ISTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

WAVE PROPAGATION IN AN ELASTIC MEDIUM: GENERALIZED DAVEY-STEWARTSON EQUATIONS

Ph.D Thesis by
Ceni BABAOĞLU, M.Sc.

Department : Mathematics

Programme : Mathematical Engineering

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(509992115)

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Supervisor (Chairman): Prof. Dr. Saadet ERBAY (Işık U.)

Members of the Examining Committee: Prof.Dr. Faruk GÜNGÖR (I.T.U.)

Prof.Dr. Ayşe H. BİLGE (I.T.U.)

Prof.Dr. Hilmi DEMİRAY (Işık U.)

Prof.Dr. Avadis HACINLIYAN (Yeditepe U.)

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June 2006 Ceni Babaoğlu

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SUMMARY

In the present study, the problem of $(2+1)$ (two spatial and one temporal) dimensional nonlinear wave propagation in a generalized elastic medium is considered. The modulation of $(2+1)$ dimensional waves is examined in an infinite, homogenous, weakly nonlinear and weakly dispersive elastic medium. The study contains five main sections.

In the first section, a general introduction is given including some information about the nonlinear systems characterizing the modulation problems and the special solutions of these systems.

In the second section, nonlinear evolution equations are derived describing the asymptotic behavior of waves for $(2+1)$ dimensional wave modulation problem. While deriving the equations an asymptotic technique called reductive perturbation method is used and it is shown that the problem of wave modulation is characterized by a system of three nonlinear partial differential equations. These equations involve interactions of a free short transverse, a free long longitudinal and a free long transverse wave modes, and is called the "generalized Davey-Stewartson equations". Under some restrictions on parameter values, it is shown that the generalized Davey-Stewartson equations reduce to the well-known Davey-Stewartson and to the nonlinear Schrödinger equations. Besides, some special solutions of the generalized Davey-Stewartson equations are calculated by two different methods. While calculating the special solutions by the help of the first method, travelling wave transformations are used in order to reduce the partial differential equations to ordinary differential equations and special solutions are given in terms of Jacobian elliptic functions. It is shown that these solutions reduce to hyperbolic functions for some cases and that they involve sech-tanh-tanh and tanh-tanh-tanh type solutions under some constraints on the parameter values. The second method is based on coupled Riccati equations and their travelling wave transformations. By using the second method, the special solutions of the generalized Davey-Stewartson equations are obtained in terms of hyperbolic functions.

In the third section, it is observed that the generalized Davey-Stewartson equations are not valid for the long-wave short-wave resonant case. In the case where the phase velocity of the long longitudinal wave is equal to the group velocity of the short transverse wave, new evolution equations are derived characterizing the problem and are called long-wave short-wave interaction equations. The special solutions of the long-wave short-wave interaction equations are obtained by Jacobian elliptic functions and tanh method.

In the fourth section, degenerate generalized Davey-Stewartson equations, which are obtained when one of the coefficients of the generalized Davey-Stewartson equations becomes zero, are considered. The special solutions of the degenerate generalized Davey-Stewartson equations are obtained by using the two methods given in the second and third sections. Besides, the bilinear form of the degenerate generalized Davey-Stewartson equations are calculated and the special solutions are obtained by using a variable separation method developed for nonlinear equations.

In the fifth section, conclusions of the study are given briefly.

ELASTİK BİR ORTAMDA DALGA YAYILIMI: GENELLEŞTİRİLMİŞ DAVEY-STEWARTSON DENKLEMLERİ

ÖZET

Bu çalışmada, genelleştirilmiş elastik bir ortamda, (2+1) (iki uzay ve bir zaman) boyutlu nonlineer dalga yayılımı problemi ele alınmıştır. Sonsuz, homojen, zayıf nonlineer ve zayıf dispersif elastik bir ortamda (2+1) boyutlu dalgaların modülasyonu incelenmiştir. Çalışma beş ana bölümden oluşmaktadır.

Birinci bölümde, konuya genel bir giriş yapılmış, dalga modülasyonu problemlerini karakterize eden bazı nonlineer sistemler ve onların özel çözümleri hakkında bilgiler verilmiştir.

İkinci bölümde, $(2+1)$ boyutlu dalga modülasyonu problemi için dalgaların asimptotik davranışını tanımlayan nonlineer evolüsyon denklemleri türetilmiştir. Denklemler türetilirken indirgeyici pertürbasyon yöntemi olarak adlandırılan bir asimptotik yöntem kullanılmış ve dalgaların modülasyonu probleminin üçlü bir nonlineer kısmi diferansiyel denklem sistemi ile karakterize edildiği gösterilmiştir. Bu denklemler, bir kısa enine dalga, bir uzun enine dalga ve bir uzun boyuna dalga olmak üzere üç dalganın etkileşimlerini içermiş ve "genelleştirilmiş Davey-Stewartson denklemleri" olarak adlandırılmıştır. Parametre değerlerinin bazı kısıtları altında, genelleştirilmiş Davey-Stewartson denklemlerinin literatürde sıkça karşılaşılan nonlineer Schrödinger denklemine veya Davey-Stewartson denklemlerine indirgendiği gösterilmiştir. Ayrıca, türetilen genelleştirilmiş Davey-Stewartson denklemlerinin bazı özel çözümleri iki farklı yöntemle hesaplanmıştır. Birinci yöntem yardımı ile özel çözümler hesaplanırken gezen dalga dönüşümleri kullanılarak kısmi diferansiyel denklemler adi diferansiyel denklemlere indirgenmiş ve özel çözümler Jacobi eliptik fonksiyonları cinsinden verilmiştir. Elde edilen özel çözümlerin bazı durumlarda hiperbolik fonksiyonlara indirgendiği gösterilmiş ve parametre değerlerinde alınan kimi kısıtlar altında sech-tanh-tanh ve tanh-tanh-tanh yapılarındaki çözümleri içerdiği ifade edilmiştir. İkinci yöntem ile bir Riccati denklem çifti ve onların gezen dalga dönüşümleri ele alınmıştır. Bu teknik yardımıyla, genelleştirilmiş Davey-Stewartson denklemlerinin özel çözümleri hiperbolik fonksiyonlar cinsinden ifade edilmiştir.

Üçüncü bölümde, uzun-dalga kısa-dalga rezonans durumu için genelleştirilmiş Davey-Stewartson denklemlerinin geçerli olmadığı gözlenmiştir. Uzun boyuna dalganın faz hızının kısa enine dalganın grup hızına eşit olduğu halde problemi karakterize eden ve uzun-dalga kısa-dalga etkileşim denklemleri olarak adlandırılan yeni evolüsyon denklemleri türetilmiştir. Uzun-dalga kısa-dalga etkileşim denklemlerinin özel çözümleri Jacobi eliptik fonksiyonları ve tanh yöntemi ile elde edilmiştir.

Dördüncü bölümde, genelleştirilmiş Davey-Stewartson denklemlerinde beliren katsayılardan birinin sıfır olması durumunda elde edilen dejenere genelleştirilmiş Davey-Stewartson denklemleri ele alınmıştır. Dejenere genelleştirilmiş Davey-Stewartson denklemlerinin özel çözümleri öncelikle ikinci ve üçüncü bölümde verilen iki yöntem kullanılarak elde edilmiştir. Ayrıca, denklemlerin bilineer formu hesaplanmış ve özel çözümler lineer olmayan deklemler için geliştirilen bir değişkenlere ayırma yöntemi yardımı ile elde edilmiştir.

Beşinci bölümde, çalışmanın sonuçları kısaca verilmiştir.

1 INTRODUCTION

1.1 Introduction

The subject of nonlinear waves in various continuous media has received great attention up to present as evidenced by the large number of studies [1-3]. However, most of the studies are confined to (1+1) (one spatial and one temporal) dimensions and several (1+1) evolution equations are derived to describe the far-field behavior of the waves. It is well-known [4] that the envelope of a (1+1) dimensional quasi-monochromatic wave train is governed by a single nonlinear Schrödinger (NLS) equation

$$iA_t + pA_{xx} + q|A|^2A = 0, \tag{1.1}$$

where t is time, x is the spatial coordinate and A denotes the complex amplitude; the coefficients p and q depend on the (dimensionless) fluid depth and acceleration due to gravity. The NLS equation appears to be a generic equation describing unidirectional wave modulation. Obviously, the far field behavior of nonlinearly modulated waves cannot be described by the (1+1) dimensional NLS equation if modulations transverse to the wave propagation direction are also allowed. Therefore, the second spatial coordinate effect should be taken into account and new (2+1) evolution equations should be derived to describe (2+1) dimensional wave modulation [5, 6]. A natural way to obtain two-dimensional modulations of nonlinear waves is simply to replace the one dimensional dispersive term with a two dimensional dispersive term

$$iA_t + pA_{xx} + sA_{xy} + rA_{yy} + q|A|^2A = 0, \tag{1.2}$$

where the coefficients p, s, r and q depend on the (dimensionless) fluid depth and accelaration due to gravity. The (2+1) dimensional form of the NLS equation (1.2) correctly describes (2+1) dimensional wave motion when there is no resonance between the main quasi-harmonic wave and zero harmonics induced by nonlinear effects [7, 8]. However, in many two dimensional systems both short waves and long waves may co-exist and a proper description of two dimensional modulation of nonlinear waves should involve both short wave and long wave modes. Such a system, called the Davey-Stewartson (DS) equations, was introduced in [9-11] as follows:

$$
\begin{aligned}
&iA_t + pA_{xx} + rA_{yy} + q|A|^2A = bA\varphi_x,\\
&\varphi_{xx} + m\varphi_{yy} = (|A|^2)_x,
\end{aligned}
\tag{1.3}
$$

where A is the complex amplitude of a short wave and φ is a long wave amplitude; the coefficients p, r, q, b and m depend on the (dimensionless) fluid depth and surface tension. The DS system is a model for the evolution of weakly nonlinear packets of water waves that travel in one direction but in which the amplitude of waves is modulated in two spatial directions. The main purpose of the present study is to extend the analysis of Davey and Stewartson to describe (2+1) dimensional wave motion in a bulk elastic medium.

In recent years, various generalized theories of elasticity have been proposed to incorporate the internal structure of the matter to the classical elasticity theory [12, 13]. These theories, e.g. higher-order gradient, couple stress, micromorphic, etc., have been proposed to study both statical problems in the case of higher stress gradients and dynamical problems in the case of high frequency wave propagation in grained composites and polycrystalline structures [14-16]. Most of these theories, unlike the classical elasticity theory, give rise to dispersive wave propagation in linear approximation. Thus, inclusion of weak nonlinear effects in the constitutive equations leads to several nonlinear evolution equations which admit solitary wave or soliton solutions [17]. Increasing interest has been given to wave propagation problems in elastic rods [18], in layered elastic media [19, 20] and in bulk elastic media [14]. Various (1+1) and (2+1) dimensional nonlinear evolution equations have been derived to describe the long time behavior of elastic waves propagating in rods, plates and bulk media [12-17]. In particular, waves in elastic rods can be regarded as (1+1) dimensional objects since they are confined by the waveguides. Therefore, the behavior of a quasi-monochromatic wave is described properly by the (1+1) dimensional

NLS equation [18]. However, in a plate and in a bulk medium there exist transverse directions, and therefore the effect of higher dimensions on the wave propagation should be considered. Recently, a (2+1) dimensional NLS equation was derived for the description of the amplitude of nonlinear waves propagating in a two-dimensional Frenkel-Kontorova model [7] and in a plate [8] when no resonance is present between the slowly varying amplitude and long wave modes. There are already works on the interaction of a quasi-monochromatic wave with zero harmonics in elastic rods [18] and on elastic substrates [19] where the Davey-Stewartson type evolution equations are derived. In both of these studies, the second spatial coordinate effect is not considered. However, in a recent study [22], (2+1) dimensional nonlinear transverse wave propagation is considered in a bulk elastic medium and it is shown that the wave motion is governed by a coupled pair of the modified Kadomtsev-Petviashvili equations in the long wave approximation.

The main purpose of the present study is to extend the analysis of Davey and Stewartson and derive a generalized form of the Davey-Stewartson equations. To this aim, the problem of (2+1) dimensional wave propagation in the above mentioned bulk medium composed of an elastic material with couple stresses is considered. In order to obtain (2+1) nonlinear model equations for the description of elastic waves in the far field, the contribution of the second spatial coordinate effect on the propagation of a quasi-monochromatic wave and zero harmonic modes is determined. That is, the direction of the wave propagation is taken along the x axis while the field variables are assumed to depend also on the transverse coordinate y. Since the nonlinear interaction of the quasi-monochromatic transverse wave and the zero harmonic transverse and longitudinal waves are considered, i.e., a free short transverse, a free long longitudinal and a free long transverse wave modes are present in the problem, in this study a generalized form of the Davey-Stewartson equations is obtained as evolution equations. Using a multi-scale expansion of quasi-monochromatic wave solutions, it is shown that the modulation of waves is governed by a system of three nonlinear evolution equations. These equations involve amplitudes of a short transverse wave, a long transverse wave and a long longitudinal wave, and are called "generalized Davey-Stewartson equations" (GDS). Under some restrictions on parameter values, the GDS equations reduce to the Davey-Stewartson and to the nonlinear Schrödinger equations.

Besides, it is observed that the GDS equations are not appropriate for a resonant case, i.e. long-wave short-wave resonant case where the phase velocity of the long longitudinal wave is equal to the group velocity of the short transverse wave. For this resonant case, it is shown that the interaction between the long longitudinal and the short transverse wave is described by two coupled nonlinear evolution equations.

On the other hand, if one of the coefficients of the GDS equations becomes infinitely large for specific values of the dependent variables, the GDS system is said to be degenerate. The system may also be degenerate if one of the coefficients vanishes. For a degenerate case of the GDS equations, three coupled evolution equations which are called degenerate GDS equations are also considered.

The investigation of the exact solutions of nonlinear evolution equations also plays an important role in the study of nonlinear waves. Many effective methods such as the inverse scattering method [3,19-21], Darboux transformation [22-25], the Hirota bilinear method [26-30] and the tanh method [31-34] have been developed. Among these, the tanh method is considered to be the most effective and direct algebraic method for solving nonlinear equations. In recent years, much work has been concentrated on the various extensions and applications of the tanh method [35,36]. As an alternative method, a Jacobian elliptic function expansion method is proposed for constructing periodic wave solutions for some nonlinear evolution equations [37-39]. Another alternative method is to obtain some special solutions by means of separation of variables for the nonlinear differential equations. Recently, some variable separation procedures have been established by using symmetry constraints [44, 45] or by solving the bilinear form of the nonlinear equations [46].

In the present study, some special solutions of the GDS equations, long-wave short-wave interaction equations and the degenerate GDS equations are investigated. The special solutions of the GDS equations are given in terms of

Jacobian elliptic functions and it is shown that these solutions involve hyperbolic type solitary wave solutions. The travelling wave solutions of the GDS equations are also given by a tanh method based on the combinations of solutions of the coupled Riccati equations. The same methods, i.e. the Jacobian elliptic function method and the tanh method are used for finding the special solutions of the long-wave short-wave interaction equations. The travelling wave solutions of the degenerate GDS equations are given in terms of Jacobian elliptic functions and by using the tanh method. Besides, some special solutions of the degenerate GDS equations are established by a variable separation approach using a prior ansatz for its bilinear form.

The present study is organized as follows: In Section 2, the derivation of the DS system is first recalled briefly. The governing equations of the above mentioned bulk medium are briefly summarized and dispersion relations are presented. The general procedure of the reductive perturbation method is given and the nonlinear wave interaction of a short transverse wave, a long transverse wave and a long longitudinal wave is considered. As nonlinear evolution equations, GDS equations involving two long wave modes and a short wave envelope are derived. Some special cases of the GDS equations are examined in order to see their connection with the well-known NLS and DS equations. Later, some special solutions of the GDS system are presented by Jacobian elliptic functions and tanh method. In Section 3, the long-wave short-wave resonant case is considered. Since the GDS equations are not valid in this resonance case, new evolution equations are derived by using the same perturbation method. Some special solutions of the interaction equations are given again by Jacobian elliptic functions and tanh method. In Section 4, a degenerate case is considered and some special solutions are obtained by Jacobian elliptic functions and tanh method as well as by a variable separation approach whose main procedure is also introduced.

2 GENERALIZED DAVEY-STEWARTSON EQUATIONS

2.1 On The Davey-Stewartson System

To describe the irrotational motion of water waves, the fluid which rests on a horizontal and impermeable bed of infinite extent is assumed to be inviscid, incompressible and homogeneous [9].

The velocity potential ϕ satisfies

$$\nabla^2\phi = 0, \qquad -h < z < \zeta(x,y,t), \tag{2.1}$$

where h (may be finite or infinite) denotes the depth of the bed and $\zeta(x,y,t)$ is the free surface.

The only boundary condition on the bottom, i.e., $z = -h$ is

$$\phi_z = 0, \tag{2.2}$$

and along the free surface $z = \zeta(x,y,t)$ the boundary conditions are

$$\begin{aligned} &\zeta_t + \phi_x\zeta_x + \phi_y\zeta_y = \phi_z, \\ &g\zeta + \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = T\,\frac{\zeta_{xx}(1+{\zeta_y}^2) + \zeta_{yy}(1+{\zeta_x}^2) - 2\zeta_{xy}\zeta_x\zeta_y}{(1+{\zeta_x}^2+{\zeta_y}^2)^{3/2}}, \end{aligned} \tag{2.3}$$

where g is the gravitational acceleration, and T is the ratio of surface-tension coefficient to fluid density. The first equation of (2.3) expresses the conservation of mass while the second one expresses the conservation of energy.

The equations governing the time evolution of the free surface envelope A and the mean flow φ are derived by performing an asymptotic power series expansion in ϵ

$$\phi = \epsilon\{\varphi(\xi,\eta,\tau) + A(\xi,\eta,\tau)\exp[i(kx-\omega t)] + c.c.\} + \epsilon^2\varphi_2 + ... \tag{2.4}$$

where c.c. stands for the complex conjugate of the preceding term and introducing the slow variables

$$\xi = \epsilon(x - c_g t), \quad \eta = \epsilon y, \quad \tau = \epsilon^2 t, \tag{2.5}$$

where c_g denotes the group velocity. In (2.5), the direction of wave propagation is taken along the x axis and it is assumed that the field variables depend on the transverse coordinate y as well. By substituting the scale transformation (2.5) together with power series solutions (2.4) into the field equations the Davey-Stewartson system is obtained as evolution equations and the dimensionless form of this system can be given as follows:

$$\begin{aligned} &iA_t + \delta A_{xx} + A_{yy} = \chi |A|^2 A + bA\,\varphi_x, \\ &\varphi_{xx} + m\,\varphi_{yy} = (|A|^2)_x. \end{aligned} \tag{2.6}$$

where A is the complex amplitude of the short wave, φ is the long wave amplitude; the coefficients δ, χ, b and m depend on the (dimensionless) fluid depth and surface tension [9-11]. The Davey-Stewartson system (2.6) is one of the classical models for (2+1) dimensional description of weakly nonlinear water waves that travel predominantly in one direction but in which the wave amplitude is modulated in two spatial directions.

2.2 Basic Equations and Dispersion Relations

The derivation of classical continuum models of solids is based on macroscopic concepts, such as energy density, observable variables (displacement, strain), etc. However, solids usually have microstructure due to the existence of crystalline structure. Hence, any theory describing the behavior of a solid should consider both the macroscopic and the microscopic properties occurring at different length scales. Internal length scales may be introduced in the constitutive theory in a natural way by assuming the hypothesis of "substructure" assigned to every material point. If this substructure deforms homogeneously the continuum is called micromorphic [47]. Thus, a material point of such a medium has 12 degrees of freedom, three for deformation of the macro volume element and nine for micro deformation of the micro elements. The deformation is described by the classical motion

$$x_k = x_k(X_K, t) \quad (k, K = 1, 2, 3), \tag{2.7}$$

where x_k and X_K are the spatial and material Cartesian coordinates of the same material point at times t and $t = 0$, respectively. Then, the deformation tensors for a micromorphic elastic medium are defined as

$$\begin{aligned} 2e_{kl} &= u_{k,l} + u_{l,k} + u_{m,k}\, u_{m,l}, \\ \varepsilon_{kl} &= \Phi_{kl} + u_{l,k} + u_{m,k}\, \Phi_{ml}, \\ \Gamma_{klm} &= \Phi_{kl,m} + u_{n,k}\, \Phi_{nl,m}, \end{aligned} \tag{2.8}$$

where subscript k after comma denotes partial differentiation with respect to space variable x_k and summation convention is valid over repeated indices. Here $u_{k,l}$ is the displacement gradient, e_{kl} is the macro deformation tensor characterizing the relative displacements of the mass center of macro volume, ϵ_{kl} and Γ_{klm} are new micro deformation tensors of the micromorphic theory [47].

An elastic medium with couple stresses (also called the Le Roux continuum), which is the subject of this study, corresponds to a special case of the micromorphic elasticity theory. If the micro deformation tensor ε_{kl} in (2.8) is assumed to be zero and Φ_{kl} and its gradient are assumed to be so small that the quadratic terms can be neglected, then Φ_{kl} is approximated by

$$\Phi_{kl} \simeq -u_{l,k}. \tag{2.9}$$

In this case the macro and micro deformation tensors take the form:

$$\begin{aligned} &2e_{kl} = u_{k,l} + u_{l,k} + u_{m,k}u_{m,l}, \\ &\Gamma_{klm} = -u_{l,km}. \end{aligned} \tag{2.10}$$

In addition, the kinetic energy density function is assumed to be the identical with that of the classical elasticity theory, i.e.

$$T = \frac{1}{2}\rho_0[(u_{1,t})^2 + (u_{2,t})^2 + (u_{3,t})^2], \tag{2.11}$$

where ρ_0 is the mass density in the reference configuration and subscript t denotes partial differentiation with respect to time. The physical nonlinearity of the medium is characterized by the presence of cubic terms with respect to the macro deformation tensor in the strain energy density function, Σ,

$$\begin{aligned}
\Sigma &= \frac{\lambda}{2} e_{kk} e_{ll} + \mu e_{kl} e_{kl} + \frac{\mathcal{A}}{3} e_{kl} e_{ml} e_{km} + \mathcal{B} e_{kl} e_{lk} e_{mm} \\
&+ \frac{\mathcal{C}}{3} e_{kk} e_{ll} e_{mm} + 2\mu m^2 (\Gamma_{klm}\Gamma_{klm} + \nu \Gamma_{klm}\Gamma_{lkm}), \qquad (2.12)
\end{aligned}$$

where λ, μ are linear elastic constants, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are second-order elastic constants, and ν and m are new constants characterizing the microstructure [48].

For the problem under consideration, it suffices to take the nonlinear terms up to the third-order in $u_{k,m}$ and the second-order in $u_{k,mn}$

$$\begin{aligned}
\Sigma &= \frac{\lambda}{2} \left(u_{k,k} u_{m,m} + u_{k,k} u_{m,n} u_{m,n}\right) + \frac{\mu}{2} \left(u_{m,n} u_{m,n} + u_{m,n} u_{n,m} + u_{k,m} u_{n,m} u_{m,k}\right. \\
&\left. + u_{n,k} u_{m,k} u_{n,m}\right) + \frac{A}{12} u_{k,n} (u_{m,n} + u_{n,m})(u_{k,m} + u_{m,k}) \\
&+ \frac{B}{2} u_{m,m} u_{n,k} (u_{n,k} + u_{k,n}) + \frac{C}{3} u_{k,k} u_{m,m} u_{n,n} \\
&+ 2\mu\, m^2 \left(u_{n,km} u_{n,km} + \nu\, u_{n,km} u_{k,nm}\right) + \cdots \qquad (2.13)
\end{aligned}$$

The governing equations of (2+1) dimensional problems are obtained from the variational problem

$$\delta \int L dt = \delta \int\int\int \mathcal{L} dx dy dt = 0, \qquad (2.14)$$

where L represents the Lagrangian, and $\mathcal{L} = T - \Sigma$. As is evident from (2.12), the Lagrangian density function will be in the following form:

$$\mathcal{L} = \mathcal{L}(u_{k,t}, u_{k,x}, u_{k,y}, u_{k,xx}, u_{k,xy}, u_{k,yy}), \quad (k = 1, 2, 3).$$

For such a case, the Euler-Lagrange equations of the variational problem are obtained as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_{k,t}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_{k,x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial u_{k,y}}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial u_{k,xx}}\right) \\
&\quad - \frac{\partial^2}{\partial y^2}\left(\frac{\partial \mathcal{L}}{\partial u_{k,yy}}\right) - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial \mathcal{L}}{\partial u_{k,xy}}\right) = 0, \quad (k = 1, 2, 3). \quad (2.15)
\end{aligned}$$

For two dimensional weakly nonlinear waves propagating in a bulk elastic medium, the displacement vector $\mathbf{u}$ is assumed to be a function of x, y and

t. Consequently, it is convenient to express the strain energy density function (2.12) as a cubic polynomial of strain components

$$
\begin{aligned}
\Sigma \;=\; & \frac{\lambda}{2}\{(u_x+v_y)^2+(u_x+v_y)[(u_x)^2+(u_y)^2+(v_x)^2+(v_y)^2+(w_x)^2+(w_y)^2]\} \\
& +\frac{\mu}{2}\{2(u_x)^2+2(v_y)^2+(u_y)^2+(v_x)^2+(w_x)^2+(w_y)^2+2v_xu_y \\
& +2u_x[(u_x)^2+(v_x)^2+(w_x)^2+(u_y)^2]+2v_y[(v_y)^2+(v_x)^2+(w_y)^2+(u_y)^2] \\
& +2u_y(w_xw_y+v_xu_x+v_xv_y)+2v_xw_xw_y\}+2\mu m^2\{(u_{xx})^2+(v_{xx})^2+(w_{xx})^2 \\
& +(u_{yy})^2+(v_{yy})^2+(w_{yy})^2+2(u_{xy})^2+2(v_{xy})^2+2(w_{xy})^2+\nu[(u_{xx})^2 \\
& +(u_{xy})^2+2u_{xy}v_{xx}+2u_{yy}v_{xy}+(v_{xy})^2+(v_{yy})^2\}+\frac{\mathcal{A}}{12}[4(u_x)^3+6u_xv_xu_y \\
& +3(u_y)^2u_x+3(u_y)^2v_y+3(v_x)^2u_x+3(v_x)^2v_y+6v_yu_yv_x+4(v_y)^3 \\
& +3(w_x)^2u_x+3w_xv_xw_y+3w_yu_yw_x+3(w_y)^2v_y]+\frac{\mathcal{B}}{2}[2(u_x)^3+2(u_x)^2v_y \\
& +(u_y)^2u_x+(u_y)^2v_y+(v_x)^2u_x+(v_x)^2v_y+2(v_y)^2u_x+2(v_y)^3+(w_x)^2u_x \\
& +(w_x)^2v_y+(w_y)^2u_x+(w_y)^2v_y+2u_yv_xu_x+2u_yv_xv_y]+\frac{\mathcal{C}}{3}(u_x+v_y)^3,
\end{aligned} \tag{2.16}
$$

where subscripts x and y denote partial differentiation with respect to spatial variables, and u is the longitudinal component whereas v and w are the transverse components of the displacement vector: $u \equiv u_1$, $v \equiv u_2$, $w \equiv u_3$.

If the strain energy density function Σ given by (2.16) and the kinetic energy density function T given by (2.11) are substituted into equation (2.15), the field equations are obtained as follows:

$$
\begin{aligned}
& u_{tt}-c_1^2u_{xx}-c_2^2u_{yy}-(c_1^2-c_2^2)v_{xy}-(\gamma_1+\gamma_4)w_yw_{xy}-\gamma_2(v_yv_{xy}+v_yu_{xx} \\
& \quad +u_xv_{xy})-\gamma_3(2u_yu_{xy}+v_xv_{xx}+w_xw_{xx}+u_xu_{yy}+u_yv_{yy}+v_yu_{yy}) \\
& \quad -\gamma_4w_xw_{yy}-\gamma_5(2v_xu_{xy}+u_yv_{xx}+u_xv_{xy}+v_yv_{xy}+v_xv_{yy})-\gamma_6u_xu_{xx} \\
& \quad +4c_2^2m^2[(1+\nu)u_{xxxx}+(2+\nu)u_{xxyy}+u_{yyyy}+\nu(v_{xxxy}+v_{xyyy})]=0, \\
& v_{tt}-c_2^2v_{xx}-c_1^2v_{yy}-(c_1^2-c_2^2)u_{xy}-(\gamma_1+\gamma_4)w_xw_{xy}-\gamma_2(u_xu_{xy}+u_xv_{yy} \\
& \quad +v_yu_{xy})-\gamma_3(2v_xv_{xy}+v_xu_{xx}+u_xv_{xx}+v_yv_{xx}+u_yu_{yy}+w_yw_{yy}) \\
& \quad -\gamma_4w_yw_{xx}-\gamma_5(2u_yv_{xy}+u_yu_{xx}+u_xu_{xy}+v_yu_{xy}+v_xu_{yy})-\gamma_6v_yv_{yy} \\
& \quad +4c_2^2m^2[v_{xxxx}+(2+\nu)v_{xxyy}+(1+\nu)v_{yyyy}+\nu(u_{xxxy}+u_{xyyy})]=0,
\end{aligned}
$$

$$
\begin{aligned}
&w_{tt} - c_2^2(w_{xx} + w_{yy}) - \gamma_1(u_x w_{yy} + w_y u_{xy} + w_x v_{xy} + v_y w_{xx}) \\
&\quad - \gamma_3(w_x u_{xx} + u_x w_{xx} + v_y w_{yy} + w_y v_{yy}) - \gamma_4(2u_y w_{xy} + 2v_x w_{xy} + w_y u_{xy} \\
&\quad + w_y v_{xx} + w_x u_{yy} + w_x v_{xy}) + 4c_2^2 m^2(w_{xxxx} + 2w_{xxyy} + w_{yyyy}) = 0.
\end{aligned} \tag{2.17}
$$

In equation (2.17) the coefficients are defined as

$$
\begin{aligned}
&c_1^2 = (\lambda + 2\mu)/\rho_0, \quad c_2^2 = \mu/\rho_0, \\
&\gamma_1 = c_1^2 - 2c_2^2 + \mathcal{B}/\rho_0, \quad \gamma_2 = c_1^2 - 2c_2^2 + (2\mathcal{B} + 2\mathcal{C})/\rho_0, \\
&\gamma_3 = c_1^2 + (\mathcal{A} + 2\mathcal{B})/2\rho_0, \quad \gamma_4 = c_2^2 + \mathcal{A}/4\rho_0, \\
&\gamma_5 = c_2^2 + (\mathcal{A} + 2\mathcal{B})/2\rho_0, \quad \gamma_6 = 3c_1^2 + (2\mathcal{A} + 6\mathcal{B} + 2\mathcal{C})/\rho_0,
\end{aligned} \tag{2.18}
$$

where c_1 and c_2 are the velocities of longitudinal and transverse waves, respectively. In order to see the dispersive character of the model, the linearized form of the field equations (2.17) are obtained as follows:

$$
\begin{aligned}
&u_{tt} - c_1^2 u_{xx} - c_2^2 u_{yy} - (c_1^2 - c_2^2) v_{xy} \\
&\quad + 4c_2^2 m^2[(1+\nu)u_{xxxx} + (2+\nu)u_{xxyy} + u_{yyyy} + \nu(v_{xxxy} + v_{xyyy})] = 0, \\
&v_{tt} - c_2^2 v_{xx} - c_1^2 v_{yy} - (c_1^2 - c_2^2) u_{xy} \\
&\quad + 4c_2^2 m^2[v_{xxxx} + (2+\nu)v_{xxyy} + (1+\nu)v_{yyyy} + \nu(u_{xxxy} + u_{xyyy})] = 0, \\
&w_{tt} - c_2^2(w_{xx} + w_{yy}) + 4c_2^2 m^2(w_{xxxx} + 2w_{xxyy} + w_{yyyy}) = 0.
\end{aligned} \tag{2.19}
$$

If the following harmonic wave solutions are assumed for equation (2.19):

$$
\mathbf{u} = \mathbf{A}\exp[i(k_1 x + k_2 y - \omega t)], \tag{2.20}
$$

where $\mathbf{A}$ denotes the complex amplitude vector, ω is the frequency, k_1 and k_2 are the wave numbers in the x and y directions, respectively; linear homogeneous equations are obtained as

$$
\begin{aligned}
&[-\omega^2 + c_1^2 k_1^2 + c_2^2 k_2^2 + 4c_2^2 m^2(\kappa^4 + \nu\kappa^2 k_1^2)]A_1 \\
&\qquad\qquad + k_1 k_2(c_1^2 - c_2^2 + 4c_2^2 m^2 \nu\kappa^2)A_2 = 0, \\
&[-\omega^2 + c_2^2 k_1^2 + c_1^2 k_2^2 + 4c_2^2 m^2(\kappa^4 + \nu\kappa^2 k_2^2)]A_2 \\
&\qquad\qquad + k_1 k_2(c_1^2 - c_2^2 + 4c_2^2 m^2 \nu\kappa^2)A_1 = 0, \\
&(-\omega^2 + c_2^2 k^2 + 4c_2^2 m^2 k^4)A_3 = 0,
\end{aligned} \tag{2.21}
$$

where $k^2 = k_1^2 + k_2^2$. The last equation of (2.21) gives the dispersion relation corresponding to the transverse displacement mode associated with w. It should be noted that the first two homogeneous linear equations of (2.21) have nontrivial solutions only if the determinant of coefficients of the system is equal to zero, i.e.,

$$\begin{aligned}\omega^4 - \omega^2[c_1^2k^2 + c_2^2k^2 + 4c_2^2m^2(2+\nu)k^4]\\ + c_1^2c_2^2k^4 + 4c_2^2m^2k^6[c_1^2 + (1+\nu)c_2^2] + 16c_2^2m^4(1+\nu)k^8 = 0 \quad (2.22)\end{aligned}$$

which can be written as a product of two factors

$$[\omega^2 - c_1^2k^2 - 4c_2^2m^2(1+\nu)k^4](\omega^2 - c_2^2k^2 - 4c_2^2m^2k^4) = 0. \quad (2.23)$$

The first factor of (2.23) indicates the dispersion relation of the longitudinal displacement mode associated with u, while the second one represents the dispersion relation corresponding to the transverse displacement mode associated with v. Hence, the dispersion relations can be written as follows:

$$\begin{aligned}D_1(k,\omega) &= \omega^2 - c_1^2k^2 - 4(1+\nu)c_2^2m^2k^4,\\ D_2(k,\omega) = D_3(k,\omega) &= \omega^2 - c_2^2k^2 - 4c_2^2m^2k^4. \quad (2.24)\end{aligned}$$

In equation (2.24), D_1 denotes the dispersion relation corresponding to u. Also, D_2 and D_3 denote the dispersion relations corresponding to v and w, respectively. As is seen from equations (2.24), both the longitudinal and the transverse modes are dispersive. Thus, it is possible to describe wave motion in bulk elastic media by the well-known evolution equations if the physical dispersion (due to inner structure) is balanced by the material nonlinearity.

2.3 Reductive Perturbation Method

Many physical systems involving nonlinear wave propagation include the effects of dispersion, dissipation or the inhomogeneous property of the medium. In general, the physical processes involved are so complex that the governing equations are very complicated and are not integrable by analytic methods. So, special attention is given to seeking mathematical methods which lead to a less complicated problem retaining all of the important physical features, and hence, several asymptotic methods have been developed. Among these, the reductive perturbation method [45-48] is a systematic method for derivation of the so-called evolution equations that describe how some dynamical variables evolve in time and space.

It is well-known that if the amplitude of a wave is small enough many nonlinear systems admit harmonic wave-train solutions in which the amplitude remains constant in time and the nonlinear terms are small enough to be neglected. If the amplitude of the wave is not small enough, nonlinear terms cannot be neglected and a variation in the amplitude occurs in both space and time. While the amplitude varies slowly over the period of the oscillation, a stretching transformation allows one to separate the system into a rapidly varying part associated with the oscillation and a slowly varying part associated with the amplitude. Then, a formal solution can be given in the form of an asymptotic expansion and an evolution equation determining the modulation of the first-order amplitude can be derived.

The main purpose of this subsection is to investigate how the amplitude of a harmonic wave is modulated by nonlinear effects in a weakly nonlinear system. In order to use the reductive perturbation method, first of all the appropriate coordinate stretching of the independent variables for the two-dimensional wave modulation problem will be determined. To this aim, two sinusoidal waves propagating in a two dimensional linear medium will be taken whose amplitudes are the same and wave numbers are very close to each other

$$\begin{aligned} u_1(x,y,t) &= A\exp\{i[(k_1+\Delta k_1)x+\Delta k_2 y-(\omega+\Delta\omega)t]\}, \\ u_2(x,y,t) &= A\exp\{i[(k_1-\Delta k_1)x-\Delta k_2 y-(\omega-\Delta\omega)t]\}. \end{aligned} \tag{2.25}$$

By superposition of the harmonic waves (2.25),

$$u(x,y,t) = u_1(x,y,t) + u_2(x,y,t), \tag{2.26}$$

the following equation is obtained:

$$u(x,y,t) = 2A\cos(\ \Delta k_1 x + \Delta k_2 y - \Delta\omega t)\ \mathrm{e}^{i(k_1 x-\omega t)}. \tag{2.27}$$

The superposition of two quasi-monochromatic waves in a linear medium separate the harmonic waves into two different parts, i.e., a slow varying part with the function $\cos(\ \Delta k_1 x + \Delta k_2 y - \Delta\omega t)$ and a fast varying part with the function $\exp[i(k_1 x - \omega t)]$. Thus, the amplitude is modulated.

On the other hand, by assuming

$$\Delta k_1/k_1 << 1,\ \ \Delta\omega/\omega << 1,$$

$\Delta\omega$, the variation in the frequency, can be written as follows:

$$\Delta\omega = \omega(k_1 + \Delta k_1, k_2 + \Delta k_2) - \omega(k_1, k_2). \tag{2.28}$$

Since the wave propagation is assumed to be only in the x direction, it is convenient to consider k_2 as zero. Besides, the effects on the second spatial direction y will be examined with Δk_2. Equation (2.28) can be written as

$$\Delta\omega = \frac{d\omega}{dk_1}\Delta k_1 + \frac{1}{2}\frac{d^2\omega}{dk_1^2}(\Delta k_1)^2 + ...$$

Hence, the phase function of the amplitude read as

$$\Delta k_1 x + \Delta k_2 y - \Delta\omega t = \Delta k_1(x - \frac{d\omega}{dk_1}t) + \Delta k_2 y - \frac{1}{2}\frac{d^2\omega}{dk_1^2}(\Delta k_1)^2 t. \tag{2.29}$$

At this step, it will be assumed that $\Delta k = \epsilon^p$ and to consider the modulation problem the slow variables will be introduced as

$$\xi = \epsilon^p(x - c_g t), \quad \eta = \epsilon^p y, \quad \tau = \epsilon^{2p} t, \tag{2.30}$$

where ϵ is a small parameter, p is a positive number which will be chosen later, and $c_g = d\omega/dk$ denotes the group velocity of the wave propagating along the direction of the x axis.

In linear systems the parameter ϵ appears to measure the weakness of dispersion while in weakly nonlinear systems it is taken into account to measure the weakness of nonlinearity. For this purpose, the dependent variables are assumed to be written in terms of an asymptotic series as follows:

$$u = \epsilon[u_1(\xi,\eta,\tau)\mathrm{e}^{i(k_1 x - \omega t)} + c.c.] + \epsilon^2[u_0(\xi,\eta,\tau) + u_2(\xi,\eta,\tau)]\mathrm{e}^{i(k_1 x - \omega t)} + c.c.] + ... \tag{2.31}$$

By substituting the stretched coordinates (2.30) and the solution functions (2.31) into the field equations, a hierarchy of equations in powers of the parameter ϵ is obtained. If the coefficients of like powers are equated the perturbation equations are found. Thus, the evolution equations characterizing the long time behavior of the original system are obtained where the effects of dispersion and nonlinearity are balanced.

2.4 Two Dimensional Wave Packets

In the present subsection, nonlinear wave interaction of a short transverse wave, a long transverse wave and a long longitudinal wave is considered. To find the evolution equations that characterize the interaction, the reductive perturbation method mentioned above is used.

The slowly varying amplitude of the short wave and amplitudes of the long waves are assumed to be functions of the slow variables

$$\xi = \epsilon(x - c_g t), \quad \eta = \epsilon y, \quad \tau = \epsilon^2 t, \tag{2.32}$$

where ϵ is a small parameter measuring the weakness of dispersion and nonlinearity, c_g is the group speed of the short transverse wave, and x, y and t are fast space and time variables; ξ, η and τ are slow space and time coordinates in a frame of reference moving with the group speed of the short transverse wave. Here the direction of the wave propagation is taken along the x axis and it is assumed that the field variables depend on the transverse coordinate y as well. Since we deal with the nonlinear interaction of the quasi-harmonic transverse wave and the zero harmonic transverse and longitudinal waves, it is convenient to expand the components of the displacement vector in asymptotic power series of ϵ as follows:

$$\begin{aligned} u &= \epsilon\phi_1(\xi,\eta,\tau) + \epsilon^2[u_2(\xi,\eta,\tau)\mathrm{e}^{2i\theta} + \text{c.c.}] + \cdots \\ v &= \epsilon\phi_2(\xi,\eta,\tau) + \epsilon^2[v_2(\xi,\eta,\tau)\mathrm{e}^{2i\theta} + \text{c.c.}] + \cdots \\ w &= \epsilon[A(\xi,\eta,\tau)\mathrm{e}^{i\theta} + \text{c.c.}] + \epsilon^2[w_2(\xi,\eta,\tau)\mathrm{e}^{2i\theta} + \text{c.c.}] + \cdots \end{aligned} \tag{2.33}$$

Here $\theta = kx - \omega t$ is the phase, k is the wave number of the carrier wave and ω is the frequency. In expansion (2.33), A is the complex amplitude of the free short transverse wave mode whereas ϕ_1 and ϕ_2 are the free long longitudinal and free long transverse wave modes, respectively. The main purpose of this subsection is to find evolution equations that describe the interaction between the long and short wave modes, and to determine the effect of second spatial coordinate on the wave propagation.

Now, the scale transformation (2.32) together with power series solutions (2.33) are substituted into the field equations (2.17). If the coefficients of like powers of ϵ are equated, a hierarchy of perturbation equations is obtained. For $\mathcal{O}(\epsilon)$, the only equation is

$$\mathrm{e}^{i\theta}: \qquad D_3(k,\omega)A = 0. \tag{2.34}$$

Assuming $D_3(k,\omega) = 0$ in equation (2.34), the first-order transverse wave amplitude, A, remains arbitrary. In order to eliminate self-resonance, it is assumed that $D_3(nk,n\omega) \neq 0$ for $n \geq 2$.

Similarly, for order ϵ^2, setting the coefficient of each mode equal to zero gives

$$\begin{aligned}
\mathrm{e}^{i\theta}: &\qquad (c_g\omega - c_2^2 k - 8c_2^2 m^2 k^3)A_\xi = 0,\\
\mathrm{e}^{2i\theta}: &\qquad D_1(2k,2\omega)u_2 = ik^3\gamma_3 A^2,\\
\mathrm{e}^{2i\theta}: &\qquad D_2(2k,2\omega)v_2 = 0,\\
\mathrm{e}^{2i\theta}: &\qquad D_3(2k,2\omega)w_2 = 0.
\end{aligned} \tag{2.35}$$

Since c_g is the group velocity of the transverse wave, the coefficient of A_ξ in equation $(2.35)_1$ is identically zero. Recalling that $D_2(2k,2\omega) \neq 0$ and $D_3(2k,2\omega) \neq 0$, we obtain $v_2 = 0$ and $w_2 = 0$. In addition, the second-order longitudinal component is expressed in terms of the first-order complex amplitude of the short transverse wave

$$u_2 = \frac{ik^3\gamma_3}{D_1(2k,2\omega)}A^2. \tag{2.36}$$

For order ϵ^3, using the results of lower order perturbation equations, the following equations are obtained:

$$\begin{aligned}
&iA_\tau + pA_{\xi\xi} + rA_{\eta\eta} = q|A|^2A + \frac{k^2}{2\omega}(\gamma_3\phi_{1,\xi} + \gamma_1\phi_{2,\eta})A,\\
&(c_g^2 - c_1^2)\phi_{1,\xi\xi} - c_2^2\phi_{1,\eta\eta} - (c_1^2 - c_2^2)\phi_{2,\xi\eta} = \gamma_3 k^2(|A|^2)_\xi,\\
&(c_g^2 - c_2^2)\phi_{2,\xi\xi} - c_1^2\phi_{2,\eta\eta} - (c_1^2 - c_2^2)\phi_{1,\xi\eta} = \gamma_1 k^2(|A|^2)_\eta,
\end{aligned} \tag{2.37}$$

where the coefficients are defined as

$$p = -\frac{1}{2\omega}(c_g^2 - c_2^2 - 24m^2c_2^2k^2), \quad r = \frac{c_2^2}{2\omega}(1 + 8m^2k^2), \quad q = \frac{k^6\gamma_3^2}{\omega D_1(2k,2\omega)}. \tag{2.38}$$

Hence, the coupled system (2.37) govern (2+1) dimensional weakly nonlinear waves in a generalized elastic solid, that travel mostly in the x direction and whose amplitudes are slowly modulating in both x and y directions [53]. Since the nonlinear interaction of the quasi-monochromatic transverse wave and the zero harmonic transverse and longitudinal waves is considered, i.e., a free short transverse, a free long longitudinal and a free long transverse wave modes are included, the evolution equations (2.37) present a generalized form of the Davey-Stewartson equations (1.3).

It is more convenient to express the GDS equations (2.37) in a dimensionless form. Thus, introducing the dimensionless variables as

$$\tau = \frac{\gamma_1^2(c_g^2 - c_1^2)^2}{\gamma_3^4 k^4 r}\, t, \quad \xi = \frac{c_g^2 - c_1^2}{\gamma_3 k^2}\, x, \quad \eta = \frac{\gamma_1(c_g^2 - c_1^2)}{\gamma_3^2 k^2}\, y,$$

the three coupled evolution equations take the form

$$\begin{aligned}
&iA_t + \delta A_{xx} + A_{yy} = \chi |A|^2 A + b(\phi_{1,x} + \phi_{2,y})A, \\
&\phi_{1,xx} + m_2\phi_{1,yy} + n\phi_{2,xy} = (|A|^2)_x, \\
&\lambda\phi_{2,xx} + m_1\phi_{2,yy} + n\phi_{1,xy} = (|A|^2)_y,
\end{aligned} \tag{2.39}$$

where $(\lambda - 1)(m_2 - m_1) = n^2$ and the non-dimensional coefficients are given in the form

$$\begin{aligned}
&\delta = \frac{p}{r}(\frac{\gamma_1}{\gamma_3})^2, \quad \chi = \frac{q\gamma_1^2(c_g^2 - c_1^2)^2}{r\gamma_3^4 k^4}, \quad b = \frac{c_g^2 - c_1^2}{2\omega r}(\frac{\gamma_1}{\gamma_3})^2, \quad m_1 = \frac{c_1^2}{c_1^2 - c_g^2}(\frac{\gamma_3}{\gamma_1})^2, \\
&m_2 = \frac{c_2^2}{c_1^2 - c_g^2}(\frac{\gamma_3}{\gamma_1})^2, \quad n = \frac{\gamma_3(c_1^2 - c_2^2)}{\gamma_1(c_1^2 - c_g^2)}, \quad \lambda = \frac{c_2^2 - c_g^2}{c_1^2 - c_g^2}.
\end{aligned} \tag{2.40}$$

Here, in order to make the further calculations simpler, it will be assumed that $\delta = 1$.

2.5 Special Cases of the GDS Equations

In this section, it will be shown that the generalized Davey-Stewartson equations reduce to the nonlinear Schrödinger and to the Davey-Stewartson equations, under some restrictions on the parameter values.

If the y dependence is dropped from system (2.43) and $Q_2 = 0$, then the evolution equations (2.43) reduce to the single NLS equation

$$iA_t + A_{xx} = \tilde{\chi}|A|^2 A, \tag{2.41}$$

where

$$\tilde{\chi} = \chi + b. \tag{2.42}$$

If the following dependent variables for the long wave components are introduced

$$Q_1 = \phi_{1,x}, \qquad Q_2 = \phi_{2,y},$$

the evolution equations (2.37) can be rewritten to eliminate the mixed derivatives of the long wave components:

$$\begin{aligned}
&iA_t + A_{xx} + A_{yy} = \chi|A|^2 A + b(Q_1 + Q_2)A,\\
&Q_{1,xx} + m_2 Q_{1,yy} + nQ_{2,xx} = (|A|^2)_{xx},\\
&\lambda Q_{2,xx} + m_1 Q_{2,yy} + nQ_{1,yy} = (|A|^2)_{yy}.
\end{aligned} \tag{2.43}$$

For some special cases, system (2.39) reduce to the DS and to the NLS equations. If we introduce a new dependent variable as

$$Q = Q_1 + Q_2 - \frac{1}{m_1}|A|^2,$$

and assume that $n = m_1 - m_2 = 1 - \lambda$ (2.39) reduce to the well-known DS system

$$\begin{aligned}
&iA_t + A_{xx} + A_{yy} = \bar{\chi}|A|^2 A + bQ_x A,\\
&Q_{xx} + m_1 Q_{yy} = \bar{m}(|A|^2)_x,
\end{aligned} \tag{2.44}$$

where

$$\bar{\chi} = \chi + \frac{b}{m_1}, \qquad \bar{m} = 1 - \frac{1}{m_1}. \tag{2.45}$$

Therefore, system (2.39) may be called “generalized Davey-Stewartson” (GDS) equations since they have a more general form than the classical DS system.

2.6 Special Solutions of the GDS System

2.6.1 Using Jacobian elliptic functions

In this subsection, some special solutions of GDS equations (2.39) will be given in terms of Jacobian elliptic functions. These solutions involve hyperbolic secant functions and hyperbolic tangent functions as special cases.

In order to find the travelling wave solutions of GDS equations (2.39), the following solutions are considered:

$$A = f(\zeta)e^{i\theta}, \quad \phi_1 = g(\zeta), \quad \phi_2 = h(\zeta), \tag{2.46}$$

where $\theta = l_1x + l_2y - l_3t$, and the amplitudes of the short wave f, and the long travelling waves g and h are real functions of $\zeta = k_1x+k_2y-k_3t$. Here we assume that all the derivatives of f, g and h tend to zero as $\zeta \to \pm\infty$. Substituting the solution functions given by (2.46) into the system (2.39) leads to a set of coupled ordinary differential equations

$$\begin{aligned}
&(2k_1l_1 + 2k_2l_2 - k_3)f^{'} = 0,\\
&(k_1^2 + k_2^2)f^{''} - (l_1^2 + l_2^2 - l_3)f = \chi f^3 + b(k_1g^{'} + k_2h^{'})f,\\
&(k_1^2 + m_2k_2^2)g^{''} + nk_1k_2h^{''} = 2k_1ff^{'},\\
&nk_1k_2g^{''} + (\lambda k_1^2 + m_1k_2^2)h^{''} = 2k_2ff^{'},
\end{aligned} \tag{2.47}$$

where $'$ denotes differentiation with respect to ζ. From equation $(2.47)_1$, we obtain a compatibility condition as $k_3 = 2(k_1l_1 + k_2l_2)$ where $f \neq 0$. Integrating equations $(2.47)_3$ and $(2.47)_4$ with respect to ζ once gives

$$\begin{aligned}
&(k_1^2 + m_2k_2^2)g^{'} + nk_1k_2h^{'} - k_1f^2 = C_1,\\
&nk_1k_2g^{'} + (\lambda k_1^2 + m_1k_2^2)h^{'} - k_2f^2 = C_2,
\end{aligned} \tag{2.48}$$

where C_1 and C_2 are integration constants. Solving the algebraic equations (2.48) for $g^{'}$ and $h^{'}$ in terms of f^2 gives

$$g^{'} = d_1f^2 + d_2, \qquad h^{'} = d_3f^2 + d_4, \tag{2.49}$$

where

$$\begin{aligned}
&d_1 = \frac{k_1}{\Delta}(\lambda k_1^2 + m_1k_2^2 - nk_2^2), \quad d_2 = \frac{1}{\Delta}[C_1(\lambda k_1^2 + m_1k_2^2) - C_2nk_1k_2],\\
&d_3 = \frac{k_2}{\Delta}(k_1^2 + m_2k_2^2 - nk_1^2), \quad d_4 = \frac{1}{\Delta}[C_2(k_1^2 + m_2k_2^2) - C_1nk_1k_2],\\
&\Delta = (k_1^2 + m_2k_2^2)(\lambda k_1^2 + m_1k_2^2) - n^2k_1^2k_2^2.
\end{aligned}$$

If $f \to f_-$ as $\zeta \to -\infty$ and $f \to f_+$ as $\zeta \to \infty$, we obtain $f_-^2 = f_+^2$ from equations (2.48), and consequently the integration constants are $C_1 = -k_1f_-^2$

and $C_2 = -k_2 f_-^2$. Then, the coefficients d_2 and d_4 in equations (2.49) are $d_2 = -d_1 f_-^2$ and $d_4 = -d_3 f_-^2$. Therefore, the solutions associated with the long wave components are expressed in terms of the short wave amplitude f

$$g^{'} = -d_1(f_-^2 - f^2), \qquad h^{'} = -d_3(f_-^2 - f^2). \tag{2.50}$$

In such a case, the second equation of (2.47) takes the form

$$f^{''} = 2\alpha f^3 + \beta f, \tag{2.51}$$

where the coefficients α and β are

$$\begin{aligned}\alpha &= \frac{1}{2(k_1^2 + k_2^2)}[\chi + b(k_1 d_1 + k_2 d_3)],\\ \beta &= \frac{1}{k_1^2 + k_2^2}[l_1^2 + l_2^2 - l_3 - b f_-^2(k_1 d_1 + k_2 d_3)].\end{aligned} \tag{2.52}$$

By multiplying (2.51) with $f^{'}$, integrating with respect to ζ once and introducing a new dependent variable $z = f^2$, (z real), the following equation is obtained

$$(z^{'})^2 = 4z(\alpha z^2 + \beta z + C) \equiv \Phi(z), \tag{2.53}$$

where C is an integration constant. The solutions of this nonlinear ordinary differential equation can be found for $\Phi \geq 0$.

First type of solutions:

Let $\alpha < 0$, $\beta > 0$ and $\Phi(z) = -4\alpha z(z - z_1)(z_2 - z)$, where $0 < z_1 \leq z < z_2$ and

$$z_1 = \frac{1}{2\alpha}(\sqrt{\beta^2 - 4\,\alpha\,C} - \beta), \quad z_2 = \frac{1}{2\alpha}(\sqrt{\beta^2 - 4\,\alpha\,C} + \beta). \tag{2.54}$$

Integrating equation (2.53) the solution for the envelope function $f(\zeta)$ is obtained in terms of Jacobian elliptic function:

$$f(\zeta) = \{z_2 - (z_2 - z_1)\mathrm{sn}^2[(-\alpha z_2)^{\frac{1}{2}}\zeta + D_1; k]\}^{\frac{1}{2}}, \tag{2.55}$$

where $k^2 = (z_2 - z_1)/z_2$ and D_1 is an integration constant. Using (2.50), and the fact that $f_-^2 = z_1$, the solutions associated with the long wave components are

$$\begin{aligned}g(\zeta) &= \frac{d_1(z_2 - z_1)}{(-\alpha z_2)^{\frac{1}{2}} k^2}[E(u) - (1 - k^2)u],\\ h(\zeta) &= \frac{d_3(z_2 - z_1)}{(-\alpha z_2)^{\frac{1}{2}} k^2}[E(u) - (1 - k^2)u],\end{aligned} \tag{2.56}$$

where $u = (-\alpha z_2)^{\frac{1}{2}}\zeta + D_1$, and $E(u)$ is the elliptic integral of the second kind [54].

In the limit $k = 1$; since $z_1 = f_- = 0$ and $z_2 = -\beta/\alpha$, solutions (2.55) and (2.56) reduce to the following solitary wave solutions in terms of hyperbolic secant function and hyperbolic tangent functions

$$\begin{aligned} f(\zeta) &= \pm(-\alpha)^{-1/2}\ \bar{\beta}\ \mathrm{sech}[\bar{\beta}\ \zeta + D_1], \\ g(\zeta) &= -\frac{d_1}{\alpha}\ \bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_1], \\ h(\zeta) &= -\frac{d_3}{\alpha}\ \bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_1], \end{aligned} \tag{2.57}$$

where

$$\bar{\beta} = (\frac{l_1^2 + l_2^2 - l_3}{k_1^2 + k_2^2})^{1/2} \tag{2.58}$$

and the solutions of the GDS equations are obtained as follows [53]:

$$\begin{aligned} A &= \pm(-\alpha)^{-1/2}\ \bar{\beta}\ \mathrm{sech}[\bar{\beta}\ \zeta + D_1]\ \mathrm{e}^{i\theta}, \\ \phi_1 &= -\frac{d_1}{\alpha}\ \bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_1], \\ \phi_2 &= -\frac{d_3}{\alpha}\ \bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_1]. \end{aligned} \tag{2.59}$$

Second type of solutions:

Let $\alpha > 0$, $\beta < 0$ and $\Phi(z) = 4\alpha z(z_1 - z)(z_2 - z)$, where $0 < z \leq z_1 < z_2$ and

$$z_1 = \frac{1}{2\alpha}\,(\sqrt{\beta^2 - 4\,\alpha\,C}\ - \beta), \quad z_2 = \frac{1}{2\alpha}\,(\sqrt{\beta^2 - 4\,\alpha\,C}\ + \beta). \tag{2.60}$$

Integrating equation (2.53) the solution for the envelope function $f(\zeta)$ is obtained in terms of Jacobian elliptic function:

$$f(\zeta) = z_1^{1/2}\ \mathrm{sn}[(\alpha z_2)^{1/2}\zeta + D_2; k], \tag{2.61}$$

where $k^2 = z_1/z_2$ and D_2 is an integration constant. Using (2.50), and the fact that $f_-^2 = z_1$, the solutions associated with the long wave components are

$$\begin{aligned} g(\zeta) &= \frac{-d_1 z_1}{(\alpha z_2)^{\frac{1}{2}} k^2}[E(u) - (1 - k^2)u], \\ h(\zeta) &= \frac{-d_3 z_1}{(\alpha z_2)^{\frac{1}{2}} k^2}[E(u) - (1 - k^2)u], \end{aligned} \tag{2.62}$$

where $u = (\alpha z_2)^{\frac{1}{2}}\zeta + D_2$.

In the limit $k = 1$; $z_1 = z_2 = -\beta/(2\alpha)$. The frequency must be chosen as $l_3 = l_1^2 + l_2^2 + \chi f_-^2$ which is evident from equations (2.51) and (2.52). Then, solutions (2.61) and (2.62) reduce to the following solitary wave solutions in terms of hyperbolic tangent functions

$$
\begin{aligned}
f(\zeta) &= \pm\,\alpha^{-1/2}\ \bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_2],\\
g(\zeta) &= -\frac{d_1}{\alpha}\,\bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_2],\\
h(\zeta) &= -\frac{d_3}{\alpha}\,\bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_2],
\end{aligned}
\tag{2.63}
$$

where

$$
\bar{\beta} = \{\frac{1}{2(k_1^2 + k_2^2)}[\chi + b(k_1 d_1 + k_2 d_3)]f_-^2\}^{\frac{1}{2}}, \tag{2.64}
$$

and the solutions of the GDS equations are obtained as follows [53]:

$$
\begin{aligned}
A &= \pm\ \alpha^{-1/2}\ \bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_2]\ \mathrm{e}^{i\theta},\\
\phi_1 &= -\frac{d_1}{\alpha}\,\bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_2],\\
\phi_2 &= -\frac{d_3}{\alpha}\,\bar{\beta}\ \tanh[\bar{\beta}\ \zeta + D_2],
\end{aligned}
\tag{2.65}
$$

This subsection will be concluded by summarizing the properties for the travelling wave solutions of GDS system (2.39). While the sech type envelope solitary wave for A can propagate for negative values of α and positive values of β, the tanh type envelope solitary wave can propagate for positive values of α and negative values of β.

2.6.2 By the tanh method

In this subsection, special solutions of GDS equations (2.39) will be given in the form of travelling waves by following the procedure given in [39]. According to this method, the solutions are expressed as polynomials in terms of tanh or/and sech functions and their combinations in which the degree of the polynomial is determined by balancing the highest nonlinear term and the highest order partial derivative term. The method given in [39], transforms the nonlinear partial differential equations

$$
P_i(u_j, u_{j,t}, u_{j,x}, u_{j,y}, ...) = 0, \quad i, j = 1, 2, 3, \tag{2.66}
$$

where $u_1 = A(x,y,t)$, $u_2 = \phi_1(x,y,t)$ and $u_3 = \phi_2(x,y,t)$; into nonlinear ordinary differential equations through a travelling wave transformation $\zeta = k_1x+k_2y+k_3t$ as

$$P_i(f,g,h,f^{'},g^{'},h^{'},...) = 0, \qquad i = 1,2,3, \tag{2.67}$$

where $A = f(\zeta)$, $\phi_1 = g(\zeta)$, $\phi_2 = h(\zeta)$ and where prime denotes differentiation with respect to ζ. Then, two types of solution functions of the nonlinear differential equations, that is,

$$U(\zeta) = \pm\text{sech}(k\zeta), \qquad V(\zeta) = \tanh(k\zeta) \tag{2.68}$$

are introduced as solutions of the coupled Riccati equations

$$U^{'} = -kUV, \qquad V^{'} = k(1-V^2). \tag{2.69}$$

Solutions of the GDS equations (2.39) is expressed in the form

$$\begin{aligned} f(\zeta) &= \sum_{i=0}^{p} a_i U^i + \sum_{i=1}^{p} \bar{a}_i V U^{i-1}, \\ g(\zeta) &= \sum_{i=0}^{r} b_i U^i + \sum_{i=1}^{r} \bar{b}_i V U^{i-1}, \\ h(\zeta) &= \sum_{i=0}^{s} c_i U^i + \sum_{i=1}^{s} \bar{c}_i V U^{i-1}, \end{aligned} \tag{2.70}$$

where $a_i, b_i, c_i, \bar{a}_i, \bar{b}_i$ and $\bar{c}_i$ are constants. The values of p, r and s are the polynomial degrees of the solution in f, g and h respectively and will be determined by balancing the highest nonlinear term with the highest derivative term.

In order to find the travelling wave solutions of GDS equations (2.39), the following solutions are considered:

$$A = f(\zeta)\,e^{i\theta}, \quad \phi_1 = g(\zeta), \quad \phi_2 = h(\zeta), \tag{2.71}$$

where $\theta = l_1x + l_2y - l_3t$, and the amplitudes of the short wave f, and the long travelling waves g and h are real functions of $\zeta = k_1x + k_2y - 2(k_1l_1 + k_2l_2)t$.

Substituting solutions (2.71) into the system (2.39) leads to a set of coupled ordinary differential equations

$$\begin{aligned} & f^{''} - \alpha_1 f + \alpha_2 f^3 + \beta_1 g^{'} f + \beta_2 h^{'} f = 0, \\ & g^{''} + \gamma_1 h^{''} + \gamma_2 f f^{'} = 0, \\ & g^{''} + \gamma_3 h^{''} + \gamma_4 f f^{'} = 0, \end{aligned} \tag{2.72}$$

with

$$\alpha_1 = \frac{l_1^2 + l_2^2 - l_3}{k_1^2 + k_2^2}, \quad \alpha_2 = \frac{-\chi}{k_1^2 + k_2^2}, \quad \beta_1 = \frac{-bk_1}{k_1^2 + k_2^2}, \quad \beta_2 = \frac{-bk_2}{k_1^2 + k_2^2},$$
$$\gamma_1 = \frac{nk_1k_2}{k_1^2 + m_2k_2^2}, \ \gamma_2 = \frac{-2k_1}{k_1^2 + m_2k_2^2}, \ \gamma_3 = \frac{\lambda k_1^2 + m_1k_2^2}{nk_1k_2}, \ \gamma_4 = \frac{-2}{nk_1}. \tag{2.73}$$

Here, it should be noted that the form of the equations (2.72) is the same as (2.47) in Section 2.6.1.

At this step, by substituting solutions (2.70) into the system (2.72) and by using (2.69) together with the relation $V^2 = 1-U^2$ repeatedly, the term f'' is balanced by the term f^3 while g'' and h'' are balanced with the term ff'. Thus, the positive integers p, r and s are found as $p = r = s = 1$. Now, solutions (2.70) can be written as follows:

$$f(\zeta) = a_0 + a_1U + \bar{a}_1V, \quad g(\zeta) = b_0 + b_1U + \bar{b}_1V, \quad h(\zeta) = c_0 + c_1U + \bar{c}_1V. \tag{2.74}$$

Substituting (2.74) into (2.72) and equating the coefficient of each power of the variables U and V leads to the following nonlinear system of algebraic equations:

$$\begin{aligned}
&U^0V^0: \quad a_0[\alpha_1 - \alpha_2(a_0^2 + 3\bar{a}_1^2)] = 0,\\
&V: \quad \bar{a}_1[\alpha_1 - \alpha_2(3a_0^2 + \bar{a}_1^2)] = 0,\\
&U: \quad a_1(k^2 - \alpha_1 + 3\alpha_2a_0^2) + \bar{a}_1[3\alpha_2a_1\bar{a}_1 - k(\beta_1b_1 + \beta_2c_1)] = 0,\\
&\qquad kb_1 + kc_1\gamma_1 - a_1\bar{a}_1\gamma_2 = 0, \quad kb_1 + kc_1\gamma_3 - a_1\bar{a}_1\gamma_4 = 0,\\
&UV: \quad a_0[6\alpha_2a_1\bar{a}_1 - k(\beta_1b_1 + \beta_2c_1)] = 0,\\
&\qquad a_0a_1\gamma_2 = 0, \quad a_0a_1\gamma_4 = 0,\\
&U^2: \quad a_0[3\alpha_2(a_1^2 - \bar{a}_1^2) + k(\beta_1\bar{b}_1 + \beta_2\bar{c}_1)] = 0,\\
&\qquad a_0\bar{a}_1\gamma_2 = 0, \quad a_0\bar{a}_1\gamma_4 = 0,\\
&U^2V: \quad a_1[3\alpha_2a_1\bar{a}_1 - k(\beta_1b_1 + \beta_2c_1)] - \bar{a}_1[\alpha_2\bar{a}_1^2 + k(2k - \beta_1\bar{b}_1 - \beta_2\bar{c}_1)] = 0,\\
&\qquad 2k(\bar{b}_1 + \bar{c}_1\gamma_1) + \gamma_2(a_1^2 - \bar{a}_1^2) = 0, \quad 2k(\bar{b}_1 + \bar{c}_1\gamma_3) + \gamma_4(a_1^2 - \bar{a}_1^2) = 0,\\
&U^3: \quad \bar{a}_1[3\alpha_2a_1\bar{a}_1 - k(\beta_1b_1 + \beta_2c_1)] - a_1[\alpha_2a_1^2 - k(2k - \beta_1\bar{b}_1 - \beta_2\bar{c}_1)] = 0,\\
&\qquad k(b_1 + c_1\gamma_1) - a_1\bar{a}_1\gamma_2 = 0, \quad k(b_1 + c_1\gamma_3) - a_1\bar{a}_1\gamma_4 = 0.
\end{aligned} \tag{2.75}$$

Solving the nonlinear system of equations (2.75) with the help of Mathematica gives the solutions.

The parameters are found as

$$
\begin{aligned}
& k = \pm\, \frac{D}{2}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2}, \qquad \bar{a}_1 = \pm\, (\frac{\alpha_1}{\alpha_2})^{1/2}, \\
& \bar{b}_1 = \pm\, \frac{\gamma_1\gamma_4-\gamma_2\gamma_3}{D}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2}, \qquad \bar{c}_1 = \pm\, \frac{\gamma_2-\gamma_4}{D}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2}, \\
& D = [2\alpha_2(\gamma_3-\gamma_1)+\beta_2(\gamma_2-\gamma_4)+\beta_1(\gamma_1\gamma_4-\gamma_2\gamma_3)]^{1/2}, \\
& a_0 = a_1 = b_1 = c_1 = 0, \qquad b_0 \text{ and } c_0 \text{ arbitrary,}
\end{aligned} \tag{2.76}
$$

and the solution functions f, g and h are found as

$$
\begin{aligned}
& f(\zeta) = \pm\, (\frac{\alpha_1}{\alpha_2})^{1/2} \tanh\{\pm\, \frac{D}{2}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2}\zeta\}, \\
& g(\zeta) = b_0 \pm\, \frac{\gamma_1\gamma_4-\gamma_2\gamma_3}{D}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2} \tanh\{\pm\, \frac{D}{2}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2}\zeta\}, \\
& h(\zeta) = c_0 \pm\, \frac{\gamma_2-\gamma_4}{D}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2} \tanh\{\pm\, \frac{D}{2}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2}\zeta\}.
\end{aligned} \tag{2.77}
$$

Finally, the solutions of the GDS equations are obtained as follows:

$$
\begin{aligned}
& A = \pm\, (\frac{\alpha_1}{\alpha_2})^{1/2} \tanh\{\pm\, \frac{D}{2}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2}\zeta\} \exp(i\theta), \\
& \phi_1 = b_0 \pm\, \frac{\gamma_1\gamma_4-\gamma_2\gamma_3}{D}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2} \tanh\{\pm\, \frac{D}{2}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2}\zeta\}, \\
& \phi_2 = c_0 \pm\, \frac{\gamma_2-\gamma_4}{D}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2} \tanh\{\pm\, \frac{D}{2}\, [\frac{\alpha_1}{\alpha_2(\gamma_1-\gamma_3)}]^{1/2}\zeta\},
\end{aligned} \tag{2.78}
$$

The solutions (2.78) are in agreement with the ones (2.65) obtained in Section 2.6.1. By considering the relation $l_3 = l_1^2 + l_2^2 + \chi f_-^2$ and assuming $D_2 = 0$ in (2.65) and $b_0 = c_0 = 0$ in (2.78), it is easily seen that these two sets of solutions coincide.

By using the tanh method only tanh-tanh-tanh type solutions have been obtained. Sech-tanh-tanh type solutions which was given in Section 2.6.1 by Jacobian elliptic functions couldn't be obtained by the tanh method.

3 LONG-WAVE SHORT-WAVE INTERACTION EQUATIONS

3.1 Derivation of the Interaction Equations

The DS equations given by (2.44) consist of two partial differential equations, first of which is a nonlinear Schrödinger equation with a forcing term and the second is a linear equation. It is easily seen that, in equation (2.44), the coefficient $\bar{\chi}$ becomes zero and the coefficient m_1 becomes singular if the group speed c_g of the transverse wave is equal to the speed c_1 of long longitudinal wave, i.e., if $c_1^2 = c_g^2$. This corresponds to a long-wave and short-wave resonance condition in which the group velocity of the transverse wave and the phase velocity of the longitudinal wave are equal to each other. This resonant case has already been studied in the context of water waves [10], and the following evolution equations have been derived to describe the resonant interaction of the waves for (1+1) dimensional case:

$$\begin{aligned} &iA_t + \frac{1}{2}\,\omega''(k)A_{xx} = BA, \\ &B_t = -\alpha(|A|^2)_x. \end{aligned} \tag{3.1}$$

Sulem and Sulem [55] have extended this result for the (2+1) dimensional case, and Colin and Lanes [56] have proved that such phenomena also occurs in a general way in nonlinear optics. Under the long-wave short-wave resonance hypothesis, the equation satisfied by free surface A is coupled with that of mean flow Q as

$$\begin{aligned} &iA_t - \frac{1}{2}\omega''(k)A_{xx} = \chi_3 A\, Q, \\ &Q_{xt} + \frac{c_g}{2}\, Q_{yy} = -\chi_4(|A|^2)_x. \end{aligned} \tag{3.2}$$

On the other hand, a similar argument can be done for the GDS equations (2.37). Simple algebra shows that, in (2.37), $c_g^2 - c_2^2 = 4m^2c_2^4k^4(3 + 16m^2k^2)/\omega^2$ is always positive. However, it is possible to show that there exists a critical wave number k_c

$$k_c^2 = \frac{1}{32c_2^2m^2}[c_1^2 - 4c_2^2 + c_1(c_1^2 + 8c_2^2)^{\frac{1}{2}}]$$

such that $c_g^2 - c_1^2 > 0$ if $k > k_c$ and $c_g^2 - c_1^2 < 0$ if $k < k_c$ because $c_1^2 > c_2^2$. The case where $k = k_c$ corresponds to longitudinal long wave and transverse short wave resonance since the phase velocity of longitudinal wave, c_1, is equal to the group velocity of the transverse wave, c_g.

In order to analyze the long-wave short-wave interaction of the so-called GDS equations for the case $c_g^2 = c_1^2$, the interaction equations will be developed for the same elastic medium by using the reductive perturbation method. Since longitudinal long wave and transverse short wave resonance will be investigated, it is convenient to introduce the scale transformation as

$$\xi = \epsilon(x - c_g t), \quad \eta = \epsilon^{3/2}y, \quad \tau = \epsilon^2 t, \tag{3.3}$$

where ϵ is a small parameter measuring the weakness of dispersion, c_g is the group velocity of the short transverse wave, and x, y and t are fast space and time variables; ξ, η and τ are slow space and time coordinates in a frame of reference moving with the group speed of the short transverse wave. Here the direction of the wave propagation is taken along the x axis and it is assumed that the field variables depend on the transverse coordinate y as well. Since we deal with the nonlinear interaction of the transverse short wave and longitudinal long wave, it is convenient to expand the components of the displacement vector in asymptotic power series of ϵ as follows:

$$\begin{aligned}
u &= \epsilon\phi_1(\xi,\eta,\tau) + \epsilon^2[u_2(\xi,\eta,\tau)\mathrm{e}^{2i\theta} + \text{c.c.}] + \epsilon^3[u_3(\xi,\eta,\tau)\mathrm{e}^{2i\theta} + \text{c.c.}] + \cdots \\
v &= \epsilon^{3/2}\phi_2(\xi,\eta,\tau) + \epsilon^{5/2}[v_2(\xi,\eta,\tau)\mathrm{e}^{2i\theta} + \text{c.c.}] + \cdots \\
w &= \epsilon^{3/2}[A(\xi,\eta,\tau)\mathrm{e}^{i\theta} + \text{c.c.}] + \epsilon^{5/2}[w_2(\xi,\eta,\tau)\mathrm{e}^{2i\theta} + \text{c.c.}] + \cdots
\end{aligned} \tag{3.4}$$

Here $\theta = kx - \omega t$ is the phase, k is the wave number of the carrier wave, ω is the frequency, c.c. stands for the complex conjugate of the preceding term. A is the complex amplitude of the free short transverse wave mode whereas ϕ_1 and ϕ_2 are the free long longitudinal and free long transverse wave modes, respectively. If the scale transformation (3.3) and expansions (3.4) are compared with the ones in Section 2.4, that is, equations (2.32) and (2.33), it is observed that the interaction occurs in a smaller scale of y coordinate and the amplitudes ϕ_2 and A in (3.4) are larger than the ones in (2.33).

Substituting expansions together with the coordinate stretching into the governing equations (2.17) and equating the coefficients of like powers of ϵ to zero, we obtain a hierarchy of perturbation equations. For $\mathcal{O}(\epsilon^{3/2})$, the only equation is

$$\mathrm{e}^{i\theta}: \qquad D_3(k,\omega)A = 0. \tag{3.5}$$

Assuming $D_3(k,\omega) = 0$ in equation (3.5), the first-order transverse wave amplitude, A, remains arbitrary. In order to eliminate self-resonance, it is assumed that $D_3(nk, n\omega) \neq 0$ for $n \geq 2$.

Similarly, for order ϵ^2, setting the coefficient of each mode equal to zero gives

$$\mathrm{e}^{2i\theta}: \qquad D_1(2k,2\omega)u_2 = 0, \tag{3.6}$$

and for order $\epsilon^{5/2}$,

$$\begin{aligned}
\mathrm{e}^{i\theta}: &\qquad i\frac{\partial}{\partial k}D_3(k,\omega)A_\xi = 0,\\
\mathrm{e}^{2i\theta}: &\qquad D_2(2k,2\omega)v_2 = 0,\\
\mathrm{e}^{2i\theta}: &\qquad D_3(2k,2\omega)w_2 = 0.
\end{aligned} \tag{3.7}$$

Since the dispersion relation of the transverse wave is identically zero, A_ξ in equation $(3.7)_1$ remains arbitrary. Recalling that $D_1(2k,2\omega) \neq 0$, $D_2(2k,2\omega) \neq 0$ and $D_3(2k,2\omega) \neq 0$, we obtain $u_2 = 0$, $v_2 = 0$ and $w_2 = 0$. For order ϵ^3:

$$\begin{aligned}
\mathrm{e}^{i0}: &\qquad (c_g^2 - c_1^2)\phi_{1,\xi\xi} = 0,\\
\mathrm{e}^{2i\theta}: &\qquad (4i\omega c_g - c_1^2 ik - 96ik^3c_2^2m^2)u_{2,\xi} + 4(k^2c_1^2 - \omega^2)u_3 + \gamma_3 ik^3A^2 = 0.
\end{aligned} \tag{3.8}$$

Since long-wave short-wave interaction equations are looked for $c_g^2 = c_1^2$ in the first equation of (3.8) ϕ_1 is obtained as a arbitrary function and since $u_2 = 0$, u_3 is written in terms of A as:

$$u_3 = \frac{\gamma_3 ik^3}{4(\omega^2 - k^2c_1^2)}A^2. \tag{3.9}$$

For order $\epsilon^{7/2}$

$$
\begin{aligned}
\mathrm{e}^{i0}: &\quad (c_g^2 - c_2^2)\phi_{2,\xi\xi} - (c_1^2 - c_2^2)\phi_{1,\xi\eta} = 0, \\
\mathrm{e}^{i\theta}: &\quad (c_g^2 - c_2^2 - 6k^2c_2^2m^2)A_{\xi\xi} - 2i\omega A_\tau + \gamma_3 k^2 \phi_{1,\xi} A = 0.
\end{aligned}
\tag{3.10}
$$

Using $c_1^2 = c_g^2$ in $(3.10)_1$ gives the following relation between the long longitudinal and long transverse wave modes:

$$
\phi_{2,\xi} = \phi_{1,\eta}. \tag{3.11}
$$

The only equation for order ϵ^4 is

$$
\mathrm{e}^{i0}: \quad -2c_g\phi_{1,\tau\xi} - c_2^2\phi_{1,\eta\eta} - (c_1^2 - c_2^2)\phi_{2,\xi\eta} - \gamma_3 k^2(|A|^2)_\xi = 0. \tag{3.12}
$$

By using (3.11) in (3.12) together with $(3.10)_2$ the following evolution equations are obtained for the description of the interaction:

$$
\begin{aligned}
&iA_\tau + pA_{\xi\xi} + \frac{\gamma_3}{2\omega}k^2\phi_\xi A = 0, \\
&2c_g\phi_{\tau\xi} + c_g^2\phi_{\eta\eta} + \gamma_3 k^2(|A|^2)_\xi = 0,
\end{aligned}
\tag{3.13}
$$

where

$$
\phi = \phi_1 \quad \text{and} \quad p = -\frac{1}{2\omega}(c_g^2 - c_2^2 - 6m^2c_2^2k^2). \tag{3.14}
$$

By introducing the dimensionless variables as

$$
\tau = \frac{2c_g}{k^2\gamma_3}\, t, \quad \xi = \frac{\omega}{c_g}\, x, \quad \eta = \frac{1}{k}\left(\frac{c_g\omega}{\gamma_3}\right)^{1/2} y, \tag{3.15}
$$

the dimensionless form of the two coupled evolution equations (3.13) are obtained as follows:

$$
\begin{aligned}
&iA_t + \delta A_{xx} = \phi_x A, \\
&\phi_{xt} + \phi_{yy} = -(|A|^2)_x,
\end{aligned}
\tag{3.16}
$$

where

$$
\delta = 2\frac{p\, c_g^3}{k^2\omega^2\gamma_3}. \tag{3.17}
$$

Here, in order to make the further calculations simpler, it will be assumed that $\delta = 1$. It is easily seen that equation (3.16) is in complete agreement with equation (3.2) which was obtained in [55].

3.2 Special Solutions of the Interaction Equations

3.2.1 Using Jacobian elliptic functions

The travelling wave solutions of (3.16) can be obtained simply by introducing the following solutions:

$$A = f(\zeta)e^{i\theta}, \quad \phi = g(\zeta), \tag{3.18}$$

where $\theta = l_1 x + l_2 y - l_3 t$, and the amplitudes of the short wave f, and the long travelling wave g are real functions of $\zeta = k_1 x + k_2 y - 2k_1 l_1 t$. Again we assume that all the derivatives of f and g tend to zero as $\zeta \to \pm\infty$. Substituting equations (3.18) into the system (3.16) leads to

$$\begin{aligned} f^{''} + \frac{l_3 - l_1^2}{k_1^2} f - \frac{1}{k_1} g^{'} f &= 0, \\ g^{''} + \frac{2k_1}{k_2^2 - 2k_1^2 l_1} f f^{'} &= 0, \end{aligned} \tag{3.19}$$

where $'$ denotes differentiation with respect to ζ. Integrating equations $(3.19)_2$ with respect to ζ once and assuming that $f \to f_-$ as $\zeta \to -\infty$ and $f \to f_+$ as $\zeta \to \infty$, we obtain

$$g^{'} = \frac{k_1}{k_2^2 - 2k_1^2 l_1}(f_-^2 - f^2) \tag{3.20}$$

and $f_-^2 = f_+^2$. The first equation of (3.19) takes the form

$$f^{''} = 2\alpha f^3 + \beta f, \tag{3.21}$$

where the coefficients α and β are

$$\alpha = \frac{1}{2(2k_1^2 l_1 - k_2^2)}, \quad \beta = \frac{l_1^2 - l_3}{k_1^2} + \frac{f_-^2}{k_2^2 - 2k_1^2 l_1} \tag{3.22}$$

By multiplying (3.21) with $f^{'}$, integrating with respect to ζ once and introducing a new dependent variable $z = f^2$, (z real), the following equation is obtained:

$$(z^{'})^2 = 4z(\alpha z^2 + \beta z + \bar{C}) \equiv \Phi(z), \tag{3.23}$$

where $\bar{C}$ is an integration constant. The solutions of this nonlinear ordinary differential equation can be found for $\Phi \geq 0$.

First type of solutions:

Let $\alpha < 0$, $\beta > 0$ and $\Phi(z) = -4\alpha z(z - z_1)(z_2 - z)$, where $0 < z_1 \leq z < z_2$ and

$$z_1 = \frac{1}{2\alpha}\left(\sqrt{\beta^2 - 4\,\alpha\,C} - \beta\right), \quad z_2 = \frac{1}{2\alpha}\left(\sqrt{\beta^2 - 4\,\alpha\,C} + \beta\right). \tag{3.24}$$

Integrating equation (3.23) the solution for the envelope function $f(\zeta)$ is obtained in terms of Jacobian elliptic function:

$$f(\zeta) = \{z_2 - (z_2 - z_1)\mathrm{sn}^2[(-\alpha z_2)^{\frac{1}{2}}\zeta + D_1; k]\}^{\frac{1}{2}}, \tag{3.25}$$

where $k^2 = (z_2 - z_1)/z_2$ and D_1 is an integration constant. Using (3.20), and the fact that $f_-^2 = z_1$, the solution associated with the long wave component is expressed as:

$$g(\zeta) = \frac{2k_1(z_1 - z_2)}{(-\alpha z_2)^{\frac{1}{2}} k^2 (2k_1^2 l_1 - k_2^2)}[E(u) - (1 - k^2)u], \tag{3.26}$$

where $u = (-\alpha z_2)^{\frac{1}{2}}\zeta + D_1$, and $E(u)$ is the elliptic integral of the second kind [54].

In the limit $k = 1$; since $z_1 = f_- = 0$ and $z_2 = -\beta/\alpha$, solutions (3.25) and (3.26) reduce to the following solitary wave solutions in terms of hyperbolic secant function and hyperbolic tangent functions

$$\begin{aligned} f(\zeta) &= \pm(-\alpha)^{-\frac{1}{2}}\, \bar{\beta}\ \mathrm{sech}\,[\bar{\beta}\ \zeta + D_1], \\ g(\zeta) &= \frac{2k_1}{\alpha\,(2k_1^2 l_1 - k_2^2)}\ \bar{\beta}\ \tanh\,[\bar{\beta}\ \zeta + D_1], \end{aligned} \tag{3.27}$$

where

$$\bar{\beta} = (\frac{l_3 - l_1^2}{k_1^2})^{1/2}. \tag{3.28}$$

Finally, the solutions of (3.16) can be given as follows:

$$\begin{aligned} A &= \pm(-\alpha)^{-\frac{1}{2}}\ \bar{\beta}\ \mathrm{sech}\,[\bar{\beta}\ \zeta + D_1]\, e^{i\theta}, \\ \phi &= 4k_1\ \bar{\beta}\ \tanh\,[\bar{\beta}\ \zeta + D_1]. \end{aligned} \tag{3.29}$$

Second type of solutions:

Let $\alpha > 0$, $\beta < 0$ and $\Phi(z) = 4\alpha z(z_1 - z)(z_2 - z)$, where $0 < z \le z_1 < z_2$ and

$$z_1 = \frac{1}{2\alpha}\,(\sqrt{\beta^2 - 4\,\alpha\,C}\ - \beta), \quad z_2 = \frac{1}{2\alpha}\,(\sqrt{\beta^2 - 4\,\alpha\,C}\ + \beta). \tag{3.30}$$

Integrating equation (3.23) leads one to the following solutions:

$$\begin{aligned} f(\zeta) &= z_1^{\frac{1}{2}}\mathrm{sn}[(\alpha z_2)^{\frac{1}{2}}\zeta + D_2; k], \\ g(\zeta) &= \frac{2z_1 k_1}{(\alpha z_2)^{\frac{1}{2}}\ k^2\ (2k_1^2 l_1 - k_2^2)}[E(u) - (1 - k^2)u], \end{aligned} \tag{3.31}$$

where $k^2 = z_1/z_2$, D_2 is an integration constant and $u = (\alpha z_2)^{\frac{1}{2}}\zeta + D_2$ with $f_-^2 = z_1$.

In the limit $k = 1$; $z_1 = z_2 = -\beta/(2\alpha)$ and the frequency must be chosen as $l_3 = l_1^2$ which is evident from equations (3.21) and (3.22). Then, solutions (3.31) reduce to the following solitary wave solutions in terms of hyperbolic tangent functions

$$
\begin{aligned}
f(\zeta) &= \pm\,\alpha^{-1/2}\,\bar{\beta}\,\tanh[\bar{\beta}\,\zeta + D_2],\\
g(\zeta) &= \frac{2k_1}{\alpha(2k_1^2 l_1 - k_2^2)}\,\bar{\beta}\,\tanh[\bar{\beta}\,\zeta + D_2],
\end{aligned} \tag{3.32}
$$

where

$$
\bar{\beta} = [\frac{f_-^2}{2(2k_1^2 l_1 - k_2^2)}]^{1/2}. \tag{3.33}
$$

Finally, solutions of (3.16) can be given as follows:

$$
\begin{aligned}
A &= \pm\,\alpha^{-1/2}\,\bar{\beta}\,\tanh[\bar{\beta}\,\zeta + D_2]\,e^{i\theta}\\
\phi &= 4k_1\,\bar{\beta}\,\tanh[\bar{\beta}\,\zeta + D_2].
\end{aligned} \tag{3.34}
$$

3.2.2 By the tanh method

In this subsection, special solutions of the long-wave short-wave interaction equations (3.16) will be given by the tanh method.

In order to find the travelling wave solutions of the interaction equations (3.16), the following solutions are considered:

$$
A = f(\zeta)\,e^{i\theta}, \quad \phi = g(\zeta), \tag{3.35}
$$

where $\theta = l_1 x + l_2 y - l_1^2 t$, and the amplitudes of the short wave f and the long travelling waves g are real functions of $\zeta = k_1 x + k_2 y - 2k_1 l_1 t$. Substituting solutions (3.35) into the system (3.16) leads to ordinary differential equations

$$
\begin{aligned}
f^{''} - \gamma_1 g^{'} f &= 0,\\
g^{''} - \gamma_2 f f^{'} &= 0,
\end{aligned} \tag{3.36}
$$

with

$$
\gamma_1 = \frac{1}{k_1}, \quad \gamma_2 = \frac{2k_1}{2k_1^2 l_1 - k_2^2}. \tag{3.37}
$$

At this step, the following solutions will be introduced:

$$f(\zeta)=\sum_{i=0}^{p}a_iU^i+\sum_{i=1}^{p}\bar{a}_iVU^{i-1},$$
$$g(\zeta)=\sum_{i=0}^{r}b_iU^i+\sum_{i=1}^{r}\bar{b}_iVU^{i-1}, \tag{3.38}$$

where $a_i, b_i, \bar{a}_i$ and $\bar{b}_i$ are constants. The values of p and r are the polynomial degrees of the solution in f and g respectively and will be determined by balancing the highest nonlinear term with the highest derivative term. By substituting (3.38) into the system (3.36) and by using (2.69) together with the relation $V^2 = 1 - U^2$ repeatedly, the term $f^{''}$ is balanced by the term $g^{'}f$ while $g^{''}$ is balanced with the term $ff^{'}$. Thus, the positive integers p and r are found as $p = r = 1$. Now, solutions (3.38) can be written as follows:

$$f(\zeta)=a_0+a_1U+\bar{a}_1V, \quad g(\zeta)=b_0+b_1U+\bar{b}_1V. \tag{3.39}$$

Substituting (3.39) into (3.36) and equating the coefficient of each power of the variables U and U leads to the following nonlinear system of algebraic equations:

$$\begin{aligned}
&U: && ka_1+\gamma_1b_1\bar{a}_1=0, \quad kb_1+\gamma_2a_1\bar{a}_1=0,\\
&UV: && \gamma_1a_0b_1=0, \quad \gamma_2a_0a_1=0,\\
&U^2: && \gamma_1a_0\bar{b}_1=0, \quad \gamma_2a_0\bar{a}_1=0,\\
&U^2V: && \gamma_1a_1b_1-\bar{a}_1(2k+\gamma_1\bar{b}_1)=0, \quad \gamma_2a_1^2-(\gamma_2\bar{a}_1^2+2k\bar{b}_1)=0,\\
&U^3: && -\gamma_1b_1\bar{a}_1-a_1(2k+\gamma_1\bar{b}_1)=0, \quad kb_1+\gamma_2a_1\bar{a}_1=0.
\end{aligned} \tag{3.40}$$

Solving the nonlinear system of equations (3.40) with the help of Mathematica gives the solutions.

The parameters are found as

$$k=\pm\frac{\sqrt{\gamma_1\gamma_2}}{2}\bar{a}_1, \qquad \bar{b}_1=\mp\sqrt{\frac{\gamma_2}{\gamma_1}}\,\bar{a}_1,$$
$$a_0=a_1=b_1=0, \qquad \bar{a}_1 \text{ and } b_0 \text{ arbitrary}, \tag{3.41}$$

and the solution functions f and g are found as

$$f(\zeta)=\pm\bar{a}_1\tanh(\pm\frac{\sqrt{\gamma_1\gamma_2}}{2}\bar{a}_1\,\zeta),$$
$$g(\zeta)=b_0\mp\sqrt{\frac{\gamma_2}{\gamma_1}}\,\bar{a}_1\tanh(\pm\frac{\sqrt{\gamma_1\gamma_2}}{2}\bar{a}_1\,\zeta). \tag{3.42}$$

Finally, the solutions of the interaction equations (3.16) are obtained as follows:

$$A = \pm\, \bar{a}_1 \;\; \tanh(\mp\zeta_0\, \zeta)\, e^{i\theta},$$
$$\phi = b_0 \mp \;\bar{b}_0 \tanh(\mp\zeta_0\, \zeta), \tag{3.43}$$

where

$$\zeta_0 = [\frac{1}{2(2k_1^2 l_1 - k_2^2)}]^{1/2}\, \bar{a}_1, \qquad \bar{b}_0 = [\frac{2k_1^2}{2k_1^2 l_1 - k_2^2}]^{1/2}. \tag{3.44}$$

The solutions (3.43) have the same structure as the solutions (3.34). Considering $b_0 = 0$ and $\bar{a}_1 = 1$ in (3.43) while $f_-^2 = 1$ in (3.33) one can easily see that two sets of solutions coincide.

By using the tanh method only tanh-tanh type solutions have been obtained. Sech-tanh type solutions which was given in Section 3.2.1 by Jacobian elliptic functions couldn't be obtained by the tanh method.

4 ON DEGENERATE GDS EQUATIONS

If one of the coefficients of the GDS system (2.37) vanishes, the system is said to be degenerate. In (2.18), if $\lambda = -\mu$, then $c_1^2 = c_2^2 = \mu/\rho_0$. This corresponds to a degenerate case since two of the coefficients of the GDS system becomes zero. In the next subsection, this degenerate case of the GDS equations will be considered.

4.1 Degenerate GDS Equations

For the case $c_1^2 = c_2^2$, the GDS equations (2.37) reduce to the following equations:

$$iA_\tau + pA_{\xi\xi} + rA_{\eta\eta} = q|A|^2 A + \frac{k^2}{2\omega}(\gamma_3\phi_{1,\xi} + \gamma_1\phi_{2,\eta})A,$$
$$(c_g^2 - c_1^2)\phi_{1,\xi\xi} - c_1^2\phi_{1,\eta\eta} = \gamma_3 k^2(|A|^2)_\xi,$$
$$(c_g^2 - c_1^2)\phi_{2,\xi\xi} - c_1^2\phi_{2,\eta\eta} = \gamma_1 k^2(|A|^2)_\eta. \tag{4.45}$$

It is easily seen from (4.45) that, considering $c_1^2 = c_2^2$ makes system (2.37) uncoupled due to the variables ϕ_1 and ϕ_2. By introducing the dimensionless variables as

$$\tau = \frac{c_1^2}{r}\, t, \quad \xi = (c_g^2 - c_1^2)^{1/2}\, x, \quad \eta = c_1\, y,$$
$$A = \frac{(2\omega)^{1/2}}{k^2}\, u, \quad \phi_1 = -\frac{2\omega r (c_g^2 - c_1^2)^{1/2}}{k^2 \gamma_3\, c_1{}^2}\, \varphi_1, \quad \phi_2 = -\frac{2\omega r}{k^2 \gamma_1\, c_1}\, \varphi_2,$$

the dimensionless form of (4.45) can be given as

$$\begin{aligned} &i u_t + \delta u_{xx} + u_{yy} = \chi |u|^2 u - (\varphi_{1,x} + \varphi_{2,y}) u, \\ &\varphi_{1,xx} - \varphi_{1,yy} = \kappa_1 (|u|^2)_x, \\ &\varphi_{2,xx} - \varphi_{2,yy} = \kappa_2 (|u|^2)_y, \end{aligned} \tag{4.46}$$

where the non-dimensional coefficients are given in the form

$$\delta = \frac{p\, c_1^2}{r(c_g^2 - c_1^2)}, \quad \chi = \frac{2\,\omega\, q\, c_1^2}{r k^4}, \quad \kappa_1 = \frac{\gamma_3^2\ c_1^2}{r(c_1^2 - c_g^2)}, \quad \kappa_2 = -\frac{\gamma_1^2}{r}, \tag{4.47}$$

and in order to make the further calculations simpler, it will be assumed that $\delta = 1$. The system (4.46) can be classified as elliptic-hyperbolic-hyperbolic by considering the linear terms of the equations.

In the next subsection, some special solutions of the degenerate GDS equations (4.46) are investigated. To this aim, travelling wave solutions of the degenerate GDS equations are given by using the Jacobi elliptic functions and the tanh method. Besides, some special solutions of the degenerate GDS equations are established by using a variable separation approach considering a prior ansatz to its bilinear form [46].

4.2 Special Solutions of the Degenerate GDS Equations

4.2.1 Using Jacobian elliptic functions

In this subsection, some special solutions of the degenerate GDS equations (4.46) are given in terms of Jacobian elliptic functions.

In order to find the travelling wave solutions of the degenerate GDS equations (4.46), the following solutions are considered:

$$u = f(\zeta)\, e^{i\theta}, \quad \varphi_1 = g(\zeta), \quad \varphi_2 = h(\zeta), \tag{4.48}$$

where $\theta = l_1 x + l_2 y - l_3 t$, and the amplitudes of the short wave f, and the long travelling waves g and h are real functions of $\zeta = k_1 x + k_2 y - 2(k_1 l_1 + k_2 l_2) t$.

Here we assume that all the derivatives of f, g and h tend to zero as $\zeta \to \pm\infty$. Substituting solution (4.48) into the system (4.46) leads to a set of coupled ordinary differential equations

$$
\begin{aligned}
&(k_1^2+k_2^2)f^{''}-(l_1^2+l_2^2-l_3)f=\chi f^3-(k_1g^{'}+k_2h^{'})f,\\
&(k_1^2-k_2^2)g^{''}=2\kappa_1k_1ff^{'},\\
&(k_1^2-k_2^2)h^{''}=2\kappa_2k_2ff^{'},
\end{aligned}
\tag{4.49}
$$

where $'$ denotes differentiation with respect to ζ. Integrating equations $(4.49)_2$ and $(4.49)_3$ with respect to ζ once gives $g^{'}$ and $h^{'}$ in terms of f^2 as

$$
g^{'}=d_1f^2+d_2, \qquad h^{'}=d_3f^2+d_4, \tag{4.50}
$$

where

$$
\begin{aligned}
d_1&=\frac{\kappa_1k_1}{k_1^2-k_2^2}, \qquad d_2=\frac{C_1}{k_1^2-k_2^2},\\
d_3&=\frac{\kappa_2k_2}{k_1^2-k_2^2}, \qquad d_4=\frac{C_2}{k_1^2-k_2^2},
\end{aligned}
$$

and C_1 and C_2 are integration constants. If $f \to f_-$ as $\zeta \to -\infty$ and $f \to f_+$ as $\zeta \to \infty$, we obtain $f_-^2 = f_+^2$ from equations (4.50), and consequently the integration constants are $C_1 = -\kappa_1k_1f_-^2$ and $C_2 = -\kappa_2k_2f_-^2$. Then, the coefficients d_2 and d_4 in equations (4.50) are $d_1 = -a_1f_-^2$ and $d_4 = -a_2f_-^2$. Therefore, the solutions associated with the long wave components are expressed in terms of the short wave amplitude f

$$
g^{'}=d_1(f^2-f_-^2), \qquad h^{'}=d_3(f^2-f_-^2). \tag{4.51}
$$

In such a case, the first equation of (4.49) takes the form

$$
f^{''}=2\alpha f^3+\beta f, \tag{4.52}
$$

where the coefficients α and β are

$$
\begin{aligned}
\alpha&=\frac{1}{2(k_1^2+k_2^2)}[\chi-(k_1d_1+k_2d_3)],\\
\beta&=\frac{1}{k_1^2+k_2^2}[l_1^2+l_2^2-l_3+(k_1d_1+k_2d_3)f_-^2].
\end{aligned}
$$

By multiplying (4.52) with $f^{'}$, integrating with respect to ζ once and introducing a new dependent variable $z = f^2$, (z real), the following equation is obtained

$$(z^{'})^2 = 4z(\alpha z^2 + \beta z + C) \equiv \Phi(z), \tag{4.53}$$

where C is an integration constant. The solutions of this nonlinear ordinary differential equation can be found for $\Phi \geq 0$.

First type of solutions:

Let $\alpha < 0$, $\beta > 0$ and $\Phi(z) = -4\alpha z(z - z_1)(z_2 - z)$, where $0 < z_1 \leq z < z_2$ and

$$z_1 = \frac{1}{2\alpha}\,(\sqrt{\beta^2 - 4\,\alpha\,C}\, - \beta), \quad z_2 = \frac{1}{2\alpha}\,(\sqrt{\beta^2 - 4\,\alpha\,C}\, + \beta). \tag{4.54}$$

Integrating equation (4.53) the solution for the envelope function $f(\zeta)$ is obtained in terms of Jacobian elliptic function:

$$f(\zeta) = \{z_2 - (z_2 - z_1)\mathrm{sn}^2[(-\alpha z_2)^{\frac{1}{2}}\zeta + D_1; k]\}^{\frac{1}{2}}, \tag{4.55}$$

where $k^2 = (z_2 - z_1)/z_2$ and D_1 is an integration constant. Using (4.51), and the fact that $f_-^2 = z_1$, the solutions associated with the long wave components are expressed as:

$$\begin{aligned} g(\zeta) &= \frac{d_1(z_2 - z_1)}{(-\alpha z_2)^{\frac{1}{2}} k^2}[E(\mu) - (1 - k^2)\mu], \\ h(\zeta) &= \frac{d_3(z_2 - z_1)}{(-\alpha z_2)^{\frac{1}{2}} k^2}[E(\mu) - (1 - k^2)\mu], \end{aligned} \tag{4.56}$$

where $\mu = (-\alpha z_2)^{\frac{1}{2}}\zeta + D_1$, and $E(u)$ is the elliptic integral of the second kind [54].

In the limit $k = 1$; since $z_1 = 0$ and $z_2 = -\beta/\alpha$, solutions (4.55) and (4.56)can be written in terms of hyperbolic secant function and hyperbolic tangent functions and u , φ_1 and φ_2 accepts the following solitary wave solutions:

$$\begin{aligned} u &= \pm u_0\,\mathrm{sech}[\,\zeta_0\,\zeta + D_1]\ \exp(i\theta), \\ \varphi_1 &= \bar{b}_0\ \tanh[\,\zeta_0\,\zeta + D_1], \\ \varphi_2 &= \bar{c}_0\ \ \tanh[\,\zeta_0\,\zeta + D_1], \end{aligned} \tag{4.57}$$

where

$$u_0 = [\frac{2(k_1^2 - k_2^2)(l_1^2 + l_2^2 - l_3)}{\kappa_1 k_1^2 + \kappa_2 k_2^2 + \chi(k_2^2 - k_1^2)}]^{1/2}, \quad \zeta_0 = [\frac{l_1^2 + l_2^2 - l_3}{k_1^2 + k_2^2}]^{1/2},$$

$$\bar{b}_0 = -2\kappa_1 k_1 \frac{[(k_1^2+k_2^2)(l_1^2+l_2^2-l_3)]^{1/2}}{\kappa_1 k_1^2+\kappa_2 k_2^2+\chi(k_2^2-k_1^2)}, \quad \bar{c}_0 = -2\kappa_2 k_2 \frac{[(k_1^2+k_2^2)(l_1^2+l_2^2-l_3)]^{1/2}}{\kappa_1 k_1^2+\kappa_2 k_2^2+\chi(k_2^2-k_1^2)}. \tag{4.58}$$

Second type of solutions:

Let $\alpha > 0$, $\beta < 0$ and $\Phi(z) = 4\alpha z(z_1 - z)(z_2 - z)$, where $0 < z \leq z_1 < z_2$ and

$$z_1 = \frac{1}{2\alpha}(\sqrt{\beta^2 - 4\,\alpha\,C} - \beta), \quad z_2 = \frac{1}{2\alpha}(\sqrt{\beta^2 - 4\,\alpha\,C} + \beta). \tag{4.59}$$

Integrating equation (4.53) the solution for the envelope function $f(\zeta)$ is obtained in terms of Jacobian elliptic function as follows:

$$f(\zeta) = z_1^{\frac{1}{2}}\,\mathrm{sn}[(\alpha z_2)^{\frac{1}{2}}\zeta + D_2; k], \tag{4.60}$$

where $k^2 = z_1/z_2$ and D_2 is an integration constant. Using (4.51), and the fact that $f_-^2 = z_1$, the solutions associated with the long wave components are expressed as:

$$\begin{aligned} g(\zeta) &= \frac{-d_1 z_1}{(\alpha z_2)^{\frac{1}{2}} k^2}[E(\mu) - (1-k^2)\mu], \\ h(\zeta) &= \frac{-d_3 z_1}{(\alpha z_2)^{\frac{1}{2}} k^2}[E(\mu) - (1-k^2)\mu], \end{aligned} \tag{4.61}$$

where $\mu = (\alpha z_2)^{\frac{1}{2}}\zeta + D_2$.

In the limit $k = 1$; $z_1 = z_2 = -\beta/(2\alpha)$ and the frequency must be chosen as $l_3 = l_1^2 + l_2^2 + \chi f_-^2$ which is evident from equation (4.49). Then, solutions (4.60) and (4.61) can be written in terms of hyperbolic tangent functions and u , φ_1 and φ_2 accepts the following solitary wave solutions

$$\begin{aligned} u &= \pm\,u_0\,\tanh[\zeta_0\,\zeta + D_2]\;\exp(i\theta), \\ \varphi_1 &= \bar{b}_0\,\tanh[\zeta_0\,\zeta + D_2], \\ \varphi_2 &= \bar{c}_0\,\tanh[\zeta_0\,\zeta + D_2], \end{aligned} \tag{4.62}$$

where

$$\begin{aligned} u_0 &= |f_-|, \quad \zeta_0 = [\frac{f_-^2}{2(k_1^2+k_2^2)}(\chi + \frac{\kappa_1 k_1^2 + \kappa_2 k_2^2}{(k_2^2-k_1^2)})]^{1/2}, \\ \bar{b}_0 &= \kappa_1 k_1 [\frac{2 f_-^2\,(k_1^2+k_2^2)}{(k_2^2-k_1^2)[\kappa_1 k_1^2+\kappa_2 k_2^2+\chi(k_2^2-k_1^2)]}]^{1/2}, \\ \bar{c}_0 &= \kappa_2 k_2 [\frac{2 f_-^2\,(k_1^2+k_2^2)}{(k_2^2-k_1^2)[\kappa_1 k_1^2+\kappa_2 k_2^2+\chi(k_2^2-k_1^2)]}]^{1/2}. \end{aligned} \tag{4.63}$$

4.2.2 By the tanh method

In this subsection, some special solutions of the degenerate GDS equations (4.46) will be given by the tanh method. In order to find the travelling wave solutions of the degenerate GDS equations (4.46), the following solutions are considered:

$$u = f(\zeta)\, e^{i\theta}, \quad \varphi_1 = g(\zeta), \quad \varphi_2 = h(\zeta), \tag{4.64}$$

where $\theta = l_1 x + l_2 y - l_3 t$, and the amplitudes of the short wave f, and the long travelling waves g and h are real functions of $\zeta = k_1 x + k_2 y - 2(k_1 l_1 + k_2 l_2)t$. Substituting solutions (4.64) into the system (4.46) leads to a set of coupled ordinary differential equations

$$\begin{aligned} & f^{''} - \alpha_1 f + \alpha_2 f^3 + \beta_1 g^{'} f + \beta_2 h^{'} f = 0, \\ & g^{''} + \gamma_1 f f^{'} = 0, \\ & h^{''} + \gamma_2 f f^{'} = 0, \end{aligned} \tag{4.65}$$

with

$$\begin{aligned} \alpha_1 &= \frac{l_1^2 + l_2^2 - l_3}{k_1^2 + k_2^2}, \quad & \alpha_2 &= \frac{-\chi}{k_1^2 + k_2^2}, \quad & \beta_1 &= \frac{k_1}{k_1^2 + k_2^2}, \\ \beta_2 &= \frac{k_2}{k_1^2 + k_2^2}, \quad & \gamma_1 &= \frac{2\kappa_1 k_1}{k_2^2 - k_1^2}, \quad & \gamma_2 &= \frac{2\kappa_2 k_2}{k_2^2 - k_1^2}. \end{aligned} \tag{4.66}$$

Here, it should be noted that equations (4.65) have the same structure as (4.49) in Section 4.2.1.

At this step, by substituting solutions (2.70) into the system (4.65) and by using (2.69) together with the relation $V^2 = 1 - U^2$ repeatedly, the term $f^{''}$ is balanced by the term f^3 while $g^{''}$ and $h^{''}$ are balanced with the term $f f^{'}$. Thus, the positive integers p, r and s are found as $p = r = s = 1$. Now, solutions (2.70) can be written as follows:

$$f(\zeta) = a_0 + a_1 U + \bar{a}_1 V, \quad g(\zeta) = b_0 + b_1 U + \bar{b}_1 V, \quad h(\zeta) = c_0 + c_1 U + \bar{c}_1 V. \tag{4.67}$$

Substituting (4.67) into (4.65) and equating the coefficient of each power of the variables U and V leads to the following nonlinear system of algebraic equations:

$$U^0 V^0 : \quad a_0[\alpha_1 - \alpha_2(a_0^2 + 3\bar{a}_1^2)] = 0,$$

$$
\begin{aligned}
&V: && \bar{a}_1[\alpha_1 - \alpha_2(3a_0^2 + \bar{a}_1^2)] = 0,\\
&U: && a_1(k^2 - \alpha_1 + 3\alpha_2 a_0^2) + \bar{a}_1[3\alpha_2 a_1\bar{a}_1 - k(\beta_1 b_1 + \beta_2 c_1)] = 0,\\
& && kb_1 - \gamma_1 a_1\bar{a}_1 = 0, \quad kc_1 - \gamma_2 a_1\bar{a}_1 = 0,\\
&UV: && a_0[6\alpha_2 a_1\bar{a}_1 - k(\beta_1 b_1 + \beta_2 c_1)] = 0,\\
& && \gamma_1 a_0 a_1 = 0, \quad \gamma_2 a_0 a_1 = 0,\\
&U^2: && a_0[3\alpha_2(a_1^2 - \bar{a}_1^2) + k(\beta_1\bar{b}_1 + \beta_2\bar{c}_1)] = 0,\\
& && \gamma_1 a_0\bar{a}_1 = 0, \quad \gamma_2 a_0\bar{a}_1 = 0,\\
&U^2V: && a_1[3\alpha_2 a_1\bar{a}_1 - k(\beta_1 b_1 + \beta_2 c_1)] - \bar{a}_1[\alpha_2\bar{a}_1^2 + k(2k - \beta_1\bar{b}_1 - \beta_2\bar{c}_1)] = 0,\\
& && 2k\bar{b}_1 + \gamma_1(a_1^2 - \bar{a}_1^2) = 0, \quad 2k\bar{c}_1 + \gamma_2(a_1^2 - \bar{a}_1^2) = 0,\\
&U^3: && \bar{a}_1[3\alpha_2 a_1\bar{a}_1 - k(\beta_1 b_1 + \beta_2 c_1)] - a_1[\alpha_2 a_1^2 - k(2k - \beta_1\bar{b}_1 - \beta_2\bar{c}_1)] = 0,\\
& && kb_1 - \gamma_1 a_1\bar{a}_1 = 0, \quad kc_1 - \gamma_2 a_1\bar{a}_1 = 0.
\end{aligned} \tag{4.68}
$$

Solving the nonlinear system of equations (4.68) with the help of Mathematica gives two types of solutions.

First type of solutions:

The parameters are found as

$$
\begin{aligned}
&k = \mp\sqrt{\alpha_1}, \quad a_1 = \mp\frac{2\sqrt{\alpha_1}}{\sqrt{2\alpha_2 - \beta_1\gamma_1 - \beta_2\gamma_2}},\\
&\bar{b}_1 = \frac{-2\alpha_1\gamma_1}{k(2\alpha_2 - \beta_1\gamma_1 - \beta_2\gamma_2)}, \quad \bar{c}_1 = \frac{-2\alpha_1\gamma_2}{k(2\alpha_2 - \beta_1\gamma_1 - \beta_2\gamma_2)},\\
&a_0 = \bar{a}_1 = b_1 = c_1 = 0, \quad b_0 \text{ and } c_0 \text{ arbitrary},
\end{aligned} \tag{4.69}
$$

and the solution functions f, g and h are found as

$$
\begin{aligned}
f(\zeta) &= \mp\frac{2\sqrt{\alpha_1}}{\sqrt{2\alpha_2 - \beta_1\gamma_1 - \beta_2\gamma_2}}\,\mathrm{sech}(\mp\sqrt{\alpha_1}\zeta),\\
g(\zeta) &= b_0 \mp \frac{2\gamma_1\sqrt{\alpha_1}}{2\alpha_2 - \beta_1\gamma_1 - \beta_2\gamma_2}\,\tanh(\mp\sqrt{\alpha_1}\zeta),\\
h(\zeta) &= c_0 \mp \frac{2\gamma_2\sqrt{\alpha_1}}{2\alpha_2 - \beta_1\gamma_1 - \beta_2\gamma_2}\,\tanh(\mp\sqrt{\alpha_1}\zeta).
\end{aligned} \tag{4.70}
$$

The solutions of the degenerate GDS equations (4.46) are obtained as [57]:

$$
\begin{aligned}
u &= \mp u_0\,\mathrm{sech}(\mp\zeta_0\,\zeta)\,\exp(i\theta),\\
\varphi_1 &= b_0 \mp \bar{b}_0\tanh(\mp\zeta_0\,\zeta),\\
\varphi_2 &= c_0 \mp \bar{c}_0\tanh(\mp\zeta_0\,\zeta),
\end{aligned} \tag{4.71}
$$

where

$$
\begin{aligned}
u_0 &= [\frac{2(k_1^2-k_2^2)(l_1^2+l_2^2-l_3)}{\kappa_1 k_1^2+\kappa_2 k_2^2+\chi(k_2^2-k_1^2)}]^{1/2}, \quad \zeta_0 = [\frac{l_1^2+l_2^2-l_3}{k_1^2+k_2^2}]^{1/2},\\
\bar{b}_0 &= 2\kappa_1 k_1 \frac{[(k_1^2+k_2^2)(l_1^2+l_2^2-l_3)]^{1/2}}{\kappa_1 k_1^2+\kappa_2 k_2^2+\chi(k_2^2-k_1^2)}, \quad \bar{c}_0 = 2\kappa_2 k_2 \frac{[(k_1^2+k_2^2)(l_1^2+l_2^2-l_3)]^{1/2}}{\kappa_1 k_1^2+\kappa_2 k_2^2+\chi(k_2^2-k_1^2)}.
\end{aligned} \tag{4.72}
$$

The solutions (4.71) coincide with (4.57) for $b_0 = c_0 = 0$ and $D_2 = 0$.

Second type of solutions:

The parameters are found as

$$
\begin{aligned}
k &= \mp\frac{1}{2}\sqrt{-2\alpha_1+\frac{\alpha_1\beta_1\gamma_1}{\alpha_2}+\frac{\alpha_1\beta_2\gamma_2}{\alpha_2}}, \qquad \bar{a}_1 = \mp\sqrt{\frac{\alpha_1}{\alpha_2}},\\
\bar{b}_1 &= \frac{\gamma_1\sqrt{\alpha_1}}{2\sqrt{\alpha_2(\beta_1\gamma_1+\beta_2\gamma_2-2\alpha_2)}}, \qquad \bar{c}_1 = \frac{\gamma_2\sqrt{\alpha_1}}{2\sqrt{\alpha_2(\beta_1\gamma_1+\beta_2\gamma_2-2\alpha_2)}},\\
a_0 &= a_1 = b_1 = c_1 = 0, \qquad b_0 \text{ and } c_0 \text{ arbitrary,}
\end{aligned} \tag{4.73}
$$

and the solution functions f, g and h are found as

$$
\begin{aligned}
f(\zeta) &= \mp\sqrt{\frac{\alpha_1}{\alpha_2}}\ \tanh(\mp\frac{1}{2}\sqrt{-2\alpha_1+\frac{\alpha_1\beta_1\gamma_1}{\alpha_2}+\frac{\alpha_1\beta_2\gamma_2}{\alpha_2}}\ \zeta),\\
g(\zeta) &= b_0 \mp \frac{\gamma_1\sqrt{\alpha_1}}{\sqrt{\alpha_2(\beta_1\gamma_1+\beta_2\gamma_2-2\alpha_2)}}\tanh(\mp\frac{1}{2}\sqrt{-2\alpha_1+\frac{\alpha_1\beta_1\gamma_1}{\alpha_2}+\frac{\alpha_1\beta_2\gamma_2}{\alpha_2}}\ \zeta),\\
h(\zeta) &= c_0 \mp \frac{\gamma_2\sqrt{\alpha_1}}{\sqrt{\alpha_2(\beta_1\gamma_1+\beta_2\gamma_2-2\alpha_2)}}\tanh(\mp\frac{1}{2}\sqrt{-2\alpha_1+\frac{\alpha_1\beta_1\gamma_1}{\alpha_2}+\frac{\alpha_1\beta_2\gamma_2}{\alpha_2}}\ \zeta).
\end{aligned} \tag{4.74}
$$

The solutions of the degenerate GDS equations (4.46) are obtained as [57]:

$$
\begin{aligned}
u &= \mp\ u_0\ \tanh(\mp\zeta_0\zeta)\ \exp(i\theta),\\
\varphi_1 &= b_0 \mp\ \bar{b}_0 \tanh(\mp\zeta_0\zeta),\\
\varphi_2 &= c_0 \mp\ \bar{c}_0 \tanh(\mp\zeta_0\zeta),
\end{aligned} \tag{4.75}
$$

where

$$
\begin{aligned}
u_0 &= [\frac{(l_3-l_1^2-l_2^2)}{\chi}]^{1/2}, \quad \zeta_0 = [\frac{l_3-l_1^2-l_2^2}{2(k_1^2+k_2^2)}(1+\frac{\kappa_1 k_1^2+\kappa_2 k_2^2}{\chi(k_2^2-k_1^2)})]^{1/2},\\
\bar{b}_0 &= \kappa_1 k_1[\frac{2(k_1^2+k_2^2)(l_3-l_1^2-l_2^2)}{\chi(k_2^2-k_1^2)[\kappa_1 k_1^2+\kappa_2 k_2^2+\chi(k_2^2-k_1^2)]}]^{1/2},\\
\bar{c}_0 &= \kappa_2 k_2[\frac{2(k_1^2+k_2^2)(l_3-l_1^2-l_2^2)}{\chi(k_2^2-k_1^2)[\kappa_1 k_1^2+\kappa_2 k_2^2+\chi(k_2^2-k_1^2)]}]^{1/2}.
\end{aligned} \tag{4.76}
$$

The solutions (4.75) coincide with (4.62) for $b_0 = c_0 = 0$ and $D_2 = 0$.

4.2.3 By variable separation approach

In this subsection, some special solutions of (4.46) will be presented by using a variable separation approach [46]. In [46], a variable separation procedure for DS system has been proposed by using a prior ansatz to its Hirota bilinear form [30, 34].

In order to obtain the bilinear form of (4.46) the following restrictions on the parameters will be made:

$$\kappa_1 - \kappa_2 = 2\,\chi, \quad \kappa_1 + \kappa_2 = 8, \tag{4.77}$$

and the degenerate GDS equations (4.46) will be rewritten as

$$\begin{aligned} &iu_t + u_{xx} + u_{yy} = \chi|u|^2u - (\varphi_{1,x} + \varphi_{2,y})u, \\ &\varphi_{1,xx} - \varphi_{1,yy} = (\chi + 4)(|u|^2)_x, \\ &\varphi_{2,xx} - \varphi_{2,yy} = (4 - \chi)(|u|^2)_y. \end{aligned} \tag{4.78}$$

Introducing new dependent variables F (real) and G (complex) by

$$u = \frac{G}{F}, \quad \varphi_1 = \frac{1}{2}(\chi + 4)(\log \mathrm{F})_x, \quad \varphi_2 = \frac{1}{2}(4 - \chi)(\log \mathrm{F})_y, \tag{4.79}$$

equations $(4.78)_2$ and $(4.78)_3$ can be written as

$$\begin{aligned} &(\log F)_{xxx} - (\log F)_{xyy} = 2(\frac{|G|^2}{F^2})_x, \\ &(\log F)_{xxy} - (\log F)_{yyy} = 2(\frac{|G|^2}{F^2})_y. \end{aligned} \tag{4.80}$$

Integrating $(4.80)_1$ in terms of x and $(4.80)_2$ in terms of y once gives

$$(\log F)_{xx} - (\log F)_{yy} = 2\,\frac{|G|^2}{F^2} \tag{4.81}$$

which implies

$$(F_{xx} - F_{yy})F - {F_x}^2 + {F_y}^2 = 2\,|G|^2. \tag{4.82}$$

On the other hand, substituting (4.79) into the equation $(4.78)_1$ leads to

$$\begin{aligned} i(\frac{FG_t - GF_t}{F^2}) + (\frac{FG_x - GF_x}{F^2})_x + (\frac{FG_y - GF_y}{F^2})_y - \chi\frac{G}{F^3}|G|^2 \\ +\frac{1}{2}[(\chi + 4)(\log F)_{xx} - (\chi - 4)(\log F)_{yy}]\frac{G}{F} = 0. \end{aligned} \tag{4.83}$$

Using (4.81) in (4.83) gives

$$\begin{aligned}
&i(FG_t - GF_t) + (FG_{xx} - GF_{xx} + FG_{yy} - GF_{yy}) \\
&+2\{[(F_x)^2 + (F_y)^2]\frac{G}{F} - F_xG_x - F_yG_y\} \\
&-\frac{\chi}{2}(FF_{xx} - FF_{yy} - (F_x)^2 + (F_y)^2)\frac{G}{F} \\
&+\frac{1}{2}[(\chi+4)(FF_{xx} - {F_x}^2) - (\chi-4)(FF_{yy} - {F_y}^2)]\frac{G}{F} = 0, \qquad (4.84)
\end{aligned}$$

which simply implies

$$i(FG_t - GF_t) + (G_{xx} + G_{yy})F + (F_{xx} + F_{yy})G - 2(F_xG_x + F_yG_y) = 0. \quad (4.85)$$

Thus, equations (4.85) and (4.81) can be rewritten as follows:

$$\begin{aligned}
&i(G_tF - GF_t) + (G_{xx}F - 2G_xF_x + GF_{xx}) + (G_{yy}F - 2G_yF_y + GF_{yy}) = 0, \\
&(F_{yy}F - 2F_yF_y + FF_{yy}) - (F_{xx}F - 2F_xF_x + FF_{xx}) = -4|G|^2, \qquad (4.86)
\end{aligned}$$

and which means

$$\begin{aligned}
&(i\mathcal{D}_t + \mathcal{D}_x^2 + \mathcal{D}_y^2)G \cdot F = 0, \\
&(\mathcal{D}_y^2 - \mathcal{D}_x^2)F \cdot F = -4|G|^2. \qquad (4.87)
\end{aligned}$$

Here, $\mathcal{D}$ is the Hirota bilinear operator defined by

$$\mathcal{D}_t^m\mathcal{D}_x^n\mathcal{D}_y^p F \cdot G \equiv (\partial_t - \partial_{t'})^m(\partial_x - \partial_{x'})^n(\partial_y - \partial_{y'})^p F(x,y,t) \cdot G(x',y',t')|_{x=x',\, y=y',\, t=t'}.$$

Applying the following characteristic transformation,

$$X = \frac{\sqrt{2}}{2}(x-y), \qquad Y = \frac{\sqrt{2}}{2}(x+y), \qquad (4.88)$$

which corresponds to the characteristics of hyperbolic equations $(4.78)_2$ and $(4.78)_3$, Hirota derivatives are obtained as

$$\mathcal{D}_X = \frac{\sqrt{2}}{2}(\mathcal{D}_x - \mathcal{D}_y), \quad \mathcal{D}_Y = \frac{\sqrt{2}}{2}(\mathcal{D}_x + \mathcal{D}_y), \qquad (4.89)$$

and (4.87) reads

$$\begin{aligned}
&(i\mathcal{D}_t + \mathcal{D}_X^2 + \mathcal{D}_Y^2)G \cdot F = 0, \\
&\mathcal{D}_X\mathcal{D}_Y F \cdot F = 2|G|^2. \qquad (4.90)
\end{aligned}$$

Now, equations (4.90) are assumed to possess the following variable separation form:

$$\begin{aligned}
&F = P(X,t) - Q(Y,t) + P(X,t)Q(Y,t),\\
&G = [P_X(X,t)\ Q_Y(Y,t)]^{1/2}\exp\ [i(R(X,t) + S(Y,t))],
\end{aligned} \tag{4.91}$$

where $P = P(X,t)$ and $Q = Q(Y,t)$ are Y and X independent, respectively.

After substituting the solution functions (4.91) into the first equation of (4.90) and vanishing both real and imaginary parts, respectively, the following equations are obtained:

$$\begin{aligned}
&P_X^2 Q_{YY}^2 + P_{XX}^2 Q_Y^2 - 2(P_X P_{XX} Q_Y^2 + Q_Y Q_{YYY} P_X^2)\\
&\qquad\qquad +4P_X^2 Q_Y^2 (R_t + R_X^2 + S_t + S_Y^2) = 0,\\
&(P - Q + PQ)P_X Q_Y^2 (P_t + 2P_X R_X)_X - 2(1+Q)P_X^2 Q_Y^2\\
&+(P - Q + PQ)P_X^2 Q_Y (Q_t + 2Q_Y S_Y)_Y - 2(1-P)P_X^2 Q_Y^2 = 0
\end{aligned} \tag{4.92}$$

which may be rewritten as

$$\begin{aligned}
&Q_Y^2[P_{XX}^2 - 2P_X P_{XXX} + 4P_X^2(R_t + R_X^2)]\\
&\qquad\qquad +P_X^2[Q_{YY}^2 - 2Q_Y Q_{YYY} + 4Q_Y^2(S_t + S_Y^2)] = 0,\\
&[-2(1+Q)P_X Q_Y + (P - Q + PQ)Q_Y \frac{\partial}{\partial X}](P_t + 2P_X R_X)\\
&+[2(1-P)P_X Q_Y + (P - Q + PQ)P_X \frac{\partial}{\partial Y}](Q_t + 2Q_Y S_Y) = 0.
\end{aligned} \tag{4.93}$$

According to the fact that P and R are only functions of X and t, Q and S are only functions of Y and t equations (4.93) are reduced to

$$\begin{aligned}
&P_t + 2P_X R_X + C_1(t)(1-P) = 0,\\
&Q_t + 2Q_Y S_Y + C_1(t)(1+Q) = 0,\\
&4P_X^2(R_t + R_X^2) + P_{XX}^2 - 2P_X P_{XXX} - C_2(t)P_X^2 = 0,\\
&4Q_Y^2(S_t + S_Y^2) + Q_{YY}^2 - 2Q_Y Q_{YYY} + C_2(t)Q_Y^2 = 0,
\end{aligned} \tag{4.94}$$

where $C_1 = C_1(t)$ and $C_2 = C_2(t)$ are arbitrary functions introduced in the variable separation procedure.

In order to be able to simplify equations (4.94), the functions P and Q will be introduced as

$$P = BU + 1, \quad Q = \frac{V}{B} - 1, \quad B = \exp\int^t C_1(t')dt', \tag{4.95}$$

and the first two equations of (4.94) will be written as

$$
\begin{aligned}
R_X(X,t) &= -\frac{U_t}{2U_X} + \frac{C_1(t)UB - UB_t}{2U_X B}, \\
S_Y(Y,t) &= -\frac{V_t}{2V_Y} - \frac{C_1(t)VB - VB_t}{2V_Y B}.
\end{aligned} \tag{4.96}
$$

By integrating first equation of (4.96) with respect to X and the second one with respect to Y the functions $R(X,t)$ and $S(Y,t)$ are found as

$$
\begin{aligned}
R(X,t) &= -\int^X \frac{U_t}{2U_X} dX + C_3(t), \\
S(Y,t) &= -\int^Y \frac{V_t}{2V_Y} dY + C_4(t),
\end{aligned} \tag{4.97}
$$

where $C_3 = C_3(t)$ and $C_4 = C_4(t)$ are integration functions.

Substituting equations (4.97) into equations $(4.94)_3$ and $(4.94)_4$ leads to the following two (1+1) dimensional equations about fields $U(X,t)$ and $V(Y,t)$, respectively,

$$
\begin{aligned}
-2U_X^2 \int^X \Big(\frac{U_t}{U_X}\Big)_t dX + U_X^2(4C_{3t} - C_2) - 2U_X U_{XXX} + U_{XX}^2 + U_t^2 = 0, \\
-2V_Y^2 \int^Y \Big(\frac{V_t}{V_Y}\Big)_t dY + V_Y^2(4C_{4t} - C_2) - 2V_Y V_{YYY} + V_{YY}^2 + V_t^2 = 0.
\end{aligned} \tag{4.98}
$$

Dividing $(4.98)_1$ by U_X^2 and differentiating the resulting equation with respect to X once gives,

$$
U_t^2 U_{XX} + U_{tt} U_X^2 - 2U_t U_{tX} U_X + U_{XX}^3 + U_X^2 U_{XXXX} - 2U_X U_{XX} U_{XXX} = 0, \tag{4.99}
$$

and similarly dividing $(4.98)_2$ by V_y^2 and differentiating the resulting equation with respect to y once gives,

$$
V_t^2 V_{YY} + V_{tt} V_Y^2 - 2V_t V_{tY} V_Y + V_{YY}^3 + V_Y^2 V_{YYYY} - 2V_Y V_{YY} V_{YYY} = 0. \tag{4.100}
$$

Five type of special solutions of equations (4.99) and (4.100) will be given and with the help of these solution functions U and V; first P, Q and B in (4.95) will be calculated and then F and G in (4.91) will be obtained. Finally, by using the transformation (4.88) backwards, the solutions of the degenerate GDS equations will be written in terms of x and y.

First type of solutions:

$$U = \exp(k_1 X + \omega_1 t), \quad V = \exp(k_2 Y + \omega_2 t), \tag{4.101}$$

where k_1, k_2, ω_1 and ω_2 are arbitrary constants. The solution functions are

$$\begin{aligned}
u &= \pm \frac{\sqrt{2(k^2 - l^2)}}{\Delta} \exp\{kx + ly + 2(k\omega + l\Omega)t \\
&\qquad\qquad -i[\omega x + \Omega y + (\omega^2 + \Omega^2 - k^2 - l^2)t]\}, \\
\varphi_1 &= \frac{(\chi + 4)k}{\Delta} \exp[2kx + 2ly + 4(k\omega + l\Omega)t], \\
\varphi_2 &= \frac{(4 - \chi)l}{\Delta} \exp[2kx + 2ly + 4(k\omega + l\Omega)t], \\
\Delta &= 1 + \exp[2kx + 2ly + 4(k\omega + l\Omega)t],
\end{aligned} \tag{4.102}$$

where

$$k = \frac{k_1 + k_2}{2\sqrt{2}}, \quad l = \frac{k_2 - k_1}{2\sqrt{2}}, \quad \omega = \frac{1}{2\sqrt{2}}(\frac{\omega_2}{k_2} + \frac{\omega_1}{k_1}), \quad \Omega = \frac{1}{2\sqrt{2}}(\frac{\omega_2}{k_2} - \frac{\omega_1}{k_1}).$$

By writing the exponential functions in terms of trigonometric functions, the solutions (4.102) can be rewritten as follows [57]:

$$\begin{aligned}
u &= \pm\, u_0 \operatorname{sech}\zeta \exp(i\theta), \\
\varphi_1 &= b\,(1 + \tanh\zeta), \\
\varphi_2 &= c\,(1 + \tanh\zeta),
\end{aligned} \tag{4.103}$$

where

$$\begin{aligned}
&\zeta = kx + ly + 2(k\omega + l\Omega)t, \quad \theta = -[\omega x + \Omega y + (\omega^2 + \Omega^2 - k^2 - l^2)t], \\
&u_0 = [(k^2 - l^2)/2]^{1/2}, \quad b = \frac{(\chi + 4)k}{2}, \quad c = \frac{(4 - \chi)l}{2}.
\end{aligned} \tag{4.104}$$

It is easily seen that the solutions (4.103) is in complete agreement with that of (4.71) obtained in Section 4.2.2. Actually, by assuming the constraints on parameters given by (4.77) in equations (4.71) and (4.72), and considering the relations

$$\begin{aligned}
&\omega = -l_1, \quad \Omega = -l_2, \quad b = b_0, \quad c = c_0, \\
&k = \pm\, k_1\,(\frac{l_1^2 + l_2^2 - l_3}{k_1^2 + k_2^2})^{1/2}, \quad l = \pm\, k_2\,(\frac{l_1^2 + l_2^2 - l_3}{k_1^2 + k_2^2})^{1/2},
\end{aligned} \tag{4.105}$$

exactly the same solutions given by (4.103) and (4.104) are obtained.

Second type of solutions:

$$U = k_1X + \omega_1 t, \quad V = k_2Y + \omega_2 t, \tag{4.106}$$

where k_1, k_2, ω_1 and ω_2 are arbitrary constants. The solutions are found as [57]:

$$\begin{aligned}
u &= \pm\ \frac{\sqrt{2k}}{\Delta}\ \exp\{-\frac{i}{4k^2}[2k(\omega x + \Omega y) + (\omega^2 + \Omega^2)t]\},\\
\varphi_1 &= \frac{(\chi + 4)}{\Delta}\ (kx + \omega t),\\
\varphi_2 &= \frac{(\chi - 4)}{\Delta}\ (ky + \Omega t),\\
\Delta &= 1 + k(x^2 - y^2) + 2(\omega x - \Omega y)t + \frac{1}{k}(\omega^2 - \Omega^2)t^2,
\end{aligned} \tag{4.107}$$

where

$$k = \frac{k_1k_2}{2}, \qquad \omega = \frac{1}{2\sqrt{2}}(k_1\omega_2 + k_2\omega_1), \quad \Omega = \frac{1}{2\sqrt{2}}(k_1\omega_2 - k_2\omega_1).$$

Third type of solutions:

$$U = (k_1X + \omega_1 t)^3, \quad V = (k_2Y + \omega_2 t)^3, \tag{4.108}$$

where k_1, k_2, ω_1 and ω_2 are arbitrary constants. The solutions are found as [57]:

$$\begin{aligned}
u &= \pm\ \frac{3k\sqrt{2k}}{\Delta}\ [x^2 - y^2 + 4(\omega x - \Omega y)t + 4(\omega^2 - \Omega^2)t^2]\\
&\qquad\qquad\qquad \times\exp\{-i[\omega x + \Omega y + (\omega^2 + \Omega^2)t]\},\\
\varphi_1 &= \frac{3(\chi + 4)k^3}{\Delta}(x + 2\omega t)[x^2 - y^2 + 4(\omega x - \Omega y)t + 4(\omega^2 - \Omega^2)t^2)]^2,\\
\varphi_2 &= \frac{3(\chi - 4)k^3}{\Delta}(y + 2\Omega t)[x^2 - y^2 + 4(\omega x - \Omega y)t + 4(\omega^2 - \Omega^2)t^2)]^2,\\
\Delta &= 1 + k^3[x^2 - y^2 + 4(\omega x - \Omega y)t + 4(\omega^2 - \Omega^2)t^2]^3,
\end{aligned} \tag{4.109}$$

where

$$k = \frac{k_1k_2}{2}, \quad \omega = \frac{1}{2\sqrt{2}}(\frac{\omega_2}{k_2} + \frac{\omega_1}{k_1}), \quad \Omega = \frac{1}{2\sqrt{2}}(\frac{\omega_2}{k_2} - \frac{\omega_1}{k_1}).$$

Fourth type of solutions:

$$U = k_1X + \omega_1 t + \sinh(k_1X + \omega_1 t), \quad V = k_2Y + \omega_2 t + \sinh(k_2Y + \omega_2 t), \tag{4.110}$$

where k_1, k_2, ω_1 and ω_2 are arbitrary constants. The solutions are found as [57]:

$$
\begin{aligned}
u &= \pm \frac{4\sqrt{2kl}}{\Delta} \cosh[k(x-y)+\omega t] \cosh[l(x+y)+\Omega t)] \\
&\quad \times \exp\{-\frac{i}{4}[(\frac{\omega}{k}+\frac{\Omega}{l})x+(\frac{\Omega}{l}-\frac{\omega}{k})y+(\frac{\omega^2}{2k^2}+\frac{\Omega^2}{2l^2}-8k^2-8l^2)t]\}, \\
\varphi_1 &= \frac{2(\chi+4)}{\Delta}\{l[2k(x-y)+2\omega t+ \ \sinh[2k(x-y)+2\omega t]\cosh^2[l(x+y)+\Omega t] \\
&\quad +k[2l(x+y)+2\Omega t+\sinh[2l(x+y)+2\Omega t]\cosh^2[k(x-y)+\omega t]\}, \\
\varphi_2 &= \frac{2(4-\chi)}{\Delta}\{l[2k(x-y)+2\omega t+ \ \sinh[2k(x-y)+2\omega t]\cosh^2[l(x+y)+\Omega t] \\
&\quad -k[2l(x+y)+2\Omega t+\sinh[2l(x+y)+2\Omega t]\cosh^2[k(x-y)+\omega t]\}, \\
\Delta &= 1+\{2k(x-y)+2\omega t+\sinh[2k(x-y)+2\omega t]\} \\
&\qquad\qquad \times\{[2l(x+y)+2\Omega t+\sinh[2l(x+y)+2\Omega t]\},
\end{aligned}
$$

where

$$
k=\frac{k_1}{2\sqrt{2}}, \quad l=\frac{k_2}{2\sqrt{2}}, \quad \omega=\frac{\omega_1}{2}, \quad \Omega=\frac{\omega_2}{2}.
$$

Fifth type of solutions:

$$
U=k_1X+\omega_1 t+\sin(k_1X+\omega_1 t), \quad V=k_2Y+\omega_2 t+\sin(k_2Y+\omega_2 t), \tag{4.111}
$$

where k_1, k_2, ω_1 and ω_2 are arbitrary constants. The solutions are found as [57]:

$$
\begin{aligned}
u &= \pm \frac{4\sqrt{2kl}}{\Delta} \cos[k(x-y)+\omega t] \cos[l(x+y)+\Omega t)] \\
&\quad \times \exp\{-\frac{i}{4}[(\frac{\omega}{k}+\frac{\Omega}{l})x+(\frac{\Omega}{l}-\frac{\omega}{k})y+(\frac{\omega^2}{2k^2}+\frac{\Omega^2}{2l^2}-8k^2-8l^2)t]\}, \\
\varphi_1 &= \frac{2(\chi+4)}{\Delta}\{l[2k(x-y)+2\omega t+ \ \sin[2k(x-y)+2\omega t]\cos^2[l(x+y)+\Omega t] \\
&\quad +k[2l(x+y)+2\Omega t+\sin[2l(x+y)+2\Omega t]\cos^2[k(x-y)+\omega t]\}, \\
\varphi_2 &= \frac{2(4-\chi)}{\Delta}\{l[2k(x-y)+2\omega t+ \ \sin[2k(x-y)+2\omega t]\cos^2[l(x+y)+\Omega t] \\
&\quad -k[2l(x+y)+2\Omega t+\sin[2l(x+y)+2\Omega t]\cos^2[k(x-y)+\omega t]\}, \\
\Delta &= 1+\{2k(x-y)+2\omega t+\sin[2k(x-y)+2\omega t]\} \\
&\qquad\qquad \times\{[2l(x+y)+2\Omega t+\sin[2l(x+y)+2\Omega t]\},
\end{aligned}
$$

where

$$
k=\frac{k_1}{2\sqrt{2}}, \quad l=\frac{k_2}{2\sqrt{2}}, \quad \omega=\frac{\omega_1}{2}, \quad \Omega=\frac{\omega_2}{2}.
$$

As a final remark, it should be noted that since the system (4.46) is elliptic-hyperbolic-hyperbolic, all of the solutions given in this subsection accepts $x+y$ and $x-y$ structures which are the form of the characteristics of hyperbolic systems.

CONCLUSIONS

In the present study, the problem of (2+1) (two spatial and one temporal) dimensional nonlinear wave modulation is considered in an infinite, homogenous, weakly nonlinear and weakly dispersive elastic medium and (2+1) dimensional nonlinear evolution equations are derived describing the asymptotic behavior as a function of the spatial coordinates. While deriving the equations an asymptotic technique called reductive perturbation method is used and it is shown that the problem of wave modulation is characterized by a system of three nonlinear partial differential equations. These equations involve interactions of a free short transverse, a free long longitudinal and a free long transverse wave modes, and are called the "generalized Davey-Stewartson" (GDS) equations [22]. Under some restrictions on parameter values, it is shown that the GDS equations reduce to the well-known Davey-Stewartson and under further restrictions to the nonlinear Schrödinger equations. The naming "generalized Davey-Stewartson" is also justified from a group theoretical point of view in a recent study [58]. In [58], the Lie symmetry algebra of the GDS equations are computed and it is shown that under certain conditions imposed on parameters in the system it is infinite-dimensional and isomorphic to that of the standard integrable Davey-Stewartson equations. The conditions imposed in this work correspond to the restrictions that reduce the GDS system to the DS system.In another recent work, further qualitative properties of the GDS system is investigated. In [59], global existence and nonexistence results of the GDS system in the elliptic-elliptic-elliptic case are given based on some physically obvious Noetherian symmetries (time-space translations and constant change of phase) .

In the thesis, it is also observed that the GDS equations are not valid for the long-wave short-wave resonant case, i.e. where the phase velocity of the long longitudinal wave is equal to the group velocity of the short transverse wave.

For this case, new evolution equations are derived characterizing the problem and are called long-wave short-wave interaction equations. Besides, degenerate GDS equations, which are obtained when one of the coefficients of the GDS equations becomes zero, are also considered.

Finally, some special solutions of the GDS equations, the long-wave short-wave interaction equations and the degenerate GDS equations are obtained by using various methods. The special solutions of the GDS and long-wave short-wave interaction equations are given by using Jacobian elliptic functions and the tanh method. These two methods are based on the travelling wave solutions. The special solutions of the degenerate GDS equations are also obtained by using the two above mentioned methods. Besides, the bilinear form of the degenerate GDS equations are calculated and the special solutions are obtained by using a variable separation approach [57].

The GDS system is one that is attacking attention recently. Two recent studies on the GDS system can be noted in this connection. In the first one, the travelling wave solutions of the GDS system are studied by deriving Pohozaev-type identities [60]. In [60], some necessary conditions on the parameters for the existence of the travelling waves are established. In the second one, two results on GDS system is given, both following from the pseudo-conformal invariance of its solutions [61]. Hence, a perturbative approach to the GDS system has cleared the way for an understanding the existence properties of the actual solution and an approximate symmetry property for a restricted parameter values.

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VITA

Ceni Babaoğlu was born in Istanbul, in 1975, with the surname Duhanyan. After she graduated from Beyoglu Anadolu Lisesi in 1993, she entered Istanbul Technical University where she graduated as a Mathematical Engineer in 1997. She received her M.Sc. degree in Mathematical Engineering from the Institute of Science and Technology of the same university. She enrolled the Ph.D program of the same institute in 1999. She has been working as a research and teaching assistant at the Mathematics Department of Istanbul Technical University since 1997. She is married and has one child.