# HIGHER ORDER DIFFERENCE APPROXIMATIONS FOR THE NUMERICAL SOLUTION OF FRACTIONAL HEAT EQUATIONS

by

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# APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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## ABSTRACT

In this study, history of fractional calculus was investigated and we consider the numerical solution of a time-fractional heat equation, which is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order ; 0< $\alpha$ <1: The main purpose of this work is to extend the idea on Crank-Nicholson method to the time-fractional heat equation. We prove that the proposed method is unconditionally stable, and the numerical solution converges to the exact one with order O($\tau^2 + h^2$). Numerical experiments are carried out to support the theoretical claims.The results are compared both numerically and graphically, computer programmes and algorithms are presented.

**Keywords:** Crank-Nicholson method, time-fractional diffusion equation, ADI, FDW, FPDEs.

# KESİRLİ ISI DENKLEMLERİNİN NUMERİK ÇÖZÜMLERİ İÇİN YÜKSEK MERTEBELİ FARK YAKLAŞIMLARI

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## ÖZ

Bu çalışmada kesirli mertebeden analizden bahsettik ve zaman üzerinden kesirli mertebeden ısı denkleminin 0< $\alpha$ <1: aralığında kesirli mertebeden türev ile birinci dereceden zaman üzerinden türevi ile yer değiştiren standart yayılım denkleminin çözümünü açıkladık. Bu çalısmanın asıl amacı zaman üzerinden kesirli mertebeden ısı denklemini Crank-Nicholson metodu ile açıklamaktır.

**Anahtar Kelimeler:** Crank-Nicholson method, zaman üzerinden kesirli yayılım denklemi, ADI, FDW, FPDEs.

# DEDICATION

To my parents and my sisters.

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# LIST OF SYMBOLS/ABBREVIATIONS

| | |
|---|---|
| $\alpha$ | Order of a fractional operator |
| $\rho$ | spectral radius |
| ADI | Alternating-directions implicit |
| FPDEs | Fractional partial differential equations |
| FDW | Fractional diffusion-wave equation |
| D | Fractional derivative |
| I | Fractional integral |

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# CHAPTER 1

# INTRODUCTION FRACTIONAL CALCULUS

The word fraction actually comes from the Latin ”fractio” which means to break. To understand how fractions have developed into the form we recognise, we’ll have to step back even further in time to discover what the first number systems were like.

From as early as 1800 BC, the Egyptians were writing fractions. Their number system was a base 10 idea (a little bit like ours now) so they had separate symbols for 1, 10, 100, 1000, 10000, 100000 and 1000000. The ancient Egyptian writing system was all in pictures which were called hieroglyphs and in the same way, they had some pictures for the numbers.

Otherwise, to consider fractional power functions, we need to use the third power law. Remember that this says

$$(x^m)^n = x^{nm}.$$

To start with we’ll just consider fractions of the form 1/n where n is a whole number. So we’ll start off by considering the expression ”$x$ to the power a half”: $x^{1/2}$. What happens if we raise this to the power 2, i.e. if we square this expression? By using the third power law we can see that

$$\left(x^{1/2}\right)^2 = x^1 = x,$$

since ”x to the power 1” is just x.

This shows that if we square our expression "$x$ to the power a half" then we get simply $x$. Therefore "$x$ to the power a half" must be the square root of $x$. The square root of a quantity is the number which when we square it gives the original quantity. Similarly the cube root of a quantity is the number which we have to cube to get the original quantity.

The integer $n$ on the bottom of the fraction can be positive or negative. Consider the function "$x$ to the power of minus a half": $x^{-1/2}$.

Now by the definition of negative powers, a quantity to the power of $-1$ means "1 over the quantity" and raising to the power a half means taking the square root, so the function $f(x)$ can be rewritten as: 1/(square root of $x$).

In this way, all fractional powers can be reduced to some power of a function of the form $x^{1/n}$.

Moreover, the traditional integral and derivative are, to say the least, a staple for the technology professional, essential as a means of understanding and working with natural and artificial systems. Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Consider the physical meaning of the exponent. According to our primary school teachers exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. This concept in itself is easy to grasp and straight forward. However, this physical definition can clearly become confused when considering exponents of non integer value. While almost anyone can verify that $x^3 = x \cdot x \cdot x$, how might one describe the physical meaning of $x^{3,4}$, or moreover the transcendental exponent $x^{\pi}$. One cannot conceive what it might be like to multiply a number or quantity by itself $3,4$ times, or $\pi$ times, and yet these expressions have a definite value for any value $x$, verifiable by infinite series expansion, or more practically, by calculator. Now, in the same way consider the integral and derivative. Although they are indeed concepts of a higher complexity by nature, it is still fairly easy to physically represent their meaning. Once mastered, the idea of completing numerous of these operations, integrations or differentiations follows naturally. Given the satisfaction of a very few restrictions (e.g.function continuity), completing $n$ integrations can become as methodical as multiplication. But the curious mind can not be restrained from asking the question what if $n$ were not restricted to an integer value? Again, at first glance, the physical meaning can become convoluted (pun intended), fractional calculus flows quite naturally from our traditional definitions. And just as fractional exponents such as the square root may find their way into innumerable equations and applications, it will become apparent that integrations of order 1/2 and beyond can find practical use in many modern problems.

Most authors on this topic will cite a particular date as the birthday of so called 'Fractional Calculus'. In a letter dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function $f(x) = x$, $\frac{D^n x}{Dx^n}$ . L'Hopital's posed the question to Leibniz, what would the result be if $n = 1/2$. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn." In these words fractional calculus was born.

Although the name "fractional calculus" is actually a misnomer, the designation "integration and differentiation of arbitrary order" being more appropriate, one usually sticks to "fractional calculus", a terminology in use since the days of L'Hospital.

Following L'Hopital's and Liebniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace are among the many that dabbled with fractional calculus and the mathematical consequences [1]. Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus (not yet the world as a whole) are the Riemann-Liouville and Grunwald-

Letnikov definition. While the shear number of actual definitions are no doubt as numerous as the men and women that study this field, they are for the most part variations on the themes of these two and so are addressed in detail in this document.

Fractional calculus has developed especially intensively since 1974 when the first international conference in the field took place. It was organized by Bertram Ross [2] and took place at the University of New Haven, Connecticut in 1974. It had an exceptional turnout of 94 mathematicians; the proceedings contain 26 papers by the experts of the time. It was followed by the conferences conducted by Adam Mc Bride and Garry Roach [3] (University of Strathclyde, Glasgow, Scotland) of 1984, by Katsuyuki Nishimoto [4] (Nihon University, Tokyo, Japan) of 1989, and by Peter Rusev, Ivan Dimovski and Virginia Kiryakova [5] (Varna, Bulgaria) of 1996. In the period 1975 to the present, about 600 papers have been published relating to fractional calculus.

Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found. The mathematics has in some cases had to change to meet the requirements of physical reality. Caputo reformulated the more 'classic' definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations [6]. As recently as 1996, Kolowankar reformulated again, the Riemann-Liouville fractional derivative in order to differentiate no-where differentiable fractal functions [7].

Leibniz's response, based on studies over the intervening 300 years, has proven at least half right. It is clear that within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found. However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult (arguably impossible) to grasp, the definitions themselves are no more rigorous than those of their integer order counterparts.

Understanding of definitions and use of fractional calculus will be made more clear by quickly discussing some necessary but relatively simple mathematical definitions that will arise in the study of these concepts. These are The Gamma Function, Beta Function, and Riemann-Liouville Fractional Operator and are addressed in the following chapter.

# CHAPTER 2

# SOME BASIC DEFINITIONS

## 2.1. The Gamma Function

The simplest interpretation of the gamma function is simply the generalization of the factorial for all real numbers. The definition of the gamma function is given by (2.1).

$$\Gamma\left(z\right)=\int_{0}^{\infty}e^{-u}u^{z-1}du, \qquad \text{for all } z\in\mathbb{R} \tag{2.1}$$

The 'beauty' of the gamma function can be found in its properties. First, as seen in (2.2), this function is unique in that the value for any quantity is, by consequence of the form of the integral, equivalent to that quantity $z$ minus one times the gamma of the quantity minus one.

$$\Gamma\left(z+1\right)=z\Gamma\left(z\right), \quad \text{also, when } z\in\mathbb{N}_{+}, \quad \Gamma\left(z\right)=\left(z-1\right)! \tag{2.2}$$

This can be shown through a simple integration by parts. The consequence of this relation for integer values of $z$ is the definition for factorial. Some equations related to the gamma function is as follows

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)...(z+n-1)},$$

$$\binom{-z}{r} = \frac{\Gamma(1-z)}{\Gamma(r+1)\Gamma(1-z-r)}.$$

## 2.2. Beta Function

Also known as the Euler Integral of the First Kind, the Beta Function is in important relationship in fractional calculus. Its solution is not only defined through the use of multiple Gamma Functions, but furthermore shares a form that is characteristically similar to the Fractional Integral/Derivative of many functions,like the Mittag-Leffler Function. Equation (2.3) demonstrates the Beta Integral and its solution in terms of the Gamma function.

$$B(p,q) := \int_0^1 (1-u)^{p-1}u^{q-1}du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(q,p), \quad \text{where } p,q \in \mathbb{R}_+. \tag{2.3}$$

## 2.3. Riemann-Liouville Fractional Operator

Let us consider a continuous function $y = f(x)$. According to the well-known definition, the first-order derivative of the function $f(t)$ is defined by

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}. \tag{2.4}$$

Applying this definition twice gives the second-order derivative:

$$\begin{aligned} f^{''}(t) = \frac{d^2 f}{dt^2} &= \lim_{h \to 0} \frac{f^{'}(t) - f^{'}(t-h)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h} \right\} \\ &= \lim_{h \to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}. \end{aligned} \tag{2.5}$$

Using (2.4) and (2.5) we obtain the third-order derivative

$$f^{'''}(t) = \frac{d^3 f}{dt^3} = \lim_{h \to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3}, \tag{2.6}$$

and, by induction, the $n - th$ order derivative can be obtained as:

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh), \tag{2.7}$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!} \tag{2.8}$$

is the usual notation for the binomial coefficients.

Let us now consider the following expression generalizing the fractions in (2.4)-(2.7):

$$f_h^{(p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t - rh), \tag{2.9}$$

where p is an arbitrary integer number; $n$ is also integer, as above.

Obviously, for $p \leq n$ we have

$$\lim_{h \to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p}, \tag{2.10}$$

because in such a case, as follows from (2.8), all the coefficients in the numerator after $\binom{p}{p}$ are equal to 0.

Let us consider negative values of $p$. For convenience, let us denote

$$\binom{p}{r} = \frac{p(p-1)...(p+r-1)}{r!} \tag{2.11}$$

Then we have

$$\binom{-p}{r} = \frac{-p(-p-1)...(-p-r+1)}{r!} = (-1)^r \binom{p}{r} \tag{2.12}$$

and replacing $p$ in (2.9) with $-p$ we can write

$$f_h^{(-p)}(t) = \frac{1}{h^p}\sum_{r=0}^{n}\binom{p}{r}f(t-rh), \tag{2.13}$$

where $p$ is a positive integer number.

If $n$ is fixed, then $f_h^{(-p)}(t)$ tends to the uninteresting limit 0 as $h \to 0$. To arrive at a non-zero limit, we have to suppose that $n \to \infty$ as $h \to 0$. We can take $h = \dfrac{t-a}{n}$, where a is a real constant, and consider the limit value, either finite or infinite, of $f_h^{(-p)}(t)$, which we denote as

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-p)}(t) =_a D_t^{-p}f(t). \tag{2.14}$$

Here ${}_aD_t^{-p}f(t)$ denotes, in fact, a certain operation performed on the function $f(t)$; $a$ and $t$ are the limits relating to the operation.

Let us consider several particular cases.

For $p = 1$ we have:

$$f_h^{(-1)}(t) = h\sum_{r=0}^{n}f(t-rh). \tag{2.15}$$

Taking into account that $t - nh = a$ and that the function $f(t)$ is assumed to be continuous, we conclude that

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-1)}(t) = {}_a D_t^{-1} f(t) = \int\limits_0^{t-a} f(t-z)dz = \int\limits_a^t f(\tau)d\tau. \tag{2.16}$$

Let us take $p = 2$. In this case

$$\binom{2}{r} = \frac{2 \cdot 3 \cdot \ldots \cdot (2+r+1)}{r!} = r+1$$

and we have:

$$f_h^{(-2)}(t) = h\sum_{r=0}^{n}(rh)f(t-rh). \tag{2.17}$$

Denoting $t + h = y$ we can write

$$f_h^{(-2)}(t) = h\sum_{r=1}^{n+1}(rh)f(t-rh), \tag{2.18}$$

and taking $h \to 0$ we obtain

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-2)}(t) = {}_aD_t^{-2}f(t) = \int\limits_0^{t-a} zf(t-z)dz = \int\limits_a^t (t-\tau)f(\tau)d\tau, \tag{2.19}$$

because $y \to t$ as $h \to 0$.

The third particular case, namely $p = 3$, will show us the general expression for ${}_aD_t^{-p}$.

Taking into account that:

$$\binom{3}{r} = \frac{3\cdot 4\cdot \ldots \cdot (3+r-1)}{r!} = \frac{(r+1)(r+2)}{1\cdot 2},$$

we have

$$f_h^{(-3)}(t) = \frac{h}{1\cdot 2}\sum_{r=0}^{n}(r+1)(r+2)h^2 f(t-rh). \tag{2.20}$$

Denoting, as above, $t + h = y$, we write

$$f_h^{(-3)}(t) = \frac{h}{1\cdot 2}\sum_{r=1}^{n+1} r(r+1)h^2 f(y-rh). \tag{2.21}$$

Expression (2.21) can be written as

$$f_h^{(-3)}(t) = \frac{h}{1\cdot 2}\sum_{r=1}^{n+1}(rh)^2 f(y-rh) + \frac{h^2}{1\cdot 2}\sum_{r=1}^{n+1} rhf(y-rh). \tag{2.22}$$

Taking now $h \to 0$, we obtain

$$_aD_t^{-3}f(t) = \frac{1}{2!} = \int\limits_0^{t-a} z^2 f(t-z)dz = \int\limits_a^t (t-\tau)^2 f(\tau)d\tau, \tag{2.23}$$

because $y \to t$ as $h \to 0$ and

$$\lim_{\substack{h\to 0\\ nh=t-a}} \frac{h^2}{1\cdot 2}\sum_{r=1}^{n+1} rhf(y-rh) = \lim_{\substack{h\to 0\\ nh=t-a}} h\int\limits_a^t (t-\tau)f(\tau)d\tau = 0.$$

Relationship (2.16)-(2.23) suggest the following general expression:

$$_aD_t^{-p}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^p\sum_{r=0}^{n}\binom{p}{r} f(t-rh) = \frac{1}{(p-1)!}\int\limits_a^t (t-\tau)^{p-1}f(\tau)d\tau. \tag{2.24}$$

To prove the formula (2.24) by induction we have to show that if it holds for some $p$, then it holds also for $p+1$.

Let us introduce the function

$$f_1(t) = \int_a^t f(\tau) d\tau, \tag{2.25}$$

which has the obvious property $f_1(a) = 0$, and consider

$$\begin{aligned} {}_aD_t^{-p-1} f(t) &= \lim_{\substack{h\to 0\\ nh=t-a}} h^{p+1} \sum_{r=0}^{n} \binom{p+1}{r} f(t-rh) \\ &= \lim_{\substack{h\to 0\\ nh=t-a}} h^{p} \sum_{r=0}^{n} \binom{p+1}{r} f_1(t-rh) - \lim_{\substack{h\to 0\\ nh=t-a}} h^{p+1} \sum_{r=0}^{n} \binom{p+1}{r} f_1(t-(r+1)\,h) \end{aligned} \tag{2.26}$$

Using (2.11) it is easy to verify that

$$\binom{p+1}{r} = \binom{p}{r} + \binom{p+1}{r-1}, \tag{2.27}$$

where we must put

$$\binom{p+1}{-1} = 0.$$

Relationship (2.27) applied to the first sum in (2.26) and the replacement of $r$ by $r-1$ in the second sum gives:

$$\begin{aligned}
{}_aD_t^{-p-1}f(t) = {} & \lim_{\substack{h\to 0\\ nh=t-a}} h^{p+1}\sum_{r=0}^{n}\binom{p}{r}f_1(t-rh) + \lim_{\substack{h\to 0\\ nh=t-a}} h^{p+1}\sum_{r=0}^{n}\binom{p+1}{r-1}f_1(t-rh) \\
& - \lim_{\substack{h\to 0\\ nh=t-a}} h^{p+1}\sum_{r=1}^{n+1}\binom{p+1}{r-1}f_1(t-rh) \\
& =_a D_t^{-p}f_1(t) - \lim_{\substack{h\to 0\\ nh=t-a}} h^{p}\sum_{r=0}^{n}\binom{p+1}{n}f_1(t-(n+1)h) \\
& =_a D_t^{-p}f_1(t) - (t-a)^p \lim_{n\to\infty}\binom{p+1}{n}\frac{1}{n^p}f_1(a-\frac{t-a}{n}).
\end{aligned}$$

It follows from the definition (2.25) of the function $f_1(t)$ that

$$\lim_{n\to\infty} f_1(a-\frac{t-a}{n}) = 0.$$

Taking into account the known limit

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)...(z+n)},$$

then we obtain

$$
\begin{aligned}
{}_aD_t^{-p-1}f(t) ={}_a D_t^{-p}f_1(t) &= \frac{1}{(p-1)!}\int\limits_a^t (t-\tau)^{p-1}f(\tau)d\tau \\
&= -\frac{(t-\tau)^p f_1(\tau)}{p!}\,\Big|_{\tau=a}^{\tau=t} + \frac{1}{p!}\int\limits_a^t (t-\tau)^p f(\tau)d\tau \\
&= \frac{1}{p!}\int\limits_a^t (t-\tau)^p f(\tau)d\tau, \qquad (2.28)
\end{aligned}
$$

which ends the proof of formula (2.24) by induction.

Riemann and Liouville continued this result with replacing the discrete factorial $(p+1)!$ with Euler's continuous gamma function

$$
{}_aD_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int\limits_a^t (t-\tau)^{p-1}f(\tau)d\tau \qquad (2.29)
$$

To obtain differentiation of fractional order, we can write

$$
\begin{aligned}
{}_aD_t^{p}f(t) &= \frac{d^n}{dt^n}\,{}_aD_t^{-(n-p)}f(t) \\
&= \frac{1}{\Gamma(n-p)}\frac{d^n}{dt^n}\int\limits_a^t (t-\tau)^{n-p-1}f(\tau)d\tau.
\end{aligned}
$$

If we take the lower limit 0, we can obtain the most used formula:

$$ {}_0D_t^p f(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int\limits_0^t (t-\tau)^{n-p-1} f(\tau) d\tau. $$

### 2.3.1. Examples

What is the fractional derivative ${}_0D_x^\alpha f(x)$ of the function $f(x) = x^b$ for $b > -1$?

Let us first evaluate the fractional derivative ${}_0D_x^\alpha f(x)$ of the function $f(x) = x^b$ for $b > -1$. In view of the evaluation of the beta integral, valid for all $p > 0$ and $q > 0$, namely

$$ B(p,q) := \int\limits_0^1 u^{p-1} (1-u)^{q-1} \, du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, $$

we have for $m-1 \leq \alpha < m$, $m \in \mathbb{N}$,

$$ \begin{aligned} {}_0D_x^\alpha x^b &= \left(\frac{d}{dx}\right)^m \left\{ \frac{1}{\Gamma(m-\alpha)} \int\limits_0^x (x-u)^{m-\alpha-1} u^b du \right\} \\ &= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \left\{ x^{m-\alpha+b} \int\limits_0^1 (1-v)^{m-\alpha-1} v^b dv \right\} \\ &= \frac{1}{\Gamma(m-\alpha)} \frac{\Gamma(b+1)\Gamma(m-\alpha)}{\Gamma(b+1+m-\alpha)} \left(\frac{d}{dx}\right)^m x^{m-\alpha+b} \\ &= \frac{\Gamma(b+1)}{\Gamma(b+1+m-\alpha)} x^{b-\alpha} \frac{\Gamma(m-\alpha+b+1)}{\Gamma(m-\alpha+b-m+1)} \\ &= \frac{\Gamma(b+1)}{\Gamma(b-\alpha+1)} x^{b-\alpha}, \end{aligned} $$

where we made use of, with $p \in \mathbb{R} \backslash \{-1, -2, -3, ...\}$,

$$\left(\frac{d}{dx}\right)^{m} x^{p} = \frac{\Gamma\left(p+1\right)}{\Gamma\left(p-m+1\right)} x^{p-m} = p(p-1)...(p-m+1)x^{p-m} \quad \left(m \in \mathbb{N}\right).$$

For $b = 0$ we obtain ${}_0D_x^{\alpha} c = cx^{-\alpha}/\Gamma(1-\alpha)$ for any $\alpha > 0$. Thus fractional differentiation of a constant $c$ is zero only for positive integral values of $\alpha = n \in \mathbb{N}$ (recall $\Gamma(1-n) = \infty$ ). On the other hand, for any $\alpha > 0$, ${}_0D_x^{\alpha} f(x) \equiv 0$ if $f(x) = x^{\alpha-k}$, $k = 1, 2, ..., 1 + [\alpha]$.

Let us consider a second example, namely ${}_0I_x^{\alpha} f$ for $f(x) = \log x$. Indeed, under the substitution $u = x(1-v)$,

$$\begin{aligned}
{}_0I_x^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-u)^{\alpha-1} \log u du \\
&= \frac{\log ux^{\alpha}}{\Gamma(\alpha)} \int\limits_0^1 v^{\alpha-1} dv + \frac{x^{\alpha}}{\Gamma(\alpha)} \int\limits_0^1 v^{\alpha-1} \log(1-v) dv \\
&= \frac{x^{\alpha}}{\Gamma(\alpha+1)} \log x - \frac{x^{\alpha}}{\Gamma(\alpha)\alpha} \int\limits_0^1 \log(1-v) d\left(1-v^{\alpha}\right) \\
&= \frac{x^{\alpha}}{\Gamma(\alpha+1)} \left\{ \log x - \left(1-v^{\alpha}\right) \log(1-v) \left.\right|_0^1 - \int\limits_0^1 \frac{1-v^{\alpha}}{1-v} dv \right\}
\end{aligned}$$

by integration by parts. Noting

$$\int\limits_0^1 \frac{v^x - v^y}{1-v} dv = \psi(y+1) - \psi(x+1) \quad (x, y > -1),$$

where the psi function, defined by $\psi(x) = [\Gamma(x)]^{-1}(d/dx)\Gamma(x)$, obey the recursion $\psi(x+1) - \psi(x) = x^{-1}$ with $-\psi(1) = \gamma = 0.5772157...$, this yields

$$ {}_0I_x^{\alpha} \log x = \frac{x^{\alpha}}{\Gamma(\alpha+1)} \left\{ \log x - \psi(\alpha+1) - \psi(1) \right\}. $$

The remainder of the article is arranged as follows. In Chapter 3, we consider the numerical solution of a time-fractional heat equation, which is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha$, $0 < \alpha < 1$. The main purpose of this work is to extend the idea on Crank-Nicholson method to the time-fractional heat equation. We prove that the proposed method is unconditionally stable, and the numerical solution converges to the exact one with order $O(\tau^2 + h^2)$. Numerical experiments are carried out to support the theoretical claims. In Chapter 4, we mention about a journal that's name is 'A compact difference scheme for the fractional diffusion-wave equation'.This article is devoted to the study of high order difference methods for the fractional diffusion-wave equation. The time fractional derivatives are described in the Caputo's sense. A compact difference scheme is presented and analyzed. It is shown that the difference scheme is unconditionally convergent and stable in $L_\infty$norm. The convergence order is $O(\tau^{3-\alpha} + h^4)$. Two numerical examples are also given to demonstrate the theoretical results. The article ends with a brief conclusion section.

# CHAPTER 3

# CRANK-NICHOLSON DIFFERENCE SCHEME FOR THE TIME FRACTIONAL HEAT EQUATIONS

Fractional calculus is one of the most popular subject in many scientific areas for decades. Many problems in applied science, physic and engineering are modeled mathematically by the fractional partial differential equations(FPDEs). We can see these models adoption in viscoelasticity [8, 9], finance [10, 11], hydrology [12, 13], engineering [6, 14], control systems [15]. FPDEs may be investigated into two fundamental types : time-fractional differential equation and space-fractional differential equation.

Several different methods have been used for solving FPDEs. For the analytical solutions to problems some methods have been proposed: the variational iteration method [16, 17], the Adomian decomposition method [17, 18, 19, 20], the Laplace transform and Fourier transform methods [21, 22, 23].

On the other hand numerical methods which based on a finite difference approximation to the fractional derivative, for solving FDPEs [25, 26, 27, 28, 29] have been proposed. A practical numerical method for solving multi-dimensional fractional partial differential equations, using a variation on the classical alternating-directions implicit (ADI) Euler method is presented in [24]. Many finite difference approximations for the FPDEs are only first-order accurate. The second order accurate numerical approximations for the **space**-fractional differential equations [30, 31, 32] was presented, but we are not aware of any published finite-difference methods for the **time**-fractional differential equations which offer better than first-order accuracy.

In this work we have developed a Crank-Nicholson method for the approximate solution of the time fractional heat equation

$$\begin{cases} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), (0 < x < 1, 0 < t < 1), \\ u(0,x) = r(x),\ 0 \le x \le 1, \\ u(t,0) = 0,\ \ u(t,1) = 0,\ \ 0 \le t \le 1. \end{cases} \tag{3.1}$$

Here, the term $\dfrac{\partial^\alpha u(t,x)}{\partial t^\alpha}$ denotes $\alpha$-order Caputo derivative, defined by [2], with the formula:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{u_t(\tau,x)}{(t-\tau)^\alpha} d\tau, \text{ where } 0 < \alpha < 1, \tag{3.2}$$

where $\Gamma(.)$ is the Gamma function.

### 3.1. Discretization of the problem

In this section we introduce the basic ideas for the numerical solution of the time fractional heat equation (3.1) by Crank-Nicholson difference scheme.

For some positive integers $M$ and $N$, the grid sizes in space and time for the finite difference algorithm are defined by $h = 1/M$ and $\tau = 1/N$ respectively. The grid points in the space interval $[0,1]$ are the numbers $x_i = ih$, $i = 0, 1, 2, ..., M$, and the grid points in the time interval $[0,1]$ are labeled $t_n = n\tau, n = 0, 1, 2, ..., N$. The values of the functions $U$ and $f$ at the grid points are denoted $U_i^n = U(t_n, x_i)$ and $f_i^n = f(t_n, x_i)$, respectively.

As in the Classical Crank-Nicholson difference scheme, a discrete approximation to the fractional derivative $\dfrac{\partial^\alpha U(t,x)}{\partial t^\alpha}$ at $(t_{n+\frac{1}{2}},x_i)$ can be obtained by the following quadrature formula:

$$\begin{aligned}\frac{\partial^\alpha U(t_{n+\frac{1}{2}},x_i)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\int\limits_0^{t_{n+1/2}} U_t(s,x)(t_{n+1/2}-s)^{-\alpha}ds\\
&= \frac{1}{\Gamma(1-\alpha)}\left[\int\limits_0^{t_n} U_t(s,x)\left((n+\frac{1}{2})\tau-s\right)^{-\alpha}ds\right.\\
&+\left.\int\limits_{t_n}^{t_{n+1/2}}\left[\frac{U_i^{n+1}-U_i^n}{\tau}+O(\tau^2)\right]\left((n+\frac{1}{2})\tau-s\right)^{-\alpha}ds\right]\\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{n}\int\limits_{(j-1)\tau}^{j\tau}\left[\frac{U_i^j-U_i^{j-1}}{\tau}+O(\tau^2)\right]\left((n+\frac{1}{2})\tau-s\right)^{-\alpha}ds+\\
&+\frac{1}{\Gamma(1-\alpha)}\int\limits_{n\tau}^{(n+1/2)\tau}\left[\frac{U_i^{n+1}-U_i^n}{\tau}+O(\tau^2)\right]\left((n+\frac{1}{2})\tau-s\right)^{-\alpha}ds.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial^\alpha U(t_{n+\frac{1}{2}},x_i)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{n}\left[\frac{U_i^j-U_i^{j-1}}{\tau}+O(\tau^2)\right]\int\limits_{(j-1)\tau}^{j\tau}\left((n+\frac{1}{2})\tau-s\right)^{-\alpha}ds+\\
&+\frac{1}{\Gamma(1-\alpha)}\left[\frac{U_i^{n+1}-U_i^n}{\tau}+O(\tau^2)\right]\int\limits_{n\tau}^{(n+1/2)\tau}\left((n+\frac{1}{2})\tau-s\right)^{-\alpha}ds\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{1}{\tau^\alpha}\frac{1}{1-\alpha}\sum_{j=1}^{n}\left[U_i^j-U_i^{j-1}\right]\left[(n-j+3/2)^{1-\alpha}-(n-j+1/2)^{1-\alpha}\right]\\
&+\frac{1}{\Gamma(1-\alpha)}\frac{1}{\tau^\alpha}\frac{1}{1-\alpha}(U_i^{n+1}-U_i^n)\frac{1}{2^{1-\alpha}}\\
&+\frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\sum_{j=1}^{n}\left[(n-j+3/2)^{1-\alpha}-(n-j+1/2)^{1-\alpha}\right]O(\tau^{3-\alpha})\\
&+\frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\frac{1}{2^{1-\alpha}}O(\tau^{3-\alpha})\end{aligned}$$

So, we obtain,

$$\begin{aligned}\frac{\partial^\alpha U(t_{n+\frac{1}{2}},x_i)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\frac{1}{\tau^\alpha}\frac{1}{1-\alpha}\sum_{j=1}^{n}\left[U_i^j - U_i^{j-1}\right]\left[(n-j+\frac{3}{2})^{1-\alpha} - (n-j+\frac{1}{2})^{1-\alpha}\right] \\ &+ \frac{1}{\Gamma(1-\alpha)}\frac{1}{\tau^\alpha}\frac{1}{1-\alpha}(U_i^{n+1} - U_i^n)\frac{1}{2^{1-\alpha}} \\ &+ \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\left[(n+1/2)^{1-\alpha}\right]O(\tau^{3-\alpha}).\end{aligned}$$

Setting $\sigma = \frac{1}{\Gamma(1-\alpha)}\frac{1}{\tau^\alpha}\frac{1}{1-\alpha}$ and $w_j = (j+1/2)^{1-\alpha} - (j-1/2)^{1-\alpha}$, finally we have obtained the following approximation;

$$\frac{\partial^\alpha U(t_{n+\frac{1}{2}},x_i)}{\partial t^\alpha} = \sigma\left[\sum_{j=1}^{n} w_j(U_i^{n-j+1} - U_i^{n-j}) + \frac{(U_i^{n+1} - U_i^n)}{2^{1-\alpha}}\right] + O(\tau^2). \qquad (3.3)$$

On the other hand, we have

$$\frac{\partial^2 U(t_{n+\frac{1}{2}},x_i)}{\partial x^2} = \frac{1}{2}\left[\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} + \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}\right] + O(h^2). \quad (3.4)$$

### 3.1.1. Crank-Nicholson Difference Scheme

Using the approximations at (3.3) and (3.4), we obtain the following difference scheme which is accurate of order $O(\tau^2 + h^2)$;

$$\begin{cases} \sigma\left[\sum_{j=1}^{n} w_j(U_i^{n-j+1}-U_i^{n-j})+\frac{(U_i^{n+1}-U_i^n)}{2^{1-\alpha}}\right]-\left[\frac{U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}}{2h^2}+\frac{U_{i+1}^{n}-2U_i^{n}+U_{i-1}^{n}}{2h^2}\right] \\ =f(t_n+\frac{\tau}{2},x_i),\quad 0\le n\le N-1,\ 1\le i\le M-1, \\ \\ U_i^0=r(x_i),\ 1\le i\le M-1, \\ U_0^n=0,\quad U_M^n=0,\ 0\le n\le N. \end{cases} \tag{3.5}$$

We can arrange the system above, to obtain

$$\begin{cases} (-\frac{1}{2h^2})(U_{i+1}^{n+1}+U_{i+1}^n) \\ +\sigma\left[\sum_{j=1}^{n} w_j(U_i^{n-j+1}-U_i^{n-j})+\frac{(U_i^{n+1}-U_i^n)}{2^{1-\alpha}}\right]+\frac{U_i^{n+1}+U_i^n}{h^2} \\ +(-\frac{1}{2h^2})(U_{i-1}^{n+1}+U_{i-1}^n)=f(t_n+\frac{\tau}{2},x_i),\quad 0\le n\le N-1,\ 1\le i\le M \\ U_i^0=r(x_i),\ 1\le i\le M, \\ U_0^n=0,\quad U_M^n=0,\ 0\le n\le N. \end{cases}$$

The difference scheme above can be written in matrix form,

$$AU_{i+1}+BU_i+AU_{i-1}=\varphi_i \tag{3.6}$$

where $\varphi_i=[\varphi_i^0,\varphi_i^1,\varphi_i^2,...,\varphi_i^N]^T$, $\varphi_i^0=r(x_i)$, $\varphi_i^n=f(t_{n+1/2},x_i)$, $1\le n\le N$, $1\le i\le M$, and $U_i=\left[U_i^0,U_i^1,U_i^2,...,U_i^N\right]^T$.

Here $A_{(N+1)\times(N+1)}$ and $B_{(N+1)\times(N+1)}$ matrices

$$A = (-\tfrac{1}{2h^2})\begin{bmatrix} 0 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & & & \\ z_0 & b_0 & & & \\ -\sigma w_1 & b_1 & b_0 & & \\ \vdots & & \ddots & \ddots & \\ -\sigma w_{N-1} & b_{N-1} & ... & b_1 & b_0 \end{bmatrix}$$

where

$b_0 = \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2}, \quad z_0 = -\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2},$

$b_1 = \sigma w_1 - \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2},$

$b_i = \sigma(-w_{i-1} + w_i), 2 <= i <= N-1,$

we note that the unspecified entries are zero at the matrices above.

Using the idea on the modified Gauss-Elimination method, we can convert the equation (3.6) into the following form;

$$U_i = \alpha_{i+1} U_{i+1} + \beta_{i+1}, \quad i = M-1, ..., 2, 1, 0. \tag{3.7}$$

This way, the two-step form of difference schemes in (3.6) is transformed to one-step method as in (3.7).

Now, we need to determine the matrices $\alpha_{i+1}$ and $\beta_{i+1}$ satisfying the last equality. Since $U_0 = \alpha_1 U_1 + \beta_1 = 0$, we can select $\alpha_1 = O_{(N+1)\times(N+1)}$ and $\beta_1 = O_{(N+1)\times 1}$. Combining the equalities $U_i = \alpha_{i+1} U_{i+1} + \beta_{i+1}$ , and $U_{i-1} = \alpha_i U_i + \beta_i$ and the matrix equation (3.6), we have

$$(A + B\alpha_{i+1} + A\alpha_i\alpha_{i+1})\,U_{i+1} + (B\beta_{i+1} + A\alpha_i\beta_{i+1} + A\beta_i) = \varphi_i.$$

Then, we write

$$\begin{cases} A + B\alpha_{i+1} + A\alpha_i\alpha_{i+1} = 0 \\ B\beta_{i+1} + A\alpha_i\beta_{i+1} + A\beta_i = \varphi_i, \end{cases}$$

where $1 \le i \le M-1$.

So, we obtain the following pair of formulas:

$$\begin{cases} \alpha_{i+1} = -(B + A\alpha_i)^{-1}A, \\ \beta_{i+1} = (B + A\alpha_i)^{-1}(\varphi_i - A\beta_i) \end{cases}$$

where $1 \le i \le M-1$.

### 3.2. Stability of The Method

The stability analysis is done by using the analysis of the eigenvalues of the iteration matrix $\alpha_i$ $(0 \le i \le M)$ of the scheme.

Let $\rho(A)$ denote the spectral radius of a matrix $A$, i.e. the maximum of the absolute value of the eigenvalues of the matrix $A$.

We will prove that $\rho(\alpha_i) < 1$, $(1 \le i \le M)$, by induction.

Since $\alpha_1$ is a zero matrix $\rho(\alpha_1) = 0 < 1$.

Moreover, $\alpha_2 = -B^{-1}A$, $\rho(\alpha_2) = \rho(-B^{-1}A) = \dfrac{-1}{\frac{1}{h^2} + \frac{\sigma}{2^{1-\alpha}}}.\dfrac{-1}{2h^2} = \dfrac{\frac{1}{h^2}}{2\left(\frac{1}{h^2} + \frac{\sigma}{2^{1-\alpha}}\right)}$, since $\alpha_1$ is of the form

$$\alpha_1 = \begin{bmatrix} 0 & & & & \\ * & \frac{\frac{1}{h^2}}{2\left(\frac{1}{h^2}+\frac{\sigma}{2^{1-\alpha}}\right)} & & & \\ * & * & \frac{\frac{1}{h^2}}{2\left(\frac{1}{h^2}+\frac{\sigma}{2^{1-\alpha}}\right)} & & \\ & & & \ddots & \\ * & * & * & & \frac{\frac{1}{h^2}}{2\left(\frac{1}{h^2}+\frac{\sigma}{2^{1-\alpha}}\right)} \end{bmatrix}.$$

Therefore $\rho(\alpha_2) < 1$.

Now, assume $\rho(\alpha_i) < 1$.

After some calculations we find that

$$\alpha_{i+1} = -(B + A\alpha_i)^{-1}A$$
$$= \left(\frac{1}{2h^2}\right)\begin{bmatrix} 0 & & & & \\ * & \frac{1}{B_{2,2}-\frac{1}{2h^2}\alpha_{i_{2,2}}} & & & \\ * & * & \frac{1}{B_{3,3}-\frac{1}{2h^2}\alpha_{i_{3,3}}} & & \\ * & * & * & \ddots & \\ * & * & * & & \frac{1}{B_{N+1,N+1}-\frac{1}{2h^2}\alpha_{i_{N+1,N+1}}} \end{bmatrix},$$

and we already know that $B_{j,j} = \dfrac{1}{h^2} + \dfrac{\sigma}{2^{1-\alpha}}$ and $\alpha_{i_{j,j}} = \rho(\alpha_i)$ for $2 \le j \le N+1$.

$\rho(\alpha_{i+1}) = \left| \dfrac{\left(\frac{1}{2h^2}\right)}{\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} - \frac{1}{2h^2}\rho(\alpha_i)} \right| = \dfrac{M^2}{2\left[M^2\left(1 - \frac{\rho(\alpha_i)}{2}\right) + \frac{\sigma}{2^{1-\alpha}}\right]}$. And since $0 \le \rho(\alpha_i) < 1$, it follows that $\rho(\alpha_{i+1}) < 1$. So $\rho(\alpha_i) < 1$ for any $i$, where $1 \le i \le M$.

### 3.3. Numerical Analysis

**Example 1:**

$$\begin{cases} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} + x\left(\frac{16t^{5/2}x^3}{5\sqrt{\pi}} - \frac{16t^{5/2}x^2}{5\sqrt{\pi}} - 12\,xt^3 - 12\,x + 6\,t^3 + 6\right), \\ (0 < x < 1, 0 < t < 1), \\ u(0,x) = (x-1)x^3,\ 0 \le x \le 1, \\ u(t,0) = 0,\ \ u(t,1) = 0,\ \ 0 \le t \le 1. \end{cases}$$

Exact solution of this problem is $U(t,x) = (x-1)x^3(t^3+1)$. The solution by the Crank-Nicholson scheme is given in Figure **3.1** . The errors when solving this problem are listed in the Table **3.1** for various values of time and space nodes. The errors in the table is calculated by the formula $\max\limits_{\substack{0\le n\le M \\ 0\le k\le N}} \left|u(t_k,x_n) - U_n^k\right|$

**Table 3.1** . The errors for some values of $M$ and $N$.

| $M$ **# space nodes** | $N$ **# time nodes** | *error* | **Error rate** |
|---|---|---|---|
| 4 | 4 | 0.02672030654000000024 | — |
| 8 | 8 | 0.00766748429999999993 | 3.48 |
| 16 | 16 | 0.0019791574999999995 | 3.87 |
| 32 | 32 | 0.00049852810000000063 | 3.97 |
| 64 | 64 | 0.00012628359999999989 | 3.96 |
| 128 | 128 | 0.00003205619999999913 | 3.93 |
| 256 | 256 | 0.00000820420000000004 | 3.91 |

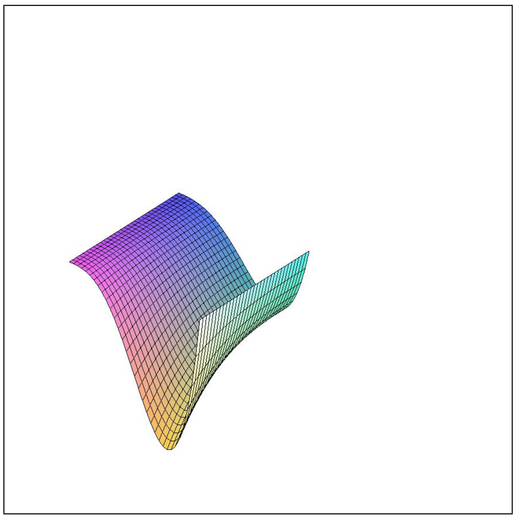

**Figure 3.1** . The solutions by the proposed method when $N$=32, $M$=32.

**Example 2:**

$$\begin{cases} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial x^2} + x\left(\frac{8t^{3/2}}{3\sqrt{\pi}} - \frac{8t^{3/2}x}{3\sqrt{\pi}}\right) + 6 + 2t^{2}, \\ (0 < x < 1, 0 < t < 1), \\ u(0,x) = 3x(1-x),\ 0 \leq x \leq 1, \\ u(t,0) = 0,\ \ u(t,1) = 0,\ \ 0 \leq t \leq 1. \end{cases}$$

Exact solution of this problem is $U(t,x) = (3+t^2)x(1-x)$. The solution by the Crank-Nicholson scheme is given in Figure **3.2** . The errors when solving this problem are listed in the Table ( **3.2** ) for various values of time and space nodes. The errors in the table is calculated by the formula $\max\limits_{\substack{0\leq n\leq M \\ 0\leq k\leq N}} \left|u(t_k, x_n) - U_n^k\right|$.

**Table 3.2** . The errors for some values of $M$ and $N$.

| $M$ **# space nodes** | $N$ **# time nodes** | *error* | **Error rate** |
|---|---|---|---|
| 4 | 4 | 0.00713528580000000190 | — |
| 8 | 8 | 0.00174166070000003615 | 4.10 |
| 16 | 16 | 0.00042257930000000332 | 4.11 |
| 32 | 32 | 0.00010213519999990872 | 4.15 |
| 64 | 64 | 0.00002464140000002502 | 4.15 |
| 128 | 128 | 0.00000594789999996959 | 4.13 |
| 256 | 256 | 0.00000153250000001215 | 3.89 |

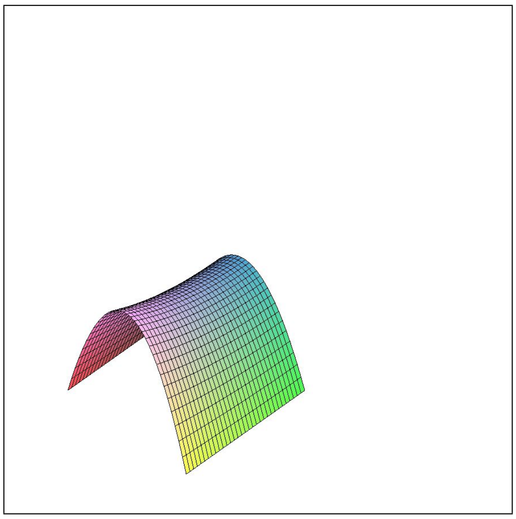

**Figure 3.2** . The solutions by the proposed method when $N$=32, $M$=32.

Consequently, in this work, the Crank-Nicholson difference scheme was successfully extended to solve the time fractional heat equations. It is proven that the time fractional Crank-Nicholson difference scheme is unconditionally stable and convergent. Numerical results are in good agreement with the theoritical results.

# CHAPTER 4

# A COMPACT DIFFERENCE SCHEME FOR THE FRACTIONAL DIFFUSION-WAVE EQUATION

## 4.1. Introduction

In recent years there has been a growing interest in the field of fractional calculus [6, 33, 34, 35, 36]. Many phenomena in engineering [37, 38], physics, chemistry and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integrals of fractional order. Fractional order partial differential equations are generalizations of classical partial differential equations. Many considerable works on the theoretical analysis [39, 40, 41, 42, 43] have been carried on, but analytic solutions of most fractional differential equations cannot be obtained explicitly. So many authors have resorted to numerical solution strategies based on convergence and stability analysis [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. Liu has carried on so many work on the finite difference method of fractional differential equations [44, 45, 46, 47, 48]. Recently, he [48] constructed a new implicit difference method for a modified anomalous sub-diffusion equation with a nonlinear source term. The stability and convergence are discussed using a new energy method. Yuste and Acedo [49, 50] presented forward-Euler scheme and weighted average finite difference methods for time fractional diffusion equation and analyzed the stability. An implicit difference scheme presented by Chen et al. [51] for the fractional diffusion equation describing sub-diffusion and the convergence and stability were obtained. Scherer et al. [52]constructed an explicit scheme and an implicit scheme for the fractional heat equation and stability, convergence and error behavior were also studied. Langlands and Henry [53] constructed the implicit finite difference using the L1 scheme to approximate the fractional derivative. The accuracy and unconditional stability were analyzed by Fourier method.

A fractional diffusion-wave equation (FDW) is a linear integro-partial differential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order $\alpha > 0$ [54], also called time-fractional diffusion-wave equation. The FDW equations can be used to model many of the universal electromagnetic, acoustic, and mechanical responses accurately [55, 56].

Consider the FDW equation [57]:

$$\frac{1}{c}\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{K} f(x,t), \;\; x \in [0, L], \;\; t > 0, \tag{4.1}$$

with the initial value conditioins

$$u(x,0) = \phi(x), \;\; \frac{\partial u(x,t)}{\partial t} = \psi(x), \;\; x \in [0, L], \tag{4.2}$$

and the boundary value conditions

$$u(0,t) = 0, \;\; u(L,t) = 0, \;\; t > 0, \tag{4.3}$$

where $c$ and $K$ are constants of dimensions $Length^2 Time^{-\alpha}$ and $Length^2$, $x \in [0, L]$ and $t > 0$ are space and time variables, $\phi(0) = \phi(L) = 0$, $u = u(x,t)$ and $f(x,t)$ are field variables, and

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\partial^2 u(x,s)}{\partial s^2}\frac{ds}{(t-s)^{\alpha-1}}, \quad \alpha \in (1,2)$$

with $\Gamma$ denoting the gamma function. When $\alpha = 1$, Eq.(4.1) represents a diffusion equation, $c$ and $f(x,t)$ are called the diffusion coefficient and the source term, respectively. When $\alpha = 2$, Eq.(4.1) represents a wave equation, $c$ and $f(x,t)$ denote the square of the wave velocity and an external force field. For $1 < \alpha < 2$ the fractional equation in Eq.(4.1) is expected to interpolate the diffusion equation and wave equation.

Let $\omega_h = \{x_i\,|x_i = ih,\, 0 \le i \le M\}$ is a uniform mesh of the interval $[0, L]$ with $h = L/M$. Let $\omega_\tau = \{t_n\,|n \ge 0\}$, where $t_n = n\tau$, $\tau > 0$. Suppose $u = \{u_i^n\,|0 \le i \le M,\, n \ge 0\}$ is a grid function on $\omega_{h\tau} = \omega_h \times \omega_\tau$. Introduce the following notations:

$$u_i^{n-\frac{1}{2}} = \frac{1}{2}\left(u_i^n + u_i^{n-1}\right), \; \delta_t u_i^{n-\frac{1}{2}} = \frac{1}{\tau}(u_i^n - u_i^{n-1}),$$

$$\delta_x u_{i-\frac{1}{2}}^n = \frac{u_i^n - u_{i-1}^n}{h}, \quad \delta_x^2 u_i^n = \frac{1}{h}(\,\delta_x u_{i+\frac{1}{2}}^n \; - \delta_x u_{i-\frac{1}{2}}^n).$$

Sun and Wu [58] proposed, by introducing two new variables to transform the original equation into a low order system of equations, the following fully discrete difference scheme

$$\frac{1}{c\Gamma(2-\alpha)}\frac{1}{\tau}\mathcal{L}(\delta_t u_i^{n-\frac{1}{2}},\psi_i)=\delta_x^2 u_i^{n-\frac{1}{2}}+\frac{1}{K}f_i^{n-\frac{1}{2}},\ \ 1\le i\le M-1,\, n\ge 1, \tag{4.4}$$

$$u_i^0=\phi_i,\ \ o\le i\le M, \tag{4.5}$$

$$u_0^n\ =0,\ \ u_M^n=0,\ \ n\ge 1, \tag{4.6}$$

where

$$\mathcal{L}(u_i^{n-\frac{1}{2}},q_i)=a_0u_i^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})u_i^{k-\frac{1}{2}}-a_{n-1}q_i, \tag{4.7}$$

$$a_l\equiv\int\limits_{t_l}^{t_{l+1}}\frac{dt}{t^{\alpha-1}}=\frac{1}{2-\alpha}[(t_{l+1})^{2-\alpha}-(t_l)^{2-\alpha}] \tag{4.8}$$

$$=\frac{\tau^{2-\alpha}}{2-\alpha}[(l+1)^{2-\alpha}-(l)^{2-\alpha}],\ \ l\ge 0. \tag{4.9}$$

and $f_i^{n-\frac{1}{2}}=f(x_i,t_{n-\frac{1}{2}\tau})$, $\psi_i=\psi(x_i)$, $\phi_i=\phi(x_i)$. The solvability unconditional stability and $L_\infty$ convergence were proved by the energy method. And the convergence order is $O(\tau^{3-\alpha}+h^2)$.

In this article, we will present the compact difference scheme (4.1)-(4.3)for as follows

$$\frac{1}{12c\Gamma(2-\alpha)}\frac{1}{\tau}[\mathcal{L}(\delta_t u_{i-1}^{n-\frac{1}{2}},\psi_{i-1})]+10\mathcal{L}(\delta_t u_i^{n-\frac{1}{2}},\psi_i)+\mathcal{L}(\delta_t u_{i+1}^{n-\frac{1}{2}},\psi_{i+1}) \tag{4.10}$$

$$=\delta_x^2 u_i^{n-\frac{1}{2}}+\frac{1}{12K}(f_{i-1}^{n-\frac{1}{2}}+f_i^{n-\frac{1}{2}}+f_{i+1}^{n-\frac{1}{2}}),\ \ 1\le i\le M-1,\ \ n\ge 1, \tag{4.11}$$

$$u_i^0=\phi_i,\ \ o\le i\le M, \tag{4.12}$$

$$u_0^n\ =0,\ \ u_M^n=0,\ \ n\ge 1, \tag{4.13}$$

then prove that the difference scheme is uniquely solvable, unconditionally stable and convergent in $L_\infty$ norm. The convergence order can reach $O(\tau^{3-\alpha}+h^4)$.

## 4.2. The derivation of the difference scheme

For the derivation of the difference scheme, we need the lemmas.

*Lemma* 4.1. For $n\ge 1$ and $t_k=k\tau$, $0\le k\le n$, we have

$$0\le\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left\{(t_n-t)^{2-\alpha}-\left[\frac{t-t_{k-1}}{\tau}(t_n-t_k)^{2-\alpha}+\frac{t_k-t}{\tau}(t_n-t_{k-1})^{2-\alpha}\right]\right\}dt$$

$$\le\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-(1+2^{1-\alpha})\right]\tau^{3-\alpha}.$$

*Proof.* Let $g(t) = (t_n - t)^{2-\alpha}$, then

$$\begin{aligned} g(t) - \left[\frac{t - t_{k-1}}{\tau} g(t_k) + \frac{t_k - t}{\tau} g(t_{k-1})\right] &= \frac{1}{2} g''(\xi_k)(t - t_k)(t - t_{k-1}) \\ &\leq \frac{1}{2}(2-\alpha)(\alpha-1)(t_n - \xi_k)^{-\alpha}(t_k - t)(t - t_{k-1}) \geq 0, \end{aligned}$$

where $\xi_k \in (t_{k-1}, t_k)$, $t \in (t_{k-1}, t_k)$. From the above inequality, we have

$$\begin{aligned} &\sum_{k=1}^{n-2} \int\limits_{t_{k-1}}^{t_k} \left\{ g(t) \left[\frac{t - t_{k-1}}{\tau} g(t_k) + \frac{t_k - t}{\tau} g(t_{k-1})\right] \right\} dt \\ &= \sum_{k=1}^{n-2} \int\limits_{t_{k-1}}^{t_k} \frac{1}{2}(2-\alpha)(\alpha-1)(t_n - \xi_k)^{-\alpha}(t_k - t)(t - t_{k-1}) dt \\ &\leq \frac{1}{2}(2-\alpha)(\alpha-1) \sum_{k=1}^{n-2} (t_n - t_k)^{-\alpha} \int\limits_{t_{k-1}}^{t_k} (t_k - t)(t - t_{k-1}) dt. \end{aligned}$$

Since $\int\limits_{t_{k-1}}^{t_k} (t_k - t)(t - t_{k-1}) dt = \tau^3/6$, we obtain

$$\begin{align} &\sum_{k=1}^{n-2} \int\limits_{t_{k-1}}^{t_k} \left\{ g(t) \left[\frac{t - t_{k-1}}{\tau} g(t_k) + \frac{t_k - t}{\tau} g(t_{k-1})\right] \right\} dt \tag{4.14} \\ &\leq \frac{1}{12}(2-\alpha)(\alpha-1)\tau^3 \sum_{k=1}^{n-2} (t_n - t_k)^{-\alpha} \leq \frac{1}{12}(2-\alpha)(\alpha-1)\tau^2 \int\limits_{t_1}^{t_{n-1}} (t_n - t)^{-\alpha} dt \notag \\ &= \frac{1}{2}(2-\alpha)\tau^2 \left[(t_n - t_{n-1})^{1-\alpha} - (t_n - t_1)^{1-\alpha}\right] \leq \frac{2-\alpha}{12}\tau^{3-\alpha}. \tag{4.15} \end{align}$$

On the other hand,

$$\begin{aligned}
&\sum_{k=1}^{n-2}\int_{t_{k-1}}^{t_k}\left\{g(t)-\left[\frac{t-t_{k-1}}{\tau}g(t_k)+\frac{t_k-t}{\tau}g(t_{k-1})\right]\right\}dt \qquad (4.16)\\
&=\int_{t_1}^{t_{n-1}}g(t)dt-\left[\frac{1}{2}g(t_{n-2})+g(t_{n-1})\right]\tau\\
&=\int_{t_{n-2}}^{t_n}(t_n-t)^{2-\alpha}dt-\left[\frac{1}{2}(t_n-t_{n-2})^{2-\alpha}+(t_n-t_{n-1})^{2-\alpha}\right]\tau\\
&=\left[\frac{2^{3-\alpha}}{3-\alpha}-(1+2^{1-\alpha})\right]\tau^{3-\alpha}. \qquad (4.17)
\end{aligned}$$

The lemma follows from (4.14)and (4.16):

$$\begin{aligned}
&\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left\{g(t)\left[\frac{t-t_{k-1}}{\tau}g(t_k)+\frac{t_k-t}{\tau}g(t_{k-1})\right]\right\}dt\\
&=\left(\sum_{k=1}^{n-2}+\sum_{k=n-2}^{n}\right)\int_{t_{k-1}}^{t_k}\left\{g(t)\left[\frac{t-t_{k-1}}{\tau}g(t_k)+\frac{t_k-t}{\tau}g(t_{k-1})\right]\right\}dt\\
&\le\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-(1+2^{1-\alpha})\right]\tau^{3-\alpha}.
\end{aligned}$$

□

*Lemma* 4.2. Suppose $g(t)\in C^2\left[0,t_n\right]$. Then

$$\begin{aligned}
&\left|\int_0^{t_n}g'(t)\frac{dt}{(t_n-t)^{\alpha-1}}-\sum_{k=1}^{n}\frac{g(t_k)-g(t_{k-1})}{\tau}\int_{t_{k-1}}^{t_k}\frac{dt}{(t_n-t)^{\alpha-1}}\right|\\
&\le\frac{1}{2-\alpha}\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-(1+2^{1-\alpha})\right]\max_{0\le t\le t_n}\left|g''(t)\right|\tau^{3-\alpha.}
\end{aligned}$$

*Proof.* For simplicity, denote

$$A \equiv \int_{0}^{t_n} g'(t)\frac{dt}{(t_n-t)^{\alpha-1}} - \sum_{k=1}^{n} \frac{g(t_k)-g(t_{k-1})}{\tau} \int_{t_{k-1}}^{t_k} \frac{dt}{(t_n-t)^{\alpha-1}}$$

Using Taylor expansion with integral remainder, we have

$$g'(t) - \frac{g(t_k)-g(t_{k-1})}{\tau} = \frac{1}{\tau}\left[\int_{t_{k-1}}^{t} g''(s)(s-t_{k-1})ds - \int_{t}^{t_k} g''(s)(t_k-s)ds\right],$$

which yields

$$\begin{aligned} A &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t} \left[g'(t) - \frac{g(t_k)-g(t_{k-1})}{\tau}\right] \frac{dt}{(t_n-t)^{\alpha-1}} \\ &= \frac{1}{\tau}\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left[\int_{t_{k-1}}^{t} g''(s)(s-t_{k-1})ds - \int_{t}^{t_k} g''(s)(t_k-s)ds\right] \frac{dt}{(t_n-t)^{\alpha-1}}. \end{aligned}$$

Exchanging the order of integration, we get

$$A = \frac{1}{2-\alpha}\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left\{(t_n-s)^{2-\alpha} - \left[\frac{s-t_{k-1}}{\tau}(t_n-t_k)^{2-\alpha} + \frac{t_k-s}{\tau}(t_n-t_{k-1})^{2-\alpha}\right]\right\} g''(s)ds.$$

Applying Lemma 4.1, we obtain the result:

$$\begin{aligned} |A| &\le \frac{1}{2-\alpha}\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left\{(t_n-s)^{2-\alpha} - \left[\frac{s-t_{k-1}}{\tau}(t_n-t_k)^{2-\alpha} + \frac{t_k-s}{\tau}(t_n-t_{k-1})^{2-\alpha}\right]\right\} \left|g''(s)\right| ds \\ &\le \frac{1}{2-\alpha} \max_{0\le t\le t_n} \left|g''(t)\right| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left\{(t_n-s)^{2-\alpha} - \left[\frac{s-t_{k-1}}{\tau}(t_n-t_k)^{2-\alpha} + \frac{t_k-s}{\tau}(t_n-t_{k-1})^{2-\alpha}\right]\right\} d \\ &\le \frac{1}{2-\alpha}\left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1+2^{1-\alpha})\right] \max_{0\le t\le t_n} \left|g''(t)\right| \tau^{3-\alpha.} \end{aligned}$$

$\square$

*Lemma* 4.3. Suppose $g(t) \in C^2\left[0, t_n\right]$. Then

$$\left| \int_0^{t_n} g'(t) \frac{dt}{(t_n - t)^{\alpha-1}} - \frac{1}{\tau}\left[ a_0 g(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) g(t_k) - a_{n-1} g(t_0) \right] \right|$$
$$\leq \frac{1}{2-\alpha}\left[ \frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1 + 2^{1-\alpha}) \right] \max_{0 \leq t \leq t_n} |g''(t)|\, \tau^{3-\alpha},$$

where $a_l$ is defined in (4.8) and $1 < \alpha < 2$.

*Proof.* Observing Lemma 4.1, it suffices to verify

$$\sum_{k=1}^{n} \frac{g(t_k) - g(t_{k-1})}{\tau} \int_{t_{k-1}}^{t_k} \frac{dt}{(t_n - t)^{\alpha-1}} = \frac{1}{\tau}\left[ a_0 g(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) g(t_k) - a_{n-1} g(t_0) \right].$$

In fact,

$$\sum_{k=1}^{n} \frac{g(t_k) - g(t_{k-1})}{\tau} \int_{t_{k-1}}^{t_k} \frac{dt}{(t_n - t)^{\alpha-1}}$$
$$= \sum_{k=1}^{n} \frac{g(t_k) - g(t_{k-1})}{\tau} \frac{1}{2-\alpha}\left[ (t_{n-k+1})^{2-\alpha} - (t_{n-k})^{2-\alpha} \right] = \frac{1}{\tau} \sum_{k=1}^{n} a_{n-k} (g(t_k) - g(t_{k-1}))$$
$$= \frac{1}{\tau}\left[ a_0 g(t_n) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) g(t_k) - a_{n-1} g(t_0) \right].$$

This completes the proof. □

*Lemma* 4.4. [59]. Suppose $p(x) \in C^6[0, L]$ and $x_i = ih$, $0 \leq i \leq M$. Then

$$\frac{1}{12}\left[ p''(x_{i+1}) + 10p''(x_i) + p''(x_{i-1}) \right] - \frac{1}{h^2}\left[ p(x_{i+1}) - 2p(x_i) + p(x_{i-1}) \right]$$
$$= \frac{h^4}{360} \int_0^1 \left[ p^{(6)}(x_i + sh) + p^{(6)}(x_i - sh) \right] (1-s)^3 \left[ 5 - 3(1-s)^2 \right] ds$$
$$= \frac{h^4}{240} p^{(6)}(\xi_i), \quad \xi_i \in (x_{i-1}, x_{i+1}), \quad 1 \leq i \leq M-1.$$

Let

$$\vartheta = \{g\,|(g_0, g_1, ..., g_M) \text{ being a grid function on } \omega_h\}\,.$$

Define the average operator $P : \vartheta \to \vartheta$ as follows

$$(Pg)_i = \begin{cases} g_0, & i = 0, \\ \frac{1}{12}(g_{i-1} + 10g_i + g_{i+1),} & 1 \le i \le M-1, \\ g_M, & i = M. \end{cases}$$

It is obvious that the operator satisfies $P\mathcal{L}(u_i^{n-\frac{1}{2}}, q_i) = \mathcal{L}(Pu_i^{n-\frac{1}{2}}, Pq_i)$. Using the operator $P$, the difference scheme (4.10)-(4.13) can be written as

$$\frac{1}{c\Gamma(2-\alpha)}\frac{1}{\tau}P\mathcal{L}(\delta_t u_i^{n-\frac{1}{2}}, \psi_i) = \delta_x u_i^{n-\frac{1}{2}} + \frac{1}{K}Pf_i^{n-\frac{1}{2}}, \;\; 1 \le i \le M-1, \;\; n \ge 1, \quad (4.18)$$

$$u_i^0 = \phi_i, \;\; o \le i \le M, \quad (4.19)$$

$$u_0^n \; = 0, \;\; u_M^n = 0, \;\; n \ge 1. \quad (4.20)$$

Observing the Crank-Nicholson scheme (4.4) and the compact difference scheme (4.18), we find that the difference between these two difference schemes is only the operator $P$. If we ignore the operator $P$ in (4.18), then (4.18) is just the same as (4.4). It is the existence of the operator $P$ in (4.18) that makes the difference scheme (4.18) have a higher accuracy than the difference scheme (4.4).

We will use Lemma 4.3 to discretize the time fractional derivative and apply Lemma 4.4 to deal with the spatial derivative.

Let

$$v(x,t) = \frac{\partial u(x,t)}{\partial t}, \tag{4.21}$$

and

$$w(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial v(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha-1}}. \tag{4.22}$$

Then (4.1) becomes

$$\frac{1}{c} w(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{K} f(x,t), \;\; x[0,L], \;\; t > 0. \tag{4.23}$$

Considering (4.23) at $(x_i, t)$, we have

$$\frac{1}{c}w(x_i,t) = \frac{\partial^2 u(x_i,t)}{\partial x^2} + \frac{1}{K}f(x_i,t), \ \ 1 \le i \le M-1. \tag{4.24}$$

Acting $P$ on the both sides of (4.24), we obtain

$$\frac{1}{c}Pw(x_i,t) = P\frac{\partial^2 u(x_i,t)}{\partial x^2} + \frac{1}{K}Pf(x_i,t), \ \ 1 \le i \le M-1. \tag{4.25}$$

According to Lemma 4.4, we have

$$P\frac{\partial^2 u(x_i,t)}{\partial x^2} = \frac{1}{h^2}[u(x_{i-1},t) - 2u(x_i,t) + u(x_{i+1},t)] + O(h^4)$$

Define the grid functions

$$U_i^n = u(x_i,t_n), \quad V_i^n = v(x_i,t_n), \quad W_i^n = w(x_i,t_n), \quad 0 \le i \le M, \ \ n \ge 0,$$

and denote

$$U_i^{n-\frac{1}{2}} = \frac{1}{2}(U_i^n + U_i^{n-1}), \quad V_i^{n-\frac{1}{2}} = \frac{1}{2}(V_i^n + V_i^{n-1}), \ \ W_i^{n-\frac{1}{2}} = \frac{1}{2}(W_i^n + W_i^{n-1}).$$

Using Taylor expansion, it follows from (4.21) and (4.25) that

$$V_i^{n-\frac{1}{2}} = \delta_t U_i^{n-\frac{1}{2}} + (r_1)_i^{n-\frac{1}{2}}, \tag{4.26}$$

$$\frac{1}{c} P W_i^{n-\frac{1}{2}} = \delta_x^2 U_i^{n-\frac{1}{2}} + \frac{1}{K} P f_i^{n-\frac{1}{2}} + (r_2)_i^{n-\frac{1}{2}}, \tag{4.27}$$

and there exists a constant $c_1$ such that

$$\left|(r_1)_i^{n-\frac{1}{2}}\right| \le c_1 \tau^2, \quad \left|(r_2)_i^{n-\frac{1}{2}}\right| \le c_1(\tau^2 + h^4). \tag{4.28}$$

Based on Lemma 4.3 and $V_i^0 = v(x_i, 0) = \psi(x_i) = \psi_i$, we have

$$W_i^n = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial v(x_i, t)}{\partial t} \frac{dt}{(t_n - t)^{\alpha-1}} = \frac{1}{\Gamma(2-\alpha)} \frac{1}{\tau} \mathcal{L}(V_i^n, \psi_i) + O(\tau^{3-\alpha}), \quad n \ge 0.$$

Then

$$W_i^{n-\frac{1}{2}} = \frac{1}{2}(W_i^n + W_i^{n-1}) = \frac{1}{\Gamma(2-\alpha)} \frac{1}{\tau} \mathcal{L}(V_i^{n-\frac{1}{2}}, \psi_i) + O(\tau^{3-\alpha}).$$

Consequently

$$PW_i^{n-\frac{1}{2}} = \frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}P\mathcal{L}(V_i^{n-\frac{1}{2}}, \psi_i) + (r_3)_i^{n-\frac{1}{2}} \tag{4.29}$$

and there exists a constant $c_2$ such that

$$\left|(r_3)_i^{n-\frac{1}{2}}\right| \leq c_2\tau^{3-\alpha}. \tag{4.30}$$

Substituting (4.26) into (4.29), we have

$$PW_i^{n-\frac{1}{2}} = \frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}P\mathcal{L}(\delta_t U_i^{n-\frac{1}{2}}, \psi_i) + \frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}P\mathcal{L}((r_1)_i^{n-\frac{1}{2}}, 0) + (r_3)_i^{n-\frac{1}{2}}.$$

Then substituting above result into (4.27), we obtain

$$\frac{1}{c}\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}P\mathcal{L}(\delta_t U_i^{n-\frac{1}{2}}, \psi_i) = \delta_x^2 U_i^{n-\frac{1}{2}} + \frac{1}{K}Pf_i^{n-\frac{1}{2}} + R_i^{n-\frac{1}{2}}, \quad 1 \leq i \leq M-1, \ \ n \geq 1 \tag{4.31}$$

where

$$R_i^{n-\frac{1}{2}} = -\frac{1}{c}\left\{\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}P\mathcal{L}((r_1)_i^{n-\frac{1}{2}},0) + (r_3)_i^{n-\frac{1}{2}}\right\} + (r_2)_i^{n-\frac{1}{2}}.$$

It is easy to know that

$$a_0 = \frac{\tau^{2-\alpha}}{2-\alpha}, \quad a_l > a_{l+1}, \quad l \geq 0, \tag{4.32}$$

and

$$\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k}) = a_0 - a_{n-1}, \;\; n \geq 2. \tag{4.33}$$

According to (4.32), (4.33), (4.28) and (4.30), we have

$$\left|R_i^{n-\frac{1}{2}}\right| \leq \frac{1}{c}\left\{\frac{2c_1}{(2-\alpha)\Gamma(2-\alpha)} + c_2 + cc_1\right\} + (h^4 + \tau^{3-\alpha}). \tag{4.34}$$

In addition, from (4.2) and (4.3), we have

$$U_i^0 = \phi(x_i), \;\; 0 \leq i \leq M, \tag{4.35}$$

and

$$U_0^n = 0,\ \ U_M^n = 0,\ \ n \geq 1. \tag{4.36}$$

Observing (4.31), (4.35) and (4.36) and omitting the small term $R_i^{n-\frac{1}{2}}$, it is natural to construct the difference scheme (4.18)-(4.20) for the problem (4.1)-(4.3).

**Example :** [61]. We consider the following problem

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + 4\sqrt{t}\frac{\sin(\pi x)}{\Gamma(0.5)} + t^2 \sin(\pi x)\pi^2, \quad (0 < x < 1,\ \ 0 < t \leq 1), \\ u(x,0) = 0,\ \ \frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = 0,\ \ 0 \leq x \leq 1, \\ u(0,t) = 0,\ \ u(1,t) = 0,\ \ 0 < t \leq 1. \end{cases}$$

Exact solution of the above problem is $u(x,t) = t^2 \sin(\pi x)$. According to this example, for $\alpha = 3/2$, the compact difference scheme as follows

$$\begin{aligned} &\frac{1}{12\Gamma(0.5)}\frac{1}{\tau}[\mathcal{L}(\delta_t u_{i-1}^{n-\frac{1}{2}}, 0)] + 10\mathcal{L}(\delta_t u_i^{n-\frac{1}{2}}, 0) + \mathcal{L}(\delta_t u_{i+1}^{n-\frac{1}{2}}, 0) \\ &= \delta_x^2 u_i^{n-\frac{1}{2}} + \frac{1}{12}(f_{i-1}^{n-\frac{1}{2}} + f_i^{n-\frac{1}{2}} + f_{i+1}^{n-\frac{1}{2}}),\ \ 1 \leq i \leq M-1,\ \ n \geq 1, \end{aligned}$$

$$u_i^0 = 0, \quad 0 \leq i \leq M,$$

$$u_0^n = 0, \quad u_M^n = 0, \quad n \geq 1.$$

where

$$f(x_i, t_{n-\frac{1}{2}}) = 4\sqrt{t_{n-\frac{1}{2}}}\frac{\sin \pi x_i}{\Gamma(0.5)} + t_{n-\frac{1}{2}}^2 \sin(\pi x_i)\pi^2$$

and the difference scheme is uniquely solvable, unconditionally stable and convergent in $L_\infty$ norm. The convergence order can reach $O(\tau^{3-\alpha} + h^4)$

# CHAPTER 5

# CONCLUSION

In this work, we mentioned about history of fractional calculus and some basic definitions to understand some terms easily. And the Crank-Nicholson difference scheme was successfully extended to solve the time fractional heat equations. It is proven that the time fractional Crank-Nicholson difference scheme is unconditionally stable and convergent. Numerical results are in good agreement with the theoritical results. Also we continued to explain with a journal.

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# APPENDIX

```
restart;
with(plots):
zalfa:=0.5:
caputo:=proc(f,salfa)
local tau,s,ss;
ss:=diff(f,t);
s:=subs(t=tau,ss);
( 1/GAMMA( 1-salfa) )* int( s / ((t-tau) salfa),tau=0..t):
end proc:
u:=(t,x)-¿ (3+t 2)*x*(1-x):
utx:= u(t,x);
utx2:=u(t,x);
P1:= subs(t=0,utx): rrr := unapply(P1,x);
P11:= subs(t=1,utx): rrr2 := unapply(P11,x);
```

```
Q:=diff(utx2,t); P2:= subs(t=0,Q);

sss:= unapply(P2,x);

P3:= subs(x=1,utx); saguc:=unapply(P3,t);

P4:= subs(x=0,utx); soluc:=unapply(P4,t);

ro:= x -¿ u(0,x)-u(1,x): rox:=ro(x);

P5:=caputo(utx,zalfa)-diff(utx,x$2):

f:=unapply(P5,t,x);

saguc(Z);

soluc(Z);

rrr(Z);

sss(Z);

f(t,x);

rrr2(Z);

N:=25: M:=10:

tau:= evalf(1/N): h:=evalf(1/M): v := evalf(-1/(2*h 2)):

A:=Matrix(N+1,N+1):

for i from 2 to N+1 do
```

```
A[[i],[i]]:=v;
A[[i],[i-1]]:=v;
end do :
A;
C:=A:
sgm:=evalf((1/GAMMA( 1-zalfa))*(1/( 1-zalfa))*(1/(tau) zalfa)):
for j from 1 to N do
w[j]:=evalf( (j+1/2) (1-zalfa)- (j-1/2) (1-zalfa) ):
end do:
B:=Matrix(N+1,N+1):
B[[1],[1]]:=1:
B[[2],[1]]:= evalf( 1/h 2 - sgm/(2 (1-zalfa))):
for i from 3 to N+1 do
B[[i],[1]]:=evalf(-sgm*w[i-2]):
end do:
for i from 2 to N+1 do
B[[i],[i]]:=evalf(sgm/(2 (1-zalfa))+1/h 2):
```

```
end do:
for i from 3 to N+1 do
B[[i],[i-1]]:=evalf(sgm*w[1]-sgm/(2 (1-zalfa))+1/h 2):
end do:
for i from 2 to N+1 do
for j from 2 to N do
if (i+j)¡=N+1 then
B[[i+j],[j]]:=evalf( sgm*(w[i]-w[i-1]) ):
fi;
end do:
end do:
B;
Dm:=Matrix(N+1,shape=identity):
alpha[1] := Matrix(N+1) ;
betha[1] := Matrix(N+1,1) ;
for k from 1 to N+1 do
betha[1,k]:=soluc( (k-1)*tau );
```

```
end do:
fii:=Matrix(N+1,M):
for j from 1 to M do
xj:=(j)*h;
fii[[1],[j]]:= evalf(rrr(xj)):
for k from 0 to N-1 do
tk:= k*tau+tau/2 ; # 'to avoid zero indices';
fii[[k+2],[j]] := evalf( f(tk,xj) ):
end do;
end do:
fii[[1..N+1],[1]];
for j from 1 to M-1 do
Q:=(B+C.alpha[j]) (-1):
alpha[j+1]:= - Q . A :
betha[j+1]:= Q . (Dm . ( fii[[1..N+1],[j]])-(C . betha[j]) ):
end do:
U:=Matrix(N+1,M+1):
```

```
for j from 1 to N+1 do ;

U[[j],[M+1] ] := saguc( (j-1)*tau);

end do:

U[[1..N+1], [M+1] ]:

for z from M-1 to 1 by -1 do ;

U[[1..N+1],[z+1]] := evalf(alpha[z+1] . U[[1..N+1],[z+2]] + betha[z+1]);

end do:

U:

matrixplot(U);

musta:=Matrix(N+1,M+1):

for j from 1 to M+1 do;

for k from 1 to N+1 do;

tk:=(k-1)*tau;

xj:=(j-1)*h;

musta[[k],[j]] := evalf(u(tk,xj));

end do;

end do;
```

```
matrixplot(musta);

matrixplot((U-musta));

theerror := max(abs(U-musta));
```

.