THOMASON'S HOMOTOPY COLIMIT THEOREM AND COHOMOLOGY OF CATEGORIES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF
MASTER OF SCIENCE
IN
MATHEMATICS

By
Mehmet Kırtışoğlu
July 2024

Thomason's Homotopy Colimit Theorem and Cohomology of Categories
By Mehmet Kırtışoğlu
July 2024

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Ergün Yalçın(Advisor)

Özgün Ünlü

Mustafa Korkmaz

Approved for the Graduate School of Engineering and Science:

Orhan Arıkan
Director of the Graduate School

ABSTRACT

THOMASON'S HOMOTOPY COLIMIT THEOREM AND COHOMOLOGY OF CATEGORIES

Mehmet Kırtışoğlu
M.S. in Mathematics
Advisor: Ergün Yalçın
July 2024

In 1978, R.W. Thomason [1] proves that there is a homotopy equivalence $\eta : \operatorname{hocolim}_{\mathfrak{C}} N(F) \to N(\int_{\mathfrak{C}} F)$ for any functor $F : \mathfrak{C} \to \mathbf{Cat}$. Here $\operatorname{hocolim}_{\mathfrak{C}} N(F)$ is the diagonalization of the bisimplicial set $\coprod_{\sigma \in N(\mathfrak{C})} N(F(\sigma(0)))$ and $\int_{\mathfrak{C}} F$ is the Grothendieck construction whose objects are given by the pairs (C, X) with $C \in \mathrm{Ob}(\mathfrak{C})$ and $X \in \mathrm{Ob}(F(C))$, and whose morphisms are given by the pairs $(\alpha, \gamma) : (C, X) \to (C', X')$ with $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(C, C')$ and $\gamma \in \mathrm{Hom}_{F(C')}(F(\alpha)(X), X')$. We prove that Grothendieck construction is a precofibred category over the canonical functor $\pi : \int_{\mathfrak{C}} F \to \mathfrak{C}$. We also prove that for any functor $\varphi : \mathfrak{D} \to \mathfrak{C}$ there is a homotopy equivalence $\lambda : \operatorname{hocolim}_{\mathfrak{C}} N(\varphi/C) \to N(D)$. We show that these two together prove Thomason's homotopy colimit theorem from a conceptual point of view. We further investigate how our conceptual approximation for the proof of Thomason's homotopy colimit theorem becomes useful for the cohomology version of Thomason's theorem which was proven by A. M. Cegarra [2] in 2020.

Keywords: Homotopy Colimit, Grothendieck Construction, Simplicial Sets, Nerve of a Category, Cohomology of Categories.

ÖZET

THOMASON'UN HOMOTOPİ EŞLİMİT TEOREMİ VE KATEGOİLERİN KOHOMOLOJİSİ

Mehmet Kırtışoğlu
Matematik, Yüksek Lisans
Tez Danışmanı: Ergün Yalçın
Temmuz 2024

1978'de R.W. Thomason [1] herhangi bir $F : \mathfrak{C} \to \mathbf{Kat}$ funktoru için $\eta : \operatorname{hocolim}_{\mathfrak{C}} N(F) \to N(\int_{\mathfrak{C}} F)$ şeklinde bir homotopi denkliğinin olduğunu kanıtladı. Burada $\operatorname{hocolim}_{\mathfrak{C}} N(F)$ ikili simpleksel küme $\bigsqcup_{\sigma \in N(\mathfrak{C})} N(F(\sigma(0)))$'nin köşegenleştirilmesi ve $\int_{\mathfrak{C}} F$, nesneleri $C \in \mathrm{Ob}(\mathfrak{C})$ ve $X \in \mathrm{Ob}(F(C))$ olacak şekilde (C, X) çiftleri ve morfizmaları $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(C, C')$ ve $\gamma \in \mathrm{Hom}_{F(C')}(F(\alpha)(X), X')$ olacak şekilde $(\alpha, \gamma) : (C, X) \to (C', X')$ çiftleri ile verilen Grothendieck yapısıdır. Grothendieck yapısının $\pi : \int_{\mathfrak{C}} F \to \mathfrak{C}$ kanonikal funktoru üzerine bir prekofiber kategori olduğunu kanıtlıyoruz. Ayrıca herhangi bir $\varphi : \mathfrak{D} \to \mathfrak{C}$ funktoru için $\lambda : \operatorname{hocolim}_{\mathfrak{C}} N(\varphi/C) \to N(D)$ şeklinde bir homotopi denkliğinin olduğunu kanıtlıyoruz. Bu ikisinin Thomason'un teoremini kavramsal bir bakış açısıyla kanıtladığını gösteriyoruz. Thomason'un ispatına geliştirdiğimiz kavramsal yaklaşımın A. M. Cegarra [2] tarafından 2020'de kanıtlanan kohomoloji versiyonunda nasıl kolaylık sağladığını inceliyoruz.

Anahtar sözcükler: Homotopi Eşlimit, Grothendieck Yapısı, Simpleksel Kümeler, Kategorilerin Sinir Uzayları, Kategorilerin Komolojisi.

Acknowledgement

First and foremost, I would like to express my sincere gratitude to my advisor, Prof. Ergün Yalçın, for his unwavering support and guidance throughout my master's degree. His expertise and insight have been priceless, and his patience and understanding have helped me overcome many challenges.

I am also incredibly grateful to the members of my thesis committee, Prof. Özgün Ünlü and Prof. Mustafa Korkmaz, for reading my thesis.

I am deeply indebted to my family for their unconditional love and support. To my parents, Sema and Hüseyin, thank you for always believing in my abilities. To my brothers, Alaittin and Harun, thank you for struggling with me against all difficulties—special thanks to my brother Alaittin and Ceren Aydın for their constant encouragement and for being a source of inspiration.

I want to extend my thanks to my colleagues and friends Atahan, Bahar, Bütün, Demir, Ege, Esat, Nira, and Serdar for their friendship that has made this journey enjoyable and fulfilling.

Thank you all for being a part of this arduous journey. Without your support, this thesis would not have been possible.

Contents

Chapter 1

Introduction

The simplex category, denoted by Δ, is the category whose objects are non-empty finite totally ordered sets and morphisms are (non-strictly or sometimes called weakly) order-preserving maps. More precisely, an object $[n]$ is a totally ordered set with $n+1$ elements

$$[n] = \{0, 1, \ldots, n\} \tag{1.1}$$

for $n = 0, 1, \ldots$, and a morphism $\theta \in \mathrm{Hom}_{\Delta}([m], [n])$ is an order-preserving set function.

Simplicial sets are defined as functors $X : \Delta^{op} \to \mathbf{Set}$ from the opposite category of the simplex category to the category of sets. They form a combinatorial object to study homotopy theory. This follows from the adjunction between the category of simplicial sets and topological spaces. That's why a common convention is to call a simplicial set space and describe its properties accordingly. For instance, we can say that a map of simplicial sets is a weak equivalence meaning that the induced map on the realizations is a weak equivalence of topological spaces.

Furthermore, the simplicial set category and category of small categories are equivalent from a homotopical point of view by the nerve functor [3]. Here the nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ from the category of small categories to simplicial

sets is defined by sending a category $\mathfrak{C}$ to the simplicial set $N(\mathfrak{C})$ whose n-simplices are given by chain of n-composable morphisms in $\mathfrak{C}$, that is,

$$N(\mathfrak{C})_n = Hom_{\mathbf{Cat}}([n], \mathfrak{C}). \tag{1.2}$$

However, given a natural transformation of two functors, the induced map of colimits may not be a weak equivalence even if we have pairwise weak equivalence between the objects of the diagrams. The homotopy colimit construction arises to handle this problem and it is defined by a simplicial replacement of a functor, i.e.,

$$\operatorname*{hocolim}_{\mathfrak{C}} F := diag\Big(\coprod_{\sigma \in N(\mathfrak{C})} F(\sigma(0)) \Big). \tag{1.3}$$

where $F : \mathfrak{C} \to \mathbf{sSet}$ is any functor. Here for any bisimplicial set X, its diagonal, diag, is defined by

$$(\mathrm{diag}X)_n := X_{n,n}. \tag{1.4}$$

Let $F : \mathfrak{C} \to Cat$ be a functor. Then the Grothendieck Construction on F is a category, denoted by $\int_{\mathfrak{C}} F$, whose objects are given by pairs (C, X) with $C \in \mathrm{Ob}(\mathfrak{C})$ and $X \in \mathrm{Ob}(F(C))$, and whose morphisms are given by pairs $(\alpha, \gamma) : (C, X) \to (C', X')$ with $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(C, C')$ and $\gamma : F(\alpha)(X) \to X'$ in $F(C')$.

In 1978, R.W. Thomason [1] proved that there is a way of interpreting the homotopy colimit construction as a nerve of this special category, namely the Grothendieck Construction.

Theorem 1.0.1 (Thomason's Homotopy Colimit Theorem)**.** *Let $\mathfrak{C}$ be a small category and $F : \mathfrak{C} \to \boldsymbol{Cat}$ be any functor. Then there is a homotopy equivalence of the spaces*

$$\eta : \operatorname*{hocolim}_{\mathfrak{C}} N(F) \to N(\int_{\mathfrak{C}} F). \tag{1.5}$$

Thomason proves his theorem using an intermediate functor $\tilde{F} : \mathfrak{C} \to \mathbf{Cat}$ defined by sending objects of the given small category to their left comma categories π/C, see Definition 4.1.1, over the canonical functor $\pi : \int_{\mathfrak{C}} F \to \mathfrak{C}$ from the Grothendieck construction to the given small category. We briefly sketch his proof by the following diagram:

$$\operatorname*{hocolim}_{\mathfrak{C}} N(F) \xleftarrow{\lambda_1} \operatorname*{hocolim}_{\mathfrak{C}} N(\tilde{F}) \xrightarrow{\lambda_2} N(\textstyle\int_{\mathfrak{C}} F). \qquad (1.6)$$

$$\eta$$

Here Thomason first shows that there exists a simplicial map η from the homotopy colimit construction to the Grothendieck construction. Then he shows that λ_1 and λ_2 are homotopy equivalences. Lastly, Thomason constructs a simplicial homotopy from the composition of the simplicial maps $\eta \circ \lambda_1$ to the simplicial map λ_2 proving that η induces a homotopy equivalence.

In this thesis, we follow the same sketch from a general point of view. For the λ_2 part, we show that the Grothendieck construction can be replaced by any small category as long as $\tilde{F}$ is defined in the same way and we can prove the following proposition.

Proposition 1.0.2. *Let $\varphi : \mathfrak{D} \to \mathfrak{C}$ be any functor and $G : \mathfrak{C} \to \boldsymbol{Cat}$ be the functor defined by sending objects $C \in Ob(\mathfrak{C})$ to their left comma categories φ/C. Then there is a natural homotopy equivalence*

$$\lambda : \operatorname*{\mathit{hocolim}}_{\mathfrak{C}} N(G) \longrightarrow N(\mathfrak{D}). \qquad (1.7)$$

Note that the above proposition is closely related to D. Quillen's Theorem A and Theorem B, [4]. Indeed, it is a corollary for simplicial sets.

Let $\pi : \mathfrak{D} \to \mathfrak{C}$ be any functor and $C \in \mathrm{Ob}(\mathfrak{C})$ be a fixed object. Note that there are canonical functors from the fiber category, see Definition 4.3.1, to left

comma categories as follows:

$$\Phi : \pi^{-1}(C) \to \pi/C, \tag{1.8}$$

$$D \mapsto (D, \mathrm{id}_C) \tag{1.9}$$

$$k \mapsto k. \tag{1.10}$$

Then we say that $\mathfrak{D}$ is a precofibred category with respect to π if for every object $C \in \mathrm{Ob}(\mathfrak{C})$, the functor Φ given above has a left adjoint $L_\Phi : \pi/C \to \pi^{-1}(C)$, that is, there are natural natural transformations $\eta : L_\Phi \circ \Phi \implies \mathrm{id}_{\pi^{-1}(C)}$ and $\epsilon : \mathrm{id}_{\pi/C} \implies \Phi \circ L_\Phi$.

Thomason proves that λ_1 is a homotopy equivalence by showing that for all $C \in \mathrm{Ob}(\mathfrak{C})$ the nerves $N(F(C))$ and $N(\tilde{F}(C))$ are weakly equivalent. Here we observe that this follows from the fact that Grothendieck construction is a precofibred category and we prove:

Proposition 1.0.3. *Let $F : \mathfrak{C} \to Cat$ be any functor and $\int_{\mathfrak{C}} F$ be the Grothendieck construction on F, then $\int_{\mathfrak{C}} F$ is a precofibred category with respect to the canonical functor $\pi : \int_{\mathfrak{C}} F \to \mathfrak{C}$ sending an object (C, X) to C and a morphism $(\alpha, \gamma) : (C, X) \to (C', X')$ to α.*

In 2020, Antonio M. Cegarra showed that the map η in the homotopy colimit theorem also induces isomorphisms on cohomologies for any natural system [2].

Theorem 1.0.4 (Cegarra's Theorem)**.** *For any given natural coefficient system $\mathcal{A}$ on $\int_{\mathfrak{C}} F$, the map η induces the isomorphisms*

$$H^n(\int_{\mathfrak{C}} F, \mathcal{A}) \cong H^n(\underset{\mathfrak{C}}{hocolim}\, NF, \eta^*\mathcal{A}) \quad \forall n \geq 0. \tag{1.11}$$

To prove his theorem Cegarra constructs a projective resolution $\mathcal{P} \to \underline{\mathbb{Z}}$ over the Grothendieck construction $\int_{\mathfrak{C}} F$ and shows that $\mathrm{Hom}_{\mathbb{Z}F\mathfrak{C}}(\mathcal{P}, \mathcal{A})$ is isomorphic to the cochain complex of $\underset{\mathfrak{C}}{\mathrm{hocolim}}\, NF$ with the induced natural coefficient system $\eta^*\mathcal{A}$. We use a similar idea for our homotopy equivalence in Proposition 1.0.2 to construct a projective resolution with

$$H^n(\mathfrak{D}, \mathcal{A}) \cong H^n(\mathrm{Hom}_{\mathbb{Z}F\mathfrak{D}}(\mathcal{P}, \mathcal{A})), \quad n \geq 0. \tag{1.12}$$

Then we use this isomorphism to prove the following theorem.

Theorem 1.0.5. *Let $\varphi : \mathfrak{D} \to \mathfrak{C}$ be any functor and $G : \mathfrak{C} \to \boldsymbol{Cat}$ be the functor defined by sending objects $C \in Ob(\mathfrak{C})$ to their left comma categories φ/C and $\lambda : \underset{\mathfrak{C}}{hocolim}\, N(G) \to N(\mathfrak{D})$ be as in Proposition 4.2.1. Then for any natural coefficient system $\mathcal{A} : F\mathfrak{D} \to \boldsymbol{Ab}$, there is an isomorphism on*

$$H^n(\mathfrak{D}, \mathcal{A}) \cong H^n(\underset{\mathfrak{C}}{hocolim}\, N(G), \lambda^*\mathcal{A}) \quad \forall n \geq 0. \tag{1.13}$$

The thesis is organized as follows:

In Chapter 2, we provide the background materials of the contents developed in later chapters. They are preliminaries of category theory, simplicial sets and realization, and nerve of a category. The main references for these sections are [5] ,[6], [7], and [8].

In Chapter 3, we introduce the simplicial spaces and homotopy colimit. We also prove two facts about simplicial spaces that are used for the proof of Thomason's homotopy colimit. The main references for this chapter are [9] and [10] in addition to those of Chapter 2.

In Chapter 4, we prove Proposition 1.0.2 and Proposition 1.0.3. Lastly, in Chapter 5 we introduce the definition of cohomology of categories and prove Theorem 1.0.5 using the conceptual ideas we developed for the proof of Thomason's homotopy colimit theorem. For these chapters main references are [2], [11], and [12].

Chapter 2

Preliminaries

In this chapter, we recall basic definitions of category theory and introduce the simplicial sets that are important as combinatorial models for topological spaces. The category theory part enables us to see the relation between topological spaces and simplicial sets. Furthermore, our proofs and definitions in the proceeding sections will mostly be in the categorical setup.

2.1 Categories

We will need basic definitions of category theory since our concerns in algebraic topology are mostly the algebraic invariants of topological spaces. For this purpose, category theory provides a language to express those invariants and a conceptual understanding.

Definition 2.1.1. A *category* $\mathfrak{C}$ consists of:

- $\mathrm{Ob}(\mathfrak{C})$, a collection of objects,
- $\mathrm{Hom}_{\mathfrak{C}}(C, C')$, a set of morphisms between any $C, C' \in \mathrm{Ob}(\mathfrak{C})$,
- $\mathrm{id}_X \in \mathrm{Hom}_{\mathfrak{C}}(C, C)$, an identity morphism for any $C \in \mathrm{Ob}(\mathfrak{C})$,

- An associative composition $\circ : \mathrm{Hom}(C', C'') \times \mathrm{Hom}(C, C') \to \mathrm{Hom}(C, C'')$ satisfying the property that for every morphism $f : C \to C'$,

$$\mathrm{id}_{C'} \circ f = f = f \circ \mathrm{id}_C.$$

Example 2.1.2. Some practical examples of categories are:

- The category of sets, commonly denoted by **Set**, in which the objects are sets and morphisms are functions,
- The category of topological spaces, commonly denoted by **Top**, in which the objects are topological spaces and morphisms are the continuous functions,
- The category of groups, commonly denoted by **Grp**, in which the objects are groups and morphisms are group homomorphisms,
- The category of abelian groups, commonly denoted by **Ab**, in which the objects are abelian groups and morphisms are group homomorphisms.

A category is called "small" if the collection of its objects is a set. There is also a notion of morphism among different categories which enables us to carry problems of a category to another category. With this notion, we introduce in the next definition, the collection of all small categories forms a category, denoted by **Cat**.

Definition 2.1.3. A (covariant) *functor* F from a category $\mathfrak{C}$ to a category $\mathfrak{D}$ written $F : \mathfrak{C} \to \mathfrak{D}$ consists of:

- For all $C \in \mathrm{Ob}(\mathfrak{C})$, an object $F(C)$ in $\mathfrak{D}$
- For all morphisms $f : C \to C'$ in $\mathfrak{C}$, a morphism $F(f) : F(C) \to F(C')$ in $\mathfrak{D}$ such that
  - For all $C \in \mathrm{Ob}(\mathfrak{C})$, $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$
  - For all $f : C \to C'$ and $g : C' \to C''$ in $\mathfrak{C}$, $F(g \circ f) = F(g) \circ F(f)$.

A contravariant functor is defined similarly, except it reverses the direction of the arrows, that is, F sends $f : C \to C'$ to $F(f) : F(C') \to F(C)$ and the distribution law becomes $F(g \circ f) = F(f) \circ F(g)$. For a given category $\mathfrak{C}$ we consider its opposite category, denoted by $\mathfrak{C}^{op}$, whose objects are the same but the morphisms are reversed, that is, $\mathrm{Hom}_{\mathfrak{C}}(C, C') = \mathrm{Hom}_{\mathfrak{C}^{op}}(C', C)$. With this setting, a contravariant functor $F : \mathfrak{C} \to \mathfrak{D}$ can be seen as just a covariant functor $\mathfrak{C}^{op} \to \mathfrak{D}$.

Example 2.1.4. Some practical examples of functors are:

- For any small category $\mathfrak{C}$ and a fixed object $C \in \mathrm{Ob}(\mathfrak{C})$, we define $h_C : \mathfrak{C} \to \mathbf{Set}$ by $h_C(C') = \mathrm{Hom}_{\mathfrak{C}}(C, C')$, called as the Hom-functor,
- $F : \mathbf{Top} \to \mathbf{Set}$, $F(X) = X$ and $F(f) = f$, the functor forgetting the topology and continuity on the topological spaces,
- $F : \mathbf{Set} \to \mathbf{Ab}$, the functor sending set X to the free abelian group whose underlying set is X

$$\begin{aligned} X \mapsto \mathbb{Z}X = \{ \sum_{finite} n_i x_i \mid x_i \in X, n_i \in \mathbb{Z} \} \\ (X \xrightarrow{f} Y) \mapsto (\mathbb{Z}X \xrightarrow{F(f)} \mathbb{Z}Y) \\ \sum n_i x_i \mapsto \sum n_i f(x_i), \end{aligned}$$

- **Homology:** $H_* : \mathbf{Top} \to \mathbf{Graded\ Ab\ Grp}$

$$\begin{aligned} X \mapsto H_*(X) \\ (X \xrightarrow{f} Y) \mapsto (H_*(X) \xrightarrow{H_*(f)} H_*(Y)), \end{aligned}$$

- **Cohomology:** $H^* : \mathbf{Top} \to \mathbf{Graded\ Rings}$

$$\begin{aligned} X \mapsto H^*(X) \\ (X \xrightarrow{f} Y) \mapsto (H^*(Y) \xrightarrow{H^*(f)} H^*(X)). \end{aligned}$$

The reason that category theory is helpful in the comparison of different categories follows from the following lemma.

Lemma 2.1.5. *Functors send isomorphisms to isomorphisms.*

Proof. Let $F : \mathfrak{C} \to \mathfrak{D}$ be any (covariant) functor between categories $\mathfrak{C}$ and $\mathfrak{D}$. Suppose that $C, C' \in \mathrm{Ob}(\mathfrak{C})$ are two objects with $f : C \to C'$ being an isomorphism. Then we have

$$F(f) : F(C) \to F(C') \qquad \& \qquad F(f^{-1}) : F(C') \to F(C)$$

where f^{-1} stands for the inverse of f, it exists since f is an isomorphism. Then by the distributive property, we have

$$\mathrm{id}_{F(C)} = F(\mathrm{id}_C) = F(f^{-1} \circ f) = F(f^{-1}) \circ F(f)$$
$$\mathrm{id}_{F(C')} = F(\mathrm{id}_{C'}) = F(f \circ f^{-1}) = F(f) \circ F(f^{-1})$$

Hence, $F(f) : F(C) \to F(C')$ is an isomorphism with inverse $F(f^{-1})$. One can use the same idea to prove the lemma for the case in which the functor is contravariant. □

The comparison of maps in categories extends to the comparison of maps between categories by the following definition.

Definition 2.1.6. Let $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{C} \to \mathfrak{D}$ be two functors. A *natural transformation* $\eta : F \to G$ is a map of functors in the sense that it consists of morphisms $\eta_C : F(C) \to G(C)$ for each object C in $\mathfrak{C}$ and the following diagram commutes for each morphism $f : C \to C'$ in $\mathfrak{C}$:

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ \Big\downarrow{\scriptstyle \eta_C} & & \Big\downarrow{\scriptstyle \eta_{C'}} \\ G(C) & \xrightarrow[G(f)]{} & G(C'). \end{array}$$

We further say that the natural transformation η is a natural isomorphism if for all $C \in \mathrm{Ob}(\mathfrak{C})$ the morphism $\eta_C : F(C) \to G(C)$ is an isomorphism.

As we saw in Example 2.1.4, there is a natural way of defining a functor from any small category to **Set** by the Hom-functor. It turns out that given any category $\mathfrak{C}$ with a functor $F : \mathfrak{C} \to \mathbf{Set}$, the image $F(C)$ of any object $C \in \mathrm{Ob}(\mathfrak{C})$ can be given by Hom-functors.

Lemma 2.1.7 (Yoneda)**.** *Let $F : \mathfrak{C} \to \boldsymbol{Set}$ be any functor. Then there is a bijection*

$$Hom_{\boldsymbol{Cat}}(h_C, F) \cong F(C) \tag{2.1}$$

for any object $C \in Ob(\mathfrak{C})$, where h_C denotes the Hom-functor $Hom_{\mathfrak{C}}(C, -) : \mathfrak{C} \to \boldsymbol{Set}$ and $Hom_{\boldsymbol{Cat}}(h_C, F)$ denotes the set of all natural transformations from the Hom-functor h_C to F.

Proof. Let $C \in \mathrm{Ob}(\mathfrak{C})$ be any object and $\mu \in \mathrm{Hom}_{\mathbf{Cat}}(h_C, F)$ be any natural transformation, then we have

$$\mu_C : \mathrm{Hom}_{\mathfrak{C}}(C, C) \to F(C). \tag{2.2}$$

Hence, we can define a function $\phi : \mathrm{Hom}_{\mathbf{Cat}}(h_C, F) \to F(C)$ by $\phi(\mu) = \mu_C(\mathrm{id}_C)$.

Now let $x \in F(C)$ be any element, then we can define a function $\psi : F(C) \to \mathrm{Hom}_{\mathbf{Cat}}(h_C, F)$ for each $C' \in \mathrm{Ob}(\mathfrak{C})$ componentwise $\psi(x)_{C'} : \mathrm{Hom}_{\mathfrak{C}}(C, C') \to F(C)$ as:

$$\psi(x)_{C'}(f) = F(f)(x). \tag{2.3}$$

Note that the following diagram commutes for each morphism $\alpha : C' \to C''$ in $\mathfrak{C}$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{C}}(C, C') & \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(C,\alpha)} & \mathrm{Hom}_{\mathfrak{C}}(C, C'') \\ \Big\downarrow{\scriptstyle \psi(x)_{C'}} & & \Big\downarrow{\scriptstyle \psi(x)_{C''}} \\ F(C') & \xrightarrow[F(\alpha)]{} & F(C'') \end{array}$$

since we have

$$\begin{aligned} F(\alpha) \circ \psi(x)_{C'}(f) &= F(\alpha) \circ F(f)(x) \\ &= F(\alpha \circ f)(x) = \psi(x)_{C''}(\alpha \circ f). \end{aligned} \tag{2.4}$$

This verifies that $\psi(x) \in \mathrm{Hom}_{\mathbf{Cat}}(h_C, F)$.

Lastly, we claim that ϕ and ψ are inverses of each other. Note that for any $x \in F(C)$ we have

$$(\phi \circ \psi)(x) = F(\mathrm{id}_C)(x) = \mathrm{id}_{F(C)}(x) = x. \tag{2.5}$$

On the other hand for any natural transformation $\eta : h_C \implies F$ and for any $f \in \mathrm{Hom}_{\mathfrak{C}}(C, C')$ we have

$$\begin{aligned}(\psi_{C'} \circ \phi(\eta))(f) &= \psi(\eta_C(\mathrm{id}_C))_{C'}(f) = F(f)(\eta_C(\mathrm{id}_C)) \\ &= \eta_{C'}(f \circ \mathrm{id}_C) = \eta_{C'}(f).\end{aligned} \tag{2.6}$$

This proves the claim, hence, the lemma

$$\mathrm{Hom}_{\mathbf{Cat}}(h_C, F) \cong F(C)$$

for any $C \in \mathrm{Ob}(\mathfrak{C})$. □

There is also an obvious notion for the isomorphism of categories. We say $F : \mathfrak{C} \to \mathfrak{D}$ is an isomorphism, if there is a functor $G : \mathfrak{D} \to \mathfrak{C}$ such that $F \circ G = \mathrm{id}_{\mathfrak{D}}$ and $G \circ F = \mathrm{id}_{\mathfrak{C}}$. However, we do not often encounter isomorphism of categories and it is not very interesting in practice. Instead, we have a more flexible notion which makes it more interesting and useful. That is the equivalence of categories.

Definition 2.1.8. For two functors $G : \mathfrak{D} \to \mathfrak{C}$ and $F\mathfrak{C} \to \mathfrak{D}$, We say that G is right-adjoint to F and F is left-adjoint to G if there exists natural transformations $\eta : F \circ G \implies \mathrm{id}_{\mathfrak{D}}$ and $\epsilon : \mathrm{id}_{\mathfrak{C}} \implies G \circ F$.

Equivalently, we say that F and G are adjoint pairs if there is a natural isomorphism of sets

$$\mathrm{Hom}_{\mathfrak{C}}(C, G(D)) \cong \mathrm{Hom}_{\mathfrak{D}}(F(C), D)$$

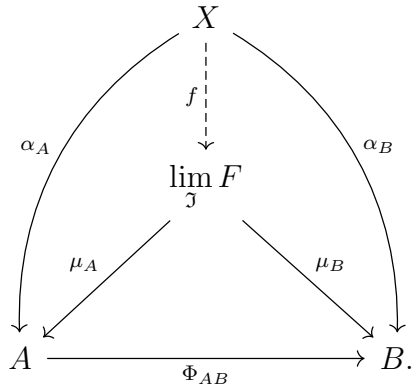
for all objects $C \in \mathrm{Ob}(\mathfrak{C})$ and $D \in \mathrm{Ob}(\mathfrak{D})$.

Definition 2.1.9. We say a functor $F : \mathfrak{C} \to \mathfrak{D}$ is an equivalence between the categories $\mathfrak{C}$ and $\mathfrak{D}$ if there exists another functor $G : \mathfrak{D} \to \mathfrak{C}$ and natural isomorphisms $\eta : F \circ G \implies \mathrm{id}_{\mathfrak{D}}$ and $\epsilon : \mathrm{id}_{\mathfrak{C}} \implies G \circ F$.

The last important concepts of categories for now are limit and colimit. These can be considered as a way of constructing new objects from an existing collection of objects in a category. The procedures are defined over a chosen collection of objects and morphisms admitting a universal property.

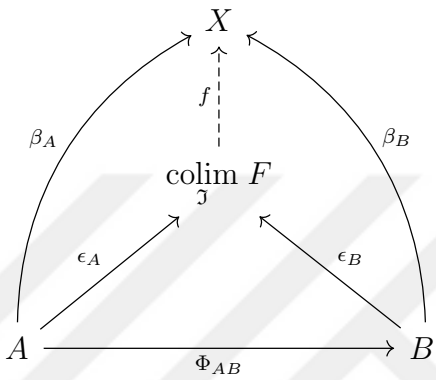
For the setup, an indexing category $\mathfrak{I}$ and a functor $F : \mathfrak{I} \to \mathfrak{C}$ are used to specify a sub-collection of morphisms and objects in a given category $\mathfrak{C}$ which we will call an $\mathfrak{I}$-shaped diagram, or just a diagram when it is obvious in the context.

Definition 2.1.10. The *limit* of a diagram $F : \mathfrak{I} \to \mathfrak{C}$ is an object $\lim\limits_{\mathfrak{I}} F$ in $\mathfrak{C}$ together with morphisms $\mu_A : \lim\limits_{\mathfrak{I}} F \to A$, for each A with $A = F(I)$ for some $I \in \mathrm{Ob}(\mathfrak{I})$, satisfying $\mu_B = \Phi_{AB} \circ \mu_A$ for every morphism $\Phi_{AB} : A \to B$ in the diagram. Moreover, these maps have the universal property that for any other object, say X, with morphisms $\alpha_A : X \to A$, for each A in the diagram, satisfying $\alpha_B = \Phi_{AB} \circ \alpha_A$ there exists a unique morphism $f : X \to \lim\limits_{\mathfrak{I}} F$ such that $\alpha_A = \mu_A \circ f$ for any A in the diagram. Diagrammatically,



Definition 2.1.11. The *colimit* of a diagram $F : \mathfrak{I} \to \mathfrak{C}$ is an object $\operatorname{colim}\limits_{\mathfrak{I}} F$ in $\mathfrak{C}$ together with morphisms $\epsilon_A : A \to \operatorname{colim}\limits_{\mathfrak{I}} F$, for each A in the diagram, satisfying $\epsilon_A = \epsilon_B \circ \Phi_{AB}$ for every morphism $\Phi_{AB} : A \to B$ in the diagram. Moreover, these maps have the universal property that for any other object, say X, with

morphisms $\beta_A : A \to X$, for each A in the diagram, satisfying $\beta_A = \beta_B \circ \Phi_{AB}$ there exists a unique morphism $f : X \to \operatorname*{colim}_{\mathfrak{J}} F$ such that $\beta_A = f \circ \epsilon_A$ for any A in the diagram. Diagrammatically,



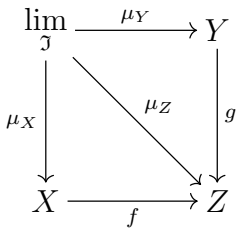
Example 2.1.12. The most basic example is to consider an empty diagram that is a diagram with no objects and no morphisms. In this case, the limit is called the terminal object, and the colimit is called the initial object if they exist.

Since the diagram is empty, any object in the category is a candidate for being limit. However, the universal property suggests that there must be a unique morphism from all failing candidates to the limit. Hence, the terminal object is an object $T \in \mathrm{Ob}(\mathfrak{C})$ with the property that for each object $C \in \mathrm{Ob}(\mathfrak{C})$ there is a unique morphism $f : C \to T$.

Similarly for the colimit, the initial object is an object $I \in \mathrm{Ob}(\mathfrak{C})$ with the property that for each object $C \in \mathrm{Ob}(\mathfrak{C})$ there is a unique morphism $f : I \to C$.

The names limit and colimit derive from being a codomain and a domain of morphisms, respectively, for each object in the diagram.

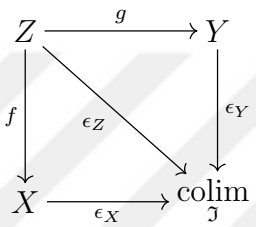
Example 2.1.13. The limit of the diagram given $\mathfrak{J}$ by $X \xrightarrow{f} Z \xleftarrow{g} Y$ is called the *Pullback* if it exists.



In the category of **Set**, the pullback exists and is given by

$$P = \{(x, y) \in X \times Y | f(x) = g(y)\} \subset X \times Y$$

Example 2.1.14. The colimit of the diagram $\mathfrak{I}$ given by $X \xleftarrow{f} Z \xrightarrow{g} Y$ is called the *Pushout* if it exists.



In the category of **Top**, the pushout exists and it is the quotient of the disjoint union $X \sqcup Y$ by $f(z) \sim g(z)$ for each $z \in Z$

$$P = X \sqcup Y \Big/ f(z) \sim g(z).$$

As we compare two groups we investigate the images of certain subgroups under group homomorphism. Likewise, the limit and colimit diagrams play a similar role in the comparison of categories with functors being the homomorphisms in **Cat**.

2.2 Simplicial Sets and Realization

In this section, we aim to introduce the simplicial sets which are combinatorial models for topological spaces, and how they produce a topological space under the realization functor. The adjunction between simplicial sets and topological spaces offers that colimits in simplicial sets are preserved by the realization functor.

Definition 2.2.1. The simplex category Δ is the category whose objects are non-empty finite totally ordered sets and morphisms are (non-strictly or sometimes called weakly) order-preserving maps. More precisely, an object $[n]$ is a totally

ordered set with $n+1$ elements

$$[n] = \{0, 1, \ldots, n\} \tag{2.7}$$

for $n = 0, 1, \ldots$, and a morphism $\theta \in \mathrm{Hom}_{\Delta}([m], [n])$ is an order-preserving set function.

Any morphism of the simplex category Δ can be described by the composition of the following two types of maps which will be proved after the relations between them are shown.

Definition 2.2.2. One type of these special maps is injective and called the co-face map $d^i : [n-1] \to [n]$, $0 \leq i \leq n$, which "skips i"

$$d^i(a) = \begin{cases} a & \text{if } a < i, \\ a+1 & \text{if } i \leq a, \end{cases} \tag{2.8}$$

for all $a \in \{0, 1, \ldots, n\}$. Namely, $d^i : [n-1] \to [n]$ assigns any element a in the set $[n-1]$ to itself if $a < i$ and to its successor $a+1$ if $i \leq a$.

Definition 2.2.3. The other type of these special maps is surjective and called the co-degeneracy map $s^i : [n+1] \to [n]$, $0 \leq i \leq n$, which "repeats i"

$$s^i(a) = \begin{cases} a & \text{if } a \leq i, \\ a-1 & \text{if } i < a, \end{cases} \tag{2.9}$$

for all $a \in \{0, 1, \ldots, n+1\}$. Namely, $s^i : [n+1] \to [n]$ assigns any element a in the set $[n+1]$ to itself if $a \leq i$ and to its predecessor $a-1$ if $i < a$.

The domain of the maps does not appear in the notation as d^i_n or s^i_n since, most of the time, it is clear from the context. However, if necessary within the context, it can be emphasized by indicating that $s^i \in \mathrm{Hom}_{\Delta}([n+1], [n])$ for some $n \in \mathbb{N}$.

Proposition 2.2.4. *The maps d^i and s^i satisfy the following obvious simplicial*

identities:

$$(1) \quad d^j d^i = d^i d^{j-1}, \quad i < j \tag{2.10}$$

$$(2) \quad s^j s^i = s^i s^{j+1}, \quad i \leq j \tag{2.11}$$

$$(3) \quad s^j d^i = d^i s^{j-1}, \quad i < j \tag{2.12}$$

$$(4) \quad s^j d^i = d^{i-1} s^j, \quad i > j+1 \tag{2.13}$$

$$(5) \quad s^i d^i = s^i d^{i+1} = 1. \tag{2.14}$$

Definition 2.2.5. A simplicial set $X : \Delta^{op} \to \mathbf{Set}$ is a functor from the opposite category of the simplex category to the category of sets. Equivalently, a simplicial set $X \in \mathbf{sSet}$, where $\mathbf{sSet}$ stands for the category of simplicial sets, consists of:

- $X_n \in \mathbf{Set}$, set of n-simplices, for each $n \in \mathbb{N}$;
- $d_i : X_n \to X_{n-1}$, the i-th face map on n-simplices for each co-face map $d^i : [n-1] \to [n]$, $n \in \mathbb{N}$ and $0 \leq i \leq n$;
- $s_i : X_n \to X_{n+1}$, the i-th degeneracy map on n-simplices for each co-degeneracy map $s^i : [n+1] \to [n]$, $n \in \mathbb{N}$ and $0 \leq i \leq n$;

such that the face and degeneracy maps satisfy the simplicial identities:

$$d_i \circ d_j = d_{j-1} \circ d_i \quad i < j \tag{2.15}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \quad i \leq j \tag{2.16}$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \text{id} & \text{if } i = j, j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases} . \tag{2.17}$$

With the definition of simplicial sets above we see that a simplicial map is indeed a natural transformation, that is, given simplicial sets $X : \Delta^{op} \to \mathbf{Set}$ and $Y : \Delta^{op} \to \mathbf{Set}$, we have $\mathrm{Hom}_{\mathbf{sSet}}(X, Y) = \mathrm{Hom}_{\mathbf{Cat}}(X, Y)$.

Example 2.2.6. Recall that there is a natural way of defining a functor from any small category to $\mathbf{Set}$ by the Hom-functor. Here we define $\Delta_n : \Delta^{op} \to \mathbf{sSet}$ by $\Delta_n([m]) = \mathrm{Hom}_{\Delta}([m], [n])$. This is called the standard n-simplicial set.

Remark 2.2.7. The above example and the Yoneda Lemma 2.1.7 provide that $\mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, X) \cong X_n$, that is, the set of n-simplices are identified with the set of natural transformations from Δ_n to X.

With this definition of simplicial sets above, we call an element $\sigma \in X_n$ degenerate if there exists a collection of degeneracy maps and $\tau \in X_m$ such that σ is the image of τ under the composition of these degeneracy maps, that is,

$$\sigma = (s_{i_1} \circ \cdots \circ s_{i_{n-m}})(\tau). \tag{2.18}$$

Otherwise, we say x is non-degenerate. For instance, all 0-simplices are non-degenerate.

Proposition 2.2.8. *Any degenerate simplex has a unique non-degenerate representative.*

Proof. Let x be a degenerate simplex and $x = Sy = Tz$ where $S = s_{i_1} \circ \cdots \circ s_{i_m}$ and $T = s_{j_1} \circ \cdots \circ s_{j_n}$ for some non-degenerate simplices y and z. Now let $D = d_{i_m} \circ \cdots \circ d_{i_1}$. Then the image of the first equality under D becomes

$$y = Dx = DTz$$

since $d_i \circ s_i = \mathrm{id}$. Then using the simplicial identities (2.17) to swap the places of degeneracy maps and face maps we can write y as

$$y = \tilde{T}\tilde{D}z.$$

However, y was non-degenerate by assumption, so $\tilde{T}$ must be empty. That implies y is a face of z. Repeating the argument with D is replaced by $D' = d_{j_n} \circ \cdots \circ d_{j_1}$ and z is replaced by y, we see that z is also a face y. This is a contradiction, so $DT = D'S = \mathrm{id}$ and $y = z$. □

Now let's see how this combinatorial object is used to generate a topological space and how the homotopical relations between the category of topological spaces **Top** and the category of simplicial sets **sSet** including nerve spaces.

Definition 2.2.9. Given a simplicial set X we define its geometric realization by the realization functor $|-| : \mathbf{sSet} \to \mathbf{Top}$, as the topological space

$$|X| = (\bigcup_{n\geq 0} X_n \times \Delta^n)/\sim \tag{2.19}$$

where X_n represents the set of n-simplices as a discrete topological space, Δ^n denotes the geometric n-simplex, that is,

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \ \sum_{i=0}^{n} t_i = 1\}$$

and the relations of quotient space is given by

1. $(d_i(x), (t_0, ..., t_n)) \sim (x, d^i(t_0, ..., t_n))$, and
2. $(s_j(x), (t_0, ..., t_n)) \sim (x, s^j(t_0, ..., t_n))$.

Here item 1 corresponds to the attachments of higher simplices with the lower ones and item 2 corresponds to the collapses of degenerate simplices onto non-degenerates. Note that for an element of $X_n \times \Delta^n$ to be non-degenerate, we now have an extra condition than (2.18) that the second component $(t_0, ..., t_n) \in \Delta^n$ is supposed to be a point in the interior of the geometric n-simplex.

Remark 2.2.10. Any degenerate element $(x, (t_0, ..., t_n)) \in X_n \times \Delta^n$ has a unique non-degenerate representative in $|X|$ by Proposition 2.2.8 applied componentwise with composition.

There is also a categorical way of defining the realization of a simplicial set X as:

$$|X| = \operatorname*{colim}_{\Delta_n \to X} \Delta^n$$

followed by the Remark 2.2.7.

Proposition 2.2.11 (Proposition 2.2 in [6])**.** *The realization functor* $|-| :$ ***sSet*** $\to$ ***Top*** *is left adjoint to the singular functor* $Sing(-) :$ ***Top*** $\to$ ***sSet*** *where* $Sing(Y)_n = Hom_{\mathbf{Top}}(\Delta^n, Y)$.

Proof. There is a natural isomorphism of sets

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Top}}(|X|, Y) &\cong \lim_{\Delta_n \to X} \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y) && (2.20)\\ &\cong \lim_{\Delta_n \to X} \mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, Sing(Y)) && (2.21)\\ &\cong \mathrm{Hom}_{\mathbf{sSet}}(X, Sing(Y)). && (2.22) \end{aligned}$$

since (2.20) follows from the definition of $Sing(Y)_n$ and Remark 2.2.7, whereas (2.21) and (2.22) follow by categorical definition of the realization. □

Proposition 2.2.12. *For any simplicial set X, the geometric realization $|X|$ is a CW-complex having one n-cell for each non-degenerate n-simplex of X.*

Proof. The proof follows by the pushout diagrams

$$\begin{array}{ccc} \bigsqcup_{x \in NX_n} \partial\Delta^n & \longrightarrow & |\, X^{(n-1)} \,| \\ \downarrow & & \downarrow \\ \bigsqcup_{x \in NX_n} \Delta^n & \longrightarrow & |\, X^{(n)} \,| \end{array}$$

where NX_n stands for the non-degenerate n-simplices of X and $X^{(n)}$ is the simplicial set generated by non-degenerate simplices of X with degree less than or equal to n. See Proposition I.2.3 in [6] for the details. □

Definition 2.2.13. Given two simplicial maps $f, g \in \mathrm{Hom}_{\mathbf{sSet}}(X, Y)$ we say $h : X \times \Delta_1 \to Y$ is a simplicial homotopy if there is a commutative diagram

$$\begin{array}{ccc} X \times \Delta_0 = X & & \\ \downarrow{\scriptstyle 1\times d^1} & \searrow^{f} & \\ X \times \Delta_1 & \xrightarrow{h} & Y, \\ \uparrow{\scriptstyle 1\times d^0} & \nearrow_{g} & \\ X \times \Delta_0 = X & & \end{array}$$

that is, $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all simplices $x \in X$.

2.3 Nerve of a Category

Recall that **Cat** denotes the category of small categories which has objects as small categories and morphisms as functors. The simplicial set notion naturally arises from **Cat** as we consider composable morphisms in a category.

Definition 2.3.1. We define the nerve functor from the category of small categories to simplicial sets

$$\begin{aligned} N : \mathbf{Cat} &\to \mathbf{sSet} \\ \mathfrak{C} &\mapsto N(\mathfrak{C}) : \Delta^{op} \to \mathbf{Set}, \end{aligned}$$

where $N(\mathfrak{C})$ associates $[n] \in \Delta$ with the set of n-composable morphisms in $\mathfrak{C}$

$$N(\mathfrak{C})_n = \mathrm{Hom}_{\mathbf{Cat}}([n], \mathfrak{C}).$$

Explicitly, an n-simplex σ of $N\mathfrak{C}$ can be represented by a chain of morphisms in the following form

$$\sigma = C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} C_{n-1} \xrightarrow{\alpha_n} C_n$$

with $C_i \in \mathrm{Ob}(\mathfrak{C})$ and $\alpha_i \in \mathrm{Hom}_{\mathfrak{C}}(C_{i-1}, C_i)$. Then the face maps for $i = 1, \ldots, n$ become composition of morphisms at i-th position

$$d_i(\sigma) = C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} C_{i-1} \xrightarrow{\alpha_{i+1} \circ \alpha_i} C_{i+1} \xrightarrow{\alpha_{i+2}} \cdots \xrightarrow{\alpha_n} C_n$$

whereas the face map for $i = 0$ is given by omitting the first object

$$d_0(\sigma) = C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} C_n.$$

On the other hand, the degeneracy maps for all $i = 0, \ldots, n$ are given by inserting an identity map at i-th position

$$s_i(\sigma) = C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} C_i \xrightarrow{\mathrm{id}_{C_i}} C_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_n} C_n.$$

Lemma 2.3.2. *Let $F_0, F_1 : \mathfrak{C} \to \mathfrak{D}$ be two functors. Then there exists a natural transformation $\mu : F_0 \implies F_1$ if and only if there exists a functor*

$$H : \mathfrak{C} \times \{0 \leq 1\} \to \mathfrak{D} \tag{2.23}$$

such that

(i) $H((C, i)) = F_i(C)$, *for all* $C \in Ob(\mathfrak{C}) \quad i = 0, 1$

(ii) $H((\alpha, id_i)) = F_i(\alpha)$, *for all* $\alpha \in Hom_{\mathfrak{C}}(-, -) \quad i = 0, 1$

(iii) $H((\alpha, <)) : F_0(C_0) \to F_1(C_1)$, *for all* $\alpha : C_0 \to C_1$ *and objects* $C_0, C_1 \in Ob(\mathfrak{C})$.

Proof. Let $H : \mathfrak{C} \times \{0 \leq 1\} \to \mathfrak{D}$ be as defined in the lemma. Then for all $C \in \mathrm{Ob}(\mathfrak{C})$ we define $\mu : F_0 \implies F_1$ by $\mu_C = H((\mathrm{id}_C, <))$, so we have a commutative diagram

$$\begin{array}{ccc} F_0(C_0) & \xrightarrow{\mu_{C_0}} & F_1(C_0) \\ {\scriptstyle F_0(\alpha)}\downarrow & & \downarrow{\scriptstyle F_1(\alpha)} \\ F_0(C_1) & \xrightarrow[\mu_{C_1}]{} & F_1(C_1). \end{array}$$

Note that this proves the "only if" part and the "if" part uses the same idea to define $H((\mathrm{id}_C, <) = \mu_C$ as the natural transformation provides $H((\alpha, <)) = H((\mathrm{id}_{C_1}, <) \circ H((\alpha, \mathrm{id}_0)) = H((\alpha, \mathrm{id}_1)) \circ H((\mathrm{id}_{C_0}, <))$. □

Proposition 2.3.3. *Let $F_0, F_1 : \mathfrak{C} \to \mathfrak{D}$ be two functors. If there exists a natural transformation $\mu : F_0 \implies F_1$, then $N(F_0)$ is simplicially homotopic to $N(F_1)$.*

Proof. Let $\mu : F_0 \implies F_1$ be a natural transformation then by Lemma 2.3.2 we have a functor $H : \mathfrak{C} \times \{0 \leq 1\} \to \mathfrak{D}$. The proof follows by a straightforward observation that $N(\{0 \leq 1\}) = \Delta_1$. Hence, $N(\mu)$ induces the required simplicial homotopy $h : N(F_0) \times \Delta_1 \to N(F_1)$ as $N(\mathfrak{C} \times \{0 \leq 1\}) \cong N(\mathfrak{C}) \times N(\{0 \leq 1\}) = N(\mathfrak{C}) \times \Delta_1$. □

Corollary 2.3.4. *Given a category $\mathfrak{C}$ if it has either an initial object or a terminal object then $N(\mathfrak{C})$ is contractible, that is, $N(\mathfrak{C})$ is weakly equivalent to $N(*)$ where $*$ denotes the trivial category with one object.*

Proof. Let $C \in \mathrm{Ob}(\mathfrak{C})$ be either an initial or a terminal object. Note that there is a unique functor $F : \mathfrak{C} \to *$ and a functor $G : * \to \mathfrak{C}$ with $G(*) = C$. Then the composition $F \circ G$ is the identity functor on the category $*$. Also, there is a natural transformation $\eta : G \circ F \implies \mathrm{id}_{\mathfrak{C}}$ or $\eta : \mathrm{id}_{\mathfrak{C}} \implies G \circ F$ if C is an initial or a terminal object, respectively. In either case, Proposition 2.3.3 implies that $N(\mathfrak{C})$ is weakly equivalent to $N(*)$. □

Chapter 3

Simplicial Spaces and Homotopy Colimit

In this chapter, we describe simplicial spaces which are also called bisimplicial sets and homotopy colimit construction. We so far discussed that the simplicial set category and category of small categories are equivalent from a homotopical point of view in the previous chapter. However, given a natural transformation of two diagrams, the induced map of colimits may not be a weak equivalence even if we have pairwise weak equivalences between the objects of the diagrams. The homotopy colimit construction arises at this stake. Lastly, we define Grothendieck Construction whose relation with homotopy colimit will be studied in Chapter 4.

3.1 Bisimplicial Sets and Simplicial Spaces

Definition 3.1.1. A bisimplicial set is defined as a functor

$$X : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}. \tag{3.1}$$

Explicitly, a bisimplicial set X consists of the following data:

- A set $X_{m,n}$ for each integer $m, n \geq 0$ which indicates the set of bisimplices of X of bidegree (m, n). Here we will call "m" as the horizontal degree and "n" as the vertical degree.

- Horizontal face and degeneracy maps,

$$d_i^h : X_{m,n} \to X_{m-1,n} \tag{3.2}$$
$$s_i^h : X_{m,n} \to X_{m+1,n} \tag{3.3}$$

- Vertical face and degeneracy maps,

$$d_i^v : X_{m,n} \to X_{m,n-1} \tag{3.4}$$
$$s_i^v : X_{m,n} \to X_{m,n+1} \tag{3.5}$$

such that the simplicial identities given in Proposition 2.2.4 hold for each type and the maps of different components commute

$$s_i^h d_i^v = d_i^v s_i^h \tag{3.6}$$
$$d_i^h s_i^v = s_i^v d_i^h. \tag{3.7}$$

Example 3.1.2. Let K and L be simplicial sets. Then $K \times L$ is a bisimplicial set in which bisimplices of bidegree (m, n) are given by

$$(K \times L)_{m,n} = K_m \times L_n. \tag{3.8}$$

Example 3.1.3. Given a simplicial set X we can see X as a bisimplicial set with $X_{m,n} = X_n$ that is constant in the horizontal direction or $X_{m,n} = X_m$ that is constant in the vertical direction.

Example 3.1.4. Let $\mathfrak{C}$ be a small category and $F : \mathfrak{C} \to \mathbf{sSet}$ a functor. Then $N(\mathfrak{C}) \times F$ is a bisimplicial set whose (m, n)-simplices are given by the set

$$\bigsqcup_{\sigma \in N(\mathfrak{C})_m} F(\sigma(0))_n. \tag{3.9}$$

Similarly, for an arbitrary functor $F : \mathfrak{C} \to \mathfrak{D}$ of small categories, we can construct a bisimplicial set by composing F with the nerve functor. Then the bisimplicial set $N(\mathfrak{C}) \times N(F)$ has (m, n)-simplices given by the set

$$\bigsqcup_{\sigma \in N(\mathfrak{C})_m} N\big(F(\sigma(0))\big)_n. \tag{3.10}$$

Definition 3.1.5. Given a bisimplicial set X we define its geometric realization on the vertical dimension as a simplicial set whose simplices are given by

$$|\tilde{X}|_m = \big(\bigcup_{n\geq 0} X_{n,m} \times \Delta_n[m]\big)/\sim . \tag{3.11}$$

Definition 3.1.6. Let X be a bisimplicial set then we define its diagonalization, diag$X$, as a simplicial set whose n-simplices are given by

$$(\mathrm{diag}X)_n = X_{n,n}. \tag{3.12}$$

We can also interpret diag$X$ as the composition of functors

$$\Delta^{op} \xrightarrow{\mathrm{diag}} \Delta^{op} \times \Delta^{op} \xrightarrow{X} Sets \tag{3.13}$$

$$[n] \longmapsto ([n],[n]) \longmapsto X_{n,n}. \tag{3.14}$$

Proposition 3.1.7. *diag$X$, is simplicially isomorphic to $|\tilde{X}|$.*

Proof. The simplicial set maps

$$\lambda_n : X_{n,m} \times \Delta_n[m] \to X_{m,m} \tag{3.15}$$

$$(x, \tau : [m] \to [n]) \mapsto \tau^*(x) \tag{3.16}$$

for all $m \in \mathbb{N}$ and $\tau \in \Delta_n[m]$ induce the isomorphism on the quotient space $|\tilde{X}|$ by Proposition 2.2.8 since

$$(x, \tau : [m] \to [n]) \sim (\tau^*(x), \mathrm{id} : [m] \to [m]) \tag{3.17}$$

is the unique representative. □

3.2 Homotopy Colimit of a Functor

Recall that given a category $\mathfrak{C}$ the nerve space $N(\mathfrak{C})$ is the simplicial set whose n-simplices are given by chain of n-composable morphisms in the following form

$$\sigma = C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} C_{n-1} \xrightarrow{\alpha_n} C_n$$

with $C_i \in \mathrm{Ob}(\mathfrak{C})$ and $\alpha_i \in \mathrm{Hom}_{\mathfrak{C}}(C_{i-1}, C_i)$.

Definition 3.2.1. Let $\mathfrak{C}$ be a small category and $F : \mathfrak{C} \to \mathbf{sSet}$ a functor. Then the homotopy colimit is the space given by the diagonal of the bisimplicial set $N(\mathfrak{C}) \times F$, that is,

$$\operatorname*{hocolim}_{\mathfrak{C}} F := \operatorname{diag}\Big(\bigsqcup_{\sigma \in N(\mathfrak{C})} F(\sigma(0)) \Big). \tag{3.18}$$

Here $\operatorname*{hocolim}_{\mathfrak{C}} F$ is sometimes called the simplicial replacement of F. Note that whenever we have a natural transformation $\eta : F \implies F'$ it induces a map between the homotopy colimits:

$$\operatorname*{hocolim}_{\mathfrak{C}} \eta : \operatorname*{hocolim}_{\mathfrak{C}} F \to \operatorname*{hocolim}_{\mathfrak{C}} F', \tag{3.19}$$

that is, the homotopy colimit is functorial.

The proof the Proposition 3.2.2 requires the gluing lemma that we refer from [6] (Lemma 8.10) and [13] (Theorem A.1).

Proposition 3.2.2 ((Theorem A.4) in [13])**.** *Let $f : X \to Y$ be a map of bisimplicial sets. If f induces pointwise weak equivalence, that is, for all $n \geq 0$ the maps*

$$f_{n*} : X_{n,*} \to Y_{n,*}$$

are weak equivalences of simplicial sets, then the induced map $diag(f) : diag(X) \to diag(Y)$ of simplicial sets is a weak equivalence.

Proof. Note that for any morphism $f : [m] \to [n]$ in Δ we have a diagram

$$\begin{array}{ccc} X_{n,*} \times \Delta_m & \xrightarrow{f^* \times 1} & X_{m,*} \times \Delta_m \\ {\scriptstyle 1 \times f_*}\downarrow & & \\ X_{n,*} \times \Delta_n & & \end{array} \tag{3.20}$$

where $f_* : \Delta_m[k] \to \Delta_n[k]$ is given by $f_*(\tau) = f \circ \tau$ for all $k \in \mathbb{N}$ and $f^* \times 1 = X(f, 1)$. Hence, all the morphisms in $\operatorname{Hom}_\Delta([m], [n])$ induce a pair of maps

$$\bigsqcup_{f \in \operatorname{Hom}_\Delta([m],[n])} X_{n,*} \times \Delta_m \underset{f^* \times 1}{\overset{1 \times f_*}{\rightrightarrows}} \bigsqcup_n X_{n,*} \times \Delta_n \ . \tag{3.21}$$

Also, we have the following maps of simplicial sets

$$\lambda_n : X_{n,m} \times \Delta_n[m] \to X_{m,m} \tag{3.22}$$

$$(x, \tau : [m] \to [n]) \mapsto \tau^*(x) \tag{3.23}$$

for all $m \in \mathbb{N}$ and $\tau \in \Delta_n[m]$.

This collection of maps together gives a coequalizer diagram in the category of simplicial sets as

$$\bigsqcup_{f \in \mathrm{Hom}_\Delta([m],[n])} X_{n,*} \times \Delta_m \underset{f^* \times 1}{\overset{1 \times f_*}{\rightrightarrows}} \bigsqcup_n X_{n,*} \times \Delta_n \xrightarrow{\lambda} \mathrm{diag}(X) \cdot$$

since for $k \in \mathbb{N}$ fixed and $(x, \tau) \in X_{n,k} \times \Delta_m[k]$ we have

$$\lambda((1 \times f_*)(x, \tau)) = \lambda(x, f \circ \tau) \tag{3.24}$$

$$= \tau^* f^*(x) \tag{3.25}$$

$$= \lambda(f^*(x), \tau) \tag{3.26}$$

$$= (f^* \times 1)(\lambda(x, \tau)). \tag{3.27}$$

Let $s^h X_{p,*} = \bigcup_{0 \le i \le p} s_i^h X_{p,*}$ be the union of images of $X_{p,*}$ under all horizontal degeneracy maps. Then we have the cofibrant inclusion

$$i_p : s^h X_{p,*} \to X_{p+1,*}. \tag{3.28}$$

Note that we have the following pushout diagrams

$$\begin{array}{ccc} s^h X_{p,*} \times \partial\Delta_{p+1} & \xrightarrow{i_p \times 1} & X_{p+1,*} \times \partial\Delta_m \\ \downarrow{\scriptstyle 1 \times i_{p*}} & & \downarrow \\ s^h X_{p,*} \times \Delta_{p+1} & \longrightarrow & (s^h X_{p,*} \times \Delta_{p+1}) \cup (X_{p+1,*} \times \partial\Delta_m) \end{array} \tag{3.29}$$

and

$$\begin{array}{ccc} (s^h X_{p,*} \times \Delta_{p+1}) \cup (X_{p+1,*} \times \partial\Delta_m) & \longrightarrow & \mathrm{diag}(X)^{(p)} \\ \downarrow & & \downarrow \\ X_{p+1,*} \times \Delta_{p+1} & \longrightarrow & \mathrm{diag}(X)^{(p+1)} \end{array} \tag{3.30}$$

where

$$\mathrm{diag}(X)^{(p)} = \big(\bigcup_{0 \le n \le p} X_{n,n} \times \Delta_n \big) / \sim . \tag{3.31}$$

The Diagram (3.30) shows that by induction on p and the Gluing Lemma, if $\mathrm{diag}(f)^{(p)} : \mathrm{diag}(X)^p \to \mathrm{diag}(Y)^{(p)}$ is a homotopy equivalence then $\mathrm{diag}(f) : \mathrm{diag}(X) \to \mathrm{diag}(Y)$ is. Note that $\mathrm{diag}(f)^{(p)}$ is a homotopy equivalence if $f_{p+1} : s^h X_{p,*} \to s^h Y_{p,*}$ is a homotopy equivalence by the Diagram (3.29). However, this is held by another induction on p and the gluing lemma again since we are given the pointwise weak equivalences.

□

Remark 3.2.3. In the above proposition, the assumption of pointwise weak equivalence in the horizontal direction is dual to the assumption of pointwise weak equivalence in the vertical direction. Hence, there is no difference in proving one direction instead of the other.

Example 3.2.4. Let $F : \mathfrak{C} \to \mathbf{sSet}$ be the constant functor, that is, F maps objects of $\mathfrak{C}$ to one-point space Δ_0, then $\underset{\mathfrak{C}}{\mathrm{hocolim}}\, F \cong N(\mathfrak{C})$.

Proposition 3.2.5. *Let $F : \mathfrak{C} \to \boldsymbol{sSet}$ and $F' : \mathfrak{C} \to \boldsymbol{sSet}$ be two functors and $\eta : F \implies F'$ be a natural transform. If $\eta_C : F(C) \to F'(C)$ is a weak equivalence for all objects $C \in Ob(\mathfrak{C})$, then the induced map $\underset{\eta}{\mathrm{hocolim}} : \underset{\mathfrak{C}}{\mathrm{hocolim}}\, F \to \underset{\mathfrak{C}}{\mathrm{hocolim}}\, F'$ is a weak equivalence.*

Proof. Note that assuming $\eta_C : F(C) \to F'(C)$ is a weak equivalence for all

objects $C \in \mathrm{Ob}(\mathfrak{C})$, we have a weak equivalence of simplicial spaces

$$\coprod_{\sigma \in N(\mathfrak{C})} F(\sigma(0)) \to \coprod_{\sigma \in N(\mathfrak{C})} F'(\sigma(0)) \tag{3.32}$$

by Proposition 3.2.2. Then by Proposition 3.2.2 we have the desired weak equivalence of diagonalizations:

$$\operatorname*{hocolim}_{\mathfrak{C}} \eta : \operatorname*{hocolim}_{\mathfrak{C}} F \to \operatorname*{hocolim}_{\mathfrak{C}} F'. \tag{3.33}$$

□

Chapter 4

Thomason's Homotopy Colimit Theorem

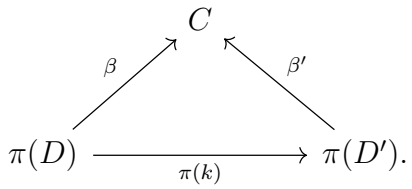
In this chapter, we give the required definitions for the statement of Thomason's homotopy colimit theorem. Then we proceed by proving parts of the theorem from a more general perspective that will imply Thomason's work at the end.

4.1 Grothendieck Construction and Statement of Thomason's Theorem

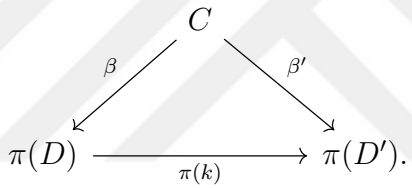
In this section, we emphasize constructions of categories from an arbitrary functor as left and right comma categories and introduce a fundamental concept in category theory which is the Grothendieck Construction. With these constructions, we will be able to state the homotopy colimit theorem.

Definition 4.1.1. Given a functor $\pi : \mathfrak{D} \to \mathfrak{C}$ and an object $C \in \mathrm{Ob}(\mathfrak{C})$ we define the left comma category π/C by setting its objects as pairs (D, γ) with $D \in \mathrm{Ob}(\mathfrak{D})$ and $\beta : \pi(D) \to C$ a morphism in $\mathfrak{C}$. A morphism in π/C from (D, β) to (D', β') is given by the morphisms $k : D \to D'$ of $\mathrm{Hom}_{\mathfrak{D}}(D, D')$ such

that $\beta' \circ \pi(k) = \beta$, that is, the following diagram commutes

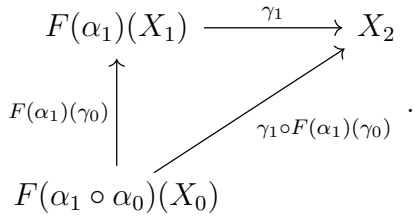


Dually, the right comma category C/π is the category whose objects are pairs (D, β) with $\beta : C \to \pi(D)$ and whose morphisms are given by $k : D \to D'$ such that $\pi(k) \circ \beta = \beta'$, that is, the following diagram commutes



Definition 4.1.2. Let $F : \mathfrak{C} \to Cat$ be a functor. The Grothendieck Construction on F is a category, denoted by $\int_{\mathfrak{C}} F$, whose objects are given by pairs (C, X) with $C \in \mathrm{Ob}(\mathfrak{C})$ and $X \in \mathrm{Ob}(F(C))$, and whose morphisms are given by pairs $(\alpha, \gamma) : (C, X) \to (C', X')$ with $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(C, C')$ and $\gamma : F(\alpha)(X) \to X'$ in $F(C')$.

The composition in this category for two morphisms $(\alpha_0, \gamma_0) : (C_0, X_0) \to (C_1, X_1)$ and $(\alpha_1, \gamma_1) : (C_1, X_1) \to (C_2, X_2)$ is given by $(\alpha_1 \circ \alpha_0, \gamma_1 \circ F(\alpha_1)(\gamma_0)) : (C_0, X_0) \to (C_2, X_2)$:



Now, we can state Thomason's Homotopy Colimit Theorem.

Theorem 4.1.3 (Thomason's Homotopy Colimit Theorem)**.** *Let $F : \mathfrak{C} \to \boldsymbol{Cat}$ be a functor. Then there is a homotopy equivalence of the spaces*

$$\eta : \operatorname*{hocolim}_{\mathfrak{C}} N(F) \to N(\int_{\mathfrak{C}} F). \tag{4.1}$$

The proof of the theorem, as in Thomason's work, consists of two main parts. The idea is to use an intermediate space that will be homotopy equivalent to both sides of the map η. Before proceeding with the proofs of transitions from each side we introduce this intermediate space as the homotopy colimit of the nerve spaces of the categories in the diagram induced by the following functor defined.

Definition 4.1.4. Let $F : \mathfrak{C} \to \mathbf{Cat}$ be a functor and $\int_{\mathfrak{C}} F$ be the Grothendieck Construction on F. Then we define $\tilde{F} : \mathfrak{C} \to \mathbf{Cat}$ to be the functor sending objects $C \in \mathrm{Ob}(\mathfrak{C})$ to their left comma categories π/C where $\pi : \int_{\mathfrak{C}} F \to \mathfrak{C}$ is the canonical functor sending objects (C, X) to C and morphisms (α, γ) to α.

Now we can explicitly emphasize the parts. We will show that there are homotopy equivalences λ_1 and λ_2 as shown in the diagram:

$$\operatorname*{hocolim}_{\mathfrak{C}} N(F) \xleftarrow{\lambda_1} \operatorname*{hocolim}_{\mathfrak{C}} N(\tilde{F}) \xrightarrow{\lambda_2} N(\textstyle\int_{\mathfrak{C}} F). \qquad \eta \tag{4.2}$$

In addition to Thomason's proof, we show that the maps λ_1 and λ_2 can be derived from a more general perspective. Then the transition between the maps will be given by a simplicial homotopy $H : \operatorname*{hocolim}_{\mathfrak{C}} N(\tilde{F}) \times \Delta_1 \to N(\int_{\mathfrak{C}} F)$ from λ_2 to $\eta \circ \lambda_1$ just as in Thomason's proof.

We proceed by the existence of such a map η that will be compatible for writing such a homotopy H.

Lemma 4.1.5. *Let $F : \mathfrak{C} \to \boldsymbol{Cat}$ be a functor. Then there is a natural simplicial map*

$$\eta : \operatorname*{hocolim}_{\mathfrak{C}} N(F) \to N(\int_{\mathfrak{C}} F). \tag{4.3}$$

Proof. Note that an n-simplex (σ, τ) of $\operatorname*{hocolim}_{\mathfrak{C}} N(F) = \operatorname{diag}\big(\bigsqcup_{\sigma \in N(\mathfrak{C})} N(F(\sigma(0)))\big)$ is given by

$$(\sigma, \tau) = (C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} C_n, X_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} X_n) \in N(\mathfrak{C})_n \times N(F(C_0))_n \tag{4.4}$$

Now we define $\eta : \operatorname{diag} \bigsqcup_{\sigma \in N(\mathfrak{C})} \big(N(F(\sigma(0)))\big) \to N(\int_{\mathfrak{C}} F)$ by sending such an n-simplex (σ, τ) to n-simplex

$$(C_0, X_0) \xrightarrow{(\alpha_1, F(\alpha_1)(\beta_1))} \big(C_1, F(\alpha_1)(X_1)\big) \xrightarrow{(\alpha_2, F(\alpha_2\alpha_1)(\beta_2))} \cdots \tag{4.5}$$

$$\cdots \xrightarrow{(\alpha_n, F(\alpha_n \cdots \alpha_1)(\beta_n))} (C_n, F(\alpha_n \cdots \alpha_1)(X_n)) \tag{4.6}$$

in $N(\int_{\mathfrak{C}} F)_n$. This defines a simplicial map, so η is a map from $\operatorname*{hocolim}_{\mathfrak{C}} N(F)$ to $N(\int_{\mathfrak{C}} F)$.

□

4.2 Homotopy Colimit of Comma Categories

In this section we show that the proposed homotopy equivalence λ_2 given in the diagram (4.2) can be proven for any category $\mathfrak{D}$ rather than the Grothendieck construction $\int_{\mathfrak{C}} F$ as long as $\tilde{F} : \mathfrak{C} \to \mathbf{Cat}$ is defined in the same way for an arbitrary functor $\varphi : \mathfrak{D} \to \mathfrak{C}$, that is, $\tilde{F}(C) = \varphi/C$.

Proposition 4.2.1. *Let $\varphi : \mathfrak{D} \to \mathfrak{C}$ be any functor and $G : \mathfrak{C} \to \boldsymbol{Cat}$ be the functor defined by sending objects $C \in Ob(\mathfrak{C})$ to their left comma categories φ/C. Then there is a natural homotopy equivalence*

$$\lambda : \operatorname*{hocolim}_{\mathfrak{C}} N(G) \longrightarrow N(\mathfrak{D}). \tag{4.7}$$

Proof. Note that an n-simplex of $\underset{\mathfrak{C}}{\operatorname{hocolim}}\, N(G) = \operatorname{diag}\big(\bigsqcup_{\sigma\in N(\mathfrak{C})} N(G(\sigma(0)))\big)$ is given by

$$(C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} C_n, \big(D_0, \beta_0\big) \xrightarrow{k_1} \cdots \xrightarrow{k_n} (D_n, \beta_n)\big) \in N(\mathfrak{C})_n \times N(\varphi/C_0)_n \quad (4.8)$$

with $k_i : D_{i-1} \to D_i$ a morphism in $\mathfrak{D}$ for $i = 1, \ldots, n$ and $\beta_i : \varphi(D_i) \to C_0$ a morphism in φ/C_0 for $i = 0, \ldots, n$ such that $\beta_i = \beta_{i+1} \circ \varphi(k_{i+1})$, that is, the following diagram in $\mathfrak{C}$ commutes

$$\begin{array}{ccccccc} & & C_0 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_n} & C_n \\ & {\scriptstyle\beta_0}\nearrow & & & & & \nwarrow{\scriptstyle\beta_n} \\ \varphi(D_0) & \xrightarrow{\varphi(k_1)} & \cdots & \xrightarrow{\varphi(k_n)} & \varphi(D_n). & & \end{array} \quad (4.9)$$

Now we define $\lambda : \operatorname{diag} \bigsqcup_{N(\mathfrak{C})} N(G) \to N(\mathfrak{D})$ by sending such an n-simplex to

$$D_0 \xrightarrow{k_1} \cdots \xrightarrow{k_n} D_n \in N(\mathfrak{D})_n \quad (4.10)$$

which is clearly a simplicial set map. To show that λ induces a homotopy equivalence we will use Proposition 3.2.2 and Remark 3.2.3 of simplicial space theory. For this purpose, we consider $N(\mathfrak{D})$ as the simplicial space whose vertical direction is constant, that is, $N(\mathfrak{D})_{p,q} = N(\mathfrak{D})_q$ for all $p \in \mathbb{N}$. Also, we consider the simplicial space map

$$\Lambda_{p,q} : \bigsqcup_{N(\mathfrak{C})_p} N(G)_q \to N(\mathfrak{D})_{p,q} \quad (4.11)$$

defined by

$$(C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_p} C_p, \big(D_0, \beta_0\big) \xrightarrow{k_1} \cdots \xrightarrow{k_q} (D_q, \beta_q)\big) \mapsto (D_0 \xrightarrow{k_1} \cdots \xrightarrow{k_q} D_q). \quad (4.12)$$

Then note that we have $\operatorname{diag}(N(\mathfrak{D})_{*,*}) = N(\mathfrak{D})_*$ and $\operatorname{diag}(\Lambda_{p,q}) = \lambda$. Hence, it is enough to show that $\Lambda_{*,q} : \bigsqcup_{N(\mathfrak{C})_*} N(G)_q \to N(\mathfrak{D})_{*,q}$ is a homotopy equivalence for each $q \in \mathbb{N}$.

Fixing $q \in \mathbb{N}$, we see that the image of $\Lambda_{*,q}$ is given by the all q-simplices

$$D_0 \xrightarrow{k_1} \cdots \xrightarrow{k_q} D_q \tag{4.13}$$

over the coproduct of all p-simplices

$$C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_p} C_p \in N(\mathfrak{C})_p \tag{4.14}$$

with $\varphi(D_q) \xrightarrow{\beta_q} C_0$ in $\mathfrak{C}$ as β_q and k_i's generate all β_i's by the commutativity relation in Diagram (4.9). Then we know that the right comma category $C_0/\mathfrak{C}$ of C_0 in $\mathfrak{C}$ has the initial object C_0 and it is contractible by Corollary 2.3.4. Hence, we conclude that

$$\Lambda_{*,q} \cong N(\mathfrak{D})_{*,q} \tag{4.15}$$

is a weak equivalence. This proves that $\mathrm{diag}(\Lambda_{p,q}) = \lambda$ induces a natural homotopy equivalence

$$\lambda : \operatorname*{hocolim}_{\mathfrak{C}} N(G) \longrightarrow N(\mathfrak{D}) \tag{4.16}$$

by Proposition 3.2.2. □

4.3 Precofibred and Prefibred Categories

In this section, we show that the proposed homotopy equivalence λ_1 given in the diagram (4.2) can be derived from a more general case. We basically show that this homotopy equivalence is an outcome of the fact that the Grothendieck construction is a precofibred category with respect to the functor π given in Definition 4.1.4.

We begin by providing the definitions of fiber, precofibred, and prefibred categories.

Definition 4.3.1. Given a functor $\pi : \mathfrak{D} \to \mathfrak{C}$ and an object $C \in \mathrm{Ob}(\mathfrak{C})$ we define the fiber category $\pi^{-1}(C)$ whose objects are given by $D \in \mathrm{Ob}(\mathfrak{D})$ such that $\pi(D) = C$ and whose morphisms are given by $k : D \to D'$ such that $\pi(k) = \mathrm{id}_C$.

Note that given a functor $\pi : \mathfrak{D} \to \mathfrak{C}$ and a fixed object $C \in \mathrm{Ob}(\mathfrak{C})$ there are canonical functors from the fiber category to both left and right comma categories as follows:

$$1.\ \Phi : \pi^{-1}(C) \to \pi/C,$$

$$D \mapsto (D, \mathrm{id}_C)$$
$$k \mapsto k$$

$$2.\ \Psi : \pi^{-1}(C) \to C/\pi,$$

$$D \mapsto (D, \mathrm{id}_C)$$
$$k \mapsto k.$$

Definition 4.3.2. Given a functor $\pi : \mathfrak{D} \to \mathfrak{C}$ we say that $\mathfrak{D}$ is a precofibred category with respect to π if for every object $C \in \mathrm{Ob}(\mathfrak{C})$ the functor Φ given above has a left adjoint $L_\Phi : \pi/C \to \pi^{-1}(C)$, that is, there are natural transformations $\eta : L_\Phi \circ \Phi \implies \mathrm{id}_{\pi^{-1}(C)}$ and $\epsilon : \mathrm{id}_{\pi/C} \implies \Phi \circ L_\Phi$.

Dually, we say that $\mathfrak{D}$ is a prefibred category with respect to π if for every object $C \in \mathrm{Ob}(\mathfrak{C})$ the functor Ψ has a right adjoint $R_\Psi : C/\pi \to \pi^{-1}(C)$.

Proposition 4.3.3. *Let $\pi : \mathfrak{D} \to \mathfrak{C}$ be a precofibred functor. Then $N(\pi^{-1}(C))$ is weakly homotopy equivalent to $N(\pi/C)$ for all objects $C \in Ob(\mathfrak{C})$.*

Proof. The proof directly follows from the Definition 4.3.2 and Proposition 2.3.3 □

Corollary 4.3.4. *Let $\pi : \mathfrak{D} \to \mathfrak{C}$ be a prefibred functor. Then $N(\pi^{-1}(C))$ is weakly homotopy equivalent to $N(C/\pi)$ for all objects $C \in Ob(\mathfrak{C})$.*

Proposition 4.3.5. *Let $F : \mathfrak{C} \to Cat$ be any functor and $\int_{\mathfrak{C}} F$ be the Grothendieck construction on F, then $\int_{\mathfrak{C}} F$ is a precofibred category with respect to the canonical functor $\pi : \int_{\mathfrak{C}} F \to \mathfrak{C}$ sending an object (C, X) to C and a morphism $(\alpha, \gamma) : (C, X) \to (C', X')$ to α.*

Proof. Let $C \in \mathrm{Ob}(\mathfrak{C})$ be any object and $\Phi : \pi^{-1}(C) \to \pi/C$ be the functor given above. Then we define the left adjoint $L_\Phi : \pi/C \to \pi^{-1}(C)$ as $L_\Phi((C', X'), \gamma) = (C, F(\gamma)(X'))$. Note that

$$(L_\Phi \circ \Phi)(C, X) = L_\Phi((C, X), \mathrm{id}_C) \tag{4.17}$$
$$= (C, X), \tag{4.18}$$

so $L_\Phi \circ \Phi = \mathrm{id}_{\pi^{-1}(C)}$. We also have

$$\begin{aligned}(\Phi \circ L_\Phi)((C', X'), \gamma) &= \Phi(C, F(\gamma)(X')) \\ &= ((C, F(\gamma)(X')), \mathrm{id}_C),\end{aligned}$$

so we define $\epsilon : \mathrm{id}_{\pi/C} \implies \Phi \circ L_\Phi$ for any morphism $\alpha : ((C', X'), \gamma) \to ((C'', X''), \beta)$ in π/C as:

$$\begin{array}{ccc} ((C', X'), \gamma) & \xrightarrow{\epsilon_\gamma = \gamma} & ((C, F(\gamma)(X')), \mathrm{id}_C) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \mathrm{id}_C} \\ ((C'', X''), \beta) & \xrightarrow[\epsilon_\beta = \beta]{} & ((C, F(\beta)(X'')), \mathrm{id}_C) \end{array} .$$

This proves the claim that $\int_{\mathfrak{C}} F$ is a precofibred category with respect to the canonical functor π. □

Theorem 4.3.6. *Let $F : \mathfrak{C} \to \boldsymbol{Cat}$ be any functor and let $\int_{\mathfrak{C}} F$ be the Grothendieck construction over F. If $\pi : \int_{\mathfrak{C}} F \to \mathfrak{C}$ is the canonical functor sending an object (C, X) to C and a morphism $(\alpha, \gamma) : (C, X) \to (C', X')$ to α, then there is a natural homotopy equivalence*

$$\lambda_1 : \underset{\mathfrak{C}}{hocolim}\, N(\tilde{F}) \longrightarrow \underset{\mathfrak{C}}{hocolim}\, N(F) \tag{4.19}$$

where $\tilde{F} : \mathfrak{C} \to \boldsymbol{Cat}$ be the functor defined by sending objects $C \in Ob(\mathfrak{C})$ to their left comma categories π/C.

Proof. By Proposition 3.2.2 it is enough to show that there is a natural homotopy equivalence

$$f_p : \bigsqcup_{\sigma \in N(\mathfrak{C})} N(\pi/\sigma(0))_p \to \bigsqcup_{\sigma \in N(\mathfrak{C})} N\big(F(\sigma(0))\big)_p, \tag{4.20}$$

so fixing $\sigma \in N(\mathfrak{C})_p$ with $\sigma(0) = C$ we observe that

$$F(\sigma(0)) = \pi^{-1}(C). \tag{4.21}$$

Then by Proposition 4.3.5 and Proposition 4.3.3 we have the weak equivalence between the realizations of $N(\tilde{F}(C))$ and $N(F(C))$, so the induced map λ_1 is a homotopy equivalence. □

4.4 Proof of Thomason's Homotopy Colimit Theorem

In this section, we finish the proof of Thomason's homotopy colimit theorem by giving the simplicial homotopy $H : \operatorname{hocolim}_{\mathfrak{C}} N(\tilde{F}) \times \Delta_1 \to N(\int_{\mathfrak{C}} F)$ from λ_2 to $\eta \circ \lambda_1$.

$$\operatorname{hocolim}_{\mathfrak{C}} N(F) \xleftarrow{\lambda_1} \operatorname{hocolim}_{\mathfrak{C}} N(\tilde{F}) \xrightarrow{\lambda_2} N(\textstyle\int_{\mathfrak{C}} F). \qquad \eta \tag{4.22}$$

Here the maps $\eta, \lambda_1, \lambda_2$ are as in Lemma 4.1.5, Theorem 4.3.6, and Proposition 4.2.1, respectively.

Theorem 4.4.1 (Thomason's Homotopy Colimit Theorem)**.** *Let $F : \mathfrak{C} \to Cat$ be a functor. Then there is a homotopy equivalence of the spaces*

$$\eta : \operatorname{hocolim}_{\mathfrak{C}} N(F) \to N(\int_{\mathfrak{C}} F).$$

Proof. We show that there is a simplicial homotopy $H : \operatorname{hocolim}_{\mathfrak{C}} N(\tilde{F}) \times \Delta_1 \to N(\int_{\mathfrak{C}} F)$ between $\eta \circ \lambda_1$ and λ_2 proving that η is a homotopy equivalence as λ_1 and λ_2 are.

Note that an n-simplex of $\operatorname{hocolim}_{\mathfrak{C}} N(\tilde{F}) \times \Delta_1$ is given by

$$\begin{aligned}(C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} C_n, \big(\beta_0, X_0\big) \xrightarrow{(k_1,\gamma_1)} \cdots \xrightarrow{(k_n,\gamma_n)} (\beta_n, X_n)\big) \\ \times (0, \ldots, 0, 1, \ldots, 1)\end{aligned} \tag{4.23}$$

with $p+1$ many 0's and $n-p$ many 1's where $\beta_i : D_i \to C_0$ a morphism in $\mathfrak{C}$ for $i = 0, \ldots, n$ and $k_i : D_i \to D_{i+1}$ a morphism in π/C_0 for $i = 1, \ldots, n$ such that $\beta_i = \beta_{i+1} \circ \pi(k_{i+1})$, and $X_i \in F(D_i)$ such that $\gamma_{i+1} : F(k_i)(X_i) \to X_{i+1}$.

Now we define $H : \operatorname{hocolim}_{\mathfrak{C}} N(\tilde{F}) \times \Delta_1 \to N(\int_{\mathfrak{C}} F)$ by sending such an n-simplex to

$$\begin{aligned}&(D_0, X_0) \xrightarrow{(k_1,\gamma_1)} \cdots \xrightarrow{(k_p,\gamma_p)} (D_p, X_p) \xrightarrow{(\alpha_{p+1}\cdots\alpha_1\beta_p, F(\alpha_{p+1}\cdots\alpha_1\beta_{p+1})(\gamma_{p+1}))} \\ &\big(C_{p+1}, F(\alpha_{p+1}\cdots\alpha_1\beta_{p+1})(X_{p+1})\big) \xrightarrow{(\alpha_{p+2}, F(\alpha_{p+2}\cdots\alpha_1\beta_{p+2})(\gamma_{p+2}))} \cdots \\ &\xrightarrow{(\alpha_n, F(\alpha_n\cdots\alpha_1\beta_n)(\gamma_n))} (C_n, F(\alpha_n\cdots\alpha_1\beta_n)(X_n)).\end{aligned} \tag{4.24}$$

Then note that $H(-, (0, \cdots, 0)) = \lambda_2$ and $H(-, (1, \cdots, 1)) = \eta \circ \lambda_1$ where the induced map λ_2 is given by sending the n-simplex in (4.23) to

$$(D_0, X_0) \xrightarrow{(k_1,\gamma_1)} \cdots \xrightarrow{(k_n,\gamma_n)} (D_n, X_n) \tag{4.25}$$

and the induced map $\eta \circ \lambda_1$ is given by sending the same n-simplex in (4.21) to

$$\begin{aligned}&\big(C_0, F(\beta_0)(X_0)\big) \xrightarrow{(\alpha_1, F(\alpha_1\beta_1)(\gamma_1))} \big(C_1, F(\alpha_1\beta_1)(X_1)\big) \xrightarrow{(\alpha_2, F(\alpha_2\alpha_1\beta_2)(\gamma_2))} \cdots \\ &\xrightarrow{(\alpha_n, F(\alpha_n\cdots\alpha_1\beta_n)(\gamma_n))} (C_n, F(\alpha_n\cdots\alpha_1\beta_n)(X_n)).\end{aligned} \tag{4.26}$$

Hence, H is a simplical homotopy from λ_2 to $\eta \circ \lambda_1$ and this proves our assertion that η is a homotopy equivalence.

□

Chapter 5

Cohomology Version of Thomason's Theorem

In this last chapter, we study the cohomology-preserving property of Thomason's Homotopy Colimit Theorem. We know by [11] that Baus-Wirsching cohomology groups $H^n(\mathfrak{C}, \mathcal{M})$ of small categories are homotopy invariant for local coefficients $M : \Pi(\mathfrak{C}) \to \mathbf{Ab}$. However, the same does not necessarily hold for natural coefficient systems $\mathcal{A} : F\mathfrak{C} \to \mathbf{Ab}$.

Antonio M. Cegarra shows in [2] that the weak equivalence in Thomason's homotopy colimit theorem, $\eta : \operatorname{hocolim}_{\mathfrak{C}} N(F) \to N(\int_{\mathfrak{C}} F)$, indeed induces isomorphisms on

$$H^n(\int_{\mathfrak{C}} F, \mathcal{A}) \cong H^n(\operatorname{hocolim}_{\mathfrak{C}} N(F), \eta^* \mathcal{A}) \quad \forall n \geq 0 \tag{5.1}$$

for every natural coefficient system $\mathcal{A} : F \int_{\mathfrak{C}} F \to \mathbf{Ab}$. We use a similar idea to prove the following theorem:

Theorem 5.0.1. *Let $\varphi : \mathfrak{D} \to \mathfrak{C}$ be any functor and $G : \mathfrak{C} \to \boldsymbol{Cat}$ be the functor defined by sending an object $C \in Ob(\mathfrak{C})$ to its left comma category φ/C, and let $\lambda : \operatorname{hocolim}_{\mathfrak{C}} N(G) \to N(\mathfrak{D})$ be as in Proposition 4.2.1. Then for any natural*

coefficient system $\mathcal{A} : F\mathfrak{D} \to \boldsymbol{Ab}$*, there is an isomorphism on*

$$H^n(\mathfrak{D}, \mathcal{A}) \cong H^n(\underset{\mathfrak{C}}{hocolim}\, N(G), \lambda^*\mathcal{A}) \quad \forall n \geq 0. \tag{5.2}$$

5.1 Modules Over a Category

Before introducing the definition for cohomology of categories we first introduce modules over a category.

Definition 5.1.1. Let R be a commutative ring with unity and $\mathfrak{C}$ be a small category. We define right $R\mathfrak{C}$-module as a contravariant functor $M : \mathfrak{C}^{op} \to R\text{-}mod$.

Here given an $R\mathfrak{C}$-module M, we consider $M(C)$ as a right $R\mathrm{Hom}_{\mathfrak{C}}(C, C)$-module where the action of $f \in \mathrm{Hom}_{\mathfrak{C}}(C, C)$ is given by the R-module homomorphism $M(f) : M(C) \to M(C)$. The category of $R\mathfrak{C}\text{-}mod$ is an abelian category whose morphisms are given by natural transformations $\mu : M_1 \to M_2$ for $R\mathfrak{C}$-modules $M_1 : \mathfrak{C} \to R\text{-}mod$ and $M_2 : \mathfrak{C} \to R\text{-}mod$.

A short exact sequence of $R\mathfrak{C}$-modules is given by a sequence

$$M_1 \xrightarrow{\mu} M_2 \xrightarrow{\xi} M_3 \tag{5.3}$$

for morphisms $\mu : M_1 \to M_2$ and $\xi : M_2 \to M_3$ satisfying the condition that the sequence

$$M_1(C) \xrightarrow{\mu_C} M_2(C) \xrightarrow{\xi_C} M_3(C) \tag{5.4}$$

is exact for all objects $C \in \mathfrak{C}$, that is, $Ker(\mu_C) = Im(\xi_C)$. Here we have

$$Ker(\mu_C) = Ker(\{\mu_C : M_1(C) \to M_2(C)\}) \quad and, \tag{5.5}$$

$$Im(\xi_C) = Im(\{\xi_C : M_2(C) \to M_3(C)\}). \tag{5.6}$$

Proposition 5.1.2. *Let* $M : \mathfrak{C}^{op} \to R\text{-}mod$ *be any* $R\mathfrak{C}$*-module. Then for all objects* $C \in Ob(\mathfrak{C})$*, there is an isomorphism of* $R\mathfrak{C}$*-modules*

$$Hom_{R\mathfrak{C}}(RHom_{\mathfrak{C}}(-, C), M) \cong M(C). \tag{5.7}$$

Moreover, the $R\mathfrak{C}$*-module* $RHom_{\mathfrak{C}}(-, C)$ *is projective.*

Proof. The first part directly follows from Yoneda Lemma 2.1.7. Now consider the short exact sequence of $R\mathfrak{C}$-modules

$$M_1 \xrightarrow{\mu} M_2 \xrightarrow{\xi} M_3. \tag{5.8}$$

Applying $\mathrm{Hom}_{R\mathfrak{C}}(R\mathrm{Hom}_{\mathfrak{C}}(-,C),-)$, for a given object $C \in \mathfrak{C}$, we get

$$M_1(C) \xrightarrow{\mu_C} M_2(C) \xrightarrow{\xi_C} M_3(C) \tag{5.9}$$

by the first part and we know this sequence is exact by definition. Hence, the $R\mathfrak{C}$-module $R\mathrm{Hom}_{\mathfrak{C}}(-,C)$ is projective. □

Remark 5.1.3. By Proposition 5.1.2 one can show that for every $R\mathfrak{C}$-module M, there is a projective module P_* with a surjective $R\mathfrak{C}$-module homomorphism $P_* \to M$. Hence, the category of $R\mathfrak{C}$-modules has enough projectives Also, this category has enough injectives (see [[14], p. 43]).

Let M and N be $R\mathfrak{C}$-modules, then we define the *n-th* ext-group as:

$$\mathrm{Ext}^n_{R\mathfrak{C}}(M,N) := H^n(\mathrm{Hom}_{R\mathfrak{C}}(P_*,M)) \tag{5.10}$$

where $P \to M$ is a projective resolution of M as an $R\mathfrak{C}$-module.

Definition 5.1.4. The constant functor $\underline{R}$ over the category $\mathfrak{C}$ is the $R\mathfrak{C}$-module sending every object $C \in \mathrm{Ob}(\mathfrak{C})$ to R and every morphism in $\mathfrak{C}$ to the identity morphism $\mathrm{id}_R : R \to R$. If M is an $R\mathfrak{C}$-module, then the *n-th* cohomology group of $\mathfrak{C}$ with coefficients in M is defined by

$$H^n(\mathfrak{C},M) = \mathrm{Ext}^n_{R\mathfrak{C}}(\underline{R},M) \quad \forall n \geq 0. \tag{5.11}$$

5.2 Baues-Wirsching Cohomology of Categories

In section 5.0 we introduced the earlier approach to the cohomology of categories by $R\mathfrak{C}$-modules. This approach assigns each object of a category to an abelian group and admits a right action. In this section we introduce another definition

for cohomology of categories, called Baues-Wirsching Cohomology, which admits a bimodule structure, by taking natural coefficient systems, see Definition 5.2.2. The difference comes from the fact that an $R\mathfrak{C}$-module associates an abelian group to each object whereas a natural system associates an abelian group to each morphism of the category.

Definition 5.2.1. Given a category $\mathfrak{C}$ we define its factorization category, denoted by $F\mathfrak{C}$, as:

- The objects of $F\mathfrak{C}$ are the morphisms of $\mathfrak{C}$.
- A morphism between two objects $\alpha : C_0 \to C_1$ and $\alpha' : C_0' \to C_1'$ of $F\mathfrak{C}$ is given by pair of morphisms $(u, v) : \alpha \to \alpha'$ of $\mathfrak{C}$ with $u : C_1 \to C_1'$ and $v : C_0' \to C_0$ such that $u\alpha v = \alpha'$, that is, the following diagram commutes

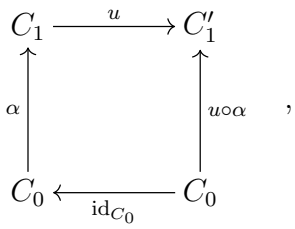
$$\begin{array}{ccc} C_1 & \xrightarrow{u} & C_1' \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha'} \\ C_0 & \xleftarrow[v]{} & C_0' \end{array} .$$

- Given two morphisms $(u_0, v_0) : \alpha \to \alpha'$ and $(u_1, v_1) : \alpha' \to \alpha''$ their composition is naturally defined by $(u_1u_0, v_0v_1) : \alpha \to \alpha''$

$$\begin{array}{ccccc} C_1 & \xrightarrow{u_0} & C_1' & \xrightarrow{u_1} & C_1'' \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha'} & & \uparrow{\scriptstyle\alpha''} \\ C_0 & \xleftarrow[v_0]{} & C_0' & \xleftarrow[v_1]{} & C_0'' \end{array} .$$

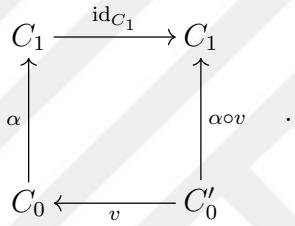
  since $u_0\alpha v_0 = \alpha'$ and $u_1\alpha' v_1 = \alpha''$ gives $u_1u_0\alpha' v_0 v_1 = \alpha''$.

Note that any morphism $u : C_1 \to C_1'$ in $\mathfrak{C}$ induces a morphism $(u, \mathrm{id}_{C_0}) : \alpha \to u \circ \alpha$ in $F\mathfrak{C}$



,

and similarly any morphism $v : C_0' \to C_0$ induces a morphism $(\mathrm{id}_{C_1}, v) : \alpha \to \alpha \circ v$ in $F\mathfrak{C}$



.

Hence, we can write any morphism $(u, v) : \alpha \to \alpha'$ between the objects $\alpha : C_0 \to C_1$ and $\alpha' : C_0' \to C_1'$ of the factorization category $F\mathfrak{C}$ as the composition $(u, \mathrm{id}_{C_0}) \circ (\mathrm{id}_{C_1}, v) = (\mathrm{id}_{C_1}, v) \circ (u, \mathrm{id}_{C_0})$.

Definition 5.2.2. Given a category $\mathfrak{C}$ we define a natural coefficient system of abelian groups on $\mathfrak{C}$ with a functor

$$\mathcal{A} : F\mathfrak{C} \to \mathbf{Ab},$$

from its factorization category to the category of abelian groups. The mapping of an object $\alpha : C_0 \to C_1$ in $F\mathfrak{C}$ is given by $\mathcal{A}(\alpha) = \mathcal{A}_\alpha$ and a morphism $(u, v) : \alpha \to \alpha'$ is given by $\mathcal{A}(u, v) = \mathcal{A}(u, 1) \circ \mathcal{A}(1, v) : \mathcal{A}_\alpha \to \mathcal{A}_{u\alpha v} = \mathcal{A}_{\alpha'}$. Here we denote $\mathcal{A}(u, \mathrm{id}) = u_*$ and $\mathcal{A}(\mathrm{id}, v) = v^*$ as they are given by composition from left and right, respectively. That is $u_* v^* : A_\alpha \to A_{u\alpha v}$.

Definition 5.2.3. Given a category $\mathfrak{C}$ and a natural coefficient system $\mathcal{A}$ on $\mathfrak{C}$ we define the cohomology $H^n(\mathfrak{C}, \mathcal{A})$ by the cochain complex of abelian groups $\mathcal{C}$

$$\mathcal{C} : \qquad \cdots \xrightarrow{\delta^{n-2}} C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} \cdots \tag{5.12}$$

where C^n is the abelian group of all functions

$$\varphi : N(\mathfrak{C})_n \to \bigsqcup_{\alpha \in \mathrm{Ob}(F\mathfrak{C})} \mathcal{A}_\alpha \tag{5.13}$$

such that $\varphi(\sigma) \in \mathcal{A}_{\alpha_n \circ \cdots \circ \alpha_1}$ where

$$\sigma = C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} C_n, \tag{5.14}$$

and the coboundary maps $\delta^{n-1} : C^{n-1} \to C^n$ are defined by sending such a function of C^{n-1} to the function

$$(\delta\varphi)(\sigma) = (\alpha_1)^*\varphi(d_0\sigma) + \sum_{i=1}^{n-1}(-1)^i\varphi(d_i\sigma) + (-1)^n(\alpha_n)_*\varphi(d_n\sigma) \tag{5.15}$$

whenever $n > 1$. Note that when $n = 1$ the coboundary map $\delta_1 : C^0 \to C^1$ for any morphism $\alpha : C_0 \to C_1$ and $\varphi \in C^0$ becomes

$$(\delta\varphi)(C_0 \xrightarrow{\alpha} C_1) = \alpha^*\varphi(C_1) - \alpha_*\varphi(C_0) \tag{5.16}$$

We denote the category of all pairs $(\mathfrak{C}, \mathcal{A})$ where $\mathfrak{C}$ is a category and $\mathcal{A} : F\mathfrak{C} \to \mathbf{Ab}$ is a natural coefficient system of abelian groups on $\mathfrak{C}$ by **Nat**. The morphisms in the category **Nat** are also given by pairs $(\phi^{op}, \eta) : (\mathfrak{C}, \mathcal{A}) \to (\mathfrak{C}', \mathcal{A}')$ such that $\phi : \mathfrak{C}' \to \mathfrak{C}$ is a functor and $\eta : \phi^*\mathcal{A} \to \mathcal{A}'$ is a natural transformation with $\phi^*\mathcal{A} : F\mathfrak{C}' \to \mathbf{Ab}$ given by $\phi^*\mathcal{A}(\alpha) = D(\phi(\alpha))$ for $\alpha \in \mathrm{Hom}_{\mathfrak{C}}(-,-)$. The composition for any two morphisms (ϕ^{op}, η) and (ψ^{op}, μ) in this category is given by

$$(\phi^{op}, \eta)(\psi^{op}, \mu) = ((\psi\phi)^{op}, \eta\phi^*\mu) \tag{5.17}$$

Remark 5.2.4. We see that the cohomology defined above is indeed a functor

$$H^n : \mathbf{Nat} \to \mathbf{Ab} \quad n \in \mathbb{Z} \tag{5.18}$$

and the induced homomorphism by $(\phi^{op}, \eta) : (\mathfrak{C}, \mathcal{A}) \to (\mathfrak{C}', \mathcal{A}')$

$$H(\phi^{op}, \eta) : H^n(\mathfrak{C}, \mathcal{A}) \to H^n(\mathfrak{C}', \mathcal{A}') \tag{5.19}$$

is given on cochains $\varphi \in C^n$ by

$$(\phi^*\eta_*\varphi)(\sigma) = \eta_\varphi \circ \varphi(\phi(\sigma)) \tag{5.20}$$

Theorem 5.2.5. *Let $F : \mathfrak{C}' \to \mathfrak{C}$ be an equivalence between the categories $\mathfrak{C}'$ and $\mathfrak{C}$. Then for any natural coefficient system $\mathcal{A}$ on $\mathfrak{C}$ there is an induced isomorphism on cohomologies*

$$F^* : H^n(\mathfrak{C}, \mathcal{A}) \cong H^n(\mathfrak{C}', F^*\mathcal{A}) \quad n \in \mathbb{Z}. \tag{5.21}$$

Proof. We consider a natural equivalence of functors $F, G : \mathfrak{C}' \to \mathfrak{C}$ with $\eta : F \cong G$. First note that there is an isomorphism of natural systems

$$\tilde{\eta} : F^*\mathcal{A} \cong G^*\mathcal{A} \tag{5.22}$$

This gives a diagram of cochain maps

$$\begin{array}{ccc} P^*(\mathfrak{C}, \mathcal{A}) & \xrightarrow{F^*} & P^*(\mathfrak{C}', F^*\mathcal{A}) \\ & \searrow^{G^*} & \downarrow{\scriptstyle \tilde{\eta}^*} \\ & & P^*(C', G^*\mathcal{A}), \end{array} \tag{5.23}$$

Now note that there is a chain homotopy $h^n : F^{n+1}(\mathfrak{C}, \mathcal{A}) \to F^n(C', G^*\mathcal{A})$ given by the formula

$$(h\varphi)(\sigma) = (\eta^*)^{-1} \sum_{i=0}^{n} (-1) i\varphi(F(\alpha_1), \cdots F(\alpha_i), \eta, G(\alpha_{i+1}), \cdots, G(\alpha_n)) \tag{5.24}$$

where $\sigma = * \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} *$ and we have the equality $\tilde{\eta}_* F^* - G^* = \delta h + h\delta$. Diagrammatically, we see that the terms in the alternating sum correspond to

$$\begin{array}{ccccccc} * & \xrightarrow{F(\alpha_1)} & * & \xrightarrow{F(\alpha_2)} & \cdots & \xrightarrow{F(\alpha_n)} & * \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & \cdots & & \downarrow{\scriptstyle \eta} \\ * & \xrightarrow{G(\alpha_1)} & * & \xrightarrow{G(\alpha_2)} & \cdots & \xrightarrow{G(\alpha_n)} & * \end{array} \tag{5.25}$$

This proves that $\tilde{\eta}_* F^* = G^*$ on $H^n(\mathfrak{C}, \mathcal{A})$. Then considering an equivalence of small categories with $F' : \mathfrak{C} \to \mathfrak{C}$, $\eta : F'F \implies 1_{\mathfrak{C}'}$, and $\epsilon : 1_{\mathfrak{C}} \implies FF'$ we reach the desired isomorphisms by

$$\tilde{\eta}_*(F'F)^* = 1^* = 1 \quad \textit{and} \quad \tilde{\epsilon}_*(FF') = 1^* = 1. \tag{5.26}$$

□

5.3 Cohomology of Homotopy Colimit of Comma Categories

In this section, we show how the Definition 5.2.3 can be derived from the Definition 5.1.4. To this end, we use the definition 5.1.4 to obtain the cohomology $H^n(\mathfrak{C}, \mathcal{A})$ of a small category $\mathfrak{C}$ with coefficients in a natural system $\mathcal{A}$ by a projective resolution of the constant functor $\underline{\mathbb{Z}}$ as an $\mathbb{Z}F\mathfrak{C} - module$. The coherence of the definitions will follow by applying $\mathrm{Hom}_{\mathbb{Z}F\mathfrak{C}}(\mathcal{A}, -)$ and showing that we get the cochain complex (5.12).

Here the constant functor

$$\underline{\mathbb{Z}} : F\mathfrak{C} \to \mathbf{Ab} \tag{5.27}$$

is given by carrying an object $\alpha \in \mathrm{Ob}(F\mathfrak{C})$ to the free abelian group with one generator α, denoted by $\mathbb{Z}\{\alpha\}$, and a morphism $(u, v) : \alpha \to \alpha'$ to the isomorphism $(u, v)_* : \mathbb{Z}\{\alpha\} \xrightarrow{\cong} \mathbb{Z}\{\alpha'\}$ such that $(u, v)_*(\alpha) = \alpha'$.

Notation 5.3.1. To switch from an element $\sigma \in N(\mathfrak{C})_n$ given by

$$\sigma = C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} C_n \tag{5.28}$$

to objects of $F\mathfrak{C}$, we will use the notation

$$\sigma(i, j) = \alpha_j \circ \cdots \circ \alpha_{i+1} : C_i \to C_j \tag{5.29}$$

for the composition of morphisms in σ. Note that we consider the case $i = j$ as

$$\sigma(i, i) = \mathrm{id}_{C_i} : C_i \to C_i. \tag{5.30}$$

Also note that considering the face maps, by

$$(d_i\sigma)(0, n-1) = \alpha_1 \circ \cdots \circ \alpha_{i+1}\alpha_i \circ \cdots \alpha_n \quad i \neq 0, n \tag{5.31}$$

we still mean the same object in $F\mathfrak{C}$ except for $i = 0$ and $i = n$. However, when we consider the direct sum in equation (5.34) and the boundary maps as in equation (5.37) then what differs is the component of the sum.

Proposition 5.3.2 (Theorem 4.4 in [11]). *Let $\underline{\mathbb{Z}} : F\mathfrak{C} \to \boldsymbol{Ab}$ be the constant functor, then for any natural coefficient system $\mathcal{A} : F\mathfrak{C} \to \boldsymbol{Ab}$ there is an isomorphism*

$$H^n(\mathfrak{C}, \mathcal{A}) = Ext^n_{\mathbb{Z}F\mathfrak{C}}(\underline{\mathbb{Z}}, \mathcal{A}) \quad \forall n \geq 0. \tag{5.32}$$

Proof. We begin by constructing the generalized bar resolution $\mathcal{P}_*$ of $\underline{\mathbb{Z}}$ as an $\mathbb{Z}F\mathfrak{C} - module$:

$$\mathcal{P}_* : \cdots \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \underline{\mathbb{Z}} \to 0 \tag{5.33}$$

where the $\mathbb{Z}F\mathfrak{C}$-modules $P_n : F\mathfrak{C} \to \mathbf{Ab}$ are defined by free abelian groups

$$P_n = \bigoplus_{\sigma \in N(\mathfrak{C})_n} \mathbb{Z}\mathrm{Hom}_{F\mathfrak{C}}(\sigma(0,n), -) \tag{5.34}$$

for any

$$\sigma = C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} C_n, \tag{5.35}$$

so they are projective. Now we need to show that $\mathcal{P}$ is a resolution, that is, there exists a contracting chain homotopy $h_* : \mathcal{P}_* \to \mathcal{P}_{*+1}$. Here, we will denote the generators in P_n for an n-simplex $\sigma \in N(\mathfrak{C})_n$ and for a morphism $(u, v) : \sigma(0,n) \to \alpha$ in $\mathrm{Hom}_{F\mathfrak{C}}(\sigma(0,n), \alpha)$ by the triple

$$(u, v, \sigma) \tag{5.36}$$

to encode its summand component in P_n since we already know that the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{u} & C' \\ {\scriptstyle \alpha_n \circ \cdots \circ \alpha_1}\uparrow & & \uparrow{\scriptstyle \alpha} \\ C_0 & \xleftarrow[v]{} & C \end{array}$$

commutes. The boundary maps $\partial_n : P_n \to P_{n-1}$, $n = 1, 2, \ldots$, for an n-simplex $\sigma \in N(\mathfrak{C})_n$ and a morphism $(u, v) : \sigma(0,n) \to \alpha$ in $\mathrm{Hom}_{F\mathfrak{C}}(\sigma(0,n), \alpha)$ are defined

on generators by

$$\begin{aligned}\partial_n(u,v,\sigma) =&(u,\alpha_1\circ v,d_0\sigma)\\ &+\sum_{i=1}^{n-1}(-1)^i(u,v,d_i\sigma)\\ &+(-1)^n(u\circ\alpha_n,v,d_n\sigma).\end{aligned}\tag{5.37}$$

The boundary map $\epsilon : P_0 \to \underline{\mathbb{Z}}$ for a 0-simplex $\sigma \in N(\mathfrak{C})_0$ and a morphism $(u,v) : \sigma(0,0) \to \alpha$ in $\mathrm{Hom}_{F\mathfrak{C}}(\sigma(0,0),\alpha)$ is defined on generators by

$$\epsilon(u,v,\sigma) = u\circ v = \alpha\tag{5.38}$$

which determines an augmentation on $\underline{\mathbb{Z}}$. Then we define a chain homotopy $h_n : P_n \to P_{n+1}$, $n = 0,1,\ldots$, for an n-simplex $\sigma \in N(\mathfrak{C})_n$ and a morphism $(u,v) : \sigma(0,n) \to \alpha$ in $\mathrm{Hom}_{F\mathfrak{C}}(\sigma(0,n),\alpha)$ and $h_{-1} : \underline{\mathbb{Z}} \to P_0$ for the same object $\alpha : C \to C'$ in $F\mathfrak{C}$ by

$$h_n(u,v,\sigma) = (u,\mathrm{id}_C, C\star_v\sigma),\quad h_{-1}(\alpha) = (\alpha,\mathrm{id}_C,C)\tag{5.39}$$

where $C\star_v\sigma$ denotes the $n+1$-simplex

$$C\star_v\sigma = C\xrightarrow{v}C_0\xrightarrow{\alpha_1}\cdots\xrightarrow{\alpha_n}C_n.\tag{5.40}$$

Then note that we have

$$\begin{aligned}(\partial_{n+1}h_n + h_{n-1}\partial_n)(u,v,\sigma) =&\partial_{n+1}(u,\mathrm{id}_C,C\star_v\sigma)+\\ &h_{n-1}(u,\alpha_1\circ v,d_0\sigma)+\\ &h_{n-1}\big(\sum_{i=1}^{n-1}(-1)^i(u,v,d_i\sigma)\big)+\\ &(-1)^n h_{n-1}(u\circ\alpha_n,v,d_n\sigma)\end{aligned}\tag{5.41}$$

where the first summand is given by

$$\begin{aligned}\partial_{n+1}(u,\mathrm{id}_C,C\star_v\sigma) =&(u,v,\sigma)+\sum_{i=1}^{n}(-1)^i(u,\mathrm{id}_C,d_i(C\star_v\sigma))+\\ &(-1)^{n+1}(u\circ\alpha_n,\mathrm{id}_C,d_{n+1}(C\star_v\sigma)),\end{aligned}\tag{5.42}$$

the second summand is given by

$$h_{n-1}(u, \alpha_1 \circ v, d_0\sigma) = (u, \mathrm{id}_C, C \star_{\alpha_1 \circ v} d_0\sigma), \tag{5.43}$$

the third summand is given by

$$h_{n-1}\big(\sum_{i=1}^{n-1}(-1)^i(u, v, d_i\sigma)\big) = \sum_{i=1}^{n-1}(-1)^i(u, \mathrm{id}_C, C \star_v d_i\sigma), \tag{5.44}$$

and the last summand is given by

$$(-1)^n h_{n-1}(u \circ \alpha_n, v, d_n\sigma) = (-1)^n(u \circ \alpha_n, \mathrm{id}_C, C \star_v d_n\sigma). \tag{5.45}$$

Note that $C \star_v d_i\sigma = d_{i+1}(C \star_v \sigma)$, so the Equation (5.45) cancels with the last summand of the Equation (5.42). Also, the second summand in (5.42) cancels with (5.44) except the first term $-(u, \mathrm{id}_C, d_1(C \star_v \sigma))$ which cancels with the summand (5.43).

Also, for the cases $\epsilon h_{-1} = \mathrm{id}_{\underline{\mathbb{Z}}}$ and $\partial_1 h_0 + h_{-1}\epsilon = \mathrm{id}_{P_0}$ we have

$$(\epsilon h_{-1})(\alpha) = \epsilon(\alpha, \mathrm{id}_C, C) = \alpha \tag{5.46}$$

and

$$\begin{aligned}(\partial_1 h_0 + h_{-1}\epsilon)(u, v, \sigma) &= \partial_1(u, \mathrm{id}_C, C \star_v \sigma) + h_{-1}(u \circ v) \\ &= (u, v, \sigma) - (u \circ v, \mathrm{id}_C, C) + (u \circ v, \mathrm{id}_C, C) \\ &= (u, v, \sigma).\end{aligned} \tag{5.47}$$

Hence, h is a contracting chain homotopy. This shows that $\mathcal{P}_*$ is a projective resolution of $\underline{\mathbb{Z}}$ as an $\mathbb{Z}F\mathfrak{C}$-module. □

In Proposition 4.2.1 we proved that there is a weak equivalence

$$\lambda : \operatorname*{hocolim}_{\mathfrak{C}} N(G) \to N(\mathfrak{D}) \tag{5.48}$$

where $G : \mathfrak{C} \to \mathbf{Cat}$ is the functor sending objects $C \in \mathrm{Ob}(\mathfrak{C})$ to their left comma categories φ/C for an arbitrary functor $\varphi : \mathfrak{D} \to \mathfrak{C}$. We now construct a projective resolution $\mathcal{P} \to \underline{\mathbb{Z}}$ over $\mathfrak{D}$ induced by the weak equivalence λ.

Consider the simplicial replacement X on $N(G)$ that is

$$X = \bigsqcup_{\sigma \in N(\mathfrak{C})} N\big(G(\sigma(0))\big). \tag{5.49}$$

Recall that a (p,q)-simplex $(\sigma,\tau) \in X_{p,q}$ is given by

$$(C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_p} C_p, \big(D_0, \beta_0\big) \xrightarrow{k_1} \cdots \xrightarrow{k_q} (D_n, \beta_q)\big) \in N(\mathfrak{C})_p \times N(\varphi/C_0)_q \tag{5.50}$$

that is, diagrammatically we have

$$\begin{array}{ccccccc} & & C_0 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_p} & C_p \\ & \overset{\beta_0}{\nearrow} & & \vdots & & \overset{\beta_q}{\nwarrow} & \\ \varphi(D_0) & \xrightarrow{\varphi(k_1)} & \cdots & & \xrightarrow{\varphi(k_q)} & \varphi(D_q). & \end{array} \tag{5.51}$$

Here the horizontal $d_i^h : X_{p,q} \to X_{p-1,q}$ and vertical $d_i^v : X_{p,q} \to X_{p,q-1}$ face operators are defined by

$$d_i^h(\sigma,\tau) = \begin{cases} (d_0\sigma, \alpha_1^*\tau) & \\ (d_i\sigma, \tau) \quad 0 < i \leq p & \end{cases} \quad , d_i^v(\sigma,\tau) = (\sigma, d_i\tau) \tag{5.52}$$

where $(d_0\sigma, \alpha_1^*\tau)$ denotes the simplex given by

$$\begin{array}{ccccccc} & & C_1 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_p} & C_p \\ & \overset{\alpha_1\circ\beta_0}{\nearrow} & & \vdots & & \overset{\alpha_1\circ\beta_q}{\nwarrow} & \\ \varphi(D_0) & \xrightarrow{\varphi(k_1)} & \cdots & & \xrightarrow{\varphi(k_q)} & \varphi(D_q). & \end{array} \tag{5.53}$$

Now let $\mathbb{P}$ be the bisimplicial natural system on $\mathfrak{D}$ defined by

$$\mathbb{P}_{p,q} = \bigoplus_{(\sigma,\tau) \in X_{p,q}} \mathbb{Z}\mathrm{Hom}_{F\mathfrak{D}}(\Lambda_{p,q}(\sigma,\tau)(0,q), -) \tag{5.54}$$

where $\Lambda_{p,q} : \bigsqcup_{N(\mathfrak{C})_p} N(G)_q \to N(\mathfrak{D})_{p,q}$ is as given in the proof of Proposition 4.2.1 with $diag(\Lambda_{p,q}) = \lambda$.

Proposition 5.3.3. *Let $\varphi : \mathfrak{D} \to \mathfrak{C}$ be any functor and $G : \mathfrak{C} \to \boldsymbol{Cat}$ be the functor defined by sending objects $C \in Ob(\mathfrak{C})$ to their left comma categories φ/C. Then $diag\mathbb{P}_{*,*}$, given in (5.54), is a projective resolution of the natural system $\underline{\mathbb{Z}}$ on $\mathfrak{D}$. Consequently, for any natural coefficient system $\mathcal{A}$ on $\mathfrak{D}$ we have*

$$H^n(\mathfrak{D}, \mathcal{A}) \cong H^n(Hom_{\mathbb{Z}F\mathfrak{D}}(diag\mathbb{P}_{*,*}, \mathcal{A})), \quad n \geq 0. \tag{5.55}$$

Proof. Clearly $\mathbb{P}_{p,q}$ is free, for all $p, q \geq 0$ so we need to show that the following chain complex is exact

$$\mathrm{diag}\mathbb{P}_{*,*} : \cdots \xrightarrow{\partial_{n+1}} \mathbb{P}_{n,n} \xrightarrow{\partial_n} \mathbb{P}_{n-1,n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \mathbb{P}_{0,0} \xrightarrow{\epsilon} \underline{\mathbb{Z}} \to 0. \tag{5.56}$$

Here, we will denote the generators in $\mathbb{P}_{p,q}$ for a (p,q)-simplex $(\sigma, \tau) \in \bigsqcup_{N(\mathfrak{C})_p} N(G)_q$ and for a morphism $(u, v) \in \mathrm{Hom}_{F\mathfrak{D}}(\Lambda_{p,q}(\sigma, \tau)(0, q), k)$ with $k : D \to D'$ by the quadruple

$$(u, v, \sigma, \tau), \tag{5.57}$$

and it is enough to show that the sequence is exact on generators for a fixed object $k : D \to D'$.

To show that $\mathrm{diag}\mathbb{P}_{*,*}$ is exact we consider the homomorphisms

$$e_{p,q} : \mathbb{P}_{p,q} \to \mathcal{P}_{p,q} \tag{5.58}$$

where $\mathcal{P}_{*,q} = \mathcal{P}_q$ of the generalized bar resolution $\mathcal{P}_*$ of $\underline{\mathbb{Z}}$ constructed in Proposition 5.3.2 as $\mathbb{Z}F\mathfrak{D}$-module that is constant in the horizontal direction, and $e_{p,q}$ is given for a (p,q)-simplex $(\sigma, \tau) \in \bigsqcup_{N(\mathfrak{C})_p} N(G)_q$ and for a morphism $(u, v) \in \mathrm{Hom}_{F\mathfrak{D}}(\Lambda_{p,q}(\sigma, \tau)(0, q), k)$ with $k : D \to D'$ on generators by

$$e_{p,q}(u, v, \sigma, \tau) = (u, v, \Lambda_{p,q}(\sigma, \tau)). \tag{5.59}$$

Now we will use Lemma 2.6 in [6], stating that pointwise weak equivalence of bisimplicial abelian groups induces a weak equivalence of the associated diagonal complexes. Note that for a fixed $q \geq 0$, the associated augmented chain complex of abelian groups

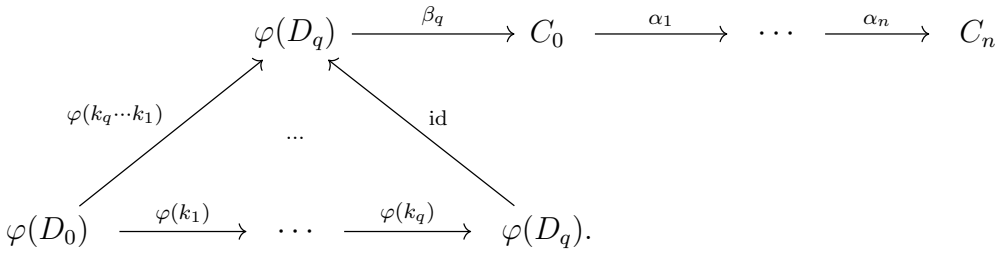
$$\cdots \to \mathbb{P}_{2,q} \xrightarrow{\partial_2} \mathbb{P}_{1,q} \xrightarrow{\partial_1} \mathbb{P}_{0,q} \xrightarrow{e} \mathcal{P}_q \to 0 \tag{5.60}$$

is exact since there is a contracting chain homotopy defined by the homomorphisms $h_{-1} : \mathcal{P}_q \to \mathbb{P}_{0,q}$ and $h_n : \mathbb{P}_{n,q} \to \mathbb{P}_{n+1,q}$, $n \geq 0$, on the generators by

$$h_n(u, v, \sigma, \tau) = (u, v, \varphi(D_q) \star_{\beta_q} \sigma, \tau) \text{ for all } n \geq 0, \tag{5.61}$$

$$h_{-1}(u, v, \tau) = (u, v, \varphi(\tau(q)), \varphi(\tau)) \tag{5.62}$$

where h_n is diagrammatically given on the (p,q)-simplex (σ,τ) by



(5.63)

To see that h is indeed a contracting homotopy, that is,

$$h_{n-1}\partial_n + \partial_{n+1}h_n = \mathrm{id}_{\mathbb{P}_{n,q}}, \tag{5.64}$$

we consider the equations

$$\begin{aligned}(\partial_{n+1}h_n)(u,v,\sigma,\tau) &= \partial_{n+1}(u,v,\varphi(D_q)\star_{\beta_q}\sigma,\tau) && (5.65)\\ &= (u,v,\sigma,\tau)\\ &\quad + \sum_{i=1}^{n+1}(-1)^i(u,v,d_i(\varphi(D_q)\star_{\beta_q}\sigma),\tau) && (5.66)\end{aligned}$$

and

$$\begin{aligned}(h_{n-1}\partial_n)(u,v,\sigma,\tau) =& h_{n-1}\big((u,v,d_0\sigma,\alpha_1^*(\tau)) + \sum_{i=1}^{n}(-1)^n(u,v,d_n\sigma,\tau)\big) && (5.67)\\ &= (u,v,\varphi(D_q)\star_{\alpha_1\circ\beta_q} d_0\sigma,\alpha_1^*(\tau))\\ &\quad + \sum_{i=1}^{n}(-1)^n(u,v,\varphi(D_q)\star_{\beta_q} d_i\sigma,\tau). && (5.68)\end{aligned}$$

Here Equation (5.64) follows from the equality

$$d_{i+1}(\varphi(D_n)\star_{\beta_n}\sigma) = \varphi(D_n)\star_{\beta_n} d_i\sigma. \tag{5.69}$$

Also, for the cases $eh_{-1} = \mathrm{id}_{\mathcal{P}_q}$ and $\partial_1 h_0 + h_{-1}e = \mathrm{id}_{\mathbb{P}_{0,q}}$ we have

$$(eh_{-1})(u,v,\tau) = e(u,v,\varphi(\tau(q)),\varphi(\tau)) = (u,v,\tau) \tag{5.70}$$

and

$$\begin{aligned}(\partial_1 h_0 + h_{-1}e)(u,v,C_0,\tau) &= \partial_1(u,v,\varphi(D_q)\star_{\beta_q} C_0,\tau) + h_{-1}(u,v,\Lambda_{0,q}(C_0,\tau))\\ &= (u,v,C_0,\tau) - (u,v,\varphi(D_q),\tau) + (u,v,\varphi(D_q),\tau)\\ &= (u,v,C_0,\tau) && (5.71)\end{aligned}$$

Then the associated chain complex of abelian groups in (5.60) induces a map $e : \mathrm{diag}\mathbb{P}_{*,*} \to \mathcal{P}$,

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial} & \mathbb{P}_{n,n} & \xrightarrow{\partial} & \mathbb{P}_{n-1,n-1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathbb{P}_{0,0} & \xrightarrow{\epsilon} & \underline{\mathbb{Z}} \longrightarrow 0 \\
\cdots & & \downarrow e & & \downarrow e & & \cdots & & \downarrow e & & \| \\
\cdots & \xrightarrow{\partial} & \mathcal{P}_n & \xrightarrow{\partial} & \mathcal{P}_{n-1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathcal{P}_0 & \xrightarrow{\epsilon} & \underline{\mathbb{Z}} \longrightarrow 0
\end{array}$$

which is a homology isomorphism. Hence, for any natural coefficient system $\mathcal{A}$ on $\mathfrak{D}$ we have

$$H^n(\mathfrak{D}, \mathcal{A}) \cong H^n(\mathrm{Hom}_{\mathbb{Z}F\mathfrak{D}}(\mathrm{diag}\mathbb{P}_{*,*}, \mathcal{A})), \quad n \geq 0. \tag{5.72}$$

by Proposition 5.3.2. □

We can now proceed with our main result of this chapter again by using Cegarra's method.

Theorem 5.3.4. *Let $\varphi : \mathfrak{D} \to \mathfrak{C}$ be any functor and $G : \mathfrak{C} \to \boldsymbol{Cat}$ be the functor defined by sending objects $C \in Ob(\mathfrak{C})$ to their left comma categories φ/C and $\lambda : \underset{\mathfrak{C}}{hocolim}\, N(G) \to N(\mathfrak{D})$ be as in Proposition 4.2.1. Then for any natural coefficient system $\mathcal{A} : F\mathfrak{D} \to \boldsymbol{Ab}$, there is an isomorphism on*

$$H^n(\mathfrak{D}, \mathcal{A}) \cong H^n(\underset{\mathfrak{C}}{hocolim}\, N(G), \lambda^*\mathcal{A}) \quad \forall n \geq 0. \tag{5.73}$$

Proof. Let $\mathcal{A} : F\mathfrak{D} \to \mathbf{Ab}$ be any natural coefficient system and (σ, τ) be an n-simplex of $\underset{\mathfrak{C}}{\mathrm{hocolim}}\, N(G)$ then the induced natural system is given by

$$\lambda^*\mathcal{A}(\sigma, \tau) = \mathcal{A}(\lambda(\sigma, \tau)(0, n)). \tag{5.74}$$

Hence, the standard cochain complex is given by the abelian groups

$$C^n(\underset{\mathfrak{C}}{\mathrm{hocolim}}\, N(G), \lambda^*\mathcal{A}) = \prod_{(\sigma,\tau)\in \underset{\mathfrak{C}}{\mathrm{hocolim}}\, N(G)} A(\lambda(\sigma, \tau)(0, n)). \tag{5.75}$$

and the coboundary maps $\delta_n : C^{n-1} \to C^n$ are given by sending a function f of C^{n-1} to the function

$$(\delta f)(\sigma, \tau) = (k_1)^* f(d_0\sigma, d_0\tau) + \sum_{i=1}^{n-1} (-1)^i f(d_i\sigma, d_i\tau) + (-1)^n (k_n)_* f(d_n\sigma, d_n\tau).$$

Now consider the bisimplicial natural system $\mathbb{P}$ on $\mathfrak{D}$ in (5.54), that is,

$$\mathbb{P}_{p,q} = \bigoplus_{(\sigma,\tau)\in X_{p,q}} \mathbb{Z}\mathrm{Hom}_{F\mathfrak{D}}(\Lambda_{p,q}(\sigma,\tau)(0,q),-).$$

Recall that we denote the generators in $\mathbb{P}_{n,n}$ for an n-simplex $(\sigma,\tau) \in \operatorname{hocolim}_{\mathfrak{C}} N(G)$ and for a morphism $(u,v) \in \mathrm{Hom}_{F\mathfrak{D}}(\Lambda_{p,q}(\sigma,\tau)(0,q),k)$ with $k : D \to D'$ by the quadruple

$$(u,v,\sigma,\tau), \tag{5.76}$$

and the boundary maps $\partial_n : \mathbb{P}_{n,n} \to \mathbb{P}_{n-1,n-1}$, $n = 1,2,\ldots$, for a morphism $(u,v) : \lambda(\sigma,\tau)(0,n) \to k$ are defined on generators by

$$\begin{aligned}\partial_n(u,v,\sigma,\tau) =& (u, k_1 \circ v, d_0\sigma, \alpha_1^*(d_0\tau)) + \\ & \sum_{i=1}^{n-1}(-1)^i(u,v,d_i\sigma,d_i\tau) + \\ & (-1)^n(u\circ k_n, v, d_n\sigma, d_n\tau)\end{aligned} \tag{5.77}$$

where the n-simplex $(\sigma,\tau) \in \operatorname{hocolim}_{\mathfrak{C}} N(G)$ is given by

$$(C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} C_n, \big(D_0,\beta_0) \xrightarrow{k_1} \cdots \xrightarrow{k_n} (D_n,\beta_n)\big). \tag{5.78}$$

Then we know by definition of $\mathbb{P}_{p,q}$ we have the isomorphism of abelian groups

$$\Gamma : C^n(\operatorname{hocolim}_{\mathfrak{C}} N(G), \lambda^*\mathcal{A}) \cong \mathrm{Hom}_{\mathbb{Z}F\mathfrak{D}}(\mathbb{P}_{n,n},\mathcal{A}) \tag{5.79}$$

with

$$\Gamma_f : \mathbb{Z}\mathrm{Hom}_{F\mathfrak{D}}(\lambda(\sigma,\tau)(0,n),k) \to \mathcal{A}(k) \tag{5.80}$$

$$(u,v,\sigma,\tau) \mapsto u_*v^*f(\sigma,\tau). \tag{5.81}$$

Clearly, we have

$$\Gamma_{\delta f}(u,v,\sigma,\tau) = \Gamma_f(\partial(u,v,\sigma,\tau)), \tag{5.82}$$

so Γ gives an isomorphism of cochain complexes. Thus, the statement

$$H^n(\mathfrak{D},\mathcal{A}) \cong H^n(\operatorname{hocolim}_{\mathfrak{C}} N(G), \lambda^*\mathcal{A}) \quad \forall n \geq 0 \tag{5.83}$$

follows from Proposition 5.3.3. □

Bibliography

[1] R. W. Thomason, "Homotopy colimits in the category of small categories," in *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 85, pp. 91–109, Cambridge University Press, 1979.

[2] A. M. Cegarra, "Cohomology of homotopy colimits of simplicial sets and small categories," *Mathematics*, vol. 8, no. 6, p. 981, 2020.

[3] R. W. Thomason, "Cat as a closed model category," *Cahiers de topologie et géométrie différentielle*, vol. 21, no. 3, pp. 305–324, 1980.

[4] D. Quillen, "Higher algebraic k-theory: I," in *Higher K-Theories: Proceedings of the Conference held at the Seattle Research Center of the Battelle Memorial Institute, from August 28 to September 8, 1972*, pp. 85–147, Springer, 2006.

[5] J. P. May, *Simplicial objects in algebraic topology*, vol. 11. University of Chicago Press, 1992.

[6] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[7] G. Friedman, "An elementary illustrated introduction to simplicial sets," *arXiv preprint arXiv:0809.4221*, 2008.

[8] B. Richter, *From categories to homotopy theory*, vol. 188. Cambridge University Press, 2020.

[9] W. G. Dwyer, H.-W. Henn, and W. Dwyer, "Classifying spaces and homology decompositions," *Homotopy theoretic methods in group cohomology*, pp. 1–53, 2001.

[10] J. Milnor, "The geometric realization of a semi-simplicial complex," *Annals of Mathematics*, vol. 65, no. 2, pp. 357–362, 1957.

[11] H.-J. Baues and G. Wirsching, "Cohomology of small categories," *Journal of Pure and Applied Algebra*, vol. 38, no. 2, pp. 187–211, 1985.

[12] I. Gálvez-Carrillo, F. Neumann, and A. Tonks, "Thomason cohomology of categories," *Journal of Pure and Applied Algebra*, vol. 217, p. 2163–2179, Nov. 2013.

[13] J. P. May, "E1-spaces, group completions and permutative categories," *New Developments in Topology, London Math. Soc. Lecture Note Series*, vol. 11, 1974.

[14] C. A. Weibel, *An introduction to homological algebra*. No. 38, Cambridge university press, 1994.