INTERPOLATING BASES IN THE SPACES OF C^P-FUNCTIONS ON CANTOR-TYPE SETS

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Necip Özfidan

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Alexander Goncharov(Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Mefharet Kocatepe

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Seçil Gergün

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet B. Baray
Director of the Institute Engineering and Science

ABSTRACT

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Necip Özfidan
M.S. in Mathematics
Supervisor: Assoc. Prof. Dr. Alexander Goncharov
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In this work by using the method of local interpolations suggested in [9] we construct topological bases in the spaces of C^p-functions defined on uniformly perfect compact sets of Cantor type. Elements of the basis are polynomials of any preassigned degree.

Keywords: spaces of p-times differentiable functions, bases,Cantor-type sets, local interpolation.

ÖZET

CANTOR-TİPİ KÜMELERDE P-KEZ TÜREVLENEBİLİR FONKSİYONLARIN UZAYINDA BAZ OLUSTURULMASI

Necip Özfidan
Matematik, Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Alexander Goncharov
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Bu tezde biz Goncharov'un [9] makalesindeki lokal interpolasyon metodunu p-kez türevlenebilir fonksiyonların uzayında kullanarak kompak Cantor-tipi kümelerde baz oluşturulabileceğini gösterdik. Bazları oluşturmak için derecesi belli polinomları kullandık.

Anahtar sözcükler: p-kez türevlenebilir fonksiyonların uzayları, bazlar, Cantor-tipi kümeler, interpolasyon.

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Contents

Chapter 1

Introduction

The scalar field $\mathbb{K}$ for all linear spaces considered in the sequel will be either the field of complex numbers or the field of real numbers.

Definition 1.0.1. A sequence $\{f_n\}_1^\infty$ in an infinite dimensional Banach space E is called a *Schauder basis* of E if for every $f \in E$ there exists a unique sequence of scalars $\{\alpha_n\} \subset \mathbb{K}$ such that

$$f = \sum_{i=1}^{\infty} \alpha_i f_i$$

(i.e. such that $\lim_{n \longrightarrow \infty} \left\| f - \sum_{i=1}^{n} \alpha_i f_i \right\| = 0$.)

An importance of the concept of basis lies in the fact that it provides a natural method of approximation of vectors and operators in the space. The notion of basis of a Banach space was introduced by Schauder [17]. But interest to on the theory of bases in topological vector spaces has grown after the publication of Banach's book on the theory of linear operators. Banach asked a question in his book that whether every separable Banach space possesses a basis or not. This problem was known as the basis problem. Up to 1970's much of the literature on the theory of basis were devoted this problem. Although the problem was not solved up to 1972, there were many good works which shows the connection between the theory of basis and the structure of the linear spaces. In 1972, Enflo [6] constructed the first example of Banach space which does not have the approximation property and hence does not possess a basis. Enflo proved that there exists a separable Banach space E, with a sequence $\{G_n\}$ of finite dimensional subspaces satisfying $\lim_{n\to\infty} \dim G_n = \infty$ and a constant C, such that for every finite operator $v \in L(E,E)$,

$$\|v|_{G_n} - I_{G_n}\| \geq 1 - \frac{C\|v\|}{\log \dim G_n} \qquad n = 1, 2, ...$$

hence, since E is separable and reflexive, E does not have the approximation property. This result was improved by Figiel and Pelczynski [8] which showed that there exists subspaces E of c_0 and l^p, for each p with $2 < p < \infty$ without the approximation property. Then Davie [5] has obtained subspaces E of L^p,for each p with $2 < p < \infty$, which do not have the approximation property. Furthermore, Szankowski [22] has obtained, for any p, q with $1 \leq q \leq \infty$, a Banach lattice which can be linearly isometrically embedded into $(E_1 \times E_2 \times ...)_{l_p}$, where $E_n = L^q([0,1])(n = 1, 2, ...)$ and which fails to have the approximation property.

The idea that how results on Banach spaces may be generalized to complete metric spaces or locally convex topological spaces enriched the theory. Interesting new directions have been developed and interplaited in the Banach space theory. For example, if we replace the elements of basis by linear subspaces of a Banach space, we can obtain decompositions of this space.Also, if we do not use only basis series expansion, if we use biorthogonal systems then we obtain the systems called dual generalized basis. Now we start the history of the basis in the spaces of continuous functions and p-times differentiable functions.

In 1910, Faber [7] showed that there exists a basis in the space $C[0,1]$ consisting of the primitives of the Haar functions. Faber used the diadic system of points in his construction. In 1927, Schauder [17] rediscovered the more general form of this result. Now we examine the Schauder system.

Let $\{a_n\}_1^\infty \subset [0,1]$ be an arbitrary dense sequence of distinct points in $[0,1]$ and for each $n \geq 2$ let v_n denote that one of the n subsegments of $[0,1]$ determined by 0, a_1, a_2, ..., a_{n-1}, 1 (rearranged in increasing order) which contains a_n. Let $f_0(t) \equiv 1$, $f_1(t) = t$. Then the sequence

$$f_n(t) = \begin{cases} 0, & \text{for } t \notin v_{n-1} \\ 1, & \text{for } t = a_{n-1} \\ \text{linear} & \text{for the other } t \end{cases} \tag{1.0.1}$$

where $n = 2, 3, ...$ constitutes a basis in the space $C[0,1]$.

We show this first for $f \in C[0,1]$ satisfying $f(0) = f(1) = 0$. Let

$$\alpha_2 = f(a_1), \qquad \alpha_n = f(a_{n-1}) - \sum_{i=2}^{n-1} \alpha_i f_i(a_{n-1}) \quad (n = 3, 4, ...) \tag{1.0.2}$$

$$S_n(f) = \sum_{i=2}^{n} \alpha_i f_i \qquad\qquad (n = 2, 3, ...) \tag{1.0.3}$$

Then, since $f_2, ..., f_n$ are linear on each subsegment of $[0,1]$ determined by $\{a_i\}_{i=1}^{n-1}$, S_n is also linear. Also, by (1.0.1) we have $f_{j+1}(a_j) = 1$ and $f_i(a_j) = 0$ for $i = j+2, j+3, ..., n \quad (j = 1, ..., n-1)$. Then

$$\begin{aligned} [S_n(f)](a_j) &= \sum_{i=2}^{j} \alpha_i f_i(a_j) + \alpha_{j+1} f_{j+1}(a_j) + \sum_{i=j+2}^{n} \alpha_i f_i(a_j) \\ &= \sum_{i=2}^{j} \alpha_i f_i(a_j) + \Big[f(a_j) - \sum_{i=2}^{j} \alpha_i f_i(a_j) \Big] = f(a_j) \end{aligned} \tag{1.0.4}$$

So $S_n(f)$ is the polygonal function interpolating the values of f in the points $0, a_1, a_2, ..., a_{n-1}, 1$. Then if a_{i_1}, a_{i_2} are any two consecutive points of $\{a_i\}_{i=1}^{n-1}$,

$$[S_n(f)](\lambda a_{i_1} + (1-\lambda) a_{i_2}) = \lambda f(a_{i_1}) + (1-\lambda) f(a_{i_2}) \quad (0 \le \lambda \le 1) \tag{1.0.5}$$

Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous on [0,1], there exists an $\delta = \delta(\varepsilon) > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ whenever $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$. Since $\{a_j\}$ is dense in [0,1], there exists a positive integer $N = N[\delta(\varepsilon)]$ such that for $n > N$ we have $\max |a_{i_1} - a_{i_2}| < \delta$, where the maximum is taken over all couples of consecutive points of $\{a_i\}_{i=1}^{n-1}$. Now, let $t \in [0,1]$ be arbitrary. Then there exists a λ with $0 \le \lambda \le 1$, such that $t = \lambda a_{i_1} + (1-\lambda) a_{i_2}$, where a_{i_1}, a_{i_2} are consecutive points of $\{a_i\}_{i=1}^{n-1}$ satisfying $t \in [a_{i_1}, a_{i_2}]$. Then, by (1.0.5),

$$\begin{aligned} |f(t) - S_n(t)| &= |f(t) - \lambda f(a_{i_1}) - (1-\lambda) f(a_{i_2})| \\ &= |\lambda [f(t) - f(a_{i_1})] + (1-\lambda)[f(t) - f(a_{i_2})]| \\ &\le \max_{t_1, t_2 \in [a_{i_1}, a_{i_2}]} |f(t_1) - f(t_2)| < \varepsilon \qquad (n > N[\delta(\varepsilon)]), \end{aligned} \tag{1.0.6}$$

and thus, since $N[\delta(\varepsilon)]$ is independent of $t \in [0,1]$,

$$\|f - S_n\| < \varepsilon \qquad\qquad (n > N[\delta(\varepsilon)]).$$

Consequently, we have

$$f \;=\; \lim_{n\to\infty} S_n \;=\; \sum_{i=2}^{\infty} \alpha_i f_i.$$

Now we drop the restriction $f(0) = f(1) = 0$. Let $f \in C[0,1]$ be arbitrary. Then for the $\bar{f} \in C[0,1]$ defined by

$$\bar{f}(t) = f(t) - f(0) - [f(1) - f(0)]t \qquad (t \in [0,1])$$

we have $\bar{f}(0) = \bar{f}(1) = 0$. By the above, $\bar{f}$ has an expansion of the form $\bar{f} = \sum_{i=2}^{\infty} \alpha_i f_i$. By putting $\alpha_0 = f(0)$, $\alpha_1 = f(1) - f(0)$, we obtain

$$f \;=\; \sum_{i=0}^{\infty} \alpha_i f_i.$$

So $(f_i)_{i=0}^{\infty}$ is a basis of $C[0,1]$.

Using basis of $C[0,1]$ it is not difficult to present a basis in the space $C^p[0,1]$. First we show this for $p = 1$. Let $f \in C^1[0,1]$ with $\|f\|^{(1)} = \max(|f(0)|, \sup_{0<x<1} |f'(x)|)$. Let $\phi :\; C^1[0,1] \mapsto C[0,1] \bigoplus \mathbb{R}$ be a linear operator such that $\phi f(x) = \int_0^x f(t)dt + f(0)$. Then for every $f \in C[0,1]$, $\phi(f(x)) \in C^1[0,1]$ since $(\phi(f(x))' = f(x) - f(0) + f(0) = f(x)$ is continuous. Also for every $\phi(g) \in C^1[0,1]$, $g \in C[0,1]$. Then $C^1[0,1]$, $C[0,1] \bigoplus \mathbb{R}$ are isomorphic. Also we look isometry.

$$\|\phi f(x)\| = \sup_{0<x<1} |(\phi f(x))'| = \sup_{0<x<1} \Big|\Big(\int\limits_0^x f(t)dt + f(0)\Big)'\Big| = \|f(x)\|.$$

Then the spaces are isometric. Now we find the basis for $C^1[0,1]$. We know that for every $g \in C^1[0,1]$, $g = \int_0^x g'(t)dt + g(0)$. Since $g' \in C[0,1]$ there exists basis f_k such that $g' = \sum_{k=0}^{\infty} \alpha_k f_k$. Then

$$g = \sum \alpha_k \int\limits_0^x f_k(t)dt + g(0).$$

Then the elements of basis are 1, $\int_0^x f_0(t)dt = x$, $\int_0^x f_1(t)dt, \ldots.$ By using the same method in the space $C^p[0,1]$, the spaces $C[0,1] \bigoplus \mathbb{R}^p$ and $C^p[0,1]$ are isomorphic. Then the elements of basis in $C^p[0,1]$ are

$$1,\ x,\ \frac{x^2}{2},\ \ldots,\ \frac{x^p}{p!},\ \int\limits_0^{x_1=x}\int\limits_0^{x_2}\cdots\int\limits_0^{x_p} f_1(t)dt,\ \int\limits_0^{x_1=x}\int\limits_0^{x_2}\cdots\int\limits_0^{x_p} f_2(t)dt,\ \ldots$$

On the other hand the basis problem for the space $C^p[0,1]^2$ is much more difficult. In 1932 Banach [1] raised this problem in his book: Let B be the space of all real-valued continuous functions on the unit square $0 \leq t \leq 1,\quad 0 \leq s \leq 1$, admitting continuous partial derivatives of order 1, endowed with the norm

$$||x|| = \max_{0\leq t\leq 1, 0\leq s\leq 1} |x(t,s)| + \max_{0\leq t\leq 1, 0\leq s\leq 1} |x_t'(t,s)| + \max_{0\leq t\leq 1, 0\leq s\leq 1} |x_s'(t,s)|;$$

does B possess a basis? This problem was solved by Ciesielski [3] and Schonefeld [18] independently only in 1969. Ciesielski and Schonefeld used the Franklin dyadic functions elements for the basis. But this system is not easily generalize to show the existence of a basis for $C^p[0,1]^2$ when $p \geq 2$.

There were many works also for the space $C(K)$ for K is different from $[0,1]$ after 1960's. In 1966 V.I.Gurariĭ [12] showed that for any dense sequence (x_n) in a metric compact set K there exists a basis in the space $C(K)$ which is interpolating with nodes (x_n). V.D.Milman [14, p119] illustrated this approach by constructing a basis from broken lines in the space of continuous functions defined on Cantor sets. In 1985 Bochkarev [2] constructed a dyadic interpolating basis (P_n) for $C[0,1]$ with moderate growth of degrees of the polynomial P_n. Grober and Bychkov showed in [11] that the Schauder system is an interpolating basis in the space Hölder type functions $C_\alpha[0,1]$. Using the method of local interpolations Goncharov [10] constructed interpolating basis in the space of continuous functions defined on compact sets of the Cantor type. I use Gonchrov's paper in my thesis many times.

After the construction of basis in $C^p[0,1]^2$ Ciesielski and Schonefeld improved this result. In 1972 Schonefeld [19] constructed a Schauder basis for the space $C^p(T^q)$ where T^q is the product of q copies of the one-dimensional torus. This basis is also a basis for $C^1(T^q)$, $C^2(T^q), ..., C^{p-1}(T^q)$ and an interpolating basis for $C(T^q)$. Schonefeld firstly proved that $C^p(T)$ has a basis.

Let the partition Δ_n be the set of points $\left\{0, \frac{1}{N}, \frac{2}{N}, ..., \frac{N-1}{N}\right\}$ and $(2p+1)$ periodic spline on Δ_n where $p = 1, 2, ...$ be an element of $C^{2p}(T)$ whose restriction to each interval $\Big(i/N,\ (i{+}1)/N\Big)$, $i = 0, 1, ..., N{-}1$ is a $(2p{+}1)$-degree polynomial. Then Schonefeld constructed the basis:
$f_1 \equiv 1$, f_{N+q} is the $(2p+1)$-periodic spline on the partition Δ_{2N} which is zero at every point of the partition Δ_{2N} except $(2q-1)/2N$ where $N = 1, 2, 4, 8, ...$ and $q = 1, 2, ..., N$ and $f_{N+q}((2q-1)/2N) = 1$.

Then he defined an operator S_n inductively by the following:

$$a_1 = f(r_1) \quad S_n f = \sum_{i=1}^{n} a_i f_i \qquad a_{n+1} = f(r_{n+1}) - S_n f(r_{n+1}) \qquad n = 1, 2, ...$$

where $\{r_n;\, n = 1, 2, ...\} = \left\{0,\, \frac{1}{2},\, \frac{1}{4},\, \frac{3}{4}, ...,\, \frac{2^{m-1}-1}{2^{m-1}},\, \frac{1}{2^m},\, \frac{3}{2^m},\, \frac{5}{2^m}, ...,\, \frac{2^m-1}{2^m}\right\}$. So $S_n f$ interpolates f on Δ_N that is $S_N f(i/N) = f(i/N)$. Then he showed that $\{f_n\}$ is an interpolating basis for $C(T)$ that is $\|f - S_N f\| \leq \varepsilon$. Then he differentiated $S_n(f) - f$ and by using the properties of divided differences he proved that $C^p(T)$ has a basis. At the end of this paper he remarked that the spaces $C^p(T^q)$, $C^p(I^q)$, $C^p(M)$ (where M is q-dimensional compact C^p-manifold) and $C^p(D)$ (where D is a domain in $\mathbb{R}^q$ with boundary such that there exists a linear extension operator $L:\ C^p(\partial D) \to C^p(D)$) are isomorphic. So with these isomorphisms there exists Schauder basis also in these spaces. Also Ciesielski and Domsta [4] showed the existence of basis for $C^p(I^q)$ in 1972. Furthermore in 1978 Ryll [16] defined a system $(\varphi_n^q)_{n\in\mathbb{N}}$ such that (φ_n^q) is an interpolating basis in $C(I^d)$, (φ_n^q) is a basis in each space $C^{(k)}(I^d)$ for $k = 1, ..., q$ and

$$diam(\sup \varphi_n^q) \to 0 \quad as \quad n \to \infty.$$

In 2004, Jonsson [13] used the method of triangulations to construct basis for the space $C^p(F)$ where F is a compact subset of $\mathbb{R}^n$ preserving Markov inequality. (We give the details of this paper in section 2.3)

In this thesis in section 2.1 and 2.2 we give some basic definitions and properties about spaces of differentiable functions and interpolation. In section 2.3 we define interpolating basis and recall some examples of interpolating basis. Also we give some details from Jonsson's paper. In section 2.4 we give the method of local interpolation. Also we give some lemmas which we use in Chapter 3. In last chapter we found some bounds for divided differences. Then by using local interpolations and this bounds we construct a basis for the space $C^p(K)$ where K is uniformly perfect Cantor-type set.

Chapter 2

Preliminary

2.1 Spaces of Differentiable Functions

Let K be an open set of $\mathbb{R}^n$. We denote $C^p(K)$ (respectively $C(K)$) the algebra of p times continuously differentiable functions in K, with the topology of uniform convergence of functions and all their partial derivatives on compact subsets of K. This is topology defined by the norm

$$|f|_p = sup|f^{(p)}(x)| : x \in K$$

We can define $f'(x)$ in the following:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This is true only for $x + h \in K$ where K is uniformly perfect. If x is an isolated point, $f'(x)$ does not depend on $f|_K$. We use jets to define derivatives on sets not uniformly perfect but we do not use jets in our work.

Let $\mathcal{E}^p(K)$ we denote the space of continuous functions f in K such that f is C^p in K and all derivatives of $f|_K$ extend continuously to K.

Theorem 2.1.1. *(Tietze Extension Theorem) If X is a normal topological space and $f : K \to \mathbb{R}$ is a continuous map from a closed subset of K of X into real* *numbers carrying the standard topology, then there exists a continuous map*

$$\tilde{f} : X \to \mathbb{R}$$

with $f(x) = \tilde{f}(x)$ *for all* x *in* K. $\tilde{f}$ *is called a* continuous extension *of* f.

For the space of continuous functions $C(K) = \mathcal{E}(K)$. But we can not say this for $p \geq 1$.

Let we denote by D^k the mapping of $\mathcal{E}^p(K)$ into $\mathcal{E}^{p-|k|}(K)$ where $|k| \leq p$ given by

$$D^k f = \frac{\partial^{|k|} f}{\partial x^k}.$$

Theorem 2.1.2. *(Whitney Theorem) Let* K *be an open subset of* $\mathbb{R}^n$ *and* X *be a closed subset of* K. *There is a continuous linear mapping*

$$W : \mathcal{E}^p(X) \to \mathcal{E}^p(K)$$

such that $D^k W(F)(x) = F^{(k)}$ *if* $F \in \mathcal{E}^p(X)$, $x \in X$, $|k| \leq m$

Then we can say that $\mathcal{E}^p(K) \subset C^p(K)$.

2.2 Interpolation

Given the values of a function $f(x)$ at two distinct values of x, say x_0 and x_1, we could approximate f by linear functions p that satisfies the conditions

$$p(x_0) = f(x_0) \quad \text{and} \quad p(x_1) = f(x_1).$$

It is geometrically obvious that such a p exists and unique. We call p a linear interpolating polynomial and this process *linear interpolation.*

We can construct the linear interpolating polynomial directly by using the above two conditions. Then we obtain

$$p(x) = \frac{x_1 f(x_0) - x_0 f(x_1)}{x_1 - x_0} + x\Big(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\Big).$$

This can also be expressed in Newton's divided difference form

$$p(x) = f(x_0) + (x - x_0)\Big(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\Big).$$

Let we denote the set of all polynomials of degree at most n by P_n. Let f be a function defined on a set of distinct points $x_0, x_1, ..., x_n$. Then there exists a unique polynomial $p_n \in P_n$ such that $p(x_j) = f(x_j)$ for $j = 0, 1, ..., n$. This polynomial is called the *interpolating polynomial.* Since

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad p(x_j) = f(x_j).$$

Then

$$f(x_j) = a_0 + a_1 x_j + \cdots + a_n x_j^n.$$

Isaac Newton (1642-1727) found another way to constructing the interpolating polynomial. Instead of using monomials $1, x, ..., x^n$, he used the polynomials $\pi_0, \pi_1, ..., \pi_n$ where

$$\pi_i(x) = \begin{cases} 1, & i = 0 \\ (x - x_0)(x - x_1)\cdots(x - x_{i-1}), & 1 \leq i \leq n. \end{cases}$$

The interpolating polynomial $p_n \in P_n$, which assumes the same values as the function f at $x_0, x_1, ..., x_n$, is then written in the form

$$p_n(x) = a_0\pi_0(x) + a_1\pi_1(x) + \cdots + a_n\pi_n(x).$$

We may determine the coefficients a_j by setting

$$p_n(x_j) = f(x_j), \quad 0 \leq j \leq n$$

giving the system of linear equations

$$a_0\pi_0(x_j) + a_1\pi_1(x_j) + \cdots + a_n\pi_n(x_j) = f(x_j), \quad for \;\; 0 \leq j \leq n.$$

Then we obtain

$$a_0 = f(x_0) \qquad \text{and} \qquad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

We will write

$$a_j = [x_0, x_1, ..., x_j]f$$

and we say a_j j-th *divided difference*. Thus we may write $p_n(x)$ in the form

$$p_n(x) = [x_0]f\pi_0(x) + [x_0, x_1]f\pi_1(x) + \cdots + [x_0, x_1, ..., x_n]f\pi_n(x),$$

which is Newton's *divided difference formula* for the interpolating polynomial.

Now we give some properties of the divided differences.

Proposition 2.2.1. *The divided difference $[x_0, x_1, ..., x_n]f$ can be expressed as the following symmetric sum of multiples of $f(x_j)$,*

$$[x_0, x_1, ..., x_n]f = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{j\neq r}(x_r - x_j)},$$

where in the above product of n factors, r remains fixed and j takes all values from 0 to n, excluding r.

Proposition 2.2.2. *Let x and the abscissas $x_0, x_1, ..., x_n$ be contained in an interval $[a, b]$ on which f and its first n derivatives are continuous, and let $f^{(n+1)}$ exists in the open interval (a, b). Then there exists $\xi_x \in (a, b)$, which depends on x, such that*

$$f(x) - p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n)\frac{f^{(n+1)}(\xi_x)}{(n+1)!}.$$

Corollary 2.2.3. *Let $f \in C^n[a, b]$ and let $x_i : i = 0, ..., n$ be a set of distinct points in $[a, b]$. Then there exists a point θ, in the smallest interval that contains the points $x_i : i = 0, ..., n$, at which the equation*

$$[x_0, x_1, ..., x_n]f = \frac{f^{(n)}(\theta)}{n!}$$

is satisfied.

Proposition 2.2.4. *Let f defined on a set of distinct points $x_0, x_1, ..., x_n$. Then we express the divided difference $[x_j, x_{j+1}, ..., x_{j+k+1}]f$ where $j, k \in \mathbb{N}$ such a way that*

$$[x_j, ..., x_{j+k+1}]f = \frac{[x_{j+1}, ..., x_{j+k+1}]f - [x_j, ..., x_{j+k}]f}{x_{j+k+1} - x_j} \quad \textit{where} \quad j + k + 1 \leq n.$$

For the proof of these propositions see e.g. [15].

2.3 Interpolating Basis

Definition 2.3.1. A basis (f_n) in an infinite dimensional Banach space E is called *interpolating basis* with nodes (x_n) if and only if for each f in E and each n we have

$$S_m(x_m) \ = \ f(x_m) \qquad for \quad m = 1, 2, ..., n$$

where $S_n = \sum_{k=1}^{n} \xi(f) f_k = S_n(f)$. Thus, the n-th partial sum S_n interpolates f at n points $x_1, ..., x_n$. Here by (ξ) we denote the functional biorthogonal to (f_n)

There are many examples of interpolating bases. The basis of unit vectors $e_1, e_2, ...$ in c_0 is interpolating. Also Faber-Schauder system is interpolating. (We showed this in chapter 1). Furthermore Gurariĭ [12], Bochkarev [2], Grober and Bychkov [11] used interpolating basis in their construction. (For more information about interpolating basis see [20])

Jonsson also used interpolating bases. In 2004, Jonsson [13] used triangulations to construct basis for the space $C^p(F)$ where F is a compact subset of $\mathbb{R}^n$ preserving Markov inequality. Now we give some information about this paper.

Definition 2.3.2. Let F be a compact subset of $\mathbb{R}^n$. A finite set T of n-dimensional closed, non-degenerated, simplices is called a triangulation of F if the following conditions hold.

A1. For each pair $\Delta_1, \Delta_2 \in T$, the intersection $\Delta_1 \cap \Delta_2$ is empty or a common face of lower dimension.

A2. Every vertex of a simplex $\Delta \in T$ is in F.

A3. $F \subset \cup_{\Delta \in T} \Delta$.

For a triangulation T, let $\delta = \max_{\Delta \in T} diam(\Delta)$ be the diameter of the triangulation. When considering a sequence $\{T_i\}_{i=0}^{\infty} \ = \ \{T_i\}$ of triangulations, we denote by δ_i the diameter of triangulation. In the sequel, we deal with sequences $\{T_i\}$ of triangulations satisfying the following conditions:

B1. For each $i \geq 0$, T_{i+1} is a refinement of T_i, i.e., for each $\Delta \in T_{i+1}$ there is $\tilde{\Delta} \in T_i$ such that $\Delta \subset \tilde{\Delta}$.

B2. $\delta_i \rightarrow 0, i \rightarrow \infty$.

It is easy to see that if a triangulation satisfies B1 and B2, then the following condition holds. We denote U_i the set of vertices of T_i.
B3. for $i \geq 0$, $U_i \subset U_{i+1}$

Definition 2.3.3. Let $F \in \mathbb{R}$, and let $\{T_i\}$ be a sequence of triangulations satisfying B1. Then $\{T_i\}$ is a *regular sequence of triangulations* if the following conditions hold.
T1. There is a constant $c_1 > 0$, independent of i, such that, for all Δ_1, $\Delta_2 \in T_i$,

$$c_1^{-1} diam(\Delta_2) \leq diam(\Delta_1) \leq c_1 diam(\Delta_2).$$

T2. There are constants $0 < c_2 < c_3 < 1$ such that, for all $i \leq 0$,

$$c_2 \delta_i \leq \delta_{i+1} \leq c_3 \delta_i.$$

T3. There exists a constant $a > 0$, independent of i, such that if $\Delta \in T_i$ and $\Delta' \in T_i$ and the distance between these intervals is less than or equal to $a\delta_i$, then the intervals intersect.

Definition 2.3.4. Let F be in (X, d). F *uniformly perfect* if there exists C, δ such that for every $y \in F$ there exists $(x_k) \subset F$ such that $dist(x_k, y) \to 0$, $dist(y, x_1) > \delta$, and

$$dist(y, x_k) \leq C dist(y, x_{k+1}) \qquad \forall k.$$

For Cantor type sets the condition T1 satisfies. For condition T2 we can take $\delta_i = l_i$ where $(l_i)_{i=0}^{\infty}$ is a sequence in the Cantor set such that $l_0 = 1$ and $0 < 2l_{s+1} < l_s$ for $s \in \mathbb{R}$. (For details of cantor set see section 2.4) Then $c_2 l_i \leq l_{i+1} \leq c_3 l_i$, i.e., our compact set K is uniformly perfect. Also the condition T3 is satisfies for Cantor type sets. Then Cantor type sets with regular triangulations are uniformly perfect.

Definition 2.3.5. Denote by P_m the set of all polynomials in n variables of total degree less than or equal to m. A closed set $F \subset \mathbb{R}^n$ *preserves Markov's inequality* if for every fixed positive integer m there exists a constant c, such that for all polynomials $P \in P_m$ and all closed balls $B = B(x_0, r)$, $x_0 \in F$, $0 < r < 1$, holds

$$\max_{F \cap B} |\nabla P| \leq \frac{c}{r} \max_{F \cap B} |P|$$

where ∇ denotes the gradient.

This definition of Markov's inequality is the definition in Jonsson's paper but in general the condition in the definition is:

$$\sup_{x\in F} |\nabla P_m(x)| \le Cm^R \sup_{x\in F} |P_m(x)|$$

where C and R are positive constants. Examples of sets preserving Markov inequality are e.g. selfsimilar fractals not contained in an n - 1-dimensional subspace of $\mathbb{R}^n$, and d-sets.

Theorem 2.3.6 (Theorem 3.1, [13]). *Let F be a compact subset of $\mathbb{R}$. Then there exists a regular triangulation of F if and only if F preserves Markov's inequality.*

Then Jonsson constructed basis for the space $C^p(F)$ where F is a compact subset of $\mathbb{R}$ preserving Markov's inequality. Now we define the functions that gave a basis in $C^p(F)$.

Let F be a compact subset of $\mathbb{R}$ preserving Markov's inequality and f be a function defined on F. Then, since F is perfect derivatives can be defined in the usual way by $Df(x_0) = \lim_{x\to x_0, x\in F}(f(x)-f(x_0))/(x-x_0)$, $x_0 \in F$. For a given $p \ge 0$, assuming that derivatives of order $\le p$ exist, denote by R_j the Taylor remainders given by

$$D^j f(x) = \sum_{l=0}^{p-j} (x-y)^l D^{j+l} f(y)/l! + R_j(x,y), \quad 0 \le j \le p.$$

A function f belongs to the space $C^p(F)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|R_j(x,y)| < \varepsilon |x-y|^{p-j}$ for $0 \le j \le p$ and $|x-y| < \delta$. The Whitney extension theorem given in Section 2.1 gives that there is a linear extension operator E from $C^p(F)$ to $C^p(\mathbb{R})$. It should be noted that in the present setting derivatives are uniquely determined by f, which means that elements in $C^p(F)$ are functions rather than families of functions as in the general Whitney extension theorem.

Let a sequence $\{T_i\}_{i=0}^{\infty}$ of regular triangulations of F be given, and let U_i be the set of vertices of T_i. Let $V_i = U_i \setminus U_{i-1}$, $i > 0$, $V_0 = U_0$, and $V = \bigcup_{i=0}^{\infty} V_i$

and let $\preceq$ be a linear order on V satisfying the following condition: if $\xi \in V_i$ and $\eta \in V_j$ with $i < j$, then $\xi \preceq \eta$.

Let $p \geq 0$, and let, for $s = 0, 1, ..., p$, P^s be the polynomial of degree $2p+1$ which satisfies $D^\nu P^s(0) = 0$ for $\nu = 0, 1, ..., p$, $D^\nu P^s(1) = 0$ for $\nu = 0, 1, ..., p$, $v \neq s$, and $D^\nu P^s(1) = 1\, if\, \nu = s$. Let $\xi \in V_i$. Then ξ is the right endpoint of one or two intervals in T_i. If ξ is the right endpoint of $\Delta \in T_i$ of length l, define $\phi^s_{i,\varepsilon}$ on Δ by $\phi^s_{i,\xi}(x) = l^s P^s((x-\xi+l)/l)$, and if ξ is the left endpoint of $\Delta' \in T_i$ of length l', define $\phi^s_{i,\xi}$ on Δ' by $\phi^s_{i,\xi}(x) = l'^s(-1)^s P^s((x-\xi+l')/l')$. If ξ is an endpoint of both Δ and Δ', this means that $\phi^s_{i,\xi}$ is a spline function defined on $\Delta \cup \Delta'$ which is p times differentiable, equals a polynomial of degree at most $2p+1$ on each of the intervals Δ and Δ', and whose derivatives of orders less than or equal to p are equal to zero at the endpoints of these intervals, except that $D^s \phi^s_{i,\xi} = 1$. Note that for $x \in \Delta$ we have $D^\nu \phi^s_{i,\xi}(x) = l^s (D^\nu P^s)((x-\xi+l)/l)$ and consequently for $x \in \Delta$ we have, with a constant c depending on ν,

$$|D^\nu \phi^s_{i,\xi}(x)| \leq c l^{s-\nu}, \qquad \nu \leq 0,$$

an estimate which clearly also holds if $x \in \Delta$, with l replaced by l'.

For $\xi \in V_i$, let $\psi^s_\xi = \phi^s_{i,\xi}$ on the one or two intervals in T_i which have ξ as an endpoint, and put $\psi^s_\xi = 0$ elsewhere on $\{\cup\Delta;\, \Delta \in T_i\}$. Note that ψ^s_ξ is defined which respect to the triangulations T_i in which ξ first appears as a vertex. We let $C^p(F)$ be normed by $\max_{0 \leq \nu \leq p} \|D^\nu f\|_{\infty,F}$, although $C^p(F)$ is, in general, not complete in this norm. Then the functions $\psi^s_\xi|_F$ form a basis in $C^p(F)$ which interpolates to f and its derivatives of degree at most p. Here $f|_F$ denotes the restriction of f to F.

Given $f \in C^p(F)$, let $S_i(F)$ denote the spline function defined on $\{\cup\Delta;\, \Delta \in T_i\}$ which coincides with a polynomial of degree at most $2p+1$ an each interval in T_i, and interpolates to f and all its derivatives of orders less than or equal to p at each point in U_i.

Proposition 2.3.7 (Prop 5.2, [13]). *Let $F \subset \mathbb{R}$ be a compact set with a regular sequence $\{\mathrm{T}_i\}_{i=0}^\infty$ of triangulations of F, and let p be a nonnegative integer. Then* *the system of functions* $\{\psi^s_\xi|_F,\ \xi \in \mathrm{V},\ s = 0, 1, ..., p\}$*, ordered by* $\preceq$*, is an interpolating Schauder basis in* $C^p(F)$*.*

More precisely, every $f \in C^p(F)$ *has a unique representation*

$$f = \sum_{i=0}^{n} \sum_{\xi\in \mathrm{V}_i} \sum_{s=0}^{p} c^s_\xi \psi^s_\xi|_F$$

in $C^p(F)$*, where* $c^s_\xi = D^s f(\xi) - D^s S_{i-1} f(\xi)$ *for* $\xi \in \mathrm{V}_i$*,* $i > 0$*, and* $c^s_\xi = D^s f(\xi)$ *for* $\xi \in \mathrm{V}_0 = \mathrm{U}_0$*. In addition, for* $N \leq 0$*,*

$$D^\nu f(\eta) = \sum_{i=0}^{n} \sum_{\xi\in \mathrm{V}_i} \sum_{s=0}^{p} c^s_\xi D^\nu \psi^s_\xi|_F$$

for $0 \leq \nu \leq p$ *and* $\eta \in \mathrm{U}_N$*.*

In the construction of Jonsson F preserves Markov's inequality. Then from theorem 2.3.6 there is a regular triangulation of F, so F is uniformly perfect. In our work we use another method to show that $C^p(F)$ has an interpolating basis where F is a uniformly perfect Cantor-type set.

2.4 Local Interpolation

Suppose K_0 and K_1 are infinite compact sets such that $K_1 \subset K_0$ and $K_0 \setminus K_1$ is closed. Let natural numbers N_0, M_1, N_1 be given with $N_0 \geq 2$, $M_1 \leq N_0, M_1 \leq N_1, N_0 - M_1 \leq N_1$. Let for $s \in \{0, 1\}$ we have a finite system of points $(x^{(s)}_k)^{N_s}_{k=1} \subset K_s$. Here we suppose that $x^{(s)}_k \neq x^{(s)}_l$ for $k \neq l$ and $(x^{(0)}_k)^{N_0-M_1}_{k=1} \subset K_0 \setminus K_1$, $x^{(0)}_{N_0-M_1+r} = x^{(1)}_r$ for $r = 1, \cdots, M_1$. We adopt the conventions that $\sum^n_{k=m}(\cdots) = 0$ and $\prod^n_{k=m}(\cdots) = 1$ for $m > n$. For $s \in \{0, 1\}$, $0 \leq n \leq N_s$ set $\tilde{e}_{ns}(x) = \prod^n_{k=1}(x - x^{(s)}_k)$, and let e_{ns} be the restriction of $\tilde{e}_{ns}$ to K_s, otherwise $e_{ns}(x) = 0$. Also for any function f defined on K_s let $\xi_{ns}(f) = [x^{(s)}_1, x^{(s)}_2, \cdots, x^{(s)}_{n+1}]f$, where $x^{(0)}_{N_0+1} := x^{(1)}_{M_1+1}$ and $x^{(1)}_{N_1+1} \in K_1$ is any point differing from $x^{(1)}_k$, $k = 1, \cdots, N_1$. By the properties of divided differences see (Propositions 2.2.1-2.2.4). We have,

Lemma 2.4.1. *If a sequence* $(x_k^{(s)})_{k=1}^{N_s}$ *of distinct points is dense on a perfect set* $K_s \subset \mathbb{R}$, *then the system* $(e_{ns}, \xi_{ns})_{n=0}^{N_s}$ *is biorthogonal and the sequence of functionals* $(\xi_{ns})_{n=0}^{\infty}$ *is total on* $X(K_s)$, *that is, whenever* $\xi_{ns}(f) = 0$ *for all* n, *it follows that* $f = 0$.

Lemma 2.4.2 (Lemma 2.2, [9]). *Let* $(x_k^{(s)})_{k=1}^{\infty}$, $s = 1, 2, 3$, *be three sequences such that for a fixed superscript* s *all points in the sequence* $(x_k^{(s)})_{k=1}^{\infty}$ *are different. Let* $e_{ns} = \prod_{k=1}^{n}(x - x_k^{(s)})$ *and* $\xi_{ns}(f) = [x_1^{(s)}, x_2^{(s)}, \cdots, x_{n+1}^{(s)}]f$, *for* $n \in \mathbb{N}_0$. *Then*

$$\sum_{p=q}^{r} \xi_{p3}(e_{q2})\xi_{q2}(e_{r1}) = \xi_{p3}(e_{r1}) \qquad \mathit{for} \quad p \leq r.$$

By means of Lemma 2.4.2 we can construct new biorthogonal systems corresponding the local interpolation of functions. For $n = M_1 + 1, ..., N_1$, $e_{n1} = \prod_{k=1}^{n}(x - x_k^{(1)})$ for $x \in K_1$ and $e_{n1} = 0$ for $x \in K_0 \setminus K_1$. Since $K_0 \setminus K_1$ is closed e_{n1} is continuous on K_0. Then $\xi_{n0}(e_{m,1}) = 0$, because the number $\xi_{n0}(f)$ is defined by values of f at some points on $K_0 \setminus K_1$ and at some points from $(x_k^{(1)})_{k=1}^{M_1}$, where the function $e_{m,1}$ is zero. Clearly, $\xi_{n,1}(e_{m0}) = 0$ for $n \geq m$. But for $n \leq m$, the functional $\xi_{n,1}$, in general, is not biorthogonal to e_{m0}. For this reason we take the functional

$$\eta_{n,1} = \xi_{n,1} - \sum_{k=n}^{N_0} \xi_{n,1}(e_{k0})\,\xi_{k0}$$

which is biorthogonal to e_{m0}.

Now we show that we can interpolate functions locally. Given f on K_0 let us denote by $Q_n(f, (x_k)_{k=1}^{n+1}, \cdot)$ (also by $Q_n(\cdot)$) the Newton interpolation polynomial of degree n for f with nodes at $x_1, \cdots, x_{n+1}$.

Let us consider the function $S_n(f, x) = Q_n(f, (x_k^{(0)})_{k=1}^{n+1}, x)$ for $n = 0, \cdots, N_0$ and

$$S_{N_0+r}(f, x) = Q_{N_0}(f, (x_k^{(0)})_{k=1}^{N_0+1}, x) + \sum_{k=M_1+1}^{M_1+r} \eta_{k,1}(f)e_{k,1}(x) \tag{2.4.1}$$

for $r = 1, \cdots, N_1 - M_1$. We see at once that $S_{N_0+r} \in \Pi_{N_0}(K_0 \setminus K_1)$ and $S_{N_0+r} \in \Pi_{\max\{N_0, M_1+r\}}(K_1)$.

Lemma 2.4.3 (Lemma 2.1, [10]). *Given function f defined on K_0 and $n = 0, 1, \cdots, N_0 + N_1 - M_1$, the function $S_n(f, \cdot)$ interpolates f at the first $n+1$ points from the set $\{x_1^{(0)}, \cdots, x_{N_0}^{(0)}, x_{M_1+1}^{(1)}, \cdots, x_{N_1+1}^{(1)}\}$.*

The point of the lemma is that it allows one to interpolate functions locally. Suppose we have a chain of compact sets $K_0 \supset K_1 \supset \cdots \supset K_s \supset \cdots$ and finite systems of distinct points $(x_k^{(s)})_{k=1}^{N_s} \subset K_s$ for $s = 0, 1, \cdots$. Some part of the knots on K_{s+1} – let $(x_k^{(s+1)})_{k=1}^{M_{s+1}}$ – belongs to the previous set $(x_k^{(s)})_{k=1}^{N_s}$. We will interpolate a given function f on K_s up to the degree N_s and then restrict the interpolation to the set K_{s+1}, where the degree of interpolation will be N_{s+1}, etc. As the diameters of K_s decrease, the approximation properties of the interpolating polynomials will improve. Since the points of interpolation are chosen independently on functions, the approach allows us to construct topological bases in spaces of functions defined on rarefied sets.

In our work all our subsequent considerations are related to Cantor-type sets. Let $\Lambda = (l_s)_{s=0}^{\infty}$ be a sequence such that $l_0 = 1$ and $0 < 2l_{s+1} \leq l_s$ for $s \in \mathbb{N}_0$. Let $K(\Lambda)$ be the Cantor set associated with the sequence Λ that is $K(\Lambda) = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0, 1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,s}$ of length l_s and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,s}$, $j = 1, 2, ...2^s$.

Given a nondecreasing sequence of natural numbers $(n_s)_0^{\infty}$, let $N_s = 2^{n_s}$, $M_s^{(l)} = N_{s-1}/2 + 1$, $M_s^{(r)} = N_{s-1}/2$ for $s \geq 1$ and $M_0 = 1$. Here, (l) and (r) mean *left* and *right* respectively. For any basic interval $I_{j,s} = [a_{j,s}, b_{j,s}]$ we choose the sequence of points $(x_{n,j,s})_{n=1}^{\infty}$ using the rule of increase of the type. we take $e_{N,1,0} = \Pi_{n=1}^{N}(x - x_{n,1,0}) = \Pi_1^N(x - x_n)$ for $x \in K(\Lambda)$, $N = 0, 1, \cdots, N_0$. For $s \geq 1$, $j \leq 2^s$ let $e_{N,j,s} = \Pi_{n=1}^{N}(x - x_{n,j,s})$ if $x \in K(\Lambda) \cap I_{j,s}$, and $e_{N,j,s} = 0$ on $K(\Lambda)$ otherwise. Here, $N = M_s^{(a)}, M_s^{(a)} + 1, \cdots, N_s$ with $a = l$ for odd j and $a = r$ if j is even. The functionals are given as follows: for $s = 0, 1, \cdots$; $j = 1, 2, \cdots, 2^s$ and $N = 0, 1, \cdots$, let $\xi_{N,j,s}(f) = [x_{1,j,s}, \cdots, x_{N+1,j,s}]f$. Set $\eta_{N,1,0} = \xi_{N,1,0}$ for $N \leq N_0$. Every basic interval $I_{j,s}$, $s \geq 1$, is a subinterval of a certain $I_{i,s-1}$ with

$j = 2i - 1$ or $j = 2i$. Let

$$\eta_{N,j,s}(f) = \xi_{N,j,s}(f) - \sum_{k=N}^{N_{s-1}} \xi_{N,j,s}(e_{k,i,s-1})\, \xi_{k,i,s-1}(f)$$

for $N = M_s^{(a)}, M_s^{(a)}+1, \cdots, N_s$. As before, $a = l$ if $j = 2i-1$, and $a = r$ if $j = 2i$.

Now we give an example to local interpolation for $N_0 = N_1 = N_2 = 4$ and $f \in C[0,1]$. Then our points are $x_1 = 0$, $x_2 = 1$, $x_3 = l_1$, $and\, x_4 = 1 - l_1$. Then the interpolating polynomial

$$\begin{aligned} Q_4 \quad &= f(x_1) + [x_1, x_2]f(x - x_1) + \cdots + [x_1, x_2, ..., x_5]f(x - x_1)\cdots(x - x_4) \\ &= f(0) + (f(1) - f(0))x + \cdots + [0, 1, l_1, 1 - l_1, l_2]fx(x-1)(x-l_1)(x-1+l_1) \end{aligned}$$

As seen in the equation we add one more point $x_5 = l_2$. For $s = 0$,

$$\begin{aligned} e_{0,1,0} &= 1, \\ e_{1,1,0} &= x, \\ e_{2,1,0} &= x(x-1), \\ e_{3,1,0} &= x(x-1)(x-l_1), \\ e_{4,1,0} &= x(x-1)(x-l_1)(x-1+l_1) \end{aligned}$$

and

$$\begin{aligned} \xi_{0,1,0} &= f(0), \\ \xi_{1,1,0} &= [0,1]f, \\ \xi_{2,1,0} &= [0,1,l_1]f, \\ \xi_{3,1,0} &= [0,1,l_1,1-l_1]f, \\ \xi_{4,1,0} &= [0,1,l_1,1-l_1,l_2]. \end{aligned}$$

Now we look this for the intervals $I_{1,1}$, $I_{2,1} \in I_{1,0} = [0,1]$. $I_{1,1}$ is the left part and $I_{2,1}$ is the right part. Then on $I_{1,1}$

$$\begin{aligned} e_{1,1,1} &= x, \\ e_{2,1,1} &= x(x-l_1) \end{aligned}$$

and on $I_{1,2}$

$$\begin{aligned} e_{1,2,1} &= x - 1, \\ e_{2,2,1} &= (x-1)(x-1+l_1). \end{aligned}$$

Also

$$\begin{aligned} \xi_{0,1,1} &= \ f(0), \\ \xi_{1,1,1} &= [0, l_1]f, \\ \xi_{2,1,1} &= [0, l_1, l_2]f \end{aligned}$$

and

$$\begin{aligned} \xi_{0,2,1} &= \ f(1-l_1), \\ \xi_{1,2,1} &= [1-l_1, 1]f, \\ \xi_{2,2,1} &= [1-l_1, 1, 1-l_1+l_2]f. \end{aligned}$$

So we add a new point $x_6 = 1 - l_1 + l_2$ to $I_{2,1}$. In this way we add new points to left and right intervals and interpolate f. Let us look this. We know that $S_5(f, x) = Q_4$ from lemma 2.4.3. Then

$$\begin{aligned} S_6 &= Q_4 + \eta_{2,2,1}(f)e_{2,2,1} \\ S_7 &= Q_4 + \eta_{2,2,1}(f)e_{2,2,1} + \eta_{3,1,1}(f)e_{3,1,1} \\ S_8 &= Q_4 + \eta_{2,2,1}(f)e_{2,2,1} + \eta_{3,2,1}(f)e_{3,2,1} \\ S_{10} &= Q_4 + \eta_{2,2,1}(f)e_{2,2,1} + \eta_{3,2,1}(f)e_{3,2,1} + \eta_{4,2,1}(f)e_{4,2,1} \end{aligned}$$

Here

$$\begin{aligned} \eta_{2,2,1}(f) &= \xi_{2,2,1}(f) - \sum_{k=2}^{4} \xi_{2,2,1}(e_{k,1,0})\xi_{k,1,0}(f) \\ \eta_{3,1,1}(f) &= \xi_{3,1,1}(f) - \sum_{k=3}^{4} \xi_{3,1,1}(e_{k,1,0})\xi_{k,1,0}(f). \end{aligned}$$

Then

$$\|f(x) - S_7(f,x)\| \le \eta_{2,2,1}(f)e_{2,2,1} + \eta_{3,1,1}(f)e_{3,1,1},$$

since Q_4 is the interpolation polynomial of $f(x)$. In chapter 3 we find bounds for $\eta(f)\, and\, \xi(f)$ and we show that $\|f(x) - S_n(f,x)\|_p \le \varepsilon$.

Finally I give one more definition which we use in thesis.

Definition 2.4.4. A basis $\{f_n\}$ of a Banach space E is said to be *unconditional* if every convergent series of the form $\sum_{i=1}^{\infty} \alpha_i\, f_i$ is unconditionally convergent, i.e., for every $f \in E$ the series $\sum_{i=1}^{\infty} f_i(x)\, x_i$ converges unconditionally. The basis $\{f_n\}$ is said to be *conditional* if it is non-unconditional, i.e., if there exists a convergent series $\sum_{i=1}^{\infty} \alpha_i\, f_i$ which is not unconditionally convergent.

Chapter 3

Bases in the spaces of C^p-functions

3.1 Estimations

Given function $f \in C(K)$ on a compact set $K \subset \mathbb{R}$, let $\omega(f, \cdot)$ be the modulus of continuity of f, that is $\omega(f,t) = \sup\{\,|\,f(x) - f(y)\,| : x, y \in K,\ |\,x - y| \le t\},\ t > 0.$

Lemma 3.1.1. *([10])For $N \ge 1$ and distinct points $x_1, \cdots, x_{N+1} \in K$ with $x_1 < x_2 < \cdots < x_{N+1}$ let $e_{N+1}(x) = \prod_{k=1}^{N+1}(x - x_k),\ \xi_N(f) = [x_1, x_2, \cdots, x_{N+1}]f$ and $t = \max_{k \le N} |\,x_{k+1} - x_k|$. Then*

$$|\,\xi_N(f)\,| \le N^2\ \omega(f,t)\ (\min_{k \le N} |\,e'_{N+1}(x_k)|\,)^{-1}.$$

Given function $f \in C^q(K)$ where $q = 1, 2, ..., p$ on a compact set $K \subset \mathbb{R}$, let $\tilde{f}$ be the extension of f on K. Let $\omega(f^{(q)}, .)$ be the modulus of continuity of $f^{(q)}$, that is $\omega(f^{(q)}, t) = \sup\{|f^{(q)}(x) - f^{(q)}(y)| : x, y \in K, |x - y| \le t\}$, $t > 0$.

Lemma 3.1.2. *For $N \ge 1$ and distinct points $x_1, ..., x_{N+1} \in K$ with $x_1 < x_2 < ... < x_{N+1}$ let $t = \max_{k \le N} |x_{k+1} - x_k|$ and $\xi_N(f) = [x_1, x_2, ..., x_{N+1}]f$. Then*

$$|\xi_N(f)| \le \frac{N^2\,\omega(\tilde{f}^{(q)}, t)}{q!} \Big(\min_{k \le N-q, j \le N} \Big| \prod_{s=1}^{k}{}_{s \ne j} (x_j - x_s) \prod_{s=k+q+1}^{N+1}{}_{s \ne j} (x_j - x_s) \Big| \Big)^{-1}.$$

Proof. First we show this for fix q, then we show this for $q = 1, 2, ..., p$. Let $e'_{N+1}(x_k) = \prod_{j=1, j\neq k}^{N+1}(x_k - x_j)$. From Proposition 2.2.1 we know that $\xi_N(f) = \sum_{k=1}^{N+1} \frac{f(x_k)}{e'_{N+1}(x_k)}$. We write $\xi_N(f)$ in terms of divided differences of q-th order.

$$\xi_N(f) = \sum_{k=1}^{N+1} \frac{f(x_k)}{e'_{N+1}(x_k)} \leq$$

$$\leq (x_1 - x_{q+2})[x_1, ..., x_{q+2}]f \frac{\prod_{j=2}^{q+1}(x_1 - x_j)}{e'_{N+1}(x_1)}$$
$$+(x_2 - x_{q+3})[x_2, ..., x_{q+3}]f \left[\frac{\prod_{j=3}^{q+2}(x_1 - x_j)}{e'_{N+1}(x_1)} + \frac{\prod_{j=3}^{q+2}(x_2 - x_j)}{e'_{N+1}(x_2)}\right]$$
$$+(x_3 - x_{q+4})[x_3, ..., x_{q+4}]f \left[\frac{\prod_{j=4}^{q+3}(x_1 - x_j)}{e'_{N+1}(x_1)} + ... + \frac{\prod_{j=4}^{q+3}(x_3 - x_j)}{e'_{N+1}(x_3)}\right] +$$
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$$+(x_N - x_{N+q+1})[x_N, ..., x_{N+q+1}]f \left[\frac{\prod_{j=N+1}^{N+q}(x_1 - x_j)}{e'_{N+1}(x_1)} + \cdots + \frac{\prod_{j=N+1}^{N+q}(x_N - x_j)}{e'_N(x_{N+q+1})}\right]$$
$$\leq \sum_{k=1}^{N}(x_k - x_{k+q+1})[x_k, ..., x_{k+q+1}]f \sum_{j=1}^{k} \frac{\prod_{s=k+1}^{k+q}(x_j - x_s)}{e'_{N+1}(x_j)}$$
$$\leq \sum_{k=1}^{N} N(x_k - x_{k+q+1})[x_k, x_{k+1}, ..., x_{k+q+1}]f$$
$$\left(\min_{j\leq N}\left|\prod_{s=1, s\neq j}^{k}(x_j - x_s)\prod_{s=k+q+1, s\neq j}^{N+1}(x_j - x_s)\right|\right)^{-1}$$

From Proposition 2.2.4 we can write $[x_k, x_{k+1}, ..., x_{k+q+1}]f$ such a way that

$$[x_k, x_{k+1}, ..., x_{k+q+1}]f = \frac{[x_{k+1}, x_{k+1}, ..., x_{k+q+1}]f - [x_k, x_{k+1}, ..., x_{k+q}]f}{x_{k+q+1} - x_k}$$

Since

$$[x_{k+1}, x_{k+1}, ..., x_{k+q+1}]f = \frac{\tilde{f}^{(q)}(\theta)}{q!}$$

where $\theta \in [k+1, k+q+1]$ (Since θ can be different from x_j we use the extension of f) then

$$(x_{k+q+1} - x_k)[x_k, x_{k+1}, ..., x_{k+q+1}]f = \frac{\tilde{f}^{(q)}(\theta_1) - \tilde{f}^{(q)}(\theta_2)}{q!} \leq \frac{\omega(\tilde{f}^{(q)}, t)}{q!} \qquad (3.1.1)$$

Then

$$\begin{aligned}|\xi_N(f)| &\leq \sum_{k=1}^{N} \frac{N\,\omega(\tilde{f}^{(q)},t)}{q!}\Big(\min_{j\leq N}\Big|\prod_{s=1,s\neq j}^{k}(x_j-x_s)\prod_{s=k+q+1,s\neq j}^{N+1}(x_j-x_s)\Big|\Big)^{-1}\\ &\leq \frac{N^2\,\omega(\tilde{f}^{(q)},t)}{q!}\Big(\min_{k\leq N-q,j\leq N}\Big|\prod_{s=1,s\neq j}^{k}(x_j-x_s)\prod_{s=k+q+1,s\neq j}^{N+1}(x_j-x_s)\Big|\Big)^{-1}.\end{aligned}$$

Let we prove this for every $q = 0, 1, ..., p$. We use induction.
For $q = 0$ Lemma 3.1.1 satisfies.
For $q = p - 1$ We know that

$$|\xi_N(f)| \leq \sum_{k=1}^{N}(x_k - x_{k+p})[x_k, ..., x_{k+p}]f \sum_{j=1}^{k} \frac{\prod_{s=k+1}^{k+p-1}(x_j - x_s)}{e'_{N+1}(x_j)}$$

Then,

$$|\xi_N(f)| \leq \frac{N^2\,\omega(\tilde{f}^{(p-1)},t)}{(p-1)!}\Big(\min_{k\leq N-p+1,j\leq N}\Big|\prod_{s=1\ \ s\neq j}^{k}(x_j-x_s)\prod_{s=k+p\ \ s\neq j}^{N+1}(x_j-x_s)\Big|\Big)^{-1}.$$

Now we show that it is true for $q = p$. We have

$$|\xi_N(f)| \leq \sum_{k=1}^{N}(x_k - x_{k+p+1})[x_k, ..., x_{k+p+1}]f \sum_{j=1}^{k} \frac{\prod_{s=k+1}^{k+p}(x_j - x_s)}{e'_{N+1}(x_j)}$$

From Proposition 2.2.4

$$[x_k, x_{k+1}, ..., x_{k+p+1}]f = \frac{[x_{k+1}, x_{k+1}, ..., x_{k+p+1}]f - [x_k, x_{k+1}, ..., x_{k+p}]f}{x_{k+p+1} - x_k}$$

Therefore,

$$\xi_N(f) \leq \sum_{k=1}^{N}([x_k, ..., x_{k+p}]f - [x_{k+1}, ..., x_{k+p+1}]f) \sum_{j=1}^{k} \frac{\prod_{s=k+1}^{k+p}(x_j - x_s)}{e'_{N+1}(x_j)}$$

$$\leq ([x_1, ..., x_{p+1}]f - [x_2, ..., x_{p+2}]f)\Big\{\frac{\prod_{s=2}^{p+2}(x_1 - x_s)}{e'_{N+1}(x_1)}\Big\}$$

$$+([x_2, ..., x_{p+2}]f - [x_3, ..., x_{p+3}]f)\Big\{\frac{\prod_{s=3}^{p+3}(x_1 - x_s)}{e'_{N+1}(x_1)} + \frac{\prod_{s=3}^{p+3}(x_2 - x_s)}{e'_{N+1}(x_2)}\Big\}$$

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$$+([x_N, ..., x_{N+p}]f - [x_{N+1}, ..., x_{N+p+1}]f)\sum_{j=1}^{N}\frac{\prod_{s=N+1}^{N+p}(x_j - x_s)}{e'_{N+1}(x_j)}$$

$$\leq (x_1 - x_{1+p})[x_1, ..., x_{1+p}]f\Big\{\frac{\prod_{s=2}^{p+2}(x_1 - x_s)}{e'_{N+1}(x_1)}\Big\}$$

$$+(x_2 - x_{2+p})[x_2, ..., x_{2+p}]f\Big\{\frac{\prod_{s=3}^{p+3}(x_1 - x_s)}{e'_{N+1}(x_1)} + \frac{\prod_{s=3}^{p+3}(x_2 - x_s)}{e'_{N+1}(x_2)}\Big\}$$

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$$+(x_N - x_{N+p})[x_N, ..., x_{N+p}]f\sum_{j=1}^{N}\frac{\prod_{s=N+1}^{N+p-1}(x_j - x_s)}{e'_{N+1}(x_j)}$$

$$-[x_{N+1}, ..., x_{N+p+1}]f\sum_{j=1}^{N}\frac{\prod_{s=N+1}^{N+p}(x_j - x_s)}{e'_{N+1}(x_j)}$$

$$\leq \sum_{k=1}^{N}(x_k - x_{k+p})[x_k, ..., x_{k+p}]f\sum_{j=1}^{k}\frac{\prod_{s=k+1}^{k+p=1}(x_j - x_s)}{e'_{N+1}(x_j)}$$

$$-[x_{N+1}, ..., x_{N+p+1}]f\sum_{j=1}^{N}\frac{\prod_{s=N+1}^{N+p}(x_j - x_s)}{e'_{N+1}(x_j)}$$

By (3.1.1)

$$(x_k - x_{k+p})[x_k, ..., x_{k+p}]f \leq \frac{w(\tilde{f}^{(p-1)}, t)}{(p-1)!}.$$

Then

$$
\begin{aligned}
|\xi_N(f)| \quad & \leq \frac{N^2\,\omega(\tilde{f}^{(p-1)},t)}{(p-1)!}\Big(\min_{k\leq N-p+1, j\leq N}\Big|\prod_{s=1\ s\neq j}^{k}(x_j-x_s)\prod_{s=k+p\ s\neq j}^{N+1}(x_j-x_s)\Big|\Big)^{-1} \\
& + \Big|[x_{N+1},...,x_{N+p+1}]f\sum_{j=1}^{N}\frac{\prod_{s=N+1}^{N+p}(x_j-x_s)}{e'_{N+1}(x_j)}\Big| \\
& \leq \frac{N^2\,\omega(\tilde{f}^{(p-1)},t)}{(p-1)!}\Big(\min_{k\leq N-p+1, j\leq N}\Big|\prod_{s=1\ s\neq j}^{k}(x_j-x_s)\prod_{s=k+p\ s\neq j}^{N+1}(x_j-x_s)\Big|\Big)^{-1} \\
& + \frac{N\,\omega(\tilde{f}^{(p-1)},t)}{(p-1)!}\Big(\min_{k\leq N-p+1, j\leq N}\Big|\prod_{s=1\ s\neq j}^{k}(x_j-x_s)\prod_{s=k+p\ s\neq j}^{N+1}(x_j-x_s)\Big|\Big)^{-1} \\
& \leq \frac{N^2\,\omega(\tilde{f}^{(p-1)},t)}{(p-1)!}\Big(\min_{k\leq N-p+1, j\leq N}\Big|\prod_{s=1\ s\neq j}^{k}(x_j-x_s)\prod_{s=k+p\ s\neq j}^{N+1}(x_j-x_s)\Big|\Big)^{-1}
\end{aligned}
$$

So it is satisfies for $q = p$. So the proof is complete.

Lemma 3.1.3. *Let $e_{N+1}(x) = \prod_{k=1}^{N+1}(x - x_k)$ and $e_{N+1}^{(q)}$ be the q-th derivative of e_{N+1}. Then*

$$|e_{N+1}^{(q)}| \leq N^q\Big(\max_{k\leq N}\Big|\prod_{s=1}^{k}(x-x_s)\prod_{s=k+q+1}^{N+1}(x-x_s)\Big|\Big),$$

for all $q = 1, 2, ..., p$.

We will consider Cantor-type sets with the restriction only that

$$\exists A: \quad l_k \leq A \cdot h_k, \ \ \forall k. \tag{3.1.2}$$

Without loss of generality we suppose $A \geq 2$. For the definition of Cantor-type sets, see section 2.4.

Let x be an endpoint of some basic interval. Then there exists the minimal number s (the *type* of x) such that x is the endpoint of some $I_{j,m}$ for every $m \geq s$.

By K_s we denote $K(\Lambda) \cap l_s$. Given K_s with $s \in \mathbb{N}_0 := \{0, 1, \cdots\}$, let us choose the sequence $(x_n)_1^\infty$ by including all endpoints of basic intervals, using the rule of increase of the type. For the points of the same type we first take the endpoints of the largest gaps between the points of this type; here the intervals $(-\infty, x), (x, \infty)$ are considered as gaps. From points adjacent to the equal gaps, we choose the left one x and then $l_s - x$. Thus, $x_1 = 0, x_2 = l_s, x_3 = l_{s+1}, \cdots, x_7 = l_{s+1} - l_{s+2}, \cdots, x_{2^k+1} = l_{s+k}, \cdots$

Let

$$\mu_{s,\,N} := \frac{\max_{x\in K_s} |\, e_N(x)|}{\min_{j\le N} |\, e'_{N+1}(x_j)|}, \qquad L_{N,\,j}(x) = \prod_{k=1,\,k\neq j}^{N} \frac{x - x_j}{x_k - x_j},$$

that is $L_{N,\,j}$ denotes the fundamental Lagrange polynomial.

Lemma 3.1.4. *(Lemma 3.2, [10]) Suppose the Cantor-type set $K(\Lambda)$ satisfies* (3.1.2) *and for $N \ge 1$ the points $(x_k)_1^{N+1} \subset K_s$ are chosen by the rule of increase of the type. Then*

$$\mu_{s,\,N} \le A^N \quad \textit{and} \quad \max_{j\le N,\, x\in K_s} |\, L_{N,\,j}(x)| \le A^{N-1}.$$

Let we set

$$\varphi_{s,N} := \frac{\max_{k\le N-p} \left| \prod_{s=1}^{k}(x - x_s) \prod_{s=k+p+1}^{N+1}(x - x_s) \right|}{\min_{k\le N-p, j\le N} \left| \prod_{s=1, s\neq j}^{k}(x_j - x_s) \prod_{s=k+p+1, s\neq j}^{N+1}(x_j - x_s) \right|}$$

Suppose the Cantor-type set $K(\Lambda)$ where $K(\Lambda)$ is uniformly perfect and satisfies (3.1.2). Since $K(\Lambda)$ is uniformly perfect, there exists $B \in \mathbb{R}$ such that $l_s \le B l_{s+1}$.

Lemma 3.1.5. *For $N \ge q+1$ the points $(x_k)_1^{N+1} \subset K_s$ are chosen by the rule of increase of the type. Then*

$$\varphi_{s,N} \le A^{N-q}\, B^{q \log_2 N}$$

where $q = 1, 2, ..., p$.

Proof. For $N = q+1$ it satisfies. Let $N = 2^n + \nu$ with $0 \le \nu < 2^n$.Then $(x_k)_1^{N+1}$ consists of all endpoints basic intervals of the type $s + n - 1$ and $\nu + 1$ points of the type $s + n$. Fix any $x \in K_s$ and $x_j,\ j \le N + 1$.

By $(y_k)_1^N$ we denote the points $(x_k)_1^N$ arranged in the order of distances $|x - x_k|$, that is $|x - y_k| = |x - x_{\sigma_k}| \uparrow$. Then $Y = (y_k)_1^N = \bigcup_{m=0}^n Y_{s+m}$ where $Y_r = \{y_k :\ h_r \le |x - y_k| \le l_r\},\ r = s+n-1,\ \cdots, s$ and $Y_{s+n} = \{y_k :\ h_{s+n} \le |x - y_k| \le l_{s+n}\}$.

Similarly, $Z = (z_k)_1^N$ consist of all points x_k $1 \le k \le N+1,\ k \ne j$ but here $|x_j - z_k| = |x_j - x_{\tau_k}| \uparrow$. As before, $Z = \bigcup_{m=0}^n Z_{s+m}$, $Z_r = \{z_k :\ h_r \le |x_j - z_k| \le l_r\}$ for $r = s+n-1,\ \cdots, s$. Let $a_q = |Y_q|,\ b_q = |Z_q|$ be the cardinalities of the corresponding sets. Since the points $(x_k)_1^{N+1}$ are uniformly distributed on K_s, it follows that the numbers of points x_k in two basic intervals $I_{i,r},\ I_{j,r}$ of equal length are the same of differ by 1. But the point x_j is not included into computation of b_r. Hence we have for $r = s+n,\ \cdots, s$ the following inequality

$$a_{s+n} + \cdots + a_r \ge b_{s+n} + \cdots + b_r.$$

Next to find the maximum of the product $\left|\prod_{s=1}^k (x - x_s) \prod_{s=k+q+1}^{N+1} (x - x_s)\right|$ we choose q points which are very close to x. So the distance between x and other $N-q$ points, is maximum. We know that $a_s + ... + a_{s+n} = N$. Then $a_s + ... + a_{s+n} - q = N - q$. Let we choose v_q, $c_{1q} \in \mathbb{N}$ such that

$$a_s + ... + a_{s+v_q} + c_{1q} = N - q, \quad c_{1q} \le a_{s+v_q+1}, \quad v_q \le n \tag{3.1.3}$$

Then

$$\max_{k \le N} \left| \prod_{s=1}^k (x - x_s) \prod_{s=k+q+1}^{N+1} (x - x_s) \right| \le l_s^{a_s} l_{s+1}^{a_{s+1}} \cdots l_{s+v_q}^{a_{s+v_q}} l_{s+v_q+1}^{c_{1q}}$$

Also to find the minimum of the product $\left|\prod_{s=1}^k (x_j - x_s) \prod_{s=k+q+1}^{N+1} (x_j - x_s)\right|$, first we fix $j = \tilde{j}$ such that

$$\min_{k \le N, j \le N} \left| \prod_{s=1}^k (x_j - x_s) \prod_{s=k+q+1}^{N+1} (x_j - x_s) \right| = \min_{k \le N} \left| \prod_{s=1}^k (x_{\tilde{j}} - x_s) \prod_{s=k+q+1}^{N+1} (x_{\tilde{j}} - x_s) \right|$$

where $\tilde{j} \in \mathbb{N}$ and $\tilde{j} \le N$. Then we choose q points which are far away from $x_{\tilde{j}}$. So the distance between other points and $x_{\tilde{j}}$ is minimum. We know that $b_s + ... + b_{s+n} = N$. Let we choose u_q, $c_{2q} \in \mathbb{N}$ such that

$$b_{s+n} + ... + b_{s+u_q} + c_{2q} = N - q, \quad c_{2q} \le b_{s+u_q-1}, \quad u_q \le n \tag{3.1.4}$$

Then

$$\min_{k \le N, j \le N} \left| \prod_{s=1, s \ne j}^{k} (x_j - x_s) \prod_{s=k+q+1, s \ne j}^{N+1} (x_j - x_s) \right| \ge l_{s+n}^{b_{s+n}} h_{s+n-1}^{b_{s+n-1}} \cdots h_{s+u_q}^{b_{s+u_q}} h_{s+u_q-1}^{c_{2q}}$$

Then

$$|\varphi_{s,N}| \le \frac{l_s^{a_s}\, l_{s+1}^{a_{s+1}} \cdots l_{s+v_q}^{a_{s+v_q}}\, l_{s+v_q+1}^{c_{1q}}}{l_{s+n}^{b_{s+n}}\, h_{s+n-1}^{b_{s+n-1}} \cdots h_{s+u_q}^{b_{s+u_q}}\, h_{s+u-1}^{c_{2q}}} \le \prod_{k=s}^{s+n} l_k^{a_k-b_k} \prod_{k=s}^{s+n-1} (l_k/h_k)^{b_k} \frac{h_s^{b_s} + ... + h_{s+u_q-1}^{b_{s+u_q-1}-c_{2q}}}{l_{s+v_q+1}^{a_{s+v_q+1}-c_{1q}} + ... + l_{s+n}^{a_{s+n}}} \tag{3.1.5}$$

Then by [9, Lemma 3.2]

$$\prod_{k=s}^{s+n} l_k^{a_k-b_k} \prod_{k=s}^{s+n-1} (l_k/h_k)^{b_k} \le A^N$$

Then

$$|\varphi_{s,N}| \le A^N \prod_{k=s}^{s+u_q-1} (h_k/l_k)^{b_k} \left(\frac{h_{s+u_q-1}}{l_{s+u_q-1}}\right)^{-c_{2q}} \frac{l_s^{l_s} + ... + l_{s+u_q-1}^{b_{s+u_q-1}-c_{2q}}}{l_{s+v_q+1}^{a_{s+v_q+1}-c_{1q}} + ... + l_{s+n}^{a_{s+n}}}$$

By (3.1.2) $h_k/l_k \ge 1/A$ and by (3.1.4) $b_s + ... + b_{s+u_q-1} - c_{2q} = q$. So $\prod_{k=s}^{s+u_q-1}(h_k/l_k)^{b_k}\left(\frac{h_{s+u_q-1}}{l_{s+u_q-1}}\right)^{-c_{2q}} \le \frac{1}{A^q}$. Since $l_{s+n} < l_{s+n-1}$ and $a_{s+v_q+1} + ... + a_{s+n} - c_{1q} = q$ by (3.1.3),

$$|\varphi_{s,N}| \le A^{N-q} \frac{l_s^q}{l_{s+n}^q}$$

Since $K(\Lambda)$ is uniformly perfect, there exists $B \in \mathbb{R}$ such that $l_s \le B l_{s+1}$. So $l_s \le B^n l_{s+n}$. Since $N \ge 2^n$, $\log_2 N \ge n$. Then

$$|\varphi_{s,N}| \le A^{N-q}\, B^{q \log_2 N}.$$

3.2 Interpolating bases

Fix $s \in \mathbb{N}$. Let natural numbers n_{s-1}, n_s be given with $n_{s-1} \le n_s$. Set $N_s = 2^{n_s}$ and $N_{s-1} = 2^{n_{s-1}}$. Given N with $1 \le N \le N_{s-1}$ we choose the points

$(x_k^{(s-1)})_{k=1}^{N_{s-1}+1}$ on K_{s-1} and $(x_k)_{k=1}^{N}$ on K_s by the rule of increase of the type. As above, $\xi_{k,\,s-1}(f) = [x_1^{(s-1)}, \cdots, x_{k+1}^{(s-1)}]f$, $e_{k,\,s-1}(x) = \prod_{j=1}^{k}(x - x_j^{(s-1)})|_{K_{s-1}}$ for $k = 1, 2, \cdots, N_{s-1}$. Also let $e_N(y) = \prod_{j=1}^{N}(y - x_j)|_{K_s}$

Lemma 3.2.1. *[10, Lemma 4.3] For fixed $f \in C(K(\Lambda))$, $x \in K_s$ let $\tilde{\xi}(f) = [x_1, \cdots, x_N, x]f$,
$\tilde{\eta}(f) = \tilde{\xi}(f) - \sum_{k=N}^{N_{s-1}} \tilde{\xi}(e_{k,\,s-1})\xi_{k,\,s-1}(f)$. Then*

$$|\, \tilde{\eta}(f)\, e_N(x)\,| \le N_{s-1}^4\, A^{2\,N_{s-1}}\, \omega(f, l_{s-1}).$$

In the case $K(\Lambda) = K^{(\alpha)}$ we have $|\, \tilde{\eta}(f)\, e_N(x)\,| \le e^6\, N_{s-1}^4\, \omega(f, l_{s-1})$, provided the condition $N_s\, l_s^{\alpha-1} \le 1$.

Proof. By $\tilde{e}$ we denote the function $\tilde{e}(y) = (y - x)\, e_N(y)$. Then by Lemma 3.1.1 $|\, \tilde{\xi}(f)\,| \le N^2\, \omega(f, l_s)\, (\min_{j \le N} |\, \tilde{e}\,'(x_j)|\,)^{-1}$. Since $e_N(x)/\tilde{e}\,'(x_j) = -L_{N,\,j}(x)$, Lemma 3.1.4 now implies

$$|\, \tilde{\xi}(f)\, e_N(x)\,| \le N^2\, A^{N-1}\, \omega(f, l_s). \tag{3.2.1}$$

The representation $\tilde{\xi}(e_{k,\,s-1}) = -e_{k,\,s-1}(x)/e_N(x) + \sum_{j=1}^{N} e_{k,\,s-1}(x_j)/\tilde{e}\,'(x_j)$ gives

$$|\, \tilde{\xi}(e_{k,\,s-1})\, \xi_{k,\,s-1}(f)\, e_N(x)\,| \le$$

$$|\, \xi_{k,\,s-1}(f)\, e_{k,\,s-1}(x)\,| + \sum_{j=1}^{N} \frac{|\, e_{k,\,s-1}(x_j)\,|}{|\, \tilde{e}\,'(x_j)\,|} \cdot \frac{|\, e_N(x)\,|}{\min_{i \le k} |\, e'_{k+1,\,s-1}(x_i)|}\;\; k^2\, \omega(f, l_{s-1}). \tag{3.2.2}$$

The first term on the right does not exceed $k^2\, A^k\, \omega(f, l_{s-1})$, by Lemma 3.1.1 and Lemma 3.1.4. The parts of the two fractions in the second sum will be considered cross-wise. Applying Lemma 3.1.4 twice we get

$$|\, \tilde{\xi}(e_{k,\,s-1})\, \xi_{k,\,s-1}(f)\, e_N(x)\,| \le (1 + N\, A^{N-1})\, k^2\, A^k\, \omega(f, l_{s-1}).$$

Clearly, $\sum_1^n k^2 A^k \le 5/8\; A^n\, n^3$ for $n \ge 2$. Summing over k and taking into account (3.2.2), we get the general estimation of $|\, \tilde{\eta}(f)\, e_N(x)\,|$.

In the same manner we obtain the desired bound in the case $K(\Lambda) = K^{(\alpha)}$.

Lemma 3.2.2. *For fixed* $f \in C^q(K(\Lambda))$, $x \in K_s$ *let* $\tilde{\xi}(f) = [x_1, \cdots, x_N, x]f$, $\tilde{\eta}(f) = \tilde{\xi}(f) - \sum_{k=N}^{N_{s-1}} \tilde{\xi}(e_{k,\,s-1})\xi_{k,\,s-1}(f)$. *Then*

$$|\,\tilde{\eta}(f)\, e_N^{(q)}(x)\,| \le \frac{N^{2q+3}\, A^{2N-2q}\, B^{2q\log_2 N}\omega(\tilde{f}^{(q)}, t)}{q!}.$$

where $q = 1, 2, ..., p$.

Proof. By Lemma 3.1.2,

$$|\,\tilde{\xi}(f)\,| \le \frac{N^2\omega(\tilde{f}^{(q)}, t)}{q!}\Big(\min_{k\le N, j\le N} |x - x_j|\Big|\prod_{s=1}^{k}(x_j - x_s)\prod_{s=k+q+1}^{N+1}(x_j - x_s)\Big|\Big)^{-1}$$

And from Lemma 3.1.3

$$|e_{N+1}^{(q)}| \le N^q\Big(\max_{k\le N}\Big|\prod_{s=1}^{k}(x - x_s)\prod_{s=k+q+1}^{N+1}(x - x_s)\Big|\Big)$$

Then by Lemma 3.1.5

$$\begin{aligned} |\,\tilde{\xi}(f)\, e_N^{(q)}(x)\,| \quad &\le \frac{N^{q+2}\omega(\tilde{f}^{(q)}, t)}{q!}\frac{|\varphi_{s,N}|}{\min_{j\le N}|x - x_j|} \\ &\le \frac{A^{N-q}\, B^{p\log_2 N} N^{q+2}\,\omega(\tilde{f}^{(q)}, t)}{q!}. \end{aligned} \tag{3.2.3}$$

The representation $\tilde{\xi}(e_{k,\,s-1}) = -e_{k,\,s-1}(x)/e_N(x) + \sum_{j=1}^{N} e_{k,\,s-1}(x_j)/\tilde{e}\,'(x_j)$ gives

$$|\,\tilde{\xi}(e_{k,\,s-1})\,\xi_{k,\,s-1}(f)\, e_N(x)\,| \le$$

$$|\,\xi_{k,\,s-1}(f)\, e_{k,\,s-1}(x)\,| + \sum_{j=1}^{N}\frac{|\,e_{k,\,s-1}(x_j)\,|}{|\,\tilde{e}\,'(x_j)\,|}\cdot\frac{|\,e_N(x)\,|}{\min_{i\le k}|\,e'_{k+1,\,s-1}(x_i)|}\;\; k^2\,\omega(\tilde{f}^{(q)}, l_{s-1}). \tag{3.2.4}$$

Then the first term

$$|\,\xi_{k,\,s-1}(f)\, e_{k,\,s-1}(x)\,| \le \frac{A^{k-q}\, B^{q\log_2 k}\, k^{q+2}\omega(\tilde{f}^{(q)}, t)}{q!}$$

by Lemma 3.1.2, Lemma 3.1.3 and Lemma 3.1.5. The parts of the two fractions in the second sum will be considered cross-wise. Applying Lemma 3.1.5 twice we get

$$|\,\tilde{\xi}(e_{k,\,s-1})\,\xi_{k,\,s-1}(f)\, e_N(x)\,| \le (1 + N^q\, A^{N-q}\, B^{q\log_2 N})\, k^{q+2}\, A^{k-q}\, B^{q\log_2 N}\omega^q(\tilde{f}, l_{s-1}).$$

Clearly, $\sum_1^n k^{q+2} A^{k-q} B^{q\log_2 k} \le A^{n-q}\, B^{q\log_2 n} n^{2q+3}$ for $n \ge 2$. Summing over k and taking into account (3.2.4), we get the general estimation of $|\,\tilde{\eta}(f)\, e_N(x)\,|$.

The task is now to show that the biorthogonal system suggested in [9] as a basis for the space $C(K(\Lambda))$ forms a topological basis in the space $C^p(K(\Lambda))$ as well, provided a suitable choice of degrees of polynomials.

Given a nondecreasing sequence of natural numbers $(n_s)_0^\infty$, let $N_s = 2^{n_s}$, $M_s^{(l)} = N_{s-1}/2+1$, $M_s^{(r)} = N_{s-1}/2$ for $s \geq 1$ and $M_0 = 1$. Here, (l) and (r) mean *left* and *right* respectively. For any basic interval $I_{j,s} = [a_{j,s}, b_{j,s}]$ we choose the sequence of points $(x_{n,j,s})_{n=1}^\infty$ using the rule of increase of the type.

As in Section 2.4 we take $e_{N,1,0} = \Pi_{n=1}^N (x - x_{n,1,0}) = \Pi_1^N (x - x_n)$ for $x \in K(\Lambda)$, $N = 0, 1, \cdots, N_0$. For $s \geq 1$, $j \leq 2^s$ let $e_{N,j,s} = \Pi_{n=1}^N (x - x_{n,j,s})$ if $x \in K(\Lambda) \cap I_{j,s}$, and $e_{N,j,s} = 0$ on $K(\Lambda)$ otherwise. Here, $N = M_s^{(a)}, M_s^{(a)}+1, \cdots, N_s$ with $a = l$ for odd j and $a = r$ if j is even. The functionals are given as follows: for $s = 0, 1, \cdots$; $j = 1, 2, \cdots, 2^s$ and $N = 0, 1, \cdots$, let $\xi_{N,j,s}(f) = [x_{1,j,s}, \cdots, x_{N+1,j,s}]f$. Set $\eta_{N,1,0} = \xi_{N,1,0}$ for $N \leq N_0$. Every basic interval $I_{j,s}$, $s \geq 1$, is a subinterval of a certain $I_{i,s-1}$ with $j = 2i-1$ or $j = 2i$. Let

$$\eta_{N,j,s}(f) = \xi_{N,j,s}(f) - \sum_{k=N}^{N_{s-1}} \xi_{N,j,s}(e_{k,i,s-1})\, \xi_{k,i,s-1}(f)$$

for $N = M_s^{(a)}, M_s^{(a)}+1, \cdots, N_s$. As before, $a = l$ if $j = 2i-1$, and $a = r$ if $j = 2i$.

In the space $C(K(\Lambda))$ has no unconditional basis. Thus we have to enumerate the elements $(e_{N,j,s})_{s=0,\, j=1,\, N=M_s}^{\infty,\ 2^s,\ N_s}$ in a reasonable way. We arrange them by increasing the level s. Elements of the same level are ordered by increasing the degree, that is with respect to N. For fixed s and N the elements $e_{N,j,s}$ are ordered by increasing j, that is from left to right. In this way we introduce a injective function $\sigma : (N,\, j,\, s) \mapsto M \in \mathbb{N}$. At the beginning we have for zero level: $\sigma(0,\, 1,\, 0) = 1, \cdots, \sigma(N_0,\, 1,\, 0) = N_0 + 1$. Since the degree of the first element on $I_{1,1}$ is greater that on $I_{2,1}$, we start the first level from $e_{N_0/2,2,1} : \sigma(N_0/2,\, 2,\, 1) = N_0+2$, $\sigma(N_0/2+1,\, 1,\, 1) = N_0+3$, $\sigma(N_0/2+1,\, 2,\, 1) = N_0+4, \cdots, \sigma(N_1,\, 2,\, 1) = N_0 + 1 + 2(N_1 - N_0/2) + 1 = 2(N_1+1)$ and we finish all elements of the first level. For $s = 2$ we have two elements $e_{N_1/2,2,2}$, $e_{N_1/2,4,2}$ of the smaller degree, so they have a priority: $\sigma(N_1/2,\, 2,\, 2) = 2(N_1+1)+1$, $\sigma(N_1/2,\, 4,\, 2) = 2(N_1+1)+2$. Then $\sigma(N_1/2+1,\, 1,\, 2) = 2(N_1+1)+3, \sigma(N_1/2+1,\, 2,\, 2) = 2(N_1+1)+4, \cdots, \sigma(N_2,\, 4,\, 2) = 2(N_1+1) + 4(N_2 - N_1/2) + 2 = 4(N_2+1)$.

Continuing in this manner after completing of the $s-$th level we get the value $\sigma(N_s\,,\,2^s,\,s) = 2^s(N_s+1)$.

By injectivity of the function σ there exists the inverse function σ_{-1}. Let $f_m = e_{\sigma_{-1}(m)}$, $m \in \mathbb{N}$.

Theorem 3.2.3. *[10, Theorem 4.1] Let a Cantor-type set $K(\Lambda)$ satisfy* (3.1.2). *Then for any bounded sequence $(N_s)_0^\infty$ the system $(f_m)_1^\infty$ forms a Schauder basis in the space $C(K(\Lambda))$.*

Proof. Given $f \in C(K(\Lambda))$ by $S_M(f,\cdot)$ we denote the M-th partial sum of the expansion of f with respect to the system $(f_m)_1^\infty$, that is $S_M(f,x) = \sum \eta_{N,j,\,s}(f)\, e_{N,j,\,s}(x)$, where the sum is taken over all $N,\ j,\ s$ with $\sigma(N,j,\,s) \le M$. If $1 \le M \le N_0+1$, then $S_M(f,x) = Q_{M-1}(f,(x_{n,1,\,0})_{n=1}^M, x)$. The next function S_{N_0+2} is not a polynomial on $I_{1,\,0}$. The restriction of S_{N_0+2} to the interval $I_{1,1}$ is Q_{N_0}, whereas $S_{N_0+2}|_{I_{2,1}} = Q_{N_0} + \eta_{N_0/2,2,\,1}(f)\, e_{N_0/2,2,\,1}$. In both cases we get the polynomials of degree N_0 that interpolate f at $N_0/2+1$ points each. And always the subscript M gives the total number of points where S_M interpolates f.

Continuing in this way we see that the restriction of the function $S_{2^p(N_p+1)}$ to any interval $I_{j,\,p}\,,\ j = 1, \cdots, 2^p$, coincides with $Q_{N_p}(f,(x_{n,j,\,p})_{n=1}^{N_p+1}, \cdot)$. Adding the next terms $\eta(f)\, e$ to $S_{2^p(N_p+1)}$ we get on the intervals $I_{j,\,p+1}\,,\ j = 1, \cdots, 2^{p+1}$ certain polynomials of degree N_p that interpolate f at some points. Increasing M we get $S_{2^{p+1}N_p+1}$ that has a degree N_p on $I_{1,\,p+1}$ and interpolates f on this interval at N_p+1 points; so here it is the usual interpolating polynomial. Then the restriction of $S_{2^{p+1}(N_p+1)}$ to the interval $I_{j,\,p+1}$ gives $Q_{N_p}(f,(x_{n,j,\,p+1})_{n=1}^{N_p+1}, \cdot)$ and $S_{2^{p+1}(N+1)}|_{I_{j,\,p+1}}$ produces $Q_N(f,(x_{n,j,\,p+1})_{n=1}^{N+1}, x)$ for $N \ge N_p$. It will continue up to the value $N = N_{p+1}$, after which we do the next splitting.

Suppose $\eta_{N,j,\,s}(f) = 0$ for all $N, j,\ s$. Then, by considering step by step all triples $\sigma_{-1}(m)$, $m \in \mathbb{N}$, we get $\xi_{N,j,\,s}(f) = 0$ for all $N,\ j,\ s$. The set of nodes of the corresponding divided differences is dense in $K(\Lambda)$. Therefore, $f = 0$ and the expansion $f = \sum \eta_{N,j,\,s}(f)\, e_{N,j,\,s}(x)$ is unique. We need to check only the convergence of $S_M(f,\cdot)$ to f in the norm of the space $C(K(\Lambda))$.

Let $N_s \le C$ for $s \in \mathbb{N}_0$. Fix $f \in C(K(\Lambda))$, $\varepsilon > 0$ and s_ε such that $\omega(f, l_{s_\varepsilon}) \le$

$C^{-4}\,A^{-2C}\,\varepsilon$. Let $M_\varepsilon = 2^{s_\varepsilon}(N_{s_\varepsilon}+1)$. For any $M \ge M_\varepsilon$ we get $2^{s-1}(N_{s-1}+1) \le M < 2^s(N_s+1)$ with $s \ge s_\varepsilon + 1$.

Fix $x \in K(\Lambda)$. Without loss of generality let $x \in K(\Lambda) \cap [0, l_s]$.

If $2^{s-1}(N_{s-1}+1) \le M \le 2^s N_{s-1}$, then

$$S_M(f,x) = Q_{N_{s-1}}(f, (x_{n,1,s-1})_{n=1}^{N_{s-1}+1}, x) + \sum \eta_{N,1,s}(f)\, e_{N,1,s}(x), \qquad (3.2.5)$$

where the sum is taken over all N, j, s with $2^{s-1}(N_{s-1}+1) < \sigma(N,j,\,s) \le M$. The degree N_{s-1} will appear for the first time when $\sigma_{-1}(2^s\,N_{s-1}+1) = (N_{s-1},1,s)$. Thus for the values N in (3.2.5) we have $N \le N_{s-1}-1$.

For the second case $2^s N_{s-1} + 1 \le M < 2^s(N_s+1)$ we get

$$S_M(f,x) = Q_N(f, (x_{n,1,s})_{n=1}^{N+1}, x)$$

with some N, $N_{s-1} \le N \le N_s$.

Let us consider at the beginning the simpler second case. With the notation $\tilde{\xi}(f) = [x_{1,1,s}, \cdots, x_{N+1,1,s}, x]f$, we have the polynomial $\tilde{Q}_{N+1}(\cdot) = Q_N(\cdot) + \tilde{\xi}(f)\, e_{N+1,1,s}(\cdot)$ that interpolates f also at the point x. Therefore here

$$f(x) - S_M(f,x) = \tilde{\xi}(f)\, e_{N+1,1,s}(x) \qquad (3.2.6)$$

and as in (3.2.1)

$$|\,\tilde{\xi}(f)\, e_{N+1}(x)\,| \le (N+1)^2\, A^N\, \omega(f, l_s) \le (C+1)^2\, A^C\, \omega(f, l_s),$$

which does not exceed ε.

For the case $2^{s-1}(N_{s-1}+1) \le M \le 2^s N_{s-1}$, we denote the last term of (3.2.5) by $\eta_{R,1,s}(f)\, e_{R,1,s}(x)$. As it was remarked before, $R \le N_{s-1}-1$. We can use Lemma 3.2.1 with $N = R+1$. Here $\tilde{\xi}(f) = [x_{1,1,s}, \cdots, x_{R+1,1,s}, x]f$ and $\tilde{\eta}(f) = \tilde{\xi}(f) - \sum_{k=R+1}^{N_{s-1}} \tilde{\xi}(e_{k,1,,s-1})\xi_{k,1,s-1}(f)$. Then by Lemma 2.4.3 the function $S_M(f,\cdot) + \tilde{\eta}(f)\, e_{R+1,1,s}(\cdot)$ interpolates f at the point x. Therefore,

$$|\,f(x) - S_M(f,x)\,| = |\,\tilde{\eta}(f)\, e_{R+1,1,s}(x)\,| \le \varepsilon \qquad (3.2.7)$$

by Lemma 3.2.1 and because of the choice of s_ε. Therefore, $|\,f(x) - S_M(f,x)\,| \le \varepsilon$ for any $M \ge M_\varepsilon$, which is the desired conclusion.

Theorem 3.2.4. *Let a Cantor-type set $K(\Lambda)$ be a uniformly perfect set which satisfies (3.1.2). Then for any bounded sequence $(N_s)_0^\infty$ the system $(f_m)_1^\infty$ forms a Schauder basis in the space $C^q(K(\Lambda))$ where $q = 1, 2, ..., p$.*

Proof. In the proof we use same system in Theorem 3.2.3 Let $N_s \leq D$ for $s \in \mathbb{N}_0$. Fix $f \in C^q(K(\Lambda))$, $\varepsilon > 0$ and s_ε such that $\omega(\tilde{f}^{(q)}, l_{s_\varepsilon}) \leq D^{-2q-3} A^{-2D+2q} B^{-2\log_2 D} \varepsilon$. Let $M_\varepsilon = 2^{s_\varepsilon}(N_{s_\varepsilon} + 1)$. For any $M \geq M_\varepsilon$ we get $2^{s-1}(N_{s-1} + 1) \leq M < 2^s(N_s + 1)$ with $s \geq s_\varepsilon + 1$.

Fix $x \in K(\Lambda)$. Without loss of generality let $x \in K(\Lambda) \cap l_s$.

Then if $2^{s-1}(N_{s-1} + 1) \leq M \leq 2^s N_{s-1}$, by (3.2.7)

$$|\, f(x) - S_M(f, x) \,| = |\, \tilde{\eta}(f)\, e_{R+1,1,\,s}(x) \,| \tag{3.2.8}$$

If we differentiate both parts q times then by lemma 3.2.2

$$\begin{aligned} |f^{(q)}(x) - S_M^{(q)}(f,x)| = |\, \tilde{\eta}(f)\, e_{N+1}^{(q)}(x) \,| \quad &\leq \frac{N_s^{2q+3} A^{2N_s-2q} B^{2q\log_2 N_s} \omega(\tilde{f}^{(q)}, t)}{q!} \\ &\leq \frac{D^{2q+3} A^{2D-2q} B^{2q\log_2 D} \omega(\tilde{f}^{(q)}, t)}{q!} \\ &\leq \varepsilon. \end{aligned}$$

For the second case $2^s N_{s-1} + 1 \leq M < 2^s(N_s + 1)$, by (3.2.6)

$$f(x) - S_M(f, x) = \tilde{\xi}(f)\, e_{N+1,1,\,s}(x) \tag{3.2.9}$$

Then in the same way if we differentiate both parts q times by (3.2.3)

$$\begin{aligned} |f^{(q)}(x) - S_M^{(q)}(f,x)| = |\, \tilde{\xi}(f)\, e_{N+1}^{(q)}(x) \,| \quad &\leq \frac{A^{N_s-q} B^{q\log_2 N_s} N_s^{q+2}\, \omega(\tilde{f}^{(q)}, t)}{q!} \\ &\leq \frac{A^{D-q} B^{q\log_2 D} D^{q+2}\, \omega(\tilde{f}^{(q)}, t)}{q!} \\ &\leq \varepsilon. \end{aligned}$$

Therefore $|f^{(q)}(x) - S_M^{(q)}(f,x)| \leq \varepsilon$ for any $M \geq M_\varepsilon$ and for any $1 \leq q \leq p$, which is the desired conclusion.

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