

**CHARACTERIZATION OF TORSION UNITS
OF $U_1(\mathbb{Z} S_4)$**

by

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

In ZS_4 , any torsion unit is conjugate to one of the element in S_4 . S_4 has 5 conjugacy classes : $\overline{(1)}$, $\overline{(12)}$, $\overline{(12)(34)}$, $\overline{(123)}$, $\overline{(1234)}$

$\overline{(1)}$ is trivial, $\overline{(12)}$ and $\overline{(12)(34)}$ consist of elements of order two, $\overline{(123)}$ has elements of order 3 and $\overline{(1234)}$ has elements of order 4. In this study, all torsion units are characterized according to the conjugacy classes in S_4 in terms of parameters.

Keywords: Torsion units, parameterization of torsion units, irreducible representation.

$U_1(ZS_4)$ DEKİ SONLU MERTEBELİ BİRİMSELLERİNİN BELİRLENİŞİ

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ÖZ

ZS_4 te bulunan sınırlı mertebeli birimlerden herbiri S_4 te bir elemana eşleniktir. S_4 te beş eşlenik sınıfı vardır: $\overline{(1)}$, $\overline{(12)}$, $\overline{(12)(34)}$, $\overline{(123)}$, $\overline{(1234)}$

$\overline{(1)}$ aşık sınıf, $\overline{(12)}$ ve $\overline{(12)(34)}$ sınıfları 2 mertebeli elemanları, $\overline{(123)}$ sınıfı 3 mertebeli elemanları, $\overline{(1234)}$ sınıfı 4 mertebeli elemanları içerir. Bu çalışmada tüm sınırlı mertebeli birimler S_4 te eşlenik sınıflarına göre parametrize edilmiştir.

Anahtar Kelimeler: Sonlu mertebeli birimseller, Sonlu mertebeli birimsellerin parametrizasyonu, indirgenemez temsiller.

To my parents

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LIST OF SYMSBOLS AND ABBREVIATIONS

SYMBOLS	MEANING
RG	Group ring over commutative ring with identity
ZG	Integral group ring
QG	Rational group algebra
$U(RG)$	The unit group of group ring RG
$U_1(RG)$	Normalized units of group ring RG
$\Delta(G)$	Augmentation ideal of RG
$GL(n, Z)$	General linear group of degree n over Z
ρ	Irreducible representation of S_4
$\overline{\rho}$	Linear extension of ρ over Z
γ_e	The coefficient of identity element of a group ring element γ
$Z(G)$	The center of a group G
$ \gamma\overline{\rho} $	The determinant of the representation of γ
$tr(\gamma\overline{\rho})$	The trace of the representation of γ

CHAPTER 1

INTRODUCTION

1.1. Basic Definitions and Properties of Group Ring RG

Definition 1.1.1. Let G be group and R be a commutative ring with identity. Let us denote the set of all finite formal linear combinations by RG . Then the set

$$RG = \{ \alpha = \sum_{finite} \alpha_g g : g \in G, \alpha_g \in R \}$$

is a ring under the following addition and multiplication:

1- Addition of two elements in RG : Let $\alpha = \sum_{g \in G} \alpha_g g$ and $\beta = \sum_{g \in G} \beta_g g \in RG$ then

$$\alpha + \beta = \sum_{g \in G} \alpha_g g + \sum_{g \in G} \beta_g g = \sum_{g \in G} (\alpha_g + \beta_g) g$$

2- Multiplication of two elements in RG : Let $\alpha = \sum_{g \in G} \alpha_g g$ and $\beta = \sum_{h \in G} \beta_h h \in RG$ then

$$\begin{aligned} \alpha \cdot \beta &= \left(\sum_{g \in G} \alpha_g g \right) \left(\sum_{h \in G} \beta_h h \right) \\ &= \sum_{g, h \in G} (\alpha_g \beta_h) (gh) \\ &= \sum_{g \in G} \left(\sum_{h \in G} \alpha_g \beta_h \right) (gh) \end{aligned}$$

If we choose $g = gh^{-1}$ then, we simply denote

$$\alpha \cdot \beta = \sum_{g \in G} \left(\sum_{h \in G} \alpha_{gh^{-1}} \beta_h \right) g$$

Some Basic Properties of RG

1- Equality of two elements in RG : For any two $\alpha = \sum_{g \in G} \alpha_g g$ and $\beta = \sum_{g \in G} \beta_g g \in RG$

$$\alpha = \beta \Leftrightarrow \alpha_g = \beta_g, \forall g \in G.$$

2- The group G can be embedded into RG by the following group homomorphism:

$$\begin{aligned} i: G &\rightarrow RG \\ g &\mapsto 1_R \cdot g \end{aligned}$$

3- The ring R can be embedded into RG by the following ring homomorphism:

$$\begin{aligned} \vartheta: R &\rightarrow RG \\ r &\mapsto r \cdot e_G \end{aligned}$$

4- The ring RG has the identity. $1_{RG} = 1_R \cdot e_G$. So we can consider Unit group of RG denoted by

$$U(RG) = \{\gamma = \sum \gamma_g g \in RG : \gamma \lambda = 1_{RG}, \exists \lambda \in RG\}$$

Clearly, $U(R), G \subseteq U(RG)$ and more generally $U(R) \cdot G \subseteq U(RG)$. If $U(RG) \subseteq U(R) \cdot G$ then, we say RG has trivial units.

5- For $N < G$, let us consider following natural group homomorphism :

$$\begin{aligned} \varphi: G &\rightarrow G/N \\ g &\mapsto gN \end{aligned}$$

If we extend the group homomorphism linearly over R , we get the following natural ring homomorphism.

$$\begin{aligned} \bar{\varphi}: RG &\rightarrow R(G/N) \\ \sum \gamma_g g &\mapsto \sum \gamma_g (gN) \end{aligned}$$

6- We can define a ring homomorphism from RG to R as follows:

$$\begin{aligned} \varepsilon: RG &\rightarrow R \\ \sum \gamma_g g &\mapsto \sum \gamma_g \end{aligned}$$

The kernel of ε is an ideal of RG , denoted by $\Delta(G) = \{\gamma \in RG : \sum \gamma_g = 0\}$.

7- If we restrict this homomorphism to unit group of RG then we get a group ephomorphism :

$$\begin{aligned} \bar{\varepsilon}: U(RG) &\rightarrow U(R) \\ \sum \gamma_g g &\mapsto \sum \gamma_g \end{aligned}$$

This map is called augmentation map and the kernel of $\bar{\epsilon}$ is a normal subgroup of $U(RG)$, and denoted by $U_1(RG) = \{\gamma \in U(RG) : \sum \gamma_g = 1_R\}$ and it is usually called normalized units (or units with augmentation 1).

8- If we take $R = \mathbb{Z}$ then the group ZG is called the Integral Group Ring. $U(ZG) = \pm U_1(ZG)$ since $U(\mathbb{Z}) = \{\pm 1\}$.

Definition 1.1.2. The set of torsion units can be defined as

$$\{\gamma = \sum \gamma_g g \in RG : \gamma^m = 1_{RG}, \exists m \in \mathbb{Z}^+\}$$

It is obvious that the set of torsion units is a subset of $U(RG)$.

1.2. The Unit Problem for Integral Group Ring

Describing the units of the integral group ring is a classical and difficult problem. Over the years, it has drawn the attention of those working in the areas of algebra, number theory and algebraic topology. For a compilation of results in the unit problem, the reader is directed to the monograph by Sehgal [1]. We will use the standard notation of that text. Most descriptions of $U(ZG)$ in the mathematical literature either give an explicit description of the units, the general structure of $U(ZG)$, or a subgroup of finite index of the unit group $U(ZG)$. These results are often obtained by using techniques from representation theory and algebraic number theory [2].

In 1940, substantial work on the unit problem was done by Graham Higman [3]. He first showed that if $U(ZG) = \pm G$, then $U(\mathbb{Z}[G \times C_2]) = \pm(G \times C_2)$. Using this, he showed that $U(ZG) = \pm G$ if and only if G is abelian of exponent 2,3,4, or 6 or $G = E \times K_8$ where K_8 is the quaternion group of order 8 and E is an elementary abelian 2-group. Furthermore, Higman gave a general structure theorem for $U(ZA)$, where A is a finite abelian group. Other results include: A_4 and S_4 by Allen-Hobby[4], D_{2p} by Passman-Smith[5], $G = C_p \rtimes C_q$, where q is a prime dividing $p-1$ by Galovich-Reiner-Ullom [6], $|G| = p^3$ by Ritter-Sehgal[7], and $U(ZS_3)$ by Hughes-Pearson[8].

Now let's define the following representations

1. $\rho_1 : S_4 \rightarrow GL(1, \mathbb{C})$

$$g \text{ a } [1]$$

2. $\rho_2 : S_4 \rightarrow GL(1, \mathbb{C})$

$$g \text{ a } \begin{cases} [1] & \text{if } g \in A_4 \\ [-1] & \text{if } g \in S_4 / A_4 \end{cases}$$

3. $\rho_3 : S_4 \rightarrow GL(2, \mathbb{C})$ where $\bar{\rho}_3 : S_3 \rightarrow GL(2, \mathbb{C})$

$$g \text{ a } (kh)\rho_3 = h\rho$$

$$(123) \text{ a } \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$(12) \text{ a } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\begin{array}{c} S_3 \\ \backslash \\ K_4 \end{array}$	(1)	(123)	(132)	(12)	(13)	(23)
(1)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$
(12)(34)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$
(13)(24)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$
(14)(23)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$

4. $\rho_4 : S_4 \rightarrow GL(3, \mathbb{C})$

$$(1234) \text{ a } \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(123) \text{ a } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

By extending these representations over \mathfrak{C} linearly we get the representations of $\mathfrak{C} S_4$:

$$*1. \overline{\rho_1} : \mathfrak{C} S_4 \rightarrow \mathfrak{C}^{1 \times 1}$$

$$\gamma = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} g_{ij} \quad \text{a} \quad \gamma \overline{\rho_1} = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} (g_{ij} \rho_1)$$

$$*2. \overline{\rho_2} : \mathfrak{C} S_4 \rightarrow \mathfrak{C}^{1 \times 1}$$

$$\gamma = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} g_{ij} \quad \text{a} \quad \gamma \overline{\rho_2} = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} (g_{ij} \rho_2)$$

$$*3. \overline{\rho_3} : \mathfrak{C} S_4 \rightarrow \mathfrak{C}^{2 \times 2}$$

$$\gamma = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} g_{ij} \quad \text{a} \quad \gamma \overline{\rho_3} = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} (g_{ij} \rho_3)$$

$$*4. \overline{\rho_4} : \mathfrak{C} S_4 \rightarrow \mathfrak{C}^{3 \times 3}$$

$$\gamma = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} g_{ij} \quad \text{a} \quad \gamma \overline{\rho_4} = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} (g_{ij} \rho_4)$$

$$*5. \overline{\rho_5} : \mathfrak{C} S_5 \rightarrow \mathfrak{C}^{3 \times 3}$$

$$\gamma = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} g_{ij} \quad \text{a} \quad \gamma \overline{\rho_5} = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} (g_{ij} \rho_5)$$

Then for every $\gamma \in \mathfrak{C} S_4$ we explicitly express these representations as follows :

$$1) \gamma \overline{\rho_1} = \left[\sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} \right]$$

$$2) \gamma \overline{\rho_2} = \left[\left(\sum_{i=1}^4 \sum_{j=1}^3 \gamma_{ij} \right) - \left(\sum_{i=1}^4 \sum_{j=4}^6 \gamma_{ij} \right) \right]$$

$$3) \gamma \overline{\rho_3} = \begin{bmatrix} \alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 & -\alpha_2 + \alpha_3 + \alpha_4 - \alpha_5 \\ \alpha_2 - \alpha_3 + \alpha_4 - \alpha_6 & \alpha_1 - \alpha_2 - \alpha_5 + \alpha_6 \end{bmatrix}, \quad \text{where } \alpha_k = \sum_{i=1}^4 \gamma_{ik}$$

4)

$$\overline{\rho_4} = \begin{bmatrix} \gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} & \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} & \gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} \\ -\gamma_{31} - \gamma_{36} - \gamma_{41} - \gamma_{46} & -\gamma_{33} - \gamma_{34} - \gamma_{43} - \gamma_{44} & -\gamma_{32} - \gamma_{35} - \gamma_{42} - \gamma_{45} \\ \gamma_{12} + \gamma_{14} - \gamma_{22} - \gamma_{24} & \gamma_{11} + \gamma_{15} - \gamma_{21} - \gamma_{25} & \gamma_{13} + \gamma_{16} - \gamma_{23} - \gamma_{26} \\ +\gamma_{32} + \gamma_{34} - \gamma_{42} - \gamma_{44} & +\gamma_{31} + \gamma_{35} - \gamma_{41} - \gamma_{45} & +\gamma_{33} + \gamma_{36} - \gamma_{43} - \gamma_{46} \\ \gamma_{13} + \gamma_{15} - \gamma_{23} - \gamma_{25} & \gamma_{12} + \gamma_{16} - \gamma_{22} - \gamma_{26} & \gamma_{11} + \gamma_{14} - \gamma_{21} - \gamma_{24} \\ -\gamma_{33} - \gamma_{35} + \gamma_{43} + \gamma_{45} & -\gamma_{32} - \gamma_{36} + \gamma_{42} + \gamma_{46} & -\gamma_{31} - \gamma_{34} + \gamma_{41} + \gamma_{44} \end{bmatrix}$$

5)

$$\overline{\mathcal{Y}\rho_5} = \begin{bmatrix} \mathcal{Y}_{11} - \mathcal{Y}_{16} + \mathcal{Y}_{21} - \mathcal{Y}_{26} & \mathcal{Y}_{13} - \mathcal{Y}_{14} + \mathcal{Y}_{23} - \mathcal{Y}_{24} & \mathcal{Y}_{12} - \mathcal{Y}_{15} + \mathcal{Y}_{22} - \mathcal{Y}_{25} \\ -\mathcal{Y}_{31} + \mathcal{Y}_{36} - \mathcal{Y}_{41} + \mathcal{Y}_{46} & -\mathcal{Y}_{33} + \mathcal{Y}_{34} - \mathcal{Y}_{43} + \mathcal{Y}_{44} & -\mathcal{Y}_{32} + \mathcal{Y}_{35} - \mathcal{Y}_{42} + \mathcal{Y}_{45} \\ \mathcal{Y}_{12} - \mathcal{Y}_{14} - \mathcal{Y}_{22} + \mathcal{Y}_{24} & \mathcal{Y}_{11} - \mathcal{Y}_{15} - \mathcal{Y}_{21} + \mathcal{Y}_{25} & \mathcal{Y}_{13} - \mathcal{Y}_{16} - \mathcal{Y}_{23} + \mathcal{Y}_{26} \\ +\mathcal{Y}_{32} - \mathcal{Y}_{34} - \mathcal{Y}_{42} + \mathcal{Y}_{44} & +\mathcal{Y}_{31} - \mathcal{Y}_{35} - \mathcal{Y}_{41} + \mathcal{Y}_{45} & +\mathcal{Y}_{33} - \mathcal{Y}_{36} - \mathcal{Y}_{43} + \mathcal{Y}_{46} \\ \mathcal{Y}_{13} - \mathcal{Y}_{15} - \mathcal{Y}_{23} + \mathcal{Y}_{25} & \mathcal{Y}_{12} - \mathcal{Y}_{16} - \mathcal{Y}_{22} + \mathcal{Y}_{26} & \mathcal{Y}_{11} - \mathcal{Y}_{14} - \mathcal{Y}_{21} + \mathcal{Y}_{24} \\ -\mathcal{Y}_{33} + \mathcal{Y}_{35} + \mathcal{Y}_{43} - \mathcal{Y}_{45} & -\mathcal{Y}_{32} + \mathcal{Y}_{36} + \mathcal{Y}_{42} - \mathcal{Y}_{46} & -\mathcal{Y}_{31} + \mathcal{Y}_{34} + \mathcal{Y}_{41} - \mathcal{Y}_{44} \end{bmatrix}$$

CHAPTER 2

SOME PRELIMINARY RESULTS

2.1. Basic Results in $\mathbf{Z}S_4$

Let us recall some basic results which are used to characterize the torsion units.

Theorem 2.1.1. (N.A. Fernandes[13]) Let $\gamma = \sum \gamma_g g \in U_1(\mathbf{Z}S_4)$ be a non trivial torsion unit.

- a) If the order of γ is 2 then γ is rationally conjugate to (12) or (12)(34). That is, $\gamma = \alpha(12)\alpha^{-1}$ or $\gamma = \alpha(12)(34)\alpha^{-1}$ for some $\alpha \in QS_4$
- b) If the order of γ is 3 then γ is rationally conjugate to (123). That is, $\gamma = \alpha(123)\alpha^{-1}$ for some $\alpha \in QS_4$.
- c) If the order of γ is 4 then γ is rationally conjugate to (1234). That is, $\gamma = \alpha(1234)\alpha^{-1}$ for some $\alpha \in QS_4$.

Proposition 2.1.2. Let $\bar{\rho}$ be a representation of $\mathbf{Z}G$ obtained by extending linearly a representation ρ of finite degree of a group G over \mathbf{Z} . If $\gamma \in U_1(\mathbf{Z}G)$ is a rational conjugate to g for some $g \in G$ then

$$i) \quad |\gamma\bar{\rho}| = |g\rho| \quad \text{and} \quad ii) \quad tr(\gamma\bar{\rho}) = tr(g\rho).$$

Proposition 2.1.3. (Berman-Higman [3]) Let G be a finite group. Suppose $\gamma \in U_1(\mathbf{Z}G)$ is a torsion unit, namely, $\gamma^n = 1$ for some natural number n . If $\gamma_e \neq 0$ then $\gamma = \pm 1$.

Corollary 2.1.4. If $\gamma \in U_1(\mathbf{Z}G)$ is a torsion unit and $\gamma \neq 1$ then $\gamma_e = 0$.

Corollary 2.1.5. If $\gamma \in U_1(\mathbf{Z}G)$ is a rational conjugate to $g \in Z(G)$, then $\gamma = g$. That is, any torsion unit which is rational conjugate to an element in the center of a group is trivial.

Definition 2.1.6. Let $f : Z^n \longrightarrow Z$
 (p_1, \dots, p_n) a $f(p_1, \dots, p_n)$

be a function. Then each p_i is called free parameter ($i = 1, \dots, n$). If q is a divisor of $f(p_1, \dots, p_n)$ then q is called dependent parameter ($i = 1, \dots, n$).

Theorem 2.1.7. (Bilgin [10]) Let $\gamma = \sum \gamma_g g \in U_1(\mathbf{Z}S_3)$ be a non trivial torsion unit

a) If the order of γ is 2 then γ can be expressed in terms of two parameters as follows;

$$\begin{aligned} \gamma = & [n - 3k - 1 + \frac{3k^2 + 2k}{n}](123) - (132) + [n - 4k - 1 + \frac{3k^2 + 2k}{n}](12) + [2k + 1 - n](23) \\ & + [2k + 1 - \frac{3k^2 + 2k}{n}](13), \end{aligned}$$

where k is a free parameter and n is a divisor of $3k^2 + 2k$.

b) If the order of γ is 3 then γ can be expressed in terms of two parameters as follows;

$$\begin{aligned} \gamma = & [n - 3k + 2 + \frac{3k^2 - 3k + 1}{n}](123) + [3k - n - 1 - \frac{3k^2 - 3k + 1}{n}](132) \\ & + [n - 4k + 2 + \frac{3k^2 - 3k + 1}{n}](12) + [2k - 1 - n](23) + [1 - 2k + \frac{3k^2 - 3k + 1}{n}](13) \end{aligned}$$

where k is a free parameter and n is a divisor of $3k^2 - 3k + 1$.

Proposition 2.1.8. Let $G = S_4$, $N = K_4$, $\gamma = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} g_{ij} \in \mathcal{C} S_4$ and $\alpha_j = \sum_{i=1}^4 \gamma_{ij}$ ($j = 1, 2, 3, 4, 5, 6$).

Consider the following natural ring homomorphism:

$$\begin{aligned} \varphi : \mathbf{Z}S_4 & \longrightarrow \mathbf{Z}(S_4 / K_4) \\ \sum \gamma_g g & \mathbf{a} \sum \gamma_g (gN). \end{aligned}$$

Then we have

a) If $\gamma : (12)(34)$ then $\alpha_1 = 1$ and $\alpha_j = 0$ for $j = 2, 3, 4, 5, 6$.

b) If $\gamma : (12)$ or $\gamma : (1234)$ then

$$\begin{aligned} \alpha_1 = 0 & & \alpha_2 = n - 3k - 1 + \frac{3k^2 + 2k}{n} & & \alpha_3 = -\alpha_2 \\ \alpha_4 = n - 4k - 1 + \frac{3k^2 + 2k}{n} & & \alpha_5 = 2k + 1 - n & & \alpha_6 = 1 - \alpha_4 - \alpha_5 \end{aligned}$$

where k is a free parameter and n is a divisor of $3k^2 + 2k$.

c) If $\gamma: (123)$ then

$$\begin{aligned} \alpha_1 &= 0 & \alpha_2 &= n - 3k + 2 + \frac{3k^2 - 3k + 1}{n} & \alpha_3 &= 3k - n - 1 - \frac{3k^2 - 3k + 1}{n} \\ \alpha_6 &= 2k - 1 - n & \alpha_5 &= 1 - 2k + \frac{3k^2 - 3k + 1}{n} & \alpha_4 &= n - 4k + 2 + \frac{3k^2 - 3k + 1}{n} \end{aligned}$$

where k is a free parameter and n is a divisor of $3k^2 - 3k + 1$.

2.2. Basic Results in Matrices over \mathbf{Z}

In this part we want to find matrices such that their 2nd, 3rd, 4th power is identity.

If $A^2 = I$ and $A \neq \pm I$ then $\text{tr}(A) = \pm 1$ since $\mathcal{D}_A(x) = x^2 - 1$ and $\Delta_A(x) = (x-1)(x^2-1)$ or

$\Delta_A(x) = (x+1)(x^2-1)$ that is $\Delta_A(x) = x^3 - x^2 - x + 1$ or $\Delta_A(x) = x^3 + x^2 - x - 1$

We'll determine matrices such that

- i) $A^2 = I$, $A \neq \pm I$, $\text{tr}(A) = -1$ and
- ii) $A^2 = I$, $A \neq \pm I$, $\text{tr}(A) = 1$

Proposition 2.2.1.

Let $A = \begin{bmatrix} a & b & x \\ c & d & y \\ u & v & -1-a-d \end{bmatrix} \in \mathbf{Z}^{3 \times 3}$. If $A^2 = I$ then

$$\begin{aligned} a &= pq - 1 & b &= qr & c &= pr & d &= r^2 - 1 \\ x &= nq & y &= nr & u &= -\frac{p}{n}(pq + r^2 - 2) & v &= -\frac{r}{n}(pq + r^2 - 2) \end{aligned}$$

where $n \mid pq + r^2 - 2$.

Proof:

$$\text{Since } A^2 = \begin{bmatrix} bc + ux + a^2 & \dots & \dots \\ \dots & bc + vy + d^2 & \dots \\ \dots & \dots & a^2 + d^2 + 2a + 2d + 2ad + ux + vy + 1 \end{bmatrix} = I,$$

we get

$$\left. \begin{array}{l} bc+ux+a^2=1 \\ bc+vy+d^2=1 \\ a^2+d^2+2a+2d+2ad+ux+vy+1=1 \end{array} \right\} \Rightarrow \left. \begin{array}{l} bc+ux=1-a^2 \\ bc+vy=1-d^2 \\ ux+vy=-a^2-d^2-2a-2d-2ad \end{array} \right\} \quad (2.1)$$

$$\left. \begin{array}{l} bc=(a+1)(d+1) \\ \Rightarrow ux=-(a+1)(a+d) \\ vy=-(d+1)(a+d) \end{array} \right\}$$

If we assign b, c, u, x in terms of m, n, p, q, r, s such that

$$b = qs \quad c = pr \quad u = -mp \quad x = nq \quad (2.2)$$

then we get the following result :

$$a = pq - 1 \quad d = rs - 1 \quad v = -mr \quad y = ns \quad (2.3)$$

and

$$a + d = mn \quad (2.4)$$

From (2.3) and (2.4) we get

$$\begin{aligned} a + d = mn &\Rightarrow pq - 1 + rs - 1 = mn \\ &\Rightarrow mn = pq + rs - 2 \\ &\Rightarrow n \mid pq + rs - 2 \quad \text{and} \quad m = \frac{pq + rs - 2}{n}. \end{aligned}$$

So, we get the matrix A as

$$A = \begin{bmatrix} pq-1 & qs & nq \\ pr & rs-1 & ns \\ -\frac{p}{n}(pq+rs-2) & -\frac{r}{n}(pq+rs-2) & 1-pq-rs \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} 1 & q(s-r) & snq(s-r) \\ p(r-s)(pq+rs-2) & 1 & qpn(r-s) \\ \frac{pr}{n}(pq+rs-2)(s-r) & \frac{pq}{n}(pq+rs-2)(r-s) & 1 \end{bmatrix}.$$

$$\text{So } A^2 = I \Leftrightarrow \begin{cases} \text{i) } s = r \text{ or} \\ \text{ii) } q = 0 \text{ and } p = 0 \text{ or} \\ \text{iii) } q = 0 \text{ and } rs = 2 \end{cases}$$

$$A = \begin{bmatrix} pq-1 & qr & nq \\ pr & r^2-1 & nr \\ -\frac{p}{n}(pq+r^2-2) & -\frac{r}{n}(pq+r^2-2) & 1-pq-r^2 \end{bmatrix}$$

Proposition 2.2.2.

$$\text{Let } A = \begin{bmatrix} a & b & x \\ c & d & y \\ u & v & 1-a-d \end{bmatrix} \in \mathbb{Z}^{3 \times 3}. \text{ If } A^2 = I \text{ then}$$

$$\begin{aligned} a &= 1-pq & b &= -qr & c &= pr & d &= 1+r^2 \\ x &= nq & y &= -nr & u &= -\frac{p}{n}(pq-r^2-2) & v &= -\frac{r}{n}(pq-r^2-2) \end{aligned}$$

where $n \mid pq - r^2 - 2$.

Proof:

$$\text{Since } A^2 = \begin{bmatrix} bc+ux+a^2 & \dots & \dots \\ \dots & bc+vy+d^2 & \dots \\ \dots & \dots & a^2+d^2-2a-2d+2ad+ux+vy+1 \end{bmatrix}$$

$$\left. \begin{aligned} bc+ux+a^2 &= 1 \\ bc+vy+d^2 &= 1 \\ a^2+d^2-2a-2d+2ad+ux+vy+1 &= 1 \end{aligned} \right\} \begin{aligned} bc+ux &= 1-a^2 \\ bc+vy &= 1-d^2 \\ ux+vy &= -a^2-d^2+2a+2d-2ad \end{aligned} \right\} \begin{aligned} bc &= (1-a)(1-d) \\ \Rightarrow ux &= (1-a)(a+d) \\ vy &= (1-d)(a+d) \end{aligned} \quad (2.5)$$

If we assign b, c, u, x in terms of m, n, p, q, r, s such that

$$b = qs \quad c = pr \quad u = mp \quad x = nq \quad (2.6)$$

then we get the following result :

$$a = 1 - pq \quad d = 1 - rs \quad v = mr \quad y = ns \quad (2.7)$$

and

$$a + d = mn \quad (2.8)$$

From (2.7) and (2.8) we get

$$\begin{aligned} a + d = mn &\Rightarrow 1 - pq + 1 - rs = mn \\ &\Rightarrow mn = 2 - pq - rs \\ &\Rightarrow n \mid pq + rs - 2 \quad \text{and} \quad m = \frac{2 - pq - rs}{n}. \end{aligned}$$

Then we get the matrix A as

$$A = \begin{bmatrix} 1 - pq & qs & nq \\ pr & 1 - rs & ns \\ \frac{p}{n}(2 - pq - rs) & \frac{r}{n}(2 - pq - rs) & -1 + pq + rs \end{bmatrix}. \text{ Then}$$

$$A^2 = \begin{bmatrix} 1 & q(2 - pq - rs)(r + s) & nqs(r + s) \\ p(2 - pq - rs)(r + s) & 1 & nqp(r + s) \\ \frac{pr}{n}(r + s)(2 - pq - rs) & \frac{pq}{n}(r + s)(2 - pq - rs) & 1 \end{bmatrix}$$

$$\text{So } A^2 = I \Leftrightarrow \begin{cases} \text{i) } s = -r \text{ or} \\ \text{ii) } q = 0 \text{ and } p = 0 \text{ or} \\ \text{iii) } q = 0 \text{ and } rs = 2 \end{cases}$$

If we choose the first case we get the general solution as follows

$$A = \begin{bmatrix} 1 - pq & -qr & nq \\ pr & 1 + r^2 & -nr \\ \frac{p}{n}(2 - pq + r^2) & \frac{r}{n}(2 - pq + r^2) & -1 + pq - r^2 \end{bmatrix}$$

Proposition 2.2.3.

Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \in Z^{3 \times 3}$ where $A^2 \neq I$. Then

$$A^4 = I \Leftrightarrow \begin{cases} i) a + q + z = k \\ ii) aq + az + qz - bp - cx - ry = 1 \\ iii) x(br - qc) + y(cp - ar) + z(aq - bp) = k \end{cases}$$

where $k = \pm 1$

Proof. Since $A^4 = I$ it satisfies the polynomial $p(x) = x^4 - 1$. Since $A \in \mathbb{Z}^{3 \times 3}$, $\Delta_A(x)$ divides $p(x)$. So the characteristic polynomial $\Delta_A(x) = (x-1)(x^2+1)$ or $\Delta_A(x) = (x+1)(x^2+1)$ or equivalently, $\Delta_A(x) = x^3 - x^2 + x - 1$ or $\Delta_A(x) = x^3 + x^2 + x + 1$. Simply we write $\Delta_A(x) = x^3 + kx^2 + x + k$, where $k = \pm 1$. Since

$$\Delta_A(x) = x^3 - \text{Tr}(A)x^2 + \sum_{1 \leq i < j \leq 3} (a_{ij}a_{ji} - a_{ii}a_{jj})x - \det(A), \quad (A = (a_{ij}))$$

we get

$$A^4 = I \Leftrightarrow \begin{cases} i) a + q + z = k \\ ii) aq + az + qz - bp - cx - ry = 1 \\ iii) x(br - qc) + y(cp - ar) + z(aq - bp) = k \end{cases}$$

where $k = \pm 1$.

Proposition 2.2.4.

$$\text{Let } A = \begin{bmatrix} a & b & u \\ c & d & v \\ x & y & z \end{bmatrix} \in \mathbb{Z}^{3 \times 3}.$$

$$A^3 = I \Leftrightarrow \begin{cases} z = -a - d \\ u = k[-(bx - ay)(a^2 + ad + d^2 + bc) - y(1 + (a + d)(ad - bc))] \\ v = k[(cx - dy)(a^2 + ad + d^2 + bc) + x(1 + (a + d)(ad - bc))] \end{cases}$$

where $bx^2 + (d - a)xy - cy^2 = \pm 1$ and $k = \pm 1$.

Proof. Since $A^3 = I$ and $A \in \mathbb{Z}^{3 \times 3}$, $\Delta_A(x) = x^3 - 1$. Since

$$\Delta_A(x) = x^3 - \text{Tr}(A)x^2 + \sum_{1 \leq i < j \leq 3} (a_{ij}a_{ji} - a_{ii}a_{jj})x - \det(A),$$

$$\text{we get } A^3 = I \Leftrightarrow \begin{cases} i) a + d + z = 0 \\ ii) ad + az + dz - bc - ux - vy = 0 \\ iii) u(cy - dx) + v(bx - ay) + z(ad - bc) = 1 \end{cases} \quad (2.9)$$

By substituting i) into ii) and iii) in (2.9) we get the following equations

$$\begin{aligned} ux + vy &= -a^2 - ad - d^2 - bc \\ u(cy - dx) + v(bx - ay) &= 1 + (a + d)(ad - bc) \end{aligned} \quad (2.10)$$

So we get the following system of linear equations

$$\begin{bmatrix} x & y \\ cy-dx & bx-ay \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -a^2 - ad - d^2 - bc \\ 1 + (a+d)(ad-bc) \end{bmatrix} \quad (2.11)$$

This equation has a solution in \mathbb{Z} if and only if $bx^2 + (d-a)xy - cy^2 = \pm 1$. In this case the solution is

$$\begin{aligned} u &= k[-(bx-ay)(a^2 + ad + d^2 + bc) - y(1 + (a+d)(ad-bc))] \\ v &= k[(cx-dy)(a^2 + ad + d^2 + bc) + x(1 + (a+d)(ad-bc))] \end{aligned}$$

where $k = \pm 1$. Hence

$$A^3 = I \Leftrightarrow \begin{cases} z = -a - d \\ u = k[-(bx-ay)(a^2 + ad + d^2 + bc) - y(1 + (a+d)(ad-bc))] \\ v = k[(cx-dy)(a^2 + ad + d^2 + bc) + x(1 + (a+d)(ad-bc))] \end{cases}$$

where $bx^2 + (d-a)xy - cy^2 = \pm 1$ and $k = \pm 1$.

CHAPTER 3

CHARACTERIZATION OF TORSION UNITS

$$S_4 = \{1\} \cup \{(12), (13), (14), (23), (24), (34)\} \cup \{(123), (132), (124), (142), (134), (143), (234), (243)\} \cup \{(12)(34), (13)(24), (14)(23)\} \cup \{(1234), (1243), (1324), (1342), (1423), (1432)\}$$

has five conjugacy classes. By theorem 2.1.1. any torsion unit of order 2 is conjugate to (12)(34) or (12), any torsion unit of order 3 is conjugate to (123) and any torsion unit of order 4 is conjugate to (1234).

3.1. Torsion Units of Order 2

Let $\gamma \in U_1(\mathcal{C}S_4)$ be a torsion unit of order 2. Then by theorem 2.1.1. either $\gamma: (12)(34)$ or $\gamma: (12)$

Case 1) Let γ be rationally conjugate to (12)(34)

i) By proposition 2.1.8. $\alpha_1 = 1$ and $\alpha_j = 0$ for $j = 2, 3, 4, 5, 6$ where $\alpha_k = \sum_{i=1}^4 \gamma_{ik}$. That is

$$\begin{aligned} \gamma_{11} + \gamma_{21} + \gamma_{31} + \gamma_{41} &= 1, \\ \gamma_{12} + \gamma_{22} + \gamma_{32} + \gamma_{42} &= 0, \\ \gamma_{13} + \gamma_{23} + \gamma_{33} + \gamma_{43} &= 0, \\ \gamma_{14} + \gamma_{24} + \gamma_{34} + \gamma_{44} &= 0, \\ \gamma_{15} + \gamma_{25} + \gamma_{35} + \gamma_{45} &= 0, \\ \gamma_{16} + \gamma_{26} + \gamma_{36} + \gamma_{46} &= 0 \end{aligned} \tag{3.1}$$

$$\text{ii) } \overline{\gamma\rho_4} = \begin{bmatrix} \gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} & \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} & \gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} \\ -\gamma_{31} - \gamma_{36} - \gamma_{41} - \gamma_{46} & -\gamma_{33} - \gamma_{34} - \gamma_{43} - \gamma_{44} & -\gamma_{32} - \gamma_{35} - \gamma_{42} - \gamma_{45} \\ \gamma_{12} + \gamma_{14} - \gamma_{22} - \gamma_{24} & \gamma_{11} + \gamma_{15} - \gamma_{21} - \gamma_{25} & \gamma_{13} + \gamma_{16} - \gamma_{23} - \gamma_{26} \\ +\gamma_{32} + \gamma_{34} - \gamma_{42} - \gamma_{44} & +\gamma_{31} + \gamma_{35} - \gamma_{41} - \gamma_{45} & +\gamma_{33} + \gamma_{36} - \gamma_{43} - \gamma_{46} \\ \gamma_{13} + \gamma_{15} - \gamma_{23} - \gamma_{25} & \gamma_{12} + \gamma_{16} - \gamma_{22} - \gamma_{26} & \gamma_{11} + \gamma_{14} - \gamma_{21} - \gamma_{24} \\ -\gamma_{33} - \gamma_{35} + \gamma_{43} + \gamma_{45} & -\gamma_{32} - \gamma_{36} + \gamma_{42} + \gamma_{46} & -\gamma_{31} - \gamma_{34} + \gamma_{41} + \gamma_{44} \end{bmatrix} \tag{3.2}$$

So

$$A = \overline{\gamma\rho_4} = \begin{bmatrix} 2(\gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26}) - 1 & 2(\gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24}) & 2(\gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25}) \\ 2(\gamma_{12} + \gamma_{14} + \gamma_{32} + \gamma_{34}) & 2(\gamma_{11} + \gamma_{15} + \gamma_{31} + \gamma_{35}) & 2(\gamma_{13} + \gamma_{16} + \gamma_{33} + \gamma_{36}) \\ -2(\gamma_{23} + \gamma_{25} + \gamma_{33} + \gamma_{35}) & -2(\gamma_{22} + \gamma_{26} + \gamma_{32} + \gamma_{36}) & 1 - 2(\gamma_{21} + \gamma_{24} + \gamma_{31} + \gamma_{34}) \end{bmatrix} \quad (3.3)$$

If we assign entries of A in (3.3)

$$\begin{aligned} \gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} &= a_1, & \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} &= b_1, & \gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} &= x_1, \\ \gamma_{12} + \gamma_{14} + \gamma_{32} + \gamma_{34} &= c_1, & \gamma_{11} + \gamma_{15} + \gamma_{31} + \gamma_{35} &= d_1, & \gamma_{13} + \gamma_{16} + \gamma_{33} + \gamma_{36} &= y_1, \\ \gamma_{23} + \gamma_{25} + \gamma_{33} + \gamma_{35} &= u_1, & \gamma_{22} + \gamma_{26} + \gamma_{32} + \gamma_{36} &= v_1, & \gamma_{21} + \gamma_{24} + \gamma_{31} + \gamma_{34} &= w_1 \end{aligned}$$

we get

$$A = \begin{bmatrix} 2a_1 - 1 & 2b_1 & 2x_1 \\ 2c_1 & 2d_1 - 1 & 2y_1 \\ -2u_1 & -2v_1 & 1 - 2w_1 \end{bmatrix}$$

By proposition 2.1.2. $tr(\overline{\gamma\rho_4}) = tr((12)(34)\rho_4) = -1$.

By proposition 2.2.1.

$$\begin{aligned} 2a_1 - 1 &= p_1q_1 - 1 &\Rightarrow & a_1 = \frac{p_1q_1}{2} \\ 2b_1 &= q_1r_1 &\Rightarrow & b_1 = \frac{q_1r_1}{2} \\ 2c_1 &= p_1r_1 &\Rightarrow & c_1 = \frac{p_1r_1}{2} \\ 2d_1 - 1 &= r_1^2 - 1 &\Rightarrow & d_1 = \frac{r_1^2}{2} \\ 2x_1 &= n_1q_1 &\Rightarrow & x_1 = \frac{n_1q_1}{2} \\ 2y_1 &= n_1r_1 &\Rightarrow & y_1 = \frac{n_1r_1}{2} \\ -2u_1 &= -\frac{p_1}{n_1}(p_1q_1 + r_1^2 - 2) &\Rightarrow & u_1 = \frac{p_1}{2n_1}(p_1q_1 + r_1^2 - 2) \\ -2v_1 &= -\frac{r_1}{n_1}(p_1q_1 + r_1^2 - 2) &\Rightarrow & v_1 = \frac{r_1}{2n_1}(p_1q_1 + r_1^2 - 2) \\ 1 - 2w_1 &= 1 - 2a_1 - 2d_1 &\Rightarrow & w_1 = a_1 + d_1 = \frac{p_1q_1 + r_1^2}{2} \end{aligned}$$

where p_1, q_1, r_1 are free parameters and n_1 is a divisor of $p_1q_1 + r_1^2 - 2$. Since $d_1 = \frac{r_1^2}{2}$ it is obvious that r_1 is even. Moreover as a special case we can take $n_1 = 1$. In this case $x_1 = \frac{q_1}{2}$ and so q_1 is even too. If we replace r_1, q_1 and n_1 by $2r_1, 2q_1$ and 1 respectively we get

$$\begin{aligned} a_1 &= p_1q_1 & b_1 &= 2q_1r_1 & c_1 &= p_1r_1 \\ d_1 &= 2r_1^2 & x_1 &= q_1 & y_1 &= r_1 \\ u_1 &= p_1(p_1q_1 + 2r_1^2 - 1) & v_1 &= r_1(2p_1q_1 + 4r_1^2 - 2) & w_1 &= p_1q_1 + 2r_1^2 \end{aligned}$$

$$\text{iii) } \overline{\gamma\rho_5} = \begin{bmatrix} \gamma_{11} - \gamma_{16} + \gamma_{21} - \gamma_{26} & \gamma_{13} - \gamma_{14} + \gamma_{23} - \gamma_{24} & \gamma_{12} - \gamma_{15} + \gamma_{22} - \gamma_{25} \\ -\gamma_{31} + \gamma_{36} - \gamma_{41} + \gamma_{46} & -\gamma_{33} + \gamma_{34} - \gamma_{43} + \gamma_{44} & -\gamma_{32} + \gamma_{35} - \gamma_{42} + \gamma_{45} \\ \gamma_{12} - \gamma_{14} - \gamma_{22} + \gamma_{24} & \gamma_{11} - \gamma_{15} - \gamma_{21} + \gamma_{25} & \gamma_{13} - \gamma_{16} - \gamma_{23} + \gamma_{26} \\ +\gamma_{32} - \gamma_{34} - \gamma_{42} + \gamma_{44} & +\gamma_{31} - \gamma_{35} - \gamma_{41} + \gamma_{45} & +\gamma_{33} - \gamma_{36} - \gamma_{43} + \gamma_{46} \\ \gamma_{13} - \gamma_{15} - \gamma_{23} + \gamma_{25} & \gamma_{12} - \gamma_{16} - \gamma_{22} + \gamma_{26} & \gamma_{11} - \gamma_{14} - \gamma_{21} + \gamma_{24} \\ -\gamma_{33} + \gamma_{35} + \gamma_{43} - \gamma_{45} & -\gamma_{32} + \gamma_{36} + \gamma_{42} - \gamma_{46} & -\gamma_{31} + \gamma_{34} + \gamma_{41} - \gamma_{44} \end{bmatrix}$$

$$B = \overline{\gamma\rho_5} = \begin{bmatrix} 2(\gamma_{11} + \gamma_{21} - \gamma_{16} - \gamma_{26}) - 1 & 2(\gamma_{13} + \gamma_{23} - \gamma_{14} - \gamma_{24}) & 2(\gamma_{12} + \gamma_{22} - \gamma_{15} - \gamma_{25}) \\ 2(\gamma_{12} + \gamma_{32} - \gamma_{14} - \gamma_{34}) & 2(\gamma_{11} + \gamma_{31} - \gamma_{15} - \gamma_{35}) & 2(\gamma_{13} + \gamma_{33} - \gamma_{16} - \gamma_{36}) \\ -2(\gamma_{23} + \gamma_{33} - \gamma_{35} - \gamma_{25}) & -2(\gamma_{22} + \gamma_{32} - \gamma_{26} - \gamma_{36}) & 1 - 2(\gamma_{21} + \gamma_{31} - \gamma_{24} - \gamma_{34}) \end{bmatrix}$$

If we take

$$\begin{aligned} \gamma_{11} + \gamma_{21} - \gamma_{16} - \gamma_{26} &= a_2, & \gamma_{13} + \gamma_{23} - \gamma_{14} - \gamma_{24} &= b_2, & \gamma_{12} + \gamma_{22} - \gamma_{15} - \gamma_{25} &= x_2, \\ \gamma_{12} + \gamma_{32} - \gamma_{14} - \gamma_{34} &= c_2, & \gamma_{11} + \gamma_{31} - \gamma_{15} - \gamma_{35} &= d_2, & \gamma_{13} + \gamma_{33} - \gamma_{16} - \gamma_{36} &= y_2, \\ \gamma_{23} + \gamma_{33} - \gamma_{35} - \gamma_{25} &= u_2, & \gamma_{22} + \gamma_{32} - \gamma_{26} - \gamma_{36} &= v_2, & \gamma_{21} + \gamma_{31} - \gamma_{24} - \gamma_{34} &= w_2 \end{aligned}$$

We get $B = \begin{bmatrix} 2a_2 - 1 & 2b_2 & 2x_2 \\ 2c_2 & 2d_2 - 1 & 2y_2 \\ -2u_2 & -2v_2 & 1 - 2w_2 \end{bmatrix}$

By proposition 2.1.2. $tr(\gamma\bar{\rho}_5) = tr((12)(34)\rho_5) = -1$.

By proposition 2.2.1.

$$\begin{aligned} 2a_2 - 1 = p_2q_2 - 1 &\Rightarrow a_2 = \frac{p_2q_2}{2} \\ 2b_2 = q_2r_2 &\Rightarrow b_2 = \frac{q_2r_2}{2} \\ 2c_2 = p_2r_2 &\Rightarrow c_2 = \frac{p_2r_2}{2} \\ 2d_2 - 1 = r_2^2 - 1 &\Rightarrow d_2 = \frac{r_2^2}{2} \\ 2x_2 = n_2q_2 &\Rightarrow x_2 = \frac{n_2q_2}{2} \\ 2y_2 = n_2r_2 &\Rightarrow y_2 = \frac{n_2r_2}{2} \\ -2u_2 = -\frac{p_2}{n_2}(p_2q_2 + r_2^2 - 2) &\Rightarrow u_2 = \frac{p_2}{2n_2}(p_2q_2 + r_2^2 - 2) \\ -2v_2 = -\frac{p_2}{n_2}(p_2q_2 + r_2^2 - 2) &\Rightarrow v_2 = \frac{p_2}{2n_2}(p_2q_2 + r_2^2 - 2) \\ 1 - 2w_2 = 1 - 2a_2 - 2d_2 &\Rightarrow w = a_2 + d_2 = \frac{p_2q_2 + r_2^2}{2}, \end{aligned}$$

where p_2, q_2, r_2 are free parameters and n_2 is a divisor of $p_2q_2 + r_2^2 - 2$. Again r_2 is even. Let's take $n_2 = 1$. In this case q_2 is even. If we replace r_2, q_2 and n_2 by $2r_2, 2q_2$ and 1 respectively we get

$$\begin{aligned} a_2 &= p_2q_2 & b_1 &= 2q_2r_2 & c_2 &= p_2r_2 \\ d_2 &= 2r_2^2 & x_2 &= q_2 & y_2 &= r_2 \\ u_2 &= p_2(p_2q_2 + 2r_2^2 - 1) & v_2 &= r_2(2p_2q_2 + 4r_2^2 - 2) & w_2 &= p_2q_2 + 2r_2^2 \end{aligned}$$

So we have the system

$\gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} = p_1 q_1$	$\gamma_{12} + \gamma_{32} - \gamma_{14} - \gamma_{34} = p_2 r_2$
$\gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} = 2q_1 r_1$	$\gamma_{11} + \gamma_{31} - \gamma_{15} - \gamma_{35} = 2r_2^2$
$\gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} = q_1$	$\gamma_{13} + \gamma_{33} - \gamma_{16} - \gamma_{36} = r_2$
$\gamma_{12} + \gamma_{14} + \gamma_{32} + \gamma_{34} = p_1 r_1$	$\gamma_{23} + \gamma_{33} - \gamma_{35} - \gamma_{25} = p_2(p_2 q_2 + 2r_2^2 - 1)$
$\gamma_{11} + \gamma_{15} + \gamma_{31} + \gamma_{35} = 2r_1^2$	$\gamma_{22} + \gamma_{32} - \gamma_{26} - \gamma_{36} = r_2(2p_2 q_2 + 4r_2^2 - 2)$
$\gamma_{13} + \gamma_{16} + \gamma_{33} + \gamma_{36} = r_1$	$\gamma_{21} + \gamma_{31} - \gamma_{24} - \gamma_{34} = p_2 q_2 + 2r_2^2$
$\gamma_{23} + \gamma_{25} + \gamma_{33} + \gamma_{35} = p_1(p_1 q_1 + 2r_1^2 - 1)$	$\gamma_{11} + \gamma_{21} + \gamma_{31} + \gamma_{41} = 1$
$\gamma_{22} + \gamma_{26} + \gamma_{32} + \gamma_{36} = r_1(2p_1 q_1 + 4r_1^2 - 2)$	$\gamma_{12} + \gamma_{22} + \gamma_{32} + \gamma_{42} = 0$
$\gamma_{21} + \gamma_{24} + \gamma_{31} + \gamma_{34} = p_1 q_1 + 2r_1^2$	$\gamma_{13} + \gamma_{23} + \gamma_{33} + \gamma_{43} = 0$
$\gamma_{11} + \gamma_{21} - \gamma_{16} - \gamma_{26} = p_2 q_2$	$\gamma_{14} + \gamma_{24} + \gamma_{34} + \gamma_{44} = 0$
$\gamma_{13} + \gamma_{23} - \gamma_{14} - \gamma_{24} = 2q_2 r_2$	$\gamma_{15} + \gamma_{25} + \gamma_{35} + \gamma_{45} = 0$
$\gamma_{12} + \gamma_{22} - \gamma_{15} - \gamma_{25} = q_2$	$\gamma_{16} + \gamma_{26} + \gamma_{36} + \gamma_{46} = 0$

If we solve this system we get

$\gamma_{11} = 0$
$\gamma_{12} = \frac{1}{4}(q_1 + q_2 + 2r_1 + p_1r_1 - 2p_1r_1q_1 - 4r_1^3 + 2r_2 + p_2r_2 - 2p_2r_2q_2 - 4r_2^3)$
$\gamma_{13} = \frac{1}{4}(p_1 + p_2 - p_1^2q_1 - p_2^2q_2 + r_1 + 2q_1r_1 - 2p_1r_1^2 + r_2 + 2q_2r_2 - 2p_2r_2^2)$
$\gamma_{14} = \frac{1}{4}(-p_1q_1 + p_2q_2 + p_1r_1 + 2q_1r_1 - 2r_1^2 - p_2r_2 - 2q_2r_2 + 2r_2^2)$
$\gamma_{15} = \frac{1}{4}(p_1 - p_2 + q_1 - p_1^2q_1 - q_2 + p_2^2q_2 + 2r_1^2 - 2p_1r_1^2 - 2r_2^2 + 2p_2r_2^2)$
$\gamma_{16} = \frac{1}{4}(p_1q_1 - p_2q_2 + 3r_1 - 2p_1q_1r_1 - 4r_1^3 - 3r_2 + 2p_2q_2r_2 + 4r_2^3)$
$\gamma_{21} = \frac{1}{2}(p_1q_1 + p_2q_2),$
$\gamma_{22} = \frac{1}{4}(q_1 + q_2 - 2r_1 - p_1r_1 + 2p_1q_1r_1 + 4r_1^3 - 2r_2 - p_2r_2 + 2p_2q_2r_2 + 4r_2^3)$
$\gamma_{23} = \frac{1}{4}(-p_1 - p_2 + p_1^2q_1 + p_2^2q_2 - r_1 + 2q_1r_1 + 2p_1r_1^2 - r_2 + 2q_2r_2 + 2p_2r_2^2)$
$\gamma_{24} = \frac{1}{4}(p_1q_1 - p_2q_2 - p_1r_1 + 2q_1r_1 + 2r_1^2 + p_2r_2 - 2q_2r_2 - 2r_2^2)$
$\gamma_{25} = \frac{1}{4}(-p_1 + p_2 + q_1 + p_1^2q_1 - q_2 - p_2^2q_2 - 2r_1^2 + 2p_1r_1^2 + 2r_2^2 - 2p_2r_2^2)$
$\gamma_{26} = \frac{1}{4}(p_1q_1 - p_2q_2 - 3r_1 + 2p_1q_1r_1 + 4r_1^3 + 3r_2 - 2p_2q_2r_2 - 4r_2^3)$

$\gamma_{31} = r_1^2 + r_2^2$
$\gamma_{32} = \frac{1}{4}(-q_1 - q_2 - 2r_1 - 2r_2 + p_1r_1 + p_2r_2 + 2p_1q_1r_1 + 2p_2q_2r_2 + 4r_1^3 + 4r_2^3)$
$\gamma_{33} = \frac{1}{4}(-p_1 - p_2 + p_1^2q_1 + p_2^2q_2 + r_1 + r_2 - 2q_1r_1 - 2q_2r_2 + 2p_1r_1^2 + 2p_2r_2^2)$
$\gamma_{34} = \frac{1}{4}(p_1q_1 - p_2q_2 + p_1r_1 - p_2r_2 - 2q_1r_1 + 2q_2r_2 + 2r_1^2 - 2r_2^2)$
$\gamma_{35} = \frac{1}{4}(-p_1 + p_2 - q_1 + q_2 + p_1^2q_1 - p_2^2q_2 + 2r_1^2 - 2r_2^2 + 2p_1r_1^2 - 2p_2r_2^2)$
$\gamma_{36} = \frac{1}{4}(-p_1q_1 + p_2q_2 - r_1 + r_2 + 2p_1q_1r_1 - 2p_2q_2r_2 + 4r_1^3 - 4r_2^3)$
$\gamma_{41} = \frac{1}{2}(2 - p_1q_1 - p_2q_2 - 2r_1^2 - 2r_2^2)$
$\gamma_{42} = \frac{1}{4}(-q_1 - q_2 + 2r_1 + 2r_2 - p_1r_1 - p_2r_2 - 2p_1q_1r_1 - 2p_2q_2r_2 - 4r_1^3 - 4r_2^3)$
$\gamma_{43} = \frac{1}{4}(p_1 + p_2 - p_1^2q_1 - p_2^2q_2 - r_1 - r_2 - 2q_1r_1 - 2q_2r_2 - 2p_1r_1^2 - 2p_2r_2^2)$
$\gamma_{44} = \frac{1}{4}(-p_1q_1 + p_2q_2 - p_1r_1 + p_2r_2 - 2q_1r_1 + 2q_2r_2 - 2r_1^2 + 2r_2^2)$
$\gamma_{45} = \frac{1}{4}(p_1 - p_2 - q_1 + q_2 - p_1^2q_1 + p_2^2q_2 - 2r_1^2 + 2r_2^2 - 2p_1r_1^2 + 2p_2r_2^2)$
$\gamma_{46} = \frac{1}{4}(-p_1q_1 + p_2q_2 + r_1 - r_2 - 2p_1q_1r_1 + 2p_2q_2r_2 - 4r_1^3 + 4r_2^3)$

Since $r_1 + r_2 = 2\gamma_{13} + 2\gamma_{33}$ and $q_1 + q_2 = 2\gamma_{12} + 2\gamma_{22}$ both $r_1 + r_2$ and $q_1 + q_2$ are even

Now let's analyse $p_1, q_1, r_1, p_2, q_2, r_2$ modulo 4.

If r_1, r_2, q_1, q_2 are odd

q_1	q_2	r_1	r_2	p_1 and p_2
1	1	1	1	From γ_{12} $p_1 + p_2$ is 2, from γ_{16} $p_1 - p_2$ is 0. so $p_1 = p_2 = 3$. However they don't satisfy γ_{13}
1	1	1	3	From γ_{12} $p_2 - p_1 = 2$, from γ_{14} $p_2 = 2$. So $p_1 = 0, p_2 = 2$ or $p_1 = 2, p_2 = 0$. In this case γ_{13} is not satisfied
1	1	3	1	Don't satisfy γ_{12}, γ_{13} and γ_{14} together
1	1	3	3	From γ_{12} $p_1 + p_2$ is 2, from γ_{16} $p_1 - p_2$ is 0. so $p_1 = p_2 = 3$. However they don't satisfy γ_{13}
1	3	1	1	From γ_{12} $p_1 + p_2$ is 0, from γ_{14} $p_1 = 2$. So $p_1 = p_2 = 2$. In this case γ_{13} is not satisfied
1	3	1	3	From γ_{16} $p_1 + p_2$ is 2, from γ_{12} $p_1 - p_2 = 0$. So $p_1 = p_2 = 1$ or $p_1 = p_2 = 3$. In this case γ_{13} is not satisfied
1	3	3	1	Don't satisfy γ_{12}, γ_{13} and γ_{16} together
1	3	3	3	From γ_{12} $p_1 + p_2$ is 0, from γ_{14} $p_2 = 2$. So $p_1 = p_2 = 2$. In this case γ_{13} is not satisfied
3	1	1	1	Don't satisfy γ_{12}, γ_{13} and γ_{14} together
3	1	1	3	Don't satisfy γ_{12}, γ_{13} and γ_{16} together
3	1	3	1	Don't satisfy γ_{12}, γ_{13} and γ_{16} together
3	1	3	3	Don't satisfy γ_{12}, γ_{13} and γ_{14} together
3	3	1	1	Don't satisfy γ_{12}, γ_{13} and γ_{16} together
3	3	1	3	Don't satisfy γ_{12}, γ_{13} and γ_{14} together
3	3	3	1	Don't satisfy γ_{12}, γ_{13} and γ_{14} together
3	3	3	3	Don't satisfy γ_{12}, γ_{13} and γ_{16} together

So when r_1, r_2, q_1, q_2 are odd we cannot find any torsion unit. Now let's have a look at what happens if r_1, r_2 are even and q_1, q_2 are odd.

q_1	q_2	r_1	r_2	p_1 and p_2
1	1	0	0	Don't satisfy \mathcal{Y}_{12} .
1	1	0	2	From \mathcal{Y}_{12} p_2 is odd, from \mathcal{Y}_{14} $p_1 + p_2 = 0$. In this case \mathcal{Y}_{15} is not satisfied.
1	1	2	0	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{14} and \mathcal{Y}_{15} together.
1	1	2	2	From \mathcal{Y}_{12} $p_1 + p_2 = 1$ or 3 , from \mathcal{Y}_{14} $p_1 - p_2 = 0$. This is impossible.
1	3	0	0	From \mathcal{Y}_{14} $p_1 + p_2 = 0$, So $p_1 = p_2 = 0$ or $p_1 = p_2 = 2$ or $p_1 = 1, p_2 = 3$ or $p_1 = 3, p_2 = 1$. In this case \mathcal{Y}_{15} is not satisfied.
1	3	0	2	From \mathcal{Y}_{12} p_2 is even, from \mathcal{Y}_{14} $p_1 - p_2 = 0$. In this case \mathcal{Y}_{15} is not satisfied.
1	3	2	0	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{14} and \mathcal{Y}_{15} together.
1	3	2	2	From \mathcal{Y}_{14} $p_1 + p_2 = 0$, So \mathcal{Y}_{15} is not satisfied.
3	1	0	0	From \mathcal{Y}_{14} $p_1 + p_2 = 0$, So \mathcal{Y}_{15} is not satisfied.
3	1	0	2	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{14} and \mathcal{Y}_{15} together.
3	1	2	0	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{14} and \mathcal{Y}_{15} together.
3	1	2	2	From \mathcal{Y}_{14} $p_1 + p_2 = 0$, So \mathcal{Y}_{15} is not satisfied.
3	3	0	0	Don't satisfy \mathcal{Y}_{12} .
3	3	0	2	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{14} and \mathcal{Y}_{15} together.
3	3	2	0	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{14} and \mathcal{Y}_{15} together.
3	3	2	2	From \mathcal{Y}_{12} $p_1 + p_2 = 1$ or 3 , from \mathcal{Y}_{14} $p_1 - p_2 = 0$. This is impossible.

Again we don't have any torsion unit.

If r_1, r_2 are odd and q_1, q_2 are even

q_1	q_2	r_1	r_2	p_1 and p_2
0	0			From γ_{15} $p_1 = p_2$, from γ_{16} $r_1 = r_2$. Then from γ_{12} p_1 is even. In this case $\gamma_{13} \notin \phi$
0	0			
0	0			
0	0			
0	2			$\gamma_{12} - \gamma_{14} \notin \phi$
0	2			
0	2			
0	2			
2	0			$\gamma_{12} + \gamma_{14} \notin \phi$
2	0			
2	0			
2	0			
2	2			From γ_{15} $p_1 = p_2$, from γ_{16} $r_1 = r_2$. Then from γ_{12} p_1 is even. In this case $\gamma_{13} \notin \phi$
2	2			
2	2			
2	2			

In this case we don't have any torsion unit either.

Hence r_1, r_2, q_1, q_2 are all even. Now let's analyze this case

q_1	q_2	r_1	r_2	p_1 and p_2
0	0	0	0	From \mathcal{Y}_{13} and \mathcal{Y}_{25} $p_1 = p_2 = 0$ or $p_1 = p_2 = 2$ and they satisfy all.
0	0	0	2	From \mathcal{Y}_{12} $p_2 = 2$, from \mathcal{Y}_{13} p_1 is 0. However they don't satisfy \mathcal{Y}_{25} .
0	0	2	0	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{13} and \mathcal{Y}_{25} together
0	0	2	2	From \mathcal{Y}_{13} and \mathcal{Y}_{25} both $p_1 = p_2 = 0$ or $p_1 = p_2 = 2$ and they satisfy all.
0	2	0	0	Don't satisfy \mathcal{Y}_{12}
0	2	0	2	From \mathcal{Y}_{12} p_2 is 1 or 3, from \mathcal{Y}_{13} p_1 is 3 or 1. However they don't satisfy \mathcal{Y}_{25} .
0	2	2	0	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{13} and \mathcal{Y}_{25} together
0	2	2	2	From \mathcal{Y}_{12} $p_1 + p_2$ is odd, however in this case \mathcal{Y}_{13} is not satisfied
2	0	0	0	Don't satisfy \mathcal{Y}_{12} .
2	0	0	2	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{13} and \mathcal{Y}_{25} together.
2	0	2	0	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{13} and \mathcal{Y}_{25} together.
2	0	2	2	Don't satisfy \mathcal{Y}_{12} and \mathcal{Y}_{13} together.
2	2	0	0	From \mathcal{Y}_{13} and \mathcal{Y}_{16} $p_1 = p_2 = 0$ or $p_1 = p_2 = 2$ and they satisfy all.
2	2	0	2	From \mathcal{Y}_{12} $p_2 = 2$, from \mathcal{Y}_{16} $p_1 = 3$. In this case \mathcal{Y}_{13} is not satisfied.
2	2	2	0	Don't satisfy \mathcal{Y}_{12} , \mathcal{Y}_{13} and \mathcal{Y}_{16} together.
2	2	2	2	From \mathcal{Y}_{13} and \mathcal{Y}_{16} $p_1 = p_2 = 0$ or $p_1 = p_2 = 2$ and they satisfy all.

As a result all possible values of $p_1, q_1, r_1, p_2, q_2, r_2$ modulo 4 are

q_1	q_2	r_1	r_2	p_1	p_2
0	0	0	0	0	0
0	0	0	0	2	2
0	0	2	2	0	0
0	0	2	2	2	2
2	2	0	0	0	0
2	2	0	0	2	2
2	2	2	2	0	0
2	2	2	2	2	2

Case 2) Let γ be rationally conjugate to (12)

i) By proposition 2.1.8. $\alpha_1 = 0$, $\alpha_2 = m - 3k - 1 + \frac{3k^2 + 2k}{m}$, $\alpha_3 = -\alpha_2$,

$\alpha_4 = m - 4k - 1 + \frac{3k^2 + 2k}{m}$, $\alpha_5 = 2k + 1 - m$ and $\alpha_6 = 1 - \alpha_4 - \alpha_5$.

$$\text{ii) } \overline{\gamma\rho_4} = \begin{bmatrix} \gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} & \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} & \gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} \\ -\gamma_{31} - \gamma_{36} - \gamma_{41} - \gamma_{46} & -\gamma_{33} - \gamma_{34} - \gamma_{43} - \gamma_{44} & -\gamma_{32} - \gamma_{35} - \gamma_{42} - \gamma_{45} \\ \gamma_{12} + \gamma_{14} - \gamma_{22} - \gamma_{24} & \gamma_{11} + \gamma_{15} - \gamma_{21} - \gamma_{25} & \gamma_{13} + \gamma_{16} - \gamma_{23} - \gamma_{26} \\ +\gamma_{32} + \gamma_{34} - \gamma_{42} - \gamma_{44} & +\gamma_{31} + \gamma_{35} - \gamma_{41} - \gamma_{45} & +\gamma_{33} + \gamma_{36} - \gamma_{43} - \gamma_{46} \\ \gamma_{13} + \gamma_{15} - \gamma_{23} - \gamma_{25} & \gamma_{12} + \gamma_{16} - \gamma_{22} - \gamma_{26} & \gamma_{11} + \gamma_{14} - \gamma_{21} - \gamma_{24} \\ -\gamma_{33} - \gamma_{35} + \gamma_{43} + \gamma_{45} & -\gamma_{32} - \gamma_{36} + \gamma_{42} + \gamma_{46} & -\gamma_{31} - \gamma_{34} + \gamma_{41} + \gamma_{44} \end{bmatrix}$$

By proposition 2.1.2. $\text{tr}(\overline{\gamma\rho_4}) = \text{tr}((12)\rho_4) = 1$. So if we take

$$\begin{aligned} \gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} - \gamma_{31} - \gamma_{36} - \gamma_{41} - \gamma_{46} &= a_1, \\ \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} - \gamma_{33} - \gamma_{34} - \gamma_{43} - \gamma_{44} &= b_1, \\ \gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} - \gamma_{32} - \gamma_{35} - \gamma_{42} - \gamma_{45} &= x_1, \\ \gamma_{12} + \gamma_{14} - \gamma_{22} - \gamma_{24} + \gamma_{32} + \gamma_{34} - \gamma_{42} - \gamma_{44} &= c_1, \\ \gamma_{11} + \gamma_{15} - \gamma_{21} - \gamma_{25} + \gamma_{31} + \gamma_{35} - \gamma_{41} - \gamma_{45} &= d_1, \\ \gamma_{13} + \gamma_{16} - \gamma_{23} - \gamma_{26} + \gamma_{33} + \gamma_{36} - \gamma_{43} - \gamma_{46} &= y_1, \\ \gamma_{13} + \gamma_{15} - \gamma_{23} - \gamma_{25} - \gamma_{33} - \gamma_{35} + \gamma_{43} + \gamma_{45} &= u_1, \\ \gamma_{12} + \gamma_{16} - \gamma_{22} - \gamma_{26} - \gamma_{32} - \gamma_{36} + \gamma_{42} + \gamma_{46} &= v_1, \\ \gamma_{11} + \gamma_{14} - \gamma_{21} - \gamma_{24} - \gamma_{31} - \gamma_{34} + \gamma_{41} + \gamma_{44} &= w_1 \end{aligned}$$

by proposition 2.2.2. we get

$$\begin{aligned} a_1 &= 1 - p_1 q_1, & b_1 &= -q_1 r_1, & c_1 &= p_1 r_1, \\ d_1 &= 1 + r_1^2, & w_1 &= p_1 q_1 - r_1^2 - 1, & x_1 &= n_1 q_1, \\ y_1 &= -n_1 r_1, & u_1 &= -\frac{p_1}{n_1} (p_1 q_1 - r_1^2 - 2), & v_1 &= -\frac{r_1}{n_1} (p_1 q_1 - r_1^2 - 2), \end{aligned}$$

where $n_1 \mid p_1 q_1 - r_1^2 - 2$.

$$\text{iii) } \overline{\gamma\rho_5} = \begin{bmatrix} \mathcal{Y}_{11} - \mathcal{Y}_{16} + \mathcal{Y}_{21} - \mathcal{Y}_{26} & \mathcal{Y}_{13} - \mathcal{Y}_{14} + \mathcal{Y}_{23} - \mathcal{Y}_{24} & \mathcal{Y}_{12} - \mathcal{Y}_{15} + \mathcal{Y}_{22} - \mathcal{Y}_{25} \\ -\mathcal{Y}_{31} + \mathcal{Y}_{36} - \mathcal{Y}_{41} + \mathcal{Y}_{46} & -\mathcal{Y}_{33} + \mathcal{Y}_{34} - \mathcal{Y}_{43} + \mathcal{Y}_{44} & -\mathcal{Y}_{32} + \mathcal{Y}_{35} - \mathcal{Y}_{42} + \mathcal{Y}_{45} \\ \mathcal{Y}_{12} - \mathcal{Y}_{14} - \mathcal{Y}_{22} + \mathcal{Y}_{24} & \mathcal{Y}_{11} - \mathcal{Y}_{15} - \mathcal{Y}_{21} + \mathcal{Y}_{25} & \mathcal{Y}_{13} - \mathcal{Y}_{16} - \mathcal{Y}_{23} + \mathcal{Y}_{26} \\ +\mathcal{Y}_{32} - \mathcal{Y}_{34} - \mathcal{Y}_{42} + \mathcal{Y}_{44} & +\mathcal{Y}_{31} - \mathcal{Y}_{35} - \mathcal{Y}_{41} + \mathcal{Y}_{45} & +\mathcal{Y}_{33} - \mathcal{Y}_{36} - \mathcal{Y}_{43} + \mathcal{Y}_{46} \\ \mathcal{Y}_{13} - \mathcal{Y}_{15} - \mathcal{Y}_{23} + \mathcal{Y}_{25} & \mathcal{Y}_{12} - \mathcal{Y}_{16} - \mathcal{Y}_{22} + \mathcal{Y}_{26} & \mathcal{Y}_{11} - \mathcal{Y}_{14} - \mathcal{Y}_{21} + \mathcal{Y}_{24} \\ -\mathcal{Y}_{33} + \mathcal{Y}_{35} + \mathcal{Y}_{43} - \mathcal{Y}_{45} & -\mathcal{Y}_{32} + \mathcal{Y}_{36} + \mathcal{Y}_{42} - \mathcal{Y}_{46} & -\mathcal{Y}_{31} + \mathcal{Y}_{34} + \mathcal{Y}_{41} - \mathcal{Y}_{44} \end{bmatrix}$$

By proposition 2.1.2. $\text{tr}(\overline{\gamma\rho_5}) = \text{tr}((12)\rho_5) = -1$. So if we take

$$\begin{aligned} \mathcal{Y}_{11} - \mathcal{Y}_{16} + \mathcal{Y}_{21} - \mathcal{Y}_{26} - \mathcal{Y}_{31} + \mathcal{Y}_{36} - \mathcal{Y}_{41} + \mathcal{Y}_{46} &= a_2, \\ \mathcal{Y}_{13} - \mathcal{Y}_{14} + \mathcal{Y}_{23} - \mathcal{Y}_{24} - \mathcal{Y}_{33} + \mathcal{Y}_{34} - \mathcal{Y}_{43} + \mathcal{Y}_{44} &= b_2, \\ \mathcal{Y}_{12} - \mathcal{Y}_{15} + \mathcal{Y}_{22} - \mathcal{Y}_{25} - \mathcal{Y}_{32} + \mathcal{Y}_{35} - \mathcal{Y}_{42} + \mathcal{Y}_{45} &= x_2, \\ \mathcal{Y}_{12} - \mathcal{Y}_{14} - \mathcal{Y}_{22} + \mathcal{Y}_{24} + \mathcal{Y}_{32} - \mathcal{Y}_{34} - \mathcal{Y}_{42} + \mathcal{Y}_{44} &= c_2, \\ \mathcal{Y}_{11} - \mathcal{Y}_{15} - \mathcal{Y}_{21} + \mathcal{Y}_{25} + \mathcal{Y}_{31} - \mathcal{Y}_{35} - \mathcal{Y}_{41} + \mathcal{Y}_{45} &= d_2, \\ \mathcal{Y}_{13} - \mathcal{Y}_{16} - \mathcal{Y}_{23} + \mathcal{Y}_{26} + \mathcal{Y}_{33} - \mathcal{Y}_{36} - \mathcal{Y}_{43} + \mathcal{Y}_{46} &= y_2, \\ \mathcal{Y}_{13} - \mathcal{Y}_{15} - \mathcal{Y}_{23} + \mathcal{Y}_{25} - \mathcal{Y}_{33} + \mathcal{Y}_{35} + \mathcal{Y}_{43} - \mathcal{Y}_{45} &= u_2, \\ \mathcal{Y}_{12} - \mathcal{Y}_{16} - \mathcal{Y}_{22} + \mathcal{Y}_{26} - \mathcal{Y}_{32} + \mathcal{Y}_{36} + \mathcal{Y}_{42} - \mathcal{Y}_{46} &= v_2, \\ \mathcal{Y}_{11} - \mathcal{Y}_{14} - \mathcal{Y}_{21} + \mathcal{Y}_{24} - \mathcal{Y}_{31} + \mathcal{Y}_{34} + \mathcal{Y}_{41} - \mathcal{Y}_{44} &= w_2 \end{aligned}$$

then by proposition 2.2.1. we get

$$\begin{aligned} a_2 &= p_2 q_2 - 1 & b_2 &= q_2 r_2 & c_2 &= p_2 r_2 \\ d_2 &= r_2^2 - 1 & w_2 &= 1 - p_2 q_2 - r_2^2 & x_2 &= n_2 q_2 \\ y_2 &= n_2 r_2 & u_2 &= -\frac{p_2}{n_2} (p_2 q_2 + r_2^2 - 2) & v_2 &= -\frac{r_2}{n_2} (p_2 q_2 + r_2^2 - 2) \end{aligned}$$

where $n_2 \mid p_2 q_2 + r_2^2 - 2$.

If we take $m = n_1 = n_2 = 1$ we have the system

$\gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} - \gamma_{31} - \gamma_{36} - \gamma_{41} - \gamma_{46} = 1 - p_1 q_1$
$\gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} - \gamma_{33} - \gamma_{34} - \gamma_{43} - \gamma_{44} = -q_1 r_1$
$\gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} - \gamma_{32} - \gamma_{35} - \gamma_{42} - \gamma_{45} = q_1$
$\gamma_{12} + \gamma_{14} - \gamma_{22} - \gamma_{24} + \gamma_{32} + \gamma_{34} - \gamma_{42} - \gamma_{44} = p_1 r_1$
$\gamma_{11} + \gamma_{15} - \gamma_{21} - \gamma_{25} + \gamma_{31} + \gamma_{35} - \gamma_{41} - \gamma_{45} = 1 + r_1^2$
$\gamma_{13} + \gamma_{16} - \gamma_{23} - \gamma_{26} + \gamma_{33} + \gamma_{36} - \gamma_{43} - \gamma_{46} = -r_1$
$\gamma_{13} + \gamma_{15} - \gamma_{23} - \gamma_{25} - \gamma_{33} - \gamma_{35} + \gamma_{43} + \gamma_{45} = -p_1(p_1 q_1 - r_1^2 - 2)$
$\gamma_{12} + \gamma_{16} - \gamma_{22} - \gamma_{26} - \gamma_{32} - \gamma_{36} + \gamma_{42} + \gamma_{46} = -r_1(p_1 q_1 - r_1^2 - 2)$
$\gamma_{11} + \gamma_{14} - \gamma_{21} - \gamma_{24} - \gamma_{31} - \gamma_{34} + \gamma_{41} + \gamma_{44} = p_1 q_1 - r_1^2 - 1$
$\gamma_{11} - \gamma_{16} + \gamma_{21} - \gamma_{26} - \gamma_{31} + \gamma_{36} - \gamma_{41} + \gamma_{46} = p_2 q_2 - 1$
$\gamma_{13} - \gamma_{14} + \gamma_{23} - \gamma_{24} - \gamma_{33} + \gamma_{34} - \gamma_{43} + \gamma_{44} = q_2 r_2$
$\gamma_{12} - \gamma_{15} + \gamma_{22} - \gamma_{25} - \gamma_{32} + \gamma_{35} - \gamma_{42} + \gamma_{45} = q_2$
$\gamma_{12} - \gamma_{14} - \gamma_{22} + \gamma_{24} + \gamma_{32} - \gamma_{34} - \gamma_{42} + \gamma_{44} = p_2 r_2$
$\gamma_{11} - \gamma_{15} - \gamma_{21} + \gamma_{25} + \gamma_{31} - \gamma_{35} - \gamma_{41} + \gamma_{45} = r_2^2 - 1$
$\gamma_{13} - \gamma_{16} - \gamma_{23} + \gamma_{26} + \gamma_{33} - \gamma_{36} - \gamma_{43} + \gamma_{46} = r_2$
$\gamma_{13} - \gamma_{15} - \gamma_{23} + \gamma_{25} - \gamma_{33} + \gamma_{35} + \gamma_{43} - \gamma_{45} = -p_2(p_2 q_2 + r_2^2 - 2)$
$\gamma_{12} - \gamma_{16} - \gamma_{22} + \gamma_{26} - \gamma_{32} + \gamma_{36} + \gamma_{42} - \gamma_{46} = -r_2(p_2 q_2 + r_2^2 - 2)$
$\gamma_{11} - \gamma_{14} - \gamma_{21} + \gamma_{24} - \gamma_{31} + \gamma_{34} + \gamma_{41} - \gamma_{44} = -p_2 q_2 - r_2^2 + 1$
$\gamma_{11} + \gamma_{21} + \gamma_{31} + \gamma_{41} = 0$
$\gamma_{12} + \gamma_{22} + \gamma_{32} + \gamma_{42} = -k + 3k^2$
$\gamma_{13} + \gamma_{23} + \gamma_{33} + \gamma_{43} = k - 3k^2$
$\gamma_{14} + \gamma_{24} + \gamma_{34} + \gamma_{44} = -2k + 3k^2$
$\gamma_{15} + \gamma_{25} + \gamma_{35} + \gamma_{45} = 1 - 3k^2$
$\gamma_{16} + \gamma_{26} + \gamma_{36} + \gamma_{46} = 2k$

Since $q_2 + q_1 = 2\gamma_{12} + 2\gamma_{22} - 2\gamma_{32} - 2\gamma_{42}$ $q_1 + q_2$ is even.

If we solve this system we get

$\gamma_{11} = 0$
$\gamma_{12} = \frac{1}{8}(6k^2 - 2k + q_1 + q_2 + 2r_1 + 2r_2 + r_1^3 - r_2^3 + p_1r_1 + p_2r_2 - p_1q_1r_1 - p_2q_2r_2)$
$\gamma_{13} = \frac{1}{8}(-6k^2 + 2k + 2p_1 + 2p_2 - r_1 + r_2 - q_1r_1 + q_2r_2 - p_1^2q_1 - p_2^2q_2 + p_1r_1^2 - p_2r_2^2)$
$\gamma_{14} = \frac{1}{8}(6k^2 - 4k - 2 - r_1^2 + r_2^2 + p_1q_1 + p_2q_2 + p_1r_1 - p_2r_2 - q_1r_1 - q_2r_2)$
$\gamma_{15} = \frac{1}{8}(-6k^2 + 4 + 2p_1 - 2p_2 + q_1 - q_2 + r_1^2 - r_2^2 - p_1^2q_1 + p_2^2q_2 + p_1r_1^2 + p_2r_2^2)$
$\gamma_{16} = \frac{1}{8}(4k + 2 - p_1q_1 - p_2q_2 + r_1 - 3r_2 + r_1^3 + r_2^3 - p_1q_1r_1 + p_2q_2r_2)$
$\gamma_{21} = \frac{1}{4}(p_2q_2 - p_1q_1)$
$\gamma_{22} = \frac{1}{8}(6k^2 - 2k + q_1 + q_2 - 2r_1 - 2r_2 - r_1^3 + r_2^3 - p_1r_1 - p_2r_2 + p_1q_1r_1 + p_2q_2r_2)$
$\gamma_{23} = \frac{1}{8}(-6k^2 + 2k - 2p_1 - 2p_2 + r_1 - r_2 - q_1r_1 + q_2r_2 + p_1^2q_1 + p_2^2q_2 - p_1r_1^2 + p_2r_2^2)$
$\gamma_{24} = \frac{1}{8}(6k^2 - 4k + 2 + r_1^2 - r_2^2 - p_1q_1 - p_2q_2 - p_1r_1 + p_2r_2 - q_1r_1 - q_2r_2)$
$\gamma_{25} = \frac{1}{8}(-6k^2 - 2p_1 + 2p_2 + q_1 - q_2 - r_1^2 + r_2^2 + p_1^2q_1 - p_2^2q_2 - p_1r_1^2 - p_2r_2^2)$
$\gamma_{26} = \frac{1}{8}(4k + 2 - p_1q_1 - p_2q_2 - r_1 + 3r_2 - r_1^3 - r_2^3 + p_1q_1r_1 - p_2q_2r_2)$

$\gamma_{31} = \frac{1}{4}(r_2^2 + r_1^2)$
$\gamma_{32} = \frac{1}{8}(6k^2 - 2k - q_1 - q_2 - 2r_1 - 2r_2 - r_1^3 + r_2^3 + p_1r_1 + p_2r_2 + p_1q_1r_1 + p_2q_2r_2)$
$\gamma_{33} = \frac{1}{8}(-6k^2 + 2k - 2p_1 - 2p_2 - r_1 + r_2 + q_1r_1 - q_2r_2 + p_1^2q_1 + p_2^2q_2 - p_1r_1^2 + p_2r_2^2)$
$\gamma_{34} = \frac{1}{8}(6k^2 - 4k + 2 + r_1^2 - r_2^2 - p_1q_1 - p_2q_2 + p_1r_1 - p_2r_2 + q_1r_1 + q_2r_2)$
$\gamma_{35} = \frac{1}{8}(-6k^2 + 4 - 2p_1 + 2p_2 - q_1 + q_2 + r_1^2 - r_2^2 + p_1^2q_1 - p_2^2q_2 - p_1r_1^2 - p_2r_2^2)$
$\gamma_{36} = \frac{1}{8}(4k - 2 + p_1q_1 + p_2q_2 - 3r_1 + r_2 - r_1^3 - r_2^3 + p_1q_1r_1 - p_2q_2r_2)$
$\gamma_{41} = \frac{1}{4}(-p_2q_2 + p_1q_1 - r_2^2 - r_1^2)$
$\gamma_{42} = \frac{1}{8}(6k^2 - 2k - q_1 - q_2 + 2r_1 + 2r_2 + r_1^3 - r_2^3 - p_1r_1 - p_2r_2 - p_1q_1r_1 - p_2q_2r_2)$
$\gamma_{43} = \frac{1}{8}(-6k^2 + 2k + 2p_1 + 2p_2 + r_1 - r_2 + q_1r_1 - q_2r_2 - p_1^2q_1 - p_2^2q_2 + p_1r_1^2 - p_2r_2^2)$
$\gamma_{44} = \frac{1}{8}(6k^2 - 4k - 2 - r_1^2 + r_2^2 + p_1q_1 + p_2q_2 - p_1r_1 + p_2r_2 + q_1r_1 + q_2r_2)$
$\gamma_{45} = \frac{1}{8}(-6k^2 + 2p_1 - 2p_2 - q_1 + q_2 - r_1^2 + r_2^2 - p_1^2q_1 + p_2^2q_2 + p_1r_1^2 + p_2r_2^2)$
$\gamma_{46} = \frac{1}{8}(4k - 2 + p_1q_1 + p_2q_2 + 3r_1 - r_2 + r_1^3 + r_2^3 - p_1q_1r_1 + p_2q_2r_2)$

3.2. Torsion Units of Order 3

Let $\gamma \in (\mathcal{C}S_4)$ be a torsion unit of order 3. Then by theorem 2.1.1. γ : (123)

$$\text{i) } \overline{\gamma\rho_1} = \left[\sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} \right] = [1]$$

$$\text{ii) } \overline{\gamma\rho_2} = \left[\left(\sum_{i=1}^4 \sum_{j=1}^3 \gamma_{ij} \right) - \left(\sum_{i=1}^4 \sum_{j=4}^6 \gamma_{ij} \right) \right] = [1]$$

iii) By proposition 2.1.8.

$$\begin{aligned} \alpha_1 &= 0 & \alpha_2 &= n - 3k + 2 + \frac{3k^2 - 3k + 1}{n} & \alpha_3 &= 3k - n - 1 - \frac{3k^2 - 3k + 1}{n} \\ \alpha_6 &= 2k - 1 - n & \alpha_5 &= 1 - 2k + \frac{3k^2 - 3k + 1}{n} & \alpha_4 &= n - 4k + 2 + \frac{3k^2 - 3k + 1}{n} \end{aligned}$$

where k is a free parameter and n is a divisor of $3k^2 - 3k + 1$.

$$\text{iv) } \overline{\gamma\rho_4} = \begin{bmatrix} \gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} & \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} & \gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} \\ -\gamma_{31} - \gamma_{36} - \gamma_{41} - \gamma_{46} & -\gamma_{33} - \gamma_{34} - \gamma_{43} - \gamma_{44} & -\gamma_{32} - \gamma_{35} - \gamma_{42} - \gamma_{45} \\ \gamma_{12} + \gamma_{14} - \gamma_{22} - \gamma_{24} & \gamma_{11} + \gamma_{15} - \gamma_{21} - \gamma_{25} & \gamma_{13} + \gamma_{16} - \gamma_{23} - \gamma_{26} \\ +\gamma_{32} + \gamma_{34} - \gamma_{42} - \gamma_{44} & +\gamma_{31} + \gamma_{35} - \gamma_{41} - \gamma_{45} & +\gamma_{33} + \gamma_{36} - \gamma_{43} - \gamma_{46} \\ \gamma_{13} + \gamma_{15} - \gamma_{23} - \gamma_{25} & \gamma_{12} + \gamma_{16} - \gamma_{22} - \gamma_{26} & \gamma_{11} + \gamma_{14} - \gamma_{21} - \gamma_{24} \\ -\gamma_{33} - \gamma_{35} + \gamma_{43} + \gamma_{45} & -\gamma_{32} - \gamma_{36} + \gamma_{42} + \gamma_{46} & -\gamma_{31} - \gamma_{34} + \gamma_{41} + \gamma_{44} \end{bmatrix}$$

If we take

$$\begin{aligned} \gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} - \gamma_{31} - \gamma_{36} - \gamma_{41} - \gamma_{46} &= a_1, \\ \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} - \gamma_{33} - \gamma_{34} - \gamma_{43} - \gamma_{44} &= b_1, \\ \gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} - \gamma_{32} - \gamma_{35} - \gamma_{42} - \gamma_{45} &= u_1, \\ \gamma_{12} + \gamma_{14} - \gamma_{22} - \gamma_{24} + \gamma_{32} + \gamma_{34} - \gamma_{42} - \gamma_{44} &= c_1, \\ \gamma_{11} + \gamma_{15} - \gamma_{21} - \gamma_{25} + \gamma_{31} + \gamma_{35} - \gamma_{41} - \gamma_{45} &= d_1, \\ \gamma_{13} + \gamma_{16} - \gamma_{23} - \gamma_{26} + \gamma_{33} + \gamma_{36} - \gamma_{43} - \gamma_{46} &= v_1, \\ \gamma_{13} + \gamma_{15} - \gamma_{23} - \gamma_{25} - \gamma_{33} - \gamma_{35} + \gamma_{43} + \gamma_{45} &= x_1, \\ \gamma_{12} + \gamma_{16} - \gamma_{22} - \gamma_{26} - \gamma_{32} - \gamma_{36} + \gamma_{42} + \gamma_{46} &= y_1, \\ \gamma_{11} + \gamma_{14} - \gamma_{21} - \gamma_{24} - \gamma_{31} - \gamma_{34} + \gamma_{41} + \gamma_{44} &= z_1 \end{aligned}$$

we get $A = \begin{bmatrix} a_1 & b_1 & u_1 \\ c_1 & d_1 & v_1 \\ x_1 & y_1 & z_1 \end{bmatrix}$

Since $A^3 = I$ by proposition 2.2.4.

$$\begin{aligned} z_1 &= -a_1 - d_1 \\ u_1 &= t[-(b_1x_1 - a_1y_1)(a_1^2 + a_1d_1 + d_1^2 + b_1c_1) - y_1(1 + (a_1 + d_1)(a_1d_1 - b_1c_1))] \\ v_1 &= t[(c_1x_1 - d_1y_1)(a_1^2 + a_1d_1 + d_1^2 + b_1c_1) + x_1(1 + (a_1 + d_1)(a_1d_1 - b_1c_1))] \end{aligned}$$

where $b_1x_1^2 + (d_1 - a_1)x_1y_1 - c_1y_1^2 = \pm 1$ and $t = \pm 1$.

$$\text{v) } \overline{\gamma\rho_5} = \begin{bmatrix} \gamma_{11} - \gamma_{16} + \gamma_{21} - \gamma_{26} & \gamma_{13} - \gamma_{14} + \gamma_{23} - \gamma_{24} & \gamma_{12} - \gamma_{15} + \gamma_{22} - \gamma_{25} \\ -\gamma_{31} + \gamma_{36} - \gamma_{41} + \gamma_{46} & -\gamma_{33} + \gamma_{34} - \gamma_{43} + \gamma_{44} & -\gamma_{32} + \gamma_{35} - \gamma_{42} + \gamma_{45} \\ \gamma_{12} - \gamma_{14} - \gamma_{22} + \gamma_{24} & \gamma_{11} - \gamma_{15} - \gamma_{21} + \gamma_{25} & \gamma_{13} - \gamma_{16} - \gamma_{23} + \gamma_{26} \\ +\gamma_{32} - \gamma_{34} - \gamma_{42} + \gamma_{44} & +\gamma_{31} - \gamma_{35} - \gamma_{41} + \gamma_{45} & +\gamma_{33} - \gamma_{36} - \gamma_{43} + \gamma_{46} \\ \gamma_{13} - \gamma_{15} - \gamma_{23} + \gamma_{25} & \gamma_{12} - \gamma_{16} - \gamma_{22} + \gamma_{26} & \gamma_{11} - \gamma_{14} - \gamma_{21} + \gamma_{24} \\ -\gamma_{33} + \gamma_{35} + \gamma_{43} - \gamma_{45} & -\gamma_{32} + \gamma_{36} + \gamma_{42} - \gamma_{46} & -\gamma_{31} + \gamma_{34} + \gamma_{41} - \gamma_{44} \end{bmatrix}$$

By assigning entries of $\overline{\gamma\rho_5}$ by parameters as follows

$$\begin{aligned} \gamma_{11} - \gamma_{16} + \gamma_{21} - \gamma_{26} - \gamma_{31} + \gamma_{36} - \gamma_{41} + \gamma_{46} &= a_2, \\ \gamma_{13} - \gamma_{14} + \gamma_{23} - \gamma_{24} - \gamma_{33} + \gamma_{34} - \gamma_{43} + \gamma_{44} &= b_2, \\ \gamma_{12} - \gamma_{15} + \gamma_{22} - \gamma_{25} - \gamma_{32} + \gamma_{35} - \gamma_{42} + \gamma_{45} &= u_2, \\ \gamma_{12} - \gamma_{14} - \gamma_{22} + \gamma_{24} + \gamma_{32} - \gamma_{34} - \gamma_{42} + \gamma_{44} &= c_2, \\ \gamma_{11} - \gamma_{15} - \gamma_{21} + \gamma_{25} + \gamma_{31} - \gamma_{35} - \gamma_{41} + \gamma_{45} &= d_2, \\ \gamma_{13} - \gamma_{16} - \gamma_{23} + \gamma_{26} + \gamma_{33} - \gamma_{36} - \gamma_{43} + \gamma_{46} &= v_2, \\ \gamma_{13} - \gamma_{15} - \gamma_{23} + \gamma_{25} - \gamma_{33} + \gamma_{35} + \gamma_{43} - \gamma_{45} &= x_2, \\ \gamma_{12} - \gamma_{16} - \gamma_{22} + \gamma_{26} - \gamma_{32} + \gamma_{36} + \gamma_{42} - \gamma_{46} &= y_2, \\ \gamma_{11} - \gamma_{14} - \gamma_{21} + \gamma_{24} - \gamma_{31} + \gamma_{34} + \gamma_{41} - \gamma_{44} &= z_2 \end{aligned}$$

we get $A = \begin{bmatrix} a_2 & x_2 & y_2 \\ b_2 & c_2 & z_2 \\ d_2 & t_2 & p_2 \end{bmatrix}$

Since $A^3 = I$ by proposition 2.2.4.

$$\begin{aligned} z_2 &= -a_2 - d_2 \\ u_2 &= t[-(b_2x_2 - a_2y_2)(a_2^2 + a_2d_2 + d_2^2 + b_2c_2) - y_2(1 + (a_2 + d_2)(a_2d_2 - b_2c_2))] \\ v_2 &= t[(c_2x_2 - d_2y_2)(a_2^2 + a_2d_2 + d_2^2 + b_2c_2) + x_2(1 + (a_2 + d_2)(a_2d_2 - b_2c_2))] \end{aligned}$$

where $b_2x_2^2 + (d_2 - a_2)x_2y_2 - c_2y_2^2 = \pm 1$ and $t = \pm 1$.

3.3. Torsion Units of Order 4

Let $\gamma \in (\mathcal{C}S_4)$ be a torsion unit of order 4. Then by theorem 2.1.1. $\gamma: (1234)$

$$\text{i) } \overline{\gamma\rho_1} = \left[\sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} \right] = [1]$$

$$\text{ii) } \overline{\gamma\rho_2} = \left[\left(\sum_{i=1}^4 \sum_{j=1}^3 \gamma_{ij} \right) - \left(\sum_{i=1}^4 \sum_{j=4}^6 \gamma_{ij} \right) \right] = [-1]$$

$$\text{iii) By proposition 2.1.8 } \alpha_1 = 0, \alpha_2 = n - 3k - 1 + \frac{3k^2 + 2k}{n}, \alpha_3 = -\alpha_2,$$

$$\alpha_4 = n - 4k - 1 + \frac{3k^2 + 2k}{n}, \alpha_5 = 2k + 1 - n \text{ and } \alpha_6 = 1 - \alpha_4 - \alpha_5 \text{ where } \alpha_k = \sum_{i=1}^4 \gamma_{ik}$$

$$\text{iv) } \overline{\gamma\rho_4} = \begin{bmatrix} \gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} & \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} & \gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} \\ -\gamma_{31} - \gamma_{36} - \gamma_{41} - \gamma_{46} & -\gamma_{33} - \gamma_{34} - \gamma_{43} - \gamma_{44} & -\gamma_{32} - \gamma_{35} - \gamma_{42} - \gamma_{45} \\ \gamma_{12} + \gamma_{14} - \gamma_{22} - \gamma_{24} & \gamma_{11} + \gamma_{15} - \gamma_{21} - \gamma_{25} & \gamma_{13} + \gamma_{16} - \gamma_{23} - \gamma_{26} \\ +\gamma_{32} + \gamma_{34} - \gamma_{42} - \gamma_{44} & +\gamma_{31} + \gamma_{35} - \gamma_{41} - \gamma_{45} & +\gamma_{33} + \gamma_{36} - \gamma_{43} - \gamma_{46} \\ \gamma_{13} + \gamma_{15} - \gamma_{23} - \gamma_{25} & \gamma_{12} + \gamma_{16} - \gamma_{22} - \gamma_{26} & \gamma_{11} + \gamma_{14} - \gamma_{21} - \gamma_{24} \\ -\gamma_{33} - \gamma_{35} + \gamma_{43} + \gamma_{45} & -\gamma_{32} - \gamma_{36} + \gamma_{42} + \gamma_{46} & -\gamma_{31} - \gamma_{34} + \gamma_{41} + \gamma_{44} \end{bmatrix}$$

By proposition 2.1.2 $tr(\overline{\gamma\rho_4}) = tr((1234)\rho_4) = -1$.

By proposition 2.2.3. if we take

$$\begin{aligned} \gamma_{11} + \gamma_{16} + \gamma_{21} + \gamma_{26} - \gamma_{31} - \gamma_{36} - \gamma_{41} - \gamma_{46} &= a_1, \\ \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} - \gamma_{33} - \gamma_{34} - \gamma_{43} - \gamma_{44} &= b_1, \\ \gamma_{12} + \gamma_{15} + \gamma_{22} + \gamma_{25} - \gamma_{32} - \gamma_{35} - \gamma_{42} - \gamma_{45} &= c_1, \\ \gamma_{12} + \gamma_{14} - \gamma_{22} - \gamma_{24} + \gamma_{32} + \gamma_{34} - \gamma_{42} - \gamma_{44} &= p_1, \\ \gamma_{11} + \gamma_{15} - \gamma_{21} - \gamma_{25} + \gamma_{31} + \gamma_{35} - \gamma_{41} - \gamma_{45} &= q_1, \\ \gamma_{13} + \gamma_{16} - \gamma_{23} - \gamma_{26} + \gamma_{33} + \gamma_{36} - \gamma_{43} - \gamma_{46} &= r_1, \\ \gamma_{13} + \gamma_{15} - \gamma_{23} - \gamma_{25} - \gamma_{33} - \gamma_{35} + \gamma_{43} + \gamma_{45} &= x_1, \\ \gamma_{12} + \gamma_{16} - \gamma_{22} - \gamma_{26} - \gamma_{32} - \gamma_{36} + \gamma_{42} + \gamma_{46} &= y_1, \\ \gamma_{11} + \gamma_{14} - \gamma_{21} - \gamma_{24} - \gamma_{31} - \gamma_{34} + \gamma_{41} + \gamma_{44} &= z_1 \end{aligned}$$

$$\text{then } A^4 = I \Leftrightarrow \begin{cases} i) a_1 + q_1 + z_1 = -1 \\ ii) a_1 q_1 + a_1 z_1 + q_1 z_1 - b_1 p_1 - c_1 x_1 - r_1 y_1 = 1 \\ iii) x_1 (b_1 r_1 - q_1 c_1) + y_1 (c_1 p_1 - a_1 r_1) + z_1 (a_1 q_1 - b_1 p_1) = -1 \end{cases}$$

$$v) \overline{\gamma \rho_5} = \begin{bmatrix} \gamma_{11} - \gamma_{16} + \gamma_{21} - \gamma_{26} & \gamma_{13} - \gamma_{14} + \gamma_{23} - \gamma_{24} & \gamma_{12} - \gamma_{15} + \gamma_{22} - \gamma_{25} \\ -\gamma_{31} + \gamma_{36} - \gamma_{41} + \gamma_{46} & -\gamma_{33} + \gamma_{34} - \gamma_{43} + \gamma_{44} & -\gamma_{32} + \gamma_{35} - \gamma_{42} + \gamma_{45} \\ \gamma_{12} - \gamma_{14} - \gamma_{22} + \gamma_{24} & \gamma_{11} - \gamma_{15} - \gamma_{21} + \gamma_{25} & \gamma_{13} - \gamma_{16} - \gamma_{23} + \gamma_{26} \\ +\gamma_{32} - \gamma_{34} - \gamma_{42} + \gamma_{44} & +\gamma_{31} - \gamma_{35} - \gamma_{41} + \gamma_{45} & +\gamma_{33} - \gamma_{36} - \gamma_{43} + \gamma_{46} \\ \gamma_{13} - \gamma_{15} - \gamma_{23} + \gamma_{25} & \gamma_{12} - \gamma_{16} - \gamma_{22} + \gamma_{26} & \gamma_{11} - \gamma_{14} - \gamma_{21} + \gamma_{24} \\ -\gamma_{33} + \gamma_{35} + \gamma_{43} - \gamma_{45} & -\gamma_{32} + \gamma_{36} + \gamma_{42} - \gamma_{46} & -\gamma_{31} + \gamma_{34} + \gamma_{41} - \gamma_{44} \end{bmatrix}$$

By representing the entries of $\overline{\gamma \rho_5}$

$$\begin{aligned} \gamma_{11} - \gamma_{16} + \gamma_{21} - \gamma_{26} - \gamma_{31} + \gamma_{36} - \gamma_{41} + \gamma_{46} &= a_2, \\ \gamma_{13} - \gamma_{14} + \gamma_{23} - \gamma_{24} - \gamma_{33} + \gamma_{34} - \gamma_{43} + \gamma_{44} &= b_2, \\ \gamma_{12} - \gamma_{15} + \gamma_{22} - \gamma_{25} - \gamma_{32} + \gamma_{35} - \gamma_{42} + \gamma_{45} &= c_2, \\ \gamma_{12} - \gamma_{14} - \gamma_{22} + \gamma_{24} + \gamma_{32} - \gamma_{34} - \gamma_{42} + \gamma_{44} &= p_2, \\ \gamma_{11} - \gamma_{15} - \gamma_{21} + \gamma_{25} + \gamma_{31} - \gamma_{35} - \gamma_{41} + \gamma_{45} &= q_2, \\ \gamma_{13} - \gamma_{16} - \gamma_{23} + \gamma_{26} + \gamma_{33} - \gamma_{36} - \gamma_{43} + \gamma_{46} &= r_2, \\ \gamma_{13} - \gamma_{15} - \gamma_{23} + \gamma_{25} - \gamma_{33} + \gamma_{35} + \gamma_{43} - \gamma_{45} &= x_2, \\ \gamma_{12} - \gamma_{16} - \gamma_{22} + \gamma_{26} - \gamma_{32} + \gamma_{36} + \gamma_{42} - \gamma_{46} &= y_2, \\ \gamma_{11} - \gamma_{14} - \gamma_{21} + \gamma_{24} - \gamma_{31} + \gamma_{34} + \gamma_{41} - \gamma_{44} &= z_2 \end{aligned}$$

we get by proposition 2.2.3

$$A^4 = I \Leftrightarrow \begin{cases} i) a_2 + q_2 + z_2 = 1 \\ ii) a_2 q_2 + a_2 z_2 + q_2 z_2 - b_2 p_2 - c_2 x_2 - r_2 y_2 = 1 \\ iii) x_2 (b_2 r_2 - q_2 c_2) + y_2 (c_2 p_2 - a_2 r_2) + z_2 (a_2 q_2 - b_2 p_2) = 1 \end{cases}$$

3.4. Conclusion

Let $\gamma = \sum_{i=1}^4 \sum_{j=1}^6 \gamma_{ij} \in U_1(\mathcal{C}S_4)$ be a torsion unit. Then the order of γ can be 2, 3 or 4 .

In this thesis we have analyzed all cases and we have characterized all torsion units in terms of parameters

Case-1 :

If the torsion unit γ of order 2 is rationally conjugate to (12)(34) then all coefficients can be expressed in terms of 6 free parameters $p_1, q_1, r_1, p_2, q_2, r_2$ and 2 dependent parameters (n_1, n_2) such that $n_1 \mid p_1 q_1 - r_1^2 - 2$ and $n_2 \mid p_2 q_2 + r_2^2 - 2$.

Case-2 :

If the torsion unit γ of order 2 is rationally conjugate to (12) then all coefficients can be expressed in terms of 7 free parameters $(p_1, q_1, r_1, p_2, q_2, r_2, k)$ and 3 dependent parameters (n_1, n_2, m) such that $m \mid 3k^2 + 2k$, $n_1 \mid p_1 q_1 + r_1^2 - 2$ and $n_2 \mid p_2 q_2 + r_2^2 - 2$.

Case-3 :

If the torsion unit γ of order 3 then γ is rationally conjugate to (123). All coefficients can be expressed in terms of 14 parameters. Five of them are free parameters; (a_1, d_1, a_2, d_2, k) , other 9 parameters are dependent satisfying following relations;

- 1) $b_1 x_1^2 + (d_1 - a_1)x_1 y_1 - c_1 y_1^2 = \pm 1$
- 2) $b_2 x_2^2 + (d_2 - a_2)x_2 y_2 - c_2 y_2^2 = \pm 1$
- 3) n is a divisor of $3k^2 - 3k + 1$.

Case-4 :

If the torsion unit γ of order 4 then γ is rationally conjugate to (1234). All coefficients can be expressed in terms of 14 parameters. Five of them are free parameters; (a_1, q_1, a_2, q_2, k) , other 9 parameters are dependent satisfying following relations;

- 1) $a_1 q_1 + a_1 z_1 + q_1 z_1 - b_1 p_1 - c_1 x_1 - r_1 y_1 = 1$
- 2) $x_1(b_1 r_1 - q_1 c_1) + y_1(c_1 p_1 - a_1 r_1) + z_1(a_1 q_1 - b_1 p_1) = -1$
- 3) $a_2 q_2 + a_2 z_2 + q_2 z_2 - b_2 p_2 - c_2 x_2 - r_2 y_2 = 1$
- 4) $x_2(b_2 r_2 - q_2 c_2) + y_2(c_2 p_2 - a_2 r_2) + z_2(a_2 q_2 - b_2 p_2) = 1$
- 5) n is a divisor of $3k^2 + 2k$.

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