CHARACTERIZATION OF
CENTRAL UNITS IN
INTEGRAL GROUP RING OF SIMPLE GROUPS

BY

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

This study consists of four chapters. In the first chapter, the problem is introduced as a historical advance of the literatures.

In the second chapter, some basic facts definitions and properties about group G and group ring RG are given.

In the third chapter, some main results about central units of integral group rings are given.

Finally, central units of integral group rings of alternating groups A_n (for small n) are characterized.

Keywords: Normalizer Problem, Normalizer of a subgroup, Centralizer of a subgroup, Central Units of a subgroup, Generators of Central Units.

BASİT GRUPLARIN İNTEGRAL GRUP HALKASINDAKİ MERKEZİ BİRİMSELLERİNİN BELİRLENİŞİ

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ÖZET

Bu çalışma dört bölümden oluşmaktadır. Birinci bölümde, problem tarihsel gelişim süreci içerisinde tanıtılmaktadır.

İkinci bölümde, G grubu ve RG grup halkası ile ilgili temel tanım, sonuç ve özellikler verilmektedir.

Üçüncü bölümde, integral grup halkalarının merkezi birimselleri ile ilgili bazı temel sonuçlar verilmektedir.

Son olarak, A_n alterne gruplarının integral grup halkasındaki merkezi birimselleri belirlenmektedir.

Anahtar Kelimeler: Normalleyen Problemi, Bir altgrubun normalleyeni, Bir altgrubun merkezleyeni, Bir altgrubun merkezi birimselleri, Merkezi birimsellerin üreteçleri.

DEDICATION

To my parents and grandparents

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CHAPTER 1

INTRODUCTION

Given a finite group G and the ring of integers, one can form the integral group ring $\mathbb{Z}G$. Since $\mathbb{Z}G$ is a ring with unity we can talk about the unit group of integral group ring $\mathbb{Z}G$. The unit group $\mathcal{U}(\mathbb{Z}G)$, consisting of all invertible elements in $\mathbb{Z}G$, plays a very important role in studying the relation between the group-theoretic structure of G and its group ring $\mathbb{Z}G$. Let us consider augmentation map:

$$\varepsilon:\ \mathbb{Z}G \to \mathbb{Z}$$
$$\sum \gamma_g g \mapsto \sum \gamma_g$$

ε is a ring homomorphism. If we restrict the augmentation map from $\mathbb{Z}G$ to $\mathcal{U}(\mathbb{Z}G)$,

$$\bar{\varepsilon}: \mathcal{U}(\mathbb{Z}G) \to \mathcal{U}(\mathbb{Z})$$
$$\sum \gamma_g g \mapsto \sum \gamma_g$$

we can get a multiplicative group homomorphism. The kernel of this homomorphism is called *normalized units* or *unit group with augmentation 1* and it is denoted by $\mathcal{U}_1(\mathbb{Z}G)$.

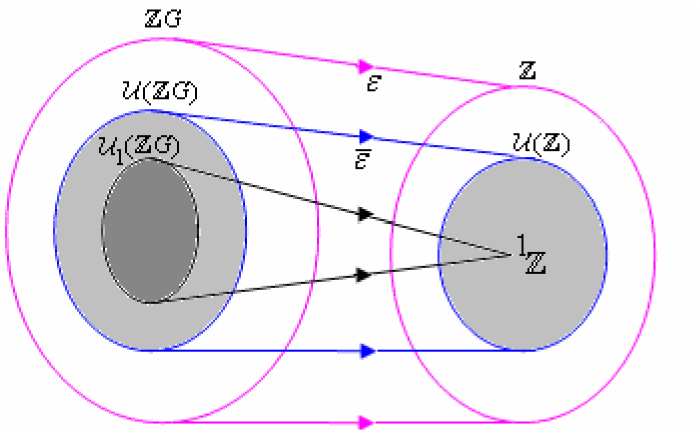

Figure 1.1. Augmentation Mapping

Let us denote $\mathcal{U}_1 = \mathcal{U}_1(\mathbb{Z}G)$ then we can define the normalizer of G in $\mathcal{U}_1$ by $\mathcal{N}_{\mathcal{U}_1}(G) = \{u \in \mathcal{U}_1(\mathbb{Z}G) : u^{-1}Gu = G\}$. Since the center of the normalized units of $\mathbb{Z}G$; $\mathcal{Z}(\mathcal{U}_1)$ is a normal subgroup of $\mathcal{U}_1$ and G is a subgroup of $\mathcal{U}_1$. Then $G.\mathcal{Z}(\mathcal{U}_1)$ is a subgroup of $\mathcal{U}_1$ which normalizes G. That's why $G.\mathcal{Z}(\mathcal{U}_1) \subseteq \mathcal{N}_{\mathcal{U}_1}(G)$. It is conjectured that the converse inclusion is also true. It is still a problem listed by Sehgal as 43rd problem out of 56 unsolved problems. This problem is known as Normalizer Problem or Normalizer Property (Sehgal, 1993) and it is denoted by (N.P.)

$$\mathcal{N}_{\mathcal{U}_1}(G) = G.\mathcal{Z}(\mathcal{U}_1) = G.\mathcal{Z}(\mathcal{U}_1(\mathbb{Z}G)) \qquad \text{(N.P.)}$$

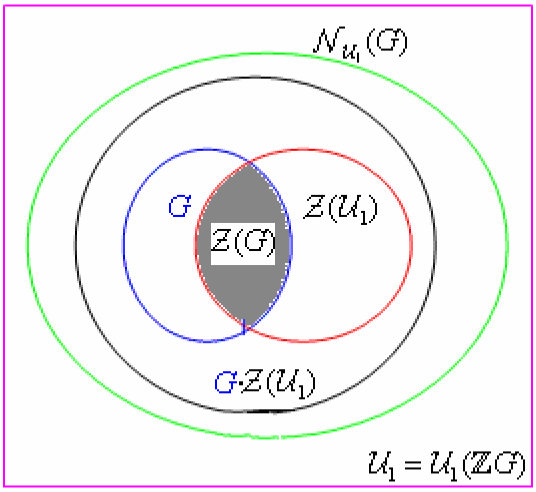

Figure 1.2. Normalizer Property

Firstly, Normalizer Property was proved for nilpotent groups (Coleman, 1964), and than this property is extended to finite groups of odd order and the groups having normal Sylow-2 subgroup (Jackowski and Marciniak, 1987). After that the normalizer property is verified for some metabelian groups (Li, 1999). Next, if the intersection of non-normal subgroups of *G* is nontrivial then it was shown that *G* satisfied the normalizer property (Li et al. 2002). Later, this conjecture is verified for Frobenius groups (Labão and Milies, 2002). The last part of the second chapter consists of these results.

Of course, in order to construct a counter example, the structure of central units of normalized units must be characterized. Then, the counter example can be investigated. Therefore, some results about the construction of central units are given as a third chapter, which consists of three articles (Ritter and Sehgal, 1990) (Jespers at al., 1996) and (Milies and Sehgal, 1999).

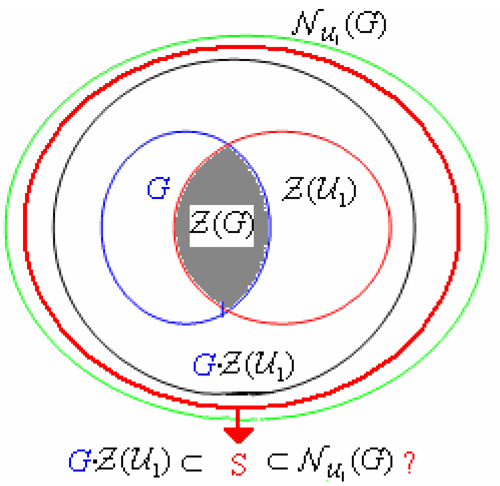

Figure 1.3. Counter Example

All groups studied on this conjecture have normal subgroups which are not simple. If G is not simple then we can consider a normal subgroup N. By using this normal subgroup we can get a natural group homomorphism.

$$\varphi : G \to G/N$$
$$g \mapsto gN$$

By extending this homomorphism to $\mathbb{Z}G$ over $\mathbb{Z}$,

$$\bar{\varphi} : \mathbb{Z}G \to \mathbb{Z}(G/N)$$
$$\sum \gamma_g g \mapsto \sum \gamma_g gN$$

we obtain a natural ring homomorphism. By using quotient ring we can construct unit group of $\mathbb{Z}G$.

The main problem arises when G is simple. In this case the construction of the unit group of $\mathbb{Z}G$ is not as easy as above.

The last chapter consists of method and construction of central units of normalized units of $\mathbb{Z}A_n$ for (n<11) by using irreducible characters. These characters are obtained by using software GAP 4.3 (Groups, Algorithm, Programming).

CHAPTER 2

BASIC FACTS, DEFINITIONS AND NOTATIONS

2.1. Basic Facts, Definition and Notation of a Group *G*

The following basic facts, definitions and notations about action, centralizer, normalizer and automorphism are taken from undergraduate text (Hungerford,1974)

Definition 2.1. An action of a group G on a set S is a function

$$G \times S \to S$$
$$(g, x) \mapsto g.x$$

such that $\forall x \in S, \forall g, h \in G$ the following conditions hold:

i) $e.x = x,$

ii) $(gh).x = g.(h.x)$

Definition 2.2. Let $G = S_n$ and $I_n = \{1, 2, ..., n\}$. Then the function

$$S_n \times I_n \to I_n$$
$$(\sigma, x) \mapsto \sigma(x)$$

is an action.

i) $\iota \in S_n$ identity permutation we have $\iota(x) = x, \ \forall x \in I_n$ so $i.x = x$.

ii) *For* $\sigma, \tau \in S_n$ $\quad (\sigma \circ \tau)(x) = \sigma(\tau(x))$

$$= \sigma.y, \quad (y = \tau(x) \in I_n)$$
$$= \sigma.(\tau(x))$$
$$= \sigma.(\tau.x)$$

the group S_n acts on the set I_n by *permutation*.

Definition 2.3. Let G be a group and $H \leq G$. Then the function

$$H \times G \to G$$
$$(h, x) \mapsto hx$$

is an action.

i) For $e \in H$ identity $e.x = x$, $(\forall x \in G)$

ii) $h, k \in K \ and \ x \in G$

$$(hk).x = h(kx)$$
$$= h.(kx)$$
$$= h.(k.x)$$

This action is called *left translation.*

Definition 2.4. Let G be a group and $K \le G,\ S = G/K$ the set of left cosets. The function

$$K \times G/K \longrightarrow G/K$$
$$(k, xK) \mapsto (kx)K$$

is an action.

i) For $e \in K,\ e.(xK) = (ex)K = xK$
ii) $k, h \in K$ and $x \in G$

$$\begin{aligned}(kh).xK &= ((kh)x)K \\ &= (k(hx))K \\ &= k.(hx)K \\ &= k.(h.(xK))\end{aligned}$$

The subgroup K acts on the set G/K by *left translation.*

Definition 2.5. Let G be a group and $H \le G$. Then the function

$$H \times G \to G$$
$$(h, x) \mapsto hxh^{-1}$$

is an action.
i) $e \in H \Rightarrow e.x = exe^{-1} = x$
ii) $k, h \in H$ and $x \in G$

$$\begin{aligned}(kh).x &= (kh)x(kh)^{-1} \\ &= (kh)x(k^{-1}h^{-1}) \\ &= k(hxh^{-1})k^{-1} \\ &= k.(hxh^{-1}) \\ &= k.(h.x)\end{aligned}$$

H acts on the set G by *conjugation.*

Definition 2.6. Let $K \le G,\ S = G/K$ the set of left cosets. The function

$$K \times G/K \longrightarrow G/K$$
$$(k, xK) \mapsto (kxk^{-1})K$$

is an action.

i) $e \in K \Rightarrow e.xK = (exe^{-1})K = xK$
ii) $g, h \in H$ and $xK \in G/K$

$$\begin{aligned}(gh).xK &= ((gh)x(gh)^{-1})K \\ &= ((gh)x(g^{-1}h^{-1}))K \\ &= (g(hxh^{-1})g^{-1})K \\ &= g.(hxh^{-1})K \\ &= g.(h.xK)\end{aligned}$$

H acts on the set G/K by *conjugation.*

Definition 2.7. $H \le G,\ S = \{K : K \le G\}$; family of subgroups of G. Then the function

$$H \times S \to S$$
$$(h, K) \mapsto hKh^{-1}$$

is an action.

i) $e \in H \Rightarrow e.K = eKe^{-1} = K$
ii) $g, h \in H$ and $K \in S$

$$\begin{aligned}(gh).K &= (gh)K(gh)^{-1} \\ &= (gh)K(g^{-1}h^{-1}) \\ &= g(hKh^{-1})g^{-1} \\ &= g.(hKh^{-1}) \\ &= g.(h.K)\end{aligned}$$

H acts on the set S by *conjugation.*

Theorem 2.8. *Let G be a group which acts on a set S.*

i) *The relation on S defined by* $'x \sim y \Leftrightarrow y = g.x,\ \exists g \in G'$ *is an equivalence relation.*

ii) $\forall x \in S, G_x = \{g \in G : g.x = x\} \le G$

Proof. i-a) For any $x \in S$ we have $e.x = ex = x \Rightarrow x \sim x \Rightarrow \sim$ is reflexive.

b) For any $x, y \in S$ let $y = g.x$ for some $g \in G$, then

$$\begin{aligned}g^{-1}.y &= g^{-1}.(g.x) \\ &= (g^{-1}g).x \\ &= e.x \\ &= x\end{aligned}$$

Hence, $x = g^{-1}y$ for some $g \in G \Rightarrow \sim$ is symmetric.

c) For $x, y, z \in S$ and $g, h \in G$. Let us assume that $y = g.x$ and $z = h.y$ then we have

$$\begin{aligned}z &= h.y \\ &= h.(g.x) \\ &= (hg).x \\ &= k.x,\ (k = hg \in G) \Rightarrow \sim \text{ is transitive.}\end{aligned}$$

So "$\sim$" is an equivalence relation on S. The equivalence class for $x \in S,\ \overline{x} = \{g.x : g \in G\}$ is called the *orbit* of $x \in S$.

ii) Let $G_x = \{g \in G : g.x = x\}$ since $e \in G$ we have $e.x = x \Rightarrow e \in G_x \ne \varnothing$.

If $g \in G_x$ then

$$\begin{aligned}x &= e.x \\ &= (g^{-1}g).x\end{aligned}$$

$$= g^{-1}.(g.x)$$
$$= g^{-1}.x, \quad (g \in G_x) \Rightarrow g^{-1} \in G_x$$

For any $g, h \in G_x$ we have

$$(gh)x = g(h.x)$$
$$= g.x$$
$$= x$$

Hence $G_x \leq G$.

G_x is called *subgroup fixing* S, *stabilizer*, *isotropy* group of x. For any $x \in S$ there is a close relation between $\overline{x}$ and G_x.

Theorem 2.9. *If a group G acts on a set S then the cardinal number of the orbit of $x \in S$ is the index $[G : G_x]$.*

Proof. Since $|\overline{x}| = [G : G_x] = |G / G_x|$, let us consider following mapping:

$$\varphi : \overline{x} \to G / G_x$$
$$a.x \mapsto a.G_x$$

Is φ well defined? One to one?

For any $a.x,\ b.x \in \overline{x}$

$$a.x = b.x \Leftrightarrow b^{-1}.(a.x) = b^{-1}.(b.x)$$
$$\Leftrightarrow (b^{-1}a).x = (b^{-1}b)x$$
$$\Leftrightarrow e.x = (b^{-1}a)x$$
$$\Leftrightarrow (b^{-1}a).x = x$$
$$\Leftrightarrow b^{-1}a \in G_x$$

$a.x = b.x \Rightarrow \varphi(ax) = \varphi(bx) \therefore \quad \varphi :$ well-defined.

$\varphi(aG_x) = \varphi(bG_x) \Rightarrow a.x = b.x \quad \therefore \varphi$ is 1-1.

$\forall aG_x \in G / G_x, aG_x = \varphi(a.x), \exists a.x \in \overline{x} \quad \therefore \varphi$ is onto.

Definition 2.10. If $G \times G \to G$

$$(g, x) \mapsto gxg^{-1}$$

then the orbit of $x \in G$ defined by $\overline{x} = \{gxg^{-1} : g \in G\}$ conjugacy class of x and subgroup fixing x is called ***Centralizer*** denoted by

$$C_G(x) \text{ or simply } C(x) = \{g \in G : gxg^{-1} = x\}$$
$$= \{g \in G : gx = xg\}$$

On the other hand, let us consider the conjugate action on $S=\{K : K \le G\}$, $H \in S$. Then

$$H \times S \to S$$
$$(h,K) \mapsto hKh^{-1}$$

the orbit $\bar{K}=\{hKh^{-1} : h \in H\}$ is the set of conjugate of K in H, and subgroup fixing K is called ***Normalizer*** *of* K *in* H and denoted by

$$N_H(K)=\{h \in H : hKh^{-1}=K\}$$

Corollary 2.11. *Let G be a group and* $K \le H \le G$. *Then*

i) $N_H(K) \subseteq N_G(K)$

ii) $K \le N_G(K) \le G$

iii) $N_G(K)=G \Leftrightarrow K \lhd G$.

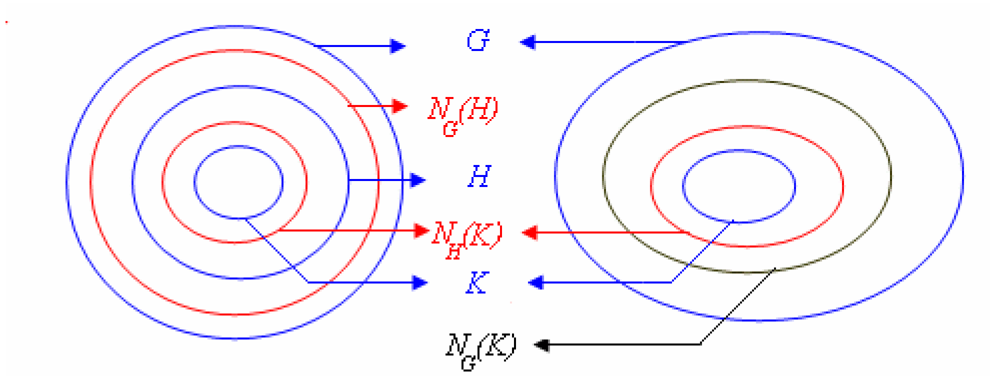

Figure 2.1 The relation among $N_H(K), N_G(K)$ and $N_G(H)$.

Proposition 2.12. $\bigcap_{x \in G} C(x)=\mathcal{Z}(G)$.

Proof.

$$(:\Rightarrow) g \in \bigcap_{x \in G} C(x) \Rightarrow (\forall x \in G), g \in C(x)$$
$$\Rightarrow gx = xg, \forall x \in G$$
$$\Rightarrow g \in \mathcal{Z}(G)$$
$$(\Leftarrow:) \quad g \in \mathcal{Z}(G) \Rightarrow \forall x \in G, gx = xg$$
$$\Rightarrow \forall x \in G, g \in C(x)$$
$$\Rightarrow g \in \bigcap_{x \in G} C(x)$$

Corollary 2.13. *Let* G *be a finite group.*

i) $|\bar{x}|=[G:C_G(x)] \mid |G|$.

ii) If $\bar{x}_1, \bar{x}_2, ..., \bar{x}_n$ are distinct conjugacy classes of G then $|G|=\sum_{i=1}^{n}[G:C_G(x_i)]$.

Proof. **i)** Let us consider the conjugate action of G to itself.

$$G \times G \to G$$
$$(g, x) \mapsto gxg^{-1}$$

Since $|G| < \infty$,

$$|\overline{x}| = [G : C_G(x)] = \frac{|G|}{|G_x|} \Rightarrow |G| = |G_x||\overline{x}|$$
$$\Rightarrow |G| = |C_G(x)||\overline{x}|$$
$$\Rightarrow |\overline{x}| \mid |G|$$

ii) Let us consider a transversal of S; $T = \{x_i \in S : x_i \cap x_j = \emptyset \text{ if } i \neq j\} \Rightarrow |T| = n$. Then

$$G = \bigcup_{x_i \in T} \overline{x}_i \Rightarrow |G| = \sum_{x_i \in T} |\overline{x}_i| = \sum_{i=1}^{n} [G : C_G(x_i)].$$

Theorem 2.14. *If a group G acts on a set S, then this action induces a homomorphism from G to $A(S)$, where $A(S)$ is the group of all permutations of S.*

Proof. By using the following action

$$G \times S \to S$$
$$(g, x) \mapsto g.x$$

let us define a function

$$\tau_g : S \to S$$
$$x \mapsto g.x$$

$\tau_g \in A(S)$ is a permutation $\Leftrightarrow \tau_g$:1−1 and onto. Now let us check

$$\forall x \in S, \quad x = e.x$$
$$=(gg^{-1}).x$$
$$= g.(g^{-1}.x)$$
$$= \tau_g(g^{-1}.x), \quad (g^{-1}.x \in S) \Rightarrow \tau_g \text{ is surjective.}$$

$$\forall x, y \in S, \tau_g(x) = \tau_g(y) \Rightarrow g.x = g.y$$
$$\Rightarrow g^{-1}.(g.x) = g^{-1}.(g.y)$$
$$\Rightarrow (g^{-1}g).x = (g^{-1}g).y$$
$$\Rightarrow x = y$$
$$\Rightarrow \tau_g \text{ is injective}$$

Hence, τ_g is a permutation. That's why, $A(S)$ is a group under composition operation . Now let us show that

$$\varphi : G \to A(S)$$
$$g \mapsto \tau_g$$

is a homomorphism. For $\forall g, h \in G$ and $\forall x \in S$

$$\tau_{gh}(x)=(gh).x=g.(h.x)=\tau_g(h.x)=\tau_g.(\tau_h)=\tau_g\circ\tau_h(x) \qquad (*)$$

So, $\varphi(gh)=\tau_{gh}\overset{(*)}{=}\tau_g\circ\tau_h=\varphi(g)\circ\varphi(h)$ then φ is a homomorphism.

Corollary 2.15. *If G is a group then there is a monomorphism from G to $A(G)$. Hence every group is isomorphic to a subgroup of group of permutations. In particular,* $|G|=n,\ G\cong H\leq S_n$.

Proof. Let us define left translation on G.

$$G\times G\to G$$
$$(g,x)\mapsto g.x=gx$$

By previous theorem we have a homomorphism.

$$\varphi: G\to A(G)$$
$$g\mapsto\tau_g$$

Let us check the kernel of the function

$$\begin{aligned} Ker\varphi &= \{g\in G:\varphi(g)=\iota\}\\ &=\{g\in G: g.x=\iota(x),\forall x\in S\}\\ &=\{g\in G: g.x=x,\forall x\in S\}\\ &=\{g\in G: g=e\}\\ &=\{e\}\end{aligned}$$

So, φ is one to one.

Corollary 2.16. *Let G be a group.*

a) *For each $g\in G$, conjugation by g induces an automorphism of G.*

b) *There is a homomorphism from G to $AutG$ whose kernel is $\mathcal{Z}(G)$*

Proof. a) Let us show that

$$\tau_g: G\to G$$
$$g\mapsto gxg^{-1}$$

is an isomorphism.

i) $\forall x,y\in G,$

$$\begin{aligned}\tau_g(xy)&=g(xy)g^{-1}\\ &=(gxg^{-1})(gyg^{-1})\\ &=\tau_g(x)\tau_g(y)\end{aligned}$$

τ_g is a homomorphism.

ii) $Ker\ \tau_g = \{x \in G : \tau_g(x) = e\}$

$= \{x \in G : gxg^{-1} = e\}$

$= \{g \in G : x = g^{-1}eg\}$

$= \{e\}$

So, τ_g is one to one.

iii) $\forall y \in G,$ y = g(g^{-1}yg)g^{-1}

$= \tau_g(g^{-1}yg),$

$= \tau_g(x),\ (x = g^{-1}yg \in G)$

So, τ_g is onto.

Consequently, τ_g is an isomorphism.

b) Let us consider the following mapping.

$$\varphi : G \to AutG$$
$$g \mapsto \tau_g$$

Since $AutG$ contained in $A(G)$ by Corollary 2.16. φ is a homomorphism.

let us check the kernel of φ

$Ker\varphi = \{g \in G : \varphi(g) = \iota\}$

$= \{g \in G : \tau_g(x) = \iota(x)\}$

$= \{g \in G : gxg^{-1} = x, \forall x \in G\}$

$= \{g \in G : gx = xg, \forall x \in G\}$

$= \mathcal{Z}(G)$

If we denote the set of all automorphisms induced by Inn G , then

$$\varphi : G \to Inn\ \mathrm{G}$$
$$\mathrm{g} \mapsto \tau_{\mathrm{g}}$$

is surjective by the first isomorphism theorem $G / \mathcal{Z}(G) \cong Inn$ G

Example. *Find inner automorphisms of* S_3

Solution.

$$Inn\ S_3 \cong S_3 / \mathcal{Z}(S_3) \Rightarrow |Inn\ S_3| = \frac{|S_3|}{|\mathcal{Z}(S_3)|} = \frac{6}{1} = 6$$

$$S_3 = \langle a, b : a^3 = b^2 = 1, bab^{-1} = a^{-1} \rangle = \langle a, b \rangle = \langle a, ba \rangle = \langle a, ba^2 \rangle$$
$$= \langle a^2, b \rangle = \langle a^2, ba \rangle = \langle a^2, ba^2 \rangle$$

$$f_1 = \begin{cases} a \mapsto a \\ b \mapsto b \end{cases} \qquad g_1 = \begin{cases} a \mapsto a^2 \\ b \mapsto b \end{cases}$$

$$f_2 = \begin{cases} a \mapsto a \\ b \mapsto ba \end{cases} \qquad g_2 = \begin{cases} a \mapsto a^2 \\ b \mapsto ba \end{cases}$$

$$f_3 = \begin{cases} a \mapsto a \\ b \mapsto ba^2 \end{cases} \qquad g_3 = \begin{cases} a \mapsto a^2 \\ b \mapsto ba^2 \end{cases}$$

$$f_2^{\,3} = f_1,\ f_3^3 = \iota,\ g_1^2 = g_2^2 = g_3^2 = f_1 .$$

$$Aut\ S_3 = \left\{ \underbrace{f_1}_{1}, \underbrace{f_2, f_3}_{3}, \underbrace{g_1, g_2, g_3}_{2} \right\} \Rightarrow S_3 \cong Aut\ \mathrm{S}_3 .$$

Since $\left| Inn\ \mathrm{S}_3 \right| = 6 \Rightarrow Inn\ \mathrm{S}_3 = Aut\ \mathrm{S}_3$

$$\begin{aligned} \tau_1 :\ & S_3 \to S_3 \\ & a \mapsto \iota a \iota^{-1} = a \qquad\qquad \tau_1 = f_1 \\ & b \mapsto \iota b \iota^{-1} = b \end{aligned}$$

$$\begin{aligned} \tau_a :\ & S_3 \to S_3 \\ & a \mapsto aaa^{-1} = a \qquad\qquad \tau_a = f_2 \\ & b \mapsto aba^{-1} = ba \end{aligned}$$

$$\begin{aligned} \tau_{a^2} :\ & S_3 \to S_3 \\ & a \mapsto a^2 a a^{-2} = a \qquad\qquad \tau_{a^2} = f_3 \\ & b \mapsto a^2 b a^{-2} = ba^2 \end{aligned}$$

$$\begin{aligned} \tau_b :\ & S_3 \to S_3 \\ & a \mapsto bab^{-1} = a^2 \qquad\qquad \tau_b = g_1 \\ & b \mapsto bbb^{-1} = b \end{aligned}$$

$$\begin{aligned} \tau_{ba} :\ & S_3 \to S_3 \\ & a \mapsto (ba)a(ba)^{-1} = a^2 \qquad\qquad \tau_{ba} = g_3 \\ & b \mapsto (ba)b(ba)^{-1} = ba^2 \end{aligned}$$

$$\begin{aligned} \tau_{ba^2} :\ & S_3 \to S_3 \\ & a \mapsto (ba^2)a(ba^2)^{-1} = a^2 \qquad\qquad \tau_{ba^2} = g_2 \\ & b \mapsto (ba^2)b(ba^2)^{-1} = ba \end{aligned}$$

Corollary 2.17. *Let A be a normal subgroup of G. Then,*

$$\begin{aligned} & G/A \times A \to A \\ & (gA, x) \mapsto (gA).x = gxg^{-1} \end{aligned}$$

is an action which induces a homomorphism from G/A to Aut A

Proof.

i) For any $x \in A$ $(A.x) = (eA.x) = exe^{-1} = x$

ii) For any $g, h \in H$

$$\begin{aligned}((gA)(hA)).x &= ((gh)A).x \\ &= (gh)x(gh)^{-1} \\ &= g(hxh^{-1})g^{-1} \\ &= gA.(hxh^{-1}) \\ &= gA.(hA.x)\end{aligned}$$

So it is an action. Now let us show that this action induces a homomorphism by the following mapping:

Claim. The following mapping induces a homomorphism

$$\varphi : G/A \to Aut\ \mathrm{A}$$
$$gA \mapsto \tau_g$$

Proof. Let $gA, hA \in G/A$

$$\begin{aligned}\varphi((gA)(hA)) &= \varphi((gh)A)) \\ &= \tau_{gh} \\ &= \tau_g \circ \tau_h \\ &= \varphi(gA) \circ \varphi(hA)\end{aligned}$$

Hence φ is a homomorphism.

Corollary 2.18. *If* H, K *are subgroups of* G *such that* $H \lhd K$ *,show that* $K < N_G(H)$

Proof. $H < K < G$, $\quad H < N_G(H) < G$ for $k \in K \Rightarrow k \in N_G(H)$?

$$\begin{aligned}H \lhd K &\Rightarrow \forall k \in K, \forall h \in H,\ khk^{-1} \in H \\ &\Rightarrow \forall k \in K,\ kHk^{-1} \subset H \\ &\Rightarrow \forall k \in K,\ k \in N_K(H) \subset N_G(H) \\ &\Rightarrow K \subset N_G(H) \\ &\Rightarrow K \leq N_G(H)\end{aligned}$$

Corollary 2.19 If a group G contains an element "a" having exactly two conjugate classes then G has a proper normal subgroup.

Proof. $G = C_1 \cup C_2$, $\forall g \in G, gag^{-1} \in C_1$ or $gag^{-1} \in C_2$

$$2 = |\overline{a}| = [G : G_a] = [G : C(a)] \Rightarrow C(a) \lhd G.$$

Corollary 2.20 **i)** $C_G(H) \leq N_G(H)$

ii) $C_G(H) \lhd N_G(H)$

Proof.

i) $a\in C_G(H)\Rightarrow a\in N_G(H)?$

$$a\in C_G(H)\Rightarrow aha^{-1}=h,\forall h\in H, a\in G$$
$$\Rightarrow aHa^{-1}\subset H$$
$$\Rightarrow a\in N_G(H)$$

$$a,b\in C_G(H)\Rightarrow ab^{-1}\in C_G(H)?$$

$$(ab^{-1})h(ab^{-1})^{-1}=(ab^{-1})h(ba^{-1})$$
$$=a(b^{-1}hb)a^{-1}$$
$$=a(b^{-1}b)ha^{-1}$$
$$=(ah)a^{-1}$$
$$=h(aa^{-1})$$
$$=h,\ \forall h\in H$$

Hence $ab^{-1}\in C_G(H)$. $C_G(H)\le N_G(H)$

ii) $C_G(H)\lhd N_G(H)?\equiv\forall a\in C_G(H)$ and $\forall x\in N_G(H)$, $xax^{-1}\in C_G(H)?$

$$a\in C_G\left(H\right)\Rightarrow aha^{-1}=h,\forall h\in H$$
$$x\in N_G(H)\Rightarrow xHx^{-1}=H\Rightarrow xhx^{-1}\in H,\forall h\in H$$

$$\forall h\in H,\forall x\in N_G(H),\ xax^{-1}\in C_G(H)?$$

$$(xax^{-1})h(xax^{-1})^{-1}=(xax^{-1})h(xa^{-1}x^{-1})$$
$$=(xa)(x^{-1}hx)a^{-1}x^{-1}$$
$$=xah'a^{-1}x^{-1}\quad (h'=x^{-1}hx)$$
$$=xh'x^{-1}$$
$$=x(x^{-1}hx)x^{-1}$$
$$=(xx^{-1})h(xx^{-1})$$
$$=h\ ,\ (xax^{-1}\in C_G(H))$$

$\therefore$ $\mathrm{C}_{\mathrm{G}}(H)\lhd N_G(H)$

2.2 Definition and Some Basic Properties of Group Ring *RG*

Definition 2.21. Let G be a group and R be a commutative ring with unity. Let us denote the set of all finite formal linear combinations by RG. Then the set

$$RG = \{\alpha = \sum_{finite} \alpha_g g : g \in G, \alpha_g \in R\}$$

is a ring under the following addition and multiplication:

Addition of two elements in RG **:** Let $\alpha = \sum_{g\in G} \alpha_g g$ and $\beta = \sum_{g\in G} \beta_g g \in RG$ then

$$\alpha + \beta = \sum_{g\in G} \alpha_g g + \sum_{g\in G} \beta_g g = \sum_{g\in G} (\alpha_g + \beta_g) g$$

Multiplication of two elements in RG **:** Let $\alpha = \sum_{g\in G} \alpha_g g$ and $\beta = \sum_{h\in G} \beta_h h \in RG$ then

$$\begin{aligned} \alpha.\beta &= (\sum_{g\in G} \alpha_g g)(\sum_{h\in G} \beta_h h) \\ &= \sum_{g,h\in G} (\alpha_g \beta_h)(gh) \\ &= \sum_{g\in G} (\sum_{h\in G} \alpha_g \beta_h)(gh) \end{aligned}$$

If we substitute $g = gh^{-1}$ then, we simply denote

$$\alpha.\beta = \sum_{g\in G} (\sum_{h\in G} \alpha_{gh^{-1}} \beta_g) g$$

Some Basic Properties of *RG*

1- Equality of two elements in RG **:** Let $\alpha = \sum_{g\in G} \alpha_g g$ and $\beta = \sum_{g\in G} \beta_g g \in RG$, then

$$\alpha = \beta \Leftrightarrow \alpha_g = \beta_g, \forall g \in G.$$

2- The group G can be embedded into RG by the following group homomorphism:

$$\begin{aligned} i &: G \to RG \\ g &\mapsto 1_R.g \end{aligned}$$

3- The ring R can be embedded into RG by the following ring homomorphism:

$$\begin{aligned} \vartheta &: R \to RG \\ r &\mapsto r.e_G \end{aligned}$$

The ring RG has the unity. $1_{RG} = 1_R.e_G$. So, we can define the unit group of RG denoted by

$$\mathcal{U}(RG) = \{\gamma \in RG : \gamma\lambda = 1_{RG}, \exists \lambda \in RG\}.$$

Clearly, $\mathcal{U}(R)$ and $G \subseteq \mathcal{U}(RG)$, more generally $\mathcal{U}(R).G \subseteq \mathcal{U}(RG)$. If $\mathcal{U}(RG) \subseteq \mathcal{U}(R).G$ then we say that RG has trivial units.

4- For $N \lhd G$, let us consider following natural group homomorphism :

$$\begin{aligned} \varphi : G &\to G/N \\ g &\mapsto gN \end{aligned}$$

If we extend the group homomorphism linearly over R, we get the following natural ring homomorphism.

$$\begin{aligned} \overline{\varphi} : RG &\to R(G/N) \\ \sum \gamma_g g &\mapsto \sum \gamma_g (gN) \end{aligned}$$

5- We can define a ring homomorphism from RG to R as follows:

$$\begin{aligned} \varepsilon : RG &\to R \\ \sum \gamma_g g &\mapsto \sum \gamma_g \end{aligned}$$

The kernel of ε is an ideal of RG, denoted by $\Delta(G) = \{\gamma \in RG : \sum \gamma_g = 0\} = \Delta_R(G,G)$.

6- If we restrict this homomorphism to the unit group of RG then we get a group ephimorphism :

$$\begin{aligned} \overline{\varepsilon} : \mathcal{U}(RG) &\to \mathcal{U}(R) \\ \sum \gamma_g g &\mapsto \sum \gamma_g \end{aligned}$$

This map is called augmentation map and the kernel of $\overline{\varepsilon}$ is a normal subgroup of

$\mathcal{U}(RG)$ and denoted by $\mathcal{U}_1(RG) = \{\gamma \in \mathcal{U}(RG) : \sum \gamma_g = 1_R\}$

and it is usually called normalized units (or units with augmentation 1).

7 - If we take $R = \mathbb{Z}$ then the group $\mathbb{Z}G$ is called Integral Group Ring. It is clear that

$\mathcal{U}(\mathbb{Z}G) = \pm \mathcal{U}_1(\mathbb{Z}G)$ since $\mathcal{U}(\mathbb{Z}) = \{\pm 1\}$.

2.3. Some Basic Results About Normalizer of Group G in $\mathcal{U}_1(\mathbb{Z}G)$

Definition 2.22. For any $u \in \mathcal{N}_{\mathcal{U}}(G)$, the function

$$\varphi : G \to G$$
$$g \mapsto u^{-1}gu$$

induces an automorphism. The set of all automorphisms induced by normalizer of *G* in $\mathcal{U}$ is a group which contains *InnG*, and denoted by $Aut_{\mathbb{Z}}G$.

Proposition 2.23. $Aut_{\mathbb{Z}}G = Inn\ G \Leftrightarrow$ Normalizer Property holds

Proof.
$$\begin{aligned}
Aut_{\mathbb{Z}}G = Inn\ G &\Leftrightarrow \left(\varphi_u \in Aut_{\mathbb{Z}}G \Rightarrow \varphi_u \in Inn\ G\right)\\
&\Leftrightarrow \forall u \in N, \varphi_u = \varphi_y, \exists y \in Inn\ G\\
&\Leftrightarrow \varphi_u(g) = \varphi_y(g), \forall g \in G\\
&\Leftrightarrow ugu^{-1} = ygy^{-1}, \forall g \in G\\
&\Leftrightarrow y^{-1}(ug) = (gy^{-1})u, \forall g \in G\\
&\Leftrightarrow (y^{-1}u)g = g(y^{-1}u), \forall g \in G\\
&\Leftrightarrow y^{-1}u \in \mathcal{Z}(\mathbb{Z}G) \cap \mathcal{U}(\mathbb{Z}G)\\
&\Leftrightarrow y^{-1}u = z \in \mathcal{U}(\mathcal{Z}(\mathbb{Z}G))\\
&\Leftrightarrow u = yz, z \in \mathcal{U}(\mathcal{Z}(\mathbb{Z}G)), \exists y \in G\\
&\Leftrightarrow \mathcal{N}_{\mathcal{U}}(G) \subseteq G.\mathcal{U}(\mathcal{Z}(\mathbb{Z}G))\\
&\Leftrightarrow \mathcal{N}_{\mathcal{U}}(G) = G\mathcal{U}(\mathcal{Z}(\mathbb{Z}G)), (\text{Converse inclusion is trivial})\\
&\Leftrightarrow \text{Normalizer Property holds}
\end{aligned}$$

Theorem 2.24. (Coleman) Suppose that P is a $p-subgroup$ of a finite group G and let $u \in \mathcal{N}_U G$. Then there exists $y \in G$ such that $ugu^{-1} = ygy^{-1}, \forall g \in P$.

Proof. For $g \in G, \varphi(g) = ugu^{-1}$ is an element of G and $u = g^{-1}u\varphi(g)$.Write $u = \sum u_x x$. Then $\sum u_x x = \sum u_x [g^{-1}x\varphi(g)]$. Thus G acts on the set G by $\sigma_g = g^{-1}x\varphi(g)$ and the function

$$u : G \to \mathbb{Z}$$
$$x \mapsto u_x$$

is a constant on orbit of this function. Let us restrict $g \in P$, so P acts on the set G. The orbits of this action are of length power of *p*. Looking at the augmentation of u we conclude

$$\pm 1 = \varepsilon(u) = \sum c_i p^{u_i}$$

where p^{u_i} is the length of the orbit of g_i and $u(g_i)=c_i$. It follows that an orbit length one, that is to say, there exists an $x\in G$ so that $\sigma_g(x)=x,\forall g\in G$. This implies that $g^{-1}x\varphi(x)=x,\varphi(g)=xgx^{-1}$ for all $g\in P$ as claimed.

Corollary 2.25. *If P is a $p-$subgroup of a finite group G. Then for*

$$\forall g\in P,\exists y\in G;\quad ugu^{-1}=ygy^{-1}$$

Corollary 2.26. *If G is a $p-$subgroup then G holds N.P.*

Proprosition 2.27. *If G is a finite nilpotent group then G holds N.P.*

Proof. Let G be a nilpotent group then G can be written as $G=\prod P_i,\ (\underbrace{P_i\le G}_{\text{sylow } p\text{-subgroup}})$

For $\forall g\in G$ and $\forall u\in \mathcal{N}_{\mathcal{U}}(G),\ ugu^{-1}=ygy^{-1},\ \exists y\in G$. By corollary 2.24

$\forall x_i\in P_i\ ,\ ux_iu^{-1}=y_ixy_i^{-1}\ ,\exists y_i\in G$

$\forall g\in G\ ,\ g=x_1x_2...x_k\ ,\ (x_i\in P_i)$

$$\begin{aligned}ugu^{-1}=u(x_1x_2...x_k)u^{-1}&=(ux_1u^{-1})...(ux_iu^{-1})...(ux_ku^{-1})\\&=(y_1x_1y_1^{-1})...(y_ix_iy_i^{-1})...(y_kx_ky_k^{-1})\\&=(y_1y_2...y_k)(x_1x_2...x_k)\left(y_k^{-1}...y_2^{-1}y_1^{-1}\right)\end{aligned}$$

If we choose $y=y_1y_2...y_k$ we get

$$\begin{aligned}ugu^{-1}&=y(x_1x_2...x_k)y^{-1}\\&=ygy^{-1},\exists y=y_1y_2...y_k\in G\end{aligned}$$

Definition 2.28. For any $\gamma=\sum_{g\in G}\gamma_g g\in\mathbb{C}G,\ \ \gamma^*=\sum_{g\in G}\overline{\gamma}_g g^{-1}$ is called involution of γ where $\overline{\gamma}_g$ is the complex conjugate of γ_g. For any $\gamma,\lambda\in\mathbb{C}G$ and $u\in\mathcal{U}(\mathbb{C}G)$

i) $(\gamma+\lambda)^*=\gamma^*+\lambda^*$

ii) $(\gamma\lambda)^*=\lambda^*\gamma^*$

iii) $(\gamma^*)^*=(\sum_{g\in G}\overline{\gamma}_g g^{-1})^*=(\sum_{g\in G}\left(\overline{\overline{\gamma_g}}\right)\left(g^{-1}\right)^{-1})=\gamma$

iv) $(u^{-1})^*=\left(u^*\right)^{-1}$

Proposition 2.29. *For any $\gamma\in\mathbb{Z}G$, $\gamma\gamma^*=1\Leftrightarrow\gamma=\pm g$, for some $g\in G$.*

Proof. $(\Leftarrow:)$ If $\gamma=g\Rightarrow\ \gamma^*=g^{-1}\Rightarrow\gamma\gamma^*=gg^{-1}=1$

or $\gamma=-g\Rightarrow\gamma^*=-g^{-1}\Rightarrow\gamma\gamma^*=(-g)(-g^{-1})=1$

$$(:\Rightarrow)\gamma=\gamma_1.1+\gamma_2.g_2+\ldots+\gamma_k.g_k \Rightarrow \gamma^*=\gamma_1.1+\gamma_2.{g_2}^{-1}+\ldots+\gamma_k.{g_k}^{-1}$$

$$\Rightarrow \gamma\gamma^*=({\gamma_1}^2+{\gamma_2}^2+\ldots+{\gamma_k}^2).1+\sum_{g.h,h\neq g^{-1}}(\gamma_h\gamma)gh=1$$

$$\Rightarrow \begin{cases} {\gamma_1}^2+{\gamma_2}^2+\ldots+{\gamma_k}^2=1 \\ \sum_{h\in G,gh=k}\gamma_g\gamma_k=0,\ \forall g\in G \end{cases}$$

$$\Rightarrow \gamma_i^2=1,\ \gamma_j=0,\ (i\neq j)$$

$$\Rightarrow \gamma_i=\pm1,\gamma_j=0,(i\neq j)$$

$$\Rightarrow \gamma=\gamma_i g=\pm g,(g\in G)$$

Proposition 2.30. Let $z\in\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. If $uu^*\in\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ then $u^*u=z$.

Proof. For any $z\in\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ let $uu^*\in\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. we have

$$\begin{aligned}(u^*u)^2&=(u^*u)(u^*u)\\&=u^*(uu^*)u\\&=u^*zu\\&=(u^*u)z\end{aligned}$$

$$\therefore u^*u=z$$

Theorem 2.31. *Let* $u\in\mathcal{U},\ u\in\mathcal{N}_{\mathcal{U}}(G)\Leftrightarrow uu^*\in\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)$

Proof. $(:\Rightarrow)$ Let $u\in\mathcal{N}_{\mathcal{U}}(G)$ then $uGu^{-1}=G.$ Let us consider the following function:

$$\varphi:G\to G$$
$$x\mapsto u^{-1}xu$$

$$\varphi(x)=u^{-1}xu\in G\Rightarrow(\varphi(x))^*=(u^{-1}xu)^*$$

$$\Rightarrow\varphi(x)^{-1}=(xu)^*(u^{-1})^*$$

$$\Rightarrow\varphi(x)^{-1}=u^*x^{-1}(u^*)^{-1}$$

By applying the involution to both sides, we have

$$\varphi(x)=u^*x(u^*)^{-1}\Rightarrow(u^*)^{-1}\varphi(x)u^*=x$$

If we put the value of $\varphi(x)$ then we obtain

$$x=(u^*)^{-1}(ux^{-1}u)u^*\Rightarrow x=\left((u^*)^{-1}u^{-1}\right)x(uu^*)$$

$$\Rightarrow x(uu^*)=(uu^*)x$$

$$\Rightarrow uu^*\in\mathcal{Z}(\mathbb{Z}G)\cap\mathcal{U}(\mathbb{Z}G)$$

$$\Rightarrow uu^*\in\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$$

$(\Leftarrow:)$ Let $uu^* \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ then by Proposition 2.30. we write

$$\begin{aligned}(u^{-1}xu)(u^{-1}xu)^* &= \left(u^{-1}xu\right)\left(u^*x^{-1}\left(u^*\right)^{-1}\right)\\ &= \left(u^{-1}x\right)\left(uu^*\right)\left(x^{-1}\left(u^*\right)^{-1}\right)\\ &= \left(u^{-1}x\right)\left(x^{-1}\left(u^*\right)^{-1}\right)\left(uu^*\right)\\ &= \left(u^{-1}\left(u^*\right)^{-1}\right)\left(uu^*\right)\\ &= \left(u^*u\right)^{-1}\left(uu^*\right)\\ &= 1\end{aligned}$$

By Proposition 2.29, $u^{-1}xu = +g$ or $u^{-1}xu = -g$ for some $g \in G$. Since $\varepsilon(u)\varepsilon(x)\varepsilon(u)^{-1} = 1$, $u^{-1}xu = g$, $(g \in G)$. Hence $u^{-1}xu \in G \Rightarrow u \in \mathcal{N}_{\mathcal{U}}(G)$.

Theorem 2.32. (Krempa) *Let* $u \in \mathcal{N}_{\mathcal{U}}(G)$, φ^2 *is inner i.e* $\varphi^2 \in Inn\ G$.

Proof. Let $v = u^*u^{-1} \in \mathcal{U}$

$$\begin{aligned}vv^* &= \left(u^*u^{-1}\right)\left(u^*u^{-1}\right)^* = \left(u^*u^{-1}\right)\left(\left(u^{-1}\right)^*\left(u^*\right)^*\right)\\ &= \left(u^*u^{-1}\right)\left(\left(u^*\right)^{-1}u\right)\\ &= u^*\left(u^{-1}\left(u^*\right)^{-1}\right)u\\ &= u^*\left(u^*u\right)^{-1}u, \quad (\text{Proposition 2.30.})\\ &= u^*zu\\ &= u^*uz^{-1}\\ &= zz^{-1}\\ &= 1\end{aligned}$$

By Proposition 2.28. we get $v = g$ for some $g \in G$

$$\begin{aligned}u^*u^{-1} = g &\Rightarrow u^* = gu\\ &\Rightarrow u^*u = gu^2\\ &\Rightarrow gu^2 = z, \quad (z \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\\ &\Rightarrow h^{-1}u^2 = z, \ (h = g^{-1})\\ &\Rightarrow u^2 = hz \in G \cdot \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\end{aligned}$$

Then

$$
\begin{aligned}
\varphi^2(x) &= \varphi(\varphi(x)) \\
&= u^{-1}(u^{-1}xu)u \\
&= u^{-2}xu^2 \\
&= (hz)^{-1}x(hz) \\
&= z^{-1}(h^{-1}xh)z \\
&= h^{-1}xh \\
&= \varphi_h(x)
\end{aligned}
$$

$$\therefore\ \varphi^2 = \varphi_h(h \in G) \Rightarrow \varphi^2 \in InnG$$

Theorem 2.33. (**Jackowski-Marciniak**) *If* G *is an odd group then* $\mathcal{N}_{\mathcal{U}}(G) = G \cdot \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$.

Proof. By the theorem 2.32, we write $\varphi^2 = \varphi_g$ for some $g \in G$. Since $|\varphi| \mid |G|$, $\varphi^s = 1$ for odd s.

So we have $(2,s) = 1$, $2k + sn = 1$, $(k, n \in \mathbb{Z})$.

$$
\begin{aligned}
\varphi &= \varphi^1 \\
&= \varphi^{2k+sn} \\
&= (\varphi^2)^k (\varphi^s)^n \\
&= (\varphi_g)^k (\iota)^n \\
&= (\varphi_g)^k \\
&= \varphi_{g^k}, (g^k \in G)
\end{aligned}
$$

$$\therefore \varphi \in Inn\ G.$$

CHAPTER 3

CHARACTERIZATION OF CENTRAL UNITS OF $\mathbb{Z}G$

3.1 Integral Group Rings with Trivial Central Units

In this chapter, finite groups G whose integral group ring $\mathbb{Z}G$ has only trivial central units were classified. This question was raised by Goodaire and Parmenter (Goodaire and Parmenter, 1986).

It was proved by Higman that all units of $\mathbb{Z}G$ are trivial if and only if

a) G is Abelian with exponent a divisor of 4 or 6, or

b) $G = K_8 \times E$ where K_8 is the quaternion group of order 8 and E is an elementary Abelian 2-group. (Higman, 1940)

It follows easily that all units of a commutative group ring $\mathbb{Z}G$ are trivial if and only if for every $x \in G$ and every natural number *j*, relatively prime to $|G|$, $x^j = x$ or $x^j = x^{-1}$.Here $\sim$ denoting conjugation in *G*. The result is stated as follows:

Theorem 3.1. *Let G be a finite group. All central units of* $\mathbb{Z}G$ *are trivial if and only if for every* $x \in G$ *and every natural number j, relatively prime to* $|G|$, $x^j \sim x$ or $x^j \sim x^{-1}$.

Proof. At first we recall that any finite group of central units of $\mathbb{Z}G$ consist of trivial units only (Sehgal, 1978). It suffices to prove that the following conditions are equivalent:

i) $\mathbb{Z}G$ has only a finite number of central units.

ii) The character field $\mathbb{Q}(\chi)$ of each absolutely irreducible character χ of *G* is either $\mathbb{Q}$ or imaginary quadratic.

iii) For every $x \in G$ and every natural number *j*, relatively prime to $|G|$, $x^j \sim x$ or $x^j \sim x^{-1}$.

a) We shall first show that (i) and (ii) are equivalent. Since the center of $\mathbb{Q}G$ is generated by the class sums, the center $\mathcal{Z}$ of $\mathbb{Z}G$ is an order in the center of $\mathbb{Q}G$, the latter being the direct sum of all character fields $\mathbb{Q}(\chi)$ (Huppert, 1967). Hence $\mathcal{Z}$ is of finite additive index in the unique maximal order $\oplus_\chi O_\chi$ of $\oplus_\chi \mathbb{Q}(\chi)$ with O_χ denoting the ring of integers in $\mathbb{Q}(\chi)$. Thus the unit group of $\mathcal{Z}$ is of finite index in the multiplicative group $\oplus_\chi (O_\chi)^\times$ (Sehgal, 1978). It follows that (i) hold precisely when $(O_\chi)^\times$ is finite for all χ which by the Drichlet unit Theorem is the same as (2).

b**)** We next prove that (iii) implies (ii). Let σ be an automorphism of $\mathbb{Q}(\chi)/\mathbb{Q}$. Extend σ to an automorphism $\xi \to \xi^j$ of $\mathbb{Q}(\xi)$ where ξ is a $|G|$th root of unity. Then $\chi^\sigma(g) = \chi(g^j) = \chi(g)$ or $\chi(g^{-1})$ by (iii). We have $\chi^\sigma(g) = \chi(g)$ or $\bar{\chi}(g)$. Since " $^-$ " commutes with σ, it follows that $\chi + \bar{\chi} = \chi^\sigma + \bar{\chi}^\sigma$. Thus $\chi^\sigma = \chi$ or $\bar{\chi}$ by the linear independence of irreducible characters. This implies that $\mathbb{Q}(\chi) = \mathbb{Q}$ or an imaginary quadratic field.

c**)** We finally show that (ii) implies (iii). The proof is dual of (b). For each $g \in G$ we define a function from irreducible characters to the complex numbers, $T(g): Irr(G) \to \mathbb{C}$ by $\chi \to \chi(g)$. It follows by the orthogonality relations that these functions, one for each conjugacy class of *G*, are linearly independent. Now let $(j, |G|) = 1$. Then we have an automorphism $\xi \to \xi^j$ of $\mathbb{Q}(\xi)$ where ξ is a $|G|$th root of unity. Let σ be the restriction of this automorphism to $\mathbb{Q}(\chi)$. Then $\chi^\sigma(g) = \chi(g)$ or $\chi(g^{-1})$ by (ii). Thus $T(g^j) + T(g^{-j}) = T(g) + T(g^{-1})$. It follows due to the linear independence of these functions that $T(g^j) = T(g)$ or $T(g^{-1})$. Thus g^j is conjugate to g or g^{-1} as desired.

An easy consequence of (ii) is:

Corollary 3.2 *If all central units of $\mathbb{Z}G$ are trivial then the same is true for $\mathbb{Z}\bar{G}$, $\bar{G}$ a homomorphic image of G.*

Examples. We close with a few examples of groups satisfying the condition of the Theorem.

a) $G = S_n$ the symmetric group on *n*-letters. In this case all the charcter fields are rational. (Huppert, 1967)

b) *G* a group of order 27. In this case, all character fields are $\mathbb{Q}$, $\mathbb{Q}(\omega)$, $\omega^3 = 1$.

c) $G = \langle x, y : x^7 = 1 = y^3, x^y = x^2 \rangle$. In this case,

$$\mathbb{Q}G = \mathbb{Q} \oplus \mathbb{Q}(\omega) \oplus \mathbb{Q}(\sqrt{-7})_{3\times 3}, \quad \omega^3 = 1$$

Observe that $V=\mathbb{Q}\left(\sqrt{-7}\right)$ is the field of index 3 in $\mathbb{Q}(\xi)$, $\xi^7=1$, and that V is a three-dimensional space over $\mathbb{Q}\left(\sqrt{-7}\right)$ on which G acts irreducibly by letting x act as multiplication by ξ and y by the automrphism $\xi \to \xi^2$. In this example, the character fields are $\mathbb{Q}$, $\mathbb{Q}(\omega)$, $\mathbb{Q}(\sqrt{-7})$.

3.2. Central Units of Integral Group Rings of Nilpotent Groups

In this part, we construct explicitly a finite set of generators for a subgroup of finite index in the center $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ of the unit group $\mathcal{U}(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$ of a finitely generated nilpotent group We first give a finite set of generators for a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ when G is a finite nilpotent group and prove that a central unit of $\mathbb{Z}G$ is a product of a trivial unit and a unit of $\mathbb{Z}T$ where T is the torsion subgroup of G. As an application we obtain that the central units of $\mathbb{Z}G$ form a finitely generated group and we are able to give an explicit set of generators for a subgroup of finite index.

3.2.1. Finite Nilpotent Groups

Throughout this section *G* is a finite group. When *G* is abelian, it was shown in (Bass, 1966) that the Bass cyclic units generate a subgroup of finite index in the unit group. Using a stronger version of this result, also proved by Bass in (Bass, 1966), we will construct a finite set of generators from the Bass cyclic units when *G* is finite nilpotent. All notations will follow that in (Sehgal, 1978).

Lemma 3.3.(Bass,1966) *The images of the Bass cyclic units of* $\mathbb{Z}G$ *under the natural homomorphism* $j:\mathcal{U}(\mathbb{Z}G)\to K_1(\mathbb{Z}G)$ *generate a subgroup of finite index.*

Let *L* denote the kernel of this map *j*, and B the subgroup of $\mathcal{U}(\mathbb{Z}G)$ generated by the Bass cyclic units. It follows that there exists an integer *m* such that $z^m \in LB$ for all $z\in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, and so we can write $z^m = lb_1b_2...b_k$ for some $l\in L$ and Bass cyclic units b_i.

Next, let $\mathcal{Z}_i$ denote the i-th center of *G*, and suppose from now on that *G* is nilpotent of class n. For any $x\in G$ and Bass cyclic unit $b\in \mathcal{Z}\langle x\rangle$, we define $b_{(1)}=b$ and for $2\le i\le n$

$$b_{(i)} = \prod_{g \in \mathcal{Z}_i} b_{(i-1)}^g$$

where $\alpha^g = g^{-1}\alpha g$ for $\alpha \in \mathbb{Z}G$. Note that by induction $b_{(i)}$ is central in $\mathbb{Z}\langle \mathcal{Z}_i, x \rangle$ and independent of the order of the conjugates in the product expression. In particular,

$$b_{(n)} \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)).$$

Recall again that if $z \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, then $z^m = lb_1b_2...b_k$ for some $l \in L$ and Bass cyclic units b_i. Since $K_1(\mathbb{Z}G)$ is Abelian, we can write

$$\begin{aligned} z^{m|\mathcal{Z}_2||\mathcal{Z}_3|..|\mathcal{Z}_n|} &= (lb_1b_2...b_k)^{m|\mathcal{Z}_2||\mathcal{Z}_3|..|\mathcal{Z}_n|} \\ &= l_1 \prod_{1 \le i \le k} b_i^{|\mathcal{Z}_2||\mathcal{Z}_3|..|\mathcal{Z}_n|} \qquad \text{for some } l_1 \in L \\ &= l_2 \prod_{1 \le i \le k} b_{i(2)}^{|\mathcal{Z}_2||\mathcal{Z}_3|..|\mathcal{Z}_n|} \qquad \text{for some } l_1 \in L \\ &= l' \prod_{1 \le i \le k} b_{i(n)} \qquad \text{for some } l' \in L \end{aligned}$$

Since each $b_{i(n)}$ is in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, we conclude that $l' \in L \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. But we shall show next that $L \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is trivial, so $l' \in \pm\mathcal{Z}(G)$. The argument uses the same idea as in (Ritter and Sehgal, 1991)

For every primitive central idempotent e in the rational group algebra $\mathbb{Q}G$, the simple ring $\mathbb{Q}Ge$ has a reduced norm which we denote by nr_e. Further, denote

$$m_e = \sqrt{[\mathbb{Q}Ge : Z(\mathbb{Q}Ge)]}$$

and let

$$r = \prod_e m_e$$

Now let $l' \in L \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. By definition of $K_1(\mathbb{Z}G)$ this means that a suitable matrix

$$\begin{bmatrix} l' & & & & & \\ & 1 & & & & \\ & & . & & & \\ & & & \cdots & & \\ & & & & . & \\ & & & & & 1 \end{bmatrix}$$

is a product of commutators. Therefore $l'e$ has reduced norm one. Since $l'e$ is also central, we obtain

$$(l'e)^{m_e} = nr(l'e)e = e$$

Hence

$$l'^{r} = 1.$$

So l' is a torsion central unit, and therfore is trivial (Sehgal ,1993)

Since $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finitely generated (Ritter and Sehgal, 1990), $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))^{m|\mathcal{Z}_2||\mathcal{Z}_3|\ldots|\mathcal{Z}_n|}$ is of finite index. But we have just seen that the latter subgroup is contained in the subgroup generated by $\pm\mathcal{Z}(G)$ and $\{b_{(n)} : b \text{ a Bass cyclic unit}\}$. We have proved

Proposition 3.4. *Let G be a finite nilpotent group of class n. Then* $\left\langle b_{(n)} \middle| b \text{ a Bass cyclic} \right\rangle$ *is of finite index in* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$.

Remark. Note that our method for constructing generators for some other classes of finite grous G. For example if $G = D_{2n} = \left\langle a,b \middle| x^n = 1, y^2 = 1, yx = x^{n-1}y \right\rangle$, the dihedral group of order 2*n*, then the only trivial Bass cyclic units *b* of $\mathbb{Z}D_{2n}$ belong to $\mathbb{Z}\langle x \rangle$. It follows that $bb^y = b^y b$ is central. Our proof now remains valid and yields that $\left\langle bb^y \middle| b \text{ a Bass cyclic in } \mathbb{Z}\langle x \rangle \right\rangle$ is of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}D_{2n}))$.

3.2.2 Finitely Generated Nilpotent Groups

We will now consider central units of an integral group ring of an arbitrary finitely generated nilpotent group *G*. The torsion subgroup of *G* is denoted by *T*. First we show that central units of $\mathbb{Z}G$ have the following decomposition.

Proposition 3.5. *Let G be a finitely generated nilpotent group. Every* $u \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ *can be written as* $u = rg, r \in \mathbb{Z}T, g \in G$.

Proof. Let $F = G/T$. Since T is finite and F acts on the set of primitive central idempotents of $\mathbb{Q}T$ by conjugation, by adding the idempotents in an orbit of this action we obtain

$$\mathbb{Q}T = \oplus(\mathbb{Q}T)e_i = \oplus R_i,$$

where e_i are primitive central idempotents of $\mathbb{Q}G$. Then $\mathbb{Q}G$ is crossed product

$$\mathbb{Q}G = \mathbb{Q}T * F = (\oplus R_i) * F = \oplus R_i * F. \tag{3.4}$$

Decompose *u* as a sum of elements in (3.4):

$$u = \bigoplus_i \left(\sum_{j=1}^{n} u_j f_j \right), \ 0 \neq u_j \in R_i,\ f_i \in G, \text{ for each j.}$$

We assume that we have put together the u_j's with the same $f_jT \in G/T$, namely for $k \neq j,\ f_kT \neq f_jT$.

We claim that $n=1$. Let us denote by "$\bar{\ }$" the projection of $\mathbb{Q}G$ onto $R_i * F$. Then since *u* is central we have $\overline{QT}\bar{u} = \bar{u}\overline{QT}$, which implies $\overline{QT}u_j f_j = u_j f_j \overline{QT}$ for all j. It follows that u_j is not a zero divisor provided R_i has only one simple (artinian) component, and so u_j is a unit. The only time u_j can be a nonunit is when it has some zero components in the simple components of R_i. However, by the construction of R_i, these latter components can be moved to any other place by conjugating suitably. But they must stay put due to the facts that *F* is ordered and $\bar{u}$ is central. It follows that u_j is a unit for all *j*. Hence, working in $R_i * F$ and using again that *F* is ordered, it follows by a classical argument that $\bar{u} = \sum_j u_j f_j$ is simply equal to $u_n f_n$ as claimed. Changing notation, we write

$$u = \bigoplus_i \alpha f,\ \alpha \in R_i, f \in G.$$

Let $k = |Aut(T)|$, so f^k commutes with *T* for $f \in G$. Hence

$$u^k = \oplus(\alpha f)^k = \oplus \beta f^k,\ \beta \in R_i$$

(note that the number of summands in u^k is the same as the number of summands in *u*, because each α is a unit in R_i), thus

$$u^k = \left(u^k\right)^{f_1^k} = \oplus\left(\beta f^k\right)^{f_1^k} = \oplus \beta t f^k,\ t \in T.$$

The last step follows from the fact that conjugation will preserve the order on the *fT*'s in the ordered group *F*. Since $\left(f^k\right)^{f_1^k} = tf^k$, we can choose k large enough so that all the f^k commute with each other and with *T*. Thus we may assume that

$$u^k = \oplus \beta f^k.$$

Again, we put together all β with the same f^kT. In other words, we assume that $u^k = \oplus \beta f^k$ with all f^kT different. Note that these new values of β all lie in $\mathbb{Z}T$. Furthermore, we now obtain for each $t \in T$,

$$u^k = \left(u^k\right)^t = \oplus(\beta)^t f^k,$$

And thus $\beta^t = \beta$. So the ring R generated by all the β is commutative. Again, is necessary, replacing k by a high enough power, we may assume that the group A generated by all the f^k in the summation of u^k is a torsion-free Abelian group, and thus a free Abelian group. Consequently

$$u^k \in RA,$$

the commutative group ring of A over R. Let $N = Rad(R)$ be the set of nilpotent elements of R. Now $\mathbb{Z}T$ has only trivial idempotents (Sehgal, 1978). Hence since $R \subseteq \mathbb{Z}T$ and since idempotents of R/N can be lifted to R, it follows that R/N also has only trivial idempotents. Therefore (Sehgal, 1978) together with an inductive argument tells us that $(R/N)A$ has only trivial units. It follows that

$$u^k = \beta f^k + \text{ nilpotent elements.}$$

But as each β is a sum of units in various R_i, it follows that the last term must be zero. Hence $u^k = \beta f^k$, and thus all f's in the original decomposition of $u = \oplus_i \alpha f$ were in the same coset of T. Thus $u = rf$ as required.

We give two important consequences of the last result. We say that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is trivial if it contains only trivial units.

Corollary 3.6. *Let G be a finitely generated nilpotent group. If* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$ *is trivial, then* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ *is trivial.*

Proof. Let $u \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ be nontrivial. Then support of u contains two different elements, say x and y. Since finitely generated nilpotent groups are residually finite, there exists a finite factor $G/N = \overline{G}$ so that $\overline{x} \neq \overline{y}$ in $\overline{G}$ (Robinson, 1982). Hence $\overline{u}$ has in its support two different elements, and thus $\overline{u}$ is of finite order (Sehgal, 1993). By Proposition 3.5 we write $u = rg, r \in \mathbb{Z}T, g \in G$. Since u is central, r commutes with g. It then follows easily that $\overline{r}$, and hence also r, is of finite order. Morover, there exists a positive integer n such that $(g^n, T) = 1$. Consequently it follows from $u^n = r^n g^n$ that r^n is a nontrivial unit of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$.

Corollary 3.7. *Let G be a finitely generated nilpotent group. Then* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$ *is finitely generated. Furthermore,* $(\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}T))\mathcal{Z}(G)$ *is of finite index in* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$.

Proof. Let $S = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$. First we show that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))/S\mathcal{Z}(G)$ is a torsion group of bounded exponent. Indeed, let $u \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. Because of Proposition 3.5 write $u = rg$, with $r \in \mathcal{Z}(\mathbb{Z}T)$ and $g \in G$. Considering the natural ephimorphism $\mathbb{Z}G \to \mathbb{Z}(G/T)$ and using the fact that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}(G/T)))$ is trivial because *G/T* is ordered, it follows that $gT \in \mathcal{Z}(G/T)$. Hence $(g^k, T) = 1$ and $g^l \in \mathcal{Z}(G)$ for $k = |Aut(T)|$ and $l = k|T|$. Now since *u* is central, *r* and *g* commute. Therefore

$$u^l = r^l g^l \text{ and } r^l \in S$$

Consequently $u^l \in S\mathcal{Z}(G)$, and the claim follows.

As a subgroup of finitely generated group $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T)$, the group *S* itself finitely generated. Hence so is $S\mathcal{Z}(G)$. Since the torion subgroup of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite (Sehgal, 1978), the above claim now yields that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is indeed finitely generated.

We will now construct finitely many generators for the central units of any finitely generated nilpotent group.

Let n be the nilpotency class of *T* and *h* the Hirch numner of *G/T*. Let $k = |Aut(T)|$. Further let $x_1, ..., x_h$ be elements of G such that for each $1 \le i \le h$ the group $G_i = \langle T, x_1, ..., x_i \rangle$ is normal in G and $G_i / G_{i-1} \cong \mathbb{Z}$, where $G_0 = T$. For any generator $b_{(n)}$ described in Proposition 3.4 define

$$b_{(n)}^{(0)} = b_{(n)}$$

and for $1 \le i \le h$

$$b_{(n)}^{(i)} = \prod_{0 \le j \le k} (b_{(n)}^{(i-1)})^{x_i^j}$$

Since each $b_{(n)}$ is in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T)$, the order of the conjugates in the product expression for $b_{(n)}^{(i)}$ is unimportant. It follows by induction that $b_{(n)}^{(i)}$ is in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G_i))$. In particular,

$$b_{(n)}^{(h)} \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)).$$

Theorem 3.8. *Let G be a finitely generated nilpotent group.Suppose n is the nilpotency class of T and h is the Hirsch number of G/T. Then* $\left\langle b_{(n)}^{(h)} \middle| b \text{ a Bass cyclic of } \mathbb{Z}T \right\rangle \mathcal{Z}(G)$ *is of finite index in* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$.

Proof. Because of Corollary 3.7 the group $S\mathcal{Z}(G)$ with $S = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$ is of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. Let $\alpha_1,...,\alpha_p$ be a finite set of generators for *S*. By Proposition 3.4 there exists a positive integer *m* such that all $\alpha_1^m,...,\alpha_p^m$ are in $\left\langle b_{(n)} \middle| b \text{ a Bass cyclic of } \mathbb{Z}T \right\rangle$. For simplicity, write $\alpha = \alpha_1^m$. Then

$$\alpha = \prod b_{(n)} ,$$

Where the product runs over a finite number of Bass cyclic units of $\mathbb{Z}T$. Since α is in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, and using the notation introduced above, we obtain

$$\alpha^k = \alpha\alpha^{x_1}...\alpha^{x_1^{k-1}} .$$

As each $b_{(n)}$ is central in $\mathbb{Z}T$, this implies

$$\alpha^k = \prod b_{(n)}^{(1)}.$$

Continuing this process one obtains that

$$\alpha^{k^h} = \prod b_{(n)}^{(h)}.$$

Since the group generated by $\alpha_1^{mk^h}...\alpha_p^{mk^h}$ is of finite index in *S*, the result follows. Note that Corollary 3.6 can now also be obtained as an easy consequence of Theorem 3.8.

We give an example showing that the converse of Corollary 3.6 does not hold.

Example. Let $G = \left\langle a, x \middle| a^x = a^3, a^8 = 1 \right\rangle$. Clearly *G* is a nilpotent group with $T = \langle a \rangle$, a cyclic group of order 8. From Higman's result (Sehgal, 1978) it follows that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, modulo the trivial units, is a free Abelian group of rank 1. We now show that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ contains only trivial units. For this suppose *u* is a nontrivial central unit in $\mathbb{Z}G$. By Proposition 3.5, we can write $u = rx^i$ for some integer *i* and $r \in \mathcal{U}(\mathbb{Z}T)$. We know from the above that *r* is of infinite order, and since *r* commutes with *x*, it must be in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$.Because the only Bass cyclic unit, up to inverses, in $\mathbb{Z}T$ is

$$b = (1 + a + a^2)^4 - 10\hat{a}, \quad \hat{a} = 1 + a + ... + a^7$$

Proposition 2 yields that

$$r^k = b^l ,$$

For some nonzero integers *k,l*. Observe, however, that $b^x = b^{-1}$. Since $b^l = r^k$ is central in $\mathbb{Z}G$, we obtain $b^l = b^{-l}$, contradicting the fact that *b* is of infinite order.

3.3 Central Units Of Integral Group Rings

3.3.1. FC- Center

There is a classical result of G.Higman (Higman, 1940) that if *A* is a finite abelian group then any torsion unit of $\mathbb{Z}A$, the integral group ring of *A*, is trivial (i.e, of the form $\pm a,\ a \in A$). This was extended by Sehgal (Sehgal, 1978) to prove that if *A* is arbitrary abelian, then any unit μ of $\mathbb{Z}A$ can be written as a product of αa, with $\alpha \in \mathbb{Z}T$, where *T* denotes the torsion subgroup of *A*, and $a \in A$. Later it was proved by Sehgal (Sehgal, 1978) that, under some stringent conditions, the last result extends to nilpotent groups. In other direction, one could ask if a similar description exists for central units in $\mathbb{Z}G$. Indeed this was done by Jespers, Parmenter and Sehgal in (Jespers, 1996) for finitely generated nilpotent groups. Using similar methods we extend this result to arbitrary groups. Recall that in a group *G*, the set:

$$\Phi(G) = \left\{ g \in G \middle| \ |G : C_G(g)| < \infty \right\}$$

is a characteristic subgroup of *G*, called the FC-centre (Passman,1977)

Theorem 3.9. *Let G be any group. Let $\Phi(G)$ denote FC-centre of G and let $T = T\Phi(G)$ be the torsion subgroup of $\Phi(G)$. Then every central unit μ of $\mathbb{Z}A$ can be written in the form $\mu = \omega g$, with $\omega \in \mathbb{Z}T$ and $g \in \Phi(G)$, morover ω and g commute.*

Further we are able to produce a finite set of generators for a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ when the FC-centre of *G* is finitely generated. We shall always use $\mathcal{Z}(H)$ to denote the centre of the group *H.*

Theorem 3.10. *Let G be a group such that its FC-centre $\Phi = \Phi(G)$ is finitely generated. Let $\{z_1, ..., z_d\}$ be a set of generators of the center of G. Then $\langle z_1, ..., z_d, \bar{b}_1 ... \bar{b}_r \rangle$ is a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$.*

The elements $\{\tilde{b}_1, ..., \tilde{b}_r\}$ will be described in section 2. They are related to a set $\{b_1, ..., b_r\}$ of generators of a subgroup of finite index $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$ which could be, for example, the one obtained by Ritter and Sehgal (Ritter and Sehgal, 1993)

Since any subgroup of a finitely generated nilpotent group is finitely generated Theorem 2 is also an extension of the results in (Jespers, 1996)

3.3.2. Central Units in $\mathbb{Z}G$ of an *FC*-Group

The form of a central unit in $\mathbb{Z}G$ has been described in (Jespers, 1996) in the case when G is finitely generated nilpotent group. In this section we shall prove that a similar result holds for arbitrary groups. To do so we firstly consider the case where G is an FC group and then show how to reduce the general case to this one.

Proposition 3.11 *Let G be an FC group. Then, every central unit* μ *of* $\mathbb{Z}G$ *can be written in the form* $\mu = \omega g$ *, with* $\omega \in \mathbb{Z}T$ *and* $g \in G$ *.*

Proof. Given a central unit $\mu \in \mathbb{Z}G$, we can work in the integral group ring of the group generated by the element in its support so, without loss of generality, we may assume that G is finitely generated and hence that T is finite. We can write $\mathbb{Q}T$ as a direct sum of simple components:

$$\mathbb{Q}T = \oplus_{j=1}^{r} A_j .$$

The central idempotents of $\mathbb{Q}T$ are not necessarily central in $\mathbb{Q}G$, so we let G act by conjugation on these idempotents. Adding the components corresoponding to all the idempotents in each orbit under this action, we can write

$$\mathbb{Q}T = \oplus_{j=1}^{n} R_i$$

where each R_i is a direct sum of simple rings and is invariant under conjugation by elements in G.

Now, set $F = G/T$. We can wrtie $\mathbb{Q}G$ as a cross product:

$$\mathbb{Q}G = (\mathbb{Q}T) * F \cong \oplus_{i=1}^{n} (R_i * F)$$

Hence, the unit μ can be viewed as a tuple $\mu = (u_1, ..., u_n)$ where $u_i \in R_i * F$, $1 \le i \le n$, is of the form $u_i = \sum_h x_h \overline{f}_h$ with $x_h \in R_h, f_h \in F$. Notice that we may assume that the elements in F have been chosen in such a way that $f_l T \ne f_k T$ if $l \ne k$.

Claim 1. With the notations above, each coefficient x_j *is a unit in R.*

Proof. Notice first that since μ is also central in $\mathbb{Q}G$, for each element $f \in F$ we have that $f\mu = \mu f$. Hence, in a component of the form $R_i * F$ we can compute

$$\overline{f} u_i = \sum \overline{f} x_j \overline{f}_j = \sum x_j^{f-1} \overline{f f_j} = \sum x_j^{f-1} \tau \overline{f f_j} \text{ with } \tau \in \mathcal{U}(R_i)$$

And taking into account the fact that $F = G/T$ is abelian, we also have

$$u_i f = \sum x_j \overline{f}_j \overline{f} = \sum x_j \overline{f f_j} = \sum x_j \tau \overline{f f_j}$$

So we see that

$$x_j^{f^{-1}} = x_j \tag{3.1}$$

Notice that if we write u_i as a tuple in the direct sum $A_{1_i} \oplus ... \oplus A_{n_i}$, the definition of R_i implies that any two components can be switched by a conjugation by an element of *G*, so equation (3.1) actually means that $\mu = (a,...,a)$ is a diagonal element.

Also, for an element $t \in T$ we have that:

$$tu_i = \sum t.x_j.\overline{f}_j,$$
$$u_i t = \sum x_j.\overline{f}_j.t = \sum x_j.t^{f_{j-1}}.\overline{f}_j,$$

Because of the choice of the elements f_j, since $tu = ut$ we have that

$$x_j t^{f-1} = tx_j. \tag{3.2}$$

Equation (3.2) shows that:

$$x_j R_i = R_i x_j.$$

So

$$R_i x_j R_i = R_i x_j.$$

Since R_i is a direct sum of simple components, it follows that, for each simple component *A* of R_i, we have that $Ax_j A = Ax_j$. Now, if $x_j \neq 0$ since x_j is diagonal, its projection in each simple component is non-zero, so $Ax_j A$ is a non trivial two-sided ideal. Then it follows that $Ax_j A = A$; consequently $Ax_j = A$ showing that x_j is invertible in each component. Hence, x_j is invertible in R_i.

Claim 2. Each component u_i *is actually of the form* $u_i = x_j \overline{f}_j$

Proof. In fact each component u_i is not a zero divisor and the group *F* is abelian, torsion free and thus ordered so, using the fact that the coefficients x_j are not zero divisors, a standart argument shows that u_i must be trivial in $R_i * F$, as desired.

Now we are ready to prove our statement. We can write μ in the form

$$\mu = \sum \alpha_i \overline{f}_i \in \oplus_i R_i * F, \quad \alpha_i \in R_i,\ f_i \in F .$$

Collecting together coefficients whenever $f_h T = f_k T$ and changing notation we can write μ in the form

$$\mu = \sum \alpha_i' \overline{f}_i \in \oplus_i R_i * F, \quad \alpha_i' \in R_i,\ f_i \in F,\ f_i T \neq f_j T \text{ if } i \neq j.$$

Since G is a finitely generated FC group, it is central-by-finite, so there exists a positive integer k such that $g^k \in \mathcal{Z}(G)$, for all $g \in G$. We compute:

$$\mu^k = \sum_i (\alpha_i' \overline{f}_i)^k = \sum_i \beta_i \overline{f}_i^k \in \oplus_i R_i * F, \quad \beta_i' \in R_i,\ f_i \in F. \tag{3.3}$$

Notice that, since G/T is a torsion-free abelian, we also have that $f_i^k T \neq f_j^k T$ whenever $i \neq j$.

Now, since each f_i^k is central, we have that:

$$t\mu^k = \sum t\beta_i \overline{f}_i^k$$
$$\mu^k t = \sum \beta_i \overline{f}_i^k t = \sum \beta_i t \overline{f}_i^k$$

So $\beta_i t = t\beta_i$, for all $t \in T$, showing that the ring R generated by all the coefficients β_i is commutative and, in fact, $R \subset \mathcal{Z}(\mathbb{Z}T)$.

Let A be the central subgroup generated by all the elements of R/N can be lifted to idempotents of R and $R \subset \mathbb{Z}T$ contains no nontrivial idempotents, it follows that R/N contains no nontrivial idempotents. So, it follows from (Sehgal,1978) that $(R/N)A$ has only nontrivial units. Hence:

$\mu^k = \beta f^k + v,$ where v is nilpotent.

Comoparing with the expression of μ^k given in equation (3.3) we see that $v = 0$. Hence, $\mu^k = \beta f^k$ as required.

We shall now show how to extend this result to the general case.

Proof of Theorem 3.9.

For each finite conjugacy class C of G consider the class sum $\gamma = \sum_{x \in C} x$. It is well known that the set of all these class sums forms a $\mathbb{Z}$-basis for the center of the group ring (. Passman, 1977). This means that μ is also a central unit in $\mathbb{Z}[\Phi(G)]$, so we can apply the proposition above and it follows immediately that $\mu = \omega g$, with $\omega \in \mathbb{Z}T$, where $T = T(\Phi(G))$ and $g \in \Phi(G)$.

Since μ is central, we have that $\mu^g = \omega^g g = \omega g$ so $\omega^g = g$ as stated. □

As an easy consequence, we can give an elementary proof for the following.

Corollary 3.12. *Any central unit of finite order in* $\mathbb{Z}G$ *is trivial.*

Proof. Let μ be a central unit of finite order *n*. Writing $\mu = \omega g$ as in Theorem 3.1, we have that

$$\mu^n = \omega^n g^n.$$

This means that $g^n = \omega^{-n} \in \mathbb{Z}T$. This implies that $g^n \in T$ and thus $g \in T$. It follows that $\mu \in \mathbb{Z}T$ where *T* is finite, so the result follows immediately from Higman's Theorem (Sehgal,1993)

We remark that, since finitely generated *FC* groups are residually finite, the same proof as in (Jespers, 1996) now gives the following.

Corollary 3.13. *Let T be the torsion subgroup of the FC-centre of a group G. If* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$ *is trivial then* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ *is also trivial.*

Lemma 3.14. *Let G be a group and* $\mathcal{Z}(G)$ *its centre. If* $\gamma \in 1+\Delta(G)\Delta(\mathcal{Z}(G))$ *is such that* $\gamma^n = 1$ *for some positive integer n, then* $\gamma = 1$.

Proof. Since γ can be written as finite sum of the form $\gamma = 1 + \sum (1-g)(1-z)$, we can assume that $\gamma \in 1+\Delta(G)\Delta(A)$, where *G* is a finitely generated group and *A* is a finitely generated central subgroup. If $A = 1$ there is nothing to prove. We use induction on the rank of *A* plus the order of its torsion subgroup to conclude that taking $z \in A$ and putting $\overline{G} = G/\langle z \rangle$ we have that $\overline{\gamma} = \overline{1}$ in $\overline{G}$.

It follows that $\gamma = 1+\delta,\ \delta \in \Delta(G, \langle z \rangle)$. Thus, there is a central element in the support of γ. Remembering that γ is a torsion element, it follows from (Sehgal, 1993) that $\gamma = z_0,\ z_0 \in \mathcal{Z}(G)$. We conclude that

$$z_0 - 1 \in \Delta(G)\Delta(\mathcal{Z}(G))$$

and thus $z_0 \in 1+\Delta(\mathcal{Z}(G))^2$ so z_0 belongs in the second dimension subgroup of $\mathcal{Z}(G)$. Hence $z_0 = 1$ and $\gamma = 1$.

Lemma 3.15. *Let G be a group and let* $\mu \in \mathbb{Z}G$ *be a central unit. If there exists a positive integer n such that* μ^n *is trivial, then* μ *itself is a trivial unit.*

Proof. Since $\mu^n \in \mathcal{Z}(G)$ we have that $\mu^n = 1$ in $\mathbb{Z}[G/\mathcal{Z}(G)]$. Thus μ is a central unit of finite order in this group ring, so it is trivial by Corollary 3.12.

Hence, there exists an element $g \in G$ such that $\overline{\mu} = \overline{g}$ in the quotient, so we can write

$$\mu \equiv g \pmod{\Delta(G, \mathcal{Z}(G))}$$

Using the Whitcomb argument (Sehgal,1993), it can be easily shown that there exists an element g_1 of the form $g_1 = gz$ with $z \in \mathcal{Z}(G)$ such that

$$\mu \equiv g_1 \pmod{\Delta(G)\Delta(\mathcal{Z}(G))}.$$

Thus

$$\mu^n \equiv g_1^{\,n} \pmod{\Delta(G)\Delta(\mathcal{Z}(G))}.$$

hence

$$\left(\mu g_1^{-1}\right)^n \equiv 1 \pmod{\Delta(G)\Delta(\mathcal{Z}(G))}.$$

Since by Lemma 3.14 the group of invertible elements in $1+\Delta(G)\Delta(\mathcal{Z}(G))$ is torsion free it follows that $\mu g_1^{-1} = 1$ and thus $\mu = g_1 \in G,$ as desired. □

Proposition 3.16. *Let g be a group such that its FC-centre* $\Phi = \Phi(G)$ *is finitely generated and let* $S = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \cap \mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$. *Then* $S\mathcal{Z}(G)$ *is a subgroup of finite index in* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ *and* $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ *itself is finitely generated.*

Proof. Since T is finite, it follows that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$ is finitely generated and thus, its subgroup S is also finitely generated.

We have that $\mathcal{Z}(G) \subset \mathcal{Z}(\Phi)$ and $\mathcal{Z}(\Phi)$ is of finite index in Φ, which is finitely generated. Hence $\mathcal{Z}(\Phi)$ is finitely generated so is $\mathcal{Z}(G)$.

We claim that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is of bounded exponent over $S\mathcal{Z}(G)$. In fact, given $\mu \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ we write it in the form $\mu = \omega g$ with $\omega \in \mathcal{U}(\mathbb{Z}T)$ and $g \in G.$ Then $gT \in \mathcal{Z}(G/T).$ If we set $k = |AutT|,$ we have that $(g^k, t) = 1,\ \forall t \in T.$ Given any element $x \in G,$ we have that:

$$\left(g^{k|T|}\right)^x = \left(g^{kx}\right)^{|T|} = \left(g^k t\right)^{|T|}, \text{ for some } t \in T.$$

Hence

$$\left(g^{k|T|}\right)^x = g^{k|T|} t^{|T|} = g^{k|T|}.$$

So, if we set $h = k|T|$ we have that $g^h \in \mathcal{Z}(G).$ □

Now $\mu^h = \omega^h g^h$ so $\omega^h = \mu^h g^{-h} \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \cap \mathcal{Z}(\mathbb{Z}T).$ Consequently, $\mu^h \in S\mathcal{Z}(G)$.

Finally, we observe that $T(\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)))$ is trivial by Corollary 3.12, so it is finite. It follows that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finitely generated, as desired.

Let $\{b_1, b_2, ..., b_r\}$ be any set of generators of a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$. For example, this could be the set of generators explicitly constructed by Ritter-Sehgal (Sehgal, 1993). Let X be a transversal of the centralizer $C_G(T)$. For each element b_i we define:

$$b_i = \prod_{x \in X} b_i^x.$$

Notice that this product is independent of the order of its factors since they belong to $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$ which is commutative and is normalized by G. Clearly $\tilde{b}_i \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)),\ 1 \le i \le r$.

Let $\{\alpha_1, ..., \alpha_s\}$ be a set of generators of S. Since $\langle b_1, ..., b_r \rangle$ is of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}T))$, there exists a positive integer m such that :

$$\alpha_i^m \in \langle b_1, ..., b_r \rangle,\ \ 1 \le i \le s.$$

Hence, each element α_i^m can be written as a product:

$$\alpha_i^m = \prod b_j \ .$$

So,

$$\alpha_i^{m|X|} = \prod \left(\alpha_i^m\right)^x = \prod \tilde{b}_j.$$

This shows that $\alpha_i^{m|X|} \in \left\langle \tilde{b}_1, ..., \tilde{b}_r \right\rangle,\ 1 \le i \le s$. It follows that $\left\langle \tilde{b}_1, ..., \tilde{b}_r \right\rangle$ is a subgroup of finite index in S.

Since we have shown that $\left|\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) : S\mathcal{Z}(G)\right|$ is finite, the proof of Theorem 3.10 follows from the consideration above.

CHAPTER 4

CHARACTERIZATION OF CENTRAL UNITS OF $\mathbb{Z}A_n$

4.1 Introduction to the Problem of Characterization of Central units of $\mathbb{Z}A_n$

For any G the integral group ring $\mathbb{Z}G$, It is trivial that $G \subseteq \mathcal{U}_1(\mathbb{Z}G)$. The units of $\mathcal{U}_1(\mathbb{Z}G)$ are called trivial if $\mathcal{U}_1(\mathbb{Z}G) = G$. On the other hand, let us $\mathcal{Z}(\mathcal{U}_1(\mathbb{Z}G))$ denote the center of the normalized units and $\mathcal{U}_1(\mathcal{Z}(\mathbb{Z}G))$ denote the normalized units of the center of the integral group ring. In fact, these two notations have the same meaning (Aleev, 2000), since

$$\mathcal{U}_1(\mathcal{Z}(\mathbb{Z}G)) = \mathcal{U}_1(\mathbb{Z}G) \cap \mathcal{Z}(\mathbb{Z}G) = \mathcal{Z}(\mathcal{U}_1(\mathbb{Z}G)).$$

Now let us recall normalizer property once more

$$\mathcal{N}_{\mathcal{U}_1}(G) = G.\mathcal{Z}(\mathcal{U}_1(\mathbb{Z}G)) \qquad \textbf{(N.P.)}$$

In order to determine the normalizer, first of all, we will investigate the central units of the normalized units of the integral group rings $\mathbb{Z}A_n$. Characterization of central units still a problem especially if G is a simple group. There are some studies to determine the central units of $\mathbb{Z}A_n$. Firstly, for $n = 5,6$ central units of $\mathbb{Z}A_n$ is characterized (Aleev, 1994). By using pell-equations, another characterization for $n = 5$ is given. Later, By using irreducible representations, a different characterization is given (Bilgin, 2004). Unfortunately, methods used up to now, can not be generalized for this problem. Increasing *n* leads to increasing the order of alternating group factorially. So for $(n \geq 7)$ the problem is open.

In this study, it is shown that central units can be determined more easily by using irreducible characters, and also for larger *n*'s, this method is applicable. Of course we will use a software GAP 4.3 to get irreducible characters of the alternating groups.

4.2 Method

GAP (Groups, Algorithms, Programming) is used to obtain the irreducible characters of A_n up to $n = 20$. Besides, the order of the conjugacy classes and the order of the centralizers are obtained by using **GAP.**

i) In order to obtain irreducible characters we used following command :

```
gap> tbl:=CharacterTable("A5");
CharacterTable( "A5" )
gap> Display(tbl);
A5

     2  2  2  .  .  .
     3  1  .  1  .  .
     5  1  .  .  1  1

       1a 2a 3a 5a 5b
    2P 1a 1a 3a 5b 5a
    3P 1a 2a 1a 5b 5a
    5P 1a 2a 3a 1a 1a

X.1     1  1  1  1  1
X.2     3 -1  .  A *A
X.3     3 -1  . *A  A
X.4     4  .  1 -1 -1
X.5     5  1 -1  .  .

A = -E(5)-E(5)^4
  = (1-ER(5))/2 = -b5
```

ii) In order to obtain size of conjugate classes we used following command :

```
gap> tbl:=CharacterTable("A5");
CharacterTable( "A5" )
gap> SizesConjugacyClasses(tbl);
[ 1, 15, 20, 12, 12 ]
```

iii) Here another way to obtain the irreducible characters from GAP program.

```
gap> A:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> Ch:=IrrDixonSchneider(A);
[ Character( CharacterTable( Alt( [ 1 .. 5 ] ) ), [ 1, 1, 1, 1, 1 ] ),
  Character( CharacterTable( Alt( [ 1 .. 5 ] ) ),
    [ 3, -1, 0, -E(5)-E(5)^4, -E(5)^2-E(5)^3 ] ),
  Character( CharacterTable( Alt( [ 1 .. 5 ] ) ),
    [ 3, -1, 0, -E(5)^2-E(5)^3, -E(5)-E(5)^4 ] ),
  Character( CharacterTable( Alt( [ 1 .. 5 ] ) ), [ 4, 0, 1, -1, -1 ] ),
  Character( CharacterTable( Alt( [ 1 .. 5 ] ) ), [ 5, 1, -1, 0, 0 ] ) ]
```

iv) To carry out algebraic operations we used Maple 10.0

- The conjugacy classes of alternating groups between A_3 and A_{10} are listed below in the order of occurance. Being in the same conjugacy class in S_n symmetric group, *n*'th conjugacy class is divided into two conjugacy classes as

C_n' and C_n'' in A_n. Considering this situation, the sum of n'th conjugacy class is $C_i = \sum_{g \in \overline{C}_i} g$, and meaning of C_i is connected to the group *G*.

For example $\overline{C}_2 = \{g(123)g^{-1} : g \in A_n\}$ means $C_2 = (123)$ when $n = 3$, and means $C_2 = (123) + (134) + (142) + (243)$ when $n = 4$. According to this, the conjugacy classes up to $n = 10$ are listed below.

$\overline{C}_1 = \{(1)\}$, $\overline{C}_2 = \{g(1,2,3)g^{-1} : g \in G\}$, $\overline{C}_3 = \{g(1,2)(3,4)g^{-1} : g \in G\}$

$\overline{C}_4' = \{g(1,2,3,4,5)g^{-1} : g \in G\}, \overline{C}_4'' = \{g(1,3,5,2,4)g^{-1} : g \in G\}$ ve $\overline{C}_4 = \overline{C}_4' \cup \overline{C}_4''$

$\overline{C}_5 = \{g(1,2,3)(4,5,6)g^{-1} : g \in G\}$ $\qquad \overline{C}_6 = \{g(1,2,3,4)(5,6)g^{-1} : g \in G\}$

$\overline{C}_7 = \{g(1,2)(3,4)(5,6,7)g^{-1} : g \in G\}$

$\overline{C}_8' = \{g(1,2,3,4,5,6,7)g^{-1} : g \in G\}, \overline{C}_8'' = \{g(1,3,5,7,2,4,6)g^{-1} : g \in G\}$ ve $\overline{C}_8 = \overline{C}_8' \cup \overline{C}_8''$

$\overline{C}_9 = \{g(1,2)(3,4)(5,6)(7,8)g^{-1} : g \in G\}$ $\qquad \overline{C}_{10} = \{g(1,2,3,4)(5,6,7,8)g^{-1} : g \in G\}$

$\overline{C}_{11} = \{g(1,2,3,4,5,6)(7,8)g^{-1} : g \in G\}$

$\overline{C}_{12}' = \{g(1,2,3,4,5)(6,7,8)g^{-1} : g \in G\}, \overline{C}_{12}'' = \{g(1,2,3,4,5)(6,8,7)g^{-1} : g \in G\}, \overline{C}_{12} = \overline{C}_{12}' \cup \overline{C}_{12}''$

$\overline{C}_{13} = \{g(1,2,3)(4,5,6)(7,8,9)g^{-1} : g \in G\}$

$\overline{C}_{14}' = \{g(1,2,3,4,5,6,7,8,9)g^{-1} : g \in G\}, \overline{C}_{14}'' = \{g(1,2,3,4,5,6,7,9,8)g^{-1} : g \in G\}, \overline{C}_{14} = \overline{C}_{14}' \cup \overline{C}_{14}''$

$\overline{C}_{15} = \{g(1,2)(3,4)(5,6,7,8,9)g^{-1} : g \in G\}$ $\qquad \overline{C}_{16} = \{g(1,2,3,4)(5,6,7)(8,9)g^{-1} : g \in G\}$

$\overline{C}_{17} = \{g(1,2,3,4)(5,6)(7,8)(9,10)g^{-1} : g \in G\}$,

$\overline{C}_{18} = \{g(1,2,3,4,5)(6,7,8,9,10)g^{-1} : g \in G\}$

$\overline{C}_{19} = \{g(1,2,3)(4,5,6)(7,8)(9,10)g^{-1} : g \in G\}$,

$\overline{C}_{20} = \{g(1,2,3,4,5,6,7,8)(9,10)g^{-1} : g \in G\}$

$\overline{C}_{21} = \{g(1,2,3,4,5,6)(7,8,9,10)g^{-1} : g \in G\}$

$\overline{C}_{22}' = \{g(1,2,3,4,5,6,7)(8,9,10)g^{-1} : g \in G\}, \overline{C}_{22}'' = \{g(1,2,3,4,5,6,7)(8,10,9)g^{-1} : g \in G\}$

$\overline{C}_{22} = \overline{C}_{22}' \cup \overline{C}_{22}''$

- Since $\mathbb{Q}A_n$ is a semi-simple ring, $\mathcal{Z}(\mathbb{Q}A_n)$ is written as the direct sum of the scalar matrices. The size of these matrices are equal to the value of the irreducible characters at the identity. The central units are going to be determined by considering this fact.

4.3 Construction of Generators of Central Units of $\mathbb{Z}A_n$

In this study, it is shown that central units can be determined by using irreducible characters. Furthermore, it is shown that this method is valid for larger *n*s.

Theorem 4.1 $\mathcal{U}_1(\mathcal{Z}(\mathbb{Z}A_5)) = < 49C_1 - 10C_3 + 26C_4' - 16C_4'' >$.

Proof. Firstly,let us consider the irreducible character table of A_5.

Table-4.1: Character Table of A_5

g	$\overline{C}_1$	$\overline{C}_2$	$\overline{C}_3$	$\overline{C}_4'$	$\overline{C}_4''$
$\lvert C(g_i)\rvert$ $\lvert g_i^{A_5}\rvert$	60 1	3 20	4 15	5 12	5 12
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	α	β
χ_5	3	0	-1	β	α

where $\alpha = -\varepsilon^2 - \varepsilon^3 = \frac{1+\sqrt{5}}{2}$, $\beta = -\varepsilon - \varepsilon^4 = \frac{1-\sqrt{5}}{2}$,$(\varepsilon = e^{2\pi i/5})$

If $\gamma \in \mathcal{U}_1(\mathcal{Z}(\mathbb{Z}A_5))$ then we can write $\gamma = \gamma_1 C_1 + \gamma_2 C_2 + \gamma_3 C_3 + \gamma_4 C_4' + \gamma_5 C_4''$. Therefore, the values of γ at the irreducible characters are $\chi_j(\gamma) = \sum_{i=1}^{5} \gamma_i \lvert C(g_i)\rvert . \chi_j(g_i)$, $(j = 1, \ldots, 5)$.

Now let us consider these values one by one.

$$\chi_1(\gamma) = \gamma_1.1.1 + \gamma_2.20.1 + \gamma_3.15.1 + \gamma_4.12.1 + \gamma_5.12.1 = \lambda_1, (\lambda_1 \in \mathcal{U}(\mathbb{Z}))$$

$$\boxed{\gamma_1 + 20\gamma_2 + 15\gamma_3 + 12\gamma_4 + 12\gamma_5 = \varepsilon(\gamma) = 1} \qquad (4.1.1)$$

$$\chi_2(\gamma) = \gamma_1.1.4 + \gamma_2.20.1 + \gamma_3.15.0 + \gamma_4.12.(-1) + \gamma_5.12.(-1) = 4\lambda_2, (\lambda_2 \in \mathcal{U}(\mathbb{Z}))$$

$$\boxed{\gamma_1 + 5\gamma_2 - 3\gamma_4 - 3\gamma_5 = \pm 1} \qquad (4.1.2)$$

$$\chi_3(\gamma) = \gamma_1.1.5 + \gamma_2.20.(-1) + \gamma_3.15.1 + \gamma_4.12.0 + \gamma_5.12.0 = 5\lambda_3, (\lambda_3 \in \mathcal{U}(\mathbb{Z}))$$

$$\boxed{\gamma_1 - 4\gamma_2 + 3\gamma_3 = \pm 1} \qquad (4.1.3)$$

$$\chi_4(\gamma)=\gamma_1.1.3+\gamma_2.20.0+\gamma_3.15.(-1)+\gamma_4.12.\alpha+\gamma_5.12.\beta=3\lambda_4,(\lambda_4\in\mathcal{U}(\mathbb{Z}[\alpha])$$

$$\boxed{\gamma_1-5\gamma_3+4\alpha\gamma_4+4\beta\gamma_5=\lambda_4\in\mathcal{U}(\mathbb{Z}[\sqrt{5}]),\ \ (\alpha=\frac{1+\sqrt{5}}{2},\beta=\frac{1-\sqrt{5}}{2})} \qquad (4.1.4)$$

$$\chi_5(\gamma)=\gamma_1.1.3+\gamma_2.20.0+\gamma_3.15.(-1)+\gamma_4.12.\beta+\gamma_5.12.\alpha=3\lambda_5,(\lambda_5\in\mathcal{U}(\mathbb{Z}[\alpha])$$

$$\boxed{\gamma_1-5\gamma_3+4\beta\gamma_4+4\alpha\gamma_5=\lambda_5\in\mathcal{U}(\mathbb{Z}[\sqrt{5}]),\ \ (\alpha=\frac{1+\sqrt{5}}{2},\beta=\frac{1-\sqrt{5}}{2})} \qquad (4.1.5)$$

Here since $d\left|\chi_j(\gamma)-\chi_1(\gamma)\right.$ for $d>2$ and $j=2,3$ we can write $\chi_j(\gamma)=1$ for $(j=2,3)$.

So the matrix of coefficients can be written as the following.

$$\begin{bmatrix}1 & 20 & 15 & 12 & 12\\ 1 & 5 & 0 & -3 & -3\\ 1 & -4 & 3 & 0 & 0\\ 1 & 0 & -5 & 4\alpha & 4\beta\\ 1 & 0 & -5 & 4\beta & 4\alpha\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\\ \gamma_4\\ \gamma_5\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 1\\ \lambda_4\\ \lambda_5\end{bmatrix}$$

If we consider the first three equations and the first three variables we get:

$$\begin{bmatrix}1 & 20 & 15\\ 1 & 5 & 0\\ 1 & -4 & 3\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix}=\begin{bmatrix}1-12(\gamma_4+\gamma_5)\\ 1+3(\gamma_4+\gamma_5)\\ 1\end{bmatrix}$$

So the parametric solution is

$$\gamma_1=1+3(\gamma_4+\gamma_5),\quad \gamma_2=0,\quad \gamma_3=-(\gamma_4+\gamma_5)$$

By substituting this solution into the equation (4.1.4) , we get

$$(1+10(\gamma_4+\gamma_5))+2(\gamma_4-\gamma_5)\sqrt{5}\in\mathcal{U}(\mathbb{Z}[\sqrt{5}])$$

For the smallest $k\in\mathbb{N}$ it is like the following

$$(1+10(\gamma_4+\gamma_5))+2(\gamma_4-\gamma_5)\sqrt{5}=(2+\sqrt{5})^k$$

We get this when $k=4$. So the solution is as follows.

$$\gamma_4=26,\gamma_5=-10\Rightarrow\gamma_3=-16,\gamma_1=49,\gamma_2=0$$

Thus, the generator is

$$\gamma=49C_1-10C_3+26C_4'-16C''$$

and its inverse is

$$\gamma^{-1}=49C_1-10C_3-16C_4'+26C_4''.$$

Theorem 4.2. $\mathcal{U}_1(\mathcal{Z}(\mathbb{Z}A_6)) = < 18433C_1 - 2304C_2 + 3728C_4' - 1424C_4'' - 2304C_5 >$.

Proof. First, let us consider the irreducible character table of A_6 .

Table-4.2: Character Table of A_6

$C(g_i)$	$\bar{C}_1$	$\bar{C}_2$	$\bar{C}_3$	$\bar{C}_4'$	$\bar{C}_4''$	$\bar{C}_5$	$\bar{C}_6$
$\lvert C(g_i)\rvert$ $\lvert g_i^{A_6}\rvert$	360 1	9 40	8 45	5 36	5 36	9 40	4 90
χ_1	1	1	1	1	1	1	1
χ_2	5	2	1	0	0	-1	-1
χ_3	10	1	-2	0	0	1	0
χ_4	9	0	1	-1	-1	0	1
χ_5	5	-1	1	0	0	2	-1
χ_6	8	-1	0	α	β	-1	0
χ_7	8	-1	0	β	α	-1	0

where $\alpha = -\varepsilon^2 - \varepsilon^3 = \frac{1+\sqrt{5}}{2}$, $\beta = -\varepsilon - \varepsilon^4 = \frac{1-\sqrt{5}}{2}$,$(\varepsilon = e^{2\pi i/5})$

If $\gamma \in U_1(Z(ZA_6)$ then we can write $\gamma = \sum_{i=1}^{7} \gamma_i C_i$. ($C_i = \sum_{g \in C(g_i)} g$). Thus

$$\chi_j(\gamma) = \sum_{i=1}^{7} \gamma_i \lvert C(g_i)\rvert \chi_j(g_i), \text{ (j=1,2,...,7)}$$

Now let us calculate the irreducible characters of γ for each $j = 1, 2, ..., 7$.

$$\chi_1(\gamma) = \gamma_1.1.1 + \gamma_2.40.1 + \gamma_3.45.1 + \gamma_4.72.1 + \gamma_5.72.1 + \gamma_6.40.1 + \gamma_7.90.1$$

$$\boxed{\gamma_1 + 40\gamma_2 + 45\gamma_3 + 72\gamma_4 + 72\gamma_5 + 40\gamma_6 + 90\gamma_7 = \varepsilon(\gamma) = 1} \quad (4.2.1)$$

$$\chi_2(\gamma) = \gamma_1.1.5 + \gamma_2.40.2 + \gamma_3.45.1 + \gamma_4.72.0 + \gamma_5.72.0 + \gamma_6.40.(-1) + \gamma_7.90.(-1) = 5\lambda_2, (\lambda_2 \in \mathcal{U}(\mathbb{Z}))$$

$$\boxed{\gamma_1 + 16\gamma_2 + 9\gamma_3 - 8\gamma_6 - 18\gamma_7 = \pm 1} \quad (4.2.2)$$

$$\chi_3(\gamma) = \gamma_1.1.5 + \gamma_2.40.(-1) + \gamma_3.45.1 + \gamma_4.72.0 + \gamma_5.72.0 + \gamma_6.40.2 + \gamma_7.90.(-1) = 5\lambda_5, (\lambda_5 \in \mathcal{U}(\mathbb{Z}))$$

$$\boxed{\gamma_1 - 8\gamma_2 + 9\gamma_3 + 16\gamma_6 - 18\gamma_7 = \pm 1} \quad (4.2.3)$$

$$\chi_4(\gamma) = \gamma_1.1.9 + \gamma_2.40.0 + \gamma_3.45.1 + \gamma_4.72.(-1) + \gamma_5.72.(-1) + \gamma_6.40.0 + \gamma_7.90.1 = 9\lambda_4, (\lambda_4 \in \mathcal{U}(\mathbb{Z}))$$

$$\boxed{\gamma_1 + 5\gamma_3 - 8\gamma_4 - 8\gamma_5 + 10\gamma_7 = \pm 1} \quad (4.2.4)$$

$$\chi_5(\gamma)=\gamma_1.1.10+\gamma_2.40.1+\gamma_3.45(-2)+\gamma_4.72.0+\gamma_5.72.0+\gamma_6.40.1+\gamma_7.90.0=10\lambda_3,(\lambda_3\in\mathcal{U}(\mathbb{Z}))$$

$$\boxed{\gamma_1+4\gamma_2-9\gamma_3+4\gamma_6=\pm 1} \qquad (4.2.5)$$

$$\chi_6(\gamma)=\gamma_1.1.8+\gamma_2.40.(-1)+\gamma_3.45.0+\gamma_4.72.\alpha+\gamma_5.72.\beta+\gamma_6.40.(-1)+\gamma_7.90.0=8\lambda_6,(\lambda_6\in\mathcal{U}(\mathbb{Z}[\alpha]))$$

$$\boxed{\gamma_1-5\gamma_2+9\alpha\gamma_4+9\beta\gamma_5-5\gamma_6=\lambda_6} \qquad (4.2.6)$$

$$\chi_7(\gamma)=\gamma_1.1.8+\gamma_2.40.(-1)+\gamma_3.45.0+\gamma_4.72.\beta+\gamma_5.72.\alpha+\gamma_6.40.(-1)+\gamma_7.90.0=\lambda_7,(\lambda_7\in\mathcal{U}(\mathbb{Z}[\alpha]))$$

$$\boxed{\gamma_1-5\gamma_2+9\beta\gamma_4+9\alpha\gamma_5-5\gamma_6=\lambda_7} \qquad (4.2.7)$$

Here since $d\,\big|\,\chi_j(\gamma)-\chi_1(\gamma)$ and $d>2$ for each j $(j=2,3,4,5)$ we get $\chi_j(\gamma)=1$.

Therefore, the matrix of coefficients can now be written as;

$$\begin{bmatrix} 1 & 40 & 45 & 36 & 36 & 40 & 90 \\ 1 & 16 & 9 & 0 & 0 & -8 & -18 \\ 1 & 4 & -9 & 0 & 0 & 4 & 0 \\ 1 & 0 & 5 & -4 & -4 & 0 & 10 \\ 1 & -8 & 9 & 0 & 0 & 16 & -18 \\ 1 & -5 & 0 & \frac{\alpha}{2} & \frac{\beta}{2} & -5 & 0 \\ 1 & -5 & 0 & \frac{\beta}{2} & \frac{\alpha}{2} & -5 & 0 \end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\\ \gamma_4\\ \gamma_5\\ \gamma_6\\ \gamma_7\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\\1\\ \lambda_6\\ \lambda_7\end{bmatrix}$$

If we consider the first five equations and the first three and last two variables, we get:

$$\begin{bmatrix} 1 & 40 & 45 & 40 & 90 \\ 1 & 16 & 9 & -8 & -18 \\ 1 & 4 & -9 & 4 & 0 \\ 1 & 0 & 5 & 0 & 10 \\ 1 & -8 & 9 & 16 & -18 \end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\\ \gamma_6\\ \gamma_7\end{bmatrix}=\begin{bmatrix}1-72(\gamma_4+\gamma_5)\\ 1\\ 1\\ 1+8(\gamma_4+\gamma_5)\\ 1\end{bmatrix}$$

So the solution is

$$\gamma_1=1+8(\gamma_4+\gamma_5),\quad \gamma_2=-(\gamma_4+\gamma_5),\quad \gamma_3=0,\quad \gamma_6=-(\gamma_4+\gamma_5),\quad \gamma_7=0.$$

Therefore, by substituting these values into the equation (4.2.6) we get

$$1+8(\gamma_4+\gamma_5)+5(\gamma_4+\gamma_5)+9(\frac{1+\sqrt{5}}{2})\gamma_4+9(\frac{1-\sqrt{5}}{2})(\gamma_5)+5(\gamma_4+\gamma_5)\in\mathcal{U}(\mathbb{Z}[\alpha])$$

Since γ is a torsion free unit and $\mathcal{U}(\mathbb{Z}[\sqrt{5}])=\{\pm1\}\times<\frac{1+\sqrt{5}}{2}>$ for the least $k\in\mathbb{N}$

$$1+\frac{45}{2}(\gamma_4+\gamma_5)+\frac{9\sqrt{5}}{2}(\gamma_4-\gamma_5)=(\frac{1+\sqrt{5}}{2})^k$$

This can be obtained when $k=24$. Then the solution is ,

$\gamma_4=3728, \gamma_5=-1424 \Rightarrow \gamma_1=18433, \gamma_2=-2304, \gamma_3=0, \gamma_6=-2304, \gamma_7=0$

So our generator is $\gamma=18433C_1-2304C_2+3728C_4'-1424C_4''-2304C_5$ and its inverse is $\gamma^{-1}=18433C_1-2304C_2-1424C_4'+3728C_4''-2304C_5$.

Theorem 4.3. $\mathcal{U}_1(\mathcal{Z}(\mathbb{Z}A_7)), \mathcal{U}_1(\mathcal{Z}(\mathbb{Z}A_8))$ and $\mathcal{U}_1(\mathcal{Z}(\mathbb{Z}A_9))$ are trivial.

Proof. In Table 4.7, Table 4.8 and Table 4.9, all values of irreducible characters are either rational or quadratic imaginary values. By Theorem 3.1 all torsion-free units are trivial.

Theorem 4.4. $\mathcal{U}_1(\mathcal{Z}(\mathbb{Z}A_{10})) = < \gamma^{180} >$

$$where\ \ \gamma=\frac{1}{1575}(1959C_1-24C_2+4C_4-C_8+C_{12}-C_{13}-C_{20}+4C_{22}'-C_{22}'').$$

Proof. If $\gamma \in \mathcal{U}_1(\mathcal{Z}(\mathbb{Z}A_{10}))$ then $\gamma \in \mathcal{Z}(\mathbb{Z}A_{10})$ and $\gamma \in \mathcal{U}_1(\mathbb{Z}A_{10})$. Let us consider both cases one by one. $\gamma \in \mathcal{Z}(\mathbb{Z}A_{10})$ means that γ can be written as a linear combination of class sums over $\mathbb{Z}$ as follows:

$$\gamma=\gamma_1C_1+\gamma_1C_1+\ldots+\gamma_{13}C_{13}+\gamma_{14}C_{14}'+\gamma_{15}C_{14}''+\gamma_{16}C_{15}+\ldots+\gamma_{22}C_{21}+\gamma_{23}C_{22}'+\gamma_{24}C_{22}''.$$

Since $\gamma \in \mathcal{U}_1(\mathbb{Z}A_{10})$ we can consider irreducible characters of γ (see table 4.10). That is,

$$\chi_j(\gamma)=\sum_{i=1}^{24}\gamma_i\left|C(g_i)\right|\chi_i(g_i),\ \text{(j=1,2,...,22)}.$$

For j=1 we get

$$\chi_1(\gamma)=\sum_{i=1}^{24}\gamma_i\left|C(g_i)\right|.\chi_1(g_i)=\sum_{i=1}^{24}\gamma_i\left|C(g_i)\right|=\varepsilon(\gamma)=1 \qquad (4.4.1)$$

For j=2,…,22 we get

$$\chi_j(\gamma)=\sum_{i=1}^{24}\gamma_i\left|C(g_i)\right|\chi_j(g_i)=\chi_j(e)\lambda_j \in \mathbb{Z} \text{ and } \lambda_j \in \mathcal{U}(\mathbb{Z}) \Rightarrow \chi_j(\gamma)=\pm\chi_j(e). \qquad (4.4.2)$$

For j=23,24 we get

$$\chi_j(\gamma)=\sum_{i=1}^{24}\gamma_i\left|C(g_i)\right|\chi_j(g_i)=\chi_j(e)\lambda_j, \lambda_j \in \mathbb{Z}[\tfrac{1+\sqrt{21}}{2}] \Rightarrow \frac{\chi_j(\gamma)}{\chi_j(e)} \in \mathcal{U}(\mathbb{Z}[\tfrac{1+\sqrt{21}}{2}]). \qquad (4.4.3)$$

If we divide the equations (4.4.2) and (4.4.3) by $\chi_j(e)$ for j = 2, . . ,24 we can write A matrix of coefficients as follows:

$A =$

1	240	630	6048	8400	18900	25200	86400	4725	18900	25200	120960	22400	201600	201600	90720	151200	56700	72576	151200	226800	151200	86400	86400
1	160	350	2688	2800	-2100	5600	19200	525	6300	-2800	13440	0	0	0	0	0	6300	-8064	16800	-25200	-16800	-9600	-9600
1	96	198	864	480	1620	1440	0	405	1620	1440	-3456	-640	-5760	-5760	2592	0	-1620	0	0	6480	0	0	0
1	100	140	1008	700	-1050	-700	2400	-525	1050	-700	0	0	0	0	-5040	-4200	0	2016	-4200	0	4200	2400	2400
1	0	90	-432	600	-1800	0	0	225	0	1800	0	-1600	0	0	2160	0	2700	3456	-3600	0	-3600	0	0
1	48	126	0	0	-756	1008	-2304	189	252	0	0	896	0	0	0	2016	-756	0	0	-3024	0	1152	1152
1	60	0	288	300	450	-900	0	-225	-450	900	1440	800	0	0	0	1800	0	-864	-1800	0	-1800	0	0
1	16	98	-336	280	840	560	-960	105	0	-280	1344	0	0	0	-1008	0	1260	0	-1680	0	1680	-960	-960
1	40	-70	48	400	0	200	0	225	-600	-400	960	0	0	0	720	-1200	-900	576	0	0	0	0	0
1	51	63	189	-105	0	-315	-540	0	0	-315	-756	-280	1260	1260	567	0	0	0	0	0	0	-540	-540
1	-24	18	144	0	0	360	0	-135	-360	0	-576	320	0	0	432	-720	540	0	0	0	0	0	0
1	15	-45	-27	75	0	225	0	0	0	225	-540	-100	1800	-900	-405	0	0	-324	0	0	0	0	0
1	15	-45	-27	75	0	225	0	0	0	225	-540	-100	-900	1800	-405	0	0	-324	0	0	0	0	0
1	16	14	0	-224	252	-112	384	189	-84	224	0	0	0	0	0	-672	252	0	0	-1008	0	384	384
1	-20	20	48	100	150	-100	0	75	-150	-100	-480	0	0	0	-720	600	0	576	600	0	-600	0	0
1	-5	35	-147	175	0	-175	300	0	0	-175	-420	0	0	0	315	0	0	-504	0	0	0	300	300
1	-12	0	0	84	-126	-252	-288	63	126	252	0	224	0	0	0	-504	0	0	504	0	504	-288	-288
1	16	38	-96	-80	-60	80	0	-75	-60	80	384	0	0	0	-288	-480	-180	0	480	720	-480	0	0
1	24	-18	0	-24	-108	-72	0	-27	-108	-72	0	-64	-576	-576	0	432	324	0	432	0	432	0	0
1	-8	14	0	-56	-84	56	384	21	-84	56	0	0	0	0	0	336	-252	0	-336	0	336	-192	-192
1	0	-18	0	-48	-36	0	0	45	108	-144	0	128	0	0	0	0	108	0	-288	432	-288	0	0
1	0	-10	-32	0	100	0	0	-75	100	0	0	0	0	0	160	0	-100	256	0	-400	0	0	0
1	-15	0	63	0	0	0	-225	0	0	0	315	-175	0	0	0	0	0	-189	0	0	0	225α	225β
1	-15	0	63	0	0	0	-225	0	0	0	315	-175	0	0	0	0	0	-189	0	0	0	225β	225α

$$\text{where } \alpha = \frac{1+\sqrt{21}}{2}, \beta = \frac{1-\sqrt{21}}{2}$$

since $d \left| \chi_j(\gamma) - \chi_1(\gamma) \right.$ and $d > 2$ for each j $(j = 2, 3, 4, ..., 22)$ we get $\chi_j(\gamma) = 1$. If we neglect two rows in the last and transfer 2 columns on the right of the matrix A to the right hand-side we get the following equation

1	240	630	6048	8400	18900	25200	86400	4725	18900	25200	120960	22400	201600	201600	90720	151200	56700	72576	151200	226800	151200
1	160	350	2688	2800	-2100	5600	19200	525	6300	-2800	13440	0	0	0	0	0	6300	-8064	16800	-25200	-16800
1	96	198	864	480	1620	1440	0	405	1620	1440	-3456	-640	-5760	-5760	2592	0	-1620	0	0	6480	0
1	100	140	1008	700	-1050	-700	2400	-525	1050	-700	0	0	0	0	-5040	-4200	0	2016	-4200	0	4200
1	0	90	-432	600	-1800	0	0	225	0	1800	0	-1600	0	0	2160	0	2700	3456	-3600	0	-3600
1	48	126	0	0	-756	1008	-2304	189	252	0	0	896	0	0	0	2016	-756	0	0	-3024	0
1	60	0	288	300	450	-900	0	-225	-450	900	1440	800	0	0	0	1800	0	-864	-1800	0	-1800
1	16	98	-336	280	840	560	-960	105	0	-280	1344	0	0	0	-1008	0	1260	0	-1680	0	1680
1	40	-70	48	400	0	200	0	225	-600	-400	960	0	0	0	720	-1200	-900	576	0	0	0
1	51	63	189	-105	0	-315	-540	0	0	-315	-756	-280	1260	1260	567	0	0	0	0	0	0
1	-24	18	144	0	0	360	0	-135	-360	0	-576	320	0	0	432	-720	540	0	0	0	0
1	15	-45	-27	75	0	225	0	0	0	225	-540	-100	1800	-900	-405	0	0	-324	0	0	0
1	15	-45	-27	75	0	225	0	0	0	225	-540	-100	-900	1800	-405	0	0	-324	0	0	0
1	16	14	0	-224	252	-112	384	189	-84	224	0	0	0	0	0	-672	252	0	0	-1008	0
1	-20	20	48	100	150	-100	0	75	-150	-100	-480	0	0	0	-720	600	0	576	600	0	-600
1	-5	35	-147	175	0	-175	300	0	0	-175	-420	0	0	0	315	0	0	-504	0	0	0
1	-12	0	0	84	-126	-252	-288	63	126	252	0	224	0	0	0	-504	0	0	504	0	504
1	16	38	-96	-80	-60	80	0	-75	-60	80	384	0	0	0	-288	-480	-180	0	480	720	-480
1	24	-18	0	-24	-108	-72	0	-27	-108	-72	0	-64	-576	-576	0	432	324	0	432	0	432
1	-8	14	0	-56	-84	56	384	21	-84	56	0	0	0	0	0	336	-252	0	-336	0	336
1	0	-18	0	-48	-36	0	0	45	108	-144	0	128	0	0	0	0	108	0	-288	432	-288
1	0	-10	-32	0	100	0	0	-75	100	0	0	0	0	0	160	0	-100	256	0	-400	0

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_8 \\ \gamma_9 \\ \gamma_{10} \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{15} \\ \gamma_{16} \\ \gamma_{17} \\ \gamma_{18} \\ \gamma_{19} \\ \gamma_{20} \\ \gamma_{21} \\ \gamma_{22} \end{bmatrix} = \begin{bmatrix} 1-151200(\gamma_{23}+\gamma_{24}) \\ 1+16800(\gamma_{23}+\gamma_{24}) \\ 1 \\ 1-4200(\gamma_{23}+\gamma_{24}) \\ 1 \\ 1-2016(\gamma_{23}+\gamma_{24}) \\ 1 \\ 1+1680(\gamma_{23}+\gamma_{24}) \\ 1 \\ 1+945(\gamma_{23}+\gamma_{24}) \\ 1 \\ 1 \\ 1 \\ 1-672(\gamma_{23}+\gamma_{24}) \\ 1 \\ 1-525(\gamma_{23}+\gamma_{24}) \\ 1+504(\gamma_{23}+\gamma_{24}) \\ 1 \\ 1 \\ 1+336(\gamma_{23}+\gamma_{24}) \\ 1 \\ 1 \end{bmatrix}$$

So the solution is

γ_1=1+672$(\gamma_{23}+\gamma_{24})$ γ_2=-42$(\gamma_{23}+\gamma_{24})$ γ_3=0 γ_4=7$(\gamma_{23}+\gamma_{24})$ γ_5=0 γ_6=0

γ_7=0 γ_8=-$\frac{7}{4}(\gamma_{23}+\gamma_{24})$ γ_9=0 γ_{10}=0 γ_{11}=0 γ_{12}=$\frac{7}{4}(\gamma_{23}+\gamma_{24})$

γ_{13}=-$\frac{21}{4}(\gamma_{23}+\gamma_{24})$ γ_{14}=0 γ_{15}=0 γ_{16}=0 γ_{17}=0 γ_{18}=0

γ_{19}=-$\frac{7}{4}(\gamma_{23}+\gamma_{24})$ γ_{20}=0 γ_{21}=0 γ_{22}=0

Therefore, by substituting these values into the equation (4.4.3) we get

$$1+\frac{7875}{2}(\gamma_{23}+\gamma_{24})+225[(\frac{1+\sqrt{21}}{2})\gamma_{23}+(\frac{1-\sqrt{21}}{2})\gamma_{24}]=\left(\frac{5+\sqrt{21}}{2}\right)^{k}$$

By simplifying we obtain

$$1+4050(\gamma_{23}+\gamma_{24})+\frac{225}{2}\sqrt{21}(\gamma_{23}-\gamma_{24})=\left(\frac{5+\sqrt{21}}{2}\right)^{k}$$

This can be obtained when $k=180$. Then the solution is ,

$\gamma_{23}=$ 1656236157651751009599670205837927977995411710301286341897146389~
84313522167030611895156430334252935161882218541589939024.
$\gamma_{24}=>$ -1282192498214662539912734861393638050624307393554577653535373~
35215654155877426963656061671174307402224668106353497553424.

But it is not useful to express the other coefficients in terms of γ_{23} and γ_{24}. Therefore we'll solve this equation for $(\frac{5+\sqrt{21}}{2})$ instead of $(\frac{5+\sqrt{21}}{2})^{180}$. Hence, the solution gives us 180th root of the solution. Now let us solve the following system of equation.

A.X=Y

where ***X*** is 24x1 type vector of coefficients and ***Y*** is the transpose of 1x24 type vector [1,1,1,…, 1, $\frac{5+\sqrt{21}}{2}$, $\frac{5-\sqrt{21}}{2}$]. The solusion is

$\gamma_1=\frac{653}{525}$	$\gamma_2=\frac{-8}{525}$	$\gamma_3=0$	$\gamma_4=\frac{4}{1575}$	$\gamma_5=0$	$\gamma_6=0$
$\gamma_7=0$	$\gamma_8=\frac{-1}{1575}$	$\gamma_9=0$	$\gamma_{10}=0$	$\gamma_{11}=0$	$\gamma_{12}=\frac{1}{1575}$
$\gamma_{13}=\frac{-1}{1575}$	$\gamma_{14}=0$	$\gamma_{15}=0$	$\gamma_{16}=0$	$\gamma_{17}=0$	$\gamma_{18}=0$
$\gamma_{19}=\frac{-1}{1575}$	$\gamma_{20}=0$	$\gamma_{21}=0$	$\gamma_{22}=0$	$\gamma_{23}=\frac{4}{1575}$	$\gamma_{24}=\frac{-1}{1575}$

Hence the 180th root of generator is

$$\gamma=\frac{1}{1575}(1959C_1-24C_2+4C_4-C_8+C_{12}-C_{13}-C_{20}+4C'_{22}-C''_{22})$$

with the inverse

$$\gamma^{-1}=\frac{1}{1575}(1959C_1-24C_2+4C_4-C_8+C_{12}-C_{13}-C_{20}-C'_{22}+4C''_{22}).$$

As a result

$$\mathcal{U}_1(\mathcal{Z}(\mathbb{Z}A_{10}))=<\gamma^{180}>$$

CHAPTER 5

CONCLUSION

In order to find a counter example for the Normalizer Property for a group G , one must know the central units of its integral group ring. As it was mentioned in the previous chapter, construction of central units of integral group rings of simple groups is not so easy, especially for alternating groups A_n . For small groups ($n \leq 6$) there are some temptations; First Aleev gave a characterization of central units of $\mathbb{Z}A_n$ for n=5 and 6. By using Pell equations a characterization of central units of $\mathbb{Z}A_5$ is given by Li and Parmenter. By using irreducible representations, another characterization of central units of $\mathbb{Z}A_5$ is given by Bilgin. Unfortunately, none of them can be generalized.

In this study, by using some softwares; GAP 4.3 to derive irreducible character tables and Maple 10, to carry out algebraic operations. By the help of these irreducible characters, the central units of $\mathbb{Z}A_n$ are characterized for n<11 as follows:

Table 5.1 Table of Generatros of A_n

n	Number of generator	Generator	Reference
3,4	0	Trivial	Theorem 3.1
5	1	$\gamma = 49C_1 - 10C_3 + 26C_4' - 16C''$	Theorem 4.1
6	1	$\gamma = 18433C_1 - 2304C_2 + 3728C_4' - 1424C_4'' - 2304C_5$	Theorem 4.2
7	0	Trivial	Theorem 4.3
8	0	Trivial	Theorem 4.3
9	0	Trivial	Theorem 4.3
10	1	$< \gamma^{180} >$ where $\gamma = \frac{1}{1575}(1959C_1 - 24C_2 + 4C_4 - C_8 + C_{12} - C_{13} - C_{20} + 4C_{22}' - C_{22}'')$	Theorem 4.4

The method used in this thesis can also be generalized for larger n's, for example ($n \leq 20$).

TABLES

Table 4.3: Character Table of A_3

Conjugacy Class →	$\bar{C}_1$	$\bar{C}_2'$	$\bar{C}_2''$
#(Centralizer) #(Conjugacy Class)	3 1	1 3	1 3
χ_1	1	1	1
χ_2	1	α	$\bar{\alpha}$
χ_3	1	$\bar{\alpha}$	α

$$\alpha = \frac{-1+i\sqrt{3}}{2}, \bar{\alpha} = = \frac{-1-i\sqrt{3}}{2} \ ,(\alpha = e^{2\pi i/3})$$

Table 4.4: Character Table of A_4

Conjugacy Class	$\bar{C}_1$	$\bar{C}_2'$	$\bar{C}_2''$	$\bar{C}_3$
#(Centralizer) #(Conjugacy Class)	12 1	3 4	3 4	4 3
χ_1	1	1	1	1
χ_2	1	α	$\bar{\alpha}$	-1
χ_3	1	$\bar{\alpha}$	α	0
χ_4	3	0	0	-1

$$\alpha = \frac{-1+i\sqrt{3}}{2}, \bar{\alpha} = = \frac{-1-i\sqrt{3}}{2} \ ,(\alpha = e^{2\pi i/3})$$

Table 4.5: Character Table of A_5

Conjugacy Class →	$\bar{C}_1$	$\bar{C}_2$	$\bar{C}_3$	$\bar{C}_4'$	$\bar{C}_4''$
#(Centralizer) #(Conjugacy Class)	60 1	3 20	4 15	5 12	5 12
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	α	β
χ_{35}	3	0	-1	β	α

$$\alpha = -\varepsilon^2 - \varepsilon^3 = \frac{1+\sqrt{5}}{2},\ \beta = -\varepsilon - \varepsilon^4 = \frac{1-\sqrt{5}}{2} \ ,(\varepsilon = e^{2\pi i/5})$$

Table 4.6: Character Table of A_6

Conjugacy Class	$\bar{C}_1$	$\bar{C}_2$	$\bar{C}_3$	$\bar{C}_4'$	$\bar{C}_4''$	$\bar{C}_5$	$\bar{C}_6$
#(Centralizer) #(Conjugacy Class)	360 1	9 40	8 45	5 72	5 72	9 40	4 90
χ_1	1	1	1	1	1	1	1
χ_2	5	2	1	0	0	-1	-1
χ_3	5	-1	1	0	0	2	-1
χ_6	9	0	1	-1	-1	0	1
χ_7	10	1	-2	0	0	1	0
χ_8	8	-1	0	α	β	-1	0
χ_9	8	-1	0	β	α	-1	0

$$\alpha = -\varepsilon^2 - \varepsilon^3 = \frac{1+\sqrt{5}}{2},\ \beta = -\varepsilon - \varepsilon^4 = \frac{1-\sqrt{5}}{2}\ ,(\varepsilon = e^{2\pi i/5})$$

Table 4.7: Character Table of A_7

Conjugacy Class $\rightarrow$	$\bar{C}_1$	$\bar{C}_2$	$\bar{C}_3$	$\bar{C}_4$	$\bar{C}_5$	$\bar{C}_6$	$\bar{C}_7$	$\bar{C}_8'$	$\bar{C}_8''$
#(Centralizer) #(Conjugacy Class)	2520 1	36 70	24 105	5 504	9 280	4 630	12 210	7 360	7 360
χ_1	1	1	1	1	1	1	1	1	1
χ_2	6	3	2	1	0	0	-1	-1	-1
χ_3	10	1	-2	0	1	0	1	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
χ_4	10	1	-2	0	1	0	1	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$
χ_5	14	2	2	-1	-1	0	2	0	0
χ_6	14	-1	2	-1	2	0	-1	0	0
χ_7	15	3	-1	1	0	-1	-1	0	0
χ_8	21	-3	1	1	0	-1	1	0	0
χ_9	35	-1	-1	0	-1	1	-1	0	0

$$\alpha = \varepsilon + \varepsilon^2 + \varepsilon^4 = \frac{-1+i\sqrt{7}}{2}, \beta = \varepsilon^3 + \varepsilon^5 + \varepsilon^6 = \frac{-1-i\sqrt{7}}{2},\quad (\varepsilon = e^{2\pi i/7})$$

Table 4.8: Character Table of A_8

Conjugacy Class	$\bar{C}_1$	$\bar{C}_2$	$\bar{C}_3$	$\bar{C}_4$	$\bar{C}_5$	$\bar{C}_6$	$\bar{C}_7$	$\bar{C}'_8$	$\bar{C}''_8$	$\bar{C}_9$	$\bar{C}_{10}$	$\bar{C}_{11}$	$\bar{C}'_{12}$	$\bar{C}''_{12}$
#(Centralizer) #(Conjugacy Class)	20160 1	180 112	192 105	15 1344	18 1120	16 1260	12 1680	7 2880	7 2880	96 210	8 2520	6 3360	15 1344	15 1344
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	4	-1	2	1	-1	0	0	0	3	1	-1	-1	-1
χ_3	14	-1	6	-1	2	2	-1	0	0	2	0	0	-1	-1
χ_4	20	5	4	0	-1	0	1	-1	-1	4	0	1	0	0
χ_5	21	6	-3	1	0	1	-2	0	0	1	-1	0	1	1
χ_6	21	-3	-3	1	0	1	1	0	0	1	-1	0	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
χ_7	21	-3	-3	1	0	1	1	0	0	1	-1	0	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$
χ_8	28	1	-4	-2	1	0	1	0	0	4	0	-2	1	1
χ_9	35	5	3	0	2	-1	1	0	0	-5	-1	0	0	0
χ_{10}	45	0	-3	0	0	1	0	γ	λ	-3	1	0	0	0
χ_{11}	45	0	-3	0	0	1	0	λ	γ	-3	1	0	0	0
χ_{12}	56	-4	8	1	-1	0	0	0	0	0	0	-1	1	1
χ_{13}	64	4	0	-1	-2	0	0	1	1	0	0	0	-1	-1
χ_{14}	70	-5	-2	0	1	-2	-1	0	0	2	0	1	0	0

$$\alpha=\varepsilon+\varepsilon^2+\varepsilon^4=\frac{-1+i\sqrt{7}}{2},\beta=\varepsilon^3+\varepsilon^5+\varepsilon^6=\frac{-1-i\sqrt{7}}{2},\quad (\varepsilon=e^{2\pi i/7} \text{ ve } \omega=e^{2\pi i/15})$$

$$\gamma=-\omega-\omega^2-\omega^4-\omega^8=\frac{-1+i\sqrt{15}}{2},\lambda=-\omega^7-\omega^{11}-\omega^{13}-\omega^{14}=\frac{-1-i\sqrt{15}}{2}.$$

Table 4.9: Character Table of A_9

Conj Class	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14'	C14''	C15	C16	C17
#Centralizer	181440	1080	480	60	81	24	24	7	192	16	6	15	15	54	9	9	20	1
#Conj Clss	1	168	378	3024	2240	7560	7560	25920	945	11340	30240	12096	12096	3360	20160	20160	9072	15120
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	8	5	4	3	-1	2	1	1	0	0	0	0	0	2	-1	-1	-1	-1
χ_3	21	-3	1	1	3	-1	1	0	-3	1	0	α	β	0	0	0	1	-1
χ_4	21	-3	1	1	3	-1	1	0	-3	1	0	β	α	0	0	0	1	-1
χ_5	27	9	7	2	0	1	1	-1	3	-1	0	-1	-1	0	0	0	2	1
χ_6	28	10	4	3	1	0	-2	0	-4	0	-1	0	0	1	1	1	-1	0
χ_7	35	5	-5	0	-1	-1	1	0	3	-1	0	0	0	2	-1	2	0	-1
χ_8	35	5	-5	0	-1	-1	1	0	3	-1	0	0	0	2	2	-1	0	-1
χ_9	42	0	6	-3	-3	0	0	0	2	2	-1	0	0	3	0	0	1	0
χ_{10}	48	6	8	-2	3	0	2	-1	0	0	0	1	1	0	0	0	-2	0
χ_{11}	56	11	-4	1	2	-2	-1	0	0	0	0	1	1	2	-1	-1	1	1
χ_{12}	84	-6	4	-1	3	0	-2	0	4	0	1	-1	-1	3	0	0	-1	0
χ_{13}	105	15	5	0	-3	-1	-1	0	1	1	1	0	0	-3	0	0	0	-1
χ_{14}	120	0	0	0	3	0	0	1	8	0	-1	0	0	-3	0	0	0	0
χ_{15}	162	0	6	-3	0	0	0	1	-6	-2	0	0	0	0	0	0	1	0
χ_{16}	168	-15	4	3	-3	-2	1	0	0	0	0	0	0	0	0	0	-1	1
χ_{17}	189	9	-11	-1	0	1	1	0	-3	1	0	-1	-1	0	0	0	-1	1
χ_{18}	216	-9	-4	1	0	2	-1	-1	0	0	0	1	1	0	0	0	1	-1

where $\alpha = -\varepsilon - \varepsilon^2 - \varepsilon^4 - \varepsilon^8 = \frac{-1+i\sqrt{15}}{2}, \beta = -\varepsilon^7 - \varepsilon^{11} - \varepsilon^{13} - \varepsilon^{14} = \frac{-1+i\sqrt{15}}{2}, (\varepsilon = e^{2\pi i/15})$

Table 4.10: Character Table of A_{10}

Conjugate Classes	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14'	C14''	C15	C16	C17	C18	C19	C20	C21	C22'	C22''
#Centra-lizer	1	240	630	6048	8400	18900	25200	86400	4725	18900	25200	120960	22400	201600	201600	90720	151200	56700	72576	151200	226800	151200	86400	86400
#Conjugate Calsses	1814400	7560	2880	300	216	96	72	21	384	96	72	15	81	9	9	20	12	32	25	12	8	12	21	21
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	9	6	5	4	3	-1	2	2	1	3	-1	1	0	0	0	0	0	1	-1	1	-1	-1	-1	-1
χ_3	35	14	11	5	2	3	2		3	3	2	-1	-1	-1	-1	1	0	-1	0	0	1	0	0	0
χ_4	36	15	8	6	3	-2	-1	1	-4	2	-1	0	0	0	0	-2	-1	0	1	-1	0	1	1	1
χ_5	42	0	6	-3	3	-4	0	0	2	0	3	0	-3	0	0	1	0	2	2	-1	0	-1	0	0
χ_6	75	15	15	0	0	-3	3	-2	3	1	0	0	3	0	0	0	1	-1	0	0	-1	0	1	1
χ_7	84	21	0	4	3	2	-3	0	-4	-2	3	1	3	0	0	0	1	0	-1	-1	0	-1	0	0
χ_8	90	6	14	-5	3	4	2	-1	2	0	-1	1	0	0	0	-1	0	2	0	-1	0	1	-1	-1
χ_9	126	21	-14	1	6	0	1	0	6	-4	-2	1	0	0	0	1	-1	-2	1	0	0	0	0	0
χ_{10}	160	34	16	5	-2	0	-2	-1	0	0	-2	-1	-2	1	1	1	0	0	0	0	0	0	-1	-1
χ_{11}	210	-21	6	5	0	0	3	0	-6	-4	0	-1	3	0	0	1	-1	2	0	0	0	0	0	0
χ_{12}	224	14	-16	-1	2	0	2	0	0	0	2	-1	-1	2	-1	-1	0	0	-1	0	0	0	0	0
χ_{13}	224	14	-16	-1	2	0	2	0	0	0	2	-1	-1	-1	2	-1	0	0	-1	0	0	0	0	0
χ_{14}	225	15	5	0	-6	3	-1	1	9	-1	2	0	0	0	0	0	-1	1	0	0	-1	0	1	1
χ_{15}	252	-21	8	2	3	2	-1	0	4	-2	-1	-1	0	0	0	-2	1	0	2	1	0	-1	0	0
χ_{16}	288	-6	16	-7	6	0	-2	1	0	0	-2	-1	0	0	0	1	0	0	-2	0	0	0	1	1
χ_{17}	300	-15	0	0	3	-2	-3	-1	4	2	3	0	3	0	0	0	-1	0	0	1	0	1	-1	-1
χ_{18}	315	21	19	-5	-3	-1	1	0	-5	-1	1	1	0	0	0	-1	-1	-1	0	1	1	-1	0	0
χ_{19}	350	35	-10	0	-1	-2	-1	0	-2	-2	-1	0	-1	-1	-1	0	1	2	0	1	0	1	0	0
χ_{20}	450	-15	10	0	-3	-2	1	2	2	-2	1	0	0	0	0	0	1	-2	0	-1	0	1	-1	-1
χ_{21}	525	0	-15	0	-3	-1	0	0	5	3	-3	0	3	0	0	0	0	1	0	-1	1	-1	0	0
χ_{22}	567	0	-9	-3	0	3	0	0	-9	3	0	0	0	0	0	1	0	-1	2	0	-1	0	0	0
χ_{23}	384	-24	0	4	0	0	0	-1	0	0	0	1	-3	0	0	0	0	0	-1	0	0	0	α	β
χ_{24}	384	-24	0	4	0	0	0	-1	0	0	0	1	-3	0	0	0	0	0	-1	0	0	0	β	α

where $\alpha=\omega+\omega^4+\omega^5+\omega^{-5}+\omega^{-4}+\omega^{-1}=\frac{1+\sqrt{21}}{2}, \beta=\omega^2+\omega^8+\omega^{10}+\omega^{-10}+\omega^{-8}+\omega^{-2}=\frac{1-\sqrt{21}}{2}, (\omega=e^{2\pi i/10})$

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