İSTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

IMBEDDING THEOREM IN VECTOR-VALUED SOBOLEV SPACES AND APPLICATIONS

M.Sc. Thesis by
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Department : Mathematics

Programme : Mathematical Engineering

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VEKTÖR DEĞERLİ SOBOLEV UZAYLARINDA GÖMME TEOREMLERİ VE UYGULAMALARI

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FOREWORD

I would like to express my deep appreciation and thanks for Prof. Dr. Veli Sahmurov.

I am very grateful to my family for their patience, love and support.

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Mathematical Engineering

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SUMMARY

In this thesis, imbedding theorems are studied for vector-valued isotropic and anisotropic Sobolev spaces. We also consider the applications of these theorems.

This thesis consists of 3 chapters.

In the first chapter, Banach spaces (which will be denoted by E) and E-valued L_p spaces are investigated. First of all, basic definitions which is used in these spaces are given. Then, Hölder's and Minkowksi's inequalities are established and the normed dual of $L_p(\Omega{:}E)$ is given. In the last part of this section, the completeness of $L_p(\Omega{:}E)$ is proved and so it is shown that this space is also a Banach space.

In the second chapter, vector-valued isotropic Sobolev spaces are defined. Also, using the completeness of $L_p(\Omega{:}E)$, the completeness of Sobolev spaces is shown. Then, the density and extension theorems, which include the continuity with conservation of differentiability and additivity of the extension operator, is given. Finally, the imbedding theorem is proved with various auxiliary lemmas. This theorem states the continuity of the imbedding operator and the boundedness of the differential operator from Sobolev space to $L_q(\Omega{:}E)$.

In the third chapter, the basic definitions and notations of vector-valued anisotropic Sobolev spaces are given. Then, the imbedding theorems for these spaces are stated. Particularly, the existence of mixed derivative and its estimates is obtained by Sobolev norm. In the proof of these theorem, we use the integral representation method in bounded domains and the Fourier analytic method in $\mathbb{R}^n$. Using these theorems, coersive estimates for solution of anisotropic elliptic equations in Banach valued L_p spaces is shown.

Consequently, we can get various results by taking concrete Banach spaces in these imbedding theorems. For instance, if $E=l_q$ then it can be shown that a class of infinite number of anisotropic differential equation has coersive property.

VEKTÖR DEĞERLİ SOBOLEV UZAYLARINDA GÖMME TEOREMLERİ VE UYGULAMALARI

ÖZET

Bu tez çalışmasında, vektör değerli izotropik ve izotropik olmayan Sobolev uzayları için gömme teoremi üzerinde çalışılmıştır. Ayrıca, bu teoremin uygulamaları incelenmiştir.

Bu çalışma 3 bölümden oluşmaktadır.

Birinci bölümde, (E ile gösterilen) Banach uzayları ve E-değerli L_p uzayları incelenmiştir. Öncelikle bu uzaylarda kullanılan temel tanımlar verilmektedir. Daha sonra, Hölder ve Minkowski eşitsizlikleri ve $L_p(\Omega:E)$ uzayının normlu dualine yer verilmiştir. Bu bölümün son kısmında, $L_p(\Omega:E)$ uzayının tamlığı ve dolayısıyla Banach uzayı olduğu gösterilmiştir.

İkinci bölümde, vektör değerli izotropik Sobolev uzayları tanımlanmıştır. Ayrıca, $L_p(\Omega:E)$ uzayının tamlığı kullanılarak, Sobolev uzaylarının tamlığı gösterilmiştir. Daha sonra, devam operatörünün sürekliliğini ve bazı hallerde difereransiyel ve toplamsallık özelliğine bağlı olmadığını gösteren devam teoremleri ve yoğunluk teoremlerine yer verilmiştir. Son olarak, çeşitli yardımcı önsavlarla gömme teoremi ispatlanmıştır. Bu teorem, gömme operatörünün sürekliliği ve diferansiyel ve toplamsallık özelliğine bağlı olmadığını gösterir.

Üçüncü bölümde, vektör değerli izotropik olmayan Sobolev uzaylarının temel kavramları ve notasyonları verilmiştir. Sonra bu uzaylar için gömme teoremi incelenmiştir. Özellikle, karma türev ve sınırlarının varlığı gösterilmiştir. Bu teoremin ispatında, sınırlı bölgelerde integral temsil metodu, $\mathbb{R}^n$'de bakıldığında ise fourier analytic metod uygulanmaktadır. Bu teoremleri kullanarak, Banach değerli L_p uzaylarında bir sınıf izotropik olmayan eliptik denklemin koersif sınırları gösterilmiştir.

Son olarak, gömme teoremlerinde somut Banach uzayları kullanılarak çeşitli sonuçlar elde edilir. Örneğin, $E=l_q$ alınarak bir sınıf sonsuz sayıda anizotropik diferansiyel denklemlerin coersif özelliğe sahip olması gösterilmektedir.

1. INTRODUCTION

1.1 Definitions

Definition 1.1: If $\alpha = (\alpha_1, \alpha_2, ..\alpha_n)$ is an n_tuple of non-negative integers α_j, we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1} x_2^{\alpha 2} .. x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^{n} \alpha_j$.

Definition 1.2: $C(\Omega : E)$ is a collection of E-valued, bounded, continuous function with norm

$$\|u\|_{C(\Omega:E)} = \sup_{x\in\Omega} \|u\|_E.$$

$C^l(\Omega : E)$ is a collection of E-valued, bounded, continuous, differentiable up to order l function with norm

$$\|u\|_{C^l(\Omega:E)} = \sum_{|\alpha|\leq l} \|D^\alpha u\|_{C(\Omega:E)}.$$

Definition 1.3: Banach spaces are defined as complete normed vector spaces. This means that a Banach space is a vector space E over the real or complex numbers with a norm $\|.\|_E$ such that every Cauchy sequence (with respect to the metric $d(x,y) = \|x-y\|_E$) in E has a limit in E.

Definition 1.4: Let E be a Banach space and p be a positive real number. We denote by $L_p(\Omega : E)$ a space of all measurable E-valued functions u, defined on the measurable subset $\Omega \subset R^n$, for which

$$\int_\Omega \|u\|_E^p \, dx < \infty,$$

with norm

$$\|u\|_{L_p} = \|u\|_{L_p(\Omega;E)} = \left(\int_\Omega \|u\|_E^p \, dx\right)^{1/p}, \; 1 \leq p < \infty.$$

Definition 1.5: A function u, measurable on Ω, is said to be essentially bounded on Ω provided there exists a constant K for which $|u(x)| \leq K$ a.e. on Ω. The greatest lower bound of such constants K is called the essential supremum of $|u|$ on Ω and is denoted by ess $\sup_{x\in\Omega}|u(x)|$. We denote by $L_\infty(\Omega : E)$ the vector space consisting of all E-valued functions u that are essentially bounded on Ω, functions being once again identified if they are equal a.e. on Ω. It is easily verified that the functional $\|\cdot\|_{L_\infty(\Omega;E)}$ defined by

$$\|u\|_{L_\infty(\Omega;E)} = \operatorname*{ess\,sup}_{x\in\Omega}|u(x)|$$

is a norm on $L_\infty(\Omega : E)$. [1]

Definition 1.6: Let Ω be an open set in $\mathbb{R}^n$. If u is a function defined on a set $G \subset \mathbb{R}^n$, we define the support of u as

$$\text{supp } u = \overline{\{x \in G : u(x) \neq 0\}}.$$

We say that u has compact support in Ω if supp $u \subset \Omega$ and supp u is compact.

Definition 1.7: If $1 < p < \infty$ we denote by $p\acute{}$ the number $p/(p-1)$ so that $1 < p\acute{} < \infty$ and

$$(1/p) + (1/p\acute{}) = 1.$$

$p\acute{}$ is called the exponent conjugate to p.

We also define $p\acute{} = \left\{\begin{array}{ll} \infty & \text{if } p = 1 \\ 1 & \text{if } p = \infty \end{array}\right\}$

Definition 1.8: Let $D(\Omega : E)$ denote the collection of E-valued infinitely differentiable functions on Ω with compact support in Ω.

If K is a compact set in Ω , we set

$$D_K(\Omega : E) = \{\varphi : \varphi \in D(\Omega : E), \varphi \text{ with support in } K\}.$$

With the norms

$$\text{þ}_j(\phi) = \sup_{\substack{x\in K \\ |\alpha|\leq j}} \|D^\alpha \varphi(x)\|_E, \qquad j = 0, 1, 2, ...$$

$D_K(\Omega : E)$ is a Fréchet -space (i.e. metrizable and complete); then if K_n is an increasing sequence of compact sets belonging to Ω and whose union is Ω, we have algebraically

$$D(\Omega : E) = \bigcup_{K_n} D_{K_n}(\Omega : E)$$

and we provide $D(\Omega : E)$ with the corresponding inductive limit topology (i.e. the finest locally convex topology which makes the injections $D_{K_n}(\Omega : E) \to D(\Omega : E)$ continuous).

We define the space of distributions on Ω, $D'(\Omega : E)$, as a collection of linear continuous transformation from $D(\Omega : E)$ to E and provide $D'(\Omega : E)$ with the strong dual topology.

If $T \in D'(\Omega : E)$ and $\varphi \in D(\Omega : E)$, the value of T at φ will be denoted by

$\langle T, \varphi \rangle$.

If $\bar{\varphi} =$ complex conjugate of φ, we shall write

$\langle T, \bar{\varphi} \rangle = (T, \varphi)$. [2]

Definition 1.9: An E-valued function u defined almost everywhere on Ω is said to be locally integrable on Ω provided $u \in L_1(A : E)$ for every measurable set A which is compact in Ω. In this case we write $u \in L_1^{loc}(\Omega : E)$. Corresponding to every $u \in L_1^{loc}(\Omega : E)$ there is a distribution $T_u \in D'(\Omega)$ defined by

$$\langle T_u, \varphi \rangle = \int_\Omega u(x)\varphi(x)dx, \ \ \varphi \in D(\Omega) .$$

We define, the derivative $\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle$.

Definition 1.10: We now define the concept of weak derivative of a locally integrable function. Let $u \in L_1^{loc}(\Omega : E)$. There may or may not exist a function $v_\alpha \in L_1^{loc}(\Omega : E)$ such that $T_{v_\alpha} = \langle D^\alpha, T_u \rangle$ in $D'(\Omega)$. If such a v_α exists, it is unique up to sets of measure zero and it is called the weak derivate of u and is denoted by $D^\alpha u$. Thus $D^\alpha u = v_\alpha$ in the weak sense provided $v_\alpha \in L_1^{loc}(\Omega : E)$ satisfies

$$\int_\Omega u(x)D^\alpha \varphi(x)dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\varphi(x)dx$$

for every $\varphi \in D(\Omega)$.

Definition 1.11: We say that the normed space X is imbedded in the normed space Y, and write $X \to Y$ to designate this imbedding, provided

(i) X is a vector subspace of Y, and

(ii) the identity operator I defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

Since I is linear, (ii) is equivalent to the existence of a constant M such that

$$\|Ix\|_Y \leq M \|x\|_X , \qquad x \in X.$$

Definition 1.12: The Fourier transform of an integrable function $f : \mathbb{R}^n \to \mathbb{C}$

$\widehat{f}(\xi) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} f(x) e^{-ix\xi} dx$, for every $x = (x_1, x_2, ..., x_n)$, $\xi = (\xi_1, \xi_2, ..., \xi_n)$ and $x\xi = \sum\limits_{k=1}^{n} x_k \xi_k$

Definition 1.13: Two Banach spaces A_0 and A_1 continuously embedded in a linear Hausdorff space A are given:

$A_0 \subset A, \qquad A_1 \subset A.$

'$\subset$' means the set theoretical and topological embedding. Such a couple $\{A_0, A_1\}$ is said to be an interpolation couple. Let $\{B_0, B_1\}$ be a second interpolation couple, and let B be linear Hausdorff space belonging to this couple. Let T be a linear operator acting from A into B, whose restrictions to A_i, $i = 0, 1$, give linear continuous operators from A_i into B_i. One asks for Banach spaces A and B,

$A \subset A \qquad B \subset B,$

such that the restriction of this operator T to A gives a linear continuous operator from A into B (interpolation property).

Clearly,

$A_0 \cap A_1$, $\|a\|_{A_0 \cap A_1} = \max(\|a\|_{A_0}, \|a\|_{A_1})$

is also a Banach space. Further we shall need the space

$A_0 + A_1 = \left\{a \mid a \in \mathcal{A}\ , \exists a_j \in A_j; j = 0, 1;\ \text{where } a = a_0 + a_1\right\},$

$\|a\|_{A_0 + A_1} = \inf\limits_{\substack{a = a_0 + a_1 \\ a_j \in A_j}} (\|a_0\|_{A_0} + \|a_1\|_{A_1}).$

The infimum is taken over all representations of $a \in A_0 + A_1$ in the described way.

Definition 1.14: For further considerations, we shall need some basic notations of the theory of category [2]. According to H.Schubert, a category can be described by the following two properties:

1. A category consists of

(a) a class of objects A, B, C...

(b) and a class of pairwise disjunctive non-empty sets $[A, B]$ where to each ordered pair (A, B) of objects there belongs a set $[A, B]$ in a unique way. The elements of $[A\ ,B]$ are said to be morphisms(from A into B).

2. For each ordered triplet $(A,\ B,\ C)$ of objects, there is defined a composition of the morhpisms,

$[B, C] \times [A, B] \to [A, C]$.

If $f \in [A, B]$ and $g \in [B, C]$, then the image of (g, f) will be denoted by gf. If $f \in [A, B]$, $g \in [B, C]$ and $h \in [C, D]$, then by assumption

$(hg)\,f = h\,(gf)$ (associative law).

Further , we shall need the notion of a (covariant) functor. Let C_1 and C_2 be two categories. A (covariant) functor F is defined as a mapping from C_2 into C_1, where the image of an object A belonging to C_2 is an object $F(A)$ of C_1 and the image of a morphism $f \in [A, B]$ belonging to C_2 is a morphism $F(f) \in [F(A), F(B)]$ of C_1. Furthermore,

$F(1_A) = 1_{F(A)}$

holds by defintion, and if $f \in [A, B]$ and $g \in [B, C]$, then

$F(gf) = F(g)F(f).$

Now we shall construct two special categories consisting of Banach spaces or interpolation couples, respectively. If A and B are two Banach spaces, then, as usual, $L(A, B)$ denotes the set of all linear continuous mappings from A into B.

The category C_1 consists of :

(a) the class of all complex Banach spaces A, B,... as objects,

(b)and the sets of morphisms $[A, B] = L(A, B)$.

If the composition of morphisms is determined as the usual product of operators, and if

$1_A = E$ (identity operator in the Banach space A), then it is easy to show that the required properties of a category are fullfilled.

The category C_2 consists of :

(a) the class of all (complex) interpolation couples $\{A_0, A_1\}$, $\{B_0, B_1\}$,... as objects,

(b) and the sets of morphisms $[A, B] = L(\{A_0, A_1\}, \{B_0, B_1\})$.

Here $L(\{A_0, A_1\}, \{B_0, B_1\})$ denotes the set of all linear operators mapping $A_0 + A_1$ into $B_0 + B_1$ such that their restrictions to A_k, $k = 0, 1$, are continuous mappings from A_k into B_k.

If the composition of morphisms and $1_{\{A_0,A_1\}} = E$ are explained in a natural way, then it is easy to show that C_2 is a category.

Let C_1 and C_2 be the categories defined above. Then a (covariant)functor F is said to be an interpolation functor if:

(a) $A_0 \cap A_1 \subset F(\{A_0, A_1\}) \subset A_0 + A_1$

(b) If $T \in L(\{A_0, A_1\}, \{B_0, B_1\})$, then $F(T)$ is the restriction of T to $F(\{A_0, A_1\})$.

Any Banach space which can be represented in the form $A = F(\{A_0, A_1\})$ with the aid of a suitable interpolation functor F is said to be an interpolation space (with respect to $\{A_0, A_1\}$).

Let $\{A_0, A_1\}$ be an interpolation couple. If $0 < t < \infty$, then

$$K(t, a) = \inf_{a=a_0+a_1} ||a_0||_{A_0} + t||a_1||_{A_1}, \qquad a \in A_0 + A_1,$$

is an equivalent norm in the space $A_0 + A_1$.

Let $\{A_0, A_1\}$ be an interpolation couple. Let $0 < \theta < 1$. If $1 \le q < \infty$, then

$$(A_0, A_1)_{\theta,q} = \left\{ a \mid a \in A_0 + A_1,\ \|a\|_{(A_0,\, A_1)_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta} K(t, a)\right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

and if $q = \infty$, then

$$(A_0, A_1)_{\theta,\infty} = \left\{ a \mid a \in A_0 + A_1,\ \|a\|_{(A_0,\, A_1)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K(t, a) < \infty \right\}$$

are interpolation spaces.

In the furhter considerations, we need the Banach spaces

$$l_p^\sigma = \left\{ \xi \mid \xi = \{\xi_j\}_{j=0}^{\infty}; \xi_j \in E; \|\xi\|_{l_p^\sigma} = \left(\sum_{j=0}^{\infty} 2^{j\sigma p} \|\xi_j\|_E^p \right)^{1/p} < \infty \right\}$$

for $1 \leq p < \infty$ and

$$l_\infty^\sigma = \left\{ \xi \mid \xi = \{\xi_j\}_{j=0}^{\infty}; \xi_j \in E; \|\xi\|_{l_\infty^\sigma} = \sup_j 2^{j\sigma p} \|\xi_j\|_E < \infty \right\}$$

for $p = \infty$. E denotes a Banach space and σ is a real number. [3]

1.2 Some Inequalities

Theorem 1.15: (Hölder´s inequality) If $1 < p < \infty$ and $u \in L_p(\Omega : E)$, $v \in L_{p'}(\Omega)$, then $uv \in L_1(\Omega : E)$ and

$$\int_\Omega \|u(x)v(x)\|_E \, dx \leq \|u\|_{L_p(\Omega:E)} \|v\|_{L_{p'}(\Omega)}, \tag{1.1}$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1.$

Proof: The function $f(t) = (t^p/p) + (1/p') - t$ has, for $t \geq 0$, the minimum value zero, and this minimum is attained only at $t = 1$. Setting $t = ab^{-p'/p}$, we conclude, for nonnegative numbers a and b, that

$$ab \leq (a^p/p) + (b^{p'}/p') \tag{1.2}$$

with equality occuring if and only if $a^p = b^{p'}$. If either $\|u\| 5_{L_p(\Omega:E)} = 0$ or $\|v\|_{L_{p'}(\Omega)} = 0$, then $u(x)v(x) = 0$ a.e. in Ω so (1) is satisfied. Otherwise we obtain (1.1) by setting $a = \|u(x)\|_E / \|u\|_{L_p(\Omega:E)}$ and $b = |v(x)| / \|v\|_{L_{p'}(\Omega)}$in (1.2) and integrating over Ω. Equality occurs in (1) if and only if $\|u(x)\|_E^p$ and $|v(x)|^{p'}$ are proportional a.e. in Ω.

Theorem 1.16: (Minkowski´s inequality) If $1 \leq p < \infty$, then

$$\|u + v\|_{L_p(\Omega:E)} \leq \|u\|_{L_p(\Omega:E)} + \|v\|_{L_p(\Omega:E)}. \tag{1.3}$$

Proof: We have already done the case in which $p = 1$ so we assume $1 < p < \infty$. We may also assume that $u, v \in L_p(\Omega : E)$, for otherwise the right side of (1.3) is infinite. Now

$$\int_{\Omega} \|u(x)+v(x)\|_E^p \, dx \le \int_{\Omega} \|u(x)+v(x)\|_E^{p-1} \left(\|u(x)\|_E + \|v(x)\|_E\right) dx$$

$$\le \left\{ \int_{\Omega} \|u(x)+v(x)\|_E^p \, dx \right\}^{1/p'} \left(\|u\|_{L_p(\Omega:E)} + \|v\|_{L_p(\Omega:E)} \right)$$

by seperate applications of Hölder´s inequality. Inequality (1.3) follows by cancellation, which is valid since $\|u+v\|_{L_p(\Omega:E)} < \infty$.

1.3 The Normed Dual of $\mathbf{L}_p(\boldsymbol{\Omega} : \mathbf{E})$

The set of all continuous linear functionals on X is called the dual of X and is denoted by X'. Under pointwise addition and scalar multiplication X' is a vector space:

$$(f+g)(x) = f(x) + g(x), \qquad (cf)(x) = cf(x), \quad f, g \in X', \quad x \in X, \quad c \in \mathbb{C}.$$

A norm on the dual X' of a normed space X can be defined by setting, for $x' \in X'$,

$$\|x'\|_{X'} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{x'(x)}{\|x'\|_X}.$$

Let $1 \le p \le \infty$ and let p' denote the exponent conjugate to p. For each element $v \in L_{p'}(\Omega : E')$ we can define a linear functional L_v on $L_p(\Omega : E)$ via

$$L_v(u) = \int_{\Omega} \langle u(x), v(x) \rangle \, dx \,, \qquad u \in L_p(\Omega : E), \qquad \langle u, v \rangle = v(u).$$

By Hölder´s inequality $\|L_v(u)\| \le \|u\|_{L_p(\Omega:E)} \|v\|_{L_{p'}(\Omega:E')}$

so that $L_v \in [L_p(\Omega : E)]'$ and

$$\|L_v\|_{[L_p(\Omega:E)]'} \le \|v\|_{L_{p'}(\Omega:E')}. \tag{1.4}$$

We show that equality must hold in (1.4). If $1 < p \le \infty$, let $u(x) = |v(x)|^{p'-2}\,\overline{v(x)}$ if $v(x) \neq 0$ and $u(x) = 0$ otherwise. Then $u \in L_p(\Omega : E)$ and

$L_v(u) = \|u\|_{L_p(\Omega:E)} \|v\|_{L_{p'}(\Omega:E')}$.

Now suppose $p = 1$ so $p' = \infty$. If $\|v\|_{L_{p'}(\Omega:E')} = 0$, let $u(x) = 0$. Otherwise let $0 < \varepsilon < \|v\|_{L_\infty(\Omega:E')}$ and let A be a measurable subset of Ω such that $0 < \mu(A) < \infty$ and $\|v(x)\|_{E'} \ge \|v\|_{L_\infty(\Omega:E')} - \varepsilon$ on A. Let $u(x) = \|v(x)\|_{E'}^{-1}\,\overline{v(x)}$ for $x \in A$; $u(x) = 0$ otherwise. Then $u \in L_1(\Omega : E)$ and $L_v(u) \ge \|u\|_1 \left(\|v\|_{L_\infty(\Omega:E')} - \varepsilon\right)$. Thus we have shown that

$\|L_v\|_{[L_p(\Omega:E)]'} = \|v\|_{L_{p'}(\Omega:E')}$

so that the operator L mapping v to L_v is an isometric isomorhism of $L_{p'}(\Omega : E')$ onto a subspace of $[L_p(\Omega : E)]'$.

Lemma 1.17: Let $1 < p < \infty$. If $L \in [L_p(\Omega : E)]'$ and $\|L\|_{[L_p(\Omega:E)]'} = 1$, then there exists unique $w \in L_p(\Omega : E)$ such that $\|w\|_{L_p(\Omega:E)} = L(w) = 1$. Dually, if $w \in L_p(\Omega : E)$ is given and $\|w\|_{L_p(\Omega:E)} = 1$, then there exists unique $L \in [L_p(\Omega : E)]'$ such that $\|L\|_{[L_p(\Omega:E)]'} = L(w) = 1$.

Proof: First assume that $L \in [L_p(\Omega : E)]'$ is given and $\|L\| = 1$. There exists a sequence $\{w_n\} \in L_p(\Omega : E)$ such that $\|w_n\| = 1$ and $\lim\limits_{n\to\infty} |L(w_n)| = 1$. We may assume that $|L(w_n)| > \frac{1}{2}$ for each n, and, replacing w_n by a suitable multiple of w_n by a complex number of unit modulus, that $L(w_n) > 0$. Suppose the sequence $\{w_n\}$ is not a Cauchy sequence in $L_p(\Omega : E)$. Then there exists $\varepsilon > 0$ such that $\|w_n - w_m\|_{L_p(\Omega:E)} \geq \varepsilon$ for some arbitrarily large values of m and n, so that by uniform convexity we have $\left\|\frac{1}{2}(w_n + w_m)\right\|_{L_p(\Omega:E)} \leq 1 - \delta$, where δ is a fixed positive number. Thus

$$1 \geq L\left(\frac{w_n+w_m}{\|w_n+w_m\|_{L_p(\Omega:E)}}\right) = \left\|\frac{w_n+w_m}{2}\right\|^{-1}_{L_p(\Omega:E)} L\left(\frac{w_n+w_m}{2}\right)$$
$$\geq \frac{1}{1-\delta}\cdot\frac{1}{2}\left[L(w_n)+L(w_m)\right]. \qquad \textbf{(1.5)}$$

Since the last expression approaches $1/(1-\delta)$ as $n, m \to \infty$, we have a contradiction. Thus $\{w_n\}$ is a Cauchy sequence in $L_p(\Omega : E)$ and so converges to an element w of that space. Clearly $\|w\|_{L_p(\Omega:E)} = 1$ and $L(w) = \lim_{n\to\infty} L(w_n) = 1$. The uniqueness w follows from (1.5) applied to two distinct candidates.

Now suppose $w \in L_p(\Omega : E)$ is given and $\|w\|_{L_p(\Omega:E)} = 1$. As noted before, the functional L_v defined by

$$L_v(u) = \int\limits_\Omega \langle u(x), v(x)\rangle\, dx\,, \qquad u \in L_p(\Omega : E),$$

where

$$v(x) = \left\{\begin{array}{ll} |w(x)|^{p-2}\,\bar{w}(x) & \text{if } w(x) \neq 0 \\ 0 & \text{otherwise} \end{array}\right\}$$

satisfies $L_v(w) = \|w\|^p_{L_p(\Omega:E)} = 1$ and

$\|L_v\|_{[L_p(\Omega:E)]'} = \|v\|_{L_{p'}(\Omega:E')} = \|w\|^{p/p'}_{L_p(\Omega:E)} = 1$. It remains to be shown, therefore, that if L_1, $L_2 \in [L_p(\Omega : E)]'$ satisfy $\|L_1\| = \|L_2\| = L_1(w) = L_2(w) = 1$, then $L_1 = L_2$. Suppose not. Then there exists $u \in L_p(\Omega : E)$ such that $L_1(u) \neq L_2(u)$. Replacing u by a suitable multiple of u, we may assume that $L_1(u) - L_2(u) = 2$. Then replacing u by its sum with a suitable multiple of w, we can arrange that $L_1(u) = 1$ and $L_2(u) = -1$. If $t > 0$, then $L_1(w + tu) = 1 + t$; since $\|L_1\| = 1$, therefore $\|w + tu\|_{L_p(\Omega:E)} \geq 1 + t$. Similarly, $L_2(w - tu) = 1 + t$ so $\|w - tu\|_{L_p(\Omega:E)} \geq 1 + t$. If $1 < p \leq 2$, Clarkson´s inequality gives

$$1 + t^p \|u\|^p_{L_p(\Omega:E)} = \left\| \tfrac{(w+tu)+(w-tu)}{2} \right\|^p_{L_p(\Omega:E)} + \left\| \tfrac{(w+tu)-(w-tu)}{2} \right\|^p_{L_p(\Omega:E)}$$

$$\geq \tfrac{1}{2} \|w + tu\|^p_{L_p(\Omega:E)} + \tfrac{1}{2} \|w - tu\|^p_{L_p(\Omega:E)} \geq (1 + t)^p. \qquad \textbf{(1.6)}$$

If $2 \leq p < \infty$, Clarkson´s inequality gives

$$1 + t^{p'} \|u\|^{p'}_{L_p(\Omega:E)} = \left\| \tfrac{(w+tu)+(w-tu)}{2} \right\|^{p'}_{L_p(\Omega:E)} + \left\| \tfrac{(w+tu)-(w-tu)}{2} \right\|^{p'}_{L_p(\Omega:E)}$$

$$\geq \left(\tfrac{1}{2} \|w + tu\|^p_{L_p(\Omega:E)} + \tfrac{1}{2} \|w - tu\|^p_{L_p(\Omega:E)} \right)^{p'-1} \geq (1 + t)^{p'}. \qquad \textbf{(1.7)}$$

Inequalities (1.6) and (1.7) are not possible for all $t > 0$ unless $\|u\|_p = 0$ which is impossible. Thus $L_1 = L_2$.

Theorem 1.18: (The Riesz representation theorem for $L_p(\Omega : E)$) Let $1 < p < \infty$ and let $L \in [L_p(\Omega : E)]'$. Then there exists $v \in L_{p'}(\Omega : E')$ such that for all $u \in L_p(\Omega : E)$

$$L(u) = \int_\Omega \langle u(x), v(x) \rangle \, dx.$$

Moreover, $\|v\|_{L_{p'}(\Omega:E')} = \|L\|_{[L_p(\Omega:E)]'}$.

Thus $[L_p(\Omega : E)]' \simeq L_{p'}(\Omega : E')$.

Proof: If $L = 0$, we may take $v = 0$. Accordingly, we assume $L \neq 0$ and, without loss of generality, that $\|L\|_{[L_p(\Omega:E)]'} = 1$. By the previous lemma there exists $w \in L_p(\Omega : E)$ with $\|w\|_{L_p(\Omega:E)} = 1$ such that $L(w) = 1$. Let

$$v(x) = \left\{ \begin{array}{ll} |w(x)|^{p-2}\, \bar{w}(x) & \text{if } w(x) \neq 0 \\ 0 & \text{otherwise} \end{array} \right\}.$$

Then L_v , defined as

$$L_v(u) = \int_\Omega \langle u(x), v(x) \rangle \, dx , \qquad u \in L_p(\Omega : E),$$

satisfies $\|L_v\|_{[L_p(\Omega:E)]'} = 1$ and $L_v(w) = 1$.By the previous lemma, again we have $L = L_v$.Since $\|v\|_{[L_{p'}(\Omega:E)]} = 1$, the proof is complete.

Theorem 1.19: (The Riesz representation theorem for $L_1(\Omega : E)$) Let $L \in [L_1(\Omega : E)]'$. Then there exists $v \in L_\infty(\Omega : E)$ such that for all $u \in L_1(\Omega : E)$

$$L(u) = \int_\Omega \langle u(x), v(x)\rangle \, dx.$$

and $\|u\|_{L_\infty(\Omega:E)} = \|L\|_{[L_1(\Omega:E)]'}$. Thus $[L_1(\Omega : E)]' \simeq L_\infty(\Omega : E)$.

1.4 Completeness of $\mathbf{L}_p(\mathbf{\Omega} : \mathbf{E})$

Theorem 1.20: Let E be a Banach space. Then $L_p(\Omega : E)$ is a Banach space if $1 \le p \le \infty$.

Proof: First assume $1 \le p < \infty$ and let $\{u_n\}$ be a Cauchy sequence in $L_p(\Omega : E)$. There is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\left\|u_{n_{j+1}} - u_{n_j}\right\|_{L_p(\Omega:E)} \le 1/2^j, \qquad j = 1, 2, \ldots.$$

Let $v_m(x) = \sum_{j=1}^{m} \left\|u_{n_{j+1}}(x) - u_{n_j}(x)\right\|_E$. Then

$$\|v_m\|_{L_p(\Omega:E)} \le \sum_{j=1}^{m} (1/2^j) < 1\,, \qquad m = 1, 2, \ldots$$

Putting $v(x) = \lim_{m\to\infty} v_m(x)$, which may be infinite for some x, we obtain by Fatou´s Lemma

$$\int_\Omega \|v(x)\|_E^p \, dx \le \liminf_{m\to\infty} \int_\Omega \|v_m(x)\|_E^p \, dx \le 1.$$

Hence $v(x) < \infty$ a.e. in Ω and the series

$$u_{n_1}(x) + \sum_{j=1}^{\infty} (u_{n_{j+1}}(x) - u_{n_j}(x)) \qquad \textbf{(1.8)}$$

converges to a limit $u(x)$ a.e. in Ω. Let $u(x) = 0$ whenever it is undefined as the limit of (1.8). Since (1.8) telescopes we have

$$\lim_{m\to\infty} u_{n_m}(x) = u(x) \qquad \text{a.e. in } \Omega.$$

For any $\varepsilon > 0$ there exists N such that if $m, n \ge N$, then

$\|u_m - u_n\|_{L_p(\Omega:E)} < \varepsilon$. Hence , by Fatou´s lemma again

$$\int_\Omega \|u(x) - u_n(x)\|_E^p \, dx = \int_\Omega \lim_{j\to\infty} \left\|u_{n_j}(x) - u_n(x)\right\|_E^p \, dx$$

$$\leq \liminf_{j\to\infty} \int_\Omega \left\|u_{n_j}(x) - u_n(x)\right\|_E^p dx \leq \varepsilon^p$$

if $n \geq M$. Thus $u = (u - u_n) + u_n \in L_p(\Omega : E)$ and $\|u - u_n\|_{L_p(\Omega:E)} \to 0$ as $n \to \infty$. Therefore $L_p(\Omega : E)$ is complete.

Finally, if $\{u_n\}$ is a Cauchy sequence in $L_\infty(\Omega : E)$, then there exists a set $A \subset \Omega$ having measure zero such that if $x \notin A$, then for every $n, m = 1, 2,$...

$\|u_n(x)\|_E \leq \|u_n\|_{L_\infty(\Omega:E)}, \qquad \|u_n(x) - u_m(x)\|_E \leq \|u_n - u_m\|_{L_\infty(\Omega:E)}.$

Since $\left\{\|u_n\|_{L_\infty(\Omega:E)}\right\}$ is bounded in $\mathbb{R}$, u_n converges uniformly on $\Omega \sim A$ to a bounded function u. Setting $u(x) = 0$ for $x \in A$, we have $u \in L_\infty(\Omega : E)$ and $\|u_n - u\|_{L_\infty(\Omega:E)} \to 0$ as $n \to \infty$. Thus $L_\infty(\Omega : E)$ is complete.

2. VECTOR-VALUED ISOTROPIC SOBOLEV SPACES

In this section we introduce abstract vector-valued Sobolev spaces of integer order and establish some of their basic properties. These spaces are defined over an arbitrary domain $\Omega \subset \mathbb{R}^n$ and are vector subspaces of various spaces $L_p(\Omega : E)$.

Definition 2.1: We define a functional $\|\cdot\|_{W_p^l(\Omega:E)}$, where l is a non-negative integer and $1 \leq p \leq \infty$, as follows :

$$\|\cdot\|_{W_p^l(\Omega:E)} = \left\{ \sum_{|\alpha| \leq l} \|D^\alpha u\|_{L_p(\Omega:E)}^p \right\}^{1/p} \quad \text{if } 1 \leq p < \infty, \tag{2.1}$$

$$\|\cdot\|_{W_\infty^l(\Omega:E)} = \max_{|\alpha| \leq l} \|D^\alpha u\|_{L_\infty(\Omega:E)} \tag{2.2}$$

for any function u for which the right side makes sense. It is clear that (2.1) or (2.2) defines a norm on any vector space of functions on which the right side takes finite values provided functions are identified in the space if they are equal almost everywhere in Ω.

Definition 2.2: We consider a space corresponding to any given values of l and p:

$$W_p^l(\Omega : E) = \left\{ \begin{array}{c} u \in L_p(\Omega : E) : D^\alpha u \in L_p(\Omega : E) \text{ for } |\alpha| \leq l, \\ \text{where } D^\alpha u \text{ is the weak parital derivative} \end{array} \right\}$$

Equipped with the appropriate norms

$$\|\cdot\|_{W_p^l(\Omega:E)} = \left\{ \sum_{|\alpha| \leq l} \|D^\alpha u\|_{L_p(\Omega:E)}^p \right\}^{1/p} \qquad \text{if } 1 \leq p < \infty \qquad \text{and}$$

$$\|\cdot\|_{W_\infty^l(\Omega:E)} = \max_{|\alpha| \leq l} \|D^\alpha u\|_{L_\infty(\Omega:E)}$$

$W_p^l(\Omega : E)$ is called Sobolev Space over Ω.

Clearly, $W_p^0(\Omega : E) = L_p(\Omega : E)$. For any l, the imbedding

$$W_p^l(\Omega : E) \to L_p(\Omega : E)$$

is also clear.

In the particular case $\Omega = \mathbb{R}^n$, it is possible to give an equivalent definition of $W_p^l(\Omega : E)$, by making use of the Fourier Transform. We will investigate the special case for $L_2(\mathbb{R}^n : H)$, i.e H-valued L_2 spaces (where H is a Hilbert space). If $u \in L_2(\mathbb{R}^n : H)$, the Fourier Transform $\hat{u}$ in $L_2(\mathbb{R}^n : H)$ is defined by

$$\hat{u}(y) = \frac{1}{(2\pi)^{\pi/2}} \int\limits_{\mathbb{R}^n} \exp(-ixy)u(x)dx,$$

$$xy = x_1y_1 + ... + x_ny_n,$$

the integral converging in the sense of L^2 and $u \to \hat{u}$ is an isomorphism of $L^2(\mathbb{R}^n : H)$ onto $L^2(\mathbb{R}^n : H)$. We set

$$\hat{u} = \mathcal{F}u$$

and

$$u = \overline{\mathcal{F}}\hat{u} = \frac{1}{(2\pi)^{\pi/2}} \int\limits_{\mathbb{R}^n} \exp(ixy)u(y)dy.$$

The Fourier Transform extends by continuity to the space S' of Schwartz´s tempered distributions, whose definition we now recall.

First of all, we define

$$S = \left\{u : x^\alpha D^\beta u \in L^2(\mathbb{R}^n : H)\ \forall\alpha \text{ and } \forall\beta\right\}.$$

With the sequence of semi-norms

$$u \to \left\|x^\alpha D^\beta u\right\|_{L^2(\mathbb{R}^n:H)},$$

S is a Frechet space: of course every $u \in S$ is (a.e. equal to a function) infinitely differentiable in $\mathbb{R}^n$ and every $u \in S$ is rapidly decreasing at infinity:

$$\forall\alpha, \forall\beta,\ |x^\alpha|\, D^\beta u(x) \to 0 \text{ if } |x| \to \infty$$

(equivalent property to the above definition).

We easily verify that

$$\mathcal{F}(D^\alpha u) = (iy)^\alpha \mathcal{F}u \qquad \forall u \in S\ ,\ \forall\alpha \tag{2.3}$$

$$D^\beta \mathcal{F}(u) = \mathcal{F}((-ix)^\beta u) \quad \forall u \in S\ ,\ \forall\beta, \tag{2.4}$$

and therefore that $\mathcal{F} \in \mathcal{L}(S; S)$. In the same way,

$\overline{\mathcal{F}} \in \mathcal{L}(S;S)$ and

$\mathcal{F}\overline{\mathcal{F}}u = \overline{\mathcal{F}}\mathcal{F}u, \ \ \forall u \in S$.

Therefore $\mathcal{F}$ is an isomorphism of S onto itself, with inverse $\bar{\mathcal{F}}$.

Because of the symmetry of the kernel

$$\frac{1}{(2\pi)^{\pi/2}}\exp(-ixy)$$

of $\mathcal{F}$, we have

$$\int\limits_{\mathbb{R}^n}(\mathcal{F}u)vdx = \int\limits_{\mathbb{R}^n}u(\mathcal{F}v)dx \quad \forall u, \quad v \in S.$$

Next, we define

$S' =$ dual space of S, with the strong dual topology and we define

$\mathcal{F},\overline{\mathcal{F}} \in \mathcal{L}(S';S')$ by transposition.

Thus $\forall u \in S'$, we have

$$\langle \mathcal{F}u, \varphi\rangle = \langle u, \mathcal{F}\varphi\rangle \quad \forall \varphi \in S$$

where the brackets denote the duality between S' and S.

The formulas (2.3) and (2.4) are still valid $\forall u \in S'$.

Theorem 2.3: If $\Omega = \mathbb{R}^n$, $l \in \mathbb{R}$, $W_2^l(\mathbb{R}^n : H)$ may be defined by

$$W_2^l(\mathbb{R}^n : H) = \left\{u : u \in \mathcal{S}', (1+|y|^2)^{l/2}\hat{u} \in L^2(\mathbb{R}^n : H)\right\}$$

(where $|y|^2 = y_1^2 + ... + y_n^2$),

the norm

$$|||u|||_{W_2^l(\mathbb{R}^n:H)} = \left\|(1+|y|^2)^{m/2}\hat{u}\right\|_{L^2(\mathbb{R}^n:H)} \tag{2.5}$$

being equivalent to the norm (11).

Proof: From (2.3) and (2.4) and Plancherel´s theorem

$$\|D^\alpha u\|_{L^2(\mathbb{R}^n:H)} = \|y^\alpha \hat{u}\|_{L^2(\mathbb{R}^n:H)},$$

so that $\|u\|_{W_p^l(\Omega:E)}$ yields (for $\Omega = \mathbb{R}^n$)

$$\|u\|^2_{W^l_2(\mathbb{R}^n:H)} = \int\limits_{\mathbb{R}^n} (\sum_{|\alpha|\le l} y^{2\alpha})\, |\hat{u}(y)|^2\, dy. \tag{2.6}$$

For a suitable constant C:

$$(1+|y|^2)^l \le \sum_{|\alpha|\le l} y^{2\alpha} \le C(1+|y|^2)^l\ ,$$

which together with (2.5) and (2.6) yields

$$|||u|||_{W^l_2(\mathbb{R}^n:H)} \le \|u\|_{W^l_2(\mathbb{R}^n:H)} \le C^{1/2}\, |||u|||_{W^l_2(\mathbb{R}^n:H)}.$$

2.1 Completeness

Theorem 2.4: $W^l_p(\Omega : E)$ is a Banach space.

Proof: Let $\{u_n\}$ be a Cauchy sequence in $W^l_p(\Omega : E)$. Then $\{D^\alpha u_n\}$ is a Cauchy sequence in $L_p(\Omega : E)$ for $|\alpha| \le l$. Since $L_p(\Omega : E)$ is complete there exist functions u and u_α, $|\alpha| \le l$, in $L_p(\Omega : E)$ such that $u_n \to u$ and $D^\alpha u_n \to u_\alpha$ in $L_p(\Omega : E)$ as $n \to \infty$. Now $L_p(\Omega : E) \subset L_1^{loc}(\Omega : E)$ and so u_n determines a distribution $T_{u_n} \in D'(\Omega)$. For any $\varphi \in D(\Omega)$ we have

$$|T_{u_n}(\varphi) - T_u(\varphi)| \le \int_\Omega \|u_n(x) - u(x)\|_E\, |\varphi(x)|\, dx \le \|\varphi\|_{Lp'(\Omega:E)}\, \|u_n - u\|_{Lp(\Omega:E)}$$

by Hölder´s inequality, where $p' = p/(p-1)$ (or $p' = \infty$ if $p = 1$, $p' = 1$ if $p = \infty$). Hence $T_{u_n}(\varphi) \to T_u(\varphi)$ for every $\varphi \in D(\Omega)$ as $n \to \infty$. Similarly, $T_{D^\alpha u_n}(\varphi) \to T_{u_\alpha}(\varphi)$ for every $\varphi \in D(\Omega)$. It follows that

$$T_{u_\alpha}(\varphi) = \lim_{n\to\infty} T_{D^\alpha u_n}(\varphi) = \lim_{n\to\infty} (-1)^{|\alpha|} T_{u_n}(D^\alpha \varphi) = (-1)^{|\alpha|} T_u(D^\alpha \varphi)$$

for every $\varphi \in D(\Omega)$. Thus $u_\alpha = D^\alpha u$ in the distributional sense on Ω for $|\alpha| \le l$ whence $u \in W^l_p(\Omega : E)$. Since $\lim\limits_{n\to\infty} \|u_n - u\|_{W^l_p(\Omega:E)} = 0$, $W^l_p(\Omega : E)$ is complete.

2.2 Density Theorem

We wish to prove that $C^\infty(\Omega : E)$ is dense in $W^l_p(\Omega : E)$. To this end we require the following standard existence theorem for infinitely differentiable partitions of unity.

Theorem 2.5: Let A be an arbitrary subset of $\mathbb{R}^n$ and let O be a collection of open sets in $\mathbb{R}^n$ which cover A, that is , such that $A \subset \bigcup\limits_{U \in \mathcal{O}} U$.

Then there exists a collection Ψ of functions $\psi \in C_0^\infty(\mathbb{R}^n)$ having the following properties:

(i) For every $\psi \in \Psi$ and every $x \in \mathbb{R}^n$, $0 \le \psi(x) \le 1$.

(ii) If $\overline{K} \subset A$ and $\overline{K}$ is compact, all but possibly finitely many $\psi \in \Psi$ vanish identically on K.

(iii) For every $\psi \in \Psi$ there exists $U \in O$ such that supp$\psi \subset U$.

(iv) For every $x \in A$, $\sum_{\psi \in \Psi} \psi(x) = 1$.

Such a collection Ψ is called a $C^\infty-$partition of unity for A subordinate to O.

Proof: Suppose first that A is compact so that $A \subset \bigcup\limits_{j=1}^{N} U_j$, where $U_1, ...U_N \in O$. Compact sets $K_1 \subset U_1, ..., K_N \subset U_N$ can be constructed so that $A \subset \bigcup\limits_{j=1}^{N} K$. For $1 \le j \le N$ there exists a non-negative-valued function $\phi_j \in C_0^\infty(U_j)$ such that $\phi_j(x) > 0$ for $x \in K_j$. A function ϕ can be constructed so as to be infinitely differentiable and positive on $\mathbb{R}^n$ and to satisfy

$\phi(x) = \sum_{j=1}^{N} \phi_j(x)$ for $x \in A$. Now $\Psi = \left\{\psi_j \mid \psi_j(x) = \phi_j(x)/\phi(x), 1 \le j \le N\right\}$ has the desired properties. Now suppose A is open. Then $A = \bigcup\limits_{j=1}^{N} A_j$, where

$A_j = \{x \in A \mid |x| \le j$ and dist(x, bdry$A$) $\ge 1/j\}$

is compact. For each j the collection

$O_j = \left\{U \cap (\text{interior } \ A_{j+1} \cap A_{j-2}^c) \mid U \in \mathcal{O}\right\}$

covers A_j and so there exists a finite C^∞-partition of unity Ψ_j for A_j subordinate to O_j. The sum $\sigma(x) = \sum_{j=1}^{\infty} \sum_{\phi \in \Psi_j} \phi(x)$ involves only finitely many nonzero terms at each point, and is positive at each $x \in A$. The collection

$\Psi = \{\psi \mid \psi(x) = \phi(x)/\sigma(x)$ for some ϕ in some Ψ_j if $x \in A$, $\psi(x) = 0$ if $x \notin A\}$ has the prescribed properties. Finally, if A is arbitrary, then $A \subset B = \bigcup\limits_{U \in \mathcal{O}} U$, where U is open. Any partition of unity for B will do for A as well. We will give the definition of the mollifier which will be used in the following lemma.

Definition 2.6: Let J be a nonnegative real-valued function belonging to $C_0^\infty(\mathbb{R}^n)$ and having the properties

(i) $J(x) = 0$ if $|x| \geq 1$, and

(ii) $\int\limits_{\mathbb{R}^n} J(x)dx = 1.$

If $\varepsilon > 0$, the function $J_\varepsilon(x) = \varepsilon^{-n}J(x/\varepsilon)$ is nonnegative, belongs to $C_0^\infty(\mathbb{R}^n)$, and satisfies

(i) $J_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$, and

(ii) $\int\limits_{\mathbb{R}^n} J_\varepsilon(x)dx = 1.$

J_ε is called a mollifier, and the convolution

$$J_\varepsilon * u(x) = \int\limits_{\mathbb{R}^n} J_\varepsilon(x-y)u(y)dy, \tag{2.7}$$

defined for functions u for which the right side of (2.7) makes sense , is called a mollification or regularization of u.

Lemma 2.7: Let u be a function which is defined on $\mathbb{R}^n$ and vanishes identically outside the domain Ω. If $u \in L_p(\Omega : E)$ where $1 \leq p < \infty$, then $J_\varepsilon * u \in L_p(\Omega : E)$. Moreover,

$\|J_\varepsilon\|_{L_p(\Omega:E)} \leq \|u\|_{L_p(\Omega:E)}$ and $\lim_{\varepsilon\to0^+} \|J_\varepsilon * u - u\|_{L_p(\Omega:E)} = 0.$

Lemma 2.8: Let J_ε be defined as above and let $1 \leq p < \infty$ and $u \in W_p^l(\Omega : E)$. If $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact, then $\lim_{\varepsilon\to0^+} J_\varepsilon * u = u$ in $W_p^l(\Omega' : E)$.

Proof: Let $\varepsilon <$dist$(\Omega', bdry\Omega)$. For any $\phi \in D(\Omega')$ we have

$$\begin{aligned}\int\limits_{\Omega'} J_\varepsilon * u(x)D^\alpha\phi(x)dx &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \widetilde{u}(x-y)J_\varepsilon(y)D^\alpha\phi(x)dxdy \\ &= (-1)^{|\alpha|}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} D_x^\alpha\widetilde{u}(x-y)J_\varepsilon(y)\phi(x)dxdy \\ &= (-1)^{|\alpha|}\int\limits_{\Omega'} J_\varepsilon * D^\alpha u(x)\phi(x)dx.\end{aligned}$$

where $\widetilde{u}$ is the zero extension of u outside Ω. Thus $D^\alpha J_\varepsilon * u = J_\varepsilon * D^\alpha u$ in the distributional sense in Ω'. Since $D^\alpha u \in L_p(\Omega : E)$ for $|\alpha| \leq l$ we have by previous lemma

$\lim_{\varepsilon\to0^+} \|D^\alpha J_\varepsilon * u - D^\alpha u\|_{L_p(\Omega:E)} = \lim_{\varepsilon\to0^+} \|J_\varepsilon * D^\alpha u - D^\alpha u\|_{L_p(\Omega:E)} = 0.$

Thus $\lim_{\varepsilon\to 0^+} \|J_\varepsilon * u - u\|_{W_p^l(\Omega:E)} = 0$.

Definition 2.9: We shall say that a domain Ω has the segment property if for every x in the boundry of Ω there exists an open set U_x and a nonzero vector y_x such that $x \in U_x$ and if $z \in \overline{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $0 < t < 1$. A domain having this property must have $(n-1)$-dimensional boundry and cannot simultaneously lie on both sides of any given part of its boundary.

The following theorem shows that this property is sufficient to guarantee that $C_0^\infty(\mathbb{R}^n : E)$ is dense in $W_p^l(\Omega : E)$, and hence in particular that $C^k(\overline{\Omega} : E)$ is dense in $W_p^l(\Omega : E)$ for any l.

Theorem 2.10: If Ω has the segment property, then the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n : E)$ is dense in $W_p^l(\Omega : E)$ for $1 \le p < \infty$.

Proof: Let f be a fixed function in $C_0^\infty(\mathbb{R}^n : E)$ satisfying

(i) $f(x) = 1$ if $|x| \le 1$,

(ii) $f(x) = 0$ if $|x| \ge 2$,

(iii) $\|D^\alpha f(x)\|_E \le M$ (constant) for all x and $0 \le |\alpha| \le l$.

Let $f_\varepsilon(x) = f(\varepsilon x)$ for $\varepsilon > 0$. Then $f_\varepsilon(x) = 1$ if $|x| \le 1/\varepsilon$ and

$\|D^\alpha f_\varepsilon(x)\|_E \le M\varepsilon^{|\alpha|} \le M$ if $\varepsilon \le 1$. If $u \in W_p^l(\Omega : E)$, then $u_\varepsilon = f_\varepsilon \cdot u$ belongs to $W_p^l(\Omega : E)$ and has bounded support. Since, for $0 < \varepsilon \le 1$ and $|\alpha| \le l$,

$$\|D^\alpha u_\varepsilon(x)\|_E = \left\| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} f_\varepsilon(x) \right\|_E \le M \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left\|D^\beta u(x)\right\|_E,$$

we have, setting $\Omega_\varepsilon = \{x \in \Omega : |x| > 1/\varepsilon\}$,

$$\begin{aligned} \|u - u_\varepsilon\|_{W_p^l(\Omega:E)} &= \|u - u_\varepsilon\|_{W_p^l(\Omega_\varepsilon:E)} \\ &\le \|u\|_{W_p^l(\Omega_\varepsilon:E)} + \|u_\varepsilon\|_{W_p^l(\Omega_\varepsilon:E)} \le \text{const}\|u\|_{W_p^l(\Omega_\varepsilon:E)}. \end{aligned}$$

The right side tends to zero as ε tends to 0. Thus any $u \in W_p^l(\Omega_\varepsilon : E)$ can be approximated in that space by functions with bounded supports.

We may now, therefore, assume $K = \{x \in \Omega : u(x) \ne 0\}$ is bounded. The set

$F = \overline{K} \sim \left(\bigcup_{x \in \text{bdry}\,\Omega} U_x \right)$ is thus compact and contained in Ω, $\{U_x\}$ being the collection of open sets referred to in the definition of the segment property. There exists an open set U_0 such that $\overline{F} \subset U_0$, $\overline{U_0} \subset U$ and $\overline{F}$, $\overline{U_0}$ is compact. Since $\overline{K}$ is compact, there exist finitely many of the sets U_x; let us rename them $U_1, ..., U_k$, such that $\overline{K} \subset U_0 \cup U_1 \cup ... \cup U_k$. Moreover, we may find other open sets $\widetilde{U}_0, \widetilde{U}_1, ..., \widetilde{U}_k$ such that $\widetilde{U}_j \subset U_j$ and the closure of $\widetilde{U}_j$ is compact, $0 \leq j \leq k$, but still
$\overline{K} \subset \widetilde{U}_0 \cup \widetilde{U}_1 \cup ... \cup \widetilde{U}_k$.

Let Ψ be a C^∞-partition of unity subordinate to $\left\{ \widetilde{U}_j : 0 \leq j \leq k \right\}$, and let ψ_j be the sum of the finitely many functions $\psi \in \Psi$ whose supports lie in $\widetilde{U}_j$. Let $u_j = \psi_j u$. Suppose that for each j we can find $\phi_j \in C_0^\infty(\mathbb{R}^n : E)$ such that

$$\left\| u_j - \phi_j \right\|_{W_p^l(\Omega:E)} < \varepsilon/(k+1). \tag{2.8}$$

Then putting $\phi = \sum_{j=0}^{k} \phi_j$, we would obtain

$$\|\phi - u\|_{W_p^l(\Omega:E)} \leq \sum_{j=0}^{k} \left\| \phi_j - u_j \right\|_{W_p^l(\Omega:E)} < \varepsilon.$$

A function $\phi_0 \in C_0^\infty(\mathbb{R}^n : E)$ satisfying (2.8) for $j = 0$ can be found via Lemma 2.8 since supp $u_0 \subset \widetilde{U}_0 \subset \Omega$. It remains, therefore, to find ϕ_j satisfying (2.8) for $1 \leq j \leq k$. For fixed such j we extend u_j to be identically zero outside Ω. Thus $u_j \in W_p^l(\mathbb{R}^n \sim \Gamma : E)$, where $\Gamma = \widetilde{U}_j \cap$ bdry Ω. Let y be the nonzero vector associated with the set U_j in the definition of the segment property. Let $\Gamma_t = \Gamma - ty$, where t is so chosen that

$$0 < t < \min(1, \text{dist}(\widetilde{U}_j, \mathbb{R}^n \sim U_j)/\,|y|).$$

Then $\Gamma_t \subset U_j$ and $\Gamma_t \cap \overline{\Omega}$ is empty by the segment property. Let
$U_{j,t}(x) = u_j(x+ty)$. Then $u_{j,t} \in W_p^l(\mathbb{R}^n \sim \Gamma_t : E)$. Translation is continuous in $L_p(\Omega : E)$ so $D^\alpha u_{j,t} \to D^\alpha u_j$ in $L_p(\Omega : E)$ as $t \to 0+$, $|\alpha| \leq l$. Thus $u_{j,t} \to u_j$ in $W_p^l(\Omega : E)$ as $t \to 0+$ and so it is sufficient to find $\phi_j \in C_0^\infty(\mathbb{R}^n : E)$ such that $\left\| u_{j,t} - \phi_j \right\|_{W_p^l(\Omega:E)}$ is sufficiently small. However, $\overline{\Omega \cap U_j} \subset \mathbb{R}^n \sim \Gamma_t$ and $\overline{\Omega \cap U_j}$ is compact and so by Lemma 2.8 we may take $\phi_j = J_\delta * u_{j,t}$ for suitably small $\delta > 0$. This comletes the proof.

2.3 Extension Theorem

Definition 2.11: Let Ω be a domain in $\mathbb{R}^n$. For given l and p a linear operator L mapping $W_p^l(\Omega : E)$ into $W_p^l(\mathbb{R}^n : E)$ is called a simple $(l,p)-$extension operator for Ω provided there exists a constant $K = K(l,p)$ such that for every $u \in W_p^l(\Omega : E)$ the following conditions hold:

(i) $Lu(x) = u(x)$ a.e. in Ω,

(ii) $\|Lu\|_{W_p^l(\mathbb{R}^n:E)} \leq K \|u\|_{W_p^l(\Omega:E)}$.

L is called a strong $l-$extension operator mapping functions defined a.e. in Ω into functions defined a.e. in $\mathbb{R}^n$ and if for every p, $1 \leq p < \infty$, and every k, $0 \leq k \leq l$, the restriction of L to $W_p^l(\Omega : E)$ is a simple $(k,p)-$extension operator for Ω. Finally, L is called a total extension operator for Ω provided L is a strong $l-$extension operator for Ω, for every l.

Theorem 2.12: Let Ω be either

(i) a half-space in $\mathbb{R}^n$, or

(ii) a domain in $\mathbb{R}^n$ having the uniform C^m- regularity property, and also having a bounded boundary.

For any positive integer l there exists a strong $l-$extension operator L for Ω. Moreover, if α and γ are multi-indices with $|\gamma| \leq |\alpha| \leq l$, there exists a linear operator $L_{\alpha\gamma}$ continuous from $W_p^j(\Omega : E)$ into $W_p^j(\mathbb{R}^n : E)$ for $1 \leq j \leq l - |\alpha|$ such that if $u \in W_p^{|\alpha|}(\Omega : E)$, then

$$D^\alpha Lu(x) = \sum_{|\gamma|\leq|\alpha|} L_{\alpha\gamma} D^\gamma u(x) \qquad \text{a.e. in } \mathbb{R}^n.$$

Proof: First let Ω be the half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. For functions u defined a.e. on $\mathbb{R}^n_+$ we define extensions Lu and $L_\alpha u$, $|\alpha| \leq l$, a.e. on $\mathbb{R}^n$ via

$$Lu(x) = \left\{ \begin{array}{ll} u(x) & \text{if } x_n > 0 \\ \sum_{j=1}^{l+1} \lambda_j u(x_1, ..., x_{n-1}, -jx_n) & \text{if } x_n \leq 0 \end{array} \right\}$$

$$L_\alpha u(x) = \left\{ \begin{array}{ll} u(x) & \text{if } x_n > 0 \\ \sum_{j=1}^{l+1} (-j)^{\alpha_n} \lambda_j u(x_1, ..., x_{n-1}, -jx_n) & \text{if } x_n \leq 0 \end{array} \right\} \tag{2.9}$$

where the coefficients $\lambda_1, ..., \lambda_{l+1}$ are the unique solutions of the $(l+1) \times (l+1)$ system of linear equations

$$\sum_{j=1}^{l+1} (-j)^k \lambda_j = 1, \qquad k = 0, 1, ..., l.$$

If $u \in C^l(\overline{\mathbb{R}^n_+})$, then it is readily checked that $Lu \in C^l(\mathbb{R}^n)$ and

$$D^\alpha Lu(x) = L_\alpha D^\alpha u(x), \qquad |\alpha| \leq l.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^n} \|D^\alpha Lu(x)\|_E^p \, dx &= \int_{\mathbb{R}^n_+} \|D^\alpha u(x)\|_E^p \, dx \\ &\quad + \int_{\mathbb{R}^n_-} \left\| \sum_{j=1}^{l+1} (-j)^{\alpha_n} \lambda_j D^\alpha u(x_1, ..., x_{n-1}, -jx_n) \right\|_E^p dx \\ &\leq K(l, p, \alpha) \int_{\mathbb{R}^n_+} \|D^\alpha u(x)\|_E^p \, dx\end{aligned}$$

The above inequality extends, by virtue of Theorem 2.10, to functions $u \in W_p^k(\mathbb{R}^n_+ : E)$, $l \geq k \geq |\alpha|$. Hence L is a strong l-extension operator for $\mathbb{R}^n_+$. Since

$D^\beta L_\alpha u(x) = L_{\alpha+\beta} u(x)$, a similar calculation shows that L_α is a strong $(l - |\alpha|)$-extension. Thus the theorem is proved for half-spaces (with $L_{\alpha\alpha} = L_\alpha$, $L_{\alpha\gamma} = 0$ otherwise).

Now suppose Ω is uniformly C^l-regular and has bounded boundry. Then the open cover $\{U_j\}$ of bdry Ω, and the corresponding l-smooth maps Φ_j from U_j onto B, are finite collections, say $1 \leq j \leq N$. Let

$$Q = \left\{ y \in \mathbb{R}^n : |y'| = \left(\sum_{j=1}^{n-1} y_j^2 \right)^{1/2} < \tfrac{1}{2}, \ |y_n| < \sqrt{3}/2 \right\}.$$

Then

$$\{y \in \mathbb{R}^n : \ |y| < 1/2\} \subset Q \subset B = \{y \in \mathbb{R}^n : \ |y| < 1\}.$$

By condition [1,(i) of Section 4.6] the open sets $\psi_j = \Psi_j(Q)$, $1 \leq j \leq N$, form an open cover of $\Omega_\delta = \{x \in \Omega : \text{ dist } (x, \text{bdry } \Omega) < \delta\}$ for some $\delta > 0$. There exists an open set ψ_0 of Ω, bounded away from bdry Ω, such that $\Omega \subset \bigcup_{j=0}^N \psi_j$. By [1,theorem 3.14] we may find infinitely differentiable functions $\omega_0, \omega_1, ..., \omega_N$ such that supp $\omega_j \subset \psi_j$ and $\sum_{j=0}^N \omega_j(x) = 1$ for all $x \in \Omega$. (Note that supp ω_0 need not to be compact if Ω is unbounded.)

Since Ω is uniformly C^l-regular it has the segment property and so restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n : E)$ are dense in $W_p^k(\Omega : E)$. If $\phi \in C_0^\infty(\mathbb{R}^n : E)$, then ϕ agrees on Ω with the function $\sum_{j=0}^N \phi_j$, where $\phi_j = \omega_j(x) \cdot \phi \in C_0^\infty(\psi_j)$

For $j \geq 1$ and $y \in B$ let $\varphi_j(y) = \phi_j(\Psi_j(y))$. Then $\varphi_j(y) \in C_0^\infty(Q : E)$. We extend φ_j to be identically zero outside Q. With L (and L_α) defined as in (2.9), we have $L\varphi_j \in C_0^l(Q : E)$, $L\varphi_j = \varphi_j$ on $Q_+ = \{y \in Q : y_n > 0\}$, and

$$\left\|L\varphi_j\right\|_{W_p^k(Q:E)} \leq K_1 \left\|\varphi_j\right\|_{W_p^k(Q_+:E)}, \qquad 0 \leq k \leq l,$$

where K_1 depends on k, l and p. If $\theta_j(x) = L\varphi_j(\Phi_j(x))$, then $\theta_j \in C_0^l(\psi_j : E)$ and $\theta_j(x) = \phi_j(x)$ if $x \in \Omega$. It may be checked by induction that if $|\alpha| \leq l$, then

$$D^\alpha \theta_j(x) = \sum_{|\beta| \leq |\alpha|} \sum_{|\gamma| \leq |\alpha|} a_{j;\alpha\beta}(x) \left[L_\beta \left(b_{j;\beta\gamma} \cdot (D^\gamma \phi_j \circ \Psi_j)\right)\right](\Phi_j(x)),$$

where $a_{j;\alpha\beta} \in C^{m-|\alpha|}(\overline{U}_j)$ and $b_{j;\beta\gamma} \in C^{m-|\beta|}(\overline{B})$ depend on the transformations Φ_j and $\Psi_j = \Phi_j^{-1}$ and satisfy

$$\sum_{|\beta| \leq |\alpha|} a_{j;\alpha\beta}(x) b_{j;\beta\gamma}(\Phi_j x) = \left\{ \begin{array}{ll} 1 & \text{if } \gamma = \alpha \\ 0 & \text{otherwise.} \end{array} \right\}$$

By [1,theorem 3.35] we have for $k \leq m$,

$$\|\theta_j\|_{W_p^k(\mathbb{R}^n:E)} \leq K_2 \left\|L\varphi_j\right\|_{W_p^k(Q:E)} \leq K_1 K_2 \left\|\varphi_j\right\|_{W_p^k(Q_+:E)} \leq K_3 \left\|\varphi_j\right\|_{W_p^k(\Omega:E)},$$

where K_3 may be chosen to be independent of j. The operator $\widetilde{L}$ defined by

$$\widetilde{L}\phi(x) = \phi_0(x) + \sum_{j=1}^N \theta_j(x)$$

clearly satisfies $\widetilde{L}\phi(x) = \phi(x)$ if $x \in \Omega$, and

$$\begin{aligned} \left\|\widetilde{L}\phi\right\|_{W_p^k(\mathbb{R}^n:E)} &\leq \|\phi_0\|_{W_p^k(\Omega:E)} + K_3 \sum_{j=1}^N \left\|\phi_j\right\|_{W_p^k(\Omega:E)} \\ &\leq K_4(1 + NK_3) \|\phi\|_{W_p^k(\Omega:E)} \end{aligned}$$

where

$$K_4 = \max_{0 \leq j \leq N} \max_{|\alpha| \leq l} \sup_{x \in \mathbb{R}^n} \|D^\alpha \omega_j(x)\|_E < \infty.$$

Thus $\widetilde{L}$ is a strong l-extension operator for Ω. Also

$$D^\alpha \widetilde{L}\phi(x) = \sum_{|\gamma| \leq |\alpha|} (L_{\alpha\gamma} D^\gamma \phi)(x),$$

where

$$L_{\alpha\gamma}v(x) = \sum_{j=1}^{N} \sum_{|\beta|\leq|\alpha|} a_{j;\alpha\beta}(x) \left[L_\beta \left(b_{j;\beta\gamma} \cdot (v \cdot \omega_j) \circ \Psi_j\right)\right] (\Phi_j(x))$$

if $\alpha \neq \gamma$, and

$L_{\alpha\alpha}v(x) = (v \cdot \omega_0)(x) + \sum_{j=1}^{N} \sum_{|\beta|\leq|\alpha|} a_{j;\alpha\beta}(x) \left[L_\beta \left(b_{j;\beta\alpha} \cdot (v \cdot \omega_j) \circ \Psi_j\right)\right] (\Phi_j(x))$ We note that if $x \in \Omega$, $L_{\alpha\gamma}v(x) = 0$ for $\alpha \neq \gamma$ and $L_{\alpha\alpha}v(x) = v(x)$. Clearly $L_{\alpha\gamma}$ is a linear operator. By the differentiability properties of $a_{j;\alpha\beta}$ and $b_{j;\beta\gamma}$, $L_{\alpha\gamma}$ is continous on $W_p^j(\Omega : E)$ into $W_p^j(\mathbb{R}^n : E)$ for $1 \leq j \leq l - |\alpha|$. This completes the proof.

Lemma 2.13: There exists a real sequence $\{a_k\}_{k=0}^{\infty}$ such that for every nonnegative integer n we have

$$\sum_{k=0}^{\infty} 2^{nk} a_k = (-1)^n \tag{2.10}$$

and

$$\sum_{k=0}^{\infty} 2^{nk} a_k < \infty. \tag{2.11}$$

Theorem 2.14: Let Ω be either

(i) a half-space in $\mathbb{R}^n$, or

(ii) a domain in $\mathbb{R}^n$ having the uniform C^l- regularity property for every l, and also having a bounded boundry.

Then there exists a total extension operator for Ω.

Proof: It is sufficient to prove the theorem for the half-space $\mathbb{R}^n_+$; the proof for Ω satisfying (ii) then follows just as in Theorem 2.12.

The restrictions to $\mathbb{R}^n_+$ of functions $\phi \in C_0^\infty(\mathbb{R}^n : E)$ being dense in $W_p^l(\mathbb{R}^n_+ : E)$ for any m and p, we define the extension operator only on such functions. Let f be a real-valued function, infinitely differentiable on $[0, \infty)$ and satisfying $f(t) = 1$ if $0 \leq t \leq \frac{1}{2}$, $f(t) = 0$ if $t \geq 1$. If $\phi \in C_0^\infty(\mathbb{R}^n : E)$, let

$$L\phi = \left\{ \begin{array}{ll} \phi(x) & \text{if } x \in \overline{\mathbb{R}^n_+} \\ \sum_{k=0}^{\infty} a_k f(-2^k x_n)\phi(x', -2^k x_n) & \text{if } x \in \overline{\mathbb{R}^n_-} \end{array} \right\} \tag{2.12}$$

where $\{a_k\}$ is the sequence constructed in the above lemma, and $x' = (x_1, ..., x_{n-1})$. Then clearly $L\phi$ is well defined on $\mathbb{R}^n$ since the sum in (2.12) has only finitely many nonvanishing terms for any particular $x \in \mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n < 0\}$. Moreover, $L\phi$ has compact support and belongs to $C^\infty(\overline{\mathbb{R}^n_+} : E) \cap C^\infty(\overline{\mathbb{R}^n_-} : E)$. If $x \in \overline{\mathbb{R}^n_-}$, we have

$$\begin{aligned} D^\alpha L\phi(x) &= \sum_{k=0}^{\infty} a_k \sum_{j=0}^{\alpha_n} \binom{\alpha_n}{j} (-2^k)^{\alpha_n} f^{(\alpha_n - j)}(-2^k x_n) D_n^j D^{\acute{\alpha}} \phi(\acute{x}, -2^k x_n) \\ &= \sum_{k=0}^{\infty} \varphi_k(x). \end{aligned}$$

Since $\varphi_k(x) = 0$ when $-x_n > 1/2^{k-1}$ it follows from (2.11) that the above series converges absolutely and uniformly as x_n tends to zero from below. Hence by (2.10)

$$\begin{aligned} \lim_{x_n \to 0_-} D^\alpha L\phi(x) &= \sum_{k=0}^{\infty} (-2^k)^{\alpha_n} a_k D^\alpha \phi(\acute{x}, 0_+) \\ &= D^\alpha \phi(\acute{x}, 0_+) = \lim_{x_n \to 0+} D^\alpha L\phi(x) = D^\alpha L\phi(0). \end{aligned}$$

Thus $L\phi \in C_0^\infty(\mathbb{R}^n : E)$. Moreover, if $|\alpha| \le l$,

$$\|\varphi_k\|_E^p \le K_1^p \, |a_k|^p \, 2^{klp} \sum_{|\beta| \le l} \left\| D^\beta \phi(\acute{x}, -2^k x_n) \right\|_E^p ,$$

where K_1 depends on $m, p, n,$ and f. Thus

$$\begin{aligned} \|\varphi_k\|_{W_p^0(\mathbb{R}^n_- : E)} &\le K_1 \, |a_k| \, 2^{kl} \left\{ \sum_{|\beta| \le l} \int_{\mathbb{R}^n_-} \left\| D^\beta \phi(\acute{x}, -2^k x_n) \right\|_E^p dx \right\}^{1/p} \\ &= K_1 \, |a_k| \, 2^{kl} \left\{ (1/2^k) \sum_{|\beta| \le l} \int_{\mathbb{R}^n_+} \left\| D^\beta \phi(y) \right\|_E^p dy \right\}^{1/p} \\ &\le K_1 \, |a_k| \, 2^{kl} \, \|\phi\|_{W_p^l(\mathbb{R}^n_+ : E)}. \end{aligned}$$

It follows by (2.11) that

$$\|D^\alpha L\phi\|_{W_p^0(\mathbb{R}^n_- : E)} \le K_1 \, \|\phi\|_{W_p^m(\mathbb{R}^n_+ : E)} \sum_{k=0}^{\infty} 2^{kl} \, |a_k| \le K_2 \, \|\phi\|_{W_p^m(\mathbb{R}^n_+ : E)}.$$

Combining this with a similar (trivial) inequality for $\|D^\alpha L\phi\|_{W_p^0(\mathbb{R}^n_+ : E)}$, we obtain

$$\|L\phi\|_{W_p^0(\mathbb{R}^n : E)} \le K_3 \, \|\phi\|_{W_p^l(\mathbb{R}^n_+ : E)}$$

with $K_3 = K_3(l, p, n)$. This completes the proof.

2.4 Imbedding Theorem

Theorem 2.15: (The Sobolev imbedding theorem) Let Ω be a domain in $\mathbb{R}^n$, E be a Banach space and let Ω^k be the $k-$dimensional domain obtained by intersecting Ω with a $k-$dimensional plane in $\mathbb{R}^n$, $1 \leq k \leq n$. (Thus $\Omega^n \equiv \Omega$.) Let j and m be non-negative integers and let p satisfy $1 \leq p < \infty$.

PART 1: If Ω has the cone property, then there exist the following embeddings:

Case A: Suppose $mp < n$ and $n - mp < k \leq n$. Then

$$W_p^{j+m}(\Omega : E) \to W_q^j(\Omega^k : E), \qquad p \leq q \leq kp/(n - mp), \tag{2.13}$$

and in particular,

$$W_p^{j+m}(\Omega : E) \to W_q^j(\Omega : E), \qquad p \leq q \leq np/(n - mp), \tag{2.14}$$

or

$$W_p^m(\Omega : E) \to L^q(\Omega), \qquad p \leq q \leq np/(n - mp). \tag{2.15}$$

Moreover, if $p = 1$, so that $m < n$, imbedding (2.13) also exists for $k = n - m$.

Case B: Suppose $mp = n$. Then for each k, $1 \leq k \leq n$,

$$W_p^{j+m}(\Omega : E) \to W_q^j(\Omega^k : E), \qquad p \leq q < \infty, \tag{2.16}$$

so that in particular

$$W_p^m(\Omega : E) \to L^q(\Omega : E), \qquad p \leq q < \infty. \tag{2.17}$$

Moreover, if $p = 1$ so that $m = n$, imbeddings (2.16) and (2.17) exist with $q = \infty$ as well; in fact,

$$W_1^{j+n}(\Omega : E) \to C_B^j(\Omega : E). \tag{2.18}$$

Case C: Suppose $mp > n$. Then

$$W_p^{j+m}(\Omega : E) \to C_B^j(\Omega : E). \tag{2.19}$$

PART 2: If Ω has the strong local Lipschitz propety, then Case C of Part 1 can be refined as follows:

Case C$'$: Suppose $mp > n > (m-1)p$. Then

$$W_p^{j+m}(\Omega : E) \to C_\lambda^j(\overline{\Omega} : E), \qquad 0 < \lambda \leq m - (n/p). \tag{2.20}$$

Case C$''$: Suppose $n = (m-1)p$. Then

$$W_p^{j+m}(\Omega : E) \to C_\lambda^j(\overline{\Omega} : E), \qquad 0 < \lambda < 1. \tag{2.21}$$

Also, if $n = m - 1$ and $p = 1$, then (2.21) holds for $\lambda = 1$ as well.

PART 3: All the conclusions of Parts 1 and 2 are valid for arbitrary domains provided the $W-$spaces undergoing imbedding are replaced with the corresponding W_0-spaces.

Remark 2.16: It is sufficient to establish each of the embeddings (2.13), (2.14), (2.16), (2.18)-(2.21) for the special case $j = 0$. For example, if $W_p^m(\Omega : E) \to L^q(\Omega : E)$ has been established, then for any $u \in W_p^{j+m}(\Omega : E)$ we have $D^\alpha u \in W_p^m(\Omega : E)$ for $|\alpha| \leq j$, whence $D^\alpha u \in L^q(\Omega : E)$; thus $u \in W_q^j(\Omega : E)$; and

$$\|u\|_{W_q^j(\Omega:E)} = \left(\sum_{|\alpha| \leq j} \|D^\alpha u\|^q_{W_q^0(\Omega:E)} \right)^{1/q} \leq K_1 \left(\sum_{|\alpha| \leq j} \|D^\alpha u\|^p_{W_p^m(\Omega:E)} \right)^{1/p}$$
$$\leq K_2 \|u\|_{W_p^{j+m}(\Omega:E)}$$

Accordingly, we will always specialize $j = 0$ in the proofs.

The proof of the imbedding theorem given here is due to Gagliardo [24]. It is based on little more than calculus combined with astute applications of Hölder's inequality. Moreover, Gagliardo's proof establishes the imbedding theorem in the greatest possible generality and is capable of generalization to produce imbedding results for some domains not having the cone property.

The proof is carried out in a chain of auxilary lemmas. In each such lemma constants K_1, K_2,... appearing in the proof are allowed to depend on the same parameters as the constant K referred to in the statement of the lemma.

Lemma 2.17: Let

$$R = \{x \in \mathbb{R}^n : a_i < x_i < b_i; 1 \leq i \leq n\}$$

and

$$R' = \{x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1} : a_i < x_i < b_i; 1 \leq i \leq n-1\}$$

be bounded open rectangles in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. If $a_n < \zeta < b_n$ and $p \geq 1$, then for every $u \in C^\infty(R:E) \cap W_p^1(\Omega:E)$ we have

$$\|u(.,\zeta)\|_{W_p^0(R':E)} \leq K \|u\|_{W_p^1(R:E)} \tag{2.22}$$

where $K = K(p, b_n - a_n)$. Thus the trace mapping $u \to u(.,\zeta)$ extends to an imbedding of $W_p^1(R:E)$ into $L_p(R_\zeta^{n-1}:E)$, where $R_\zeta^{n-1} = R \cap \{x \in \mathbb{R}^n : x_n = \zeta\}$.

Proof: By theorem 2.10, $C^\infty(\overline{R}:E)$ is dense in $W_p^1(R:E)$ so we assume $u \in C^\infty(\overline{R}:E)$. Thus $\int\limits_{R'} \|u(x',\cdot)\|_E^p \, dx'$ belongs to $C^\infty([a_n,b_n]:E)$ and by the mean value theorem for integrals we have

$$\|u\|_{W_p^0(R:E)}^p = \int\limits_{a_n}^{b_n} \left(\int\limits_{R'} \|u(x',x_n)\|_E^p \, dx' \right) dx_n = (b_n - a_n) \int\limits_{R'} \|u(x',\sigma)\|_E^p \, dx'$$

for some $\sigma \in [a_n, b_n]$. Now

$$\|u(x',\zeta)\|_E^p = \left\| u(x',\sigma) + \int\limits_\sigma^\zeta D_n u(x',t) dt \right\|_E^p$$

$$\leq 2^{p-1} \left[\|u(x',\sigma)\|_E^p + |\zeta - \sigma|^{p-1} \int\limits_\sigma^\zeta \|D_n u(x',t) dt\|_E^p \, dt \right]$$

by Hölder's inequality. Integration over R' leads to

$$\|u(\cdot,\zeta)\|_{W_p^0(R':E)}^p \leq 2^{p-1} \left[\|u(\cdot,\sigma)\|_{W_p^0(R':E)}^p + (b_n - a_n)^{p-1} \|D_n u\|_{W_p^0(R:E)}^p \right]$$

$$\leq 2^{p-1} \left[(b_n - a_n)^{-1} \|u\|_{W_p^0(R:E)}^p + (b_n - a_n)^{p-1} \|D_n u\|_{W_p^0(R:E)}^p \right]$$

which yields (2.22) with $K = \left[2^{p-1} \max((b_n - a_n)^{-1}, (b_n - a_n)^{p-1})\right]^{1/p}$. We note that K depends continuously on $b_n - a_n$ but may tend to infinity if $b_n - a_n$ tends to zero or infinity.

Lemma 2.18: Let R be as in the previous lemma. Then

$W_1^n(R:E) \to C(\overline{R}:E)$.

The imbedding constant depends only on n and the dimensions of R.

Proof: Let x be any point of R, and let R' be as in the previous lemma. If $u \in C^\infty(\overline{R}:E)$ and $|\alpha| \leq n-1$, we have by that lemma that

$$\|D^\alpha u(\cdot,x_n)\|_{W_1^0(R':E)} \leq K_1 \|D^\alpha u\|_{W_1^1(R:E)}.$$

Thus

$$\|u(\cdot, x_n)\|_{W_1^{n-1}(R:E)} \leq K_2 \|u\|_{W_1^n(R:E)}$$

with K_2 depending on $b_n - a_n$. Iteration of this argument over succesively lower-dimensional rectangles leads to

$$\|u(\cdot, x_2, x_3, ..., x_n)\|_{W_1^1((a_1,b_1):E)} \leq K_3 \|u\|_{W_1^n(R:E)}$$

with K_3 depending on $b_j - a_j$, $2 \leq j \leq n$. By the mean value theorem for integrals there exists $\sigma \in [a_1, b_1]$ such that

$$\|u(\cdot, x_2, x_3, ..., x_n)\|_{W_1^0((a_1,b_1):E)} = (b_1 - a_1) \|u(\sigma, x_2, ..., x_n)\|_E.$$

Hence

$$\begin{aligned} \|u(x)\|_E &\leq \|u(\sigma, x_2, ..., x_n)\|_E + \int_\sigma^{x_1} \|D_1 u(t, x_2, ..., x_n)\|_E \, dt \\ &\leq [1/(b_1 - a_1)] \|u(\cdot, x_2, ..., x_n)\|_{W_1^0((a_1,b_1):E)} + \|D_1 u(\cdot, x_2, ..., x_n)\|_{W_1^0((a_1,b_1):E)} \\ &\leq K_1 \|u\|_{W_1^n(R:E)} \qquad \textbf{(2.23)} \end{aligned}$$

Now suppose $u \in W_1^n(R : E)$. By theorem 2.10, u is the limit in $W_1^n(R : E)$ of the sequence of functions belonging to $C^\infty(\overline{R} : E)$. It follows from (2.23) that this sequence converges uniformly on $\overline{R}$ to a function $\widetilde{u} \in C(R : E)$. Since $\widetilde{u}(x) = u(x)$ a.e. in R, the lemma is proved.

We now turn our attention to more general domains. The following lemma of Gagliardo, which is essentially combinatorial in nature, is the foundation on which his proof of the embedding theorem rests.

Lemma 2.19: Let Ω be a domain in $\mathbb{R}^n$ where $n \geq 2$. Let k be an integer satisfying $1 \leq k \leq n$, and let $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n)$ denote a k-tuple of integers satisfying $1 \leq \kappa_1 < \kappa_2 < ... < \kappa_n \leq n$. Let S be the set of all $\binom{n}{k}$ such $k-$tuples. Also, given $x \in \mathbb{R}^n$, let x_κ denote the point $(x_{\kappa_1}, x_{\kappa_2}, ..., x_{\kappa_k}) \in \mathbb{R}^k$; $dx_\kappa = dx_{\kappa_1}...dx_{\kappa_k}$.

For given $\kappa \in S$ let E_κ be the k-dimensional plane in $\mathbb{R}^n$ spanned by the coordinate axes corresponding to the components of x_κ :

$$E_\kappa = \{x \in \mathbb{R}^n : x_i = 0 \text{ if } i \notin \kappa\}$$

and for any set $G \subset \mathbb{R}^n$ let G_κ be the projection of G onto E_κ; in particular

$$\Omega_\kappa = \{x \in E_\kappa : \exists y \in \Omega \text{ such that } y_\kappa = x_\kappa\}.$$

Let F_κ be a function depending on the k components of x_κ and belonging to $L_\lambda(\Omega_\kappa : E)$, where $\lambda = \binom{n-1}{k-1}$. Then the function F defined on Ω by

$$F(x) = \prod_{\kappa \in S} F_\kappa(x_\kappa)$$

belongs to $L_1(\Omega : E)$, and $\|F\|_{L_1(\Omega:E)} \le \prod_{\kappa \in S} \|F_\kappa\|_{L_\lambda(\Omega_\kappa:E)}$, that is,

$$\left[\int_\Omega \|F(x)\|_E \, dx\right]^\lambda \le \prod_{\kappa \in S} \int_{\Omega_\kappa} \|F_\kappa(x_\kappa)\|_E^\lambda \, dx_\kappa. \tag{2.24}$$

Lemma 2.20: Let Ω be a bounded domain in $\mathbb{R}^n$ having the cone property. If $1 \le p < n$, then $W_p^1(\Omega : E) \to L_q(\Omega : E)$, where $q = np/(n-p)$. The imbedding constant may be chosen to depend only on m, p, n and the cone C determining the cone property for Ω.

Proof: We must show that for any $u \in W_p^1(\Omega : E)$,

$$\|u\|_{W_q^0(\Omega:E)} \le K \|u\|_{W_p^1(\Omega:E)}, \tag{2.25}$$

with $K = K(m, p, n, C)$. By [1,theorem 4.8], Ω may be expressed as a union of finitely many subdomains each of which has the strong local Lipschitz property (and therefore the segment property), and aech of which is itself a union of parallel translates of a corresponding parallelepiped. A review of the proof of that theorem shows that the number of subdomains and the dimension s of the corresponding parallelepipeds depend on n and C. It is therefore sufficient to establish (2.25) for one of these subdomains.

By [1,theorem 3.35] and a suitable nonsingular linear transformation we may assume that the parallelepiped involved is, in fact, a cube Q having edge length 2 units, and having edges parallel to the coordinate axes. Accordingly we assume hereafter that $\Omega = \bigcup_{x \in A}(x + Q)$ with $A \subset \Omega$, and that Ω has the segment property. By theorem 2.10 it is sufficient to establish (2.25) for $u \in C^\infty(\overline{\Omega} : E)$.

For $x \in \Omega$ let $w_i(x)$ denote the intersection of Ω with the straight line through x parallel to the x_i coordinate axis. Clearly, $w_i(x)$ contains a segment of unit length with one endpoint at x, say the segment $x + te_i$, $0 \le t < 1$, where e_i is a unit vector along the x_i-axis.

Let $\gamma = (np - p)/(n - p)$ so that $\gamma \geq 1$. Integration by parts gives, for $u \in C^\infty(\overline{\Omega : E})$,

$$\int_0^1 \|u(x + (1-t)e_i)\|_E^\gamma \, dt = \|u(x)\|_E^\gamma$$
$$-\gamma \int_0^1 t \, \|u(x + (1-t)e_i)\|_E^{\gamma - 1} \frac{d}{dt} \|u(x + (1-t)e_i)\|_E \, dt. \qquad \textbf{(2.26)}$$

Let $\widehat{x}_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ and set

$$F_i(\widehat{x}_i) = \sup_{y \in w_i(x)} \|u(y)\|_E^{p/(n-p)}.$$

Then (2.26) gives

$$\|F_i(\widehat{x}_i)\|_E^{n-1} \leq \int_{w_i(x)} \|u(x)\|_E^\gamma \, dx_i + \gamma \int_{w_i(x)} \|u(x)\|_E^{\gamma - 1} \|D_i u(x)\|_E \, dx_i. \qquad \textbf{(2.27)}$$

Integration over Ω_i, the projection of Ω onto the plane $x_i = 0$, now leads to

$$\int_{\Omega_i} \|F_i(\widehat{x}_i)\|_E^{n-1} \, d\widehat{x}_i \leq \int_\Omega \|u(x)\|_E^\gamma \, dx + \gamma \int_\Omega \|u(x)\|_E^{\gamma - 1} \|D_i u(x)\|_E \, dx.$$

If $p > 1$, then $\gamma > 1$ and an application of Hölder's inequality gives

$$\|F_i\|_{W_{n-1}^0(\Omega_i : E)}^{n-1} \leq \gamma \left[\int_\Omega \left(\|u(x)\|_E + \|D_i u(x)\|_E \right)^p dx \right]^{1/p} \left[\int_\Omega \|u(x)\|_E^{(\gamma - 1)p'} dx \right]^{1/p'}$$
$$\leq 2^{(p-1)/p} \gamma \, \|u\|_{W_p^1(\Omega : E)} \|u\|_{W_q^0(\Omega : E)}^{q/p'}$$

since $(\gamma - 1)p' = q$.

We now apply Lemma 2.19 to the functions F_i, $1 \leq i \leq n$, noting that $k = n - 1$ so that the exponent λ of that lemma is itself $n - 1$:

$$\|u\|_{W_q^0(\Omega : E)}^q = \int_\Omega \|u(x)\|_E^{np/(n-p)} \, dx \leq \int_\Omega \prod_{i=1}^n F_i(\widehat{x}_i) dx \leq \prod_{i=1}^n \|F_i\|_{W_{n-1}^0(\Omega_i : E)}$$
$$\leq (2^{(p-1)/p}) \gamma \, \|u\|_{W_p^1(\Omega : E)} \|u\|_{W_q^0(\Omega : E)}^{q/p'}$$

since $(n-1)q/n - q/p' = 1$, (2.25) follows by cancellation. The cancellation is justified, for since $u \in C^\infty(\overline{\Omega : E})$ and Ω is bounded, $\|u\|_{W_q^0(\Omega : E)}$ is finite. Since $C^\infty(\overline{\Omega : E})$ is dense in $W_p^1(\Omega : E)$, (2.25) extends by continuity to all of $W_p^1(\Omega : E)$.

Remark 2.21: Let $u \in C_0^\infty(\mathbb{R}^n)$ and let q, r be as in the above proof. From the identity

$$\int_0^\infty \frac{d}{dt} \|u(x + te_i)\|_E^\gamma \, dt = -\|u(x)\|_E^\gamma,$$

we obtain

$$\sup_{y \in w_i(x)} \|u(y)\|_E^\gamma \leq \gamma \int_{-\infty}^\infty \|u(x)\|_E^{\gamma - 1} \|D_i u(x)\|_E \, dx_i,$$

where $w_i(x)$ is the line through x parallel to the x_i-axis. Comparing this with (2.27), we see that the computations of the above proof can be reproduced to yield in this case

$$\|u\|_{W_q^0(\mathbb{R}^n:E)} \leq K \|u\|_{W_p^1(\mathbb{R}^n:E)}, \qquad \textbf{(2.28)}$$

where the seminorm is defined as

$$|u|_{W_p^1(\mathbb{R}^n:E)} = \left\{ \sum_{|\alpha|=1} \int_{\mathbb{R}^n} \|D^\alpha u(x)\|_E^p \, dx \right\}^{1/p}$$

Inequality (2.28) is known as Sobolev's inequality.

Lemma 2.22: Let Ω be a bounded domain in $\mathbb{R}^n$ having the cone property. If $mp < n$, then $W_p^m(\Omega : E) \to L_q(\Omega : E)$ for $p \leq q \leq np/(n - mp)$. The imbedding constant may be chosen to depend only on m, p, n, q and the cone C determining the cone property for Ω.

Proof: Let $q_0 = np/(n - mp)$. We first prove by induction on m that

$W_p^m(\Omega : E) \to L_{q_0}(\Omega : E)$. Note that Lemma 2.20 establishes the case $m = 1$.

Assume, therefore, that $W_p^{m-1}(\Omega : E) \to L_r(\Omega : E)$ for $r = np/(n - mp + p)$ whenever $n > (m-1)p$. If $u \in W_p^m(\Omega : E)$, where $n > mp$, then u and $D_j u$ $(1 \leq j \leq n)$ belong to $W_p^{m-1}(\Omega : E)$. It follows that $u \in W_r^1(\Omega : E)$ and

$$\|u\|_{W_r^1(\Omega:E)} \leq K_1 \|u\|_{W_p^m(\Omega:E)} .$$

Since $mp < n$, we have $r < n$ and so by Lemma 2.20 we have $W_r^1(\Omega : E) \to L_{q_0}(\Omega : E)$ where $q_0 = nr/(n - r) = np/(n - mp)$ and

$$\|u\|_{W_{q_0}^0(\Omega:E)} \leq K_2 \|u\|_{W_r^1(\Omega:E)} \leq K_3 \|u\|_{W_p^m(\Omega:E)} \qquad \textbf{(2.29)}$$

This completes the induction.

Now suppose $p \leq q \leq q_0$. We set

$$s = (q_0 - q)p/(q_0 - p) \quad \text{and} \quad t = p/s = (q_0 - p)/(q_0 - q)$$

and obtain by Hölder's inequality

$$\begin{aligned} \|u\|_{W_q^0(\Omega:E)}^q &= \int_\Omega \|u(x)\|_E^s \|u(x)\|_E^{q-s} \, dx \\ &\leq \left[\int_\Omega \|u(x)\|_E^{st} \, dx \right]^{1/t} \left[\int_\Omega \|u(x)\|_E^{(q-s)t'} \, dx \right]^{1/t'} \end{aligned}$$

$$= \|u\|^{p/t}_{W^0_p(\Omega:E)} \|u\|^{q_0/t'}_{W^0_{q_0}(\Omega:E)} \le K_3^{q_0/t'} \|u\|^q_{W^m_p(\Omega:E)} \quad \textbf{(2.30)}$$

by (2.29).

Corollary 2.23: If $mp = n$, then $W^m_p(\Omega : E) \to L_q(\Omega : E)$ for $p \le q < \infty$. The imbedding constant here may also depend on vol Ω.

Proof: If $q \ge p' = p/(p-1)$, then $q = ns/(n-ms)$, where $s = pq/(p+q)$ satisfies $1 \le s < p$. By [1,theorem 2.8], $W^m_p(\Omega : E) \to W^m_s(\Omega : E)$ with the imbedding constant depend on vol Ω. Since $ms < n$, $W^m_s(\Omega : E) \to L_q(\Omega : E)$ by Lemma 41. If $p \le q \le p'$ the desired imbedding follows by interpolation between

$W^m_p(\Omega : E) \to L_p(\Omega : E)$ and $W^m_p(\Omega : E) \to L_{p'}(\Omega : E)$ as in (2.30).

For $mp = n$ and $q \ge p$ the dependence of the imbedding constant on vol Ω may be removed as we show in the following lemma which removes the restriction of boundedness of Ω from Lemma 2.22 and Corollary 2.23.

Lemma 2.24: Let Ω be an arbitrary domain in $\mathbb{R}^n$ having the cone property. If $mp < n$, then $W^m_p(\Omega : E) \to L_q(\Omega : E)$ for $p \le q \le np/(n-mp)$. If $mp = n$, then $W^m_p(\Omega : E) \to L_q(\Omega : E)$ for $p \le q < \infty$. If $p = 1$ and $m = n$, then $W^m_p(\Omega : E) \to C^0(\Omega : E)$. The constants for these imbeddings may depend on m, p, n, q, and the cone C determining the cone property for Ω.

Proof: We tesselate $\mathbb{R}^n$ by cubes of unit side. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is an $n-$tuple of integers, let $H = \{x \in \mathbb{R}^n : \lambda_i \le x_i \le \lambda_i + 1; 1 \le i \le n\}$. Then $\mathbb{R}^n = \bigcup_\lambda H_\lambda$.

As remarked on the first paragraph of the proof of [1,Theorem 4.8] , even an unbounded domain Ω with the cone property can be expressed as a union of finitely many subdomains, say $\Omega = \bigcup_{j=1}^N \Omega_j$, such that $\Omega_j = \bigcup_{x \in A_j} (x + P_j)$, where $A_j \subset \Omega$ and P_j is an parallelepiped with one vertex at the origin. The number N and the dimensions of the parallelepipeds P_j depend on n and the cone C determining the cone property for Ω. For each λ and for $1 \le j \le N$ let

$\Omega_{\lambda,j} = \bigcup_{x \in A_j \cap H_\lambda} (x + P_j)$.

This domain $\Omega_{\lambda,j}$ evidently posesses the following properties:

(i) $\Omega = \bigcup_{\lambda,j} \Omega_{\lambda,j}$;

(ii) $\Omega_{\lambda,j}$ is bounded;

(iii) there exists a finite cone C' depending only on $P_1, ..., P_N$ (and hence only on n and C) such that each $\Omega_{\lambda,j}$ has the cone property determined by C';

(iv) there exists a positive integer R depending on n and C such that any $R+1$ of the domains $\Omega_{\lambda,j}$ have empty intersection;

(v) there exists constants K' and K' depending on n and C such that foe each $\Omega_{\lambda,j}$,

$K' \leq \text{vol}\Omega_{\lambda,j} \leq K'$.

Suppose $mp < n$ and let $u \in W_p^m(\Omega : E)$. If $p \leq q \leq np/(n-mp)$, then by (ii),(iii), and lemma 2.22, we have

$$\|u\|_{W_q^0(\Omega_{\lambda,j}:E)} \leq K \|u\|_{W_p^m(\Omega_{\lambda,j}:E)}, \tag{2.31}$$

where $K = K(m,p,n,q,C)$is independent of λ and j. Hence by (i) and (iv) and since $q \geq p$

$$\|u\|^q_{W_q^0(\Omega:E)} \leq \sum_{\lambda,j} \|u\|^q_{W_q^0(\Omega_{\lambda,j}:E)} \leq K^q \sum_{\lambda,j} \left[\|u\|^p_{W_p^m(\Omega_{\lambda,j}:E)}\right]^{q/p}$$

$$\leq K^q \left[\sum_{\lambda,j} \|u\|^p_{W_p^m(\Omega_{\lambda,j}:E)}\right]^{q/p} \leq K^q R^{q/p} \|u\|^q_{W_p^m(\Omega:E)}$$

Thus $W_p^m(\Omega : E) \to L_p(\Omega : E)$ with the imbedding constant $KR^{1/p}$.

If $mp = n$, (2.31) holds for any q such that $p \leq q < \infty$ by virtue of Corollary 2.23, and the constant K can be chosen independent of λ and j thanks to (v). The rest of the above proof then carries over to this case.

Finally, if $p = 1$ and $m = n$, we have by Lemma 2.18 and a nonsingular linear transformation that $W_1^n(P : E) \to C^0(\overline{P} : E)$ for any parallelepiped $P \subset \Omega$, the imbedding constant depending only on n and the dimensions of P. Hence $W_1^n(\Omega : E) \to C_B^0(\Omega : E)$ by the virtue of the decomposition $\Omega = \bigcup_{\lambda,j} \Omega_{\lambda,j}$.

We have now proved Part 1, Cases A and B of Theorem 2.15 for the case $k = n$. Before completing these cases by considering the trace imbedding ($k < n$), we establish the continuous function space imbeddings, Part 1, Case C, and Part 2.

Lemma 2.25: Let Ω be a domain in $\mathbb{R}^n$ having the cone property. If $mp > n$, then $W_p^m(\Omega : E) \to C_B^0(\Omega : E)$, the imbedding constant depending only on $m, p, n,$ and the cone C determining the cone property for Ω.

Proof: Suppose that we can prove that for any $\phi \in C^\infty(\Omega : E)$,

$$\sup_{x\in\Omega} \|\phi(x)\| \leq K \|\phi\|_{W_p^m(\Omega:E)}, \tag{2.32}$$

where $K = K(m, p, n, C)$. If $u \in W_p^m(\Omega : E)$, then by [1,Theorem 3.16] there exists a sequence $\{\phi_n\}$ in $C^\infty(\Omega : E)$ converging to u in norm in $W_p^m(\Omega : E)$. Since $\{\phi_n\}$ is a Cauchy sequence in $W_p^m(\Omega : E)$, (2.32) implies that $\{\phi_n\}$ converges to a continuous function on Ω. Thus u must coincide a.e. with an element of $C_B^0(\Omega : E)$. It is therefore sufficient to establish (2.32).

First suppose $m = 1$ so that $p > n$. Let $x \in \Omega$ and let $C_x \subset \Omega$ be a finite cone congruent to C and having vertex at x. Let h be the height of C. Let (r, θ) denote spherical polar coordinates in $\mathbb{R}^n$ with origin at x so that C_x is specified by $0 < r < h$, $\theta \in A$. the volume element in this system is denoted by $r^{n-1}\omega(\theta)drd\theta$. We have

$$\phi(x) = \phi(0, \theta) = \phi(r, \theta) - \int_0^r \frac{d}{dt}\phi(t, \theta)dt,$$

from which we conclude , for $0 < r < h$,

$$\|\phi(x)\|_E \leq \|\phi(r, \theta)\|_E + \int_0^h \|grad\phi(t, \theta)\|_E \, dt.$$

Multiplying this inequality by $r^{n-1}\omega(\theta)$ and integrating r over $(0, h)$ and θ over A, we obtain

$$\begin{aligned}(\text{vol } C_x) \|\phi(x)\|_E &\leq \int_{C_x} \|\phi(y)\|_E \, dy + \frac{h^n}{n} \int_{C_x} \frac{\|grad\phi(y)\|_E}{|x-y|^{n-1}} dy \\ &\leq (\text{vol } C_x)^{1/p'} \|\phi\|_{W_p^0(C_x:E)} \\ &\qquad + \frac{h^n}{n} \|grad\phi(y)\|_{W_p^0(C_x:E)} \left| \int_{C_x} |x-y|^{-(n-1)p'} \, dy \right|^{1/p'},\end{aligned}$$

the last inequality following from two applications of Hölder's inequality.

Since $p > n$ we have $(n-1)(1-p') > -1$ and so

$$\int_{C_x} |x-y|^{-(n-1)p'}\,dy = \int_A \omega(\theta)d\theta \int_0^h r^{(n-1)(1-p')}dr < \infty.$$

Hence

$$\|\phi(x)\|_E \leq K\,\|\phi\|_{W_p^1(C_x:E)} \leq K\,\|\phi\|_{W_p^1(\Omega:E)}$$

with $K = K(m,p,n,C_x) = K(m,p,n,C)$. Thus (2.32) is proved for $m = 1$.

If $m > 1$ but $p > n$, we still have

$$\|\phi(x)\|_E \leq K\,\|\phi\|_{W_p^1(C_x:E)} \leq K\,\|\phi\|_{W_p^m(C_x:E)} \leq K\,\|\phi\|_{W_p^1(\Omega:E)}.$$

If $p \leq n < mp$, there exists an integer j satisfying $1 \leq j \leq m-1$ such that $jp \leq n < (j+1)p$. If $jp < n$, set $r = np/(n-jp)$; if $jp = n$, choose $r > \max(n,p)$. In either case we have by the result proved above and by Lemma 2.24 that

$$\|\phi(x)\|_E \leq K_1\,\|\phi\|_{W_r^1(C_x:E)} \leq K_1\,\|\phi\|_{W_r^{m-j}(C_x:E)} \leq K\,\|\phi\|_{W_p^m(C_x:E)} \leq K\,\|\phi\|_{W_p^m(C:E)},$$

the constants depending only on $m, p, n,$ and C. This completes proof.

Corollary 2.26: If $mp > n$, then $W_p^m(\Omega : E) \to L_q(\Omega : E)$ for $p \leq q \leq \infty$. The imbedding constants depend only on m, p, n, q and the cone C.

Proof: We have already established that

$$\|u\|_{W_\infty^0(\Omega:E)} = ess\sup_{x\in\Omega} \|u(x)\|_E \leq K\,\|u\|_{W_p^m(\Omega:E)}$$

for all $u \in W_p^m(\Omega : E)$. If $p \leq q < \infty$, we have

$$\begin{aligned}\|u\|^q_{W_q^0(\Omega:E)} &= \int_\Omega \|u(x)\|_E^p\,\|u(x)\|_E^{q-p}\,dx \\ &\leq K^{q-p}\,\|u\|^{q-p}_{W_p^m(\Omega:E)}\,\|u\|^p_{W_p^0(\Omega:E)} \leq K^{q-p}\,\|u\|^q_{W_p^m(\Omega:E)}.\end{aligned}$$

Lemma 2.27: Let Ω be a domain in $\mathbb{R}^n$ having the strong Lipschitz property, and suppose that $mp > n \geq (m-1)p$. Then $W_p^m(\Omega : E) \to C_\lambda^0(\overline{\Omega} : E)$ for :

(i) $0 < \lambda \leq m - n/p$ if $n > (m-1)p$, or

(ii) $0 < \lambda < 1$ if $n = (m-1)p$, or

(iii) $0 < \lambda \leq 1$ if $p = 1, n = m-1$.

In particular $W_p^m(\Omega : E) \to C^0(\overline{\Omega} : E)$. The imbedding constants depend on $m, p, n,$ and the parameters δ, M specified in the description of the strong local Lipschitz property for Ω.

Proof: Let $u \in W_p^m(\Omega : E)$. The strong local Lipschitz property implies the cone property so by Lemma 2.25 we may assume that u is continuous on Ω and satisfies

$$\sup_{x\in\Omega} \|u(x)\|_E \leq K_1 \|u\|_{W_p^m(\Omega:E)}. \tag{2.33}$$

It is therefore sufficient to establish further that for suitable λ,

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|u(x)-u(y)\|_E}{|x-y|^\lambda} \leq K_2 \|u\|_{W_p^m(\Omega:E)}. \tag{2.34}$$

Since $mp > n \geq (m-1)p$ we have by Lemma 2.24 that $W_p^m(\Omega : E) \to W_r^1(\Omega : E)$ where:

(i) $r = np/(n-mp+p)$ and $1-(n/r) = m-(n/p)$ if $n > (m-1)p$, or

(ii) r is arbitrary, $p < r < \infty$ and $0 < 1-(n/r) < 1$ if $n = (m-1)p$, or

(iii) $r = \infty$, $1-(n/r) = m-(n/p) = 1$ if $p = 1$ and $n = m-1$.

It is therefore sufficient to establish (2.38) for $m = 1$; that is , we wish to prove that if $n < p \leq \infty$ and $0 < \lambda \leq 1-(n/p)$, then

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|u(x)-u(y)\|_E}{|x-y|^\lambda} \leq K_3 \|u\|_{W_p^1(\Omega:E)}. \tag{2.35}$$

Suppose, for the moment, that Ω is a cube, which we may also assume without loss of generality to have unit edge. For $0 < t < 1$, Ω, will denote a cube of edge t with faces parallel to those of Ω and such that $\overline{\Omega}_t \subset \Omega$. Let $u \in C^\infty(\Omega : E)$.

Let $x, y \in \Omega$, $|x-y| = \sigma < 1$. Then there exists a fixed cube Ω_σ with $x, y \in \overline{\Omega}_\sigma \subset \Omega$. If $z \in \Omega_\sigma$, then

$$u(x) = u(z) - \int_0^1 \frac{d}{dt} u(x+t(z-x))dt,$$

so that

$$\|u(x)-u(z)\|_E \leq \sqrt{n}\sigma \int_0^1 \|\text{grad } u(x+t(z-x))\|_E \, dt.$$

Hence

$$\left\| u(x) - \frac{1}{\sigma^n} \int_{\Omega_\sigma} u(z)dz \right\|_E \leq \left\| \frac{1}{\sigma^n} \int_{\Omega_\sigma} (\mathrm{u}(x)-u(z))dz \right\|_E$$

$$\leq \frac{\sqrt{n}}{\sigma^{n-1}} \int\limits_{\Omega_\sigma} dz \int\limits_0^1 \|\text{grad } u(x+t(z-x))\|_E \, dt$$

$$= \frac{\sqrt{n}}{\sigma^{n-1}} \int\limits_0^1 t^{-n} dt \int\limits_{\Omega_\sigma} dz \, \|\text{grad } u(z)\|_E \, dz$$

$$\leq \frac{\sqrt{n}}{\sigma^{n-1}} \|\text{grad } u\|_{W_p^0(\Omega:E)} \int\limits_0^1 (\text{vol}\Omega_\sigma)^{1/p'} t^{-n} dt \tag{2.36}$$

$$\leq K_4 \sigma^{1-(n/p)} \|\text{grad } u\|_{W_p^0(\Omega:E)},$$

where $K_4 = K_4(n,p) = \sqrt{n} \int\limits_0^1 t^{-n/p} dt < \infty$. A similar inequality holds with y in place of x and so

$$\|u(x) - u(y)\|_E \leq 2K_4 |x-y|^{1-(n/p)} \|\text{grad } u\|_{W_p^0(\Omega:E)}.$$

It follows for $0 < \lambda \leq 1 - (n/p)$ that (2.35) holds for Ω a cube, and so, via a nonsingular linear transformation, for Ω a parallelepiped.

Now suppose that Ω has the strong local Lipschitz property. Let δ, M, Ω_δ, U_j, and ψ_j be specified in [1,section 4.5]. There exists a parallelepiped P of diameter δ whose dimensions depend only on δ and M such that to each j there corresponds a parallelepiped P_j congruent to P and havingone vertex at the origin, such that for every $x \in \psi_j \cap \Omega$ we have $x + P_j \subset \Omega$. Furthermore there exist constants δ_0 and δ_1 depending only on δ and P, with $\delta_0 \leq \delta$,such that if $x, y \in \psi_j \cap \Omega$ and $|x-y| < \delta_0$, then there exists $z \in (x+P_j) \cap (y+P_j)$ with $|x-z| + |y-z| \leq \delta_1 |x-y|$. It follows from application of (2.35) to $x + P_j$ and $y + P_j$ that if $u \in C^\infty(\Omega : E)$, then

$$\|u(x) - u(y)\|_E \leq \|u(x) - u(z)\|_E + \|u(y) - u(z)\|_E$$

$$\leq K_5 |x-z|^\lambda \|u\|_{W_p^1(\Omega:E)} + K_5 |y-z|^\lambda \|u\|_{W_p^1(\Omega:E)}$$

$$\leq 2^{1-\lambda} K_5 \delta_1^\lambda |x-y|^\lambda \|u\|_{W_p^1(\Omega:E)}. \tag{2.37}$$

Now let $x, y \in \Omega$ be arbitrary.If

$|x-y| < \delta_0 \leq \delta$ and $x, y \in \Omega_\delta$, then $x, y \in \psi_j$ for some j and estimate (2.37) holds. If $|x-y| < \delta_0$, $x \in \Omega_\delta$, $y \in \Omega \sim \Omega_\delta$, then $x \in \psi_j$ for some j and (2.37) follows by application of (2.35) to $x + P_j$ and $y + P_j$ again. If $|x-y| < \delta_0$ and $x, y \in \Omega \sim \Omega_\delta$ then

(2.37) follows from application of (2.35) to $x+P'$, $y+P'$, where P' is any parallelepiped congruent to P and having one vertex at origin. Finally, if $|x-y| \geq \delta_0$, then we have

$$\|u(x)-u(y)\|_E \leq \|u(x)\|_E + \|u(y)\|_E \leq K_6 \|u\|_{W_p^1(\Omega:E)}$$
$$\leq K_6 \delta_0^{-\lambda} |x-y|^{\lambda} \|u\|_{W_p^1(\Omega:E)} .$$

This completes the proof of (2.35) for $u \in C^\infty(\Omega : E)$, and so by [1,Theorem 3.16], for all continuous u.

Lemma 2.28: Let Q be a cube of edge length k, having edges parallel to the coordinate axes in $\mathbb{R}^n$. If $p > 1$, $q \geq 1$ and $mp - p < n < mp$, then there exists a constant $K = K(p, q, l, n, k)$ such that for every $u \in W_p^l(Q : E)$ we have (a.e. in Q)

$$\|u(x)\|_E \leq K \|u\|_{W_q^0(Q:E)}^s \|u\|_{W_p^m(Q:E)}^{1-s}, \qquad \textbf{(2.38)}$$

where $s = (mp-n)q / [np + (mp-n)q]$.

Proof: It is sufficient to establish (2.38) for $u \in C^\infty(\overline{Q})$. Since each point of $\overline{Q}$ is a corner point of a cube contained in $\overline{Q}$, having edges parallel to those of Q, and having edge length $k/2$, we may assume without loss of generality that x is itself a corner point of Q, say $Q = \{y \in \mathbb{R}^n : x_i < y_i < x_i + k : 1 \leq i \leq n\}$.

By Lemma 2.27 we have for $y \in Q$,

$$\|u(x\|_E - \|u(y)\|_E \leq \| \mathrm{u}(x) - (y)\|_E \leq K_1 \|x-y\|_E^{l-(n/p)} \|u\|_{W_p^l(Q:E)}. \qquad \textbf{(2.39)}$$

Let $U = \|u\|_{W_p^l(Q:E)}$, which we may assume to be positive; let $\rho = \|x-y\|_E$ and $\zeta = [\|u(x)\| / K_1 U]^{p/(lp-n)}$. Suppose for the moment that $\zeta \leq k$. We have for $\rho \leq \zeta$,

$$\|u(y)\|_E \geq \|u(x)\|_E - K_1 U \rho^{m-(n/p)} \geq 0.$$

Raising the above inequality to the power q and integrating y over Q, we obtain

$$\int_\Omega \|u(y)\|_E^q \, dy \geq K_2 \int_0^\zeta (\|u(x)\|_E - K_1 U \rho^{l-(n/p)})^q \rho^{n-1} d\rho$$
$$= K_2 \zeta^n \|u(x)\|_E^q \int_0^1 (1 - \sigma^{m-(n/p)}) \sigma^{n-1} d\sigma$$
$$= K_3 \|u(x)\|_E^{q+(np/(lp-n))} U^{-np/(lp-n)},$$

from which (2.38) follows at once.

If, on the other hand, $\zeta > k$, then from (2.39) we obtain

$$\|u(y)\|_E \geq \|u(x)\|_E - K_1 U \rho^{m-(n/p)} \geq \|u(x)\|_E - \|u(x)\|_E (\rho/k)^{l-(n/p)}$$

$$\geq 0 \qquad \text{if } \rho \leq m.$$

If $t > 0$, then

$$\int\limits_Q \|u(y)\|_E^t \, dy \geq K_2 \int\limits_0^k \|u(x)\|_E^t (1 - (\rho/k)^{l-(n/p)})^t \rho^{n-1} d\rho = K_4 \|u(x)\|_E^t.$$

Set $t = [(lp - n) q + np] / lp$. Then

$$\|u(x)\|_E^{[(lp-n)q+np]/lp} \leq (1/K_4) \int\limits_\Omega [\|u(y)\|_E^q]^{(lp-n)/lp} [\|u(y)\|_E^p]^{n/lp} \, dy$$

$$\leq (1/K_4) \|u\|_{W_q^0(\Omega:E)}^{q(lp-n)/lp} \|u\|_{W_q^0(\Omega:E)}^{n/l},$$

by an application of Hölder's inequality. Since $\|u\|_{W_p^0(\Omega:E)} \leq \|u\|_{W_p^l(\Omega:E)}$, (2.38) follows at once.

We remark that the above lemma also holds for the case $p = 1$, $l = n$. In this case we have from Lemma 2.24 that $W_1^n(\Omega : E) \to L_\infty(\Omega : E)$ so that $\|u(x)\|_E \leq K \|u\|_{W_1^n(\Omega:E)}$ a.e. in Q, which is (2.38) in this case.

Lemma 2.29: Let Ω be a domain in $\mathbb{R}^n$ having the cone property, and let Ω^k denote the intersection of Ω with some $k-$dimensional plane, where $1 \leq k \leq n$ ($\Omega^n \equiv \Omega$). If $n \geq lp$ and $n - lp < k \leq n$, then

$$W_p^l(\Omega : E) \to L_q(\Omega^k : E) \qquad \textbf{(2.40)}$$

for $p \leq q \leq kp/(n - lp)$ if $n > lp$, or $p \leq q < \infty$ if $n = lp$. If $p = 1$, $n > l$ and $n - l \leq k \leq n$, then (2.40) holds for $1 \leq q \leq k/(n - l)$.

The imbedding constants depend only on p, k, l , n, q, and the cone C determining the cone property for Ω.

Proof: It is sufficient to establish the above conclusions for Ω bounded, $n > lp$, and $q = kp/(n - lp)$, as extension to the other cases can be carried out in the same manner as was described for the case $k = n$ in Corollary 2.23 and Lemma 2.24. We may also assume, as in Lemma 2.20, that Ω is a union of coordinate cubes of edge 2 units.

Let $\mathbb{R}_0^k$ be a k-dimensional coordinate subspace of $\mathbb{R}^n$ on which Ω^k has a one-to-one projection Ω_0^k. Suppose, for the moment, that $p > 1$. Let v be the largest integer less than lp. Then $lp - p < v < lp$ and since $n - lp < k$ we have $n - v \leq k$. (Note that if $p = 1$, the same conclusion holds with $k = n - l$, $v = l$.) Let $\mu = \begin{pmatrix} k \\ n-v \end{pmatrix}$ and let E_i $(1 \leq i \leq \mu)$ denote the various coordinate subspaces of $\mathbb{R}_0^k$ having dimension $n - v$. Let Ω_i denote the projection of Ω_0^k (and hence of Ω^k) onto E_i. Also, for each $x \in \Omega_i$ let $\Omega_{i,x}$ denote the intersection of Ω with the v-dimensional plane through x perpendicular to E_i. Then $\Omega_{i,x}$ contains a v-dimensional coordinate cube of unit edge with one vertex at x. By Lemma 2.28, with $q = q_0 = np/(n - lp)$, we have for $u \in C^\infty(\Omega : E)$

$$\sup_{y\in\Omega_{i,x}} \|u(y)\|_E^{(n-v)p/(n-lp)} \leq K_1 \|u\|_{W_{q_0}^0(\Omega_{i,x}:E)}^{(lp-v)q_0/lp} \|u\|_{W_p^l(\Omega_{i,x}:E)}^{v/l}. \tag{2.41}$$

Let dx^i and dx_*^i denote the volume elements in E_i and the orthogonal complement of E_i, respectively. Integration of (2.41) over Ω_i leads to

$$\begin{aligned}
&\int_{\Omega_i} \sup_{y\in\Omega_{i,x}} \|u(y)\|_E^{(n-v)p/(n-lp)}\, dx^i \\
&\qquad \leq K_1 \int_{\Omega_i} \left[\int_{\Omega_{i,x}} \|u(x)\|_E^{q_0}\, dx_*^i\right]^{(lp-v)/lp} \times \left[\int_{\Omega_{i,x}} \sum_{|\alpha|\leq l} \|D^\alpha u(x)\|_E^p\, dx_*^i\right]^{v/lp} dx_i \\
&\qquad \leq K_1 \left[\int_{\Omega} \|u(x)\|_E^{q_0}\, dx\right]^{(lp-v)/mp} \left[\int_{\Omega} \sum_{|\alpha|\leq l} \|D^\alpha u(x)\|_E^p\, dx\right]^{v/lp} \\
&\qquad = K_1 \|u\|_{W_{q_0}^0(\Omega:E)}^{q_0(lp-v)/lp} \|u\|_{W_p^l(\Omega:E)}^{v/l}
\end{aligned} \tag{2.42}$$

by Hölder's inequality.

Finally, we apply Lemma 2.19 to the subspaces E_i of $\mathbb{R}_0^k$. Note that the constant λ of that lemma is here equal to $\begin{pmatrix} k-1 \\ n-v-1 \end{pmatrix}$. Letting $dx^{(k)}$ denote the volume element in $\mathbb{R}_0^k$ and setting $q = kp/(n - lp)$, we obtain

$$\begin{aligned}
\|u\|_{W_q^0(\Omega^k:E)}^q &\leq K_2 \int_{\Omega_0^k} \prod_{i=1}^{\mu} \sup_{y\in\Omega_{i,x}} \|u(y)\|_E^{q/\mu}\, dx^{(k)} \\
&\leq K_2 \prod_{i=1}^{\mu} \left[\int_{\Omega_i} \sup_{y\in\Omega_{i,x}} \|u(y)\|_E^{q\lambda/\mu}\, dx^i\right]^{1/\lambda}.
\end{aligned} \tag{2.43}$$

Since $q\lambda/\mu = (n-v)p/(n-lp)$, it follows from (2.42) and (2.43) and from Lemma 2.24 that

$$\begin{aligned}
\|u\|_{W_q^0(\Omega^k:E)} &\leq K_3 \prod_{i=1}^{\mu} \|u\|_{W_{q_0}^0(\Omega:E)}^{q_0(lp-v)/lp\lambda_q} \|u\|_{W_p^l(\Omega:E)}^{v/l\lambda_q} \\
&\leq K_4 \left[\|u\|_{W_p^m(\Omega:E)}^{q_0(lp-v)/lp} \|u\|_{W_p^l(\Omega:E)}^{v/l} \right]^{\mu/\lambda q} = K_4 \|u\|_{W_p^m(\Omega:E)}
\end{aligned}$$

This establishes the desired embedding.

We have now completed the proof of Theorem 2.15.

3. VECTOR-VALUED ANISOTROPIC SOBOLEV SPACES

Let E_0 and E be two Banach spaces and $l = (l_1, l_2, \ldots, l_n)$, where $l_n \in (0, \infty)$, i=1,...,n. Suppose E_0 is continuously and densely imbedded into E. We introduce an E-valued anisotropic Sobolev space $W_p^l(\Omega; E_0, E)$ that consist of functions $u \in L_p(\Omega; E_0)$ such that have generalized derivatives $D_k^{l_k} u \in L_p(\Omega; E)$ and with the norm

$$\|u\|_{W_p^l(\Omega;E_0,E)} = \|u\|_{L_p(\Omega;E_0)} + \sum_{k=1}^{n} \left\| D_k^{l_k} u \right\|_{L_p(\Omega;E)} < \infty, 1 \leq p < \infty. [5]$$

3.1 Continuous Imbedding on Vector-Valued Anisotropic Sobolev Spaces

Let R be the set of real numbers, C be the set of complex numbers. Let E_1 and E_2 be two Banach spaces and $L(E_1, E_2)$ denotes a space of bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ we denote $L(E, E)$ by $L(E)$; let I denote the identity operator in E. We will sometimes use $A + \xi$ or A_ξ instead of $A + \xi I$ for a scalar ξ and $(A + \xi I)^{-1}$ will denote an inverse of the operator $A + \xi I$ or the resolvent of the operator A.

Let

$$S_\varphi = \{\xi : \xi \in \mathbf{C}, |\arg \xi| \leq \varphi\} \cup \{0\}, 0 \leq \varphi < \pi.$$

Let $S = S(R^n; E)$ denote the E-valued Schwartz space of rapidly decreasing smooth functions. For $E = C$ this space will be denoted by $S = S(R^n)$. $S^{'}(E) = S^{'}(R^n; E)$ denotes the space of linear continuous mapping from S into E and is called E- valued Schwartz distributions. For any $r = (r_1, r_2, ... r_n)$, $r_i \in [0, \infty)$ the function $(i\xi)^r$ will be defined such that

$$(i\xi)^r = \left\{ \begin{array}{cc} (i\xi_1)^{r_1} ... (i\xi_n)^{r_n}, & \xi_1\xi_2, ..., \xi_n \neq 0 \\ 0, & \xi_1\xi_2, ..., \xi_n = 0 \end{array} \right\}$$

where

$(i\xi_k)^{r_k} = \exp\left[r_k\left(\ln|\xi_k| + i\pi\text{sgn}\,\xi_k/2\right)\right],\, k = 1, 2, ..., n.$

The Liouville derivatives $D^r u$ of an E- valued function u are defined the same as a in scalar functions [6] or E-valued functions case (see e.g. [7]).

Let E_0 and E be two Banach spaces. Suppose E_0 is continuously and densely embedded into E and let $l = (l_1, l_2, ..., l_n)$, where $l_i \in (0, \infty)$, $i = 1, 2, ..., n$. We introduce an E-valued space

$$H_p^l(R^n; E) = \{\, u,\ u \in S'(R^n; E),\, F^{-1}\left(1 + |x_k|^{l_k}\right) Fu \in L_p\left(R^n; E\right),$$

$$\|u\|_{H_p^l(R^n;E)} = \|u\|_{L_p(R^n;E)} + \sum_{k=1}^{n} \left\| F^{-1}\left(1 + |x_k|^{l_k}\right) Fu \right\|_{L_p(R^n;E)} < \infty,\ \},$$

$$H_p^l\left(R^n; E_0, E\right) = H_p^l(R^n; E) \cap L_p\left(R^n; E_0\right),$$

$$\|u\|_{H_p^l(R^n;E_0,E)} = \|u\|_{L_p(R^n;E_0)} + \|u\|_{H_p^l(R^n;E)} < \infty, p \in (1, \infty)$$

Let Ω be a domain in R^n; $H_p^l\left(\Omega; E_0, E\right)$ denotes the space of restrictions to Ω of all functions in $H_p^l\left(R^n; E_0, E\right)$ with the norm given by

$$\|u\|_{H_p^l(\Omega;E_0,E)} = \inf_{g \in H_p^l(R^n;E_0,E), g|_\Omega = u} \|g\|_{H_p^l(R^n;E_0,E).}$$

Suppose $S\left(R^n; E_1\right)$ is dense in $L_p\left(R^n; E_1\right)$. A function $\Psi \in C\left(R^n; L\left(E_1, E_2\right)\right)$ is called a multiplier from $L_p\left(R^n; E_1\right)$ to $L_q\left(R^n; E_2\right)$ if the map $u \rightarrow Du = F^{-1}\Psi\left(\xi\right)Fu$, $u \in S\left(R^n; E_1\right)$ are well defined and extends to a bounded linear operator

$$D:\ L_p\left(R^n; E_1\right) \rightarrow\ L_q\left(R^n; E_2\right).$$

We denote a set of all multipliers fom $L_p\left(R^n; E_1\right)$ to $L_q\left(R^n; E_2\right)$ by $M_p^q\left(E_1, E_2\right)$. For $E_1 = E_2 = E$ we denote $M_p^q\left(E_1, E_2\right)$ by $M_p^q\left(E\right)$. Let Q be domain in R^σ and $h = (h_1, h_2, ..., h_\sigma) \in Q$. Let

$$\Phi\left(h\right) = \left\{\Psi_h \in M_p^q\left(E_1, E_2\right),\ h \in Q\right\}$$

be a collection of multipliers in $M_p^q\left(E_1, E_2\right)$ depending on the parameter h. We say that Ψ_h is a uniformly bounded multiplier with respect to h if there exists a constant $C > 0$, independent on $h \in Q$ such that

$$\|F^{-1}\Psi_h Fu\|_{L_q(R^n,E_2)} \leq C\,\|u\|_{L_p(R^n,E_1)}$$

for all $h \in Q$ and $u \in S\left(R^n; E_1\right)$.

The exposition of the theory of L_p-multipliers of the Fourier transformation, and some related references, can be found in 6[3, §2.2.1-§2.2.4]. In the weighted L_p spaces Fourier multipliers have been investigated in several studies like $[8-9]$. On the other hand, in vector-valued function spaces, integral transforms and Fourier multipliers have been studied by $[10-18]$. By virtue of $[19]$, Mikhlin conditions $[20]$ are not sufficient for operator-valued multiplier theorems in Banach-valued L_p-spaces.

A set $K \subset B(E_1, E_2)$ is called R-bounded (see e.g. $[21-22]$, $[18]$) if there is a constant C (depending only on E_1, E_2) such that

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy$$

for all $T_1, T_2, ..., T_m \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in N$, where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$ and N denotes the set of natural numbers. The smallest C for which the above estimate holds is called an R-bound of the collection K and denoted by $R(K)$.

A set $K(h) \subset B(E_1, E_2)$ depending on the parameter $h \in Q$ is called uniformly R-bounded with respect to h if there is a positive constant C such that for all $T_1(h), T_2(h), ..., T_m(h) \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in N$

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y) T_j(h) u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,$$

where the constant C is independent on h.

Let

$$V_n = \left\{\xi : \xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n,\ \xi_j \neq 0\right\},$$

$$U_n = \{\beta = (\beta_1, \beta_2, ..., \beta_n),\ |\beta| \leq n\},\ \xi^\beta = \xi_1^{\beta_1} \xi_2^{\beta_2} ... \xi_n^{\beta_n}.$$

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be nonnegative integer numbers and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n} = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}.$$

Definition 3.1: A Banach space E is said to be a space satisfying a multiplier condition with respect to $p,\ q \in (1, \infty)$, (with respect to p and q if $q = p$), if for any $\Psi \in C^{(n)}(R^n; B(E))$, the R-boundedness of the set

$\left\{ \xi^{\beta+\frac{1}{p}-\frac{1}{q}} D^{\beta}\Psi\left(\xi\right) : \xi \in R^n \backslash \left\{0\right\}, \beta \in U_n \right\}$ implies that Ψ is a Fourier multiplier, i.e. $\Psi \in M_p^q\left(E\right)$ for any $p, q \in \left(1, \infty\right).$

A Banach space E has a property (α), (see e.g. $[14]$) if there exists a constant α such that

$$\left\| \sum_{i,j=1}^{N} \alpha_{ij}\varepsilon_i\varepsilon'_j x_{ij} \right\|_{L_2(\Omega\times\Omega';E)} \leq \alpha \left\| \sum_{i,j=1}^{N} \varepsilon_i\varepsilon'_j x_{ij} \right\|_{L_2(\Omega\times\Omega';E)}$$

for all $N \in N$, $x_{i,j} \in E$, $\alpha_{ij} \in \{0,1\}$, $i,j = 1,2,...,N$, and all choices of independent, symmetric, $\{-1,1\}$- valued random variables $\varepsilon_1, \varepsilon_2, ..., \varepsilon_N$, $\varepsilon'_1, \varepsilon'_2, ..., \varepsilon'_N$ on probability spaces Ω, Ω'. For example the spaces $L_p\left(\Omega\right)$, $1 \leq p < \infty$ has the property (α).

Remark 3.2: If E is an UMD space with property (α) then these spaces are satisfy the multiplier condition with respect to $p \in (1,\infty)$ (see. $[22]$).

It is well known (see e.g. $[23]$) that any Hilbert space satisfies the multiplier condition. There are, however, Banach spaces which are not Hilbert spaces but satisfy the multiplier condition, for example UMD spaces ($[22]$, $[14]$, $[18]$).

Definition 3.3: A positive operator A is said to be an $R-$positive in a Banach space E if there exists $\varphi \in [0,\pi\ )$ such that the set

$L_A = \left\{ \xi \left(A + \xi I\right)^{-1} : \xi \in S_\varphi \right\}$

is R-bounded.

Note that in Hilbert spaces every norm bounded set is R-bounded. Therefore, in Hilbert spaces all positive operators are R-positive. If A is a generator of a contraction semigroup on L_q, $1 \leq q \leq \infty$ $[24]$, A has bounded imaginary powers with $\|(-A^{it})\|_{B(E)} \leq Ce^{\nu|t|}$, $\nu < \frac{\pi}{2}$ [25], [22] or if A is generator of a semigroup with Gaussian bound [26] in E $\in$UMD then those operators are R-positive.

Let

$r = (r_1, r_2, ...r_n)$, $r + a = (r_1 + a, r_2 + a, ..., r_n + a)$,

$l = (l_1, l_2, ..., l_n)$, $|(r + a) : l| = \sum\limits_{k=1}^{n} \frac{r_k + a}{l_k}$, $\xi^r = \xi_1^{r_1}\xi_2^{r_2}...\xi_n^{r_n}.$

Let $\bar{\Omega}$ denote a closure of the region Ω.

In this section we prove that the generalized differential operator D^r gives a continuous embedding of Sobolev-Lions spaces. Multiplier theorems in operator-valued L_p spaces, played an important role in the theory of embedding of function spaces and differential operator equations.

From [27] we obtain:

Lemma 3.4: Let A be a positive number and $r = (r_1, r_2, ..., r_n)$, where $r_k \in \{0, b\}$, $b > 0$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $l = (l_1, l_2, ..., l_n)$, $\alpha_k \in [\,0, \infty\,)$, $l_k \in (0, \infty)$ such that $\varkappa = |(\alpha + r) : l| \leq 1$, and $0 < h \leq h_0 < \infty$. Then for $0 \leq \mu \leq 1 - \varkappa$ an operator-function

$$\Psi = |\xi_1|^{\alpha_1 + r_1} ... |\xi_n|^{\alpha_n + r_n} A^{1-\varkappa-\mu} h^{-\mu} \left[A + \sum_{k=1}^{n} \left(1 + |\xi_k|^{l_k}\right) + h^{-1}\right]^{-1}$$

is bounded operator in E uniformly with respect to ξ and h i.e there is a constant C_μ such that

$$\|\Psi(\xi)\|_{L(E)} \leq C_\mu. \qquad \textbf{(3.1)}$$

Theorem 3.5: Suppose the following conditions hold:

(1) E is a Banach space satisfying the multiplier condition with respect to p, q, $1 < p \leq q < \infty$ and A is an R-positive operator in E;

(2) $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $l = (l_1, l_2, ..., l_n)$, where $\alpha_k \in [\,0, \infty\,)$, $l_k \in (0, \infty)$ are such that $\varkappa = \left|\left(\alpha + \frac{1}{p} - \frac{1}{q}\right) : l\right| \leq 1$, and let $0 \leq \mu \leq 1 - \varkappa$,and $0 < h \leq h_0 < \infty$.

Then an embedding

$$D^\alpha H_p^l(R^n, E) \subset L_q(R^n, E(A^{1-\varkappa-\mu}))$$

is continuous and there exists a constant $C_\mu > 0$, depending only on μ, such that

$$\|D^\alpha u\|_{L_q(R^n,E)} \leq C_\mu \left[h^\mu \|u\|_{H_p^l(R^n,E)} + h^{-(1-\mu)} \|u\|_{L_p(R^n,E)}\right] \qquad \textbf{(3.2)}$$

for all $u \in H_p^l(R^n, E)$.

It is possible to state Theorem 3.5 in a more general setting. For this, we use the concept of extension operator.

Condition 3.6: Let E be a Banach spaces satisfying the multiplier condition with respect to p, q and A be a $\varphi-$ positive operator in E. Let a region $\Omega \subset R^n$ such that there exists a bounded linear extension operator from $H_p^l(\Omega, E)$ to $H_p^l(R^n, E)$, for $1 < p < \infty$.

Remark 3.7: If $\Omega \subset R^n$ is a region satisfying the strong $l-$horn condition (see [28], §8), $E = R$, then there exists a bounded linear extension operator from $H_p^l(\Omega) = H_p^l(\Omega; R)$ to $H_p^l(R^n) = H_p^l(R^n; R)$.

Theorem 3.8: Suppose all conditions of Theorem 3.5 and Condition 3.6 are hold. Then an embedding

$$D^\alpha H_p^l(\Omega, E) \subset L_q(\Omega, E)$$

is continuous and there exists a constant C_μ depending only on μ such that

$$\|D^\alpha u\|_{L_q(\Omega,E)} \leq C_\mu \left[h^\mu \|u\|_{H_p^l(\Omega,E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega,E)}\right] \quad \textbf{(3.3)}$$

for all $u \in H_p^l(\Omega; E(A), E)$ and $0 < h \leq h_0 < \infty$.

Result 3.9: Let all conditions of Theorem 3.8 hold. Then we have the multiplicative estimate

$$\|D^\alpha u\|_{L_q(\Omega,E)} \leq C_\mu \|u\|^{1-\mu}_{H_p^l(\Omega,E)} \|u\|^{\mu}_{L_p(\Omega,E)}$$

for all $u \in H_p^l(\Omega; E(A), E)$. Indeed setting $h = \|u\|_{L_p(\Omega;E)} \cdot \|u\|^{-1}_{H_p^l(\Omega;E(A),E)}$ in estimate (3.3) we obtain the above estimate.

Theorem 3.10: Suppose all conditions of Theorem 3.8 are hold. Then for $0 < \mu < 1 - \varkappa$ an embedding

$$D^\alpha H_p^l(\Omega; E(A), E) \subset L_q\left(\Omega; (E(A), E)_{\varkappa+\mu,p}\right)$$

is continuous and there exists a constant C_μ depending only on μ such that

$$\|D^\alpha u\|_{L_q\left(\Omega,(E(A),E)_{\varkappa+\mu,p}\right)} \leq C_\mu \left[h^\mu \|u\|_{H_p^l(\Omega;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega;E)}\right] \quad \textbf{(3.4)}$$

for all $u \in H_p^l(\Omega; E(A), E)$ and $0 < h \leq h_0 < \infty$.

3.2 Application of Imbedding Theorem

Let $s \in R$, $s > 0$. Define the following space,when $E = \mathbb{C}$, i.e,

$l_\sigma^s = \{u; \ \ u = \{u_i\}, i = 1, 2, ..., \infty, u_i \in \mathbf{C}\}$

with the norm

$$\|u\|_{l_\sigma^s} = \left(\sum_{i=1}^{\infty} 2^{ips} |u_i|^\sigma\right)^{1/\sigma} < \infty.$$

Note that, $l_\sigma^0 = l_\sigma$. Let A be an infinite matrix defined in the space l_σ such that $D(A) = l_\sigma^s, 1 < \sigma < \infty$, $A = [\delta_{ij} 2^{si}]$, where $\delta_{ij} = 0$, when $i \neq j$, $\delta_{ij} = 1$, when $i = j$, $i, j = 1, 2, ..., \infty$. It is clear to see that, this operator A is $R-$positive in l_σ. Then by Theorem 3.10 we obtain that for $0 \leq \mu \leq 1 - \varkappa$ an embedding

$$D^\alpha H_p^l(\Omega; l_\sigma^s, l_\sigma) \subset L_q\left(\Omega; l_\sigma^{s(1-\varkappa-\mu)}\right)$$

is continuous and also the estimate of type (3.3) is hold.

It should be not that the above embedding haven't obtained with classical methods so far.

Consider the following differential-operator equation

$$(L + \lambda) u = \sum_{|\alpha:l|=1} a_\alpha(x) D^\alpha u + A_\lambda(x) u = f \qquad \textbf{(3.5)}$$

in $L_p(R^n; E)$, where $A(x)$ and a_k are complex-valued functions, $A_\lambda = A(x) + \lambda$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $l = (l_1, l_2, ..., l_n)$, l_i are positive and α_i are nonnegative integers, $\lambda \in S(\varphi_0)$. The maximal regularity for elliptic BVP were studied e.g. in $[29 - 30]$ and for DOE in Banach spaces were investigated e.g. in [31], [26], [22], [32], [27], $[33 - 34]$, $[18]$. At first, we consider the following DOE with constant coefficients

$$(L_0 + \lambda) u = \sum_{|\alpha:l|=1} b_\alpha D^\alpha u + A_\lambda u = f, \qquad \textbf{(3.6)}$$

where A_λ and b_α are complex numbers.

Theorem 3.11: Suppose the following condition hold :

E is a Banach space satisfying the multiplier condition with respect to p;

$B(\xi) = -\sum_{|\alpha:l|=1} b_\alpha (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} ... (i\xi_n)^{\alpha_n} \in S(\varphi_1)$,

$|B(\xi)| \geq C \sum_{k=1}^{n} |\xi_k|^{l_k}, \xi \in R^n, \varphi_0 + \varphi_1 \leq \varphi, 1 < p < \infty.$

Then for all $f \in L_p(R^n; E)$, $\lambda \in S(\varphi_0)$ and for sufficiently large $|\lambda|$, the equation (3.6) has a unique solution $u(x)$ that belongs to space $W_p^l(R^n, E)$ and the coercive uniform estimate

$$\sum_{|\alpha:l| \leq 1} |\lambda|^{1-|\alpha:l|} \|D^\alpha u\|_{L_p(R^n;E)} + \|Au\|_{L_p(R^n;E)} \leq C \|f\|_{L_p(R^n;E)} \quad \textbf{(3.7)}$$

holds with respect to λ.

Theorem 3.12: Suppose the condition of Theorem 3.11 holds and let:

(1) $A(x) A^{-1}(x_0) \in C_b(R : E)$, $x_0 \in (-\infty, \infty)$, $a_\alpha \in C_b(R)$, $\varphi \in [0, \pi)$;

(2) $A_\alpha(x) A^{-(1-|\alpha:l|-\mu)} \in L_\infty(R^n : E)$, $0 < \mu < 1 - |\alpha : l|$, $1 < p < \infty$;

(3) $B(x, \xi) \in S(\psi)$, $|B(x, \xi)| \geq C \sum_{k=1}^{n} |\xi_k|^{l_k}$, $\varphi_0 + \psi \leq \varphi$, $x, \xi \in R^n$.

Then for all $f \in L_p(R^n; E)$, $\lambda \in S(\varphi_0)$ and for sufficiently large $|\lambda|$ the equation (3.5) has a unique solution $u(x)$ belonging to $W_p^l(R^n, E)$ and the coercive uniform estimate

$$\sum_{|\alpha:l| \leq 1} |\lambda|^{1-|\alpha:l|} \|D^\alpha u\|_{L_p(R^n;E)} \leq C \|f\|_{L_p(R^n;E)} \quad \textbf{(3.8)}$$

holds with respect to λ.

Proof: Consider in $L_p(R^n; E)$ differential operators L_0 and L generated by problems (3.6) and (3.5), respectively, that is

$$D(L_0) = D(L) = W_p^l(R^n, E), \quad L_0 u = \sum_{|\alpha:l|=1} a_\alpha(x) D^\alpha u + Au,$$

$$Lu = L_0 u + L_1 u, \; L_1 u = \sum_{|\alpha:l|<1} A_\alpha(x) D^\alpha u.$$

Let $\varphi_j \in C_0^\infty(R^n)$, $j = 1, 2, ...$ a partition of unity such that, $0 \leq \varphi_j \leq 1$ and supp$\varphi_j \subset G_j$, $\sum_j \varphi_j(x) = 1$. Then for all $u \in W_p^l(R^n, E)$ we have $u(x) = \sum_j u_j(x)$, where $u_j(x) = u(x) \varphi_j(x)$. From the equality (3.5) for $u \in W_p^l(R^n, E)$ we have

$$(L + \lambda) u_j = \sum_{|\alpha:l|=1} a_\alpha(x) D^\alpha u_j + A_\lambda(x) u_j(x) = f_j(x), \quad \textbf{(3.9)}$$

where

$$f_j = f\varphi_j - \sum_{|\alpha:l|<1} b_{\alpha j}(x) D^\alpha u - \sum_{|\alpha:l|<1} A_\alpha(x) D^\alpha u_j \quad \textbf{(3.10)}$$

and $b_{\alpha j}(x)$ are continuous and uniformly bounded functions contain derivatives of φ_j. Choose a large ball $B_{r_0}(0)$ such that $|a_\alpha(x) - a_\alpha(\infty)| \leq \delta$ for all $|x| \geq r_0$ and $G_0 = R^n \setminus \bar{B}_{r_0}(0)$. Cover $\bar{B}_{r_0}(0)$ by finitely many balls $G_j = B_{r_j}(x_{0j})$ such that

$|a_\alpha(x) - a_\alpha(x_{0j})| \leq \delta$ for all $|x - x_{0j}| \leq r_j$, $j = 1, 2, .., N$. Define coefficients of local operators L_j as in [16, Theorem 5.7] i.e.

$$a_\alpha^0(x) = \left\{ \begin{array}{cc} a_\alpha(x) & : x \notin \bar{B}_{r_0}(0) \\ a_\alpha\left(r_0^2 \frac{x}{|x|^2}\right) & : x \in \bar{B}_{r_0}(0) \end{array} \right\},$$

$$a_\alpha^j(x) = \left\{ \begin{array}{cc} a_\alpha(x) & : x \in \bar{B}_{r_j}(x_{0j}) \\ a_\alpha\left(x_{0j} + r_0^2 \frac{x - x_{0j}}{|x - x_{0j}|^2}\right) & : x \notin \bar{B}_{r_j}(x_{0j}) \end{array} \right\}$$

for each $j = 1, 2, ...$ Then $|a_\alpha(x) - a_\alpha(x_{0j})| \leq \delta$ for all $x \in R^n$ and $j = 0, 1, 2, ...$ Freezing coefficients in the equation (3.10) obtain that

$$\sum_{|\alpha:l|=1} a_\alpha(x_{0j}) D^\alpha u_j + A_\lambda(x_{0j}) u_j(x) = F_j(x), \tag{3.11}$$

where

$$F_j = f_j + \sum_{|\alpha:l|=1} [a_\alpha(x_{0j}) - a_\alpha(x)] D^\alpha u_j + [A(x_{0j}) - A(x)] u_j. \tag{3.12}$$

By virtue of Theorem 3.11 we obtain that the problem (3.11) has a unique solution u_j and for $\lambda \in S(\varphi_0)$ and sufficiently large $|\lambda|$ the coercive estimate holds:

$$\sum_{|\alpha:l|\leq 1} |\lambda|^{1-|\alpha:l|} \|D^\alpha u_j\|_{G_j,p} + \|Au_j\|_{G_j,p} \leq C \|F_j\|_{G_j,p}. \tag{3.13}$$

Whence, using properties of the smoothness of coefficients of equations (3.10), (3.12) and choosing diameters of G_j sufficiently small, we get that

$$\|F_j\|_{G_j,p} \leq \varepsilon \|u_j\|_{W_p^l(G_j,E)} + C(\varepsilon) \|u_j\|_{G_j,p} + C \|f\|_{G_j,p}, \tag{3.14}$$

where ε is a sufficiently small and $C(\delta)$ is a continuous function. Consequently, from (3.13) and (3.14) we get

$$\sum_{|\alpha:l|\leq 1} |\lambda|^{1-|\alpha:l|} \|D^\alpha u_j\|_{G_j,p} \leq C \|f\|_{G_j,p} + \delta \|u_j\|_{W_p^2} + C(\delta) \|u_j\|_{G_j,p}.$$

Choosing $\delta < 1$ from the above inequality we have

$$\sum_{|\alpha:l|\leq 1} |\lambda|^{1-|\alpha:l|} \|D^\alpha u_j\|_{G_j,p} \leq C \left[\|f\|_{G_j,p} + \|u_j\|_{G_j,p}\right]. \tag{3.15}$$

Then using an equality $u(x) = \sum_j u_j(x)$ and by virtue of the estimate (3.15) for $u \in W_p^l(R^n, E)$ we have

$$\sum_{|\alpha:l|\leq 1} |\lambda|^{1-|\alpha:l|} \|D^\alpha u_j\|_{G_j,p} \leq C \left[\|(L + \lambda) u\|_p + \|u\|_p\right]. \tag{3.16}$$

Let $u \in W_p^l(R^n, E)$ be solution of the problem (3.5). Then for $\lambda \in S(\varphi_0)$ we have

$$\|u\|_p = \|(L+\lambda)u - Lu\|_p \leq \frac{1}{\lambda}\left[\|(L+\lambda)u\|_p + \|u\|_{W_p^l}\right]. \quad \textbf{(3.17)}$$

Then by Theorem 3.11 and by virtue of (3.16) and (3.17) for sufficiently large $|\lambda|$ we have

$$\sum_{|\alpha:l|\leq 1} |\lambda|^{1-|\alpha:l|} \|D^\alpha u_j\|_p \leq C\|(L+\lambda)u\|_p. \quad \textbf{(3.18)}$$

The estimate (3.18) implies that the problem (3.5) has only a unique solution and the operator $(L+\lambda)$ has an invertible operator in its rank space. We need to show that this rank space coincide with the space $L_p(R^n; E)$. Let us construct for all j the function u_j, that are defined on the regions $G_j$and satisfying the problem (3.5). Let $g_j \in C_0^\infty(R^n)$ such that $g_j(x) \equiv 1$ on suppφ_j. The problem (3.5) can be express in the form

$$\sum_{|\alpha:l|=1} a_\alpha(x_{0j}) D^\alpha u_j + A_\lambda(x_j) u_j(x)$$

$$= \left\{g_j f + [A(x_{0j}) - A(x)] u_j - \sum_{|\alpha:l|<1} A_\alpha(x) D^\alpha u_j\right\}, j = 1, 2, .., \quad \textbf{(3.19)}$$

Since the functions u_j and g_j have the same compact support, after zero extension of the both sides of equation (3.19) we can reconsider the problem (3.19) as BVP in R^n. Let $O_{j\lambda}$ denote the operators in $L_p(R^n; E)$ generated by the DOE with constant coefficients

$$\sum_{|\alpha:l|=1} a_\alpha(x_{0j}) D^\alpha u_j + A_\lambda(x_{0j}) u_j(x) = g_j f,$$

i.e.

$$D(O_{j\lambda}) = W_p^l(R^n; E(A), E), O_{j\lambda}u = \sum_{|\alpha:l|=1} a_\alpha(x_{0j}) D^\alpha u_j + A_\lambda(x_{0j}) u_j(x).$$

By virtue of the Theorem 3.11 for all $f \in L_p(G_j; E)$, for $\lambda \in S(\varphi_0)$ and sufficiently large $|\lambda|$ we have

$$\sum_{|\alpha:l|\leq 1} |\lambda|^{1-|\alpha:.l|} \left\|D^\alpha O_{j\lambda}^{-1} f\right\|_p + \left\|A O_{j\lambda}^{-1} f\right\|_p \leq C\|f\|_p. \quad \textbf{(3.20)}$$

Extending u_j zero on the outside of suppφ_j in equalities (3.19) and passing substitutions $u_j = O_{j\lambda}^{-1} \upsilon_j$ obtain operator equations with respect to υ_j;

$$\upsilon_j = K_{j\lambda}\upsilon_j + g_j f, , j = 1, 2, ... \quad \textbf{(3.21)}$$

By virtue of Theorem 3.6 and the estimate (3.20), in view of the smoothness of the coefficients of the expression $K_{j\lambda}$ for $\lambda \in S(\varphi_0)$ and sufficiently large $|\lambda|$ we have $\|K_{j\lambda}\| < \varepsilon$, where ε is sufficiently small. Consequently, equations (3.21) have unique solutions $\upsilon_j = [I - K_{j\lambda}]^{-1} g_j f$. Moreover,

$$\|\upsilon_j\|_p = \left\|[I - K_{j\lambda}]^{-1} g_j f\right\|_p \leq \|f\|_p .$$

Whence, $[I - K_{j\lambda}]^{-1} g_j$ are bounded linear operators from $L_p(R^n; E)$ to $L_p(G_j; E)$. Thus, we obtain that the functions $u_j = U_{j\lambda} f = O_{j\lambda}^{-1} [I - K_{j\lambda}]^{-1} g_j f$ are solutions of the equations (3.21). Consider a linear operator $(U + \lambda)$ in the space $L_p(R^n; E)$ such that

$$(U + \lambda) f = \sum_j \varphi_j(y) U_{j\lambda} f.$$

It is clear from the constructions U_j and the estimate (3.20) that operators $U_{j\lambda}$ are bounded linear from $L_p(R^n; E)$ to $W_p^l(R^n, E)$ and

$$\sum_{|\alpha:l|\leq 1} |\lambda|^{1-|\alpha:l|} \left\|D^\alpha U_{j\lambda}^{-1} f\right\|_p + \left\|A U_{j\lambda}^{-1} f\right\|_p \leq C \|f\|_p , \quad \textbf{(3.22)}$$

for $\lambda \in S(\varphi_0)$ and sufficiently large $|\lambda|$. Therefore, $(U + \lambda I)$ is a bounded linear operator from L_p to L_p. Then act of $(L + \lambda)$ to $u = \sum_j \varphi_j U_{j\lambda} f$ gives $(L + \lambda) u = f + \sum_j \Phi_{j\lambda} f$, where $\Phi_{j\lambda}$ are linear combination of $U_{j\lambda}$ and $\frac{d}{dy} U_{j\lambda}$. By virtue of Theorem 3.6, the estimate (3.22) and from the expression $\Phi_{j\lambda}$ obtain that operators $\Phi_{j\lambda}$ are bounded linear from L_p to $L_p(G_j; E)$ and $\|\Phi_{j\lambda}\| < \delta$. Therefore, there exists a bounded linear invertible operator

$$\left(I + \sum_j \Phi_{j\lambda}\right)^{-1} .$$

Whence, we obtain that for all $f \in L_p(R^n; E)$ the boundary value problem (3.5) have a unique solution

$$u = (U + \lambda) \left(I + \sum_j \Phi_{j\lambda}\right)^{-1} f$$

i.e. we obtain assertion of the Theorem 3.12.

Result 3.13: Theorem 3.12 implies that the differential operator L has a resolvent operator $(L + \lambda)^{-1}$ for $\lambda \in S(\varphi_0)$ and the estimate holds

$$\sum_{|\alpha:l|\leq 1} |\lambda|^{1-|\alpha:l|} \left\|D^\alpha (L+\lambda)^{-1}\right\|_{B(L_p(R^n;E))} + \left\|A (L + \lambda)^{-1}\right\|_{B(L_p(R^n;E))} \leq C.$$

Let $E = l_q$, then we obtain maximal regularity system of infinite number of anisotropic equations.

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