THE EXTENSION CLASS OF A SUBSET COMPLEX

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Aslı Güçlükan

August, 2006

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Ergün Yalçın (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Laurence J. Barker

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Yıldıray Ozan

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet B. Baray
Director of the Institute Engineering and Science

ABSTRACT

THE EXTENSION CLASS OF A SUBSET COMPLEX

Aslı Güçlükan
M.S. in Mathematics
Supervisor: Asst. Prof. Dr. Ergün Yalçın
August, 2006

The subset complex $\Delta(X)$ of a G-set X is defined as the simplicial complex whose simplices are subsets of X. The oriented chain complex of $\Delta(X)$ gives a $\mathbb{Z}G$-module extension of $\mathbb{Z}$ by $\widetilde{\mathbb{Z}}$ (a copy of $\mathbb{Z}$ on which G acts via the sign representation of $\mathbb{R}X$) hence a class $\zeta_X \in \mathrm{Ext}^{|X|-1}_{\mathbb{Z}G}(\mathbb{Z}, \widetilde{\mathbb{Z}})$. This class was first introduced by Reiner and Webb in [21], and they asked in which cases it is non-zero. In this thesis, we consider the mod 2 reduction $\overline{\zeta}_X$ of ζ_X and prove that $\overline{\zeta}_X$ is equal to the top Stiefel-Whitney class of the augmentation module $I_X = \mathrm{Ker}\{\mathbb{R}X \to \mathbb{R}\}$. Then, we study the case where $X = G/1$, the transitive G-set with point stabilizer, and G is a 2-group. First, we show that $\overline{\zeta}_{G/1}$ is zero when G is non-abelian. Then, we calculate the top Stiefel-Whitney class explicitly for abelian 2-groups using the norm map and the tensor product formulas for Stiefel-Whitney classes, and decide when $\overline{\zeta}_{G/1}$ is non-zero using the structure of group cohomology for abelian groups. The main result of the thesis is that if G is 2-group, then $\overline{\zeta}_{G/1}$ is non-zero if and only if $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$.

Keywords: augmentation module, cohomology, vector bundles, Stiefel-Whitney classes, Euler class, extension class, spectral sequences.

ÖZET

ALTKÜME KOMPLEKSİNİN GENİŞLEME SINIFI

Aslı Güçlükan
Matematik, Yüksek Lisans
Tez Yöneticisi: Yrd. Doç. Dr. Ergün Yalçın
Ağustos, 2006

X bir G-kümesi olsun. X kümesinin altkümelerini simpleks olarak kabul eden simpleksler kompleksine altküme kompleksi diyoruz ve $\Delta(X)$ ile gösteriyoruz. Altküme kompleksi $\Delta(X)$'in zincir kompleksi bize bir $\mathbb{Z}G$-genişlemesi veriyor. Bu genişleme sınıfını ζ_X ile gösterelim. Bu sınıf ilk olarak Reiner ve Webb tarafından tanımlanmıştır. Bu tezde, ζ_X'in mod 2 indirgemesi olan $\overline{\zeta}_X$'i inceledik ve onun X'in augmentation modülünün en yüksek dereceli Stiefel-Whitney sınıfı olduğunu gösterdik. Daha sonra, G'nin bir 2-grubu olduğu ve X'in de bir yörüngeden oluşan serbest G-kümesi olduğu durumu irdeledik. Bu durumda X, G-kümesi olarak $G/1$'e izomorfiktir. Bu tezin en önemli sonucu olarak $\overline{\zeta}_{G/1}$'in $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$ iken sıfır olmadığını ve diğer durumlarda sıfır olduğunu ispatladık. Bu ispatı yapabilmek için, tümevarım yöntemiyle, grubun değişmeli olmadığı durumlarda $\overline{\zeta}_{G/1}$'ın sıfır olduğunu gösterdik. Daha sonra, değişmeli 2-gruplarının en yüksek dereceli Stiefel-Whitney sınıflarını hesapladık ve bu grupların kohomolojilerinin yapısını kullanarak en yüksek dereceli Stiefel-Whitney sınıfının sıfırdan farklı olduğu durumları bulduk.

Anahtar sözcükler: augmentation modülü, kohomoloji, vektör yığınları, Stiefel-Whitney sınıfı, Euler sınıfı, genişleme sınıfı, spektrel diziler.

Acknowledgements

First of all, I would like to express my deepest gratitude to my supervisor Asst. Prof. Dr. Ergün Yalçın for his excellent guidance, valuable suggestions, encouragement, patience, and conversations full of motivation. I am glad to have the chance to study with him.

I would like to thank Assoc. Prof. Dr. Laurence J. Barker for his help on various occasions and for his suggestions.

I would like to thank Assoc. Prof. Dr. Yıldıray Ozan for reading this thesis.

I am grateful to my family, especially my brother Yunus Emre, for their encouragement, support, understanding, love and trust.

I would like to thank my closest friends Ahmet İlhan, Seha Gülnar, and Gülnur Tekkeşin, who have always been with me in any situation, for their encouragement and patience.

I would like to thank Hüseyin Acan and Hasan İşlek who spared time to listen my talks on my work and also for their help to solve all kinds of problem that I had.

I would like to thank Fatma Altunbulak and Olcay Coşkun for their valuable conversations about mathematics and for their help resolving my confusions.

Finally, I would like to thank all my friends in the department for the warm atmosphere that they create.

Contents

1 Introduction **1**

2 Group Cohomology **5**

2.1 Simplicial Complexes 5

2.2 Homology and Cohomology 7

2.3 Group Cohomology 11

2.4 Cohomology of Abelian Groups 14

2.5 Spectral Sequences 17

2.5.1 Double Complex 18

2.5.2 Leray-Serre Spectral Sequences 20

3 Vector Bundles and Characteristic Classes **24**

3.1 Vector Bundles 24

3.2 Universal Bundles 28

3.3 Characteristic Classes 29

3.3.1 Stiefel-Whitney Classes 30

3.3.2 The Euler Class 33

3.4 Characteristic Classes of Real Representations 35

4 The Ext Class of A Subset Complex 39

4.1 The Ext Class of The Subset Complex 40

4.2 The HyperCohomology Spectral Sequence 42

4.3 The Top Stiefel-Whitney Class of the Augmentation Module . . . 44

5 Calculations for Nonabelian 2-Groups 46

5.1 Basic Properties 46

5.2 Calculations for Non-abelian 2-groups 50

6 Calculations for Abelian 2-Groups 56

6.1 Abelian 2-groups 56

Chapter 1

Introduction

Let Δ be an abstract simplicial complex with vertex set V which has order $v_1 < v_2 < \cdots < v_n$. Then its *oriented chain complex* C^{ori} has $\mathbb{Z}$-basis given by the symbols $[v_{i_0}, \ldots, v_{i_r}]$ where $\{v_{i_0}, \ldots, v_{i_r}\}$ is a r-simplex in Δ and where $v_{i_0} < \cdots < v_{i_r}$ [[22], Chapter 1 §5]. If the terms $v_{i_0}, \ldots, v_{i_r}$ are occurred in the wrong order, then the symbol $[v_{i_0}, \ldots, v_{i_r}]$ will be identified with $\text{sign}(\sigma)[v_{i_0}, \ldots, v_{i_r}]$ for any permutation σ, and if the terms are repeated, then we will take it to be zero.

For any group G, let Δ to be an $(n+1)$-simplex whose vertex set is a G-set X with order $|X| = n+1$. Then the oriented chain complex of Δ augmented by a copy of the trivial module is an exact sequence of $\mathbb{Z}G$-modules of the form

$$\varepsilon^X : \quad 0 \to \tilde{\mathbb{Z}} \to C^{ori}_{n-1}(\Delta) \to \ldots \cdots \to C^{ori}_0(\Delta) \to \mathbb{Z} \to 0$$

which represents a class $\zeta_X \in \text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}})$. This class was first introduced by Reiner and Webb in [21] and they call it the *Ext class of the subset complex*.

In [21], it is shown that the extension class ζ_X is an essential class when $X = G/1$ is a transitive G-set with point stabilizer. Therefore, when $\zeta_{G/1}$ is non-zero, it gives a non-zero essential class in cohomology. On the other hand, when $\zeta_{G/1}$ does happen to be zero, it gives an information about the spectral sequence which is defined in [21]. As a result, it is of interest to find what are the finite

groups G for which $\zeta_{G/1}$ is non-zero.

It has been pointed out to Reiner and Webb by Michael Mandell that ζ_X is the (twisted) Euler class of the augmentation module $I_X = \mathrm{Ker}\{\mathbb{R}X \to \mathbb{R}\}$, however, they did not give the proof of this fact in [21]. A consequence of this would be that the mod 2 reduction of ζ_X is the top Stiefel-Whitney class of the augmentation module. In Chapter 4, we consider the mod 2 reduction of ε^X which represents an element $\overline{\zeta}_X \in \mathrm{Ext}^n_{\mathbb{F}_2[G]}(\mathbb{F}_2, \mathbb{F}_2)$ and we show that this class is the top Stiefel-Whitney class of I_X. The main theorem of this thesis is the following:

Theorem 1.0.1 *Let G be a 2-group. Then, the extension class $\overline{\zeta}_{G/1}$ is non-zero if and only if $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$.*

We restrict ourselves to 2-groups because it has already been shown in [21] that ζ_X is zero when G is a composite group and when G is a p-group where p is odd, the cohomology ring $H^*(G, \mathbb{F}_2)$ is zero, so in these cases there is nothing to study.

We now describe the parts involved in the proof of Theorem 1.0.1. Let $\langle\ ,\ \rangle$ be an inner product on I_X and let $\langle\ ,\ \rangle_G$ be a G-invariant inner product associated to $\langle\ ,\ \rangle$ obtained by taking average over G-orbits. Let us define the G-sphere $S(I_X)$ of I_X as the set of points in I_X which has norm 1 under the G-invariant inner product $\langle\ ,\ \rangle_G$. In Chapter 4, we establish a G-homeomorphism between sphere $S(I_X)$ of the augmentation module I_X and the boundary $\partial\Delta(X)$ of $\Delta(X)$. Hence we show that the exact sequence

$$0 \to H_{n-1}(C_*(S(I_X))) \to C_{n-1}(S(I_X)) \to \cdots \to C_0(S(I_X)) \to H_0(C_*(S(I_X))) \to 0$$

also represents ζ_X. Then, we show that the hypercohomology spectral sequence for the double complex with chains $C^{r,s} = \mathrm{Hom}_G(P_r, \mathrm{Hom}(C_s(S(I_X)), \mathbb{F}_2))$ has differential d_n given by the multiplication with $\overline{\zeta}_X$. By using the spectral sequence isomorphism between this spectral sequence and the Leray-Serre spectral sequence for fibration $\pi : EG \times_G S(I_X) \to BG$, we conclude that

Proposition 1.0.2 *The extension class $\overline{\zeta}_X$ equals to the top Stiefel-Whitney class $w_{\mathrm{top}}(I_X)$ of the augmentation module of X.*

As a result of the above proposition, in the last two chapters of this thesis, we consider the top Stiefel-Whitney class of the augmentation module $I_{G/1}$. First we prove that if G is a non-abelian 2-group, then $w_{\text{top}}(I_{G/1})$ is zero. For the proof, we use induction. Note that if G is a minimal example with the property that G is non-abelian and $w_{\text{top}}(I_{G/1})$ is non-zero, then G will be a non-abelian 2-group whose all subquotients are abelian. We classify all such 2-groups and prove that in this case G is a quaternion group Q_8 of order 8 or G is a modular 2-group M_{2^k} with $k \geq 3$. However, $w_{\text{top}}(I_{G/1})$ is zero when G is Q_8 or M_{2^k}. Hence, we conclude that $w_{\text{top}}(I_{G/1}) = 0$ for non-abelian 2-groups by induction on the order of the group.

For abelian groups, we first observe that if $G \cong \mathbb{Z}/8$ or if $G \cong \mathbb{Z}/4 \times \mathbb{Z}/4$, then $w_{\text{top}}(I_{G/1})$ is zero. This gives that if G has subquotient isomorphic to $\mathbb{Z}/8$ or $\mathbb{Z}/4 \times \mathbb{Z}/4$ then the top Stiefel-Whitney class $w_{\text{top}}(I_{G/1})$ is zero. So, the only abelian groups that can have non-zero $w_{\text{top}}(I_{G/1})$ are the groups which are of the form $(\mathbb{Z}/2)^n \times \mathbb{Z}/4$ for some n. This proves the one direction of Theorem 1.0.1. For the other direction, we calculate the top Stiefel-Whitney class $w_{\text{top}}(I_{G/1})$ when $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$ and we show that it is non-zero. In fact in [25], Turygin calculates the top Stiefel-Whitney class of the augmentation module for elementary abelian 2-groups by using the tensor product formula and the Whitney product formula. In Section 6, we use similar argument to generalize his calculations to the case where $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$ and we obtain the following:

Proposition 1.0.3 *Let $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$. Then*

$$w_{\text{top}}(I_{G/1}) = f(x_1, \ldots, x_n, t)\frac{f(x_1^2, \ldots, x_n^2, s)}{f(x_1^2, \ldots, x_n^2)}$$

where $H^*(G, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_n, s] \otimes \wedge_{\mathbb{F}_2}[t]$ and where

$$f(a_1, a_2, \ldots, a_m) = \prod_{(\alpha_1, \ldots, \alpha_m) \in (\mathbb{F}_2)^m \setminus \{0\}} (\alpha_1 a_1 + \cdots + \alpha_m a_m)$$

for any tuple $(a_1, \ldots, a_m)$ of independent variables. In particular, $w_{\text{top}}(I_{G/1}) \neq 0$ in $H^*(G, \mathbb{F}_2)$.

Then by using the structure of the group cohomology $H^*(G, \mathbb{F}_2)$ for abelian groups, we conclude that $w_{\text{top}}(I_{G/1})$ is non-zero in these cases.

As an application of our calculations, a Borsuk-Ulam type theorem can be obtained. The Borsuk-Ulam theorem states that given a map from an n-sphere into Euclidean n-space there exists antipodal points which are mapped to the same point. This theorem was generalized by a number of authors. For example in [25], Turygin gives such a theorem by using his calculations for elementary abelian 2-groups. A similar theorem can be obtained for $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$ by using Proposition 1.0.3.

The thesis is organized as follows:

In Chapter 2, we give some background material from homological algebra including the general theory of simplicial complexes and the definition of the cohomology of chain complex and the definition of group cohomology. We also explain the spectral sequences associated to double complexes and we discuss Leray-Serre spectral sequences.

In Chapter 3, we explain the standard theory of vector bundles and we give the definition of characteristic classes.

Chapter 4 is devoted to the proof of Proposition 1.0.2.

In Chapter 5, we prove that the top Stiefel-Whitney class of the augmentation module of non-abelian 2-groups is zero.

Finally, in Chapter 6, we calculate the top Stiefel-Whitney class of the augmentation module for abelian 2-groups and we show that it is non-zero if and only if $G = (\mathbb{Z}/4)^n \times \mathbb{Z}/4$. This completes the proof of our main theorem.

Chapter 2

Group Cohomology

Let G be a finite group and M be a RG-module where RG is the group algebra and the ground ring R is the commutative ring with unity. In this chapter, we give the necessary definitions and background material on the simplicial complexes and the homology of chain complexes of R-modules. We also define the cohomology of group G with coefficients in M by applying $\mathrm{Hom_{RG}(-,M)}$ functor to the projective resolutions of R over RG. Moreover, we introduce the Leray-Serre spectral sequences and the spectral sequences arising from double complexes. The content of this chapter is standard and can be found in any books on group cohomology (see for example [8], [9], [17], [18]).

2.1 Simplicial Complexes

In this section, we restrict our attention to the simplicial complexes and we give some background material on this topic. We start with the definition of a k-simplex.

Definition 2.1.1 *Let $v_0, v_1, \ldots, v_k$ be points in $\mathbb{R}^m$ such that the set of vectors $\{v_1 - v_0, \ldots, v_k - v_0\}$ is a linearly independent set. Then the smallest convex set containing $k+1$ points $v_0, v_1, \ldots, v_k$ is called a k-simplex in $\mathbb{R}^m$ and the points*

$v_0, \ldots, v_k$ are called the vertices of the simplex.

The *standard k-simplex* Δ_k is the subset of $\mathbb{R}^{n+1}$ defined by

$$\Delta_k = \{(t_0, \ldots, t_k) | \sum_{i=0}^{k} t_i = 1,\ t_i \geq 0,\ i = 0, \ldots, k\}.$$

It has vertices $e_0, \ldots, e_k$ where $e_i = (0, \ldots, \underbrace{1}_{i}, \ldots, 0)$. Every k-simplex is homeomorphic to the standard simplex.

An *n-face* of k-simplex is an n-simplex generated by the $(n+1)$-points of the vertex set. The 0-faces are called vertices, the 1-faces are called the edges and the maximal face of a simplicial complex is called a facet.

Definition 2.1.2 *A simplicial complex K is the finite collection of simplices satisfying following properties:*

1. *When a simplex σ is in K, then every face of σ is also in K;*
2. *When simplices σ, τ are in K, then their intersection $\sigma \cap \tau$ is a face of each simplex, or is empty.*

Let V denote the set of vertices of simplicial complex K, and S be the collection of the subsets of V which generates K. Then the pair $\{V, S\}$ is called the *vertex scheme* of K. The vertex scheme of a simplicial complex completely describes it.

Definition 2.1.3 *An abstract simplicial complex is the pair $\{V, S\}$ satisfying the following properties:*

1. *S consists of nonempty subsets of V;*
2. *For every $v \in V$, the singleton $\{v\}$ is in S;*
3. *If $A \in S$ and $B \subseteq A$, then $B \in S$.*

Theorem 2.1.4 *Any abstract simplical complex can be realized as a vertex scheme of a simplicial complex.*

A simplicial complex can be topologized by the subspace topology. The space that we obtain by this way is called the *realization* (or *polyhedra*) of K and denoted by $|K|$.

Definition 2.1.5 *A triangulation of a topological space X is the pair (K, h) consisting of a simplicial complex K and a homeomorphism $h : |K| \to X$.*

Definition 2.1.6 *Let K and L be simplicial complexes. A map $f : |K| \to |L|$ is called simplicial if the restriction of f to any simplex is a linear map whose image is a simplex of L.*

We conclude this section with the simplicial approximation theorem. First, we give the definition of simplicial approximation.

Definition 2.1.7 *Let K and L be simplicial complexes and let map $g : |K| \to |L|$ be an arbitrary map. Then the simplicial map $f : |K| \to |L|$ is called a simplicial approximation to g if for every $x \in |K|$, $g(x) \in \sigma$ implies $f(x) \in \sigma$.*

Theorem 2.1.8 (Simplicial Approximation Theorem) *Any map between two polyhedra can be approximated by a simplicial map.*

2.2 Homology and Cohomology

In this section, we give some background material on homology of chain complexes and cohomology of cochain complexes of R-modules. Actually, the only difference between the homology theory and the cohomology theory is that homology is a covariant functor while cohomology is contravariant.

Definition 2.2.1 *A chain complex C of R-modules is a family $\{C_n, \partial_n\}_{n\geq 0}$ where C_n are R-modules and $\partial_n : C_n \to C_{n-1}$ are R-module homomorphisms such that $\partial_n \circ \partial_{n+1} = 0$*

We usually express a chain complex C as a sequence

$$C: \quad \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0$$

where each composite maps $\partial_n \circ \partial_{n+1}$ are zero. This gives Im $\partial_{n+1} \subseteq$ Ker ∂_n, for each n. We can define the nth homology of C as the quotient

$$H_n(C) = \text{Ker } \partial_n / \text{Im } \partial_{n+1}.$$

Definition 2.2.2 *The homology of the chain complex C, written $H_*(C)$, is the family of modules $\{H_n(C)\}$ defined above.*

A chain complex is said to be exact if Ker $\partial_n =$ Im ∂_{n+1} for each n. Therefore, the homology groups measures how far the chain complex is from being exact.

Definition 2.2.3 *A cochain complex C is a family $\{C^n, \delta^n\}$ of R-modules C^n and R-module homomorphisms $\delta^n : C^n \to C^{n+1}$ satisfying the relation $\delta^{n+1} \circ \delta^n = 0$.*

We express a cochain complex C by

$$C: \quad 0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \longrightarrow \cdots$$

where each composite map $\delta^n \circ \delta^{n-1}$ is zero. The maps δ and ∂ are called *differentials*, because they satisfy the relations $\partial \circ \partial = \delta \circ \delta = 0$.

Definition 2.2.4 *The cohomology of the cochain complex C, written $H^*(C)$, is the family of modules $\{H^n(C)\}_{n\geq 0}$ where*

$$H^n(C) = \text{Ker } \delta^n / \text{Im } \delta^{n-1}.$$

An element of Ker ∂_n is called an *n-cycle* and an element of Im ∂_n is called an *n-boundary*. Analogously, *n-cocycles* and *n-coboundaries* are defined as the elements of Ker δ^n and Im δ^n, respectively. From now on, we will denote the set of n-cycles and n-cocycles of any chain complex C by $Z_n(C)$ and $Z^n(C)$, and the set of n-boundaries or n-coboundaries of it by $B_n(C)$ and $B^n(C)$, respectively.

Definition 2.2.5 *Let $C = \{C_n, \partial_n\}$ and $D = \{D_n, \partial'_n\}$ be chain complexes. A chain map f is a family $\{f_n\}$ of module homomorphisms $f_n : C_n \to D_n$ which makes the following diagram commute:*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[d, "f_0"'] & 0 \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r] & \cdots \arrow[r] & D_0 \arrow[r] & 0.
\end{tikzcd}$$

Definition 2.2.6 *The chain maps f and g between the complexes $C = \{C_n, \partial_n\}$and $D = \{D_n, \partial'_n\}$ are said to be chain homotopic, written $f \cong g$, if there is a family $\{h_n\}$ of R-module homomorphisms $h_n : C_n \to D_{n+1}$ such that $f_n - g_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$ for all n.*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1} - g_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n - g_n"] \arrow[dl, "h_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1} - g_{n-1}"] \arrow[dl, "h_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 2.2.7 *The chain complex C and D are said to be chain homotopic if there are chain maps $f : C \to D$ and $g : D \to C$ such that $f \circ g \cong \mathrm{id}_D$ and $g \circ f \cong \mathrm{id}_C$. The maps f and g are called chain equivalences.*

A cochain complex can be considered as a chain complex by choosing appropriate indices so similar definitions exists for cochain complexes.

Lemma 2.2.8 *Let $f : C \to D$ be a chain map. Then it induces a chain homomorphism $f_* : H_*(C) \to H_*(D)$ defined by*

$$f_n([x]) = [f_n(x)]$$

for all $x \in \text{Ker } \partial_n$. *Similarly, any cochain map induces a morphism* f^* *between cohomoloy of cochain complexes defined by*

$$f^n([x]) = [f^n(x)]$$

for all $x \in \text{Ker } \delta^n$.

Definition 2.2.9 *A sequence of homomorphism of R-modules* $\{E_i\}$

$$E_1 \xrightarrow{\alpha_1} E_2 \xrightarrow{\alpha_2} \cdots \longrightarrow E_{n-1} \xrightarrow{\alpha_{n-1}} E_n$$

is called exact if for $i = 1, \ldots, n-2$, *we have* $\text{Im } \alpha_i = \text{Ker } \alpha_{i+1}$.

Definition 2.2.10 *Let* $f : C' \to C$ *and* $g : C \to C''$ *be chain maps. If for each* n,

$$0 \longrightarrow C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \longrightarrow 0$$

is exact, then the sequence of chain complexes

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0 \tag{2.1}$$

is called a short exact sequence of chain complexes.

Proposition 2.2.11 *For each short exact sequence* (2.1) *of chain complexes, there is a long exact sequence*

$$\cdots \longrightarrow H_{n+1}(C'') \xrightarrow{\partial} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(C'') \xrightarrow{\partial} \cdots$$

where ∂ *is called the connecting homomorphism.*

Proof: See [5], page 27.

□

Definition 2.2.12 *A resolution of an R-module M is an exact sequence of R-modules*

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0. \tag{2.2}$$

It is denoted by $\varepsilon : F \to M$

Definition 2.2.13 *An R-module P is said to be projective if for each epimorphism $\alpha : M \twoheadrightarrow N$ and a map $\beta : P \to N$, there exists a map $\gamma : P \to M$ such that the following diagram commute,*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\gamma"'] \arrow[d, "\beta"] & \\ M \arrow[r, "\alpha"] & N \arrow[r] & 0 \end{tikzcd}$$

that is $\alpha \circ \gamma = \beta$.

The resolution (2.2) is called projective if for each n, F_n is a projective module.

2.3 Group Cohomology

Throughout this section, let R be the ring of integers or a field of characteristic p. Let M and N be RG-modules and let $\mathrm{Hom}_G(M, N)$ denote the set of RG-homomorphisms between M and N. Note that, G act on $\mathrm{Hom}(M, N)$ by

$$(g \cdot f)(m) = gf(g^{-1}m),$$

for $g \in G$, $f \in \mathrm{Hom}(M, N)$ and $m \in M$. Then $f \in \mathrm{Hom}(M, N)$ commutes with $g \in G$ if and only if $gf = f$ which means

$$\mathrm{Hom}_G(M, N) = \mathrm{Hom}(M, N)^G.$$

Let $\varepsilon : P \to N$ be a projective resolution of N over RG. By applying $\mathrm{Hom}_G(-, M)$ functor to this complex, we obtain a cochain complex $\mathrm{Hom}_G(P, M) = (\mathrm{Hom}(P_n, M), \delta^n)$ where δ^n is defined by

$$\delta^n(f)(x) = (-1)^{n+1} f(\partial_n(x))$$

for $f \in \mathrm{Hom}_G(P_n, M)$ and $x \in P_n$.

Definition 2.3.1 *The cohomology of G with coefficients in M is defined as the cohomology of cochain complex*

$$0 \to \mathrm{Hom}_G(P_0, M) \to \mathrm{Hom}_G(P_1, M) \to \cdots \to \mathrm{Hom}_G(P_n, M) \to \cdots$$

that is $H^n(G, M) = H^n(\mathrm{Hom}_G(P, M))$.

Notice that, the group cohomology in the degree zero is given by

$$H^0(G, M) = \mathrm{Hom}_G(\mathbb{Z}, M) = M^G.$$

Remark 2.3.2 *If* $M = \mathbb{Z}$*, the cohomology group* $H^*(G, \mathbb{Z})$ *is called the cohomology of the group* G *and denoted by* $H^*(G)$.

Lemma 2.3.3 *Let* G *be an abelian group. Then*

$$H^1(G, \mathbb{Z}/2) \cong \mathrm{Hom}(G, \mathbb{Z}/2) = \mathrm{Rep}(G, \mathbb{R}^\times)$$

where $\mathrm{Rep}(G, \mathbb{R}^\times)$ *denotes the set of* 1*-dimensional real representations of* G.

Proof: See [5], page 86.

□

Definition 2.3.4 *Let* M *and* N *be* R*-modules. A* n*-fold extension of* M *by* N *is an exact sequence of the form*

$$E : 0 \longrightarrow N \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow M \longrightarrow 0.$$

The n-fold extension E is said to be congruent to E', written $E \vdash E'$, if there is a chain map $f : E \to E'$ which is identity on both ends, that is

$$\begin{tikzcd}
E : 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & F_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & F_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, equal] & 0 \\
E' : 0 \arrow[r] & N \arrow[r] & F'_{n-1} \arrow[r] & \cdots \arrow[r] & F'_0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Since it is not symmetric, the relation " $\vdash$ " is not an equivalence relation. In order to have an equivalence relation, we say E is congruent to E', written $E \sim E'$, if there is a sequence $D_0, \ldots, D_{2t}$ of n-fold extensions of N by M such that

$$E \vdash D_0 \dashv D_1 \vdash \cdots \vdash D_{2t} \dashv E'.$$

Definition 2.3.5 *The set of equivalence classes of n-fold extensions of M by N is denoted by* $\mathrm{Ext}^n_R(M,N)$

Theorem 2.3.6 *Let $\varepsilon : P \to M$ be a projective resolution of M. Then, for any $n > 0$,*

$$\mathrm{Ext}^n_R(M,N) \cong H^n(\mathrm{Hom}_R(P,N))$$

Proof: We only give a sketch of the proof here. The details can be found in [5]. Given an extension E representing a class α in $\mathrm{Ext}^n_R(M,N)$, we have the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "0"'] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, equal] & 0 \\
E: & 0 \arrow[r] & N \arrow[r] & M_{n-1} \arrow[r] & \cdots \arrow[r] & M_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Since, $f = \{f_i\}$ is a chain map, $0 = f_n \circ \partial_{n+1} = \delta_{n+1} \circ f_n$ which means f_n is a cocycle. We define the homomorphism

$$\varphi : \mathrm{Ext}^n_R(M,N) \to H^n(\mathrm{Hom}_R(P,N))$$

by $\varphi(\alpha) = [f_n]$. This is a well-defined as explained in [5]. The inverse of φ is defined as follows:

Let $\xi : P_n \to N$ be a cocycle representing $z \in H^n(\mathrm{Hom}_R(P,N))$. Since $\delta_{n+1} \circ \xi = \xi \circ \partial_{n+1} = 0$, we have the following commutative diagram

$$\begin{tikzcd}
 & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "0"'] & P_n \arrow[r, "\partial_n"] \arrow[d, "\xi"'] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & M \arrow[r] \arrow[d, equal] & 0 \\
E: & 0 \arrow[r] & N \arrow[r] & F \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where F is push-out of

$$\begin{tikzcd}
N & P_n \arrow[l, "\xi"] \arrow[r, "\partial_n"] & P_{n-1}
\end{tikzcd}$$

We set $\varphi^{-1}(\xi) = [E]$.

$\square$

2.4 Cohomology of Abelian Groups

For a hereditary ring R, the Künneth Theorem describes the cohomology of cartesian product of groups $G_1 \times G_2$ in terms of the cohomology of G_1 and G_2. In this chapter, we give necessary definitions to state the Künneth Theorem and we show how the Künneth Theorem applies to abelian groups.

Definition 2.4.1 *Let M be a right R-module and $P \to N$ be a projective resolution of a left R-module N. Then, we have the following chain complex*

$$\dots \xrightarrow{Id\otimes\partial_{n+2}} M \otimes_R P_{n+1} \xrightarrow{Id\otimes\partial_{n+1}} M \otimes_R P_n \xrightarrow{Id\otimes\partial_n} M \otimes_R P_{n-1} \longrightarrow \dots$$

and $Tor_n^R(M,N)$ is defined as the homology of the complex that is

$$Tor_n^R(M,N) = H_n(M \otimes P, Id \otimes \partial_*).$$

Remark 2.4.2 *Note that if we tensor any long exact sequence of R-modules from left by any projective right R-module P, then the sequence we obtained is still exact. Any left module with this property is called flat.*

Proposition 2.4.3 *If either M or N is a flat R-module, then $\mathrm{Tor}_n^R(M,N) = 0$ for all $n > 0$.*

Proof: See [5], Page 34.

□

Proposition 2.4.4 *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of right R-modules and M' be a left R-module. Then there is a long exact sequence

$$\cdots \to \mathrm{Tor}_n^R(L,M') \to \mathrm{Tor}_n^R(M,M') \to \mathrm{Tor}_n^R(N,M') \to \dots$$
$$\cdots \to \mathrm{Tor}_1^R(N,M') \to L \otimes_R M' \to M \otimes_R M' \to N \otimes_R M' \to 0$$

Proof: See [5], Page 33.

□

Let C be a chain complex of right R-modules and let D be a chain complex of left R-modules. Then, we can form a new complex with chains

$$(C \otimes_R D)_n = \bigoplus_{i+j=n} C_i \otimes_R D_j$$

and with differentials $\partial_n : (C \otimes_R D)_n \to (C \otimes_R D)_{n-1}$ defined by

$$\partial_n(x \otimes y) = \partial_i(x) \otimes y + (-1)^i x \otimes \partial_j(y)$$

for $x \in C_i$ and $y \in D_j$. It follows from the construction that if x and y are cycles then $x \otimes y$ is a cycle and if one is a cycle and the other is a boundary, then $x \otimes y$ is a boundary. Therefore, we have a well defined homomorphism,

$$H_i(C) \otimes_R H_j(D) \to H_{i+j}(C \otimes_R D)$$

which sends $[x] \otimes [y]$ to $[x \otimes y]$ for the classes in $H_i(C)$ and $H_j(D)$, respectively.

Theorem 2.4.5 ***(Künneth Theorem)*** *Let C and D be a chain complexes of right and left R-modules, respectively. Then, if the cycles $Z_n(C)$ and the boundaries $B_n(C)$ are flat for all n, then there is a short exact sequence of R-modules*

$$0 \to \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(D) \to H_n(C \otimes_R D) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i(C), H_j(D)) \to 0$$

Proof: See [5], page 39.

□

Definition 2.4.6 *A ring R is called hereditary if all submodules of projective R-modules are again projective.*

If R is hereditary, then if the modules C_n are projective, then the cycles Z_n and the boundaries B_n are projective. Hence, we have the following version of the Künneth theorem.

Theorem 2.4.7 ***(Künneth theorem for cohomology of groups)*** *Let R be a hereditary ring and G_1, G_2 be groups. For RG_1-module M_1 and RG_2-module M_2, one can regard $M_1 \otimes M_2$ as a $R(G_1 \times G_2)$-module via the multiplication $(g_1, g_2)(m_1 \otimes m_2) = g_1m_1 \otimes g_2m_2$. Then there is a short exact sequence of the form*

$$0 \to \bigoplus_{i+j=n} H^i(G_1, M_1) \otimes H^j(G_1, M_1) \to H^n(G_1 \times G_2, M_1 \otimes M_2)$$
$$\to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H^i(G_1, M_1), H^j(G_2, M_2)) \to 0.$$

Proof: See [5], page 62.

□

If R is field, then every R-module is flat and hence $H^n(G_1 \times G_2, M_1 \otimes M_2) \cong \bigoplus_{i+j=n} H^i(G_1, M_1) \otimes H^j(G_1, M_1)$. Recall that, every finite abelian group can be expressed as a cartesian product of cyclic groups by the fundamental theorem of finite abelian groups. Hence, the cohomology of abelian groups can be calculated from the cohomology of cyclic groups when the ground ring R is field. In [5], the cohomology of cyclic group G with coefficients in the field of characteristic p is given. We quote the result here.

Proposition 2.4.8 *Let G be a cyclic group of order n and let p^m be the p-part of n. Suppose k is a field of characteristic p. Then,*

1. *If $p^m = 2$, then $H^*(G, k) \cong k[x]$ with $\deg(x) = 1$*
2. *If $p^m > 2$, then $H^*(G, k) \cong k[s] \otimes \wedge(t)$ with $\deg(s) = 2$ and $\deg(t) = 1$.*

Proof: See [5], page 61.

□

In Chapter 6, to express the top Stiefel-Whitney class of the augmentation module explicitly, we use the structure of cohomology of abelian 2-groups with

coefficients in $\mathbb{F}_2$. As an immediate corollary of Proposition 2.4.8, we obtain the following:

Corollary 2.4.9 *Let $G \cong (\mathbb{Z}/2)^{m_1} \times (\mathbb{Z}/2^2)^{m_2} \times \cdots \times (\mathbb{Z}/2^n)^{m_n}$. Then the cohomology of G with coefficients in $\mathbb{F}_2$ is given by*

$$H^*(G, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_{m_1}, s_1, \ldots, s_k] \otimes \wedge_{\mathbb{F}_2}(t_1, \ldots, t_k)$$

where $k = m_2 + \cdots + m_n$ and deg $t_i =$ deg $x_i = 1$ *and* deg $s_i = 2$.

2.5 Spectral Sequences

The spectral sequences are used as a way of calculation cohomology ring by taking successive approximations. In this section, we give the necessary definitions and explain the Leray-Serre spectral sequences and the spectral sequence arising from double complexes. The content of this section can be found in [8] and [18].

Definition 2.5.1 *A spectral sequence is a collection $\{E_r^{*,*}, d_r\}$ of abelian groups $E_r^{p,q}$ and derivation $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that*

$$E_{r+1}^{p,q} \cong \text{Ker } d_r/\text{Im } d_r.$$

The maps d_r are called differentials and the term E_r is called the rth-page of the spectral sequence.

A spectral sequence is said to collapse at the Nth term if the differentials $d_r = 0$ for $r \geq N$.

Definition 2.5.2 *A filtration of an R-module M is a family of submodules $\{F^pM\}_{p\in\mathbb{Z}}$ such that $F^{p+1}M \subseteq F^pM$. The filtration F on M is said to be bounded if $F^pM = 0$ and $F^qM = M$ for some $p, q \in \mathbb{Z}$.*

Definition 2.5.3 *Let H^* be graded R-module. A spectral sequence $\{E_r^{*,*}, d_r\}$ converge to H^* if there is a filtration F on H^* such that*

$$E_\infty^{p,q} \cong E_0^{p,q}(H^*, F)$$

where $E_0^{p,q} = F^p H^{p+q}/F^{p-1}H^{p+q}$.

Definition 2.5.4 *Let M be a graded R-module that is $M = \oplus_{n\geq 0} M^n$. Then, M is said to be a filtered differential graded module if there is a differential $d : M \to M$ of degree 1 and a filtration F on M such that $d(F^pM) \subseteq F^pM$ for all p.*

Theorem 2.5.5 *Let M be a filtered differential graded module with bounded filtration F and differential d. Then, there is a spectral sequence $\{E_r^{*,*}, d_r\}$ with*

$$E_1^{p,q} \cong H^{p+q}(F^pM/F^{p+1}M)$$

which converges to $H^*(M, d)$.

Proof: See [18], page 34.

□

There are several ways of defining spectral sequences, however we only explain the Leray-Serre spectral and the spectral sequences arising from double complexes. We refer reader to the [8] and [18] for more details about spectral sequences.

2.5.1 Double Complex

Definition 2.5.6 *A double complex is a bigraded module $C = \{C^{p,q}\}_{p,q\in\mathbb{Z}}$ with differentials $\partial' : C^{*,*} \to C^{*,*}$ and $\partial'' : C^{*,*} \to C^{*,*}$ of bidegree $(1, 0)$ and $(0, 1)$ satisfying $\delta' \circ \delta'' + \delta'' \circ \delta' = 0$.*

$$\begin{tikzcd} C^{p,q} \arrow[r, "\delta'"] \arrow[d, "\delta''"'] & C^{p+1,q} \arrow[d, "\delta''"'] \\ C^{p,q+1} \arrow[r, "\delta'"] & C^{p+1,q-1} \end{tikzcd}$$

Associated to each double complex C, there is a chain complex TC, called the *total complex of* C with chains

$$(TC)_n = \bigoplus_{p+q=n} C^{p,q}$$

and with total differential

$$\delta = \delta' + \delta''.$$

One can filter total complex in two different ways each of which gives different spectral sequence converging to $H^*(TC)$. The filtrations are

$$F^p(TC)_n = \oplus_{i\geq p} C^{i,n-i} \text{ and } F^p(TC)_n = \oplus_{i\geq p} C^{n-i,i}.$$

Since they are both bounded, by Theorem 4.5, they converge to $H^*(TC)$ and have E_1-page

$$E_1^{p,q} \cong H^{p+q}(F^pTC/F^{p+1}TC) \cong H^q(C^{p,*})$$

and

$$E_1^{p,q} \cong H^{p+q}(F^pTC/F^{p+1}TC) \cong H^q(C^{*,p})$$

where the differentials d_1 are induced by chain maps δ' and δ'', respectively. Indeed, E_0-pages are $E_0^{p,q} = C^{p,q}$ and $E_0^{p,q} = C^{q,p}$ with differentials $d_0 = \pm\delta''$ and $d_0 = \pm\delta'$, respectively.

Moreover, for the first filtration, the set of cocycles $Z_n^{r,s}$ is characterized as follows: The element $a \in E_n^{r,s}$ belongs to $Z_n^{r,s}$ if and only if $\delta''(a) = 0$ and there exists sequence of elements $a_0, \ldots, a_n$ with $a_i \in E_n^{r+i,s-i}$ such that

$$a_0 = a \text{ and } \delta'(a_{i-1}) + \delta''(a_i) = 0$$

for each $i = 1, \ldots, n$. And, it belongs to $B_n^{r,s}$ if and only if there exists sequence of elements $a_0, \ldots, a_n$ with $a_i \in E_n^{r-i,s+i-1}$ such that

$$\delta''(a_n) = 0 \text{ and } \delta''(a_i) + \delta'(a_{i+1}) = 0$$

for $i = 1, \ldots, n-1$ and $\delta''(a_0) + \delta'(a_1) = a$. Then the differential

$$d_{n+1} : E_n^{r,s} \to E_n^{r+n,s-n+1}$$

is given by

$$d_{n+1}(a + B_n^{r,s}) = \delta'(a_n) + B_n^{r+n,s-n+1}$$

where $a_0 = a, \ldots, a_n$ is defined as above.

The similar characterization can be done for the second type of filtration as follows. Let $a \in E_n^{r,s}$. Then $a \in Z_n^{r,s}$ if and only if there exists sequence of elements $a_0, \ldots, a_n$ with $a_i \in E_n^{r+i,s-i}$ such that

$$a_0 = a \text{ and } \delta'(a_{i-1}) + \delta''(a_i) = 0$$

for $i = 1, 2, \ldots, n$. It belongs to $B_n^{r,s}$ if and only if there exists elements $a_0, \ldots, a_n$ with $a_i \in E_n^{r-i,s+i-1}$ such that

$$\delta''(a_n) = 0 \text{ and } \ \delta''(a_i) + \delta'(a_{i+1}) = 0$$

for $i = 1, \ldots, n-1$ and $\delta''(a_0) + \delta'(a_1) = a$. The differential d_{n+1} is defined by

$$d_{n+1}(a + B_n^{r,s}) = \delta''(a_n) + B_n^{r+n,s-n+1}.$$

We will use this construction later in Chapter 4 in the proof of Theorem 4.2.1. For more information about this material, we refer reader to [8] and [18].

2.5.2 Leray-Serre Spectral Sequences

In this part, we introduce the Leray-Serre spectral sequence for fibrations $\pi : E \to B$ with fiber F which relates the cohomology of F, X and B under certain conditions. This spectral sequence was first introduced by Leray but the algebraic formalism is done by Serre in his doctoral dissertation.

Definition 2.5.7 ([6], Definition 1.6.8) *A fibre bundle with base B, total space E and fibre F consists of a surjective map $\pi : E \to B$ such that there is an open covering $\{U_\alpha\}$ of B and corresponding homeomorphisms ϕ_α which make the following square commute:*

$$\begin{array}{ccc} U_\alpha \times F & \xrightarrow{\phi_\alpha} & p^{-1}(U_\alpha) \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi}\downarrow \\ U_\alpha & = & U_\alpha. \end{array}$$

Definition 2.5.8 ([18], Definition 5.19) *Let G be a collection of groups $\{G_b; b \in B\}$ with the homomorphisms $h[\lambda] : G_{b_1} \to G_{b_0}$ for every element in the loop space $\Omega(B, b_0, b_1)$, which satisfy the following properties:*

1. *Let c_b be the constant loop in $\Omega(B, b_0, b_1)$. Then $h_{c_b} = \mathrm{id}$.*
2. *If $\lambda'(0) = \lambda(0)$, $\lambda'(1) = \lambda(1)$ and $\lambda' \cong \lambda$ relative the end points, then $h[\lambda] = h[\lambda']$.*
3. *If $\lambda * \mu$ is the product path of λ and μ , then $h[\lambda * \mu] = h[\lambda] \circ h[\mu]$.*

Then, G is called the system of local coefficients on B.

For any group G, the trivial system of local coefficients on B, denoted by G, is the system of local coefficients G with the groups $\mathsf{G}_b = G$ for each $b \in B$ and with the homomorphisms $h[\lambda]$ equals to identity map for each loop $\lambda \in \Omega(B, b_0, b_1)$. The system of local coefficients G on B is called *simple* if there is an isomorphism $\Xi = (\Xi_b)$ between G and G in the sense that the isomorphisms $\Xi_b : \mathsf{G}_b \to G$ and $\Xi_{b_0} : \mathsf{G}_{b_0} \to G$ satisfy the relation $\Xi_b = \Xi_{b_0} h[\lambda]$ for each $\lambda \in \Omega(B, b_0, b)$.

For the fiber bundle $\pi : E \to B$, the system of local coefficients induced by fiber F is given by the collection of groups

$$H^*(F; R) = \{H^*(\pi^{-1}(b_1); R) |\ b \in B\},$$

and the collection of isomorphisms

$$\{h[\lambda] : \{H^*(\pi^{-1}(b_1); R) \to H^*(\pi_0^{-1}(b); R) |\ \lambda \in \Omega(B, b_0, b_1)\}.$$

In particular, if we take $R = \mathbb{F}_2$ and if the fiber is a sphere, then the system of local coefficients induced by the fiber is always simple.

Theorem 2.5.9 ([18], Theorem 5.2) *Let $\pi : E \to B$ be a fibration with connected fiber F where B is path-connected. Then, there is a spectral sequence converging to $H^*(E, R)$ as an algebra, with*

$$E_2^{p,q} \cong H^p(B, H^q(F, R)),$$

the cohomology of the space B with local coefficients in the cohomology of fibre.

Proposition 2.5.10 ([18], Page 143) *Let $\pi : S^n \hookrightarrow E \to B$ be a fibration with B is path connected for some $n \geq 1$. Then there is a long exact sequence:*

$$\begin{aligned} \cdots \longrightarrow H^k(B,R) &\xrightarrow{\cup d_{n+1}(\mu)} H^{n+k+1}(B,\mathbb{F}_2) \\ &\xrightarrow{\pi^*} H^{n+k+1}(E,\mathbb{F}_2) \xrightarrow{Q} H^{k+1}(B,\mathbb{F}_2) \longrightarrow \cdots \end{aligned} \tag{2.3}$$

where d_{n+1} is the $(n+1)^{th}$ differential in the Leray-Serre spectral sequence of this fibration and μ is the nontrivial class in $H^n(S^n,\mathbb{F}_2) \cong \mathbb{F}_2$.

Proof: Let $\pi : S^n \hookrightarrow E \to B$ be a fibration satisfying the above conditions. Then there is a Leray-Serre spectral sequence with E_2 page

$$E_2^{p,q} \cong H^p(B, H^q(S^n,\mathbb{F}_2))$$

where $E_2^{p,q} = \{0\}$ unless $q = 0, n$. Hence, the only possible non-zero differential is

$$d_{n+1} : E_2^{p,n} \to E_2^{p+n+1,0}$$

which means $E_2 \cong E_3 \cong \cdots \cong E_{n+1}$ and $E_\infty \cong E_{n+2} \cong H(E_{n+1}, d_{n+1})$. Since $E_\infty^{*,0} \cong E_2^{*,0}/\mathrm{im}d_{n+1}$ and $E_\infty^{*,n} \cong \ker d_{n+1}$, for each p, we have following exact sequence:

$$0 \longrightarrow E_\infty^{p,n} \longrightarrow E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \longrightarrow E_\infty^{p+n+1,0} \longrightarrow 0. \tag{2.4}$$

Since $E_\infty^{p,q} \neq 0$ unless $q = 0$ or n, the filtration on H^* is of the form:

$$H^{p+n} = F^0 H^{p+n} = \cdots = F^p H^{p+n} \supseteq F^{p+1} H^{p+n} = \cdots = F^{p+n} H^{p+n} \supseteq \{0\}$$

Moreover,

$$E_\infty^{p+n,0} = F^{p+n} H^{p+n} / F^{p+n+1} H^{p+n} = F^{p+n} H^{p+n} = F^{p+1} H^{p+n},$$

and hence

$$E_\infty^{p,n} = F^p H^{p+n} / F^{p+1} H^{p+n} = H^{p+n} / F^{p+1} H^{p+n} \cong H^{p+n} / E_\infty^{p+n,0}.$$

Therefore, we have the short sequence:

$$0 \longrightarrow E_\infty^{p+n,0} \longrightarrow H^{p+n} \longrightarrow E_\infty^{p,n} \longrightarrow 0$$

which gives the following long exact sequence with (2.4),

$$\begin{tikzcd}
 & & H^{p+n-1} \arrow[d] \arrow[dr, dashed] & & 0 \arrow[d] & & \\
0 \arrow[r] & E_\infty^{p-1,n} \arrow[r] \arrow[d] & E_2^{p-1,n} \arrow[r] & E_2^{p+n,0} \arrow[r] \arrow[dr, dashed] & E_\infty^{p+n,0} \arrow[r] \arrow[d] & 0 & \\
 & 0 & & & H^{p+n} \arrow[d] \arrow[dr, dashed] & & \\
 & & & 0 \arrow[r] & E_\infty^{p,n} \arrow[r] \arrow[d] & E_2^{p,n} & \\
 & & & & 0 & &
\end{tikzcd}$$

which we can express in the following ways

$$\begin{tikzcd}
\cdots \arrow[r] & E_2^{p,n} \arrow[r, "d_{n+1}"] & E_2^{p+n+1,0} \arrow[r] & H^{p+n+1}(E,R) & \\
 & \arrow[r] & E_2^{p+1,n} \arrow[r, "d_{n+1}"] & E_2^{p+n+2,0} \arrow[r] & \cdots
\end{tikzcd}$$

Let $z \in H^{n+1}(B, \mathbb{F}_2)$ satisfy the equation $d_{n+1}(1 \otimes \mu) = z \otimes 1$ where μ is the nontrivial class of $H^n(S^n, \mathbb{F}_2) \cong \mathbb{F}_2$. Then

$$\begin{aligned}
d_{n+1}(x \otimes \mu) &= d_{n+1}((1 \otimes \mu)(x \otimes 1)) = d_{n+1}(1 \otimes \mu)(x \otimes 1) + (1 \otimes \mu) d_{n+1}(x \otimes 1) \\
&= (z \otimes 1)(x \otimes 1) = (zx) \otimes 1.
\end{aligned}$$

On the other hand, we have

$$E_2^{p,q} \cong H^p(B, H^q(S^n, \mathbb{F}_2)) \cong H^p(B, \mathbb{F}_2) \otimes H^q(S^n, \mathbb{F}_2)$$

Therefore, if we identify $E_2^{p,n} = H^p(B, \mathbb{F}_2) \otimes \mu$ and $E_2^{n+p-1,0} = H^{n+p+1}(B, \mathbb{F}_2) \otimes 1$, we obtain the desired sequence.

$\square$

Chapter 3

Vector Bundles and Characteristic Classes

This chapter will start with the basic definition of a real vector bundle $\xi = (\pi : E \to B)$. We also describe the classification of real vector bundles over paracompact space B. Any $\mathbb{R}^n$-bundle ξ over a paracompact space B and any cohomology class c in the ith cohomology of the base space of universal bundle with coefficient ring R gives a cohomology class $c(\xi)$ in the $H^i(B, R)$. This class is called the characteristic class. The main purpose of this chapter is to introduce the most fundamental characteristic classes: the Euler class and the Stiefel-Whitney classes. To get more details about the materials in this chapter, we refer the reader to [19], [20], [23].

3.1 Vector Bundles

Definition 3.1.1 *Let B be a connected space. A real vector bundle ξ is given by the projection map $\pi : E \to B$ of topological spaces and the real vector space structure in $\pi^{-1}(b)$ for every $b \in B$ which satisfies the following condition of local triviality:*

For each point $b \in B$, there exists neighborhood $U \subseteq B$ and a homeomorphism

$$h : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

so that, for each $b \in U$, the correspondence

$$x \to h(b, x)$$

defines an isomorphism between vector space $\mathbb{R}^n$ and the vector space $\pi^{-1}(b)$ for some n.

The pair (U, h) is called a *local coordinate system* for ξ about b. The vector space $\pi^{-1}(b)$ is called the *fiber* over b and generally denoted by $F_b(\xi)$. Also, topological spaces B and E are called the base space and the total space respectively.

The *trivial bundle* is a bundle with total space $B \times \mathbb{R}^n$ and with projection map $\pi(b, x) = b$ where the vector space structures in fibers defined by

$$t_1(b, x_1) + t_2(b, x_2) = (b, t_1x_1 + t_2x2).$$

The trivial bundle over B is denoted by ε^n_B or simply by ε^n.

Definition 3.1.2 *Let $\xi = (\pi : E \to B)$ be a vector bundle. The cross-section of ξ is a continuous function $s : B \to E$ which sends each $b \in B$ to a point in the fiber $F_b(\xi)$.*

Theorem 3.1.3 *An $\mathbb{R}^n$-bundle ξ possesses n linearly independent cross section if and only if ξ is trivial.*

Proof: See [19], page 18.

□

Definition 3.1.4 *Let ξ and η be two vector bundles over same base space B. If there exists a homeomorphism*

$$f : E(\xi) \to E(\eta)$$

between the total spaces which maps every vector space $F_b(\xi)$ isomorphically to the vector space $F_b(\eta)$, ξ is said to be isomorphic to η. In this case, we write $\xi \cong \eta$.

An $\mathbb{R}^n$-bundle over B is called *trivial* if and only if it is isomorphic to ε_B^n. The canonical line bundle defined below is an example of a non-trivial $\mathbb{R}^n$-bundle:

Let P^n denote the n-dimensional projective space and let $E(\gamma_n^1)$ be the subset of $P^n \times \mathbb{R}^{n+1}$ consisting of all pairs $(\{\pm x\}, v)$ such that the vector v is a multiple of x. Then the vector bundle $\gamma_n^1 = \{\pi : E(\gamma_n^1) \to P^n\}$ with projection map given by

$$\pi(\{\pm x\}, v) = \{\pm x\}$$

is called *canonical line bundle*. Notice that each fiber of the canonical line bundle can be identified with the line through $+x$ and $-x$ in $\mathbb{R}^{n+1}$.

Theorem 3.1.5 *Let ξ and η be vector bundles over B and let $f : E(\xi) \to E(\eta)$ be a continuous function that maps every vector space $F_b(\xi)$ isomorphically to the vector space $F_b(\eta)$. Then f is necessarily a homeomorphism and $\xi \cong \eta$.*

Proof: For any $b \in B$, let (U, g) and (V, h) be corresponding coordinate systems for ξ and η respectively. Then the composition

$$(U \cap V) \times \mathbb{R}^n \xrightarrow{h^{-1} \circ f \circ g} (U \cap V) \times \mathbb{R}^n$$

is a homeomorphism, so does f.

□

For any given vector bundle ξ, we can construct a new one as follows.

Definition 3.1.6 *Let $\xi = (\pi : E \to B)$ be a vector bundle over B and $f : B_1 \to B$ be a map of base spaces. Then one can define $f^*(\xi)$ to be the* induced bundle *over B_1 with total space*

$$E_1 = \{(b, e) \in B_1 \times E \mid f(b) = \pi(e)\}$$

and with projection map $\pi_1 : E_1 \to B_1$ defined by $\pi_1((b, e)) = b$.

Definition 3.1.7 *Let ξ and η be vector bundles. A bundle map from η to ξ is a map*

$$g : E(\eta) \to E(\xi)$$

which defines an isomorphism between the vector spaces $F_b(\eta)$ and $F_{b'}(\xi)$ for some $b \in B(\eta)$ and $b' \in B(\xi)$.

Let $\bar{g} : B(\eta) \to B(\xi)$ be a map defined by $\bar{g}(b) = b'$. The the following diagram commutes:

$$\begin{array}{ccc} E(\eta) & \xrightarrow{g} & E(\xi) \\ {\scriptstyle \pi_\eta}\downarrow & & {\scriptstyle \pi_\xi}\downarrow \\ B(\eta) & \xrightarrow{\bar{g}} & B(\xi) \end{array} \tag{3.1}$$

Lemma 3.1.8 *Let $g : E(\eta) \to E(\xi)$ be a bundle map and let $\bar{g}$ be defined as above. Then η is isomorphic to the induced bundle $\bar{g}^*(\xi)$.*

Proof: Recall that

$$E(\bar{g}^*(\xi)) = \{(b, e) \in B(\eta) \times E(\xi) \mid \bar{g}(b) = \pi_\xi(e)\}$$

Since (3.1) is a commutative diagram, for all $e \in E(\eta)$, $(\pi_\eta(e), g(e))$ is an element of $E(\bar{g}^*(\xi))$. So, let us define $h : E(\eta) \to E(\bar{g}^*(\xi))$ by

$$h(e) = (\pi_\eta(e), g(e))$$

Obviously, h is a continuous function. Note that for any $b \in B(\eta)$,

$$\begin{aligned} F_b(\bar{g}^*(\xi)) &= \{(b, e) \in B(\eta) \times E(\xi) \mid e \in \pi_\xi^{-1}(\bar{g}(b))\} \\ &\cong F_{\bar{g}(b)}(\xi) \end{aligned}$$

But by definition of g, $F_b(\eta)$ is isomorphic to $F_{g(b)}(\xi)$. Hence, by Theorem (3.1.5), h is an isomorphism.

$\square$

Let $\xi = (\pi_\xi : E_1 \to B)$ and $\eta = (\pi_\eta : E_2 \to B)$ be n and m dimensional real vector bundles. Then the induced bundle $\pi_\xi^*(\eta)$,

$$\begin{tikzcd} E(\pi_\xi^*(\eta)) \arrow[r] \arrow[d] & E_2 \arrow[d, "\pi_\eta"] \\ E_1 \arrow[r, "\pi_\xi"] & \mathrm{B} \end{tikzcd}$$

is a $(n+m)$-dimensional vector bundle, called the *Whitney sum* of ξ and η. It is denoted by $\xi \oplus \eta$. Note that each fiber $F_b(\xi \oplus \eta)$ is canonically isomorphic to the direct sum $F_b(\xi) \oplus F_b(\eta)$.

Remark 3.1.9 *Let ξ and η be vector bundles over the same base space B which have an Euclidean metric. Then ξ is said to be a* subbundle *of η, denoted by $\xi \subseteq \eta$, if $E(\xi) \subseteq E(\eta)$ and each fiber $F_b(\xi)$ is a vector subspace of the fiber $F_b(\eta)$. If ξ is a subbundle of η, then $\eta = \xi \oplus \xi_1$ for some vector bundle ξ_1. The vector bundle ξ_1 is called* orthogonal complement *of ξ and denoted by $\xi^\perp$.*

For given two vector bundles $\xi = (\pi_\xi : E(\xi) \to B)$ and $\eta = (\pi_\eta : E(\eta) \to B)$ over same base space B, the *tensor product* of them, denoted by $\xi \otimes \eta$, is defined to be a vector bundle over B with total space

$$E(\xi \otimes \eta) = \coprod_{b \in B} F_b(\xi) \otimes F_b(\eta)$$

and with the projection map π given by $\pi(F_b(\xi) \otimes F_b(\eta)) = b$.

3.2 Universal Bundles

An $\mathbb{R}^n$-bundle γ is said to be universal if every n-bundle over a paracompact space can be obtained as an induced bundle of it. Moreover, if f and g from any $\mathbb{R}^n$-bundle ξ to γ are bundle maps, then they are bundle homotopic. Recall that, the bundle maps $f, g : \xi \to \eta$ are called *bundle homotopic* if there exists a map $F : E(\xi) \times [0,1] \to E(\eta)$ such that $F(e,0) = f$ and $F(e,1) = g$. In this section, we construct an universal $\mathbb{R}^n$-bundle over the Grassman manifold.

Definition 3.2.1 *The collection of n-dimensional subspaces of $\mathbb{R}^{n+k}$ is called the* Grassman manifold *and it is denoted by $G_n(\mathbb{R}^{n+k})$.*

Remark 3.2.2 *The* Stiefel manifold*, denoted by $V_n(\mathbb{R}^{n+k})$, is the set of all n-frames in $\mathbb{R}^{n+k}$ which is an open subset of the n-fold product $\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$. Analogously, the set of all orthonormal n-frames in $\mathbb{R}^{n+k}$ is denoted by $V_n^o(\mathbb{R}^{n+k})$.*

We can similarly define an *infinite Grassman manifold* $G_n(\mathbb{R}^\infty) = G_n$ where $\mathbb{R}^\infty$ consists of infinite series

$$x = (x_1, x_2, \dots)$$

of real numbers for which all but finitely many number of x_i's are zero.

Let $E(\gamma^n) \subseteq G_n \times \mathbb{R}^\infty$ be the set of all pairs (X, x) where X is an n-plane and x is a vector in that plane topologized as a subset of the Cartesian product. Let us define $\pi : E(\gamma^n) \to G_n$ given by $\pi(X, x) = X$ and define the vector space structure in the fiber by

$$t_1\pi(X, x_1) + t_2\pi(X, x_2) = (X, t_1x_1 + t_2x_2).$$

Then $\gamma^n = \pi : E(\gamma^n) \to G_n$ is a vector bundle called canonical bundle (see [19]).

Theorem 3.2.3 *The canonical bundle γ^n is an universal $\mathbb{R}^n$-bundle.*

Proof: See [19], page 65.

3.3 Characteristic Classes

Any $\mathbb{R}^n$-bundle ξ over a paracompact space B and any cohomology class c in the ith cohomology of base space of universal bundle with coefficient ring R gives a cohomology class $c(\xi)$ in the $H^i(B, R)$. The class $c(\xi)$ is called the *characteristic cohomology class* of ξ determined by c. In this section we restrict our attention to the certain type of characteristic classes called Stiefel-Whitney classes and the Euler class.

3.3.1 Stiefel-Whitney Classes

Definition 3.3.1 *For each $\mathbb{R}^n$-bundle ξ, the Stiefel-Whitney Classes, $\{w_i(\xi)\}_{i\geq 0}$, are the cohomology classes in $H^n(B(\xi), \mathbb{F}_2)$ satisfying following axioms:*

Axiom 1: *The class $w_0(\xi)$ is the unit element in $H^0(B(\xi), \mathbb{F}_2)$ and $w_i(\xi) = 0$ for $i > n$.*

Axiom 2: *For each continuous function $f : B' \to B(\xi)$,*

$$w_i(f^*(\xi)) = f^*(w_i(\xi)).$$

Axiom 3: *(The Whitney Product Formula) For the vector bundles ξ and η over the same base space,*

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) w_{k-i}(\eta).$$

Axiom 4 *For the canonical line bundle γ_1^1, we have $w_1(\gamma_1^1) = a$ where a is the generator of the group $H^1(P^1, \mathbb{F}_2)$.*

Theorem 3.3.2 *The cohomology classes satisfying the Axioms 1-4 exits and they are unique.*

Proof: See [19], Section 8.

□

Definition 3.3.3 *The total Stiefel-Whitney class of an $\mathbb{R}^n$-bundle $\xi = (\pi : E \to B)$ is the element of $H^{**}(B, \mathbb{Z}_2)$ defined by $w(\xi) = \sum_{i=0}^{n} w_i(\xi)$.*

It follows from the Whitney product formula that we have

$$w(\xi)w(\eta) = w(\xi \oplus \eta).$$

Proposition 3.3.4 *If ε is trivial vector bundle then $w_i(\varepsilon) = 0$ for $i > 0$. Therefore, $w_i(\varepsilon \oplus \xi) = w_i(\xi)$.*

Proof: It follows from Axiom 2.

□

Proposition 3.3.5 *Let a be the non-zero element of $H^1(P^n, \mathbb{F}_2)$. Then*

$$w(\gamma_n^1) = 1 + a.$$

Proof: Let $f : P^1 \to P^n$ be the inclusion defined by $i(\{\pm x\}) = \{\pm(x, 0, \ldots, 0)\}$. Then, $f^*(\gamma_n^1) = \gamma_1^1$. By Axiom 3,

$$f^*(w_1(\gamma_n^1)) = w_1(\gamma_1^1) \neq 0$$

so does $w_1(\gamma_n^1)$. Since $H^1(P^n, \mathbb{F}_2) = \{0, a\}$, we get $w_1(\gamma_n^1) = a$. So, we can conclude that $w(\gamma_n^1) = 1 + a$.

□

Proposition 3.3.6 *Let $\mathbb{R}^n$-bundle ξ admits k cross section which are linearly independent. Then, $w_i(\xi) = 0$ for $i \in \{n - k + 1, \ldots, n\}$.*

Proposition 3.3.7 (Splitting Principle) *Let $\xi_1, \ldots, \xi_n$ be a finitely many vector bundles over the same base space B. Then, there is a map $f : B' \to B$ such that each of the pullback bundles $f^*(\xi_1), \ldots, f^*(\xi_n)$ over B' is a Whitney sum of line bundles, and that the induced map $f^* : H^*(B) \to H^*(B')$ is injective.*

Proof: See [20], page 203.

□

The Stiefel-Whitney classes of a tensor product can be computed as follows:

Proposition 3.3.8 ([20], Proposition 7.5) *Let ξ and η be one dimensional vector bundles over a paracompact space B. Then*

$$w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta).$$

Proof: Let $f, g : B \to P^\infty$ be the maps such that $f^*(\gamma^1) = \xi$ and $g^*(\gamma^1) = \eta$. Let $h : P^\infty \times P^\infty \to P^\infty$ characterize the bundle $\gamma^1 \times \gamma^1$. Then the composition

$$B \xrightarrow{\Delta} B \times B \xrightarrow{f \times g} P^\infty \times P^\infty \xrightarrow{h} P^\infty$$

classifies $\xi \otimes \eta$ where Δ is the diagonal map. Therefore, if a denotes the generator of the cohomology $H^*(P^\infty, \mathbb{F}_2)$ which means $w_1(\gamma^1) = a$, then

$$w_1(\xi \otimes \eta) \quad = \quad \Delta^*(f \times g)^* h^*(a). \tag{3.2}$$

Hence to calculate $w_1(\xi \otimes \eta)$, it is necessary to find $h^*(a)$.

Let $\pi_i : P^\infty \times P^\infty \to P^\infty$ be the projection maps into i^{th} coordinate for $i = 1, 2$. Let a_1, a_2 be the generators of the first and second copy of $H^*(P^\infty, \mathbb{F}_2)$, respectively. By the Künneth theorem,

$$H^1(P^\infty \times P^\infty, \mathbb{F}_2) \cong H^0(P^\infty, \mathbb{F}_2) \otimes H^1(P^\infty, \mathbb{F}_2) \oplus H^0(P^\infty, \mathbb{F}_2) \otimes H^1(P^\infty, \mathbb{F}_2)$$

and hence

$$h^*(a) = \alpha_1 \pi_1^*(a_1) + \alpha_2 \pi_2^*(a_2)$$

for some $\alpha_1, \alpha_2 \in \mathbb{F}_2$. Let us choose some base point $x_0 \in P^\infty$ and let us define $i_1 : P^\infty \to P^\infty \times P^\infty$ by $i_1(x) = (x, x_0)$. Since the compositions $\pi_1 \circ i_1$ and $\pi_2 \circ i_2$ equal to the identity and the constant map, respectively, we have

$$i_1^*(\alpha_1 \pi_1^*(a_1) + \alpha_2 \pi_2^*(a_2)) = \alpha_1 a_1.$$

On the other hand, the pullback $i_1^*(\gamma^1 \times \gamma^1)$ is the bundle γ^1 over the first copy of P^∞. But then,

$$i_1^* h^* a = w_1(i_1^* h^* \gamma^1) = w_1(i_1^*(\gamma^1 \times \gamma^1)) = a_1$$

and hence $\alpha_1 = 1$. Similarly one can show that $\alpha_2 = 1$. Therefore,

$$h^*(a) = \pi_1^*(a_1) + \pi_2^*(a_2).$$

Now, (3.2) becomes

$$\begin{aligned} w_1(\xi \otimes \eta) &= \Delta^*(f \times g)^*(\pi_1^*(a_1) + \pi_2^*(a_2)) \\ &= \Delta^*(f \times g)^* \pi_1^*(a_1) + \Delta^*(f \times g)^* \pi_2^*(a_2) \end{aligned}$$

But for each $b \in B$,

$$\pi_1(f \times g)\Delta(b) = \pi_1(f(b) \times g(b)) = f(b)$$

and

$$\pi_2(f \times g)\Delta(b) = g(b)$$

and hence

$$w_1(\xi \otimes \eta) = f^*(a_1) + g^*(a_2) = f^*(w_1(\gamma^1)) + g^*(w_1(\gamma^1)) = w_1(\xi) + w_1(\eta).$$

□

Proposition 3.3.9 *Let ξ and η be vector bundles over a paracompact base space with dimensions m and n respectively. Let $t_1, \ldots t_m$ and $t'_1, \ldots t'_n$ be indeterminates of elementary symmetric functions $w_1(\xi), \ldots w_m(\xi)$ and $w_1(\eta), \ldots w_m(\eta)$, respectively. Then*

$$w(\xi \otimes \eta) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + t_i + t'_j)$$

Proof: This follows from Proposition 3.3.8 and the splitting principle for vector bundles.

□

3.3.2 The Euler Class

Let V be an vector space and let B_1 and B_2 be two ordered bases for V. Then the basis B_1 and B_2 are said to have the same orientation if the change-of-base matrix has positive determinant and otherwise we say that they have the opposite orientation. There are two equivalence classes determined by this relation and an *orientation* of V is an assignment of $+1$ to the one equivalence class and -1 to the other.

A vector bundle $\xi = (\pi : E \to B)$ is said to be *oriented* if for every point $b_0 \in E$ there exists a local coordinate system (N, h) such that for each fiber

$F = \pi^{-1}(b)$ over N the homomorphism $x \to h(b, x)$ from $\mathbb{R}^n$ to F is orientation preserving.

Let E_0 be the set of all nonzero elements of E and F_0 be the set of all nonzero elements of F. For each oriented $\mathbb{R}^n$-plane bundle ξ, the cohomology group $H^n(E, E_0; \mathbb{Z})$ contains a unique cohomology class u such that

$$u|_{(F,F_0)} \in H^n(F, F_0; \mathbb{Z})$$

is the generator of $H^n(F, F_0, \mathbb{Z})$ corresponding to the orientation. This class is called the Thom class. For each integer k, the group $H^k(E, \mathbb{Z})$ can be mapped isomorphically onto $H^{k+n}(E, E_0; \mathbb{Z})$ by sending each y to $y \cup u$ (see [19]). From this, we get an isomorphism $\phi : H^k(B; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z})$ defined by

$$\phi(x) = (\pi^* x) \cup u,$$

which is called the Thom isomorphism.

Definition 3.3.10 *The cohomology class $e(\xi) \in H^n(B; \mathbb{Z})$ satisfying the relation*

$$\pi^*(e(\xi)) = u|_E$$

is called the Euler class *of an oriented n-plane bundle ξ.*

Some properties of the Euler class can be listed as follows:

Proposition 3.3.11 *Let $\xi = (\pi : E \to B)$ and $\xi' = (E' \to B')$ be a vector bundles with dimensions n and m, respectively. Then, the followings hold:*

1. *If $f : B \to B'$ is covered by an orientation preserving map $\xi \to \xi'$, then $e(\xi) = f^*(e(\xi'))$;*
2. *If the orientation of ξ is reversed then the Euler class changes sign.*
3. $e(\xi \times \xi') = e(\xi) \times e(\xi')$
4. $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$

The relation between the Euler class and the top Stiefel-Whitney class is given by the following proposition.

Proposition 3.3.12 *The top Stiefel-Whitney class is the mod 2 reduction of the Euler class.*

Proof: See [19], page 99.

□

Remark 3.3.13 *In this section, we considered only the Euler class of an orientable vector bundle. In fact, the Euler class $e(\xi) \in H^n(B, \widetilde{\mathbb{Z}})$ can also be defined for a non-orientable vector bundle ξ, usually referred as twisted Euler class. Proposition 3.3.12 still holds for non-orientable vector bundles under this extended definition of the Euler class.*

3.4 Characteristic Classes of Real Representations

Let V be an n-dimensional $\mathbb{R}G$-module. An n dimensional representation of V is a homomorphism

$$\rho : G \to \mathrm{Aut}(V)$$

where $\mathrm{Aut}(V)$ is the set of automorphisms of V.

Let $B = \{v_1, \ldots, v_n\}$ be a basis of V and $T \in \mathrm{Aut}(V)$. Since $Tv_i \in V$ for all $i \in \{1, \ldots n\}$, there exists real numbers $a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{nn}$ such that

$$Tv_i = \sum_{j=1}^{n} a_{ij} v_j.$$

Let us define a map $\Phi_B : \mathrm{Aut}(V) \to GL_n(\mathbb{R})$ by

$$\Phi_B(T) = [T]_B = \begin{pmatrix} a_{11} & . & . & a_{1n} \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & . & . & a_{nn} \end{pmatrix}$$

where $GL_n(\mathbb{R})$ is the set of invertible $n \times n$ matrices. The map Φ_B is an isomorphism between $\mathrm{Aut}(V)$ and $GL_n(\mathbb{R})$. So, each choice of basis gives ρ as a homomorphism

$$\rho : G \to GL_n(\mathbb{R}) \text{ where } \rho(\mathrm{g}) = [\rho(\mathrm{g})]_{\mathrm{B}}.$$

However, with choice of suitable basis, ρ takes values in the subgroup $O_n(\mathbb{R})$ of $GL_n(\mathbb{R})$. Indeed, let $\langle\ ,\ \rangle$ be an inner product on V and $B = \{b_1, \ldots, b_n\}$ be a basis of V, with respect to the inner product $\langle\ ,\ \rangle$. Then, the inner product $\langle\ ,\ \rangle_G$ defined by

$$\langle v, w\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle gv, gw\rangle$$

is G-invariant. Since B spans V, by Gram-Schmidt orthonormalization process, we can find an orthonormal basis $S = \{s_1, \ldots, s_n\}$ with respect to $\langle\ ,\ \rangle_G$.

Recall that, for given orthonormal basis W, every inner product can be written in the form

$$\langle v, w\rangle = [v]_W^T [w]_W$$

where $[v]_W$ is a coordinate matrix with respect to W so that

$$\langle v, w\rangle_G = [v]_S^T [w]_S$$

Since S is an orthonormal basis and $\langle , \rangle_G$ is G-invariant, we have

$$\langle gs_i, gs_j\rangle_G = [gs_i]_S^T [gs_j]_S = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

But then,

$$[\rho(g)]_S^T[\rho(g)]_S = \begin{pmatrix} \underline{[gs_1]_S^T} \\ . \\ . \\ . \\ \underline{[gs_n]_S^T} \end{pmatrix} \begin{pmatrix} & & & \\ [gs_1]_S & . & . & [gs_n]_S \\ & & & \end{pmatrix} = I \tag{3.3}$$

which means $\rho(g) \in O_n(\mathbb{R})$.

Therefore, without loss of generality, we can consider an n-dimensional real representation ρ as a group homomorphism from G to $O_n(\mathbb{R})$.

Definition 3.4.1 *For a topological group G, a contractible space which has a free G-action is called a universal G-space and denoted by EG. The quotient space $EG/G = BG$ is called a classifying space for G.*

Let V be an n-dimensional $\mathbb{R}G$-module and ρ be an n-dimensional real representation. Then, we can form an $\mathbb{R}^n$-bundle over BG as follows:

Let $EG \times_G V$ be a space consists of tuples (e, v) satisfying the equivalence relation $(eg, v) = (e, gv)$ where $e \in EG,\ v \in V$. This construction is called Borel construction.

Proposition 3.4.2 *Let $\pi : EG \times_G V \to BG$ be a map defined by $\pi(e, v) = e$. Then, $\xi = (\pi : EG \times_G V \to BG)$ is an $\mathbb{R}^n$ bundle with fibers V.*

Each representation V is given by a group homomorphism $\rho : G \to O(n)$. Therefore, associated to each bundle ξ, there is a map:

$$B\rho : BG \to BO_n(\mathbb{R})$$

which is covered by a bundle map from ξ to the vector bundle

$$\eta = (\pi_\eta : V_n^o(\mathbb{R}^\infty) \times \mathbb{R}^n \to BO_n(\mathbb{R}))$$

which means, we have a commuting diagram:

$$\begin{tikzcd} EG \times_G V \arrow[r, "E\rho"] \arrow[d] & V_n^o(\mathbb{R}^\infty) \times \mathbb{R}^n \arrow[d] \\ BG \arrow[r, "B\rho"] & BO_n(\mathbb{R}) \end{tikzcd}$$

The characteristic classes of a representation V with homomorphism $\rho : G \to O_n(\mathbb{R})$, are defined as the characteristic classes of a real vector bundle $\xi = (\pi : EG \times_G V \to BG)$.

Note that, we have an injective homomorphism

$$\phi : H^*(BO(n), \mathbb{F}_2) \to H^*((\mathbb{R}P^\infty)^n, \mathbb{F}_2) = \mathbb{F}_2[x_1, \ldots, x_n]$$

since there is an inclusion $(\mathbb{Z}/2)^n = (O(1))^n \subseteq O(n)$ and $BO(1) \cong \mathbb{R}P^\infty$. Also, the image is the subring $\mathbb{F}_2[\sigma_1, \ldots, \sigma_n]$ generated by the elementary symmetric polynomials $\sigma_i(x_1, \ldots, x_n)$ of degree i. Then by letting $w_i = \phi^{-1}(\sigma_i) \in H^i(BO(n), \mathbb{F}_2)$, we have

$$H^*(BO(n), \mathbb{F}_2) = \mathbb{F}_2[w_1, \ldots, w_n]$$

Therefore, the ith Stiefel-Whitney class $w_i(V)$ of the representation V given by a group homomorphism $\rho : G \to O_n(\mathbb{R})$ is

$$w_i(V) = (B\rho)^*(w_i) \in H^i(G, \mathbb{F}_2).$$

Similarly, we can define the Euler class of the given representation. For more details about this construction, we refer reader to [6], page 52.

Chapter 4

The Ext Class of A Subset Complex

Throughout this chapter, let G be a finite group and let X be a G set with order $n+1$. Let Δ be an n-simplex whose vertex set is X. In [21], Webb and Reiner define an exact sequence ε^X associated to Δ called *subset complex* which represents the class $\zeta_X \in \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}})$. In the first section of this chapter, we show that ε^X is equivalent to the extension class

$$0 \to H_{n-2}(C_*(S(I_X))) \to C_{n-2}(S(I_X)) \to \cdots \to C_0(S(I_X)) \to H_0(C(S(I_X))) \to 0$$

where $S(I_X)$ is the G-sphere of the augmentation module I_X. Then we consider the mod 2 reduction $\overline{\zeta}_X$ of ζ_X and show that it is equal to the top Stiefel-Whitney class of the augmentation module I_X. For the proof, first we introduce the hypercohomology spectral sequence arising from the double complex $C^{r,s} = \mathrm{Hom}_G(P_r, \mathrm{Hom}(C_s(S_{I_X}), \mathbb{F}_2))$. This spectral sequence collapse at page E^{n+1} and has differential d_n given by multiplication with $\overline{\zeta}_X$. Then, by using spectral sequence isomorphism between Leray-Serre spectral sequence of the fibration $\pi : EG \times_G S(I_X) \to BG$, we conclude that $\overline{\zeta}_X$ is equal to the top Stiefel-Whitney class $w_{\mathrm{top}}(I_X)$.

4.1 The Ext Class of The Subset Complex

Let Δ be an abstract simplicial complex with vertex set V which has order $v_1 < v_2 < \cdots < v_{n+1}$. Then its *oriented chain complex* C^{ori} is constructed by taking

$$C_r^{\text{ori}}(\Delta) = \oplus\ \mathbb{Z} \cdot [v_{i_0}, \ldots, v_{i_r}]$$

where $\{v_{i_0}, \ldots, v_{i_r}\}$ is a r-simplex in Δ and where $v_{i_0} < \cdots < v_{i_r}$ [[22], Chapter 1 §5]. If the terms $v_{i_0}, \ldots, v_{i_r}$ are occurred in the wrong order, the symbol $[v_{i_0}, \ldots, v_{i_r}]$ will be identified with $\text{sign}(\sigma)[v_{i_0}, \ldots, v_{i_r}]$ for any permutation σ, and if the terms are repeated, we will take it to be zero.

Let $\Delta(X)$ be an n-simplex whose vertex set is X. Then, the oriented chain complex of $\Delta(X)$ augmented by a copy of the trivial module is the following exact sequence of $\mathbb{Z}G$-modules

$$\varepsilon^X : \quad 0 \to \tilde{\mathbb{Z}} \to C_{n-1}(\Delta(X)) \to \ldots \cdots \to C_0(\Delta(X)) \to \mathbb{Z} \to 0$$

which represents a class $\zeta_X \in \text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \tilde{\mathbb{Z}})$. In [21], the exact sequence ε^X is called the *subset complex* of X.

Let us consider the boundary $\partial\Delta$ of the simplex $\Delta(X)$ which is a simplicial complex whose simplices are the proper subsets of X. Since Δ is contractible, the following exact sequence is equivalent to ε^X:

$$0 \to H_{n-2}(C_*(\partial\Delta)) \to C_{n-1}(\partial\Delta) \to \cdots \to C_0(\partial\Delta) \to H_0(C(\partial\Delta)) \to 0.$$

Now we will show that the polyhedra obtained from $\partial\Delta$ is G-homeomorphic to the $(n-1)$-sphere $S(I_X)$ of I_X which means that their chain complexes are chain homotopic. The following observation is due to L. Barker and E. Yalçın [4]:

Lemma 4.1.1 ([4], Lemma 2.2) *Let $<,>_G$ be the G-invariant inner product for I_X where $I_X = \text{Ker}\{\mathbb{R}X \to \mathbb{R}\}$ as before. Then there is a G-homeomorphism between $S(I_X)$ and $|\partial\Delta|$ where $S(I_X)$ is the G-sphere consisting of elements v in I_X with $||v||_G = 1$.*

Proof: Let $X = \{x_1, \ldots, x_{n+1}\}$. Then,

$$\begin{aligned} I_X &= \mathrm{Ker}\{\epsilon : \mathbb{R}X \to \mathbb{R}\} = \mathrm{Ker}\{(\alpha_1, \ldots, \alpha_{n+1}) \to \sum_{i=1}^{n+1} \alpha_i,\ \alpha_i \in \mathbb{R}\} \\ &= \{v \in \mathbb{R}X \text{ such that } v = (\alpha_1, \ldots, \alpha_n,\ -(\alpha_1 + \cdots + \alpha_n))\} \end{aligned}$$

Since the map sending each x_i to $e_i = (0, \ldots, 1, 0, \ldots, 0)$ defines a homeomorphism between $\partial\Delta(X)$ and the boundary $\partial\Delta^n$ of the standard n-simplex, we can regard I_X as a space with normal $(1, \ldots, 1)$ and x_i as $(0, \ldots, \underbrace{1}_{i}, 0, \ldots, 0)$.

Let v_i be the unit vector of the projection of e_i into I_X, that is

$$v_i = (-\frac{1}{\sqrt{n^2+n}}, \ldots, \underbrace{\sqrt{\frac{n}{n+1}}}_{i}, \ldots, -\frac{1}{\sqrt{n^2+n}})$$

for each i. Then, the set $\{v_1, \ldots, v_{n+1}\}$ is an affinely independent set of vectors in $S(I_X)$. Let $\Delta'(X)$ be the n-simplex with vertex set $\{v_1, \ldots, v_{n+1}\}$. Let us define a map $\phi : \Delta(X) \to \Delta'(X)$ by $\phi(x_i) = v_i$. Obviously, the map ϕ is a homeomorphism. To show that it is a G-map, let $g \cdot x_i = x_j$ for arbitrary g in G and $x_i \in X$. Then

$$\begin{aligned} g \cdot v_i &= \frac{1}{\sqrt{n^2+n}}(g \cdot [(-1, \ldots, \underbrace{n+1}_{i}, \ldots, -1)]) \\ &= \frac{1}{\sqrt{n^2+n}}[-1, \ldots, \underbrace{n+1}_{j}, \ldots, -1] = v_j, \end{aligned}$$

as desired. Therefore, the map ϕ sends $\partial\Delta(X)$ to the boundary of $\Delta'(X)$ and it induces a G-homeomorphism between $S(I_X) = |\partial\Delta'(X)|$ and $|\partial\Delta(X)|$.

□

It follows from the above lemma that the exact sequence ε^X is equivalent to the following extension

$$0 \to H_{n-1}(S(I_X)) \to C_{n-1}(S(I_X)) \to \cdots \to C_1(S(I_X)) \to H_0(S(I_X)) \to 0. \tag{4.1}$$

The aim of this thesis is to determine the cases where the mod 2 reduction of the extension class ε^X is non-zero. For this purpose, from now on, we restrict our

attention to the mod 2 reduction of ζ_X which we denote by $\overline{\zeta}_X$. From the above equality, $\overline{\zeta}_X$ is represented by the following exact sequence

$$\overline{\varepsilon}^X : \quad 0 \to \mathbb{F}_2 \to \overline{C}_{n-1}(S(I_X)) \to \cdots \to \overline{C}_0(S(I_X)) \to \mathbb{F}_2 \to 0$$

which is the mod 2 reduction of (4.1). Note that the dual of the above extension

$$(\overline{\varepsilon}^X)^* : \quad 0 \to \mathbb{F}_2 \to \overline{C}^0(S(I_X)) \to \cdots \to \overline{C}^{n-1}(S(I_X)) \to \mathbb{F}_2 \to 0$$

also represents $\zeta_X \in \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{F}_2, \mathbb{F}_2)$.

4.2 The HyperCohomology Spectral Sequence

Let C and D be chain complexes of R-modules. Let P be a projective resolution of C. Then by regarding $\mathrm{Hom}(P, D)$ as a chain complex, we obtain a spectral sequence with E_2-page

$$E_2^{r,s} = \mathrm{Ext}^r_{RG}(C, H^s(D)) \Rightarrow \mathrm{Ext}^{r+s}_{RG}(C, D).$$

This spectral sequence is called *the hypercohomology spectral sequences.*

Let $\varepsilon : P \to \mathbb{F}_2$ be a projective resolution of $\mathbb{F}_2$ over $\mathbb{F}_2[G]$ with differential ∂. Then there is a double complex $C = (C^{r,s})_{r,s\geq 0}$ with chains

$$C^{r,s} = \mathrm{Hom}_G(P_r, \mathrm{Hom}(C_s(S(I_X)), \mathbb{F}_2))$$

and with differentials

$$\delta' : \mathrm{Hom}_G(P_r, \mathrm{Hom}(C_s(S(I_X)), \mathbb{F}_2)) \to \mathrm{Hom}_G(P_r, \mathrm{Hom}(C_{s+1}(S(I_X)), \mathbb{F}_2))$$

is given by $\delta'(f) = \delta \circ f$ and

$$\delta'' : \mathrm{Hom}_G(P_r, \mathrm{Hom}(C_s(S(I_X)), \mathbb{F}_2)) \to \mathrm{Hom}_G(P_{r+1}, \mathrm{Hom}(C_s(S(I_X)), \mathbb{F}_2))$$

is given by $\delta''(f) = f \circ \partial$ where δ is the dual of the differential in $\overline{\varepsilon}_X$.

Proposition 4.2.1 *There is a hypercohomology spectral sequence obtained from the above double complex with E_2 page*

$$\mathrm{E}_2^{*,*} = H^*(G, H^*(C^*(S(I_X), \mathbb{F}_2)) \Rightarrow H^*(G, C^*)$$

which collapses at page E_{n+1} *and which has differential* d_n *given by multiplication with* $\overline{\zeta}_X \in \mathrm{Ext}^n_{\mathbb{F}_2[G]}(\mathbb{F}_2, \mathbb{F}_2)$.

Proof: Let us denote $\mathrm{Hom}(C_i(S(I_X)), \mathbb{F}_2)$ by C^i for convenience. By filtering the column of the double complex, we obtain a spectral sequence with 0-page $E_0^{r,s} = \mathrm{Hom}_G(P_r, C^s)$ and with differential $d_0 := \pm\delta'$. Then,

$$B_0^{r,s} = \delta'(\mathrm{Hom}_G(P_r, C^{s-1}))$$

and

$$Z_0^{r,s} = \mathrm{Ker}(\delta' : \mathrm{Hom}_G(P_r, C^s) \to \mathrm{Hom}_G(P_r, C^{s+1})).$$

Hence $E_1^{r,s} = H^s(\mathrm{Hom}_G(P_r, C^*)) \cong \mathrm{Hom}_G(P_r, H^s(C^*))$ and has differential $d_1 : f \to f \circ \partial$. Therefore,

$$E_2^{r,s} \cong H^r(\mathrm{Hom}_G(P_r, H^s(C^s))) \cong H^r(G, H^s(C^*)).$$

Since $H^s(C^*) = 0$ unless $s = 0$ or $n-1$, we have $E_2 \cong \ldots\cdots \cong E_n$.

Let $a \in Z_{n-1}^{r,n-1}$, then there exist elements $a_0, \ldots, a_{n-1}$ with $a_i \in E_0^{r+i,s-i}$ such that $a_0 = a$ and the following diagram commutes

$$\begin{tikzcd}
\cdots\ P_{r+n} \arrow[r] & P_{r+n-1} \arrow[r] \arrow[d, "a_{n-1}"'] & \cdots \arrow[r] & P_{r+1} \arrow[r] \arrow[d, "a_1"'] & P_r \arrow[r] \arrow[d, "a_0=a"'] & P_{r-1}\ \cdots \\
0 \to H^0 \arrow[r] & C^0 \arrow[r] & \cdots \arrow[r] & C^{n-2} \arrow[r] & C^{n-1} \arrow[r] & H^{n-1} \to 0
\end{tikzcd}$$

and $d_n(a) = \delta''(a_{n-1}) = a_{n-1} \circ \partial$.

Let $\phi : H^*(G, H^{n-1}(C^*)) \to H^{*+n}(G, H^0(C^*))$ be the map which makes following diagram commutes:

$$\begin{tikzcd}
H^*(G, H^{n-1}(C^*)) \arrow[r, "\phi"] \arrow[d, equal, "\varphi"'] & H^{*+n}(G, H^0(C^*)) \arrow[d, equal, "\varphi"'] \\
\mathrm{Ext}^*_{\mathbb{Z}_2[G]}(\mathbb{F}_2, H^{n-1}(C^*)) \arrow[r, "\cup(\overline{\varepsilon}^X)^*"] & \mathrm{Ext}^{*+n}_{\mathbb{Z}_2[G]}(\mathbb{F}_2, H^0(C^*)).
\end{tikzcd}$$

where the bottom map is given by Yoneda splice with $(\overline{\varepsilon}^X)^*$. We claim that $\phi = d_n(a)$. To prove the claim, let us take an arbitrary element $a \in Z_{n-1}^{r,n-1}$. Since $\delta'(a) = 0$, we can regard $a \in \mathrm{Hom}(P_r, H^{n-1})$. Let $\alpha \in \mathrm{Ext}^r_{\mathbb{F}_2[G]}(\mathbb{F}_2, H^{n-1}(C^*))$ be a class with the corresponding exact sequence

$$E: \quad 0 \to H^{n-1} \to F_{r-1} \to \cdots \to F_0 \to \mathbb{F}_2 \to 0$$

which means a is the lift of the identity in the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & P_r \arrow[r] \arrow[d, "a"'] & P_{r-1} \arrow[r] \arrow[d, "a_{-1}"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "a_{-r}"'] & \mathbb{F}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^{n-1} \arrow[r, "\beta"] & F_{r-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & \mathbb{F}_2 \arrow[r] & 0
\end{tikzcd}$$

Therefore, $\phi(a) = \varphi(E \cdot \overline{\varepsilon_1}^X) = a_n$ where a_n is the lifting of identity in the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{r+n} \arrow[r] \arrow[d, "a_n"'] & P_{r+n-1} \arrow[r] \arrow[d, "a_{n-1}"'] & \cdots \arrow[r] & P_r \arrow[r] \arrow[d, "a=a_0"'] & P_{r-1} \arrow[r] \arrow[d, "a_{-1}"'] & \cdots \\
0 \arrow[r] & H^0 \arrow[r] & C^0 \arrow[r] & \cdots \arrow[r] & C^{n-1} \arrow[r] & F_{r-1} \arrow[r] & \cdots
\end{tikzcd}$$

However, since the map between H^0 and C^0 is inclusion, $a_n = \delta''(a_{n-1}) = d_n(a)$. Therefore $d_n = \phi$ and hence d_n is given by multiplication with $\overline{\zeta}_X$ as we claimed.

□

4.3 The Top Stiefel-Whitney Class of the Augmentation Module

It is well known fact that the hypercohomology spectral sequence defined in the previous section is canonically isomorphic to the Leray-Serre spectral sequence of the Borel fibration $\pi : EG \times_G S(I_X) \to BG$ (see [1], page 158). In this section, by using this isomorphism, we show that $\overline{\zeta}_X$ equals to the top Stiefel-Whitney class $w_{\mathrm{top}}(I_G)$.

Proposition 4.3.1 *(Gysin Sequence) Let $\xi = (\pi : E \to B)$ be a vector bundle. Then there is an exact sequence of the form*

$$\begin{aligned} \cdots \longrightarrow H^k(B,\mathbb{F}_2) &\xrightarrow{\cup w_{\text{top}}(\xi)} H^{k+n+1}(B,\mathbb{F}_2) \xrightarrow{\pi^*} H^{k+n+1}(E,\mathbb{F}_2) \\ &\longrightarrow H^{k+1}(B,\mathbb{F}_2) \xrightarrow{\cup w_{\text{top}}(\xi)} H^{k+n+2}(B,\mathbb{F}_2) \longrightarrow \cdots \end{aligned}$$

which is called the Gysin sequence.

Proof: See [19], Page 144.

□

Theorem 4.3.2 *We have $\overline{\zeta}_X = w_{\text{top}}(I_X)$.*

Proof: Since the hypercohomology spectral sequence defined in the previous section is canonically isomorphic to the Leray-Serre spectral sequence of the Borel fibration $\pi : EG \times_G S(I_X) \to BG$, we have $\overline{\zeta}_X = d_n(\mu)$ where μ is the nontrivial class in $H^{n-1}(S(I_X),\mathbb{F}_2)$ and d_n is the nth differential of the Leray-Serre spectral sequence. On the other hand, we have the following long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^k(BG,\mathbb{F}_2) &\xrightarrow{\cup d_n(\mu)} H^{k+n}(BG,\mathbb{F}_2) \\ &\xrightarrow{\pi^*} H^{k+n}(EG \times_G S(I_X),\mathbb{F}_2) \xrightarrow{Q} H^{k+1}(BG,\mathbb{F}_2) \longrightarrow \cdots \end{aligned}$$

by Proposition 2.3. Let us consider $\text{id}_B \in H^0(BG,\mathbb{F}_2)$. Then $\pi^*(\text{id}_B \cdot d_n(\mu)) = 0$, hence from the above Gysin sequence,

$$d_n(\mu) = m \cdot w_{\text{top}}(I_X)$$

for some $m \in H^0(BG,\mathbb{F}_2)$. Similarly, the top Stiefel-Whitney class of the augmentation module equals to

$$w_{\text{top}}(I_G) = t \cdot d_n(\mu)$$

for some $t \in H^0(BG,\mathbb{F}_2)$. Since $H^0(BG,\mathbb{F}_2) \cong \mathbb{F}_2$, we have $w_{\text{top}}(I_G) = d_n(\mu)$ and hence $\overline{\zeta}_X = w_{\text{top}}(I_X)$.

□

Chapter 5

Calculations for Nonabelian 2-Groups

The main purpose of this chapter is to prove that the top Stiefel-Whitney class of an augmentation module $I_{G/1}$ is non-zero when G is a non-abelian 2-group. In the first section, we show that if $w_{\text{top}}(I_{H/K})$ is zero for any subquotient H/K of G, then $w_{\text{top}}(I_{G/1})$ is zero. This provides us an inductive argument to find the 2-groups for which $w_{\text{top}}(I_{G/1})$ is zero. For this purpose, in Section 5.2, we classify 2-groups whose all proper subgroups are abelian and we find that these groups are abelian except the quaternion group Q_8 of order 8 and the modular group M_{2^k} of order $2^k \geq 8$. Then we show that the top Stiefel-Whitney class of the augmentation module $I_{G/1}$ is zero when G is Q_8 or M_{2^k}.

5.1 Basic Properties

In this section, we examine the cohomology class $\overline{\zeta}_X = w_{\text{top}}(I_X)$. After stating some results on general case, we restrict our attention to the case where $X = G/1$ is a transitive G-set with point stabilizer. The main goal of this section is to show that if $w_{\text{top}}(I_{G/1})$ is nonzero, then $w_{\text{top}}(I_{H/K})$ is nonzero for all subquotients H/K of G.

We start with two general observations about the top Stiefel-Whitney class.

Proposition 5.1.1 *Let V and W be $\mathbb{R}G$-modules. Then*

$$w_{\text{top}}(V \oplus W) = w_{\text{top}}(V)w_{\text{top}}(W).$$

Proof: It follows from the Whitney sum formula for the total Stiefel-Whitney class.

□

Proposition 5.1.2 *Let H be a subgroup of G and let V be $\mathbb{R}H$-module. Then*

$$w_{\text{top}}(\text{Ind}_H^G V) = \text{Norm}_H^G(w_{\text{top}}(V))$$

where Norm_H^G *is the Even's Norm map*

$$\text{Norm}_H^G : H^k(H, \mathbb{F}_2) \to H^{|G:H|k}(G, \mathbb{F}_2)$$

in group cohomology.

Proof: See [12], page 333.

□

As a consequence of Proposition 5.1.1, we obtain:

Proposition 5.1.3 ([21], Corollary 7.3) *If X is not a transitive G-set, then* $w_{\text{top}}(I_X) = 0$.

Proof: If X is not a transitive G-set, then the augmentation module I_X contains a copy of the trivial module, hence its top Stiefel-Whitney class is zero.

□

As an immediate corollary, we obtain the following consequence:

Corollary 5.1.4 ([21], Corollary 7.4) *If X is a G-set for which $|X|$ is not a power of some prime, then* $w_{\text{top}}(I_X) = 0$.

Proof: By a standard result in group cohomology, the map $\bigoplus_P \mathrm{Res}_P^G$: $H^*(G, \tilde{\mathbb{F}}_2) \to \bigoplus_P H^*(P, \mathbb{F}_2)$ is injective, where the sum is taken over Sylow p-subgroups P of G (see [[10], 12.10.1]). Since $|X|$ is not a prime power, $X \downarrow_P^G$ can not be transitive for any p-subgroup P of G. This implies that

$$\mathrm{Res}_P^G(w_{\mathrm{top}}(I_X)) = w_{\mathrm{top}}(I_{X\downarrow_P^G}) = 0$$

for every Sylow p-subgroup P of G. By the above injectivity result, we obtain $w_{\mathrm{top}}(I_X) = 0$.

□

As a consequence of Corollary 5.1.4, we see that in order to determine the cases where $\overline{\zeta}_X = w_{\mathrm{top}}(I_X)$ is nonzero, we should consider the transitive G-sets. Recall that if X is a transitive G-set, then it is isomorphic to a coset space G/K for some subgroup K of G.

Definition 5.1.5 *An element ζ of $\mathrm{Ext}^n_{\mathbb{Z}G}(M, N)$ is called an essential class if $\mathrm{Res}_H^G(\zeta) = 0$ for every proper subgroup H of G.*

Theorem 5.1.6 ([21], Theorem 7.5) *Let H and K be a subgroups of G. Then*

$$\mathrm{Res}_H^G(\zeta_{G/K}) = \begin{cases} \zeta_{H/H\cap K} & \text{if } HK{=}G; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if K is contained in the Frattini subgroup G, then $\zeta_{G/K}$ is essential.

Proof: Notice that $(G/K) \downarrow_H^G$ is transitive if and only if $HK = G$ and in that case the stabilizer of G/K equals to $H \cap K$ as a H-set.

The second part of the theorem follows from the fact that for any subgroup H of G, we have $H\Phi(G) \neq G$ where $\Phi(G)$ denotes the Frattini subgroup.

□

In particular, $w_{\mathrm{top}}(I_{G/1})$ is always essential. Thus, the above theorem gives us a way to construct essential classes in cohomology.

From now on we assume that $X = G/1$ is a G-set with point stabilizer. In this case, we denote I_X by I_G and ζ_X by ζ_G.

Proposition 5.1.7 *Let H be a subgroup of G. Then,*

$$I_G \cong \mathrm{Ind}_H^G(I_H) \oplus I_{G/H}$$

where $I_{G/H} = \mathrm{Ker}\{\mathbb{R}[G/H] \to \mathbb{R}\}$ is the augmentation module of $\mathbb{R}[G/H]$.

Proof: Let us define $\phi : I_G \twoheadrightarrow I_{G/H}$ by

$$\phi(\sum_{g\in G} \alpha_g g) = \sum_{t\in T} \beta_t \; tH$$

where T is the set of left transversal of H in G and $\beta_t = \sum_{g\in tH} \alpha_g$. Then,

$$\begin{aligned}
\mathrm{Ker}\ \phi &= \{\sum \alpha_g g \mid \sum_{g\in tH} \alpha_g = 0 \text{ for all } t \text{ in T}\} \\
&= \bigoplus_{t\in T} \{\sum_{g\in tH} \alpha_g g \mid \sum_{g\in tH} \alpha_g = 0\} \\
&= \bigoplus_{t\in T} \{t \sum_{h\in H} \alpha_h h \mid \sum_{h\in H} \alpha_h = 0\} \\
&= \mathrm{Ind}_H^G I_H.
\end{aligned}$$

This gives us an exact sequence of the form

$$0 \to \mathrm{Ind}_H^G I_G \to I_G \to I_{G/H} \to 0$$

which splits since we work over $\mathbb{R}$.

□

Corollary 5.1.8 *If $w_{\mathrm{top}}(I_G)$ is nonzero, then $w_{\mathrm{top}}(I_H)$ is nonzero for every subgroup H of G.*

Proof: The top Stiefel-Whitney class of an augmentation module I_G equals to

$$w_{\mathrm{top}}(I_G) = \mathrm{Norm}_H^G(w_{\mathrm{top}}(I_H)) \cdot w_{\mathrm{top}}(I_{G/H})$$

for any subgroup H of G. Hence, if $w_{\mathrm{top}}(I_H)$ is zero then $w_{\mathrm{top}}(I_G)$ is also zero.

□

Proposition 5.1.9 *Let $\phi : G \twoheadrightarrow G'$ be a surjection between finite groups G and G'. Then*

$$I_G = \mathrm{Ind}_N^G I_N \oplus I_{G'}$$

where N is the kernel of ϕ.

Proof: By Proposition 5.1.7,

$$I_G \cong \mathrm{Ind}_N^G(I_N) \oplus I_{G/N}$$

By the First Isomorphism Theorem in group theory, $I_{G/N} \cong I_{G'}$ as a $\mathbb{R}G$-module.

□

Corollary 5.1.10 *If the top Stiefel-Whitney class of augmentation module I_G is nonzero, then the top Stiefel-Whitney class of augmentation module $I_{H/K}$ of any subquotient H/K of G is nonzero.*

Proof: It follows from Proposition 5.1.7 and Proposition 5.1.9.

□

5.2 Calculations for Non-abelian 2-groups

As it has been shown in the previous section, the top Stiefel-Whitney class of an augmentation module I_G is zero when G is not a p-group. Therefore, to determine the groups for which $\overline{\zeta}_G$ is not equal to zero, one should examine the p-groups. Since $H^*(G, \mathbb{F}_2) = 0$ if G is p-group with $p > 2$, we consider the case where $p = 2$.

The main purpose of this section is to show that the top Stiefel-Whitney class of the augmentation module I_G is zero when G is a non-abelian 2-group. In order to obtain this result, we classify the 2-groups whose all proper subquotients are abelian and we obtain that the only non-abelian 2-groups with this property are

the quaternion group Q_8 of order 8 and modular 2-groups M_{2^k} of order $k \geq 3$. Then, we show that $w_{\text{top}}(I_G)$ is zero in these cases and hence we conclude this section with a theorem which says that the top Stiefel-Whitney class of I_G is zero when G is a non-abelian 2-group.

Theorem 5.2.1 *Let G be a 2-group such that all its proper subquotients are abelian. Then either G is abelian or G is isomorphic to either the quaternion group Q_8 of order 8 or a modular 2-group M_{2^k} of order $k \geq 3$.*

To prove the theorem, first we consider the case where G contains a normal subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ and we show that in this case G is a modular group M_{2^k} with $k \geq 3$. Then, we consider the case where G does not have a normal subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. In this case, G is said to have normal 2-rank one and G is called *Roquette group*. After reducing the proof to this case, we use the classification of Roquette groups given in [11]. We quote this classification here:

Theorem 5.2.2 ([11], Theorem 4.10) *Let G be a 2-group with normal rank one, i.e. if G does not include a normal subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, then G is isomorphic to one of the following groups:*

a) *Cyclic group;*

b) *Dihedral group D_{2^n} $(n \geq 4)$;*

c) *Generalized quaternion group Q_{2^n} $(n \geq 3)$;*

d) *Semi-dihedral group SD_{2^n} $(n \geq 4)$.*

Proof: (***of Theorem 5.2.1***) Let G be a non-abelian 2-group whose all proper subquotients are abelian. We will show that in this case $G \cong Q_8$ or M_{2^k} with $k \geq 3$. Let c be central involution of G. Since $G/\langle c \rangle$ is abelian and G is nonabelian, $\langle c \rangle$ is the commutator group of G. Similarly, any central subgroup of order 2 is

the commutator group and hence G has only one central subgroup of order 2. Therefore, the center $Z(G)$ of G is cyclic.

Now, we will show that G has no normal subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ unless $G \cong M_{2^k}$. Let $E \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ be normal subgroup of G. Since $E \cap Z(G) \neq \{1\}$ and $Z(G)$ contains only one central involution, E has central element c and non-central element a. Let $H = C_G(a)$. Then H is maximal. Indeed, any proper subgroup containing H is abelian and contains a. Hence, it must centralize a. Now, we consider two cases:

Case 1: Assume that $E = H$. Then the order of G equals to

$$|G| = \left|\frac{G}{E}\right| \cdot |E| = 8.$$

The only nonabelian groups of order 8 are D_8 and Q_8. But, Q_8 does not have normal $\mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup. Hence, in this case, G is D_8 which is the modular group of order 8.

Case 2: Assume that $E < H$. Since G is not cyclic, there exists another maximal subgroup, say S, in G. Since $H \neq S$, there is an element $s \in S$ which does not commute with a. Then the subgroup generated by s and a is nonabelian and hence isomorphic to G.

Since the abelian group S is the maximal subgroup of G containing s, we have $S = C_G(s)$ and hence $S \cap H = C_G(a) \cap C_G(s) = Z(G)$. We have

$$\left|\frac{S}{S \cap H}\right| = \left|\frac{SH}{H}\right| = \left|\frac{G}{H}\right| = 2.$$

Since S is abelian and has maximal cyclic subgroup $Z(G)$ of index 2, either S is cyclic or $S \cong \mathbb{Z}_2 \times Z(G)$.

Suppose that $S \cong \mathbb{Z}_2 \times Z(G)$. Let g be a non-central element of order 2 in S and let $K = \langle g \rangle$. Since $g \notin Z(G) = H \cap S$, g can not lies in $C_G(a) = H$, and hence can not commute with a. Therefore the subgroup EK of G is nonabelian, and hence, by assumption, $G = EK$. Let $h \in H \backslash E$. Since $h \in EK$, $h = eg$ for some $e \in E$. Then the commutativity of h and e with a implies the commutativity of g with a. This gives a contradiction. Therefore we can assume

that S is a cyclic subgroup of G which is generated by s and hence $Z(G) = \langle s^2 \rangle$. Let the order of s be 2^{k-1}. Then $k > 2$. Since s does not commute with a, we have $asa = s^m$ for some $1 < m < 2^{k-1}$. Then, since $s^2 \in Z(G)$, we get $s^2 = as^2a = s^{2m}$ which means $2 \equiv 2m \pmod{2^{k-1}}$ and hence $m = 2^{k-2} + 1$. This gives that $G = \langle a, s |\ a^2 = s^{2^{k-1}} = 1,\ asa = s^{2^{k-2}+1} \rangle$ and hence G is the modular group M_{2^k} of order 2^k. Let us show that all proper subquotients of M_{2^k} are abelian. The group M_{2^k} has three maximal subgroups: two cyclic subgroups of order 2^{k-1} generated by s and as and a subgroup isomorphic to $C_{2^{k-2}} \times C_2$ which is given by $\langle s^2, a \rangle$. Since all maximal subgroup of G are abelian, if $H \neq G$ then H/K is abelian for any $K \trianglelefteq H$. Therefore, it is sufficient to show that G/N is abelian for any normal subgroup $N \trianglelefteq G$. Let N be a normal subgroup of G. Since $Z(G) \cap N \neq \{1\}$, we have $\langle s^{2^{k-2}} \rangle \subseteq N$. However, $\langle s^{2^{k-2}} \rangle$ is the commutator group and hence the quotient G/N is abelian.

It remains to consider the case where G has normal 2-rank one, i.e when G does not have any normal subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. The only nonabelian groups of this type are the following 2-groups:

a) Dihedral group $D_{2^n} (n \geq 4)$;

b) Generalized quaternion group $Q_{2^n} (n \geq 3)$;

c) Semi-dihedral group $SD_{2^n} (n \geq 4)$.

by Theorem 5.2.2. But, except Q_8, all these groups have a unique central involution $\langle c \rangle$ where $G/\langle c \rangle$ is a dihedral group. Thus, except Q_8 all these groups have a nonabelian quotient. Therefore, the only non-abelian 2-groups whose all subquotients are abelian are the quaternion group Q_8 of order 8 and modular groups M_{2^k} with $k \geq 3$.

□

Now, we show that $w_{\text{top}}(I_G)$ is zero for the group D_8 and Q_8.

Lemma 5.2.3 *The top Stiefel-Whitney class of the augmentation module I_G is*

zero when $G = D_8$ or Q_8.

Proof: In [1], the cohomology of groups D_8 and Q_8 with $\mathbb{F}_2$ coefficients are given as follows:

$$H^*(D_8, \mathbb{F}_2) \cong \mathbb{F}_2[x, y, w]/(xy)$$

and

$$H^*(Q_8, \mathbb{F}_2) \cong \mathbb{F}_2[e, x, y]/(x^2 + xy + y^2, x^2y + xy^2)$$

where deg x = deg $y = 1$ in both cases and the degree of w and e are 2 and 4 respectively.

Both of these groups have three one dimensional non-trivial $\mathbb{R}G$-modules. Let us denote them by V_1, V_2, V_3 for the group D_8 and V_1', V_2', V_3' for Q_8. Let W be the four dimensional $\mathbb{R}G$-module of D_8 and W' be the four dimensional $\mathbb{R}G$-module of Q_8. Then the top Stiefel-Whitney classes of V_1, V_2, V_3 and W are $x, y, x + y$ and w^2 respectively. Hence,

$$w_{\mathrm{top}}(I_{D_8}) = xy(x + y)w^2 = 0.$$

Similarly, the top Stiefel-Whitney classes of V_1, V_2, V_3 and W are $x, y, x + y$ and e, respectively. Therefore,

$$w_{\mathrm{top}}(I_{Q_8}) = xy(x + y)e = (x^2y + xy^2)e = 0,$$

as desired.

□

Proposition 5.2.4 *Let G be a 2-group which has a subquotient isomorphic to $\mathbb{Z}/8$, then $w_{\mathrm{top}}(I_G)$ is zero.*

Proof: It is sufficient to prove the result for $G \cong \mathbb{Z}/8$. Indeed, if $w_{\mathrm{top}}(I_{\mathbb{Z}/8})$ is zero, then by Corollary 5.1.10, $w_{\mathrm{top}}(I_G)$ is zero for any group which has a subquotient isomorphic to $\mathbb{Z}/8$.

Now let $G = \langle g \rangle$ be a cyclic group of order 8. Let V be the irreducible two dimensional representation of $\mathbb{R}G$ with kernel $\langle g^4 \rangle$. Since

$$H^*(G, \mathbb{F}_2) = \mathbb{F}_2[s] \otimes \wedge(t)$$

where deg $s = 2$ and deg $t = 1$, we have $w_{\text{top}}(V) = \alpha s$ for some $\alpha \in \mathbb{F}_2$. Notice that

$$\text{res}^G_{<g^4>} w_{\text{top}}(V) = w_{\text{top}}(res^G_{<g^4>} V) = w_{\text{top}}(\mathbb{R}) = 0$$

On the other hand, it equals to α $\text{res}^G_{g^4} s$. But $\text{res}^G_{g^4} s \neq 0$ since $\text{Ker} s = 1$ by a standard result in group cohomology. Hence, $w_{\text{top}}(V) = 0$. Since $I_G \cong V \oplus W$ for some $\mathbb{R}G$-module W, we conclude that $w_{\text{top}}(I_G) = 0$.

□

For $k \geq 4$, the modular group M_{2^k} has at least one cyclic subgroup of order 8, hence $w_{\text{top}}(I_{M_{2^k}})$ is zero by the above proposition. We conclude the following:

Theorem 5.2.5 *If G is non-abelian 2-group, then $w_{\text{top}}(I_G) = 0$.*

Proof: Let G be the minimal non-abelian 2-group such that $w_{\text{top}}(I_G)$ is non-zero. Then, all the subquotients of G must be abelian and by Theorem 5.2.1, G is either Q_8 or M_{2^k} with $k \geq 3$. This gives a contradiction since $w_{\text{top}}(I_G)$ is zero for these groups.

□

Chapter 6

Calculations for Abelian 2-Groups

In this chapter, we calculate the top Stifel-Whitney class of the augmentation module I_G when G is an abelian 2-group. In Chapter 5, we have shown that $w_{\text{top}}(I_G)$ is zero when G has a subquotient isomorphic to $\mathbb{Z}/8$. We show that if $G = \mathbb{Z}/4 \times \mathbb{Z}/4$, then $w_{\text{top}}(I_G)$ is again zero and hence for every group which has a subquotient isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/4$, the top Stiefel-Whitney clas is zero. This gives that if $G \neq (\mathbb{Z}/2)^n \times \mathbb{Z}/4$ then $w_{\text{top}}(I_G)$ is zero. Then, we calculate the top Stiefel-Whitney class for the group $G = (\mathbb{Z}/2)^n \times \mathbb{Z}/4$ and we show that $w_{\text{top}}(I_G)$ is non-zero for these groups. As a result, we obtain the main theorem of this thesis:

Theorem 6.0.6 *Let G be a 2-group. Then $w_{\text{top}}(I_G)$ is non-zero if and only if $G = (\mathbb{Z}/2)^n \times \mathbb{Z}/4$.*

6.1 Abelian 2-groups

In [25], Turygin calculates the top Stiefel-Whitney class of the augmentation module for elementary abelian 2-groups by using the Whitney product formula

and the tensor product formula. We apply similar argument to find the top Stiefel-Whitney class of an augmentation module of all abelian 2-groups.

Let f be the polynomial

$$f(a_1, a_2, \ldots, a_m) = \prod_{(\alpha_1,\ldots,\alpha_m)\in(\mathbb{F}_2)^m\setminus\{0\}} (\alpha_1 a_1 + \cdots + \alpha_m a_m)$$

for tuples $(a_1, \ldots, a_n)$ where a_i's are independent variables. For example, for $m = 3$, we have

$$f(a_1, a_2, a_3) = a_1a_2a_3(a_1 + a_2)(a_1 + a_3)(a_2 + a_3)(a_1 + a_2 + a_3).$$

Then, the top Stiefel-Whitney class of elementary abelian 2-groups is given by the following formula.

Lemma 6.1.1 ([25], Lemma 2.1) *Let $G = (\mathbb{Z}/2)^n$, then*

$$w_{\text{top}}(I_G) = f(x_1, \ldots, x_n)$$

where $\{x_1, \ldots, x_n\}$ is a set of generators of the cohomology ring $H^*(BG, \mathbb{F}_2)$.

Proof: Let $\{g_1, \ldots, g_n\}$ be the generators of G that corresponds to the cohomology classes $\{x_1, \ldots, x_n\}$. Let $\pi_i : G \to \langle g_i\rangle$ be the projection to the ith coordinate. Note that, $\mathbb{R}[\mathbb{Z}_2] = V \oplus (V \otimes_{\mathbb{R}[\mathbb{Z}_2]} V) = V \otimes V^2$ where V is a representation given by $\rho(-1)v = -v$ and hence $V^2 \cong \mathbb{R}$. For the ith coordinate, let us write $\mathbb{R}[\mathbb{Z}_2] = V_i \otimes V_i^2$ with the corresponding representation ρ_i. Therefore, we have the following decompositions

$$\mathbb{R}G \cong \bigotimes_{i=1}^{n}(V_i \oplus V_i^2) \cong \bigoplus_{(\gamma_1,\ldots,\gamma_n)\in S} (V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n})$$

where $S = \{(\gamma_1, \ldots, \gamma_n)|\ \gamma_i \in \{1, 2\}$ for all $i\}$. Let $S^* = S\setminus(2, \ldots, 2)$. Then

$$I_G \cong \bigoplus_{(\gamma_1,\ldots,\gamma_n)\in S^*} (V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n})$$

By the Whitney product formula and the tensor product formula, we have

$$w_{\text{top}}(I_G) = \prod_{(\alpha_1,\ldots,\alpha_n)\in(\mathbb{F}_2)^n\setminus\{0\}} (\alpha_1 w_1(V_1) + \cdots + \alpha_n w_1(V_n)).$$

where $\alpha_i = \gamma_i$ in mod 2. On the other hand, $w_1(V_i) = x_i$ since the map $B\rho_i : BG \to \mathbb{R}P^\infty$ is just the projection to the ith coordinate. Therefore,

$$w_{\text{top}}(I_G) = f(x_1, \ldots, x_n).$$

□

When we calculate $w_{\text{top}}(I_G)$, we use the following combinatorial lemma frequently.

Lemma 6.1.2 *For any indeterminates $x_1, \ldots, x_n$ and z, we have*

$$\prod_{i=1}^{n}(1 + x_i + z) = \sum_{i=0}^{n}\left[\sum_{j=0}^{i}\binom{n-j}{i-j}y_j\right]z^{n-i} \tag{6.1}$$

where $y_i = \sum_{1\leq j_1<\cdots<j_i\leq\ n} x_{j_1}\ldots x_{j_i}$ for $1 \leq i \leq n$ and $y_0 = 1$.

Proof: We will argue by induction on the number n. The case $n = 2$ is trivial. Now, for the induction step, we assume

$$\prod_{i=1}^{n-1}(1 + x_i + z) = \sum_{i=0}^{n-1}\left[\sum_{j=0}^{i}\binom{n-1-j}{i-j}y'_j\right]z^{n-1-i}$$

where $y'_i = \sum_{1\leq j_1<\cdots<j_i\leq\ n-1} x_{j_1}\cdots x_{j_i}$. Therefore, the left hand side of the (6.1) equals to

$$\left[\sum_{i=0}^{n-1}\left[\sum_{j=0}^{i}\binom{n-1-j}{i-j}y'_j\right]z^{n-1-i}\right]\cdot\left(1 + x_n + z\right)$$

Then, the coefficients at z^{n-i} equals to

$$\begin{aligned}
&= (1 + x_n)\left[\sum_{j=0}^{i-1}\binom{n-1-j}{i-1-j}y'_j\right] + \left[\sum_{j=0}^{i}\binom{n-1-j}{i-j}y'_j\right] \\
&= \left[\sum_{j=0}^{i-1}\binom{n-j}{i-j}y'_j\right] + \binom{n-1-i}{0}y'_i + x_n\left[\sum_{j=0}^{i-1}\binom{n-1-j}{i-1-j}y'_j\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{j=0}^{i}\binom{n-j}{i-j}y'_j\right] + \left[\sum_{j=0}^{i-1}\binom{n-1-j}{i-1-j}x_n y'_j\right] \\
&= \left[\sum_{j=1}^{i}\binom{n-j}{i-j}(y'_j + y'_{j-1}x_n)\right] + \binom{n}{i}y_0 \\
&= \left[\sum_{j=0}^{i}\binom{n-j}{i-j}y_j\right]
\end{aligned}
$$

as desired.

□

Lemma 6.1.3 *Let V_i be a two dimensional real representation of G with total Stiefel-Whitney class $w(V_i) = 1 + s_i$ where s_i has degree 2, for $1 \leq i \leq n$. Then,*

$$w(V_1 \otimes \cdots \otimes V_n) = 1 + (s_1 + \cdots + s_n)^{2^{n-1}}.$$

Proof: For each V_i, let us denote the corresponding vector bundles by ξ_i. Let $\xi = \xi_1 \otimes \cdots \otimes \xi_{n-1}$. Then, by inductive hypothesis

$$w(\xi) = 1 + s_1^{2^{n-2}} + \cdots + s_{n-1}^{2^{n-2}}.$$

By Splitting principle, we can regard the kth Stiefel-Whitney class of the vector bundle ξ as the kth elementary symmetric function of some indeterminates, say $a_1, \ldots, a_{2^{n-1}}$. Therefore, by assumption, we have

$$w_k(\xi) = \sum_{i_1 < \cdots < i_k} a_{i_1} \ldots a_{i_k} = 0$$

for $1 \leq k \leq 2^{n-1} - 1$ and,

$$w_{2^{n-1}}(\xi) = a_1 \ldots a_{2^{n-1}} = s_1^{2^{n-2}} + \cdots + s_{n-1}^{2^{n-2}}.$$

Similarly, for the indeterminates b_1, b_2, we have

$$w_1(\xi_n) = b_1 + b_2 = 0 \text{ and } s_n = w_2(\xi_n) = b_1 b_2.$$

Then, by the tensor product formula,

$$\begin{aligned} w(\xi \otimes \xi_n) &= \prod_{i=1}^{2^{n-1}} (1 + a_i + b_1)(1 + a_i + b_2) \\ &= \prod_{i=1}^{2^{n-1}} \big(1 + a_i^2 + b_1 b_2 + \underbrace{(b_1 + b_2)}_{0}(1 + a_i) + \underbrace{2a_i}_{0}\big) \\ &= \prod_{i=1}^{2^{n-1}} (1 + a_i^2 + s_n) \end{aligned}$$

By Lemma 6.1.2,

$$w(\xi \otimes \xi_n) = \sum_{i=0}^{2^{n-1}} \left[\sum_{j=0}^{i} \binom{2^{n-1} - j}{i - j} c_j \right] s_n^{2^{n-1}-i} \tag{6.2}$$

where

$$c_j = \sum_{1 \le k_1 < \cdots < k_j \le\ 2^{n-1}} a_{k_1}^2 \dots a_{k_j}^2.$$

Since we are working with $\mathbb{F}_2$ coefficients ,

$$c_j = \Big(\sum_{1 \le k_1 < \cdots < k_j \le\ 2^{n-1}} a_{k_1} \dots a_{k_j} \Big)^2 = w_j^2.$$

Hence it is nonzero only for $j = 0$ and $j = 2^{n-1}$. Therefore, the equation (6.2) becomes

$$\begin{aligned} w(\xi \otimes \xi_n) &= s_n^{2^{n-1}} + \left[\sum_{i=1}^{2^{n-1}-1} \binom{2^{n-1}}{i} s_n^{2^{n-1}-i} \right] + \sum_{j=0}^{2^{n-1}} c_j \\ &= s_n^{2^{n-1}} + \left[\sum_{i=1}^{2^{n-1}-1} \binom{2^{n-1}}{i} s_n^{2^{n-1}-i} \right] + 1 + c_{2^{n-1}}. \end{aligned}$$

Since $\binom{2^{n-1}}{i} \equiv 0 \bmod 2$ for $1 \le i \le 2^{n-1} - 1$, we have

$$\begin{aligned} w(\xi \otimes \xi_n) &= 1 + c_{2^{n-1}} + s_n^{2^{n-1}} = 1 + a_1^2 \dots a_{2^{n-1}}^2 + x_n^{2^{n-1}} \\ &= 1 + (a_1 \dots a_{2^{n-1}})^2 + s_n^{2^{n-1}} = 1 + (s_1 + \cdots + s_{n-1})^{2^{n-1}} + s_n^{2^{n-1}} \\ &= 1 + (s_1 + \cdots + s_n)^{2^{n-1}}. \end{aligned}$$

□

Lemma 6.1.4 *Let V_i be a two dimensional real representation with total Stiefel-Whitney class $w(V_i) = 1 + s_i$ where s_i has degree 2 for $1 \leq i \leq k$. If V_j is a non-trivial one dimensional real representation with $w(V_j) = 1 + s_j$ for $k+1 \leq j \leq n$, then*

$$w_{\text{top}}(V_1 \otimes \cdots \otimes V_n) = \left(s_1 + \cdots + s_k + x_{k+1}^2 + \cdots + x_n^2\right)^{2^{k-1}}.$$

Proof: Let us denote the tensor product $V_1 \otimes \cdots \otimes V_n$ by X, for convenience. Let $\eta = \xi_1 \otimes \cdots \otimes \xi_k$ and $\theta = \xi_{k+1} \otimes \cdots \otimes \xi_n$ where ξ_i is the vector bundle corresponding to the module V_i. From previous calculations, we know that

$$w(\eta) = 1 + (s_1 + \cdots + s_k)^{2^{k-1}} \text{ and } w(\theta) = 1 + x_{k+1} + \cdots + x_n.$$

Let $a_1, \ldots, a_{2^k}$ be the indeterminates of the elementary symmetric functions $w_1(\eta), \ldots, w_{2^k}(\eta)$. Then by Proposition 3.3.9,

$$w(X) = w(\eta \otimes \theta) = \prod_{i=1}^{2^k} (1 + x_{k+1} + \cdots + x_n + a_i)$$

and hence

$$\begin{aligned} w_{\text{top}}(X) &= \prod_{i=1}^{2^k} (x_{k+1} + \cdots + x_n + a_i) = \sum_{i=0}^{2^k} (x_{k+1} \cdots + x_n)^i \; w_{2^k - i}(\eta) \\ &= w_{2^k}(\eta) + (x_{k+1} \cdots + x_n)^{2^k} = w_{2^k}(\eta) + x_{k+1}^{2^k} + \cdots + x_n^{2^k} \\ &= (s_1 + \cdots + s_k)^{2^{k-1}} + x_{k+1}^{2^k} + \cdots + x_n^{2^k} \\ &= \left(s_1 + \cdots + s_k + x_{k+1}^2 + \cdots + x_n^2\right)^{2^{k-1}}. \end{aligned}$$

□

Now, let us consider the group $\mathbb{Z}/4$. The group algebra $\mathbb{R}(\mathbb{Z}/4)$ has decomposition

$$\mathbb{R}(\mathbb{Z}/4) = \mathbb{R} \oplus V \oplus W$$

where V is the nontrivial one dimensional module with $V^2 = \mathbb{R}$ and W is the unique two dimensional module. By Proposition 2.4.8,

$$H^*(\mathbb{Z}/4, \mathbb{F}_2) \cong \mathbb{F}_2[s] \otimes \wedge(t)$$

with $\deg(s) = 2$ and $\deg(t) = 1$. Then, the total Stiefel-Whitney classes of modules V and W are

$$w(V) = 1 + t \text{ and } w(W) = 1 + s.$$

For $G = (\mathbb{Z}/4)^2$, let V_i and W_i be the non-trivial representations with dimensions one and two respectively, which correspond to the ith term in the product. Let θ_i and η_i^* be the vector bundles associated to the representations V_i and W_i, respectively. By Corollary 2.4.9,

$$H^*(G, \mathbb{F}_2) \cong \mathbb{F}_2[s_1, s_2] \otimes \wedge(t_1, t_2)$$

where $w(V_i) = 1 + t_i$ and $w(W_i) = 1 + s_i$ for $i = 1, 2$.

Proposition 6.1.5 *If G is an abelian 2-group which has $\mathbb{Z}/4 \times \mathbb{Z}/4$ subquotient, then $w_{\text{top}}(I_G)$ is zero.*

Proof: If $w_{\text{top}}(I_G)$ is zero when $G = \mathbb{Z}/4 \times \mathbb{Z}/4$, then $w_{\text{top}}(I_G)$ is zero for a group which has $G = \mathbb{Z}/4 \times \mathbb{Z}/4$ subquotient. Therefore, it is sufficient to prove result for $G = \mathbb{Z}/4 \times \mathbb{Z}/4$. The group algebra $\mathbb{R}G$ has the following decomposition

$$\mathbb{R}G \cong \bigotimes_{i=1}^{2} V_i \oplus V_i^2 \oplus W_i \cong \bigoplus_{X_i \in \{V_i, V_i^2, W_i\}} X_1 \otimes X_2.$$

when $G = \mathbb{Z}/4 \times \mathbb{Z}/4$. Let V be the following submodule of $\mathbb{R}G$

$$V = \bigoplus_{\{\gamma_1, \gamma_2\} \in S \setminus \{(2,2)\}} V_1^{\gamma_1} \otimes V_2^{\gamma_2},$$

where $S = \{(\gamma_1, \gamma_2) |\ \gamma_i \in \{1, 2\} \text{ for } i = 1, 2\}$. Then $I_G \cong V \oplus W$ for some submodule W. We have that

$$w_{\text{top}}(V) = f(t_1, t_2) = t_1 t_2 (t_1 + t_2) = t_1^2 t_2 + t_1 t_2^2 = 0$$

and hence $w_{\text{top}}(I_G) = w_{\text{top}}(V) w_{\text{top}}(W) = 0$ as claimed.

□

We have shown that the top Stiefel-Whitney class of the augmentation module of G is zero when G is an abelian 2-group which is not isomorphic to the group

$(\mathbb{Z}/2)^n \times \mathbb{Z}/4$. Now, we will show that the converse is also true. For $G \cong (\mathbb{Z}/2)^n \times (\mathbb{Z}/4)$, let V_i be the one dimensional non-trivial representation corresponding to the ith term in the product and let W_{n+1} be the irreducible two dimensional representation of $\mathbb{Z}/4$ corresponding to the $(n+1)^{th}$ term of the product. By Corollary 2.4.9, the group G has the following cohomology with coefficients in $\mathbb{F}_2$,

$$H^*(G, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_n, s] \otimes \wedge[t]$$

where $w_{\text{top}}(V_i) = x_i$ for $1 \leq i \leq n$, $w_{\text{top}}(V_{n+1}) = t$ and $w_{\text{top}}(W_{n+1}) = s$. Then the top Stiefel-Whitney class of the augmentation module of G is given by the following proposition.

Proposition 6.1.6 *Let $G \cong (\mathbb{Z}/2)^n \times \mathbb{Z}/4$. Then*

$$w_{\text{top}}(I_G) = f(x_1, \ldots, x_n, t)\frac{f(x_1^2, \ldots, x_n^2, s)}{f(x_1^2, \ldots, x_n^2)}.$$

Proof: Let S_k be the set of k-tuples $(\gamma_1, \cdots, \gamma_k)$ such that $\gamma_i \in \{1, 2\}$ for all i and let $S_k^* = S_k \backslash \{(2, \ldots, 2)\}$. Let V be the following submodule of I_G

$$V \cong \bigoplus_{(\gamma_1, \cdots, \gamma_{n+1}) \in S_{n+1}^*} V_1^{\gamma_1} \otimes \cdots \otimes V_{n+1}^{\gamma_{n+1}}.$$

Then, $w_{\text{top}}(I_G) = w_{\text{top}}(V) w_{\text{top}}(W)$ where W is the complement of V in I_G. Since V is the direct sum of tensor products of one dimensional modules, the top Stiefel-Whitney class of V is $f(x_1, \ldots, x_n, t)$. Notice that the complement W is the following submodule of I_G

$$W \cong \bigoplus_{(\gamma_1, \cdots, \gamma_n) \in S_n} V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n} \otimes W_{n+1}.$$

Therefore, by Proposition 6.1.4, the top Stiefel-Whitney class of W is given by

$$\begin{aligned} w_{\text{top}}(W) &= \prod_{(\gamma_1, \cdots, \gamma_n) \in S_n} w_{\text{top}}(V_1^{\gamma_1} \otimes \cdots \otimes V_n^{\gamma_n} \otimes W_{n+1}) \\ &= \prod_{(\alpha_1, \cdots, \alpha_n) \in (\mathbb{F}_2)^n} (\alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 + s) \\ &= \frac{f(x_1^2, \ldots, x_n^2, s)}{f(x_1^2, \ldots, x_n^2)} \end{aligned}$$

and hence

$$w_{\text{top}}(I_G) = f(x_1, \ldots, x_n, t)\frac{f(x_1^2, \ldots, x_n^2, s)}{f(x_1^2, \ldots, x_n^2)}.$$

□

Corollary 6.1.7 *The top Stiefel-Whitney class of augmentation module I_G is non-zero when $G = (\mathbb{Z}/2)^n \times \mathbb{Z}/4$*

Proof: Since $t^2 = 0$, we have

$$f(x_1, \ldots, x_{n-1}, t) = t\big(f(x_1, \ldots, x_{n-1})\big)^2.$$

This gives

$$\begin{aligned} w_{\text{top}}(I_G) &= f(x_1, \ldots, x_{n-1}, t)\frac{f(x_1^2, \ldots, x_{n-1}^2, s)}{f(x_1^2, \ldots, x_{n-1}^2)} \\ &= t\big(f(x_1, \ldots, x_{n-1})\big)^2\frac{f(x_1^2, \ldots, x_{n-1}^2, s)}{f(x_1^2, \ldots, x_{n-1}^2)}. \end{aligned}$$

It is easy to see that $w_{\text{top}}(I_G)$ is not zero since all the polynomials involved in this product are non-zero.

□

We have shown that the converse of the above corollary is also true and hence this completes the proof of our main theorem.

Bibliography

[1] A. Adem and R. J. Milgram, *Cohomology of Finite Groups,* Springer-Verlag, Berlin Heidelberg, 2004.

[2] F. Altunbulak, *L_ζ-modules and a Theorem of Jon F. Carlson*, thesis 2004.

[3] M. A. Armstrong, *Basic Topology,* Springer-Verlag, New York , 1983.

[4] L. Barker and E. Yalçın, *Units in the Burnside Ring and Group Actions on Spheres,* unpublished notes.

[5] D. J. Benson, *Representations and Cohomology I: Basic Representation Theory of Finite Groups and Associative Algebras*, Cambridge University Press, Cambridge, 1991.

[6] D. J. Benson, *Representations and Cohomology II: Cohomology of Groups and Modules.*, Cambridge University Press, Cambridge, 1991.

[7] D. J. Benson, J. F. Carlson, *Projective Resolutions and Poincare Duality Complexes.*, Trans. A.M.S 342 (1994), no. 2, 447-488.

[8] K. S. Brown, *Cohomology of Groups,* Springer-Verlag, Berlin, 1982.

[9] J. F. Carlson, L. Townsley, L. Valeri-Elizondo, and M. Zhang, *Cohomology Rings of Finite Groups*, Kluwer Academic Publisher, Dordrecht; The Netherlands, 2003.

[10] H. Cartan and S. Eilenberg, *Homological Algebra,* Princeton University Press, 1956.

[11] D. Gorenstein, *Finite Groups,* Harper and Row, Chelsea, 1968.

[12] J. Gunarwardena, B. Kahn and C. Thomas, *Stiefel-Whitney Classes of Real Representations of Finite Groups,* J. Algebra 126 (1989), no.2, 327-347.

[13] A. Hatcher, *Algebraic Topology,* Cambridge University Press, Cambridge, 2002.

[14] D. Husemoller, *Fibre Bundles,* Springer-Verlag, NewYork; Berlin, 1975.

[15] I. M. Isaacs, *Algebra,* Brooks/Cole Publishing Company, Pasific Grove; California, 1994.

[16] G. D. James and M. Liebeck, *Representations and Characters of Groups,* Cambridge University Press, Cambridge, 1993.

[17] S. Mac Lane, *Homology,* Springer-Verlag, Berlin Heidelberg , 1975.

[18] J. McCleary, *A User's Guide to Spectral Sequences,* Cambridge University Press, Cambridge, 2001.

[19] J. W. Milnor and J. D. Stasheff, *Characteristic Classes,* Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1974.

[20] H. Osborn, *Vector Bundles: Foundations and Stiefel-Whitney Classes,* Academic Press, New York, 1982.

[21] V. Reiner and P. Webb, *The Combinatorics of the Bar Resolution in Group Cohomology,* J. Pure Appl. Algebra 190 (2004), no. 1-3, 291-327.

[22] B. E. Sagan, *The Symmetric Group, Representations, Combinatorial Algorithms, and Symmetric Functions,* Spring-Verlag, New York, 1966.

[23] E. H. Spanier, *Algebraic Topology,* Wadsworth and Brooks/Cole, Pacific Grove, CA, 1991.

[24] C. B. Thomas, *Characteristic Classes and the Cohomology of Finite Groups,* Cambridge University Press, Cambridge, 1986.

[25] Y. A. Turygin, *A Borsuk-Ulam Type Theorem for* $(\mathbb{Z}/p)^k$*-Actions on Products of (mod p) Homology Spheres,* preprint 2005.