INTEGRATION OF THE DETERMINISTIC FUNCTIONS WITH RESPECT TO FRACTIONAL BROWNIAN MOTION

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[section]

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ABSTRACT

INTEGRATION OF THE DETERMINISTIC FUNCTIONS WITH RESPECT TO FRACTIONAL BROWNIAN MOTION

In this thesis, definition and the characteristic properties of fractional Brownian motion are presented and the general idea for the integration of deterministic functions is discussed with a specific class of integrands.

First, some notions and facts from probability theory are introduced. The definition and basic properties of Gaussian random variables and processes are discussed and their relation with the self similar, stationary processes is given. Moreover, covariance function of the self similar Gaussian processes with stationary increments is characterized as in Embrechts and Maejima's book.

Next, we give two representations of fractional Brownian motion. One is defined as a stochastic integral with respect to Brownian motion as in Embrechts and Maejima's book and the other with the fractional integral as Pipiras and Taqqu do. Then we consider a class of deterministic integrands for the case $H > 1/2$ which is given by Kleptsyna, LeBreton and Roubaud, and we discuss its completeness.

Finally, an example of a complete class of integrands for the case $H < 1/2$ is introduced as Pipiras and Taqqu do.

ÖZET

DETERMİNİSTİK FONKSİYONLARIN KESİRLİ BROWN HAREKETİNE GÖRE İNTEGRALLERİNİ ALMA

Bu tezde kesirli Brown hareketinin tanımı ve karakteristik özellikleri sunulduktan sonra deterministik fonksiyonların kesirli Brown hareketine göre integrallerini almada uygulanan bir yöntem tartışıldı.

Öncelikle, olasılık teorisinden bazı tanım ve teoremler verildi. Normal dağılıma sahip rastgele değişkenler ve stokastik süreçlerin tanımı ve temel özellikleri verilerek kendine benzer ve durağan süreçlerle ilişkileri belirtildi. Ayrıca, kendine benzer durağan normal dağılımlı süreçlerin kovaryans fonksiyonu karakterize edildi.

Daha sonra, kesirli Brown hareketinin iki temsil edilişi verildi. Biri Embrechts ve Maejima'nın kitabında verildiği gibi Brown hareketine göre stokastik integralle olan bir temsildir, diğeri ise Pipiras ve Taqqu tarafından verilen kesirli integrale göre olandır. Bundan sonra, $H > 1/2$ için Kleptyna, LeBreton ve Roubaud tarafından verilen deterministik integrand kümesi ve kümenin tamlık problemi incelendi.

Son olarak, $H < 1/2$ için tam olan bir integrand kümesinin varlığı Pipiras ve Taqqu' nun çalısmasında ki gibi gösterildi.

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LIST OF SYMBOLS/ABBREVIATIONS

$B^H = \{B_t^H\}_{t\in T}$	Fractional Brownian motion with Hurst index H
$B^\kappa = \{B_t^\kappa\}_{t\in T}$	Fractional Brownian motion with parameter κ
$Cov_X(s,t)$	The covariance function of the process $\mathbf{X}$
$\mathcal{C}$	A set of deterministic functions on the interval $[0,T]$
$D_{T-}^\alpha \phi$	The right sided fractional derivative of order $0 < \alpha < 1$ on interval $[0,T]$ of a function ϕ
$E[\xi]$	Expectation of the random variable ξ
$E[\xi\|\mathcal{G}]$	The conditional expectation of ξ given $\mathcal{G}$
$\mathbf{F} = \{\mathcal{F}_t\}_{t\in T}$	A filtration
$\mathcal{F}_t$	The σ-algebra generated by $\{X_s; s \leq t\}$
$\mathcal{F}_t^+$	$\bigcap_{u>t} \mathcal{F}_u, t \geq 0$
$(f,g)_{\mathcal{C}}$	The inner product on the set $\mathcal{C}$
$(f,g)_{\Lambda_T^\kappa}$	The inner product on the space Λ_T^κ
H	Hurst index
I_A	The characteristic function of a set A
$I_{T-}^\alpha f$	The right sided fractional integral of order $\alpha > 0$ on an interval $[0,T]$ of a function $f \in L^1[0,T]$
$L^1[0,T]$	The set of all integrable functions on $[0,T]$
$L^2[0,T]$	The set of all square integrable functions on $[0,T]$
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\overline{\mathbb{R}}_+$	The set of extended non-negative real numbers
$R_H(s,t)$	Covariance function of the fBm with Hurst index H
$\mathbb{R}^n$	n dimensional Euclidean space
$\overline{sp}_R(B^\kappa)$	The closure of all linear combination of increments of B^κ
T	The closed interval from 0 to T
$v_p(f;\pi)$	p-variation of f along the partition π.
$\mathbf{W} = \{W_t, t \geq 0\}$	standard Brownian motion
$\mathbf{X} = \{X_t\}_{t\in T}$	A stochastic process

X_t	A random variable

$\beta(p,q)$	The Beta function
$\Gamma(p)$	The Gamma function
κ	The modification of Hurst index $\kappa = H - 1/2$
Λ_T^κ	An inner product space on $[0,T]$
ξ_T	Elementary functions on $[0,T]$
π	a partition of T
$\lvert\pi\rvert$	The mesh of the partition π
τ	Stopping time
$\phi_X(t)$	The characteristic function of the random variable X
$(\Omega, \mathcal{F}, P)$	A probability space

a.e.	Almost every
a.s.	Almost surely
fBm	fractional Brownian motion
H-ss	H-self similar
r.v.	Random variable
PDE	Partial Differential Equation
SDE	Stochastic Differential Equation

1. INTRODUCTION

The fractional Brownian motion is a generalization of the well-known process Brownian motion. The fractional Brownian motion was originally introduced by Kolmogorov[1], in 1940 when he was interested in modelling turbulence. Kolmogorov did not used the name 'fractional Brownian motion'. He called the process 'Wiener spiral'. Kolmogorov studied the fractional Brownian motion within a Hilbert space framework and deduce its covariance function from a scaling property that we now call self-similarity.[2]. This process firstly was called 'fractional Brownian motion' by Mandelbrot and Van Ness.[3]. They defined the fractional Brownian motion by using a fractional integral of Weyl type.

Since any Gaussian process is characterized by its covariance function, we can easily deduce its basic properties such as being self similar, Markovian, etc., from this function. The covariance function of fractional Brownian motion includes a parameter H. The notion for the index H and current parametrization with range $(0, 1)$ are due to Mandelbrot and Van Ness also[3]. The parameter H is called the Hurst index after an English hydrologist who studied the memory of Nile River maxima in connection of designing water reservoirs.[2].

The fBm model is widely applied in telecommunications and is also of interest in finance via the stochastic differential equations driven by fractional Brownian motion.[4]. Integration with respect to fBm has many potential applications.[5]. However, there are two important difficulties on the integration theory which differs from the classical stochastic integration theory. On one hand, the paths of fBm are of unbounded variation and hence the usual Lebesque-Stieljets integration cannot be applied. One cannot use the usual Itô's stochastic calculus because fBm is not a semimartingale. The most important constructions, for general H, to be found in literature either use a specific class of integrands, use path wise integration or base the definition on Malliavin calculus[6]. In each of these approaches, a version of the Itô formula is deduced which allows for a calculus of stochastic differentials[7]. As far as we know, no general consensus on the best approach exists.

Brownian motion has been well established in finance. Indeed, introduction of the Brownian motion based Black-Scholes formulation of vanilla options by Black, Scholes and Merton marked the advent of mathematical finance. Nevertheless, classical mathematical models of financial assets are far from perfect. Two apparent problem exist in the Black-Scholes formulation, namely financial processes are not wholly Gaussian and Markovian in distribution.[8]. After the 1987 market crash, industry and researchers began to take note of the heavy-tail distribution of financial assets and a series of models has been developed using more general and heavy-tailed processes.[9].

The second problem leads to long-range dependence. For a couple of decades, the general consensus is to assume that all information is contained within current asset price and hence it is reasonable to assume a Markovian process. However, technical traders have consistently beaten the market using long-term memory strategies. This motivated a series of academic studies further purporting the existence of a non-Markovian market.[10]. To compensate, stochastic volatility models have been developed that can produce quasi long-range dependence . However, these models are highly intractable both analytically and numerically as they lead to high dimensional PDE's with variable coefficients. Fractional Brownian motion deals with the second problem while still assuming a Gaussian process. Nevertheless, it offers the promise of giving simple, tractable solutions to pricing financial options and presents a natural way of modeling long-range dependence.

The numerous properties of fBm easily illuminate both benefits and shortcomings of using fBm in modeling financial instruments. H-ss, where H can be between 0 and 1, makes fBm more flexible as a modeling tool than standard Brownian motion that only allows 1/2-ss. The existence of long-range dependence and positive correlation of future and past increments makes fBm especially an attractive pricing tool. Statistical analysis has indicated that most markets are monofractal have Hurst parameter between 0.5 and 1.[11]. Thus, for financial purposes, Hurst parameters are usually assumed in $(1/2, 1)$. However, if $H \neq 1/2$, we can not use semimartingale and non-Markovian properties to construct an integration theory. This makes the task infinitely harder since few results from classical stochastic calculus can be directly used.

A myriad of methods have been developed for integrating fBm. Two systems bear the most important in finance: The *fractional pathwise integral* and *the fractional Wick-Ito integral*. In the second integration method Wick product is used and H is assumed between 1/2 and 1.[12].

In relations to Brownian motion calculus, fractional pathwise integral mirrors a Stratonovich integral[13], whereas fractional Wick -Ito integral emulates an Ito type calculus. Fractional pathwise integral was developed by Lin[14] during 1995 and found to produce an arbitrage market by Rogers[15] in 1997 due to misbehavior of Gaussian kernel near 0. Apparently, pricing in an arbitrageable system is undesirable. And the use of fBm has been deterred till only recently in 2000 when Duncan and Hu[12] and Øksendal[16] utilized the Wick-Ito integral. Wick introduced the Wick product in 1950.[17]. Hida and Ikeda introduced Wick product in analyzing stochastic processes in 1965.[18]. This new type of integration suprisingly produced a no-arbitrage market. Cheridito also constructed arbitrage strategies for a financial market that consist of a money market account and a stock whose discounted price follows a fractional Brownian motion with drift or an exponential fractional Brownian motion with drift. Then he showed how arbitrage can be exluded from these models by restricting the class of trading strategies.[19].

The history of the stochastic integration and the modelling of a risky asset prices both begin with Brownian motion, so let us briefly remind them. The earliest attempt to model Brownian motion mathematically in finance was made by Bachelier who created a model while deriving the dynamic behavior of the Paris stock market in 1990[6]. This date is considered as the beginning of mathematical finance. The pioneering analysis of the stock and option markets contains several ideas of enormous value in both finance and probability. In particular, the theory of Brownian motion, one of the most important mathematical discoveries of the twentieth century, was initiated and used for mathematical modeling of price movements and evaluation of contingent claims in financial markets. Following from the Jarrow and Protter's article[20] *"The thesis of Louis Bachelier, together with his subsequent works, deeply influenced the whole development of stochastic calculus and mathematical finance. The first part of the Bachelier's thesis contains a detailed description of products available at that time in French stock market, such as forward contract and options. After the financial preliminaries, Bachelier begins the mathematical modeling of stock price movements and formulates the principle that "the expectation of the speculator is zero." Obviously, he understands here by expectation the conditional expectation given the past information. In other words, he implicitly accepts as an axiom that the market evaluates assets using a martingale measure. The further hypothesis is that the price evolves as a continuous Markov process, homogeneous in time and space. Exploiting the ideas of Central Limit Theorem, and realizing that market noise should be without memory, he reasoned that increments of stock prices should be independent and normally distributed. He combined his reasoning with Markov property and semigroups, and connected Brownian motion with heat equation, using that the Gaussian kernel is the fundamental solution of the heat equation. The thesis can be viewed as the origin of mathematical finance and of several important branches of stochastic calculus such as the theory of Brownian motion, Markov processes, diffusion processes, and even weak convergence in in functional space. Of course, the reasoning was not rigorous but it was, on the intuitive level, basically correct. This is really astonishing, because at the beginning of the century the mathematical foundations of probability did not exist. A. Markov started his studies on what are now called Markov chains only in 1906, and the concept of conditional expectations with respect to an arbitrary random variable or σ-algebra was developed only in 1930s."*

In 1913 Daniell's approach to measure theory appeared, and it was these ideas, combined with Fourier series, that N. Wiener used in 1923 to construct Brownian motion.[21]. Then Wiener and others others proved many properties of the paths of the Brownian motion. Two key properties relating to stochastic integration are that

(1) the paths of Brownian motion have a non zero finite quadratic variation, such that on an interval (s, t), the quadratic variation is $(t - s)$.

(2) the paths of Brownian motion have infinite variation on compact intervals.[20].

The next step in the ground work for stochastic integration lay with A. N. Kolmogorov. The beginnings of the theory of stochastic integration, from the non-finance perspective were motivated by the theory of Markov processes in which Kolmogorov played a fundamental role. Indeed, in 1931 two years before his famous book establishing a rigorous mathematical basis for probability theory using measure theory, Kolmogorov refers to and briefly explains Bachelier's construction of Brownian motion.[22].

As for the contributions of Kiyosi Itô, when he studied to model Markov processes, Itô constructed a stochastic differential equation of the form[23]:

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt,$$

where W_t represents a standard Wiener process. He had two problems: one was to make sense of the stochastic differential $\sigma(X_t)dW_t$ and the other was to connect Kolmogorov's work on Markov processes with his interpretation. His efforts resulted in his paper[24] in 1935, where he stated an proved what is known as as **Itô's formula**:

$$f(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X,X]_t,$$

Since Brownian motion has paths of unbounded variation almost surely on any finite time interval, Itô knew that it was not possible to integrate all continuous stochastic processes. One of his key insights was to to limit his space of integrands to those that were, as he called it, non-anticipating. That is, he only allows integrands that are adapted to the underlying filtration of σ-algebras generated by the Brownian motion. This allows him to make use of the independence of the increments of Brownian motion to establish the L^2 isometry

$$E[(\int_0^t H_s dW_s)^2] = E[\int_0^t H_s^2 ds]$$

Then J. L. Doob extended Itô's stochastic integral for Brownain motion to martingales.[25]. One of the aims the ongoing research on fBm is to obtain the theory of integration with respect to fBm that has similar power to the standard Brownian motion.

In this thesis, we start with some basic definitions and theorems from probability theory. Since fractional Brownian motion is also a Gaussian process, we discuss the basic properties of Gaussian processes. Following[7], we characterize the covariance function of a H-self similar(H-ss) Gaussian process with stationary increments. We call these special class of Gaussian processes as the 'fractional Brownian motion' as in [7]. After that we introduce the basic properties of fBm such as being H-ss with stationary increments and being non-Markovian. In the next chapter, we introduce the fractional integrals and derivatives of Riemann-Lioville type and give the relation between these two operators as in [26]. Then we relate them with the representation of fBm as Pipiras and Taqqu do in [5]. After that we consider the class of deterministic functions with an inner product which is defined in [27] and stochastic integral definition for this class. At the end of the thesis, we give the Pipiras and Taqqu's proof[5] of the incompleteness of this class of functions. We also give a complete class of functions for the case $H < 1/2$ as in [5].

2. PRELIMINARIES

2.1. Basic Definitions and Notations

To fix our notation, we begin with some elementary notions from measure theory. Let Ω be an abstract set. A *σ-algebra or σ-field* in Ω is defined as a nonempty collection $\mathcal{F}$ of subsets of Ω such that $\mathcal{F}$ is closed under the countable unions and intersections as well as complementation. A *measurable space* is a pair $(\Omega, \mathcal{F})$, where Ω is a set and is $\mathcal{F}$ a σ-field in Ω. Given two measurable spaces $(\Omega_1, \mathcal{F})$ and $(\Omega_2, \mathcal{T})$, a mapping $\xi : \Omega_1 \to \Omega_2$ is said to be *$\mathcal{F}/\mathcal{T}$-measurable or simply measurable* if $\xi^{-1}B \in \mathcal{F}$ for every $B \in \mathcal{T}$.

A *measure* on $(\Omega, \mathcal{F})$ is defined as a countably additive set function $\mu : \mathcal{F} \to \overline{\mathbb{R}}_+$ with $\mu(\emptyset) = 0$. The triple $(\Omega, \mathcal{F}, \mu)$ is called a *measure space.*

Throughout the thesis, $(\Omega, \mathcal{F}, P)$ will denote a given probability space that means $(\Omega, \mathcal{F})$ is a measurable space and P is a probability measure, has total mass 1, on $(\Omega, \mathcal{F})$. A measurable mapping ξ of Ω into some measurable space $(S, \mathcal{S})$ is called a *random element* in S. A random element is called a *random variable* when $S = \mathbb{R}$, a *random vector* when $S = \mathbb{R}^n$, a *random sequence* when $S = \mathbb{R}^\infty$. A metric or topological space S will be endowed with its Borel σ-field $\mathcal{B}(S)$, generated by its open sets, unless a σ-field is otherwise specified .

If $B \in \mathcal{S}$, then $\xi^{-1}B \in \mathcal{F}$, we may consider the associated probabilities

$$P(\xi^{-1}B) = P \circ \xi^{-1}(B), \; B \in \mathcal{S}.$$

The set function $P \circ \xi^{-1}$ is again a probability measure, defined on the range space S and called the *(probability) distribution* of ξ. For any random vector $\xi = (\xi_{t_1}, ..., \xi_{t_n})$ in

$\mathbb{R}^n$, we define the associated joint distribution function $\mathbf{F}$ by

$$\mathbf{F}(x_1, ..., x_n) = P \bigcap_{k \le n} \{\xi_{t_k} \le x_k\},\ x_1, ..., x_n \in \mathbb{R}.$$

Fix any measurable function $f \ge 0$ on some measure space $(\Omega, \mathcal{F}, \mu)$, and define a set function $f \cdot \mu$ on $\mathcal{F}$ by

$$(f \cdot \mu)(B) = \int_B f d\mu,\ B \in \mathcal{F},$$

where the last relation defines the integral over a set B. Then $\nu = f \cdot \mu$ is again a measure on $(\Omega, \mathcal{F})$. Here f is referred to as the *μ-density of ν.*

Definition 2.1.1. *The characteristic function of a random vector $X : \Omega \to \mathbb{R}^n$ is the function $\phi_X : \mathbb{R}^n \to \mathbf{C}$(where $\mathbf{C}$ denotes the complex numbers) defined by*

$$\phi_X(u_1, ..., u_n) = E[\exp(i(u_1X_1 + ... + u_nX_n))] = \int_{\mathbb{R}^n} \exp(i < u, x >) P \circ X^{-1} dx_1 dx_2 ... dx_n$$

where $< u, x >= u_1x_1 + \cdots + u_nx_n$. In other words, ϕ_X is the Fourier transform of X.

The characteristic function has the following important property :

Theorem 2.1.2. [28] *The characteristic function of X determines the distribution of X uniquely.*

Definition 2.1.3. *A stochastic process $\mathbf{X} = \{X_t\}_{t \in \mathrm{T}}$ is a collection of random variables defined on Ω, called the sample space, and taking values in another measurable space $(S, \mathcal{T})$, here S is called the state space. Throughout this thesis, state space will be the $\mathbb{R}$ equipped with the Borel σ-algebra and T will denote the interval $[0, T]$ for some fixed $T > 0$.*

For each $\omega \in \Omega$, the function

$$t \to X_t(\omega), t \in \mathrm{T},$$

is called *a path* of the process $\mathbf{X}$ associated with ω.

The process $\{X_t\}_{t \in T}$ is *left/right continuous* if for each fixed $\omega \in \Omega$, the function,

$$t \to X_t(\omega),\ t \in \mathrm{T}$$

is *left/right continuous* on T.

For any random elements ξ and ζ in a common measurable space, the equality $\xi \cong \zeta$ means that ξ and ζ have the same distribution, or $P \circ \xi^{-1} = P \circ \zeta^{-1}$. If $\mathbf{X}$ is a random process on some index set T, the associated *finite-dimensional distributions* are given by

$$P \circ (X_{t_1}, ..., X_{t_n})^{-1},\ \forall\ t_1, ..., t_n \in \mathrm{T}, \forall\ n \in \mathbb{N}.$$

The following result shows that the distribution of a process is determined by the set of its finite-dimensional distributions.

Proposition 2.1.4. [29] *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two processes with the same index set* T. *Then* $\mathbf{X} \cong \mathbf{Y}$ *iff*

$$(X_{t_1}, ..., X_{t_n}) \cong (Y_{t_1}, ..., Y_{t_n}),\ \forall\ t_1, ..., t_n \in \mathrm{T},\ \forall\ n \in \mathbb{N}.$$

The *expected value, expectation*, or *mean* of a random variable ξ is defined as

$$E[\xi] = \int_\Omega \xi dP = \int_{\mathbb{R}} xP \circ \xi^{-1}(x)dx$$

whenever

$$\int_\Omega |\xi| dP < \infty.$$

Given a measure space $(\Omega, \mathcal{F}, P)$ and some $p > 0$, we write $L^p = L^p(\Omega, \mathcal{F}, \mu)$ for the class of all measurable functions $\xi : \Omega \to \mathbb{R}$ with

$$||\xi||_p \equiv (\int_\Omega |\xi|^p dP)^{1/p} < \infty.$$

The *covariance* of two random variables $\xi, \zeta \in L^2$ is given by

$$cov(\xi, \zeta) = E[(\xi - E[\xi])(\zeta - E[\zeta])] = E[\xi\zeta] - E[\xi]E[\zeta].$$

We may further define the *variance* of a random variable $\xi \in L^2$ by

$$var[\xi] = cov(\xi, \xi) = E[\xi^2] - E[\xi]^2.$$

Two random variables are said to be *uncorrelated* if $cov(\xi, \zeta) = 0$.

For a given collection of random variables $\xi_t \in L^2, \ t \in \mathrm{T}$, we note that the associated *covariance function* $R(s,t) = cov(\xi_t, \xi_s), \ s, t \in \mathrm{T}$, is *nonnegative definite*, in the sense that

$$\sum_{i,j} R(t_i, t_j)\lambda_i\lambda_j \geq 0$$

for any $n \in \mathbb{N}$, $\forall \ t_1, ..., t_n \in \mathrm{T}, \ \forall \ \lambda_i, ..., \lambda_n \in \mathbb{R}$. This is clear if we write

$$\sum_{i,j} R(t_i, t_j)\lambda_i\lambda_j = \sum_{i,j} cov(\xi_{t_i}, \xi_{t_j})\lambda_i\lambda_j = var[\sum_i \xi_{t_i}\lambda_i] \geq 0.$$

The last inequality follows from the *Jensen's inequality*. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if

$$f(px + (1-p)y) \leq pf(x) + (1-p)f(y), \; x, y \in \mathbb{R}^n, \; p \in [0,1].$$

Lemma 2.1.5. [29] *(Jensen)Let ξ be an integrable random vector in $\mathbb{R}^n$, and fix any convex function $f : \mathbb{R}^n \to \mathbb{R}$. Then*

$$E[f(\xi)] \geq f(E[\xi]).$$

Therefore, by the above lemma, for any random variable ξ, $var[\xi] \geq 0$; since $f(x) = x^2$ is a convex function.

A *filtration* $\mathbf{F} = \{\mathcal{F}_t\}_{t \in \mathrm{T}}$ is defined as a nondecreasing family of σ-algebras $\mathcal{F}_t, \; t \in \mathrm{T}$. One says that a process $\{X_t\}_{t \in \mathrm{T}}$ is *adapted to* the filtration $\mathbf{F}$ if X_t is $\mathcal{F}_t$ -measurable for every $t \in \mathrm{T}$. The smallest filtration with this property is the *induced* or *generated* filtration given by $\mathcal{F}_t = \sigma(X_s; 0 \leq s \leq t), \; s, \; t \in \mathrm{T}$.

Given an arbitrary filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \in T}$, we may define a new filtration $\mathbf{F}^+$ by $\mathcal{F}_t^+ = \bigcap_{u>t} \mathcal{F}_u, t \geq 0$, and we say that $\mathbf{F}$ is *right continuous* if $\mathbf{F}^+ = \mathbf{F}$

By a *random time* we shall mean a random element in $\overline{\mathrm{T}} = \mathrm{T} \cup \sup\mathrm{T}$. Such a time is said to be $\mathbf{F}$- *optional* or an $\mathbf{F}$- *stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \in \mathrm{T}$.

Suppose $\xi \in L^1$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-field. Then there exists a random variable $E[\xi|\mathcal{G}]$, called the *conditional expectation* of ξ with respect to $\mathcal{G}$, such that

$$\begin{aligned} &(i) \quad E[\xi|\mathcal{G}] \; \textit{is} \; \mathcal{G} - \textit{measurable and integrable.} \\ &(ii) \quad \textit{For all} \; G \in \mathcal{G}, \; \textit{we have} \int_G \xi dP = \int_G E[\xi|\mathcal{G}] dP. \end{aligned}$$

To motivate the introduction of *martingales*, we may fix a random variable $\xi \in L^1$ and a filtration **F** on the index set T, and put

$$M_t = E[\xi|\mathcal{F}_t],\ t \in \mathrm{T}.$$

The process **M** is clearly integrable (for each t) and adapted, and chain rule for conditional expectations we note that

$$M_s = E[M_t|\mathcal{F}_s]\ a.s.,\ s \leq t, \quad s, t \in \mathrm{T}.$$

Any integrable and adapted process **M** satisfying

$$M_s = E[M_t|\mathcal{F}_s]\ a.s.,\ s \leq t.$$

is called a *martingale* with respect to **F**.

For a right-continuous filtration $\{\mathcal{F}_t\}_{t\in\mathrm{T}}$, a process $\{M_t\}_{t\in\mathrm{T}}$ is said to be a *local martingale* if it is adapted to filtration $\{\mathcal{F}_t\}_{t\in\mathrm{T}}$ and such that the stopped and shifted processes $M^{\tau_n} - M_0$ are martingales for suitable optional times $\tau_n \uparrow \infty$.

Definition 2.1.6. [7] *A stochastic process $\{X_t\}_{t\in\mathrm{T}}$ is Hölder continuous of order $\gamma \in (0,1)$ if*

$$P\{\omega \in \Omega : \sup_{0<t-s<h(\omega)} \frac{|X_t(w) - X_s(w)|}{|t-s|^\gamma} \leq \delta\} = 1$$

where h is an almost surely positive random variable and $\delta > 0$ is an appropriate constant.

Lemma 2.1.7. [13] *(A general version of Kolmogorov's criterion): If a stochastic process $\{X_t\}_{t\in\mathrm{T}}$ satisfies*

$$E[|X_t - X_s|^\delta] \leq C|t-s|^{1+\epsilon}, \forall t, s,$$

for some $\delta > 0, \epsilon > 0$ and $C > 0$,then $\{X_t\}_{t\in\mathrm{T}}$ has a modification whose sample paths of are Hölder continuous of order $\gamma \in [0, \epsilon/\delta)$.

The Markov property of a process states that if we know the present state of the process, then the future behavior of the process is independent of its past.The process $\{X_t\}_{t\in\mathrm{T}}$ has the Markov property if the conditional distribution of X_{t+s} given $X_t = x$, does not depend on the past values(but it may depend on the present value x).The process "does not remember" how it got to the present state x.

Definition 2.1.8. *$\{X_t\}_{t\in\mathrm{T}}$ is a Markov process if for any t and $s > 0$, the conditional distribution of X_{t+s} given $\mathcal{F}_t$ is the same as the conditional distribution of X_{t+s} given X_t, that is,*

$$P(X_{t+s} \leq y|\mathcal{F}_t) = P(X_{t+s} \leq y|X_t),\ a.s.$$

Proposition 2.1.9. [29] *Let $\{X_t\}_{t\in\mathrm{T}}$ be a Gaussian process, and define $R(s,t) = cov(X_s, X_t)$. Then $\{X_t\}_{t\in\mathrm{T}}$ is Markov iff*

$$R(s,u) = \frac{R(s,t)R(t,u)}{R(t,t)}, s \leq t \leq u,$$

where $0/0 = 0$.

Definition 2.1.10. *A process $\{X_t\}_{t\in\mathrm{T}}$ is said to be a continuous semimartingale if it can be written as a sum $X_t = M_t + A_t$ for $t \in \mathrm{T}$, where $\{M_t\}_{t\in\mathrm{T}}$ is a continuous local martingale and $\{A_t\}_{t\in\mathrm{T}}$ is a continuous, adapted process of locally finite variation with $A_0 = 0$.*

2.2. Brownian Motion

The methodology used here are related to chapter 4 of [30].

Definition 2.2.1. *A Brownian motion which is also called a Weiner process is a continuous process* $\boldsymbol{W} = \{W_t, t \geq 0\}$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *with the properties that* $W_0 = 0$ *a.s., and for* $0 \leq s < t$*, the increment* $W_t - W_s$ *is independent of* W_s *and is normally distributed with mean zero and variance* $t - s$*.*

Now we state some some basic properties of Brownian motion:

(i) $\mathbf{W}$ is a zero-mean Gaussian process.
Its covariance function $Cov(W_t, W_s) = s$, for $s < t$.[13].

(ii) $\mathbf{W}$ has independent increments, i.e.

$$W_{t_1}, W_{t_2} - W_{t_1}, ..., W_{t_k} - W_{t_{k-1}}$$

are independent for all $0 \leq t_1 < t_2 ... < t_k$. [13].

(iii) Since $E[W_t^4] = 3t^2$ and $E[|W_t - W_s|^4] = 3|t-s|^2$, It has a continuous version by the Lemma 2.1.7.[13].

(iv) Almost every path of $\mathbf{W} = \{W_t, t \geq 0\}$ is nowhere differentiable.[13].

(v) Almost every path has infinite total variation on $[0, T]$.[13].

Stochastic integration with respect to Brownian motion can be considered as an extension of Stieltjes integration. In this section, the stochastic integrals of the form $\int_a^b f(t,\omega)dW(t,\omega)$ or in a short form $\int_a^b f(t)dW_t$ will be defined for an appropriate class of integrands. First, the integral is defined for simple integrands and then it is extended to more general functions by limiting process.

Definition 2.2.2. *A real valued function* f *on* $[a, b] \times \Omega$ *is said to be in the class*

$\nu = \nu(a,b)$ *if it satisfies:*

(i) f *is nonanticipating, that is,* f *is* $\mathcal{B}[a,b] \times \mathcal{F}-$ *measurable and adapted to* $\{\mathcal{F}_t\}$,

(ii) $f(t) \in L^2(\Omega, \mathcal{F}, P)$ *and* $\displaystyle\int_a^b E[|f(t)|^2]dt < \infty.$

Definition 2.2.3. *A function* $f \in \nu$ *is called elementary if there exist a partition*

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

with associated random variables $f_0, f_1, ..., f_n$ *such that*

$$f(t,\omega) = \sum_{i=0}^{n-1} f_i(\omega) I_{[t_i,t_{i+1})}(t),$$

$\mathcal{S} = \mathcal{S}(a,b)$ *will denote the space of these elementary functions. Then the integral of* $f \in \mathcal{S}$ *is a random variable defined by*

$$\int_a^b f(t,\omega)dW(t,\omega) = \sum_{i=0}^{n-1} f_i(\omega)[W(t_{i+1},\omega) - W(t_i,\omega)].$$

Lemma 2.2.4. [30] *If* f *and* g *are elementary functions, then* $E[\int_a^b f(t)dW_t] = 0$ *and*

$$E[\int_a^b f(t)dW_t \int_a^b g(t)dW_t] = \int_a^b E[f(t)g(t)]dt$$

Corollary 2.2.5. [30] *If* f *is bounded and elementary, then*

$$E[(\int_a^b f(t,\omega)dW(t,\omega))^2] = E[\int_a^b f^2(t,\omega)dt]$$

In [30], it is shown that the set ν is a closed and dense subspace of the Hilbert space $L^2([a,b] \times \Omega)$.

Therefore if $f \in \nu$, then there exist elementary functions $f_n \in \nu$ such that

$$E[\int_a^b |f_n(t) - f(t)|^2 dt] \to 0,\ n \to \infty.[30].$$

Then define

$$\Phi[f](\omega) = \int_a^b f(t,\omega)dW_t(\omega) = \lim_{n\to\infty} \int_a^b f_n(t,\omega)dW_t(\omega)$$

in L^2-sense.[30]. This limit exists since $\{\int_a^b f_n(t,\omega)dW_t(\omega)\}$ forms a Cauchy sequence in L^2.[30].

Corollary 2.2.6. [30]***(The Itô Isometry)***

$$E[(\int_a^b f(t,\omega)dW_t(\omega))^2] = E[\int_a^b f^2(t,\omega)dt]$$

for all $f \in \nu(a,b)$.

3. GAUSSIAN PROCESSES AND FRACTIONAL BROWNIAN MOTION

3.1. Definition and the Characteristic Properties of Fractional Brownian Motion

Definition 3.1.1. *Let* $(\Omega, \mathcal{F}, P)$ *be a given probability space. A random variable* $X : \Omega \to \mathbb{R}$ *is Gaussian or normal if the distribution of* X *has a density of the form*

$$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}}.\exp(-\frac{(x-m)^2}{2\sigma^2}),$$

where $\sigma > 0$ *and* m *are constants. In other words,*

$$P[X \in G] = \int_G p_X(x)dx,$$

for all Borel sets $G \subset \mathbb{R}$.

If this is the case, then

$$E[X] = \int_\Omega X dP = \int_\mathbb{R} x p_X(x)dx = m,$$

and

$$Var[X] = E[(X-m)^2] = \int_\mathbb{R} (x-m)^2 p_X(x)dx = \sigma^2.$$

Hence the probability distribution of this random variable X is completely determined by its mean value m and its variance σ^2.

A random vector $X : \Omega \to \mathbb{R}^n$ is called *(multi) Gaussian* if the distribution of X

has a density of the form

$$p_X(x_1, \cdots, x_n) = \frac{\sqrt{|A|}}{(2\pi)^{n/2}}.\exp(-\frac{1}{2}.\sum_{i,j}(x_i - m_i)a_{ij}(x_j - m_j))$$

where $m = (m_1, \cdots, m_n) \in \mathbb{R}^n$ and $C^{-1} = A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a positive definite matrix.

If this is the case then

$$E[X] = m$$

and

$$A^{-1} = C = [c_{ij}]$$

is the covariance matrix of X.

The *covariance matrix* of X is the $n \times n$ matrix $C = [c_{ij}]$, where

$$c_{ij} = E[(X_j - m_j)(X_i - m_i)]$$

and

$$m_i = E[X_i].$$

It is clear that C is *symmetric.* Moreover, it is *non-negative definite,* i.e.

$$\sum_{i,j=1}^{n} c_{ij}\lambda_i\lambda_j \geq 0$$

for all $\lambda_i \in \mathbb{R}, i = 1, \cdots, n$, since

$$\sum_{i,j=1}^{n} c_{ij}\lambda_i\lambda_j = E[\sum_{i=1}^{n}(X_i - E[X_i])\lambda_i]^2 \geq 0.$$

The following lemma shows that the converse is also true.

Lemma 3.1.2. [31] *A necessary and sufficient condition that an nxn matrix C is the covariance matrix of a vector $X = (X_1, \cdots, X_n)$ is that the matrix is symmetric and non-negative definite, or, equivalently that there is a an nxk matrix $A, (1 \leq k \leq n)$ such that*

$$C = AA^{\mathbf{T}}$$

where $\mathbf{T}$ *denotes the transpose.*

Proof. Let C be a symmetric and non-negative definite matrix. We know from the matrix theory that corresponding to every symmetric, non-negative definite matrix C there is an orthogonal matrix Θ(i.e., $\Theta\Theta^{\mathbf{T}} = I$) such that

$$\Theta^{\mathbf{T}}C\Theta = D \quad where \quad D = \begin{pmatrix} d_1 & . & 0 \\ 0 & . & d_n \end{pmatrix}$$

is a diagonal matrix with nonnegative elements $d_i,\ i = 1, ..., n$.
It follows that

$$C = \Theta D\Theta^{\mathbf{T}} = (\Theta B)(B^{\mathbf{T}}\Theta^{\mathbf{T}}),$$

where B is the diagonal matrix with elements $b_i = +\sqrt{d_i},\ i = 1, ..., n$. Consequently, if we put $A = \Theta B$ we have the required representation $C = AA^{\mathbf{T}}$ for C.
It is clear that every matrix $AA^{\mathbf{T}}$ is symmetric and nonnegative definite. Consequently we have only to show that C is the covariance matrix of some random vector.
Let $\eta_1, \eta_2, ..., \eta_n$ be a sequence of independent normally distributed random variables.

Then the random vector $X = A\eta$ has the required properties. In fact,

$$E[XX^{\mathbf{T}}] = E[(A\eta)(A\eta)^{\mathbf{T}}] = A \cdot E[\eta\eta^{\mathbf{T}}] \cdot A^{\mathbf{T}} = AA^{\mathbf{T}}$$

This completes the proof of the lemma. $\square$

Definition 3.1.3. *A process $\{X_t\}_{t \in \mathrm{T}}$ is Gaussian if the random variable $c_1 X_{t_1} + c_2 X_{t_2} + \cdots + c_n X_{t_n}$ is Gaussian for any choice of $n \in \mathbb{N}, t_1, t_2, \ldots, t_n \in \mathrm{T}$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$.*

Theorem 3.1.4. [28] *Let $X_i : \Omega \to \mathbb{R}$ be random variables; $1 \leq i \leq n$. Then*

$$X = (X_1, \cdots, X_n)$$

is Gaussian if and only if

$$Y = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

is Gaussian for any $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

Proof. If X is normal, then

$$\begin{aligned} E[\exp(iu(c_1 X_1 + c_2 X_2 + \cdots + c_n X_n))] &= \exp(-\frac{1}{2} \sum_{j,k} uc_j \lambda_{jk} uc_k + i \sum_j uc_j m_j) \\ &= \exp(-\frac{1}{2} u^2 \sum_{j,k} c_j \lambda_{jk} c_k + iu \sum_j c_j m_j) \end{aligned}$$

so Y is Gaussian with $E[Y] = \sum c_j m_j,\ var[Y] = \sum c_j \lambda_{jk} c_k$.
Conversely, if $Y = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$ is Gaussian with $E[Y] = m$ and $var[Y] = \sigma^2$, then

$$E[\exp(iu(c_1 X_1 + c_2 X_2 + \cdots + c_n X_n))] = \exp(-\frac{1}{2} u^2 \sigma^2 + ium),$$

where $m = \sum_j c_j E[X_j]$, and

$$\begin{aligned} \sigma^2 &= E[(\sum_j c_j X_j - \sum_j c_j E[X_j])^2] \\ &= E[(\sum_j c_j (X_j - m_j))^2] \\ &= \sum_{j,k} c_j c_k E[(X_j - m_j)(X_k - m_k)], \end{aligned}$$

where $m_j = E[X_j]$. Hence X is Gaussian. □

A function $R(s,t) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is called *nonnegative definite* if

$$\sum_{i,j=1}^{n} R(t_i, t_j)\lambda_i \lambda_j \geq 0$$

for all $t_i,\ \lambda_i \in \mathbb{R}, i = 1, \cdots, n$ and $n \in \mathbb{N}$.

By the Lemma 3.1.2, any non-negative definite function defines a unique zero mean Gaussian process.

Lemma 3.1.5. [32] *The function $R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ is non-negative definite if $H \in (0,1]$.*

Proof. We want to show that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} R_H(t_i, t_j) u_i u_j \geq 0$$

for $t_1, ..., t_n \geq 0$ and $u_1, ..., u_n \in \mathbb{R}$.
Set $t_0 := 0$ and add a value $u_0 := -\sum_{i=1}^{n} u_i$. Then $\sum_{i=0}^{n} u_i = 0$ and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} R_H(t_i, t_j) u_i u_j = -\sum_{i=0}^{n} \sum_{j=0}^{n} |t_i - t_j|^{2H} u_i u_j.$$

Since for any $\varepsilon \geq 0$ we have

$$\sum_{i=0}^{n}\sum_{j=0}^{n} e^{-\varepsilon|t_i-t_j|^{2H}} u_i u_j = \sum_{i=0}^{n}\sum_{j=0}^{n}(e^{-\varepsilon|t_i-t_j|^{2H}} - 1)u_i u_j = -\varepsilon \sum_{i=0}^{n}\sum_{j=0}^{n} |t_i - t_j|^{2H} u_i u_j + o(\varepsilon)$$

as $\varepsilon \to 0$ it is sufficient to show that

$$\sum_{i=0}^{n}\sum_{j=0}^{n} e^{-\varepsilon|t_i-t_j|^{2H}} u_i u_j \geq 0.$$

But this follows from the fact that the mapping $\theta \to e^{-\varepsilon|\theta|^{2H}}$ is a characteristic function for $H \in (0, 1]$. It is known that a real-valued random variable ξ is symmetric α-stable,$0 < \alpha < 2$, if and only if its characteristic function satisfies

$$E[\exp(i\theta\xi)] = \exp(-c|\theta|^{\alpha}), \qquad \theta \in \mathbb{R},$$

for some $c > 0$.[7].

Now, we will show that

$$\exp(-|t|^{\alpha}), \quad 0 < \alpha < 2,$$

is a characteristic function. Here we will use the exercise 4 of section 6.5 of [33]. Firstly, let us show that

$$p(x) = \frac{\alpha}{2(\alpha+1)}(1 \wedge |x|^{-\alpha-1}) \tag{3.1}$$

is a probability density function. It is clear that $p(x) \geq 0$. Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} p(x)dx &= \frac{\alpha}{2(\alpha+1)}[\int_{-\infty}^{-1} |x|^{-\alpha-1}dx + \int_{-1}^{1} dx + \int_{1}^{\infty} |x|^{-\alpha-1}dx] \\
&= \frac{\alpha}{2(\alpha+1)}[2 + 2\int_{1}^{\infty} |x|^{-\alpha-1}dx] \\
&= \frac{\alpha}{2(\alpha+1)}[2 + \frac{2}{\alpha}] \\
&= 1.
\end{aligned}$$

Then let us show that a random variable ξ with the probability density function

$$p(x) = \frac{\alpha}{2(\alpha+1)}(1 \wedge |x|^{-\alpha-1}) \tag{3.2}$$

has a characteristic function $\phi(t)$ of the form $1 - C_\alpha|t|^\alpha + O(t^2)$ in the neighborhood of $t = 0$.

$$\begin{aligned}
E[\exp(it\xi)] &= \frac{\alpha}{2(\alpha+1)}\int_{-\infty}^{\infty} \exp(it\xi)(1 \wedge |x|^{-\alpha-1})dx \\
&= \frac{\alpha}{2(\alpha+1)}\int_{-\infty}^{\infty} [\cos(tx) + i\sin(tx)](1 \wedge |x|^{-\alpha-1})dx \\
&= \frac{\alpha}{(\alpha+1)}\int_{0}^{\infty} \cos(tx)(1 \wedge |x|^{-\alpha-1})dx \\
&= \frac{\alpha}{(\alpha+1)}\int_{0}^{1} \cos(tx)dx + \frac{\alpha}{(\alpha+1)}\int_{1}^{\infty} \cos(tx)x^{-\alpha-1}dx \\
&= \frac{\alpha}{(\alpha+1)}\frac{\sin t}{t} + \frac{\alpha}{(\alpha+1)}I_1
\end{aligned}$$

where

$$I_1 = \int_1^\infty \cos(tx)x^{-\alpha-1}dx.$$

We have

$$\int_0^\infty \frac{1-\cos(tx)}{x^{\alpha+1}}dx = C_\alpha|t|^\alpha. \tag{3.3}$$

If we apply the Fubini's theorem, we will get (3.3)

$$\begin{aligned}\int_0^\infty \frac{1-\cos(tx)}{x^{\alpha+1}}dx &= \int_0^\infty \frac{1}{x^{\alpha+1}}[\int_0^x t\sin(tu)du]dx \\ &= \int_0^\infty \int_u^\infty \frac{t\sin(tu)}{x^{\alpha+1}}dxdu \\ &= C_\alpha|t|^\alpha.\end{aligned}$$

We can write integral (3.3) in the following way :

$$\int_0^\infty \frac{1-\cos(tx)}{x^{\alpha+1}}dx = \int_0^1 \frac{1-\cos(tx)}{x^{\alpha+1}}dx + \int_1^\infty \frac{1-\cos(tx)}{x^{\alpha+1}}dx$$

we denote

$$I_2 = \int_1^\infty \frac{1-\cos(tx)}{x^{\alpha+1}}dx.$$

Let us consider the order of

$$I_3 = \int_0^1 \frac{1-\cos(tx)}{x^{\alpha+1}}dx.$$

Since $\cos(x)$ has a Taylor expansion, we have

$$\begin{aligned}
\frac{1-\cos(tx)}{x^{\alpha+1}} &= -\frac{1}{x^{\alpha+1}}\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n}x^{2n}}{(2n)!} \qquad \Rightarrow \\
\int_0^1 \frac{1-\cos(tx)}{x^{\alpha+1}}dx &= -\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\int_0^1 x^{2n-\alpha-1}dx \\
&= -\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\frac{1}{2n-\alpha} \\
&= -t^2\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n-2}}{(2n)!}\frac{1}{2n-\alpha}
\end{aligned}$$

If we can show that

$$\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n-2}}{(2n)!}\frac{1}{2n-\alpha}$$

is bounded for small t, then we can conclude that

$$\int_0^1 \frac{1-\cos(tx)}{x^{\alpha+1}}dx$$

is $O(t^2)$. Since $0<\alpha<2$, we have

$$2n-2<2n-\alpha<2n, \qquad \frac{1}{2n}<\frac{1}{2n-\alpha}<\frac{1}{2n-2}.$$

Hence for t is in the neighborhood of 0,

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n-2}}{(2n)!}\frac{1}{2n-\alpha} &\le \sum_{n,\ even}\frac{t^{2n-2}}{(2n)!}\frac{1}{2n-2}-\sum_{n,\ odd}\frac{t^{2n-2}}{(2n)!}\frac{1}{2n} \\
&< \sum_{n,\ even}\frac{1}{(2n)!}\frac{1}{2n-2} < M
\end{aligned}$$

for some M, $0 < M < \infty$. Therefore I_3 is $O(t^2)$.

Thus,

$$\begin{aligned} I_2 &= \int_1^\infty \frac{dx}{x^{\alpha+1}} - I_1 \\ C_\alpha |t|^\alpha - O(t^2) &= \frac{1}{\alpha} - I_1 \\ I_1 &= \frac{1}{\alpha} - C_\alpha |t|^\alpha + O(t^2). \end{aligned}$$

Therefore for the neighborhood of $t = 0$,

$$\begin{aligned} E[\exp(it\xi)] &= \frac{\alpha}{(\alpha+1)} + \frac{\alpha}{(\alpha+1)}(\frac{1}{\alpha} - C_\alpha |t|^\alpha + O(t^2)) \\ \phi(t) = & \; 1 - c_\alpha |t|^\alpha + O(t^2). \end{aligned}$$

Now, let $\xi_1, \xi_2, ...$ be independent identically distributed random variables with the above density function and $S_n = \sum_{i=1}^n \xi_i$.

Let $\psi_n(t)$ be the characteristic function of $\frac{S_n}{n^{1/\alpha}}$. Then

$$\begin{aligned} \psi_n(t) &= E[\exp(it\frac{S_n}{n^{1/\alpha}})] \\ &= \phi(\frac{t}{n^{1/\alpha}})^n = (1 - c_\alpha \frac{|t|^\alpha}{n} + O(\frac{t^2}{n^{2/\alpha}}))^n \\ &\longrightarrow \exp(-c_\alpha |t|^\alpha), \qquad n \longrightarrow \infty. \end{aligned}$$

This completes the proof of the Lemma 3.1.5.

The stable distributions with characteristic functions of the form

$$\exp(-\lambda|t|^{\alpha})$$

were called Cauchy laws. The tradition later died out and distribution of this type became known as symmetric stable distributions. In our times there has been a sharp increase in the interest in stable laws, due to their appearance in certain socio-economic models.[34]. $\square$

Definition 3.1.6. *An $\mathbb{R}$-valued stochastic process $\boldsymbol{X} = \{X_t\}_{t\in\mathrm{T}}$ is said to be self similar if for any $a > 0$, there exists $b > 0$ such that $\{X_{at}\}_{t\in\mathrm{T}} \cong \{bX_t\}_{t\in\mathrm{T}}$*

Definition 3.1.7. *An $\mathbb{R}$-valued stochastic process $\boldsymbol{X} = \{X_t\}_{t\in\mathrm{T}}$ is said to be H-self similar(H-ss) if*

$$\{X_{at}\}_{t\in\mathrm{T}} \cong \{a^H X_t\}_{t\in\mathrm{T}}$$

for all $a > 0$. The parameter $H > 0$ is called the Hurst index.

Definition 3.1.8. *An $\mathbb{R}$-valued stochastic process $\{X_t\}_{t\in\mathrm{T}}$ is said to have independent increments, if for any $m \geq 1$ and for any partition $0 \leq t_0 < t_1 < ... < t_m$, $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_m} - X_{t_{m-1}}$ are independent and is said to have stationary increments, if any joint distribution of $\{X_{t+h} - X_t, t \geq 0\}$is independent of t. This means that $X_t - X_s$ has the same distribution as $X_{t+h} - X_{s+h}$ for all $s, t, h \geq 0,\ s < t$.*

Theorem 3.1.9. [7] *Let $\{X_t\}_{t\in\mathrm{T}}$ be real-valued H-self similar process with stationary increments and suppose that $E[X_1^2] < \infty$. Then*

$$E[X_t X_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})E[X_1^2].$$

Proof. By selfsimilarity and stationarity of the increments,

$$\begin{aligned} E[X_tX_s] &= \frac{1}{2}\{E[X_t^2]+E[X_s^2]-E[(X_t-X_s)^2]\} \\ &= \frac{1}{2}\{E[X_t^2]+E[X_s^2]-E[X_{|t-s|}^2]\} \\ &= \frac{1}{2}\{t^{2H}+s^{2H}-|t-s|^{2H}\}E[X_1^2] \end{aligned}$$

□

Combining the Lemma 3.1.2. and Lemma 3.1.5. we arrive at a definition of "fractional Brownian motion".

Definition 3.1.10. *Let $0 < H \leq 1$. A real-valued Gaussian process $\{B_t^H\}_{t\in \mathrm{T}}$ is called fractional Brownian motion if $E[B_t^H] = 0$ and*

$$E[B_t^H B_s^H] = \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})E[(B_1^H)^2].$$

For the special case $H = \frac{1}{2}$, the process is the the well-known process 'Brownian motion'. It has independent and stationary increments. Fractional Brownian motion has a stochastic integral representation with respect to Brownian motion. For this result we will use the Corollary 2.2.6:

Theorem 3.1.11. [7] *A fractional Brownian motion $\{B_t^H\}_{t\in \mathrm{T}}$ is H-ss with stationary increments. When $0 < H < 1$, it has a stochastic integral representation*

$$C_H\{\int_{-\infty}^0((t-u)^{H-1/2}-(-u)^{H-1/2})dB(u)+\int_0^t(t-u)^{H-1/2}dB(u)\},$$

where

$$C_H = E[(B_1^H)^2]^{1/2}\{\int_{-\infty}^0((t-u)^{H-1/2}-(-u)^{H-1/2})^2du+\frac{1}{2H}\}^{-1/2}.$$

If $H = 1$, then $B_t^1 = tB_1^1$ almost surely. Fractional Brownian motion is unique in *the sense that the class of all fractional Brownian motions coincides with that of all Gaussian selfsimilar processes with stationary increments.* $\{B_t^H\}_{t\in\mathrm{T}}$ *has independent increments if and only if* $H = 1/2$.

Proof. **(i)** Selfsimilarity

$$\begin{aligned} E[B_{at}^H B_{as}^H] &= \frac{1}{2}((at)^{2H} + (as)^{2H} - (a|t-s|)^{2H})E[(B_1^H)^2] \\ &= a^{2H}E[B_t^H B_s^H] \\ &= E[(a^H B_t^H)(a^H B_s^H)]. \end{aligned}$$

Since all processes here are mean zero Gaussian, this equality in covariance implies that $\{B_{at}^H\}_{t\in\mathrm{T}} \cong \{a^H B_{at}^H\}_{t\in\mathrm{T}}$.

(ii) Stationary increments. Again, it is enough to consider only covariances. We have

$$\begin{aligned} E[(B_{t+h}^H - B_h^H)(B_{s+h}^H - B_h^H)] &= E[B_{t+h}^H B_{s+h}^H] - E[B_{t+h}^H B_h^H] \\ &- E[B_{s+h}^H B_h^H] + E[(B_h^H)^2] \\ &= \frac{1}{2}\{(t+h)^{2H} + (s+h)^{2H} - (|t-s|)^{2H} \\ &- ((t+h)^{2H} + h^{2H} - t^{2H}) \\ &- ((s+h)^{2H} + h^{2H} - s^{2H}) + 2h^{2H}\}.E[(B_1^H)^2] \\ &= \frac{1}{2}((t)^{2H} + (s)^{2H} - (|t-s|)^{2H})E[(B_1^H)^2] \\ &= E[B_t^H B_s^H], \end{aligned}$$

concluding that

$$\{B_{t+h}^H - B_h^H\}_{t\in\mathrm{T}} \cong \{B_t^H\}_{t\in\mathrm{T}}.$$

(iii) We need to show that the Wiener integral in the theorem is well-defined.

Let $v = -u$, and $\alpha = H - \frac{1}{2}$ then we will show that

$$I_1 = \int_0^{\infty} [(t+v)^{\alpha} - v^{\alpha}]^2 dv < \infty, \quad 0 \le \alpha < 1/2,$$

and

$$I_2 = \int_0^{1} [(t+v)^{\alpha} - v^{\alpha}]^2 dv < \infty, \quad -1/2 < \alpha < 0$$

and

$$I_3 = \int_1^{\infty} [(t+v)^{\alpha} - v^{\alpha}]^2 dv < \infty, \quad -1/2 < \alpha < 0$$

and

$$I_4 = \int_{-t}^{0} [(t+v)^{2\alpha} dv < \infty \quad -1/2 < \alpha < 1/2.$$

For $I_1,\ I_3$ and $g_1(v) = \alpha^2 v^{2\alpha-2}$, since

$$\lim_{v \to \infty} \frac{[(v+t)^{\alpha} - v^{\alpha}]^2}{\alpha^2 v^{2\alpha-2}} = t^2,$$

and

$$\lim_{l \to \infty} \int_0^{l} \alpha^2 v^{2\alpha-2} dv = \frac{\alpha^2}{2\alpha - 1} \lim_{l \to \infty} (l^{2\alpha-1}) = 0, \qquad -1/2 \le \alpha < 1/2,$$

then I_1 and I_3 are convergent.

For I_2 and $g_2(v) = (v+t)^{2\alpha}$, since

$$\lim_{v\to 0} \frac{[(v+t)^\alpha - v^\alpha]^2}{(v+t)^{2\alpha}} = \lim_{v\to 0}[1 - \frac{v^\alpha}{(v+t)^\alpha}]^2 = 1,$$

and

$$\int_0^1 (v+t)^{2\alpha} dv < \infty,$$

then I_2 is convergent.

For I_4, since

$$I_4 = \int_{-t}^0 [(t+v)^{2\alpha} = \frac{(t+v)^{2\alpha+1}}{2\alpha+1}|_{v=-t}^0 = \frac{t^{2\alpha+1}}{2\alpha+1} < \infty,$$

then I_4 is convergent.

Therefore the Wiener integral is well-defined. Denote the integral by X_t.

We then have that

$$\begin{aligned} E[X_t^2] &= C_H^2[\int_{-\infty}^0 ((t-u)^{H-1/2} - (-u)^{H-1/2})^2 du + \int_0^t (t-u)^{2H-1} du] \\ &= E[(B_1^H)^2] t^{2H}. \end{aligned}$$

Moreover,

$$\begin{aligned}
E[(X_{t+h} - X_h)^2] &= C_H^2 E[(\int_{-\infty}^{h} ((t+h-u)^{H-1/2} - (h-u)^{H-1/2}) dB(u) \\
&\quad + \int_{h}^{t+h} (t+h-u)^{H-1/2} dB(u))^2] \\
&= C_H^2 [\int_{-\infty}^{h} ((t+h-u)^{H-1/2} - (h-u)^{H-1/2})^2 du \\
&\quad + \int_{h}^{t+h} (t+h-u)^{2H-1} du] \\
&= C_H^2 [\int_{-\infty}^{0} ((t-u)^{H-1/2} - (-u)^{H-1/2})^2 du + \int_{0}^{t} (t-u)^{2H-1} du] \\
&= E[(B_1^H)^2] t^{2H}.
\end{aligned}$$

Hence,

$$\begin{aligned}
E[X_t X_s] &= \frac{1}{2}\{E[X_t^2] + E[X_s^2] - E[(X_t - X_s)^2]\} \\
&= \frac{1}{2}((t)^{2H} + (s)^{2H} - (|t-s|)^{2H}) E[(B_1^H)^2]
\end{aligned}$$

Therefore, $\{X_t\}_{t\in \mathrm{T}}$ is fractional Brownian motion for $0 < H < 1$.

(iv) For the case $H = 1$, first note that $E[B_t^1 B_t^1] = ttE[(B_1^1)^2]$. Then

$$\begin{aligned}
E[(B_t^1 - tB_1^1)^2] &= E[(B_t^1)^2] - 2tE[B_t^1 B_1^1] + t^2 E[(B_t^1)^2] \\
&= (t^2 - 2t^2 + t^2) E[(B_1^1)^2] = 0,
\end{aligned}$$

so that $B_t^1 = tB_1^1$ almost surely.

(v) For the uniqueness, first note that once $\{X_t\}_{t\in \mathrm{T}}$ is H-ss and has stationary increments, then by the theorem above, it has the same covariance structure as $\{B_t^H\}_{t\in \mathrm{T}}$. Since $\{X_t\}_{t\in \mathrm{T}}$ is mean zero Gaussian, it is the same as $\{B_t^H\}_{t\in \mathrm{T}}$ in terms of their distribution.

□

Proposition 3.1.12. *The fractional Brownian motion* $\{B_t^H\}_{t\in\mathrm{T}}$ *with Hurst index* H *is Markov process if and only if* $H = \frac{1}{2}$.

Proof. Assume $H = \frac{1}{2}$ and $s \le t \le u$. Then $R_{1/2}(s,u) = s,\ R_{1/2}(s,t) = s,\ R_{1/2}(t,u) = t,\ R_{1/2}(t,t) = t$. Therefore by proposition 2.1.9, $\{B_t^{1/2}\}_{t\in\mathrm{T}}$ is a Markov process.

Now, assume that $\{B_t^H\}_{t\in\mathrm{T}}$ is a Markov process. Again by proposition 2.1.9, the covariance function $R_H(s,t)$ of fBm must satisfy the following condition:

$$R_H(s,u) = \frac{R_H(s,t)R_H(t,u)}{R_H(t,t)}, s \le t \le u,$$

Hence the above equality must be satisfied by $s = m,\ t = 2m,\ u = 3m$. If we put these values in the above equality, we get

$$\begin{aligned}
[m^{2H} + (3m)^{2H} - (2m)^{2H}] \cdot 2 \cdot (2m)^{2H} &= (2m)^{2H} \cdot [(2m)^{2H} + (3m)^{2H} - m^{2H}] \\
3 &= 3 \cdot 2^{2H} - 3^{2H}
\end{aligned}$$

and therefore $H = 1/2$. □

3.2. Sample Path Properties of Fractional Brownian Motion

The notation and methodology used here are related to [2].

Theorem 3.2.1. [2] *Fractional Brownian motion* $\{B_t^H\}_{t\in\mathrm{T}}$, $0 < H < 1$ *has a modification, the sample paths of which are Hölder continuous of order* $\beta \in [0, H)$.

Proof. Choose $0 < \gamma < H$. Then we have by selfsimilarity and stationary increments of $\{B_t^H\}_{t\in\mathrm{T}}$

$$\begin{aligned}
E[|B_t^H - B_s^H|^{1/\gamma}] &= E[|B_{|t-s|}^H|^{1/\gamma}] \\
&= |t-s|^{H/\gamma} E[|B_1^H|^{1/\gamma}]
\end{aligned}$$

Then Kolmogorv's criterion is satisfied with $\delta = 1/\gamma$ and $\epsilon = H/\gamma - 1$. Thus there exists a modification which is Hölder continuous of order $\beta < (H/\gamma - 1)\gamma$. Since γ can be arbitrarily, the result follows. $\square$

Now, we shall introduce another notion, the so-called p-variation which is related to the stochastic integration.

Consider partitions $\pi := \{t_k : 0 = t_0 < t_1 < ... < t_n = T\}$ of $[0, T]$. Denote by $|\pi|$ the mesh of π, i.e. $|\pi| := max_{t_k \in \pi} \Delta t_k$ where $\Delta t_k := t_k - t_{k-1}$. Then for $p \in [1, \infty)$

$$v_p(f; \pi) := \sum_{t_k \in \pi} |\Delta f(t_k)|^p$$

where $\Delta f(t_k) := f(t_k) - f(t_{k-1})$ is the p- *variation of* f *along the partition* π.

Definition 3.2.2. *Let* f *be function over the interval* $[0, T]$. *If*

$$v_p^0(f) := \lim_{|\pi| \to 0} v_p(f; \pi)$$

exists we say that f *has a finite* p*-variation. If*

$$v_p(f) := sup_\pi v_p(f; \pi)$$

is finite then f *has a bounded* p*-variation. The variation index of* f *is*

$$v(f) := inf\{p > 0 : v_p(f) < \infty\}$$

where the infimum of the empty set is ∞.

Proposition 3.2.3. [2] *Let* $\{B_t^H\}_{t \in T}$ *be a fractional Brownian motion with Hurst index* H. *Then* $v_p^0(B_t^H) = 0$ *almost all surely if* $p > 1/H$. *For* $p < 1/H$ *we have* $v_p(B_t^H) = \infty$ *and* $v_p^0(B_t^H)$ *does not exist.*

Proof. Let K be the Hölder constant of the fractional Brownian motion. Let $p > 1/H$ and π be the partition of $[0,1]$. Then by the Theorem 3.2.1

$$\begin{aligned}
\sum_{t_k \in \pi} |\Delta B_{t_k}|^p &\leq \sum_{t_k \in \pi} |K|\Delta t_k|^{\beta}|^p \\
&= K^p \sum_{t_k \in \pi} |\Delta t_k|^{\beta p} \\
&\leq K^p |\pi| \sum_{t_k \in \pi} |\Delta t_k|^{\beta p - 1}
\end{aligned}$$

almost surely for any $\beta < H$. Letting $|\pi|$ tend to zero we see that $v_p^0(B_t^H) = 0$ almost surely.

Now, suppose that $p < 1/H$. Then we can choose a subsequence $(\pi_n^{'})_{n\in\mathbb{N}}$ of the sequence of equidistant partitions $(\pi_n)_{n\in\mathbb{N}}$ such that $v_{1/H}(B_t^H; \pi_n^{'})$ converges almost surely to $\gamma_{1/H}$. Consequently along this subsequence we have $\lim_{n\to\infty} v_p(B_t^H; \pi_n^{'}) = \infty$ almost surely. Since $|\pi_n^{'}|$ tends to zero as n increases $v_p^0(B_t^H)$ can not exist. This also shows that $v_p(B_t^H) = \infty$ almost surely for $p < 1/H$. $\square$

Now we are ready to prove the fact that makes stochastic integration with respect to the fractional Brownian motion a challenging task.

Corollary 3.2.4. [2] *The fractional Brownian motion with Hurst index $H \neq \frac{1}{2}$ is not a semimartingale.*

Proof. For $H < \frac{1}{2}$ we know that the fractional Brownian motion has no quadratic variation.So it can not be a semimartingale.

Suppose that $H > \frac{1}{2}$ and the fractional Brownian motion is a semimartingale with decomposition $B_t^H = M_t + A_t$. Now, Proposition 3.2.3 states that B^H has zero quadratic variation. So the martingale $M_t = B_t^H - A_t$ has zero quadratic variation. Since B^H is continuous we know by the properties of semimartingale decomposition, definition 2.1.10, that M is also continuous. But a continuous martingale with zero quadratic variation is a constant[3]. So $B_t^H = M_t + A_t$ and B^H must have bounded variation. This is a contradiction since $v_1(B_t^H) \geq v_p(B_t^H) = \infty$ for all $p < 1/H$. $\square$

4. INTEGRATION OF THE DETERMINISTIC FUNCTIONS WITH RESPECT TO FRACTIONAL BROWNIAN MOTION

The notation and methodology used in this chapter are related to [5] of Pipiras and Taqqu.

4.1. The Standard Approach to The Integration of Deterministic Functions

In this chapter we deal with questions related to the L^2- integration of *deterministic* functions with respect to fBm when $H \in (1/2, 1)$. As it is done in [5], we will use in the sequel another parametrization of fBm. Let

$$\kappa = H - \frac{1}{2},$$

so that the range $H \in (0,1)$ now correspond to the range $\kappa \in (-1/2, 1/2)$. We will denote the fBm $B^H = \{B_t^H\}_{t \in [0,T]}$ in terms of the parameter κ as $B^\kappa = \{B_t^\kappa\}_{t \in [0,T]}$. We will also assume that fBm is standard, i.e. $E[(B_1^\kappa)^2] = 1$. In terms of κ, covariance function of B^κ will be given in this form:

$$R(s,t) = E(B_s^\kappa B_t^\kappa) = \frac{1}{2}\{s^{2\kappa+1} + t^{2\kappa+1} - |s-t|^{2\kappa+1}\}, s, t \in [0,T].$$

When $\kappa = 0$, fBm $B^\kappa = B^0$ is the usual Brownian motion. [5].

To define integral for deterministic functions, one typically starts with an inner product space $(\mathcal{C}, (\cdot,\cdot)_{\mathcal{C}})$ of functions on a region of integration $R(R = \mathbb{R}, R = [0,T], T > 0)$ such that $(1_{[0,\ t)}, 1_{[0,\ s)})_{\mathcal{C}} = E(B_s^\kappa B_t^\kappa)$ for all $s, t \in R$. Let $\overline{sp}_R(B^\kappa)$ be the closure in $L^2(\Omega)$ of all possible linear combination of the increments of fBm on R. If the map extends to the isometry between this class of functions $\mathcal{C}$ and the space $\overline{sp}_R(B^\kappa)$, then the resulting isometry map is called the integral in the $L^2(\Omega)$-sense with respect to fBm of functions from $\mathcal{C}$.

The function f is called an *elementary function(simple)* if it is given by

$$f(u) = \sum_{k=1}^{n} f_k I_{[u_k, u_{k+1})}(u), u \in [0, T].$$

An integral for simple functions will be defined as

$$J(f) = \int_R f dB^\kappa = \sum_{k=1}^{n} f_k \Delta B^\kappa_{u_k} = \sum_{k=1}^{n} f_k (B^\kappa_{u_{k+1}} - B^\kappa_{u_k}),$$

which is a Gaussian random variable with zero mean and variance that can be calculated by using covariance function of B^κ. Since B^κ has stationary increments, the expected value and the variance of the defined Gaussian random variable is independent of the choice of representation of f.

We recall, in the proposition below, how to construct classes of integrands $\mathcal{C}$.

Proposition 4.1.1. [35] *Let ξ_T be the set of elementary functions on $[0, T]$, $J(f)$ be an integral of $f \in \xi_T$ with respect to fBm B^κ and $-1/2 < \kappa < 1/2$. Suppose that $\mathcal{C}$ is a set of deterministic functions on the $[0, T]$ such that*

(1) *$\mathcal{C}$ has an inner product $(f, g)_{\mathcal{C}}, f, g \in \mathcal{C}$.*

(2) *$\xi_T \subset \mathcal{C}$ and $(f, g)_{\mathcal{C}} = E(J(f)J(g))$ for $f, g \in \xi_T$,*

(3) *the set ξ_T is dense in $\mathcal{C}$.*

Then

(a) *there is an isometry $\widetilde{J}$ between the space $\mathcal{C}$ and a linear subspace of $\overline{sp}_R(B^\kappa)$ which is an extension of the map $f \mapsto J(f)$, for $f \in \xi_T$.*

(b) *$\mathcal{C}$ is isometric to $\overline{sp}_R(B^\kappa)$ itself if and only if $\mathcal{C}$ is complete.*

Proof. **(a)** Let $f \in \mathcal{C}$. By (3), there is a sequence $(f_n) \subset \xi_T$ such that $f_n \to f$ in $\mathcal{C}$.

In particular (f_n) is Cauchy in $\mathcal{C}$ and hence, by (2), $(J(f_n))$ is a Cauchy sequence in $L^2[0,T]$. Since the space $L^2[0,T]$ is complete, there is $\widetilde{J}(f) \in L^2[0,T]$ such that

$$\widetilde{J}(f) = \lim_n J(f_n),$$

in the L^2-sense. Let us show that $\widetilde{J}(f)$ does not depend on the sequence (f_n). Let (g_n) be also a sequence in ξ_T such that $g_n \to f$ in $\mathcal{C}$. Since

$$||f_n - g_n||_{\mathcal{C}} = ||f_n - f + f - g_n||_{\mathcal{C}} \leq ||f_n - f||_{\mathcal{C}} + ||g_n - f||_{\mathcal{C}} \to 0$$

Hence by (2),

$$\lim_n J(f_n - g_n) = 0$$

in $L^2([0,T])$. Thus $\widetilde{J}(f) = \lim_n J(f_n) = \lim_n J(g_n)$.

Moreover, since $(J(f_n)) \subset \overline{sp}_R(B^\kappa)$ and $\overline{sp}_R(B^\kappa)$ is a closed subset of $L^2[0,T]$, we obtain that $\widetilde{J}(f) \in \overline{sp}_R(B^\kappa)$. We can thus define the map $\widetilde{J}$ from the space $\mathcal{C}$ into the space $\overline{sp}_R(B^\kappa)$. This construction of $\widetilde{J}$ and (2) imply that, for $f, g \in \mathcal{C}$,

$$(f,g)_{\mathcal{C}} = E(\widetilde{J}(f)\widetilde{J}(g)),$$

and, since the map is linear, we conclude that $\widetilde{J}$ is an isometry between the space $\mathcal{C}$ and a linear subspace of $\overline{sp}_R(B^\kappa)$.

(b) If $\mathcal{C}$ is isometric to $\overline{sp}_R(B^\kappa)$ itself, then $\mathcal{C}$ is complete because the space $\overline{sp}_R(B^\kappa)$ is complete(it is a closed subset of a complete space $L^2[0,T]$.)

Conversely, If $\mathcal{C}$ is complete, then the map $\widetilde{J}$ is onto because ξ_T is dense in $\mathcal{C}$ and hence $\mathcal{C}$ is isometric to $\overline{sp}_R(B^\kappa)$ itself. $\square$

4.2. Fractional Integrals, Derivatives and a Representation of Fractional Brownian Motion on [0,T]

For completeness, we provide below definitions of those fractional operators that are used throughout this thesis.

Consider the interval $[0, T]$ and let $s \in [0, T]$. An integral over $[0, s]$ is called left-sided and one over $[s, T]$ is called right-sided. The right sided fractional integral of order $\alpha > 0$ on an interval $[0, T]$ of a function $f \in L^1[0, T]$ is defined by

$$(I^{\alpha}_{T-}f)(s) = \frac{1}{\Gamma(\alpha)} \int_0^T f(u)(u-s)_+^{\alpha-1} du = \frac{1}{\Gamma(\alpha)} \int_s^T f(u)(u-s)^{\alpha-1} du, s \in (0, T).$$

The right sided fractional derivative of order $0 < \alpha < 1$ on interval $[0, T]$ of a function ϕ is defined by

$$(D^{\alpha}_{T-}\phi)(u) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_0^T \phi(s)(s-u)_+^{-\alpha} ds, u \in (0, T).$$

The integral equation

$$\frac{1}{\Gamma(\alpha)} \int_0^x \varphi(t)(x-t)^{\alpha-1} dt = f(x),\ x > 0$$

where $0 < \alpha < 1$, is called Abel's equation. [26].

Let $a > -\infty$ and suppose that equation is considered on a finite interval $[a, T]$.

Equation may be solved in the following way which is done in [26]: Changing x to t and t to s respectively in the equation, multiplying both sides of the equation by

$(x-t)^{-\alpha}$ and integrating we have

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)}\int_a^t \varphi(s)(t-s)^{\alpha-1}ds &= f(t), \\
\frac{1}{\Gamma(\alpha)}\frac{1}{(x-t)^\alpha}\int_a^t \varphi(s)(t-s)^{\alpha-1}ds &= \frac{f(t)}{(x-t)^\alpha}, \\
\int_a^x (x-t)^{-\alpha}dt\int_a^t \varphi(s)(t-s)^{\alpha-1}ds &= \Gamma(\alpha)\int_a^x \frac{f(t)}{(x-t)^\alpha}dt.
\end{aligned}$$

Interchanging the order of integration on the left hand side by Fubini's theorem we arrive at

$$\int_a^x \varphi(s)ds\int_s^x (x-t)^{\alpha}(t-s)^{\alpha-1}dt = \Gamma(\alpha)\int_a^x \frac{f(t)}{(x-t)^\alpha}dt.$$

Let $t = s+\tau(x-s)$. Then $\tau = (t-s)/(x-s)$ and $dt = (x-s)d\tau$. Therefore

$$\begin{aligned}
\int_s^x (x-t)^{-\alpha}(t-s)^{\alpha-1}dt &= \int_0^1 \tau^{\alpha-1}(1-\tau)^{-\alpha}d\tau \\
&= \beta(\alpha, 1-\alpha) \\
&= \Gamma(\alpha)\Gamma(1-\alpha),
\end{aligned}$$

and hence

$$\begin{aligned}
\int_a^x \varphi(s)ds\Gamma(\alpha)\Gamma(1-\alpha) &= \Gamma(\alpha)\int_a^x \frac{f(t)}{(x-t)^\alpha}dt \\
\int_a^x \varphi(s)ds &= \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{f(t)}{(x-t)^\alpha}dt.
\end{aligned}$$

After differentiation we have :

$$\varphi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^\alpha}dt.$$

If $0 < \alpha < 1$ and

$$\phi(s) = (I^{\alpha}_{T-} f)(s), s \in (0, T)$$

then by the argument above

$$f(u) = (D^{\alpha}_{T-} \phi)(u), u \in (0, T). \tag{4.1}$$

Hence D^{α}_{T-} can be viewed as an inverse of I^{α}_{T-}. For this reason, we will often denote in the sequel the fractional derivative D^{α}_{T-} with $\alpha \in (0, 1)$ by $I^{-\alpha}_{T-}$ and also use the notation $I^{0}_{T-} f = f$.

The following proposition relates fractional Brownian motion and the fractional integral and derivative operators on an interval $[0, T]$. It will be used in this chapter to construct classes of integrands for fBm on an interval $[0, T]$.

Proposition 4.2.1. [5] *Let $T > 0$ and B^{κ} be a standard fBm with the parameter $\kappa \in (-1/2, 1/2)$. Then*

$$\{B^{\kappa}_t\}_{t \in [0,T]} \cong \{\sigma_1(\kappa) \int_0^T s^{-\kappa} (I^{\kappa}_{T-} u^{\kappa} 1_{[0,t)}(u))(s) dB^0(s)\}_{t \in [0,T]},$$

where

$$\sigma_1(\kappa)^2 = \frac{\Gamma(\kappa)\kappa(2\kappa + 1)}{\beta(\kappa, 1 - 2\kappa)} = \frac{\pi\kappa(2\kappa + 1)}{\Gamma(1 - 2\kappa)\sin(\pi\kappa)},$$

and $\beta(p, q)$ is the Beta function.

Proof.

$$R(s, t) = E(B^{\kappa}_s B^{\kappa}_t) = \frac{1}{2}\{s^{2\kappa+1} + t^{2\kappa+1} - |s - t|^{2\kappa+1}\}.$$

Then

$$\begin{aligned} R(s,t) &= \kappa(2\kappa+1)\int_0^\infty \int_0^\infty I_{[0,\ s)}(u) I_{[0,\ t)}(v)|u-v|^{2\kappa-1} du dv. \\ &= \kappa(2\kappa+1)\int_0^s \int_0^t |u-v|^{2\kappa-1} du dv \end{aligned}$$

for $s > t$.

Let's show that

$$|u-v|^{2\kappa-2} = \frac{(uv)^\kappa}{\beta(\kappa, 1-2\kappa)} \int_0^T q^{-2\kappa}(v-q)_+^{\kappa-1}(u-q)_+^{\kappa-1} dq$$

We will use a similar argument in [36]. Without loss of generality, assume $u > v$, then we will show that

$$(u-v)^{2\kappa-2} = \frac{(uv)^\kappa}{\beta(\kappa, 1-2\kappa)} \int_0^v q^{-2\kappa}(v-q)^{\kappa-1}(u-q)^{\kappa-1} dq.$$

Let

$$z = \frac{u-q}{v-q}, \quad then \quad q = \frac{zv-u}{z-1} \quad and \quad dz = \frac{u-v}{(v-q)^2} dq.$$

So

$$\begin{aligned} &= \int_0^v q^{-2\kappa}(v-q)^{\kappa-1}(u-q)^{\kappa-1} dq \\ &= \int_{u/v}^\infty \left(\frac{zv-u}{z-1}\right)^{-2\kappa}\left(\frac{u-v}{z-1}\right)^{\kappa-1}\left(\frac{u-v}{z-1}\right)^{\kappa-1} z^{\kappa-1} \frac{(v-q)^2}{u-v} dz \\ &= (u-v)^{2\kappa-2} \int_{u/v}^\infty (zv-u)^{-2\kappa} z^{\kappa-1} dz \end{aligned}$$

Now, let

$$x = \frac{u}{vz}, \quad then \quad dx = -\frac{u}{vz^2}dz.$$

Since

$$vz = \frac{u}{x}, \quad we \quad have \quad dz = -\frac{u}{vx^2}dx.$$

Hence

$$\begin{aligned}
\int_0^v q^{-2\kappa}(v-q)^{\kappa-1}(u-q)^{\kappa-1}dq &= (u-v)^{2\kappa-2}\int_0^1 (\frac{u}{x}-u)^{-2\kappa}(\frac{u}{vx})^{\kappa-1}\frac{u}{vx^2}dx. \\
&= (u-v)^{2\kappa-2}\int_0^1 u^{-2\kappa}(1-x)^{-2\kappa}(\frac{u}{v})^{\kappa}x^{\kappa-1}dx. \\
&= \frac{(u-v)^{2\kappa-2}}{(uv)^{\kappa}}\int_0^1 (1-x)^{-2\kappa}x^{\kappa-1}dx \\
&= \frac{(u-v)^{2\kappa-2}}{(uv)^{\kappa}}\beta(\kappa, 1-2\kappa).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{(uv)^{\kappa}}{\beta(\kappa,1-2\kappa)}\int_0^v q^{-2\kappa}(v-q)^{\kappa-1}(u-q)^{\kappa-1}dq &= \frac{(uv)^{\kappa}}{\beta(\kappa,1-2\kappa)}\frac{(u-v)^{2\kappa-2}}{(uv)^{\kappa}}\beta(\kappa,1-2\kappa) \\
&= (u-v)^{2\kappa-2}.
\end{aligned}$$

Therefore we get the result.

Let $\mathcal{V} = \kappa(2\kappa+1)/\beta(\kappa, 1-2\kappa)$, then we have:

$$\begin{aligned}
R(s,t) &= \kappa(2\kappa+1)\int_0^s\int_0^t |u-v|^{2\kappa-1}dudv \\
&= \kappa(2\kappa+1)\int_0^s\int_0^t \frac{(uv)^\kappa}{\beta(\kappa,1-2\kappa)}\int_0^T q^{-2\kappa}(v-q)_+^{\kappa-1}(u-q)_+^{\kappa-1}dqdudv \\
&= \mathcal{V}\int_0^T q^{-2\kappa}\int_0^s u^\kappa(u-q)^{\kappa-1}du\int_0^t v^\kappa(v-q)^{\kappa-1}dvdq \\
&= \mathcal{V}\int_0^T [(\int_0^T q^{-\kappa}u^\kappa I_{[0,\ s)}(u)(u-q)_+^{\kappa-1}du)(\int_0^T q^{-\kappa}v^\kappa I_{[0,\ t)}(v)(v-q)_+^{\kappa-1}dv)]dq \\
&= \mathcal{V}\Gamma(\kappa)^2\int_0^T [q^{-\kappa}(I_{T-}^\kappa u^\kappa I_{[0,\ s)}(u))(q)][q^{-\kappa}(I_{T-}^\kappa v^\kappa I_{[0,\ t)}(v))(q)]dq.
\end{aligned}$$

So they have the same covariance structure. □

4.3. A Class of Integrands for Fractional Brownian Motion on [0,T]

Let $\{B_t^\kappa\}_{t\in[0,T]}$ be a standard fBm with parameter $\kappa \in (0, 1/2)$. Let also ξ_T denote the set of all elementary functions on an interval $[0, T]$.

For an elementary function $f \in \xi_T$, define the integral with respect to fBm B^κ in a natural way by

$$J(f) = \int_0^T f(u)dB^\kappa(u) = \sum_{k=1}^n f_k(B_{u_{k+1}}^\kappa - B_{u_k}^\kappa)$$

When $\kappa \in (0, 1/2)$ and the region of integration R is $[0, T]$ with $T > 0$ defined the class of integrants for fBm

$$\Lambda_T^\kappa = \{f : [0,T] \mapsto \mathbb{R} \mid \int_0^T [s^{-\kappa}(I_{T-}^\kappa u^\kappa f(u))(s)]^2 ds < \infty\}$$

are assumed to be inner product space with the inner product

$$(f,g)_{\Lambda_T^\kappa} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)} \int_0^T [s^{-2\kappa}(I_{T-}^\kappa u^\kappa f(u))(s)(I_{T-}^\kappa u^\kappa g(u))(s)]ds$$

where

$$\Gamma(p),\ p>0$$

is the gamma function. [5]. (See appendix)

It follows by the Proposition 4.2.1 that, for $f \in \xi_T$,

$$\begin{aligned}
J(f) &= \int_0^T f(u)dB^\kappa(u) = \sum_{k=1}^n f_k(B^\kappa_{u_{k+1}} - B^\kappa_{u_k}) \\
&\cong \sum_{k=1}^n f_k\sigma_1(\kappa)[\int_0^T s^{-\kappa}(I_{T-}^\kappa u^\kappa I_{[0,\ u_{k+1})}(u))(s)dB^0(s) \\
&- \int_0^T s^{-\kappa}(I_{T-}^\kappa u^\kappa I_{[0,\ u_k)}(u))(s)dB^0(s)] \\
&= \sum_{k=1}^n f_k\sigma_1(\kappa)[\int_0^T s^{-\kappa}[(I_{T-}^\kappa u^\kappa I_{[0,\ u_{k+1})}(u))(s) - (I_{T-}^\kappa u^\kappa I_{[0,\ u_k)}(u))(s)]dB^0(s)] \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=1}^n f_k\sigma_1(\kappa) \int_0^T [s^{-\kappa} \int_{u_k}^{u_{k+1}} u^\kappa(u-s)^{\kappa-1}du]dB^0(s) \\
&= \frac{1}{\Gamma(\alpha)}\sigma_1(\kappa) \int_0^T s^{-\kappa}[\sum_{k=1}^n f_k \int_{u_k}^{u_{k+1}} u^\kappa(u-s)^{\kappa-1}du]dB^0(s) \\
&= \sigma_1(\kappa)\int_0^T s^{-\kappa}(I_{T-}^\kappa u^\kappa f(u))(s)dB^0(s)
\end{aligned}$$

and hence for all $f,g \in \xi_T$,

$$E[J(f)J(g)] = \sigma_1(\kappa)^2 \int_0^T s^{-2\kappa}(I_{T-}^\kappa u^\kappa f(u))(s)(I_{T-}^\kappa u^\kappa g(u))(s)ds$$

Theorem 4.3.1. [5] *For $\kappa \in (0, 1/2)$, the class of functions Λ_T^κ is a linear space with the inner product $(f, g)_{\Lambda_T^\kappa}$. Moreover, the set of elementary functions ξ_T is dense in the space Λ_T^κ.*

Proof. To show that the map $(f, g)_{\Lambda_T^\kappa}$ defines an inner product, we have to show that $(f, g)_{\Lambda_T^\kappa}$ satisfies the following conditions:

$$
\begin{aligned}
&(1) \quad (f+g, h)_{\Lambda_T^\kappa} = (f, h)_{\Lambda_T^\kappa} + (g, h)_{\Lambda_T^\kappa} \quad f, g, h \in \Lambda_T^\kappa. \\
&(2) \quad (\alpha f, g)_{\Lambda_T^\kappa} = \alpha (f, g)_{\Lambda_T^\kappa} \quad f, g \in \Lambda_T^\kappa,\ \alpha \in \mathbb{R}. \\
&(3) \quad (f, g)_{\Lambda_T^\kappa} = (g, f)_{\Lambda_T^\kappa} \quad f, g \in \Lambda_T^\kappa. \\
&(4) \quad (f, f)_{\Lambda_T^\kappa} \geq 0, \quad (f, f)_{\Lambda_T^\kappa} = 0 \Leftrightarrow f = 0 \quad a.e.
\end{aligned}
$$

The first three conditions are obvious since the integral operator is linear. We check the least obvious condition (4): $(f, f)_{\Lambda_T^\kappa} \geq 0$ since

$$
[(I_{T-}^\kappa u^\kappa f(u))(s)]^2 \geq 0, \qquad \sin(\pi\kappa) \geq 0
$$

when $\kappa \in (0, 1/2)$. If $(f, f)_{\Lambda_T^\kappa} = 0$ and $\kappa \in (0, 1/2)$, then $(I_{T-}^\kappa u^\kappa f(u))(s) = 0$ a.e. $s \in [0, T]$. Since the homogeneous Abel integral equation $(I_{T-}^\kappa \phi(u))(s) = 0$ has only the trivial solution $\phi(x) \equiv 0$ a.e.[26], it follows that $u^\kappa f(u) = 0$ a.e. $u \in [0, T]$ and hence that $f(u) = 0$ a.e. $u \in [0, T]$.

Let us show that the set of elementary functions ξ_T is dense in Λ_T^κ. Assume without loss of generality that $T > 1$. Since any function $\phi \in L^2[0, T]$ can be approximated in $L^2[0, T]$ by the elementary functions, we may also approximate by functions

$$
s^{-\kappa} \sum_{k=1}^{n} b_k I_{[c_k,\ d_k)}(s) = s^{-\kappa} \sum_{k=1}^{n} b_k (I_{[0,\ d_k)}(s) - I_{[0,\ c_k)}(s))
$$

with $b_k \in \mathbb{R}$ and $0 < c_k < d_k < T$, $k = 1, ..., n$. It is enough to show that there is a sequence of elementary functions $f_n \in \xi_T \subset \Lambda_T^\kappa$ such that

$$||I_{[0,1)} - f_n||^2_{\Lambda_T^\kappa} = \int_0^T s^{-2\kappa}|I_{[0,1)}(s) - (I_{T-}^\kappa u^\kappa f_n(u))(s)|^2 ds \to 0,$$

as $n \to \infty$. Let us define the functions

$$f_n(u) = \frac{1}{\Gamma(1-\kappa)} \sum_{l=2}^{n-2} f(l/n) I_{[\frac{l}{n}, \frac{l+1}{n})}, \qquad u \in [0,1].$$

By using the inequality $2|f(u)||f(v)| \leq |f(u)|^2 + |f(v)|^2$ and symmetry below, we get

$$\begin{aligned} \int_0^a \int_0^a |f(u)||f(v)||u-v|^{2\kappa-1} du dv &\leq \int_0^a \int_0^a |f(u)|^2 |u-v|^{2\kappa-1} du dv \\ &\leq \frac{a^{2\kappa}}{\kappa} \int_0^a |f(u)|^2 du \end{aligned}$$

Hence

$$\begin{aligned} \int_0^T s^{-2\kappa}|I_{[0,1)}(s) - (I_{T-}^\kappa u^\kappa f_n(u))(s)|^2 ds &\leq \int_0^1 s^{-2\kappa}[(I_{1-}^\kappa u^\kappa |f(u) - f_n(u)|)(s)]^2 ds \\ &\leq c(\kappa)||f - f_n||^2_{L^2[0,1]} \to 0, \end{aligned}$$

as $n \to \infty$. □

Now, we will give the proof of Pipiras and Taqqu in [5] which shows that the space Λ_T^κ is not a complete inner product space. They give a proof by first providing an equivalent criteria for the completeness and then showing that it does not hold. We begin with a number of lemmas which will be used in the sequel.

Lemma 4.3.2. [5] *Let $0 \leq c < b \leq T$ and $\kappa \in (0, 1/2)$. Then there is a function $f_{c,b}$*

such that

$$s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}f_{c,b}(u))(s) = 1_{[c,b)}(s), \qquad 0 \le s \le T.$$

Proof. Let us define a function

$$f_{c,b}(u) = u^{-\kappa}(I^{-\kappa}_{T-}t^{\kappa}I_{[c,b)}(t))(u) = u^{-\kappa}(D^{\kappa}_{T-}t^{\kappa}I_{[c,b)}(t))(u).$$

Since $I^{-\kappa}_{T-}$ is an inverse operator of I^{κ}_{T-}

$$\begin{aligned} s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}f_{c,b}(u))(s) &= s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}u^{-\kappa}(I^{-\kappa}_{T-}t^{\kappa}I_{[c,b)}(t))(u))(s) \\ &= s^{-\kappa}(I^{0}_{T-}t^{\kappa}I_{[c,b)}(t))(s) \\ &= s^{-\kappa}s^{\kappa}I_{[c,b)}(s) \\ &= I_{[c,b)}(s). \end{aligned}$$

We need to check whether $f_{c,b} \in \Lambda^{\kappa}_{T}$.

$$\begin{aligned} \int_0^T [s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}f_{c,b}(u))(s)]^2 ds &= \int_0^T [s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}u^{-\kappa}(I^{-\kappa}_{T-}t^{\kappa}I_{[c,b)}(t))(u))(s)]^2 ds \\ &= \int_0^T [s^{-\kappa}(I^{\kappa}_{T-}(I^{-\kappa}_{T-}t^{\kappa}I_{[c,b)}(t))(u))(s)]^2 ds \\ &= \int_0^T [s^{-\kappa}s^{\kappa}I_{[c,b)}(s)]^2 ds < \infty. \end{aligned}$$

□

Lemma 4.3.3. [5] *Let $\kappa \in (0, 1/2)$. The inner product space Λ^{κ}_{T} is complete if and only if, for every $\phi \in L^2[0,T]$, there is a function $f_{\phi} \in \Lambda^{\kappa}_{T}$ such that*

$$s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}f_{\phi}(u))(s) = \phi(s) \ a.e. \tag{4.2}$$

Proof. Suppose that the inner product space Λ^{κ}_{T} is complete and let $\phi \in L^2[0,T]$. There is a sequence of elementary functions ϕ_n such that $\phi_n \to \phi$ in $L^2[0,T]$. By the above lemma, we can express the elementary functions ϕ_n as $\phi_n = s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}f_n(u))(s)$, for some $f_n \in \Lambda^{\kappa}_T$. Then, let $h_{nm}(u) = f_n(u) - f_m(u)$,

$$\begin{aligned}
||f_n - f_m||^2_{\Lambda^{\kappa}_T} &= \sigma_1(\kappa)\int_0^T [s^{-2\kappa}(I^{\kappa}_{T-}u^{\kappa}h_{nm}(u)))(s)(I^{\kappa}_{T-}u^{\kappa}h_{nm}(u)))(s)]ds \\
&= \sigma_1(\kappa)\int_0^T [s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}h_{nm}(u))(s)][s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}h_{nm}(u))(s)]ds \\
&= \sigma_1(\kappa)\int_0^T (\phi_n - \phi_m)^2 ds \\
&\to 0
\end{aligned}$$

as $m, n \to \infty$ since $\{\phi_n\}_{n\geq 1}$ is Cauchy in $L^2[0,T]$. Thus $\{f_n\}_{n\geq 1}$ is Cauchy in Λ^{κ}_T. Then, the completeness of Λ^{κ}_T implies that there is a function $f \in \Lambda^{\kappa}_T$ such that

$$\phi_n = s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}f_n(u))(s) \to s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}f(u))(s)$$

in $L^2[0,T]$. Defining $\phi(s)$ as the limit, then the relation

$$s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}f_{\phi}(u))(s) = \phi(s)$$

holds for $f_{\phi} = f$.

Conversely, suppose that (4.2) holds and let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in Λ^{κ}_T. Then the sequence $\phi_n = s^{-\kappa}(I^{\kappa}_{T-}u^{\kappa}f_n(u))(s)$ is Cauchy in $L^2[0,T]$. Since $L^2[0,T]$ is complete, there is a $\phi \in L^2[0,T]$ such that $\phi_n \to \phi$ in $L^2[0,T]$. By the assumption, there is $f_{\phi} \in \Lambda^{\kappa}_T$ such that (4.2) holds. Since $\phi_n \to \phi$ in $L^2[0,T]$ implies $f_n \to f_{\phi}$ in Λ^{κ}_T, the space Λ^{κ}_T is complete. □

Lemma 4.3.4. [5] *Let $\kappa \in (0, 1/2)$. There are continuous functions ψ on $[0,T]$ such that the equation*

$$(I^{\kappa}_{T-}g)(s) = \psi(s) \quad a.e. \quad on \quad [0,T] \tag{4.3}$$

has no solution in $g \in L^1[0,T]$.

Proof. (Sketch) The proof is by contradiction. Suppose that the equation (4.3) has a solution $g_\psi \in L^1[0,T]$ for any $\psi \in L^2[0,T]$. By (4.1)

$$g_\psi(u) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{du}\int_0^T \psi(s)(s-u)_+^{-\kappa}ds,\ u \in (0,T).$$

Since $g_\psi(u)$ is expressed as a derivative, the function

$$U_\psi(u) = \int_0^T \psi(s)(s-u)_+^{-\kappa}ds$$

is differentiable almost everywhere on $[0,T]$. However, as shown in [35], there are functions $\psi \in L^2[0,T]$ such that U_ψ is not differentiable on a set of positive Lebesque measure. So we obtain the result. $\square$

Theorem 4.3.5. [5] *Let* $\kappa \in (0,1/2)$. *The inner product space* Λ_T^κ *is not complete.*

Proof. By the lemma, there is a continuous function function ψ such that the equation $(I_{T-}^\kappa g)(s) = \psi(s)$ has no solution in g. Then the function $\phi(s) = s^{-\kappa}\psi(s)$ is in $L^2[0,T]$, since $\kappa < 1/2$, and the the equation

$$s^{-\kappa}(I_{T-}^\kappa u^\kappa f_\phi(u))(s) = \phi(s)$$

has no solution in f. Thus, by the lemma 4.3.3., the inner product space Λ_T^κ is not complete. $\square$

5. CONCLUSIONS

In this thesis, we have studied the general idea of the integration of the deterministic functions with respect to the fractional Brownian motion. This idea is also used for general Gaussian processes.[37] As mentioned in [5], to define such integrals, one generally starts with an inner product space $(\mathcal{C}, (\cdot,\cdot)_{\mathcal{C}})$ of functions on a region of integration, R. Then one defines a map between $\overline{sp}_R(B^\kappa)$, the closure of all linear combinations of the increments of fBm in $L^2(\Omega)$, and the specified inner product space of functions on a region of integration such that $1_{[0,t)} \mapsto B^\kappa_t$ under this map. If the map $1_{[0,t)} \mapsto B^\kappa_t$ extends to the isometry between these two spaces, then the resulting isometry map is called the integral in the $L^2(\Omega)$-sense with respect to fBm of the functions from our inner product space. The extension step is usually assumed correct but not proved in the studies on this subject. For example, when $\kappa \in (0, 1/2)$ and region of integration R is $[0,T]$ with $T > 0$, Carmona et al.[38] and Kleptsyna et al.[27], defined the class of integrands for fBm which we studied on our thesis

$$\Lambda^\kappa_T = \{f : [0,T] \mapsto \mathbb{R} \mid \int_0^T [s^{-\kappa}(I^\kappa_{T-} u^\kappa f(u))(s)]^2 ds < \infty\}$$

with the inner product

$$(f,g)_{\Lambda^\kappa_T} = \frac{\pi\kappa(2\kappa+1)}{\Gamma(1-2\kappa)\sin(\pi\kappa)} \int_0^T [s^{-2\kappa}(I^\kappa_{T-} u^\kappa f(u))(s)(I^\kappa_{T-} u^\kappa g(u))(s)] ds.$$

Another class of integrands, considered by Duncan et al. [12], Kleptsyna et al. [27] and Norros et al. [39], is given by

$$|\Lambda|^\kappa_T = \{f : [0,T] \mapsto \mathbb{R} \mid \int_0^T \int_0^T |f(u)||f(v)||u-v|^{2\kappa-1} du dv < \infty\}$$

with the inner product

$$(f,g)_{|\Lambda|_T^\kappa} = \kappa(2\kappa-1)\int_0^T\int_0^T f(u)g(v)|u-v|^{2\kappa-1}dudv.$$

All the authors claimed that both Λ_T^κ and $|\Lambda|_T^\kappa$ are isometric to the Gaussian space $\overline{sp}_R(B^\kappa)$. However, Pipiras and Taqqu showed that this assumption is not correct.[5]. They argued that since $\overline{sp}_R(B^\kappa)$ is a complete inner product space, both Λ_T^κ and $|\Lambda|_T^\kappa$ have necessarily to be complete inner product spaces as well. On the other hand, they showed that, when $\kappa \in (0, 1/2)$, neither the space of functions Λ_T^κ nor the space of functions $|\Lambda|_T^\kappa$ is a complete inner product spaces. Thus, these two spaces cannot be isometric to $\overline{sp}_R(B^\kappa)$ itself. However this result does not affect the integration idea, because they are isometric only to proper linear subspaces of $\overline{sp}_R(B^\kappa)$. Consequently, there are random elements in $\overline{sp}_R(B^\kappa)$ which can not be represented by functions f belonging to either Λ_T^κ or $|\Lambda|_T^\kappa$. [5].

An application of integrals, where completeness of the function spaces is relevant, involves prediction problems. Consider, for example, the problem of prediction the value of fBm at some future time $t > 0$ given its past from time 0 to a (with $a < t$), or mathematically, to compute the conditional expectation $X = E[B_t^\kappa | B_s^\kappa,\ s \in [0,a]]$. It is known that $X \in \overline{sp}_R(B^\kappa)$. One would expect that $X = \int_0^T f dB^\kappa$. But, when $\kappa > 0$, as we mentioned above, there may be no $f \in \Lambda_T^\kappa$ or $|\Lambda|_T^\kappa$ such that $X = \int_0^T f dB^\kappa$. [5].

Is there a class of integrands for fBm with $\kappa \in (0, 1/2)$ on an interval $[0,T]$ which is a complete inner product space? However, we do not know the answer to this question.

It is natural to ask whether there is a class of integrands for fBm with $\kappa \in (-1/2, 0)$ on an interval $[0,T]$ which is a complete inner product space. It turns out that such a class of integrands exists. It is an inner product space

$$\Lambda_T^\kappa = \{f : [0,T] \mapsto \mathbb{R} \mid \exists \phi_f \in L^2[0,T]\ s.t.\ f(u) = u^{-\kappa}(I_{T-}^{-\kappa} s^\kappa \phi_f(s))(u)\}$$

with the inner product

$$(f,g)_{\Lambda_T^\kappa} = \frac{\pi\kappa(2\kappa+1}{\Gamma(1-2\kappa)\sin(\pi\kappa)}\int_0^T [s^{-2\kappa}(I_{T-}^\kappa u^\kappa f(u))(s)(I_{T-}^\kappa u^\kappa g(u))(s)]ds.$$

We want to emphasize that the inner product space Λ_T^κ is *complete* when $\kappa \in (-1/2, 0)$ and *incomplete* when $\kappa \in (0, 1/2)$. This difference in incompleteness is explained by Pipiras and Taqqu in their article [5] as the consequence of the following two facts:

(i) If $\kappa \in (-1/2, 0)$, then the equation

$$s^{-\kappa}(I_{T-}^\kappa u^\kappa f(u))(s) = \phi(s)$$

has a solution $f(u) = u^{-\kappa}(I_{T-}^{-\kappa} s^\kappa \phi(s))(u)$ for every $\phi \in L^2[0,T]$,

(ii) If $\kappa \in (0, 1/2)$, there are functions $\phi \in L^2[0,T]$ for which the equation

$$s^{-\kappa}(I_{T-}^\kappa u^\kappa f(u))(s) = \phi(s)$$

is not solvable. The idea here is that, since $I_{T-}^{-\kappa}$ is the integral operator, the left hand side of (5.1) most satisfy some smoothness conditions whereas such conditions need not hold for a general $\phi \in L^2[0,T]$.

APPENDIX A: GAMMA AND BETA FUNCTIONS

A.1. Gamma Function

The gamma function denoted by $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

which is convergent for $n > 0$.

A recurrence formula for the gamma function is

$$\Gamma(n+1) = n\Gamma(n)$$

where $\Gamma(1) = 1$. Therefore $\Gamma(n)$ can be determined for all $n > 0$ when the values for $1 \leq n < 2$ (or any other interval of unit length) are known. In particular, if n is a positive integer, then

$$\Gamma(n+1) = n!.$$

We also have another important formula which involves the Gamma function:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \qquad 0 < x < 1.$$

In particular, if $x = \frac{1}{2}$, $\Gamma(1/2) = \sqrt{\pi}$.

For the proof of the above formulas see [40].

A.2. Beta Function

The beta function denoted by $\beta(m,n)$ is defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

which is convergent for $m > 0,\ n > 0$.

The Beta function is connected with the Gamma function according to the relation

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m, n > 0.$$

Many integrals can be evaluated in terms of Beta and Gamma functions. Two useful results are

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta = \frac{1}{2}\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

valid for $m > 0$ and $n > 0$ and

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)} \quad 0 < p < 1.$$

For the proof of the above formulas [40].

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