|h(t)\|$ *for sufficiently small positive constant* a*;*

iii) There exist constants M_1, M_2 *and* M_3 *such that*

$$\int_{\tau_0}^{t} L(s) \Delta s \le M_1, \int_{\tau_0}^{t} N(s) \Delta s \le M_2, \int_{\tau_0}^{t} \|h(s)\| \Delta s \le M_3 \tag{4.24}$$

$for \;\; t \ge \tau_0$.

iv) $L(t) \in \mathcal{R}^+$, $L(t) > 0$ *and* $N(t) > 0$ *for* $t \in \mathbb{T}$.

Then

$$\|y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)\| \le \rho, \qquad for\ all \quad t \ge \tau_0 \tag{4.25}$$

provided that y_0 *and* τ_0 *are chosen sufficiently close to* x_0 *and* t_0*, respectively.*

Proof 4.2: We have

$$y(t,\tau_0,y_0) = y_0 + \int_{\tau_0}^{t} f\big(s,y(s)\big) \vartriangle s + \int_{\tau_0}^{t} R(s,y(s)) \vartriangle s \tag{4.26}$$

and

$$x(t-\eta,t_0,x_0) = x_0 + \int_{\tau_0}^{t} f(s-\eta,x(s-\eta)) \vartriangle s. \tag{4.27}$$

As a consequence;

$$\begin{aligned}
&\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| \\
&\qquad \le \|y_0 - x_0\| \\
&\qquad + \int_{\tau_0}^{t} \big\|f\big(s,y(s,\tau_0,y_0)\big) - f\big(s-\eta,x(s-\eta,t_0,x_0)\big)\big\| \vartriangle s \\
&\qquad + \int_{\tau_0}^{t} \big\|R\big(s,y(s)\big)\big\| \vartriangle s \\
&\qquad \le \|y_0 - x_0\| + a\int_{\tau_0}^{t} \|h(s)\| \vartriangle s + |\eta| \int_{\tau_0}^{t} N(s) \vartriangle s \\
&\qquad + \int_{\tau_0}^{t} L(s)\|y(s) - x(s-\eta)\| \vartriangle s \\
&\qquad \le \|y_0 - x_0\| + aM_3 + |\eta|M_2 \\
&\qquad + \int_{\tau_0}^{t} L(s)\|y(s) - x(s-\eta)\| \vartriangle s
\end{aligned} \tag{4.28}$$

Set $m(t) = \|y(t) - x(t-\eta)\|$ *and* $A = \|y_0 - x_0\| + aM_3 + |\eta|M_2$. *Then*

$$m(t) \le A + \int_{\tau_0}^{t} L(s)m(s) \vartriangle s \tag{4.29}$$

Then by Corollary 2.2, *we obtain the inequality*

$$\|y(t) - x(t-\eta)\| \leq Aexp\left(\int_{\tau_0}^{t} \frac{Log\big(1+\mu(s)L(s)\big)}{\mu(s)} \Delta s\right) \\ \leq Aexp\left(\int_{\tau_0}^{t} L(s) \Delta s\right) \leq Aexp(M_1) \tag{4.30}$$

which can be made smaller than any given ρ by choosing the constant a sufficiently small and by choosing y_0 and τ_0 sufficiently close to x_0 and t_0, respectively.□

Theorem 4.3: Assume that

i) $V \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, $V(t,u)$ is locally Lipschitzian in u and

$$D^+V^{\Delta}(t,u,\eta) \leq g(t,V(t,u),|\eta|) \tag{4.31}$$

for $(t,u,\eta) \in \mathbb{T} \times S(\rho) \times \mathbb{R}$, where $u = u(t) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$, $\eta = \tau_0 - t_0 > 0$, $g \in C_{rd}[\mathbb{T} \times \mathbb{R}^2, \mathbb{R}_+]$;
ii) $g(t,w,|\eta|)\mu(t)$ is non-decreasing in $w \in \mathbb{R}$ for each $\eta \in \mathbb{R}$ and $t \in \mathbb{T}$;
iii) There exist a function $b \in \mathcal{K}$ such that

$$V(t,u) \geq b(\|u\|) \quad for \quad (t,u) \in \mathbb{T} \times S(\rho); \tag{4.32}$$

iv) The scalar equation $w^{\Delta} = g(t,w,|\eta|)$ $w(\tau_0) = w_0 \geq 0$ $t \geq \tau_0$, $t,\tau_0 \in \mathbb{T}$ is stable;
v) The maximal solution $r(t,\tau_0,w_0,|\eta|)$ of $w^{\Delta} = g(t,w,|\eta|)$ $w(\tau_0) = w_0 \geq 0$ exists for $t \geq \tau_0 \geq 0$.

Then the solutions of the perturbed dynamic system are stable with respect to unperturbed dynamic system with ITD, provided that

$$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| \leq \rho, \qquad for\ all \quad t \geq \tau_0. \tag{4.33}$$

Proof 4.3: Let $0 < \varepsilon < \rho$ *and* $\tau_0 \in \mathbb{T}$ *be given. Since the scalar dynamic equation is equistable, we have for a given* $b(\varepsilon) > 0$ *there exists a* $\delta_1 = \delta_1(\varepsilon, \tau_0) > 0$ *and* $\tilde{\delta} = \tilde{\delta}(\varepsilon, \tau_0) > 0$ *such that*

$$w_0 < \delta_1 \ \ and \ \ |\eta| < \delta \ \ implies \ \ w(t, \tau_0, w_0, |\eta|) < b(\varepsilon) \ \ for \ \ t \geq \tau_0 \qquad (4.34)$$

Choose $w_0 = V(\tau_0, y_0 - x_0)$*. Since* $V(t,u)$ *is rd-continuous and* $V(t,0) = 0$ *it is possible to find a positive function* $\delta = \delta(\varepsilon, \tau_0)$ *that is rd-continuous in* τ_0 *for each* $\varepsilon > 0$*, satisfying the inequalities*

$$\|y_0 - x_0\| < \delta, \qquad V(\tau_0, y_0 - x_0) < \delta_1 \qquad (4.35)$$

simultaneously. We claim that

$$\begin{gathered} \|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| < \varepsilon \ \ for \ \ t \geq \tau_0 \\ provided\ that \ \ \|y_0 - x_0\| < \delta \ \ and \ \ |\eta| < \delta \end{gathered} \qquad (4.36)$$

Suppose that this is not true. Then there would exists a solution $y(t, \tau_0, y_0)$ *of* (4.2) *with* $\|y_0 - x_0\| < \delta$, $|\eta| < \tilde{\delta}$ *and a* $t_1 > \tau_0$ *such that*

$$\begin{gathered} \|y(t_1, \tau_0, y_0) - x(t_1 - \eta, t_0, x_0)\| = \varepsilon \\ \|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| \leq \varepsilon, \qquad for \ \ t \in [\tau_0, t_1] \end{gathered} \qquad (4.37)$$

The choice $w_0 = V(\tau_0, y_0 - x_0)$ *and condition (i) give, as a consequence of the Theorem* 4.1, *the estimate*

$$V(t, y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)) \leq r(t, \tau_0, w_0, |\eta|), \qquad t \in [\tau_0, t_1] \qquad (4.38)$$

where $r(t, \tau_0, w_0, |\eta|)$ *is the maximal solution of the comparison equation. Then condition (iii) and the relations* (4.34), (4.37), (4.38) *lead to the contradiction*

$$b(\varepsilon) \leq V(t_1, y(t_1, \tau_0, y_0) - x(t_1 - \eta, t_0, x_0)) \leq r(t_1, \tau_0, w_0, |\eta|) < b(\varepsilon). \qquad (4.39)$$

This proves that the solutions of the perturbed system is stable with respect to the unperturbed system with ITD.□

Theorem 4.4: Assume that

i) $V \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, $V(t,u)$ *is locally Lipschitzian in* u *and*

$$D^+V^{\Delta}(t,u,\eta) \le g(t,V(t,u),|\eta|), \qquad for \ \ (t,u,\eta) \in \mathbb{T} \times S(\rho) \times \mathbb{R} \tag{4.40}$$

where $u = u(t) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$, $\eta = \tau_0 - t_0 > 0$, $g \in C_{rd}[\mathbb{T} \times \mathbb{R}^2, \mathbb{R}_+]$;

ii) $g(t,w,|\eta|)\mu(t)$ *is non-decreasing in* $w \in \mathbb{R}$ *for each* $\eta \in \mathbb{R}$ *and* $t \in \mathbb{T}$;

iii) There exist a function $b \in \mathcal{K}$ *such that*

$$V(t,u) \ge b(\|u\|) \ \ for \ \ \ (t,u) \in \mathbb{T} \times S(\rho); \tag{4.41}$$

iv) The scalar equation $w^{\Delta} = g(t,w,|\eta|)$ $w(\tau_0) = w_0 \ge 0$ $t \ge \tau_0$, $t,\tau_0 \in \mathbb{T}$ *is asymptotically stable;*

v) the maximal solution $r(t,\tau_0,w_0,|\eta|)$ *of* $w^{\Delta} = g(t,w,|\eta|)$ $w(\tau_0) = w_0 \ge 0$ *exists for* $t \ge \tau_0 \ge 0$.

Then the solutions of the perturbed dynamic system are asymptotically stable with respect to unperturbed dynamic system with ITD, provided that

$$\|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| \le \rho \quad for \ all \quad t \ge \tau_0. \tag{4.42}$$

Proof 4.4: Since the scalar system is asymptotically stable it is also stable. Hence by Theorem 4.3, *the solution of perturbed system is stable with respect to unperturbed system with ITD. Therefore, we can choose that* $\varepsilon = \rho > 0$, $\delta_0 = \delta_0(\rho,\tau_0) > 0$ *and* $\tilde{\delta}_0 = \tilde{\delta}_0(\rho,\tau_0) > 0$ *such that*

$$\begin{gathered} \|y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)\| < \rho \ \ for \ \ t \ge \tau_0 \\ provided\ that \ \ \|y_0 - x_0\| < \delta_0 \ \ \ and \ \ |\tau_0 - t_0| < \tilde{\delta}_0 \end{gathered} \tag{4.43}$$

To prove quasi-asymptotic stability, let $0 < \varepsilon < \rho$ *and* $\tau_0 \in \mathbb{T}$ *be given. It then follow from quasi-asymptotic stability of scalar equation that given* $b(\varepsilon) > 0$, $\tau_0 \in \mathbb{T}$ *there exist positive numbers* $\delta_1 = \delta_1(\tau_0)$, $\tilde{\delta}_1 = \tilde{\delta}_1(\tau_0)$ *and* $T = T(\varepsilon, \tau_0)$ *such that*

$$\begin{gathered} w(t, \tau_0, w_0, |\eta|) < b(\varepsilon) \;\; for \quad t \geq \tau_0 + T \\ provided \; that \; w_0 < \delta_1 \;\; and \quad |\eta| < \delta_1 \end{gathered} \tag{4.44}$$

Since $V(t,u)$ *is rd-continuous and* $V(t,0) = 0$, *we can find a positive number* $\delta_2 = \delta_2(\varepsilon, \tau_0)$ *satisfying the inequalities*

$$\|y_0 - x_0\| < \delta_2, \qquad V(\tau_0, y_0 - x_0) < \delta_1 \tag{4.45}$$

simultaneously. The choice $w_0 = V(\tau_0, y_0 - x_0)$, *assumption (i) and the relation* (4.43) *gives as a consequence of Theorem* 4.1, *the estimate*

$$V(t, y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)) \leq r(t, \tau_0, w_0, |\eta|), \qquad for \quad t \geq \tau_0. \tag{4.46}$$

Set $\delta = min\{\delta_0, \delta_2\}$ *and* $\tilde{\delta} = min\{\tilde{\delta}_0, \tilde{\delta}_1\}$. *Suppose now that there exists a sequence* $\{t_k\} \in \mathbb{T}$, $t_k \geq \tau_0 + T$, $t_k \to \infty$ *as* $k \to \infty$ *and a solution* $y(t, \tau_0, y_0)$ *of perturbed system with* $\|y_0 - x_0\| < \delta$ *and* $|\eta| < \tilde{\delta}$ *such that*

$$\|y(t_k, \tau_0, y_0) - x(t_k - \eta, t_0, x_0)\| \geq \varepsilon. \tag{4.47}$$

This lead to the contradiction

$$b(\varepsilon) \leq V(t_k, y(t_k, \tau_0, y_0) - x(t_k - \eta, t_0, x_0)) \leq r(t_k, \tau_0, w_0, |\eta|) < b(\varepsilon) \tag{4.48}$$

because of (4.43), (4.44), (4.46) *and (iii). Thus the solutions of the perturbed system is asymptotically stable with respect to unperturbed system with ITD.* □

Finally, we conclude this section with a criterion for the solution of the perturbed dynamic system (4.2) to be unstable with respect to the original unperturbed dynamic system (4.1) with ITD.

Theorem 4.4: Assume that there exist functions $V(t,u)$ and $g(t,w,|\eta|)$ satisfying the following properties:

i) $V \in C_{rd}[\bar{G},\mathbb{R}_+]$, $V(t,u)$ is locally Lipschitzian in u on $\bar{G}$, $V(t,u) = 0$ for all $(t,u) \in \bar{G} - G$ and $V(t,u)$ is positive and bounded on G, where $G \subset \mathbb{T} \times S(\rho)$ is some open set such that G has at least one boundary point $(T,0), T > 0$;
ii)

$$D^+V^{\Delta}(t,u,\eta) \geq g(t,V(t,u),|\eta|) \geq 0, \qquad for \quad (t,u,\eta) \in G \times \mathbb{R} \tag{4.49}$$

where $u = u(t) = y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)$, $\eta = \tau_0 - t_0 > 0$, $g \in C_{rd}[\mathbb{T} \times \mathbb{R}^2,\mathbb{R}_+]$;
iii) $g(t,w,|\eta|)\mu(t)$ is non-decreasing in $w \in \mathbb{R}$ for each $\eta \in \mathbb{R}$ and $t \in \mathbb{T}$;
iv) For $\tau_0 > T$, the solutions $w(t,\tau_0,w_0,|\eta|)$ of comparison equation, for arbitrarily small $w_0 > 0$, are either unbounded or indeterminate, for $t \geq \tau_0$.

Then the solutions of the perturbed dynamic system are unstable with respect to unperturbed dynamic system with ITD.

Proof 4.4: Let $x(t,t_0,x_0)$ be any solution of the unperturbed system. Choose a point $(\tau_0,y_0 - x_0)$ in the vicinity of $(T,0)$. Consider the solution $y(t,\tau_0,y_0)$ of the perturbed system. Then the lipschitzian nature of $V(t,u)$ and condition (ii) yield

$$V(t,y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) \geq V(\tau_0,y_0 - x_0) = w_0 > 0 \tag{4.50}$$

for all $t \geq 0$, for which $(t,y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) \in G$. Since $V(t,u) = 0$ for all $(t,u) \in \bar{G} - G$, it follows from (4.50) *that $(t,y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) \in G$ for $t \geq \tau_0$. Moreover, we also have*

$$D^+V^{\Delta}(t,y(t) - x(t-\eta)) \geq g(t,V(t,y(t) - x(t-\eta)),|\eta|) \tag{4.51}$$

which in view of Remark 4.1, *implies that*

$$V(t,y(t) - x(t-\eta)) \geq \rho(t,\tau_0,w_0,|\eta|), \qquad t \geq \tau_0 \tag{4.52}$$

where $\rho(t,\tau_0,w_0,|\eta|)$ *is the minimal solution of the comparison equation. Since V(t,u) is bounded by assumption the estimate* (4.52) *leads to a absurdity, if we assume the solutions of the perturbed system is stable with respect to unperturbed system with ITD. This proves the theorem.*□

5. CONCLUDING REMARKS

In this thesis, in Chapter 3, we develop dynamic inequalities and a new comparison principle for dynamic equations on time scales relative to initial time difference, then we prove several stability and boundedness criteria relative to initial time difference by employing comparison method. In Chapter 4, we develop a new approach to determine behavior of solutions of perturbed dynamic system relative to original unperturbed dynamic system which have different initial time on arbitrary time scales. We give some stability properties. It is obvious that the notions introduced here can be extended to include in addition all of the various refinements of the stability properties such as uniform stability with ITD, uniform asymptotic stability with ITD and so forth.

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BIOGRAPHY

His name is Bülent Oğur. He was born in Ağrı, Turkey. He was graduated from the Mathematics Department, Faculty of Sciences, Dokuz Eylül University in 2011, İzmir, Turkey. He started to study his master's (second cycle) at Gebze Institute of Technology in 2012. At the same time he started to work as a research assistant at Department of Mathematics, Gebze Institute of Technology.