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ON SOME BEST PROXIMITY POINT RESULTS VIA R-FUNCTIONS WITH APPLICATION

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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Taif Hameed Saadoon SAADOON

ABSTRACT

ON SOME BEST PROXIMITY POINT RESULTS VIA R-FUNCTIONS WITH APPLICATION

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January 2024

This thesis explores the intricate domains of best proximity point and fixed point theories, structured into five comprehensive chapters. The first chapter introduces the thesis by outlining its motivation and providing essential background knowledge on the relevant mathematical theories. In the second chapter, we delve into the basic terminology, notations, and theorems that underpin our subsequent findings. Our primary contributions are presented in the third chapter, where we introduce a novel class of functions, including R-functions, and develop a new type of R-contraction. We also redefine the P-property and establish new results for best proximity points and fixed points, supported by an illustrative example to showcase the applicability of our results. The fourth chapter concludes the thesis by summarizing our findings and suggesting directions for future research. This work not only contributes to the existing mathematical literature but also opens new avenues for further theoretical and applied investigations.

2024, 35 pages

Keywords: Metric spaces, Fixed point, Best proximity point, R-contractions

ÖZET

R-FONKSİYONLARI YARDIMIYLA BAZI EN İYİ YAKINLIK NOKTASI SONUÇLARI VE UYGULAMASI ÜZERİNE

Taif Hameed Saadoon SAADOON
Matematik, Yüksek Lisans
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Bu tez, beş kapsamlı bölümde yapılandırılmış olan en iyi yakınlık noktası ve sabit nokta teorilerinin karmaşık alanlarını araştırmaktadır. İlk bölüm, tezin motivasyonunu özetlemekte ve ilgili matematik teorileri hakkında temel bilgiler sağlamaktadır. İkinci bölümde, sonraki bulgularımızı destekleyen temel terimler, notasyonlar ve teoremler ele alınmaktadır. Ana katkılarımız üçüncü bölümde sunulmaktadır; burada R-fonksiyonlarını da içeren yeni bir fonksiyon sınıfı tanıtılmakta ve yeni bir R-kontraksiyon tipi geliştirilmektedir. Ayrıca P-özelliği yeniden tanımlanmakta ve en iyi yakınlık noktaları ve sabit noktalar için yeni sonuçlar ortaya konmaktadır, sonuçlarımızın uygulanabilirliğini sergilemek için bir örnek de sunulmaktadır. Dördüncü bölüm, bulgularımızı özetlemekte ve gelecekteki araştırmalar için yönergeler önermektedir. Bu çalışma, mevcut matematik literatürüne katkıda bulunmakla kalmamakta, aynı zamanda daha fazla teorik ve uygulamalı araştırmalar için yeni yollar açmaktadır.

2024, 35 sayfa

Anahtar Kelimeler: Metrik uzaylar, Sabit nokta, En iyi yakınlık noktası, R-büzülmeler

PREFACE AND ACKNOWLEDGEMENTS

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CONTENTS

LIST OF SYMBOLS

$\geq$	Greater than or equal
$\leq$	Less than or equal
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{R}$	The set of real numbers

LIST OF ABBREVIATIONS

max	Maximum
min	Minimum

1. INTRODUCTION

One of the key ideas in the study of topology is the concept of metric space. French mathematician Maurice Fréchet (1906), who also contributed to the disciplines of Functional Analysis, Probability and Computation, first expressed this idea in his doctoral thesis. The term "distance" between any two elements in the space under consideration must be defined. In this way, the idea of metric space has formed the basis of much research in a wide variety of disciplines, including algebra, differential geometry, statistics, physics, and engineering. The equations $S(\varkappa)=0$ and $T(\varkappa)=\varkappa$ are frequently encountered in analysis and functional analysis. We solve these equations in many ways. One such approach is fixed point theory. The fixed point theorem is extensively used in the theory of existence for partial differential equations, differential equations, integral equations, and a number of interconnected disciplines. Let T be a transformation: $\Lambda \rightarrow \Lambda$, where Λ is a non-empty set. If there is a point $\varkappa_0$ at Λ such that $T(\varkappa_0) = \varkappa_0$, T is said to have a fixed point, and the point $\varkappa_0$ is called the fixed point of T.

Banach (1922), considered one of the most important mathematicians of the 20th century, conducted one of the most notable studies on functions defined in metric spaces. In his doctoral thesis, he first proposed the "Fixed Point Theorem in Metric Spaces" as follows.

In the context of metric spaces, if T satisfies the condition that for all $x,y \in \Lambda$,

$$d(T\varkappa, Ty) \leq kd(\varkappa, y)$$

where k is a constant in [0,1), then T is called a contraction mapping. According to the Banach Fixed Point Theorem, if T is a contraction mapping in a complete metric space, then T will have precisely one fixed point in that space.

This theory not only provides a method for identifying the fixed points of such transformations but also establishes the uniqueness and existence of fixed points in self-mappings. Utilizing this approach, numerous subjects within dynamic programming, game theory, ordinary and partial differential equations, and integral equations can be efficiently tackled .The Banach contraction principle was later applied in many different ways (Abbas *et al.* 2019, Berinde and Pacurar 2021, Karapınar 2012, Karapınar *et al.* 2022, Guran *et al.* 2021, Garodia and Uddin 2020,Muhammed and Kumam 2019, O'Regan 2019, Reich and Zalas 2017, Shukla *et al.* 2014, Vetro and Vetro). One of these was obtained by (Khojasteh *et al.* 2016) who introduced a new concept so-called simulation function, and so Z-contractions by using these functions. Moreover, with the help of these functions a nice fixed-point result was obtained which generalized many famous results (Aydi *et al.* 2020, Aydi *et al.* 2021a, Aydi *et al.* 2021b, Berinde and Berinde 2007, Roldan and Shahzad 2015, Roldan *et al.* 2015).

If T:Λ→Λ is a function on a complete metric space (Λ,d), and ζ:[0,∞)×[0,∞)→R is a simulation function, with T being a ζ-contraction according to ζ—meaning it fulfills

$$\zeta\ (d(T\varkappa, T\eta), d(\kappa, \eta))\ \geq\ 0$$

for each κ,η∈Λ—then T possesses a unique fixed point uu in Λ. Additionally, the Picard iteration sequence$\{Tn\kappa\}$ starting from any point κ∈Λ will converge to u.

Best proximity point theory serves as an innovative extension of fixed-point theory results. Consider AA and BB, which are non-empty subsets of a metric space (Λ, d), and a mapping T: A→B. Given that A∩B=∅, it's clear that T cannot have a fixed point. Thus, it becomes relevant to explore the existence of a point κ∈A such that d (κ, $T\kappa$) =d (A, B); this point is referred to as the best proximity point of T.

It is evident that if A=B=Λ, then each best proximity point of T is inherently a fixed points of T. Additionally, finding a best point of T is equivalent to solving the optimization problem $\min_{\kappa\in A}$ d (κ,$T\kappa$), which seeks to minimize the distance between each point in A

and its image under T in B. This approach not only provides a theoretical framework for addressing such optimization problems but also expands the application of metric space theory beyond traditional fixed points.

There are many papers on this topic in the literature due to these facts (Abkar and Gabeleh 2013a, Abkar and Gabeleh 2013b, Altun *et al.* 2020, Aslantas *et al.* 2021a, Aslantas *et al.* 2021b, Aslantas 2021, Basha and Veeramani 1997, Basha 2010, Gabeleh 2014, Gabeleh 2015, Gabeleh and Otafudu 2017, Imdad *et al.* 2018, Raj 2013, Reich 1978, Reich *et al.* 2021, Sahin *et al.* 2021, Sahin 2022, Sahin *et al.* 2023).

This thesis consists of five chapters. The first chapter presents the rationale for the thesis and discusses fundamental principles in both best approach point theory and fixed point theory, acting as contextual information. We review the basic terminology, notations, and theorems in the second chapter, which are connected to our findings regarding fixed points and best proximity points. In chapter three, our findings are presented. In this research, we introduce a novel class of functions, which encompass what we term R-functions. Building on this foundation, we present a new type of R-contraction and propose modifications to existing R-contractions, which have garnered significant attention in recent studies. Furthermore, we introduce a new definition of the P-property. Through these innovations, we establish several results concerning best proximity points, including fixed point outcomes for our newly defined R-contractions. To demonstrate the practical implications and effectiveness of our theoretical results, we provide a specific example. The culmination of our findings and discussions is presented in the fourth chapter, which contains our conclusions. This framework not only advances the field of proximity point and fixed point theory but also broadens the potential for applying these concepts to more complex and varied mathematical and applied scenarios.

2. PRELIMNARIES

In this section, some definitions and theorems that we use in other parts of our study are given.

2.1 Metric Spaces

This section deals with metric spaces so we will recall some relative definitions and their properties.

Definition 2.1. Consider that Λ is a nonempty set and the mapping $d: \Lambda \times \Lambda \to [0, \infty)$ filling the next axioms for all $\varkappa, y, z \in \Lambda$:

$M_1 : d(\varkappa, y) = 0$ iff $\varkappa = y$;
$M_2 : d(\varkappa, y) = d(y, \varkappa)$;
$M_3 : d(\varkappa, z) \leq d(\varkappa, y) + d(y, z)$.

Then *d* is referred to as a metric on Λ. The ordered pair (*Λ*,*d*) is called a metric space.

Next, we give some illustrative examples of metric-spaces.

Example 2.2. Assume $d: \Lambda \times \Lambda \to [0, \infty)$ as

$$d(\varkappa, y) = \begin{cases} 0 & \text{if } \varkappa = y \\ 1 & \text{if } \varkappa \neq y \end{cases}$$

for all$\varkappa, y \in \Lambda$. The pair of numbers (Λ, d) is a metric space that is often referred to as a discrete metric space.

Example 2.3. Let us define $d: \mathbb{R} \times \mathbb{R} \to [0, \infty)$ as

$$d(\varkappa, y) = |\varkappa - y|$$

for all $\varkappa, y \in \mathbb{R}$. Therefore, the pair $(\mathbb{R}, d)$ represents a metric space often referred to as the normal metric-space.

Example 2.4. The function $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by:

$$d((\varkappa_1, \varkappa_2), (y_1, y_2)) = \sqrt{(\varkappa_1 - y_1)^2 + (\varkappa_2 - y_2)^2}$$

the Euclidean metric on $\mathbb{R}^2$ is a metric defined on the two-dimensional Euclidean space $\mathbb{R}^2$.

Definition 2.5. Assuming (Λ,d) denotes a metric space with a point κ0∈Λ and a real number ι>0, the open and closed balls are defined as follows:

i) $B(\varkappa_0; \iota) = \{\varkappa \in \Lambda: d(\varkappa, \varkappa_0) < \iota\}$ is known as the open ball.

ii) $\bar{B}(\varkappa_0; \iota) = \{\varkappa \in \Lambda: d(\varkappa, \varkappa_0) \leq \iota\}$ is known as the closed ball.

In both cases, κ0 is referred to as the center and ι as the radius. The open ball B(κ0;ι) consists of all points in Λ whose distance from the center is less than ι

Theorem 2.6. Consider a metric space $(\Lambda, d\ )$

i) Every $B(\varkappa_0; \iota)$is an open set.

ii) Every $\bar{B}(\varkappa_0; \iota)$ is a closed set.

Example 2.5. Consider $(\mathbb{R}, d)$ be a usual metric space and $\varkappa_0 \in \mathbb{R}$. In this case, for $r \in \mathbb{R}^+$,

$$B(\varkappa_0, r) = (\varkappa_0 - r, \varkappa_0 + r).$$

Example 2.6. Consider $(\mathbb{R}, d)$ be a usual metric space and $\varkappa_0 \in \mathbb{R}$. In this case, for $r \in \mathbb{R}^+$,

$$\bar{B}(\varkappa_0, r) = [\varkappa_0 - r, \varkappa_0 + r].$$

Definition 2.7. In the metric space (Λ,d), a sequence {κn} in Λ is defined as converging to κκ if every ϵ>0, there exists a natural number N∈N such that d(κn,κ)<ϵ for all $n \geq N$. In this case, we write limn→∞κn=κ or κn→κ as n approaches infinity. If limn→∞κn=κ for some κ∈Λ, then the sequence{κn} is considered convergent; otherwise, it is divergent.

Definition 2.8. Assume (Λ, d) be a metric space and $\{\varkappa_n\}$ be a sequence in Λ. The sequence $\{\varkappa_n\}$ is named a Cauchy sequence iff, for all $\varepsilon > 0$, there exists $n, m \in \mathbb{N}$ such that $d(\varkappa_m, \varkappa_n) < \varepsilon$ for all $m, n \geq \mathbb{N}$.

Remark 2.9. In the metric space (Λ,d), consider $\{\kappa n\}$ as a sequence within Λ. Then

i) $\{\varkappa_n\}$ is a converge sequence iff $d(\varkappa_n, \text{x}) \to 0$ as $n \to \infty$.

ii) $\{\varkappa_n\}$ is a Cauchy sequence iff $d(\varkappa_n, \varkappa_m) \to 0$ as $n, m \to \infty$.

Theorem 2.10. Consider (Λ, d), a metric space:

i. **Uniqueness of Limit:** The limit of a convergent sequence within any metric space (Λ,d) is unique. If a sequence{κn} converges to some limit x∈Λ, and it is supposed that it converges to another limit y∈Λ, then by the properties of the metric, d(x,y)=0 would imply x=y. Hence, a sequence cannot converge to two distinct points, affirming the uniqueness of the limit.

ii. Convergence Implies Cauchy: Every convergent sequence in a metric space (Λ,d) is a Cauchy sequence. If a sequence $\{\kappa n\}$ converges to some x∈Λ, for any given ϵ>0 there exists a natural number N such that for all $n \geq N$n, $d(\kappa n, x) < \epsilon/2$. For any $m,n \geq N$, using the triangle inequality, $d(\kappa n,\kappa m) \leq d(\kappa n,x)+d(x,\kappa m) < \epsilon/2+\epsilon/2=\epsilon$, which shows that the sequence is Cauchy.

Definition 2.15. Let (Λ, d) be metric space and $\{\varkappa_n\}$ be a sequence in Λ. For $n_k < n_{k+1}$, the sequence $\{\varkappa_{n_k}\}$ is called the subsequence of $\{\varkappa_n\}$.

Definition 2.11. Let (Λ, d) be a metric space. We say that $(\Lambda, d\ )$ is called complete metric space if for every Cauchy sequence in Λ is convergent sequence.

Example 2.12. i) Consider the metric space *(R, d)*, which is a standard metric space. It is also a complete metric space.

ii) The metric space (Λ, d), which is defined with a discrete metric, is a complete metric space.

iii) Define a function d: *C([a, b])* × *C([a, b])* → R by the rule *d(f, g) = sup {|f(ϰ) - g(ϰ)| : ϰ ∈*[0, 1]}, where *C([a, b])* denotes the set of all continuous functions on the interval *[a, b]*. The space *(C ([a, b]), d)* constitutes a complete metric space.

iv) The metric space *(Q, d),* which is a conventional metric space, is not a complete metric space.

Definition 2.13. Suppose f: Λ → Y is a function where both (Λ, d_1) and (Y, d_2) are defined as metric spaces. f is continuous at $c \in \Lambda$ iff, for any $\varepsilon > 0$, there exists $\delta > 0$, and $\varkappa \in \Lambda$ with $d_1(\varkappa, c) < \delta$ implies that $d_2(f\varkappa, f\ c) < \varepsilon$. If f is continuous at every point on Λ, then it is continuous on Λ.

Definition 2.14. Let (Λ, d_1) and (Y, d_2) be two metric spaces, $f: \Lambda \to Y$ be a function and $\varkappa_0 \in \Lambda$. If the following statement hold:

$$\lim_{n\to\infty} \varkappa_n = \varkappa \Longrightarrow \lim_{n\to\infty} f(\varkappa_n) = f(\varkappa),$$

then f function is called as sequentially continuous.

Theorem 2.15. Let (Λ, d_1) and (Y, d_2) be two metric spaces, $f: \Lambda \to Y$ be a function. The function f is continuous on Λ if and only if the function f is sequentially continuous.

2.2 Some Properties of Fixed Points And Best Proximity Points

This section provides essential definitions and findings regarding the fixed point of a mapping T on the metric space Λ, including the necessary conditions for the existence and uniqueness of fixed points.

Definition 2.14. A mapping T defined on a metric space (Λ, d) is said to have a fixed point at $\varkappa \in \Lambda$ if it satisfies $T\varkappa = \varkappa$.

To illustrate this definition, consider the following example.

Example 2.15. Consider the mapping $T: \mathbb{R} \to [0,1]$ is defined by $T\varkappa = sin\varkappa$ its clear that 0 is fixed point of T but $(\pi/2)$ is not fixed point of T.

Remarks 2.16. i) Not necessary all mappings defined on metric spaces has fixed point, see the mapping $T: (\Lambda, d) \to (\Lambda, d)$ defined by $T\varkappa = \varkappa + a$, where $a \neq 0$ is a real number not has fixed point.

ii) Some types of mappings defined on metric space has only one fixed point such as $T: \mathbb{R} \to [0,1]$ defined by $T\varkappa = sin\varkappa$, and the fixed point of this mapping is only zero.

iii) The identity mapping on a metric has infinite set of fixed points.

iv) If $\varkappa$ is a fixed point of $T:(\Lambda, d) \to (\Lambda, d)$, then $\varkappa$ is not necessary by fixed point of $T|_{Ᵽ}: Ᵽ \to \Lambda$, where $Ᵽ \subseteq \Lambda$. To illustrate that see, $T: \mathbb{R} \to \mathbb{R}$ defined by $T\varkappa = \frac{\varkappa}{2}$ has a fixed point at $\varkappa = 0$, but $T|_{(0,1)}: (0,1) \to \Lambda$, has no fixed point.

Now we introduce some important theorem of existence and uniqueness fixed point for contraction mapping, but before that we are recalling the definition of contraction mapping.

Definition 2.17. A mapping $T : \Lambda \to \Lambda$ on a metric space (Λ, d) is said to be contraction mapping if there exists $0 \leq k < 1$ such that $d\,(T\varkappa, Ty) \leq kd\,(\varkappa, y)$ for all $\varkappa, y \in \Lambda$.

Theorem 2.18. Given a complete space *(Λ, d)* and a contraction mapping *T: Λ → Λ*, it follows that T possesses a unique fixed point.

The condition appeared in above theorem is necessary but not sufficient, that means sometimes the mapping is not contraction but has fixed point. Indeed, in a usual metric space the mapping $T: \mathbb{R} \to \mathbb{R}$ defined by $T\varkappa = 2 - \varkappa$ is not contraction mapping but has fixed point $\varkappa = 1$.

Assume *(Λ, d)* as metric space

$$P(\Lambda) = \{U \subseteq \Lambda : U\ nonempty\ subset\}$$

$$C(\Lambda) = \{U \subseteq \Lambda : U\ \ nonempty\ closed\ subset\}$$

$$CB(\Lambda) = \{U \subseteq \Lambda : U\ is\ nonempty\ closed\ and\ \ bounded\ subset\}$$

Definition 2.45. Consider (Λ, d) as a metric space and let U, V be elements of CB(Λ). Define the function $H: CB(\Lambda) \times CB(\Lambda) \to \mathbb{R}$ defined by

$$H(U,V) = max\{sup_{\varkappa\in U} d(\varkappa,V), sup_{y\in V} d(U,y)\}$$

where $d(\varkappa,V) = inf_{y\in V} d(\varkappa,y)$ is called Pompeiu-Hausdorff metric on $CB(\Lambda)$.

Theorem 2.46. Let (Λ, d) be a complete metric space and $T: \Lambda \to CB(\Lambda)$ be a mapping. If T is a multivalued contraction mapping, that is, there exists $k \in [0,1)$ such that

$$H(T\varkappa, Ty) \leq kd(\varkappa, y)$$

for all $\varkappa, y \in \Lambda$, then T has a unique fixed point.

Recently, Khojasteh *et al.* (2015) introduced a new concept so-called simulation function, and so Z-contractions by using these functions. Now, we recall the definition of Z-contractions and a related fixed-point result.

If the function ζ*: [0, ∞) × [0, ∞) → R* meets the specified conditions, it is characterized as a simulation function.:

$(\zeta 1)$ $\zeta\ (0,0) = 0,$

$(\zeta 2)$ $\zeta(p,q) < q - p$ for all $q,p > 0,$

$(\zeta 3)$ If $\{p_n\},\{q_n\} \subseteq (0,\infty)$ are sequences satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$, then $\lim_{n\to\infty} sup\ \zeta\ (p_n, q_n) < 0.$

Theorem 2.21. Consider *T: Λ → Λ* as a mapping defined on a complete metric space *(Λ, d)*, and let *ζ: [0, ∞) × [0, ∞) → R* be designated as a simulation function. If the mapping $T$is Z-contraction w.r.t. ζ, that is, it is satisfied

$$\zeta\ (d(T\varkappa, T\eta), d(\kappa,\eta)) \geq 0$$

for each $\varkappa, \eta \in \Lambda$, then T has a unique fixed point u in Λ. Also, the Picard sequence $\{T^n \varkappa\}$ for any initial point $\varkappa \in \Lambda$ converges to u.

Following, many authors have studied to extend the family of simulation functions. In this direct, Argoubi *et al.* (2015) noticed that the condition $(\zeta 1)$ can be removed because of the fact that it is not used in the proof of Theorem 1. Another approach to these expansion efforts was achieved by Roldan *et al.* (2015) by modifying the condition $(\zeta 3)$ as follows:

$(\zeta 3)'$ If $\{p_n\}, \{q_n\} \subseteq (0, \infty)$ are sequences satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n > 0$ and $p_n < q_n$ for $n \in N$, then $\lim_{n\to\infty} sup \zeta(p_n, q_n) < 0$

Later, surprisingly it was proved that every Z-contraction in the sense of Rold´an-L´opez-de-Hierro et al. is a Meir-Keeler contraction. To obtain a real larger family of contractions than the family of Meir-Keeler contractions, Roldán-López-de-Hierro *et al.* [23] introduced R-functions, and hence R-contractions with the help of these functions:

Assume $\emptyset \neq A \subseteq R$ and a function $\rho: A \times A \to R$ meets the specified criteria, it is then referred to as an R-function on *A:*

(ρ_1) for any sequence $\{p_n\}$ contained in $A \cap (0, \infty)$ where $\rho(p_n, p_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$, it follows that $p_n \to 0$ as $n \to \infty$.

(ρ_2) for sequence $\{p_n\}, \{q_n\} \subseteq A \cap (0, \infty)$ where both converge to $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = L \geq 0$, $L < p_n$ and $\rho(p_n, q_n) > 0$ for each $n \in \mathbb{N} \cup \{0\}$, it must be that $L = 0$.

The following property for an R-function ρ on A is useful in some cases:

(ρ_3) If $\{p_n\}, \{q_n\} \subseteq A \cap (0, \infty)$ are sequences satisfying $\rho(p_n, q_n) > 0$ for each $n \in \mathbb{N} \cup \{0\}$ and $q_n \to 0$ as $n \to \infty$, then we have $p_n \to 0$ as $n \to \infty$.

Definition 2.22. Suppose *T: Λ → Λ* is a mapping on a complete metric space (Λ, d). Assume there exists an R-function on A that satisfies the relevant conditions.

$$rand(d, \Lambda) = \{d(\varkappa, \eta) : \varkappa, \eta \in \Lambda\} \subseteq A$$

and

$$\rho(d(T\varkappa, T\eta), d(\kappa, \eta)) \geq 0$$

for each $\varkappa, \eta \in \Lambda$ with $\varkappa \neq \eta$, then T is called R-contraction with respect to ρ.

Best proximity point theory is viewed as an innovative extension of fixed-point theory. Suppose *A* and *B* are non-empty subsets of a metric space (Λ, d), and let *T*: *A*→*B* be a mapping. If *A* and *B* are disjoint sets *(A ∩ B = ∅)*, it is evident that the mapping *T* cannot have a fixed point. Consequently, it is relevant to investigate whether there exists a point *κ ∈ A* such that *d(κ, Tκ)* equals *d(A, B)*; this point is termed the best proximity point of *T*.

In the special case where *A*=*B*=Λ, every best proximity point of *T* is actually a fixed point of *T*. Additionally, finding a best proximity point of *T* corresponds to solving the optimization problem: $\min_{\kappa \in A} d(\kappa, T\kappa)$. This pursuit not only identifies the point in *A* closest to its image under TT in BB but also highlights the application of proximity point theory in optimization. We will now review some fundamental concepts and definitions relevant to this theory.

Definition 2.27. In a metric space *(Λ, d)* with nonempty subsets *A* and *B*, a mapping *T* from *A* to *B* has a best proximity point *x* in *A* if the distance between x and *T(x)*, denoted *d(x, T(x))*, equals the distance between the sets *A* and *B*, *d(A, B)*.

Let (Λ, d) be a metric space, A, B be nonempty subsets of Λ and $T: A \to B$ be a mapping. We regard the subsets of A and B, respectively.

$$A_0 = \{\varkappa \in A : d(\varkappa, y) = d(A, B) for\ some\ \varkappa \in B\}$$

and

$$B_0 = \{\varkappa \in B : d(\varkappa, y) = d(A, B) for\ some\ \varkappa \in A\}$$

where $d(A, B)$ $=inf\{\ d(\varkappa, y): \varkappa \in A\ and\ y \in B\}$.

Definition 2.37 ([15]). In a metric space (Λ, d) where P and Q are non-empty subsets, the pair (P, Q) is considered to have the P-property if this condition is met.

$$\left.\begin{matrix} d(\varkappa_1, y_1) = d(P, Q) \\ d(\varkappa_2, y_2) = d(P, Q) \end{matrix}\right\} \Longrightarrow d(\varkappa_1, \varkappa_2) = d(y_1, y_2)$$

for all $\varkappa_1, \varkappa_2 \in P\ and\ y_1, y_2 \in Q$.

Theorem 2.48. Let (Λ, d) be a complete metric space, A, B be nonempty closed subsets of Λ and $T: A \to B$ be a mapping. Suppose that $A_0 \neq \emptyset, T(A_0) \subseteq B_0$ and the pair (A, B) has the P-Property. If there exists $k \in [0,1)$ such that

$$d(T\varkappa, Ty) \leq kd(\varkappa, y)$$

for all $\varkappa, y \in A$, then T has a best proximity point in A.

3. SOME RESULTS VIA R-FUNCTIONS

In the proof of the main result (Theorem 27) in [20], we notice that the authors need $L \leq p_n$ for all $n \in N$ instead of $L < p_n$ for all $n \in N$ in the condition (ϱ_2) to show the Picard sequence $\{\varkappa_n\}$ is a Cauchy sequence. To address this issue, we make a slight alteration to condition (ρ_2), revising it to $(\rho_2)'$. Therefore, throughout the remainder of the paper and specifically in Definition 2.36, it is logical to consider an R-function on A as a function $\rho: A \times A \rightarrow R$ that fulfills the conditions (ρ_1) and the revised $(\rho_2)'$.Also, taking into account the condition $(\varrho_1)'$ which is weaker than (ϱ_1) we introduce a new concept so-called the modified R-function as follows.

Definition 3.1. Assume $\emptyset \neq A \subseteq R$ and consider a function $\rho: A \times A \rightarrow R$. If it meets the specified criteria, it is termed a modified R-function on A. Specifically, condition (ρ_1)' states that if $\{p_n\}$ is a sequence in $(0,\infty) \cap A$ with $\rho(p_{(n+1)}, p_n) > 0$ for all $n \in N \cup \{0\}$, then there must be a subsequence $\{p_{(nk)}\}$ of $\{p_n\}$ such that:

$$p_{n_k} \rightarrow 0 \; as \; k \rightarrow \infty.$$

$(\varrho_2)'$ If $\{p_n\}, \{q_n\} \subseteq (0,\infty) \cap A$ are sequences satisfying $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = L \geq 0$, $L \leq p_n$ and $\varrho\,(p_n, q_n) > 0$ for each $n \in N$, then we have $L = 0$. It is clear that every R-function on A is a modified R-function on A.

Using modified R-functions and taking into account the best proximity point theory we introduce a new type contraction called generalized R-contraction. Hence, we enlarge the family of R-contractions. Before this new concept, we present the following proposition that is important for our main result without proof since the proof is similar to Proposition 19 in [20].

Proposition 3.2. Assume $\varrho : A \times A \rightarrow \mathbb{R}$ be a modified $\mathcal{R}$-function on A. Then, we have $\varrho(\varkappa, \varkappa) \leq 0 \; for \; all \; \varkappa \in (0,\infty) \cap A.$

Now, we state our new concept.

Definition 3.3. Let $T : P \to Q$ be a mapping on a metric space (Λ, d) where $\emptyset \neq P, Q \subseteq \Lambda$. If there is a modified R-function $\varrho : A \times A \to \mathbb{R}$ on A satisfying

$$ran(d, P \cup Q) = \{d(\varkappa, y) : \varkappa, y \in P \cup Q\} \subseteq A$$

and

$$\varrho\,(d(T\varkappa, Ty), d(\varkappa, y)) > 0$$

for each $\varkappa, y \in P \; with \; \varkappa \neq y$, then T is called generalized $\mathbb{R}$-contraction with respect to ϱ.

In exploring best proximity points, it has been demonstrated that the existence of such points under the P-property can be derived from corresponding fixed point theorems by Abkar and Gabeleh. To further generalize these fixed point results, we propose the following modification to the definition of the P-property [2]:

Definition 3.4. Let $\emptyset \neq P, Q$ be subsets of a metric space (Λ, d). Then, the pair (P, Q) is said to have generalized P-property if it is satisfied

$$\left.\begin{matrix} d(\varkappa_1, y_1) = d(P, Q) \\ d(\varkappa_2, y_2) = d(P, Q) \end{matrix}\right\} \Longrightarrow d(\varkappa_1, \varkappa_2) = d(y_1, y_2)$$

for all $\varkappa_1, \varkappa_2 \in P$ with $\varkappa_1 \neq \varkappa_2 \; and \; y_1, y_2 \in Q$. Now, we present our main result:

Theorem 3.5. Consider a mapping $T\colon P \to Q$ on a complete metric space (Λ, d), where P and Q are closed subsets of Λ. Let $P_0 \neq \emptyset$ and $T(P_0)$ be included within Q_0, and assume that the pair (P, Q) possesses a generalized P-property. Furthermore, suppose T is characterized as a generalized R-contraction relative to the function ϱ. If any of the following conditions are met:

- T is continuous,
- The function ϱ meets condition (ϱ_3),

then it follows that T has a unique best proximity point within P.

Proof. Let $\varkappa_0 \in P_0$ be an arbitrary point. Since $T\varkappa_0 \in T(P_0) \subseteq Q_0$ there exists $\varkappa_1 \in P_0$ satisfying

$$d(\varkappa_1, T\varkappa_0) = d(P, Q).$$

Also, since $T\varkappa_1 \in T(P_0) \subseteq Q_0$, there exists $\varkappa_2 \in P_0$ satisfying

$$d(\varkappa_2, T\varkappa_1) = d(P, Q)$$

By this way, we can construct a sequence $\{\varkappa_n\}$ in P_0 such that

$$d(\varkappa_{n+1}, T\varkappa_n) = d(P, Q), \tag{3.1}$$

for all $n \; in \; N \cup \{0\}. If \; \varkappa_{n_0} = \varkappa_{n_{0+1}}$ for a particular $n_0 \in N \cup \{0\}$, then from (3.1) we have

$$d(\varkappa_{n_0}, T\varkappa_{n_0}) = d(P, Q),$$

and so the proof is completed. Hence, we assume that $\varkappa_n \neq \varkappa_{n+1}$ for all $n \in N \cup \{0\}$. Then, since the pair (P, Q) has generalized P-property, from (3.1) we get

$$d(\varkappa_n, T\varkappa_{n+1}) = d(\varkappa_{n-1}, T\varkappa_n) \tag{3.2}$$

for all $n \in N$. Also, since T is a generalized $\mathcal{R}$-contraction with respect to ϱ, then we obtain

$$\varrho(d(\varkappa_{n-1}, T\varkappa_n), d(\varkappa_{n-1}, \varkappa_n)) > 0$$

for all $n \in N$, and so from (3.2), we get

$$\varrho(d(\varkappa_n, T\varkappa_{n+1}), d(\varkappa_{n-1}, \varkappa_n)) > 0 \tag{3.3}$$

for all $n \in N$. Therefore, if we denote a sequence $\{p_n\}$ by $\{p_n\} = d(\varkappa_{n-1}, \varkappa_n)$ for all $n \in N$, then from (3.3) we have $p_n > 0$ and $\varrho(p_{n+1}, p_n) > 0$ for all $n \in N$. Also, since $\{p_n\} \subseteq (0, \infty) \cap A$, using the condition $(\varrho_1)'$ we can say that there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$such that

$$\lim_{k\to\infty} p_{n_k} = \lim_{k\to\infty} d\left(\varkappa_{n_{k-1}}, \varkappa_{n_k}\right) = 0 \tag{3.4}$$

Now, we want to show that $\{\varkappa_{n_k}\}$ is a Cauchy sequence. To ease, let's denote a sequence $\{y_k\}\, as\, y_k = \varkappa_{n_k}$ for all $k \in N$. Assume the contrary, that is, $\{y_k\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and two subsequences of natural numbers $\{k_r\}, \{\ell_r\}$ with $\ell_r > k_r \geq r$ such that

$$d(y_{k_r}, y_{\ell_r}) \geq \varepsilon \tag{3.5}$$

for all $r \in N$ where ℓ_r is the least integer satisfying (3.5), that is, $d\left(y_{k_r}, y_{\ell_{r-1}}\right) < \varepsilon$ for all $r \in N$. Hence, using the triangular inequality we have

$$\begin{aligned} \varepsilon &\leq d(y_{k_r}, y_{\ell_r}) \\ &\leq d\left(y_{k_r}, y_{\ell_{r-1}}\right) < + d(y_{\ell_{r-1}}, y_{\ell_r}) \\ &< \varepsilon + d(y_{\ell_{r-1}}, y_{\ell_r}) \end{aligned}$$

for all $r \in N$. Taking limit as $r \to \infty$ we get

$$\lim_{r\to\infty} d(y_{k_r}, y_{\ell_r}) = \varepsilon \tag{3.6}$$

Also, since

$$|d(y_{k_{r-1}}, y_{\ell_{r-1}}) - d(y_{k_r}, y_{\ell_r})| \leq d(y_{k_r-1}, y_{k_r}) + d(y_{\ell_{r-1}}, y_{\ell_r})$$

for all $r \in N$, from (3.6) we have

$$\lim_{r\to\infty} d\left(y_{k_r}, y_{\ell_{r-1}}\right) = \varepsilon. \tag{3.7}$$

Because of the fact that T is a generalized $\mathfrak{R}$-contraction $w.r.t.\,\varrho$, we obtain

$$\varrho\left(d\left(Ty_{k_r-1}, Ty_{\ell_{r-1}}\right), d(y_{k_r-1}, y_{\ell_{r-1}})\right) > 0$$

for all $r \in N$. So, since the pair (P, Q) has generalized P-property, we have

$$\varrho\left(d\left(y_{k_r}, y_{\ell_r}\right), d(y_{k_r-1}, y_{\ell_{r-1}})\right) > 0$$

For every r in N. Considering that as r approaches infinity, the limits lim$r$→∞d(ykr−1 ,ylr−1) and lim$r$→∞d(ykr,ylr) both approach ϵ, and given the condition ρ2 from Equations (3.5) and (3.8), we find that ϵ=0, which leads to a contradiction. Therefore, the sequence {yk}={κnk} forms a Cauchy sequence in P. Utilizing equation (3.2), it follows that the sequence{$T\kappa nk$−1} is also a Cauchy sequence in Q. Owing to the closed nature of the subsets P and Q within the complete metric space (Λ,d), there must exist elements κ∈P and y∈Q such that:

$$\lim_{k\to\infty} \varkappa_{n_k} = \varkappa \; and \; \lim_{k\to\infty} T\varkappa_{n_{k-1}} = y. \tag{3.9}$$

From (3.1), taking limit as $k \to \infty$ we have

$$d(\varkappa, y) = d(P, Q). \tag{3.10}$$

Also, we obtain

$$d(\varkappa_{n_{k-1}}, \varkappa) \leq d(\varkappa_{n_{k-1}}, \varkappa_{n_k}) + d(\varkappa_{n_k}, \varkappa)$$

for each $k \in N$. Hence, considering (3.4) and (3.9) we get

$$\lim_{k\to\infty} \varkappa_{n_{k-1}} = \varkappa \tag{3.11}$$

If there exists a subsequence of $\{\varkappa_{n_{k-1}}\}$ whose each terms equal to $\varkappa$ then from (3.9) it can be seen that $y = T\varkappa$. So, from (3.10) the proof is complete. Therefore, suppose that $\varkappa_{n_{k-1}} \neq \varkappa$ for all $k \in N$ and for some $r \in N$ with $k \geq r$. Now, we have the following cases:

Case (i): Suppose that T is a continuous mapping. Then, we obtain

$$\lim_{k\to\infty} T\varkappa_{n_{k-1}} = T\varkappa,$$

and so $y = T\varkappa$. From (3.10), we conclude that $\varkappa \in P$ is a best proximity point of T.

Case (ii): Suppose that the condition (ϱ_3) is satisfied. Since T is a generalized R-contraction mapping, we have

$$\varrho\left(d\left(T\varkappa_{n_{k-1}}, T\varkappa\right), d(\varkappa_{n_{k-1}}, \varkappa)\right) > 0$$

Hence, considering the condition (ϱ_3), from (3.11) we have

$$\lim_{k\to\infty} T\varkappa_{n_{k-1}} = T\varkappa.$$

Therefore, we get $y = T\varkappa$. Hence, from (3.10), we conclude that $\varkappa \in P$ is a best proximity point of T.

For the uniqueness, suppose that there exists $\varkappa, y \in P \; with \; \varkappa \neq y$ such that

$$d(\varkappa, T\varkappa) = d(P, Q)$$

and

$$d(y, Ty) = d(P, Q).$$

Hence, considering the generalized P-property we have

$$d(\varkappa, y) = d(T\varkappa, Ty).$$

Also, because of the fact that T is a generalized R-contraction with respect to ζ, we obtain

$$\varrho\,(d(T\varkappa, Ty), d(\varkappa, y)) > 0$$

Building on the contradiction to Proposition 1, we establish that the function T indeed has a unique best near point within P. This insight leads us to a broader implication, given that every R-function ρ defined on A can be considered as a modified R-function on A, enhancing its applicability. Thus, we can extend our understanding and formulate the following corollary, which encapsulates and expands upon the primary finding of [20]:

Corollary 1. Assume T:$P\to Q$ be a a function in a complete metric space (Λ, d), where both P and Q are closed subsets of Λ. Assume $P0\neq\emptyset$, $T(P0) \subseteq Q0$, and the pair (P, Q) holds a generalized P-property. This setting not only assures the existence of a unique best proximity point due to the specific properties of P and Q but also aligns with the enhanced definitions of R-functions, providing a more robust framework for proving best proximity point results in metric spaces. This corollary significantly broadens the implications of fixed-point theories, illustrating a practical application of theoretical advancements in metric space mappings. Assume there exists an R-function ϱ: A × A → R on A such that the range of distances between elements of P and Q, defined as ran(d,P∪Q)={$d(\kappa,y)$:$\kappa,y\in P\cup Q$} is contained within A, and for each distinct pair κ,y in P, $\rho(d(T\kappa,Ty),d(\kappa,y))>0$. If any of the following conditions is met:

T is a continuous mapping, The R-function ϱ fulfills condition (ϱ_3), then T is guaranteed to have a unique best proximity point within P.

The following example shows that Theorem 3.5 is real generalization of

Corollary: Let $T : P \rightarrow Q$ be a mapping on a complete metric space (Λ, d) where P and Q are closed subsets of Λ. Assume that $P_0 \neq \emptyset, T(P_0) \subseteq Q_0$ and the pair (P, Q) has generalized P-property. Should there be an R-function in accordance with ϱ: A × A → R such that the range of dd, when applied to P ∪ Q, which is

$$ran(d, P \cup Q) = \{d(\varkappa, y) : \varkappa, y \in P \cup Q\} \subseteq A$$

falls within A, and

$$\varrho\ (d(T\varkappa, Ty), d(\varkappa, y)) > 0$$

for each $\varkappa, y \in P\ with\ \varkappa \neq y$. If it is satisfied one of the following conditions

i. T operates continuously,
ii. The function ϱ adheres to condition (ϱ_3),

then it is concluded that T secures a unique best proximity point in P.

Example 3.6. Let $\Lambda = R_2$ be a complete metric space with the taxi-cab metric d. Consider the closed subsets of Λ

$$P = \left\{0, \frac{1}{n} : n \in N\right\} \times \{0\}$$

$$Q = \left\{0, \frac{1}{n} : n \in N\right\} \times \{1\}$$

Then, let $d(P,Q)$=1, with P_0=P and Q_0=Q. Furthermore, the pair (P, Q) possesses a generalized P-property. Define the mapping T:P→Q and the function ρ:[0,∞)×[0,∞)→R such that $T(\kappa,0)$=(0,1) and

$$\varrho(p,q) = \begin{cases} 1, & p = \dfrac{1}{n+1} \; and \; q = 1 + \dfrac{1}{n}, n \geq 1 \\ & or \\ & p = 1 + \dfrac{1}{n} \; and \; q = \dfrac{1}{n}, n \geq 1 \\ 0, & p \notin \left\{0, \dfrac{1}{n+1}\right\} and \; q = 1 + \dfrac{1}{n}, n \geq 1, \\ & or \\ & p \notin \left\{0, 1 + \dfrac{1}{n}\right\} and \; q = \dfrac{1}{n}, n \geq 1 \\ \dfrac{q}{2} - p, & otherwise \end{cases}$$

respectivelyConsequently, it is apparent that (P0) ⊆Q0 and that T is a continuous mapping. Define A as the range of distances between elements of P and Q, specifically, A=ran(d,P∪Q).

$$A = \left\{0, \frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{\left|\frac{1}{n} - \frac{1}{m}\right| : n, m \in \mathbb{N}\right\}$$

$$\cup \left\{1 + \left|\frac{1}{n} - \frac{1}{m}\right| : n, m \in \mathbb{N}\right\} \cup \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}.$$

In this case, T is a modified $\mathcal{R}$-function on A. Indeed, to show that the condition $(\varrho_1)'$ holds, let's take a sequence $\{p_n\} \subseteq (0,\infty) \cap A$ satisfying $\varrho(p_{n+1}, p_n) > 0$ for all

$n \in N$. If there is $n_O \in N$ such that $, p_{n_0} = \frac{1}{n_O}$ or $p_{n_0} = 1 + \frac{1}{n_O}$, then we have $p_{n_0+2n} \to 0 \; or \; p_{n_0+(2n-)} \to 0$ for all $n \in \mathbb{N}$. Otherwise, since $\varrho(p_{n+1}, p_n) > 0$ for each $n \in \mathbb{N}$, we have

$$\frac{p_n}{2} - p_{n+1} > 0 \tag{3.12}$$

for $n \in \mathbb{N}$, and so $\{p_n\}$ is decreasing. Hence, there exists $L \geq 0$ such that $p_n \to L \; as \; n \to \infty$. Assume that $L > 0$. Taking limit as $n \to \infty$ in inequality 3.12 we obtain $L \leq \frac{L}{2} < L$ which is contradiction. So, $L = 0$. In similar to $(\varrho_1)'$ it can be shown that the condition $(\varrho_2)'$ holds. Now, we want to show that T is a generalized $\mathcal{R}$-contraction $w.r.t. \varrho$. For this, we have the following conditions:

Case 1: Let $\varkappa = (0,0), y = \left(\frac{1}{n}, 0\right), n \geq 1$. Then, we have $T\varkappa = (0,1)$ and $Ty = (0,1)$. In this case, we obtain

$$d(T\varkappa, Ty) = 0$$

and

$$d(\varkappa, y) = \frac{1}{n}$$

Hence, we obtain:

$$\varrho\big(d\,(T\varkappa, Ty), d(\varkappa, y)\big) = \varrho\left(0, \frac{1}{n}\right) = \frac{1}{2n} > 0$$

Case 2: Let $\varkappa = \left(\frac{1}{n}, 0\right), y = \left(\frac{1}{m}, 0\right), n, m \geq 1$ (without loss of the generality, we assume that $n < m$). Then, we have $T\varkappa = (0,1)$ and $Ty = (0,1)$. In this case, we get

$$d(T\varkappa, Ty) = 0$$

and

$$d(\varkappa, y) = \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{1}{n} - \frac{1}{m}.$$

Hence, we obtain

$$\varrho\big(d\,(T\varkappa, Ty), d(\varkappa, y)\big) = \varrho\left(0, \frac{1}{n} - \frac{1}{m}\right) = \frac{1}{2n} - \frac{1}{2m} > 0$$

However, we cannot apply Corollary 1 to this example since ϱ is not a $\mathcal{R}$-function on any subset E of ℝ satisfying $ran(d, P \cup Q) \subseteq E$. Assume the contrary, that is, there exists a subset E of ℝ satisfying $ran(d) \subseteq E$ and ϱ is a $\mathcal{R}$-function on E. Now, consider the sequence

$$(p_n) = \left(\frac{1}{2}, 1 + \frac{1}{2}, \frac{1}{3}, 1 + \frac{1}{4}, \frac{1}{5} \dots\right)$$

in $(0, \infty) \cap E$. Then, we have $\varrho(p_{n+1}, p_n) = 1 > 0$ for all $n \in$ ℕ, but $p_n \nrightarrow 0$ which contradicts the condition (ϱ_1). If we take $P = Q = \Lambda$ in Corollary 1, we have the main result of [20]:

Corollary 3.7. Consider T: Λ → Λ as a mapping within a complete metric space (Λ, d). Assume there is an R-function ϱ: A × A → R defined on A such that

$$ran(d, \Lambda) = \{d(\varkappa, y) : \varkappa, y \in \Lambda\} \subseteq A$$

and

$$\varrho\,(d(T\varkappa, Ty), d(\varkappa, y)) > 0$$

for all $\varkappa, y \in \Lambda \, with \, \varkappa \neq y$. If it is satisfied one of the following conditions

i. T functions continuously,

ii. The $\mathcal{R}$ -function ϱ meets condition (ϱ_3),

then T is guaranteed to have a unique fixed point in Λ.

4. APPLICATION

Homotopy theory has significant connections to other branches of mathematics, and as a result, several authors have recently successfully applied their fixed point results to it [1, 13, 14, 24, 25]. Therefore, in this part, we apply our most effective proximity point result, Theorem 2, to homotopy theory, taking inspiration from the technique utilized by Vetro et al. [25]. In this case, we prove that if the ideal proximity point is held by one homotopic mapping, then another mapping must also have it. The idea of homotopy will be reviewed now.

Definition 4.1. Consider the topological spaces (Λ_1, τ_1) and (Λ_2, τ_2), with T and F as continuous mappings from Λ_1 to Λ_2. Suppose a continuous function h from $\Lambda_1 \times [0,1]$ to Λ_2 exists such that $h(\kappa,_0)$ equals Tκ and $h(\kappa,_1)$ equals Fκ for every κ in Λ_1. Under these conditions, T and F are recognized as homotopic mappings. The function h is identified as a homotopy. This concept is vital for the insights discussed in this section.

Definition 4.2. Let $\emptyset \neq P, Q$ be subsets of a metric space (Λ, d) and $h : P \times [0,1] \to Q$ be a mapping. If $G_d(h) \subseteq (\Lambda \times [0,1] \times \Lambda, d^*)$ is closed, then h is said to be d-closed mapping, where

$$G_d(h) = \{(\varkappa, \beta, y) : \varkappa \in P, y \in \Lambda \text{ and } \beta \in [0,1] \text{ with } d(y, h(\varkappa, \beta)) = d(P, Q)\}$$

and

$$d^*((\varkappa_1, \beta_1, y_1), (\varkappa_2, \beta_2, y_2)) = d(\varkappa_1, \varkappa_2) + |\beta_1 - \beta_2| + d(y_1, y_2)$$

for all $(\varkappa_1, \beta_1, y_1), (\varkappa_2, \beta_2, y_2) \in \Lambda \times [0,1] \times \Lambda$.

Note that, in case of $d(P, Q) = 0$, Definition 8 turns to the definition of closed mappings defined from P to Q.

Now, let's introduce the key finding of this section:

Theorem 4.3. Assume that (Λ, d) is a complete metric space with P and Q as nonempty, closed subsets of Λ, and that there is a non-empty subset U within P. Suppose further that the pair (P, Q) possesses the generalized P-property. Consider a mapping h: P × [0,1] → Q which is continuous and d-closed. Under these conditions,

i. $d(\varkappa, h(\varkappa, \lambda)) > d(P, Q)$ for each $\varkappa \in P \backslash U$ *and* $\lambda \in [0, 1]$,

ii. there exists a modified $\mathcal{R}$-function $\varrho : A \times A \to \mathbb{R}$ on A satisfying

$$ran(d, P \cup Q) = \{d(\varkappa, y) : \varkappa, y \in P \cup Q\} \subseteq A$$

and

$$\varrho\,(d(h(\varkappa, \lambda), h(y, \mu)), d(\varkappa, y)) > 0$$

for each $\varkappa, y \in P$ with $\varkappa \neq y$ and $\lambda, \mu \in [0, 1]$,

iii. for all $\varkappa \in A, \beta, r \in [0, 1]$ and $\varkappa_0 \in \bar{B}(\varkappa, r) \cap P_0$ there exists $\varkappa_1 \in \bar{B}(\varkappa, r)$ such that $d(\varkappa_1, h(\varkappa_0, \beta)) = d(P, Q)$ where

$$\bar{B}(\varkappa, r) = \{\bar{\varkappa} \in P : d(\varkappa, \bar{\varkappa}) \leq r\}.$$

Then, $h(\,\cdot\,, 1)$ has a best proximity point in P if $h(\,\cdot\,, 0)$ has a best proximity point in P.

Proof. Consider the following subset

$$K = \{(\beta, \varkappa) : d(\varkappa, h(\varkappa, \beta)) = d(P, Q)\}.$$

From the hypothesis and the condition (i), there is a point κ in P such that $d(\varkappa, h(\varkappa, 0)) = d(P, Q)$, that is, we have $(0, \varkappa) \in K$. Hence, we get $K \neq \emptyset$. Define a partial order on K by

$$(\beta, \varkappa) \preceq (\mu, y) \Leftrightarrow \beta \leq \mu \text{ and } d(\varkappa, y) \leq \mu - \beta.$$

Consider LL as an arbitrary totally ordered subset of K, with $\beta*$ defined as $\beta*=\sup\{\beta:(\beta,\kappa)\in L\}$. Suppose we have a sequence $\{(\beta n,\kappa n)\}$ within L such that $(\beta n,\kappa n)\preceq(\beta n+1,\kappa n+1)$ for all $n\in\mathrm{N}\cup\{0\}$, and βn converges to $\beta*$ as $n\to\infty$. In this scenario, the distance $d(\kappa n,\kappa m)$ satisfies $d(\kappa n,\kappa m)\leq\beta m-\beta n$ for each $n,m\in\mathrm{N}\cup\{0\}$ with $m>nm>n$, thereby making $\{\kappa n\}$ a Cauchy sequence in P. Given that P is a closed subset of Λ and (Λ,d) is a complete metric space, there exists a $\kappa*$ in P such that $d(\kappa n,\kappa*)$ converges to zero as $n\to\infty n\to\infty$. Additionally, the sequence $\{(\kappa n,\beta n,\kappa n)\}$ is included in $Gd(h)$, and the limit of $d*((\kappa n,\beta n,\kappa n),(\kappa*,\beta*,\kappa*))$ reaches zero as nn approaches infinity.

$$\lim_{n\to\infty} d^*((\varkappa_n,\beta_n,\varkappa_n),(\varkappa^*,\beta^*,\varkappa^*))=0).$$

Since h is a d-closed mapping, we get

$$d(\varkappa^* h(\varkappa^*,\beta^*)) = d(P,Q).$$

From (i), we determine that $\kappa*$ is in U, and thus, $(\beta*,\kappa*)$ is part of K. Given that L is a totally ordered set, every $(\beta,\kappa)\in L$ is less than or equal to $(\beta*,\kappa*)$. Therefore, $(\beta*,\kappa*)$ acts as an upper bound for L. Utilizing Zorn's Lemma, it is deduced that K possesses a maximal element $(\beta 0,\kappa 0)$. The objective now is to demonstrate that $\beta_0=1$. Suppose otherwise, that $\beta 0<1$. Then, a real number β exists such that $\beta_0<\beta<1$. Define $r=\beta-\beta_0$. According to (ii), the mapping $H(\cdot,\beta):B(\kappa_0,r)\to Q$ is a generalized R-contraction. Considering condition (iii) and referencing Theorem 2, it is established that there exists $\kappa\beta\in B(\kappa_0,r)$ such that $(\kappa\beta,h(\kappa\beta,\beta))=d(P,Q)$. Since from (i) $\kappa\beta$ is in U, and thus $(\beta,\kappa\beta)$ is included in K, this contradicts the assumption that (β_0,κ_0) is a maximal element in K. Consequently, $\beta_0=1$ and $h(\cdot,1)$ yields a best proximity point κ_0 in P. Applying Theorem 3 with $Q=\Lambda$, the following corollary is derived::

From point (i), it follows that $\kappa*\kappa*$ belongs to U, implying $(\beta*,\kappa*)\in K$. Given that L is totally ordered, every pair $(\beta,\kappa)\in L$ satisfies $(\beta,\kappa)\preceq(\beta*,\kappa*)$. Thus, $(\beta*,\kappa*)$ serves as an upper bound for L. By applying Zorn's Lemma, it can be concluded that K contains a maximal element (β_0,κ_0). We next aim to demonstrate that $\beta_0=1$. Suppose, to the contrary, that $\beta_0<1$. In this scenario, one can find a real number β such that $\beta_0<\beta<1$. Define $r=\beta-\beta_0$. According to point (ii), the mapping $H(\cdot,\beta)$ from $B(\kappa_0,r)$ to Q is a generalized R-contraction. Considering condition (iii) and invoking Theorem 2, we find that there exists $\kappa\beta\in B(\kappa_0,r)$ satisfying $(\kappa\beta,\beta)=d(P,Q)$. From (i), $\kappa\beta$ is within U, hence $(\beta,\kappa\beta)\in K(\beta,\kappa\beta)\in K$, which contradicts the assumed maximality of (β_0,κ_0). Consequently, $\beta_0=1$ and $h(\cdot,1)$ has a best proximity point κ_0 in P. If we set $Q=\Lambda Q=\Lambda$ in Theorem 3, the following corollary is obtained:

Theorem 4.4. Consider (Λ, d) is a complete metric space, P be a nonempty closed subset of Λ and $\emptyset \neq U \subseteq P$. Assume that $h : P \times [0, 1] \rightarrow \Lambda$ is a continuous closed mapping such that:

$d(\varkappa, h(\varkappa, \lambda)) > 0$ for each $\varkappa \in P\backslash U \; and \; \lambda \in [0, 1]$,

there exists a modified $\mathcal{R}$-function $\varrho : A \times A \rightarrow \mathbb{R}$ on A satisfying

$ran(d, \Lambda) = \{d(\varkappa, y) : \varkappa, y \in \Lambda\} \subseteq A$ and

$$\varrho\ (d(h(\varkappa, \lambda), h(y, \mu)), d(\varkappa, y)) > 0$$

for each $\varkappa, y \in P$ with $\varkappa \neq y$ and $\lambda, \mu \in [0, 1]$,

for all $\varkappa \in A, \beta, r \in [0, 1]$ and $\varkappa_0 \in \bar{B}(\varkappa, r) \cap P_0$ there exists $\varkappa_1 \in \bar{B}(\varkappa, r)$ such that $\varkappa_1 =, h(\varkappa_0, \beta)$.

If $h(\cdot, 0)$ has a fixed point in P, then $h(\cdot, 1)$ has a fixed point P.

Proof: Let's start by assuming $h(\cdot,0)$ has a fixed point κ in P. Based on Theorem 3, we can assert the existence of $\kappa*$ in P such that $d(\kappa*,h(\kappa*,1))=d(P,\Lambda)=0$. This implies that $\kappa*$ is also a fixed point of $h(\cdot,1)$.

5. CONCLUSIONS

In this thesis, we present our innovative findings. We introduce a novel class of functions, including R-functions, and propose a new type of R-contraction while also refining the commonly studied R-contractions. Additionally, we redefine the P-property to broaden its application. From these theoretical advancements, we derive several significant results, both for best proximity points and for fixed points related to our new R-contractions. To demonstrate the practical relevance and effectiveness of our contributions, we include a specific example.

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