İSTANBUL KÜLTÜR UNIVERSITY
INSTITUTE OF GRADUATE STUDIES

GENERALIZATION OF HARMONIC UNIVALENT CONVEX FUNCTIONS

THESIS OF DOCTOR OF PHILOSOPHY
BY

ASENA ÇETİNKAYA

Department: Mathematics and Computer Science
Programme: Mathematics

Supervisor : Assist. Prof. Dr. Yaşar POLATOĞLU

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I hereby declare that the thesis entitled "Generalization of Harmonic Univalent Convex Functions" submitted to Deparment of Mathematics and Computer Science, İstanbul Kültür University, for the award of the degree of Doctor of Philosophy in Mathematics, is an original research work carried out by me under the guidance of Assist. Prof. Dr. Yaşar POLATOĞLU and has not been submitted earlier in part or full or in any other form to any university or institute.

Asena ÇETİNKAYA

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İstanbul, June 2020 Asena ÇETİNKAYA

Varolmamın nedeni Canım Anneme...

Hüsna ÇETİNKAYA
(1953–2000)

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List of Symbols

$\mathbb{C}$	:	Set of complex numbers
$\mathcal{C}$	:	Class of convex functions
$\mathcal{C}(\alpha)$	:	Class of convex functions of order α, $0 \leq \alpha < 1$
$\mathcal{C}_q(b)$	:	Class of $q-$convex functions of complex order b
$\mathcal{C}_{\mathcal{H}}$	:	Class of harmonic convex functions
$\mathcal{C}_{\mathcal{H}}(\alpha)$	:	Class of harmonic convex functions of order α, $0 \leq \alpha < 1$
$\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$	:	Class of $q-$harmonic convex functions of complex order b and type α, $0 \leq \alpha < 1$
$\mathbb{D}$	:	Open unit disc $\{z : \|z\| < 1\}$
$\mathcal{D}$	:	Simply connected domain
$\mathcal{H}_q$	:	Class of $q-$harmonic univalent functions
J_f	:	Jacobian of a function f
$J_{q,f}$	:	$q-$Jacobian of a function f
$k(z)$	:	Koebe function
$\mathcal{P}_q$	:	$q-$Carathéodory class
$\mathbb{R}$	:	Set of real numbers
$\mathcal{S}$	:	Class of univalent functions
$\mathcal{S}^*$	:	Class of starlike functions
$\mathcal{S}^*(\alpha)$	:	Class of starlike functions of order α, $0 \leq \alpha < 1$
$\mathcal{S}_{\mathcal{H}}$	:	Class of harmonic univalent functions
$\mathcal{S}^*_{\mathcal{H}}$	:	Class of harmonic starlike functions
$\mathcal{S}^*_{\mathcal{H}}(\alpha)$	:	Class of harmonic starlike functions of order α, $0 \leq \alpha < 1$
$\mathcal{S}_{\mathcal{H}\mathcal{C}_q}(b)$	:	Class of $q-$harmonic mappings for which analytic part is $q-$convex functions of complex order
w	:	Second dilatation
w_q	:	$q-$Second dilatation
Δ	:	Laplace operator
Ω	:	Class of Schwarz functions
$\mathbb{Z}^+$	:	Set of positive integers

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Anabilim Dalı	**:**	**Matematik ve Bilgisayar Bilimleri**
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ÖZET

HARMONİK YALINKAT KONVEKS FONKSİYONLARIN GENELLEŞTİRİLMESİ

Asena ÇETİNKAYA

Kuantum kalkülüsün harmonik yalınkat fonksiyonlarda uygulamaları oldukça yenidir. Bu çalışmada, $q-$türev operatörü kullanılarak tanımlanan $q-$harmonik yalınkat fonksiyonların bazı alt sınıflarının incelenmesine odaklanılmıştır. Bu amaç için, harmonik fonksiyonların bazı temel terimlerini $q-$harmonik fonksiyonlara genelleştirmek gerekmektedir. İkinci bölümde, Jakobiyen ve ikinci dilatasyon $q-$harmonik fonksiyonlara genelleştirilmiştir. Bu yeni terimlerin ışığında, analitik kısmı komplex mertebeden $q-$konveks fonksiyon olan $q-$harmonik dönüşümlerin $\mathcal{S}_{\mathcal{HC}_q}(b)$ sınıfı tanımlanmıştır. Ayrıca $q-$Jack Lemma da ispatlanmıştır. Subordinasyon tekniği ve $q-$Jack Lemma kullanılarak, bu sınıfa ait fonksiyonların distorsiyon sınırları elde edilmiştir. Üçüncü bölümde, komplex mertebeden α tipinde $q-$harmonik yalınkat konveks fonksiyonların $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ sınıfı tanımlanmıştır, ve bu sınıfa ait fonksiyonların katsayı tahminleri ve yalınkatlık kriteri incelenmiştir. Katsayı tahminlerinden yararlanarak, bu sınıfa ait fonksiyonların kısmi toplamları, distorsiyon sınırları, kapsama teoremi ve konvolüsyon koşulları elde edilmiştir.

Anahtar Kelimeler: Kuantum kalkülüs, $q-$türev operatörü, yalınkat fonksiyon, analitik fonksiyon, harmonik fonksiyon.

University	: İstanbul Kültür University
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SUMMARY

GENERALIZATION OF HARMONIC UNIVALENT CONVEX FUNCTIONS

Asena ÇETİNKAYA

Applications of quantum calculus in harmonic univalent functions are quite new. In the present work, it is focused on investigating several subclasses of $q-$harmonic univalent functions defined by $q-$difference operator. For this purpose, it is necessary to extend some basic terms of the harmonic functions to the $q-$harmonic functions. In second chapter, the Jacobian and the second dilatation are extended for $q-$harmonic functions. In the light of these new terms, the class $\mathcal{S}_{\mathcal{HC}_q}(b)$ of "$q-$harmonic mappings for which analytic part is $q-$convex functions of complex order" is introduced. Also, the $q-$Jack's Lemma is proved. By using subordination technique and the $q-$Jack's Lemma, distortion bounds of the functions in this class are obtained. In third chapter, the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ of "$q-$harmonic univalent convex functions of complex order and type α" is introduced, and coefficient estimates and univalence criteria of the functions in this class are investigated. Making use of these coefficient estimates, partial sums, distortion bounds, covering theorem and convolution conditions of the functions in this class are obtained.

Keywords: Quantum calculus, $q-$difference operator, univalent function, analytic function, harmonic function.

Chapter 1

INTRODUCTION

Geometric Function Theory (GFT) is a branch of complex analysis consisting of analysis and geometry. The history of this branch dates back to 19^{th} century. After Bernard Riemann proved the Riemann Mapping Theorem, which states that every simply connected domain maps conformally onto the open unit disc, analytic functions got attraction of researchers in 1851. This remarkable theorem led to the birth of Geometric Function Theory.

The theory of univalent functions is one of the important subjects in Geometric Function Theory. In 1907, Koebe [31] gave more useful concept of the Riemann Mapping Theorem because he generalized this theorem to conformal analytic univalent functions. At the same time, in 1914/15, Gronwall [20] proved the Area Theorem, and in 1916, Bieberbach [3] proved an estimate for the second coefficient of a normalized univalent function. All these developments gave rise to appear of the theory of univalent functions.

The theory of univalent functions is quite complicated so that some simplifying are necessary. Therefore, in view of Riemann Mapping Theorem, researchers used the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ instead of an arbitrary simply connected domain, made some normalizations in the Taylor series coefficients and investigated the useful class $\mathcal{S}$ of univalent functions.

The first serious study on univalent functions began in 1907 with a paper by Koebe [31]. In paper, apart from giving the correct version of the Riemann

Mapping Theorem, he also proved an existence of a positive fixed number c for a function f belongs to the class $\mathcal{S}$ of univalent functions in $\mathbb{D}$ such that

$$\cap_{f \in \mathcal{S}} f(\mathbb{D}) \supset \{w : |w| \leq c\}.$$

Until Bieberbach [3] proved that $c = 1/4$ in 1916, this result did not find any applications. Bieberbach proved this result by using the second coefficient estimate $|a_2| \leq 2$ given in [3]. The second coefficient estimate also encouraged Bieberbach to put forward his famous conjecture in the same paper. This conjecture challenged many mathematicians all over the world about 70 years. The Bieberbach's Conjecture led to the birth of several developments in Geometric Function Theory and gave rise to appear of many new methods and related problems. Some of these problems are still open while some of them were solved completely.

After Louis de Branges [11], a French mathematician, solved the Bieberbach's Conjecture in 1984, in the same year, harmonic mappings have taken attention of researchers thank to a paper by Clunie and Sheil-Small [6]. These researchers generalized the analytic univalent functions to the conformal harmonic univalent functions and defined the class $\mathcal{S}_{\mathcal{H}}$ of harmonic univalent functions. After this famous article, in 1986, Hengartner and Schober [21, 22] handled to get a suitable statement of the Riemann Mapping Theorem for harmonic univalent functions. Due to these researchers, interesting problems, conjectures, and many questions were generated. Some of these problems, conjectures and questions were solved, but many conjectures and questions are still unsolved and need to be solved.

Quantum calculus (or $q-$ calculus) is an extension of ordinary calculus where we do not need limits. The history of quantum calculus dates back to the studies of Leonhard Euler (1707-1783), Carl Friedrich Gauss (1777-1855), Eduard Heine (1821-1881) and Carl Gustav Jacobi (1804-1851). In 1908, Jackson [25, 27] started a comprehensive study in $q-$calculus and introduced the first systematic formula of $q-$derivative. In 1910, Jackson [26] also defined the $q-$integral, which was earlier defined by Thomae [55]. After these developments, $q-$calculus has been studied in different topics of physics and mathematics; for example, in the basic hypergeometric functions, orthogonal polynomials, combinatorics, ordinary fractional calculus, and more recently in the Geometric Function Theory.

In 1990, Ismail, Merkes and Styer [23] applied to quantum calculus and introduced some interesting results of the analytic univalent functions. Although a $q-$version of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{D}$ was investigated by these researchers [23], in fact the first step of $q-$calculus in Geometric Function Theory was presented by Srivastava in [52]. After these important steps, $q-$calculus has attracted a great deal of attention and become a popular area for complex analysts.

Although there are many papers on applications of $q-$calculus in the theory of analytic univalent functions, but the impact of $q-$calculus in harmonic univalent functions is new and there are very little data on harmonic univalent functions with $q-$calculus. Therefore, this situation encouraged us to explore some new results for harmonic univalent functions related to $q-$claculus in our thesis.

This section is devoted to some fundamental definitions, theorems and conjectures of the analytic and harmonic univalent functions.

1.1 Univalent Functions

Let $\mathbb{C}$ be the complex plane. A domain $\mathcal{D}$ is an open connected non-empty set in the complex plane. The domain "$\mathcal{D}$ is said to be simply connected if its complement is connected. A neighbourhood of a set $V \subset \mathbb{C}$ is an open set which contains V. If $z_0 \in \mathbb{C}$ and $r > 0$, then $\mathbb{D}(z_0, r) = \{z : |z - z_0| < r\}$ denotes an open disc of radius r centered at z_0. The open disc $\mathbb{D}(0, r)$ will be denoted by $\mathbb{D}_r$ and the open unit disc $\mathbb{D}_1$ will be denoted by $\mathbb{D}$.

A complex-valued function f of a complex variable is *differentiable* at a point $z_0 \in \mathbb{C}$ if it has a derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at z_0. Such a function f is *analytic* at z_0 if it is differentiable at every point in some neighbourhood of z_0. Analytic functions are also called *holomorphic* functions. If f is analytic, then f have derivatives of all orders at z_0, and thus f has a Taylor series expansion given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

which is convergent in some open disc centered at z_0.

A continuous, complex-valued function $w = f(z) = u(z) + iv(z),\ z = (x, y)$ is a one-to-one mapping from a region in the $z-$plane to a region in the $w-$plane. If $w = f(z)$ is analytic, then u and v necessarily hold the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

in $\mathcal{D}$. A conformal mapping preserves angle and direction when it maps intersection curves in a domain to intersection curves in its range. An analytic mapping $w = f(z)$ in a domain $\mathcal{D}$ that preserves angles at a point z_0 in $\mathcal{D}$ is called *conformal* at z_0.

The Jacobian of a function $f = u + iv$ is defined by

$$J_f(z) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x. \tag{1.1}$$

If f is analytic, then in view of Cauchy-Riemann equations its Jacobian takes the form

$$J_f(z) = (u_x)^2 + (v_x)^2 = |f'(z)|^2.$$

The function $f = u + iv$ is conformal, if its Jacobian is nonvanishing.

A function f is said to be *univalent* or *schlicht* in $\mathcal{D}$ if it doesn't take the same value twice in $\mathcal{D}$. Mathematically, we can say that for two distinct points z_1 and z_2 in $\mathcal{D}$, $f(z_1) \neq f(z_2)$. Every analytic univalent function defined in $\mathcal{D}$ has nonvanishing derivative $f'(z) \neq 0$ for each $z \in \mathcal{D}$ and defines a conformal mapping therein. If a function f is univalent in some neighbourhood of a point $z_0 \in \mathcal{D}$, then f is called *locally univalent* at $z_0 \in \mathcal{D}$. Locally univalence of the function f at z_0 requires the condition $f'(z_0) \neq 0$. The condition $f'(z_0) \neq 0$ in a domain $\mathcal{D}$ is necessary, but not sufficient for univalence of f. For example; $f(z) = z^2$ is locally univalent in a domain $\mathcal{D} = \{z \in \mathbb{C} : 0 < |z| < 2, 0 < \arg z < 3\pi/2\}$, because $f'(z_0) = 2z \neq 0$ for all z_0 in $\mathcal{D}$. However, f is not univalent in $\mathcal{D}$ since

$$f\left(\frac{3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}i\right) = f\left(-\frac{3}{2\sqrt{2}} - \frac{3}{2\sqrt{2}}i\right) = \frac{9}{4}i.$$

The basic theorem, which leads to the birth of univalent functions, is the Riemann Mapping Theorem. In 1851, Riemann asserted that any simply connected domain can be mapped conformally onto the open unit disc $\mathbb{D}$. After this assertion, it was the German mathematician Koebe [31], who gave the correct version of" the Riemann Mapping Theorem.

Theorem 1.1.1. (*Riemann Mapping Theorem*) *Let $\mathcal{D} \subset \mathbb{C}$ be a simply connected domain and $z_0 \in \mathcal{D}$. Then there exists a unique analytic function $f : \mathcal{D} \to \mathbb{C}$ such that:*

1) $f(z_0) = 0$ and $f'(z_0) > 0$,

2) f is univalent,

3) $f(\mathcal{D}) = \mathbb{D}$.

1.1.1 The Class S

The theory of univalent functions is quite complicated so that some simplifying are necessary. In view of Riemann Mapping Theorem, it is possible to study the problems including univalence in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$. For this purpose, let $\mathcal{A}$ be the class of all analytic functions defined in the open unit disc $\mathbb{D}$ with the normalization condition $f(0) = f'(0) - 1 = 0$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Thus, a function $f \in \mathcal{S}$ has power series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ..., \quad (|z| < 1). \tag{1.2}$$

Let Σ be the class of functions φ which are univalent on $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}$, $(\mathbb{D}^* = \mathbb{C} \backslash \mathbb{D})$ with a simple pole at ∞ and normalized so that the Laurent series expansion of φ at ∞ has the form

$$\varphi(z) = z + \alpha_0 + \frac{\alpha_1}{z} + ... + \frac{\alpha_n}{z^n} + ..., \quad (|z| > 1). \tag{1.3}$$

The classes $\mathcal{S}$ and Σ are closely related as follows:

1) $f \in \mathcal{S} \Rightarrow \varphi \in \Sigma$, *where* $\varphi(z) = \frac{1}{f(1/z)}$.

2) $\varphi \in \Sigma$ *and* $c \in \mathbb{C} \backslash \varphi(\mathbb{D}^*) \Rightarrow f \in \mathcal{S}$, *where* $f(z) = \frac{1}{\varphi(1/z) - c}$.

The most important example of the class $\mathcal{S}$ is the Koebe function. There are also another examples for the classes $\mathcal{S}$ and Σ. Some of these examples can be listed as below:

i) The Koebe function given by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n = z + 2z^2 + 3z^3 + ..., \ (z \in \mathbb{D})$$

belongs to the class $\mathcal{S}$. This function maps the open unit disc $\mathbb{D}$ conformally onto the complex plane except for a real slit along $(-\infty, -1/4]$, and plays an extremal role in many problems in the theory of univalent functions (see Figure 1.1).

ii) The rotation of the Koebe function given by

$$k_\theta(z) = \frac{z}{(1 - e^{i\theta} z)^2}, \ (\theta \in \mathbb{R}; z \in \mathbb{D})$$

belongs to the class $\mathcal{S}$. The image of the open unit disc $\mathbb{D}$ is the complex plane except for a radial slit along $(-\infty, e^{-i\theta/4}]$.

iii) The generalized Koebe function given by

$$f(z) = \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^{\alpha} - 1\right], \ (z \in \mathbb{D}; \alpha \in (0,2])$$

belongs to the class $\mathcal{S}$.

iv) The function $f(z) = \frac{z}{1-z}$ is a linear fractional transformation which maps $\mathbb{D}$ onto the half plane $Re\omega(z) > -1/2$. This function belongs to the class $\mathcal{S}$ and plays an extremal role in problems for the subclass of $\mathcal{S}$ consisting of functions with convex image.

v) The function $\varphi(z) = 1/k(1/z)$ given by

$$\varphi(z) = z - 2 + 1/z$$

belongs to the class Σ. This function maps $\mathbb{D}^*$ onto the domain consisting of the entire complex plane minus a slit along the line segment $[-4, 0]$.

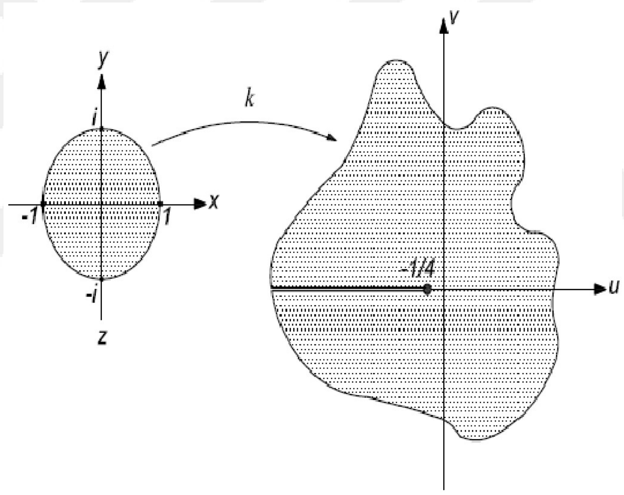

Figure 1.1: Koebe function maps $\mathbb{D}$ onto $\mathbb{C}\backslash(-\infty, -1/4]$

In 1914, Gronwall [20] proved the Area Theorem, which states that the area of the complement of the image of a domain under mapping a function is non-negative. The Area Theorem acts an important role in univalent functions.

Theorem 1.1.2. (*Area Theorem,* [*20*]) *"Let* $\varphi \in \Sigma$ *be given by* (1.3) *and let* $E = E(\varphi) = \mathbb{C}\backslash\varphi(\mathbb{D}^*)$ *denotes the complement of the image domain. Then the area of the domain* E *is*

$$Area(E) = \pi\left(1 - \sum_{n=1}^{\infty} n|\alpha_n|^2\right) \geq 0.$$

Since $Area(E) \geq 0$, *it follows that* $\sum_{n=1}^{\infty} n|\alpha_n|^2 \leq 1$ *as required."*

The Area Theorem led Bieberbach [3] to investigate the second coefficient estimate for the functions $f \in \mathcal{S}$.

Theorem 1.1.3. (*Bieberbach's Theorem*, [*3*]) *"If a function f defined by* (1.2) *is in $\mathcal{S}$, then*

$$|a_2| \leq 2.$$

Equality $|a_2| = 2$ occurs if and only if f is a rotation of the Koebe function."

By using the estimate $|a_2| \leq 2$, Bieberbach proved a covering theorem, which was earlier proposed by Koebe. In 1907, Koebe [31] asserted that the range of every function f in $\mathcal{S}$ contains a disc $|w| \leq c$, where c is a constant. Koebe also conjectured that the value of c is $1/4$ in the same paper. In 1916, Bieberbach [3] proved that $c = 1/4$ using Bieberbach's Theorem. In view of the results of Koebe and Bieberbach, Koebe One-Quarter Theorem is given as the following.

Theorem 1.1.4. (*Koebe One Quarter Theorem*, [*31*]) *"The range of every function in the class $\mathcal{S}$ contains the disc $\{w : |w| \leq 1/4\}$. The result is sharp for rotations of the Koebe function."*

Bieberbach's Theorem also led to prove the Koebe distortion theorem, which provides sharp upper and lower bounds for $|f'(z)|$ as f ranges over the class $\mathcal{S}$. Distortion theorem led to obtain growth theorem which gives the sharp bounds for $|f(z)|$. The correct proof of the growth and distortion theorems were given by Bieberbach [3, 4].

Theorem 1.1.5. ([*3, 4*]) *"If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then growth and distortion theorems are given by*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2},$$

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3},$$

$$\frac{1-|z|}{1+|z|} \leq \left| z\frac{f'(z)}{f(z)} \right| \leq \frac{1+|z|}{1-|z|}.$$

For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function."

The most important consequence of the second coefficient estimate was the Bieberbach's Conjecture. The estimate $|a_2| \leq 2$ encouraged Bieberbach to formulate his grateful conjecture in [3] for the n^{th} coefficient of the functions in the class $\mathcal{S}$.

Theorem 1.1.6. (*Bieberbach's Conjecture*, [3]) *"If a function f defined by (1.2) is in $\mathcal{S}$, then $|a_n| \leq n$ for all $n \geq 2$. Equality $|a_n| = n$ occurs for all $n \geq 2$ if and only if f is a rotation of the Koebe function."*

After Bieberbach put forward this conjecture, a good number of researchers tried to solve this problem by using different techniques. In 1923, Loewner [35] showed that $|a_3| \leq 3$. In 1955, Garabedian and Schiffer [16] proved that $|a_4| \leq 4$. In 1968, Pedersen [41] and in 1969, Ozawa [39] separately proved that $|a_6| \leq 6$. Ozawa and Kubota [40] showed that $Rea_8 \leq 8$ if $Rea_2 \geq 0$. In 1972, Pedersen and Schiffer [42] found that $|a_5| \leq 5$, and so on. The Bieberbach's Conjecture remained as an unanswered problem for all mathematicians about 70 years. In the mid-summer of 1984, Louis de Branges [11] solved the Bieberbach's Conjecture, and gave rise to appear a large number of new developments in GFT.

1.1.2 The Carathédory Class and Subordination

The functions with positive real part and subordination are very useful tools in univalent functions theory. We here make mention of several fundamental properties of the functions in the Carathédory Class; and discuss on the concept of subordination.

Definition 1.1.7. "Denote by $\mathcal{P}$ the class of analytic functions $p(z) = 1 + p_1 z + p_2 z^2 + ...$ in $\mathbb{D}$ with the conditions $Rep(z) > 0$ and $p(0) = 1$ is called the *Carathédory Class*. The function $(1+z)/(1-z)$ belongs to the class $\mathcal{P}$. This function maps the unit disc $\mathbb{D}$ conformally onto the right half plane. The class $\mathcal{P}$ acts an important role in univalent functions, since many extremal problems in the classes of univalent functions can be formulated in terms of members of $\mathcal{P}$ (see [5, 18])."

Definition 1.1.8. "Denote by Ω the class of analytic functions $\phi(z) = c_1 z + c_2 z^2 + c_3 z^3 ...$ in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ is called the *Schwarz functions*. The class Ω consists of analytic functions which hold the hypotheses of the Schwarz lemma."

The classes $\mathcal{P}$ and Ω are closely related and this relation is given by:

$$p \in \mathcal{P} \Leftrightarrow p = (1+\phi)/(1-\phi), \ (\phi \in \Omega).$$

The Schwarz Lemma is a consequence of the maximum modulus principle. The maximum modulus principle asserts that if f is analytic and nonconstant in

$\mathbb{D}$, then $|f(z)|$ cannot attain a maximum in $\mathbb{D}$; that is, there is no point z_0 in $\mathbb{D}$ such that $|f(z)| \leq |f(z_0)|$ for all points z in $\mathbb{D}$. The Schwarz lemma is a result that analytic functions map the unit disc onto itself. This lemma was used for proving many theorems including Riemann Mapping Theorem.

Lemma 1.1.9. (*Schwarz Lemma*) *"If an analytic function $f : \mathbb{D} \to \mathbb{D}$ holds the conditions $f(0) = 0$ and $|f(z)| \leq 1$ for every $z \in \mathbb{D}$, then*

1) $|f(z)| \leq |z|$.

2) $|f'(0)| \leq 1$.

Moreover, if either $|f(z)| = |z|$ for some non-zero; or if $|f'(0)| = 1$, then f is a rotation $f(z) = kz$ for some complex fixed number k with $|k| = 1$."

Definition 1.1.10. "Let f and g be analytic functions in $\mathbb{D}$. We say that f is *subordinate* to g denoted by $f \prec g$ if there is a Schwarz function ϕ given in $\mathbb{D}$ satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$ such that $f = g \circ \phi$. This concept was first found by Littlewood in 1925 (see [33])."

If $f \prec g$ in $\mathbb{D}$, then $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. It follows from the Schwarz lemma that $|f'(0)| \leq |g'(0)|$ and $f(\mathbb{D}_r) \subset g(\mathbb{D}_r)$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

In particular, an analytic function p in $\mathbb{D}$ with $p(0) = 1$ belongs to $\mathcal{P}$ if and only if $p(z) \prec (1 + z)/(1 - z)$ in $\mathbb{D}$. Also, a function ϕ in $\mathbb{D}$ with $\phi(0) = 0$ belongs to Ω if and only if $\phi(z) \prec z$ in $\mathbb{D}$.

Lemma 1.1.11. (*Jack's Lemma*, [24]) *"Let ϕ be an analytic function with the condition $\phi(0) = 0$ in $\mathbb{D}$. If $|\phi(z)|$ takes its maximum value on the circle $|z| = r$ at a point z_0 , then*

$$z_0\phi'(z_0) = m\phi(z_0),$$

for some $m \geq 1$. This lemma was proved by Jack in 1971."

Comprehensive details of the functions with positive real part are given by Graham and Kohr [19]. Also, for more details on the concept of subordination, we refer to the book by Miller and Mocanu [36].

1.1.3 Subclasses of Univalent Functions

The studies on the Bieberbach's Conjecture led to introduce and investigate several subclasses of $\mathcal{S}$. We will give the definitions and properties of the starlike and convex functions.

The domain $\mathcal{D} \subset \mathbb{C}$ is called *starlike* with respect to the point $z_0 \in \mathcal{D}$ if the closed line segment joining a point $z_0 \in \mathcal{D}$ to each point $z \in \mathcal{D}$ lies entirely in $\mathcal{D}$. Denote by $\mathcal{S}^*$ a function $f \in \mathcal{S}$ is called starlike with respect to the origin in the disc $\mathbb{D}$ if the domain $f(\mathbb{D})$ is starlike with respect to the origin (see Figure 1.2). This class was introduced by Alexander [1]. The analytic characterization of starlike functions is given in the following theorem.

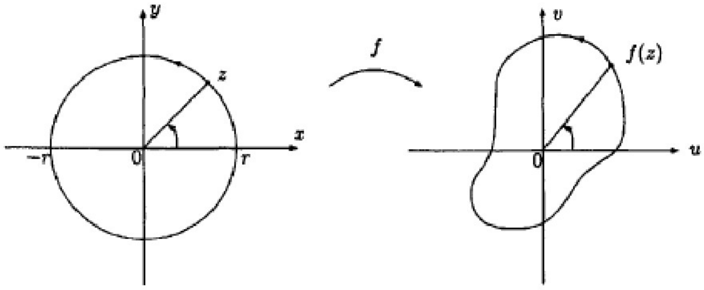

Figure 1.2: Starlikeness of the image $\mathbb{D}$

Theorem 1.1.12. *"Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$. Then, $f \in \mathcal{S}^*$ if and only if $f'(0) \neq 0$ and*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad (z \in \mathbb{D}).$$

The Koebe function acts an extremal role for the class $\mathcal{S}^*$."

Let $z_1, z_2 \in \mathcal{D}$. The domain $\mathcal{D} \subset \mathbb{C}$ is called *convex* if the closed line segment between z_1 and z_2 lies entirely in the domain $\mathcal{D}$. Denote by $\mathcal{C}$ a function $f \in \mathcal{S}$ is called convex in $\mathbb{D}$ if it maps the disc $\mathbb{D}$ onto a convex domain (see Figure 1.3). This class was introduced by Study [53]. The analytic characterization of convex functions is given in the following theorem.

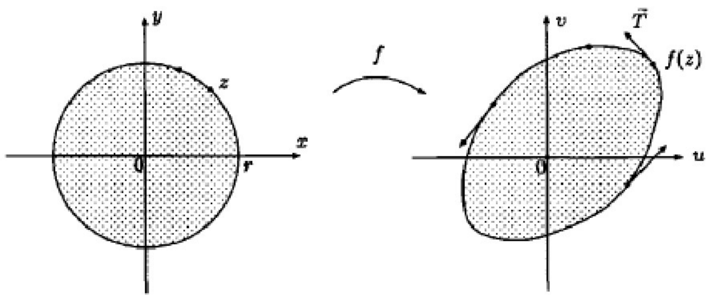

Figure 1.3: Convexity of the image $\mathbb{D}$

Theorem 1.1.13. *"Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$. Then, $f \in \mathcal{C}$ if and only if $f'(0) \neq 0$ and*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad (z \in \mathbb{D}).$$

The function $f(z) = z/(1-z)$ acts an extremal role for the class $\mathcal{C}$."

The classes $\mathcal{S}^*(\alpha)$ of starlike function of order α and $\mathcal{C}(\alpha)$ of convex function of order α were defined and studied by Robertson in [47].

Theorem 1.1.14. *"Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$. Then, $f \in \mathcal{S}^*(\alpha)$ if and only if $f'(0) \neq 0$ and*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{D})$$

and $f \in \mathcal{C}(\alpha)$ if and only if $f'(0) \neq 0$ and

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{D})$$

where $\alpha \in [0, 1)$. In particular, $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{C}(0) \equiv \mathcal{C}$."

Additionaly, we can say that the classes $\mathcal{S}^*$ and $\mathcal{C}$ are closely related to the class $\mathcal{P}$ of all functions p analytic and having positive real part. Thus, Theorems 1.1.12 and 1.1.13 can be rewritten as $f \in \mathcal{S}^*$ if and only if $zf'(z)/f(z) \in \mathcal{P}$; and $f \in \mathcal{C}$ if and only if $1 + zf''(z)/f'(z) \in \mathcal{P}$. We also note that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}$.

Connection between the classes $\mathcal{S}^*$ and $\mathcal{C}$ were first found by Alexander [1] in 1915.

Theorem 1.1.15. (*Alexander's Theorem,* [*1*]) *"Let f be holomorphic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$."*

In view of Bieberbach's Conjecture, Nevanlinna in [38] introduced coefficient bounds for the functions in the class $\mathcal{S}^*$.

Theorem 1.1.16. ([*38*]) *"If $f \in \mathcal{S}$ given by (1.2) is starlike in $\mathbb{D}$, then $|a_n| \leq n$ for each $n \geq 2$. Equality $|a_n| = n$ occurs for all $n \geq 2$ if f is a rotation of the Koebe function."*

Nevanlinna's result together with Alexander's result gives the following coefficient estimates introduced by Loewner [34] for the functions in the class $\mathcal{C}$.

Theorem 1.1.17. ([34]) *"If $f \in \mathcal{S}$ given by* (1.2) *is convex in $\mathbb{D}$, then $|a_n| \leq 1$ for all $n \geq 2$. Equality $|a_n| = 1$ occurs for a given $n \geq 2$ if $f(z) = z/(1 - \lambda z)$, ($z \neq 0$, $\lambda \in \mathbb{C}$ and $|\lambda| = 1$)."*

Koebe function and its rotation belong to the class $\mathcal{S}^*$, so growth and distortion inequalities for the functions in the class $\mathcal{S}$ are also sharp for the class $\mathcal{S}^*$.

Theorem 1.1.18. *"If $f \in \mathcal{S}^*$ and $z \in \mathbb{D}$ with $|z| = r < 1$, then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2},$$

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

These inequalities are sharp. Equality occurs in each case of these relations for a suaitable rotation of the Koebe function."

For convex functions, we have the following growth and distortion theorems.

Theorem 1.1.19. ([20]) *"If $f \in \mathcal{C}$ and $z \in \mathbb{D}$ with $|z| = r < 1$, then*

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r},$$

$$\frac{1}{(1+r)^2} \leq |f'(z)| \leq \frac{1}{(1-r)^2}.$$

These bounds are sharp. Equality occurs for $f(z) = z/(1 - \lambda z)$, ($z \neq 0$, $\lambda \in \mathbb{C}$ and $|\lambda| = 1$)."

More details on the univalent functions are given in the book by Graham and Kohr [19]. Furthermore, Goodman [18], Duren [12] and Pommerenke [44] gave comprehensive details of the univalent functions theory.

1.2 Harmonic Univalent Functions

A continuous, real-valued function $u(x, y)$ is *harmonic* if it holds the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

If a complex-valued function $w = f(z) = u(z) + iv(z)$ is analytic, then it holds the Cauchy-Riemann equations. It follows from the Cauchy-Riemann equations that every analytic function is harmonic. A pair of functions (u, v) that satisfy the Cauchy-Riemann equations is said to be a *conjugate pair*, and v is called the *harmonic conjugate* of u. Thus, $-u$ is the harmonic conjugate of v. A *critical point* of $u(x, y)$ is a point where u_x and u_y both vanish.

A continuous, complex-valued function $w = f(z) = u(z) + iv(z)$ defined in $\mathcal{D}$ is a *harmonic mapping* if both u and v are real-valued harmonic functions in $\mathcal{D}$; that is, u and v satisfy the Laplace equations. Here, u and v are not necessarily harmonic conjugate. Even though every analytic function is a complex-valued harmonic function, but not every complex-valued harmonic function is analytic. For example; the function $f(z) = -2xy + i(y^2 - x^2)$ is a harmonic function, but not analytic. Behaviour of the analytic and harmonic functions are different. As for example, analytic functions are preserved under composition, but composition of the harmonic functions may not be harmonic. Though the analytic functions form an algebra, but the harmonic functions do not.

Relation between complex-valued harmonic functions and analytic functions is shown by the next theorem.

Theorem 1.2.1. ([6]) *"If a function $f = u + iv$ is harmonic in $\mathcal{D}$, then f has a canonical representation $f = h + \overline{g}$ such that both h and g are analytic."*

Proof. If u and v are real harmonic in $\mathcal{D}$, then there are two analytic functions E and F such that $u = ReE$ and $v = ImF$. Thus,

$$f = u + iv = \frac{E + \overline{E}}{2} + i\frac{F - \overline{F}}{2i} = \frac{E + F}{2} + \overline{\frac{E - F}{2}} := h + \overline{g}.$$

Here, we call that h the analytic and g the co-analytic part of f. □

A harmonic function $f = h + \overline{g}$ can also be written in the form

$$f(z) = Re\{h(z) + g(z)\} + iIm\{h(z) - g(z)\}. \tag{1.4}$$

Locally univalent and sense-preserving property of the harmonic mappings are determined by Jacobian. We now give the definition of Jacobian for harmonic mappings. Partial defivatives of the function $f = u+iv$ with respect to $z = x+iy$ and $\overline{z} = x - iy$ are obtained as

$$\begin{aligned} f_z &= \frac{1}{2}\big(f_x - if_y\big) = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y), \\ f_{\overline{z}} &= \frac{1}{2}\big(f_x + if_y\big) = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y). \end{aligned} \tag{1.5}$$

Thus, using the relations given in (1.5) we get

$$|f_z|^2 - |f_{\overline{z}}|^2 = \frac{1}{4}\big((u_x+v_y)^2 + (v_x-u_y)^2 - (u_x-v_y)^2 - (v_x+u_y)^2\big) = u_x v_y - u_y v_x. \tag{1.6}$$

In view of the Jacobian formula given in (1.1) and the relation in (1.6), the Jacobian of the function $f = u + iv$ is defined as

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 = |h'(z)|^2 - |g'(z)|^2 .$$

Before giving Lewy's Theorem, we need the following two theorems.

Theorem 1.2.2. ([*13*]) *"All critical points of a nonconstant harmonic function are isolated."*

Theorem 1.2.3. ([*13*]) *"The level-set of a nonconstant harmonic function through a critical point z_0 consists locally of two or more analytic arcs intersecting with equal angles at z_0."*

In the light of these two theorems, Lewy [32] proved that the Jacobian of locally univalent harmonic mappings is different from zero.

Theorem 1.2.4. (*Lewy's Theorem*, [*32*]) *"A function $f = h + \overline{g}$ is locally univalent in $\mathcal{D} \subset \mathbb{C}$ if its Jacobian $J_f(z) \neq 0$ for each $z \in \mathcal{D}$."*

An another important property for harmonic mappings is sense-preserving. A continuous function f is *sense-preserving* if it preserves orientation. For example, consider the functions $f_1(z) = 1/z$ and $f_2(z) = \overline{z}$ defined on the punctured disc $\mathbb{D}\backslash\{0\}$. Both functions map the points $A = 1, B = e^{i\pi/4}, C = i$ to the points $A' = 1, B' = e^{-i\pi/4}, C' = -i$ (see Figure 1.4). Geometrically, in the domain as we travel along the unit circle in a counterclockwise direction, the region to the left of this path is $\mathbb{D}$. This region $\mathbb{D}$ is called the left hand side domain (LHS) and the region to the right of the path is called right hand side domain (RHS). In this case, $\mathbb{C}\backslash\mathbb{D}$ is RHS. As we travel along $\mathbb{D}$ in a counterclockwise direction, the image under both functions will be clockwise direction. Thus, in the image, $\mathbb{D}$ is RHS and $\mathbb{C}\backslash\mathbb{D}$ is LHS. The function f_1 maps the LHS onto the LHS, while the function f_2 maps the LHS onto the RHS.

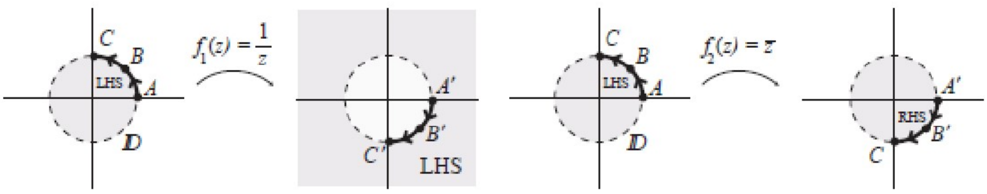

Figure 1.4: $f_1(z) = 1/z$ is sense-preserving, $f_2(z) = \overline{z}$ is sense-reversing

A function f is sense-preserving if the LHS domain is mapped to the LHS domain. A function f is sense-reversing if the LHS domain is mapped to the RHS domain. All analytic functions are sense-preserving, but some harmonic mappings are sense-preserving and some are sense-reversing. Conformal harmonic mappings are sense-preserving. Due to the Lewy's Theorem, if $|h'(z)| > |g'(z)|$, then harmonic mappings are sense-preserving; if $|h'(z)| < |g'(z)|$, then harmonic mappings are sense-reversing in $\mathbb{D}$. The most simple example of the harmonic mappings which need not be conformal are the affine mappings $f(z) = \lambda z + \gamma + \delta\overline{z}$ with $|\lambda| \neq |\delta|$, which map the whole complex plane $\mathbb{C}$ onto itself. The function $w(z) = g'(z)/h'(z)$ is called the *second dilatation* of harmonic mappings. For locally univalent, sense-preserving harmonic mappings, the second dilatation holds the following condition.

Theorem 1.2.5. ([*32*]) *"The function $f = h + \overline{g}$ is sense-preserving and locally univalent in $\mathcal{D}$ if and only if the second dilatation $|w(z)| = |g'(z)/h'(z)| < 1$ for each $z \in \mathcal{D}$."*

Earlier, harmonic mappings were studied by differential geometers due to their role in minimal surfaces. After Louis de Branges [11] proved the Bieberbach's Conjecture in 1984, in the same year, Clunie and Sheil-Small [6] explored the class of sense-preserving, harmonic univalent functions and generalized some properties of the analytic univalent functions to the harmonic univalent functions.

Definition 1.2.6. "The class $\mathcal{S}_{\mathcal{H}}$ consists of all sense-preserving, univalent harmonic mappings $f = h + \overline{g}$ with the normalization conditions $f(0) = a_0 = 0$ and $f_z(0) = a_1 = 1$ in $\mathbb{D}$. The series expansion of $f = h+\overline{g}$ in $\mathcal{S}_{\mathcal{H}}$ is given of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n, \quad (z \in \mathbb{D}). \tag{1.7}$$

Due to sense-preserving property that $|b_1| < 1$. By restricting with a condition of normalization that $f_{\overline{z}}(0) = b_1 = 0$, the class $\mathcal{S}_{\mathcal{H}}^0$ can be obtained. Also, if the co-analytic part $g(z) \equiv 0$, then we get the class $\mathcal{S}$. Hence, $\mathcal{S} \subset \mathcal{S}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}$."

In [6], harmonic Koebe function $k_0 = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^0$ is defined by

$$k_0(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}}. \tag{1.8}$$

The function k_0 maps $\mathbb{D}$ conformally onto complex plane except the real slit $(-\infty, -1/6]$.

In the same paper, Clunie and Sheil-Small discovered a distortion theorem for functions in $\mathcal{S}_{\mathcal{H}}^0$.

Theorem 1.2.7. ([6]) *"If $f \in \mathcal{S}_{\mathcal{H}}^0$, then*

$$|f(z)| \geq \frac{1}{4}\frac{|z|}{(1+|z|)^2}, \quad (z \in \mathbb{D}).$$

In particular, $\{w \in \mathbb{C} : |w| < 1/16\} \subset f(\mathbb{D})$."

This theorem is non-sharp. For k_0 the constant can be increased to 1/6.

Conjecture 1.2.8. ([6]) *"If $f \in \mathcal{S}_{\mathcal{H}}^0$, then $\{w \in \mathbb{C} : |w| < 1/6\} \subset f(\mathbb{D})$."*

Clunie and Sheil-Small [6] extended the Bieberbach's Conjecture for the class $\mathcal{S}_{\mathcal{H}}^0$.

Conjecture 1.2.9. ([6]) *"If $f \in \mathcal{S}_{\mathcal{H}}^0$, then*

1) $\big||a_n| - |b_n|\big| \leq n$

2) $|a_n| \leq \frac{(2n+1)(n+1)}{6}$

3) $|b_n| \leq \frac{(2n-1)(n-1)}{6}$

for every $n \geq 2$. Equality occurs for the function k_0 given by (1.8)."

The bounds given in Conjecture 1.2.9 were proved for the class $\mathcal{S}_{\mathcal{H}}^{*^0}$. Moreover, the coefficients bounds for the class $\mathcal{C}_{\mathcal{H}}^0$ are given as below.

Theorem 1.2.10. ([6]) *If $f = h + \overline{g} \in \mathcal{C}_{\mathcal{H}}^0$, then we have*

$$\big||a_n| - |b_n|\big| \leq 1, \quad |a_n| \leq \frac{n+1}{2}, \quad |b_n| \leq \frac{n-1}{2}.$$

for all $n \geq 2$. The sharp function of these estimates is

$$f_0(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \overline{\frac{\frac{1}{2}z^2}{(1-z)^2}}. \tag{1.9}$$

In addition to the results given by Clunie and Sheil-Small [6], in 1986, Hengartner and Schober [21, 22] get a suitable statement of the Riemann Mapping Theorem for harmonic mappings. Analogues to the theory of analytic univalent functions various properties of the harmonic mappings were investigated. But behaviour of the harmonic mappings is different from analytic functions, thus there are many properties of the analytic univalent functions which cannot extend to the harmonic mappings. However, since 1984 the studies of harmonic mappings have become a popular area for complex analysts. For comprehensive theory of the harmonic mappings, we refer to the book by Duren [13]. We also refer to the book chapter by Dorff and Rolf [10].

1.2.1 Subclasses of Harmonic Univalent Functions

Because of challenging to obtain sharp estimates for the class $\mathcal{S}_{\mathcal{H}}$ (or $\mathcal{S}_{\mathcal{H}}^0$), researchers attempt to investigate various subclasses of the class $\mathcal{S}_{\mathcal{H}}$ (or $\mathcal{S}_{\mathcal{H}}^0$). We here discuss on several important subclasses of $\mathcal{S}_{\mathcal{H}}$ (or $\mathcal{S}_{\mathcal{H}}^0$).

Definition 1.2.11. "Let $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ and let the class $\mathcal{S}_{\mathcal{H}}^*$ be the subclass of $\mathcal{S}_{\mathcal{H}}$. A harmonic function f in $\mathcal{S}_{\mathcal{H}}^*$ maps the disc $\mathbb{D}$ onto starlike domain with respect to the origin. Such functions are called harmonic starlike functions. Geometrically, this means that every point of the range can be connected to the origin by a radial line that entirely lies in the domain; that is, $f \in \mathcal{S}_{\mathcal{H}}^*$ if and only if

$$\frac{\partial}{\partial\theta}\{\arg f(re^{i\theta})\} > 0, \ \ (\theta \in [0, 2\pi], |z| = r < 1).$$

In general form, if $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^*(\alpha)$, then

$$Re\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right) > \alpha,$$

where $\alpha \in [0, 1)$ and $z \in \mathbb{D}$. Such functions are called harmonic starlike functions of order α. In particular, $\mathcal{S}_{\mathcal{H}}^*(0) \equiv \mathcal{S}_{\mathcal{H}}^*$.

We note that the subclasses of $\mathcal{S}_{\mathcal{H}}^0$ are, respectively, denoted by $\mathcal{S}_{\mathcal{H}}^{*^0}$ and $\mathcal{S}_{\mathcal{H}}^{*^0}(\alpha)$."

Definition 1.2.12. "Let $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ and let the class $\mathcal{C}_{\mathcal{H}}$ be the subclass of $\mathcal{S}_{\mathcal{H}}$. A harmonic function f in $\mathcal{C}_{\mathcal{H}}$ maps the disc $\mathbb{D}$ onto convex domains. Such functions are called harmonic convex functions. In geometric view of the range of $f(\mathbb{D})$, this means that $f \in \mathcal{C}_{\mathcal{H}}$ if and only if

$$\frac{\partial}{\partial\theta}\{\arg(\frac{\partial}{\partial\theta}\arg f(re^{i\theta}))\} > 0, \ \ (\theta \in [0, 2\pi], |z| = r < 1).$$

In general form, if $f = h + \overline{g} \in \mathcal{C}_{\mathcal{H}}(\alpha)$, then

$$Re\left(\frac{z(zh'(z))' + \overline{z(zg'(z))'}}{zh'(z) - \overline{zg'(z)}}\right) > \alpha,$$

where $\alpha \in [0, 1)$ and $z \in \mathbb{D}$. Such functions are called harmonic convex functions of order α. In particular, $\mathcal{C}_{\mathcal{H}}(0) \equiv \mathcal{C}_{\mathcal{H}}$.

We note that the subclasses of $\mathcal{S}_{\mathcal{H}}^0$ are, respectively, denoted by $\mathcal{C}_{\mathcal{H}}^0$ and $\mathcal{C}_{\mathcal{H}}^0(\alpha)$."

In [2], Avcı and Zlotkiewicz proved the coefficient estimates for the classes $\mathcal{S}_{\mathcal{H}}^{*^0}$ and $\mathcal{C}_{\mathcal{H}}^0$ as follows.

Theorem 1.2.13. ([2]) *"Let $f = h + \overline{g}$ be defined by* (1.7)*.*

i) If $f \in \mathcal{S}^{*0}_{\mathcal{H}}$, then $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1$,

ii) If $f \in \mathcal{C}^{0}_{\mathcal{H}}$, then $\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq 1$,

with $b_1 = 0$."

In [28, 29], Jahangiri proved the coefficient conditions of the classes $\mathcal{S}^{*}_{\mathcal{H}}(\alpha)$ and $\mathcal{C}_{\mathcal{H}}(\alpha)$ as follows.

Theorem 1.2.14. ([28, 29]) *"Let $f = h + \overline{g}$ be defined by* (1.7)*.*

i) If $f \in \mathcal{S}^{*}_{\mathcal{H}}(\alpha)$, then $\sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 2$,

ii) If $f \in \mathcal{C}_{\mathcal{H}}(\alpha)$, then $\sum_{n=1}^{\infty} \left(\frac{n(n-\alpha)}{1-\alpha}|a_n| + \frac{n(n+\alpha)}{1-\alpha}|b_n|\right) \leq 2$,

where $a_1 = 1$ and $\alpha \in [0, 1)$."

1.2.2 Convolution of Harmonic Functions

For two analytic functions $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and "$F(z) = z + \sum\limits_{n=2}^{\infty} A_n z^n$, *Hadamard product* or *convolution* of these functions is given by

$$f(z) * F(z) = (f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n.$$

There are the following properties for convolution of two analytic functions:

1) Convolution has commutative property. If f and F are analytic, then $f * F = F * f$.

2) If f and F are analytic, then $(f(z) * F(z))' = zf'(z) * F(z)$.

3) The right half plane mapping, $f(z) = \frac{z}{1-z}$, plays as convolution identity. If F is analytic function, then $F(z) * \frac{z}{1-z} = F(z)$.

4) The Koebe function, $f(z) = \frac{z}{(1-z)^2}$, plays as convolution identity. If F is analytic function, then $F(z) * \frac{z}{(1-z)^2} = zF'(z)$.

Let two harmonic functions be given by

$$\begin{aligned} f(z) &= z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}, \\ F(z) &= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n}, \end{aligned} \tag{1.10}$$

then their convolution is defined as

$$f(z) * F(z) = (f * f)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n z^n}, \quad (z \in \mathbb{D}).$$

Convolution of the univalent functions may not be univalent. An example can be given below:

$$\frac{z}{(1-z)^2} * \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n * \sum_{n=1}^{\infty} nz^n = \sum_{n=1}^{\infty} n^2 z^n \notin \mathcal{S}.$$

But there are some results on convolution of the analytic functions; as for example, Ruscheweyh and Sheil-Small [49] discovered that if $f \in \mathcal{C}$, then

i) $f * g \in \mathcal{C}$ *for all* $g \in \mathcal{C}$.

ii) $f * g \in \mathcal{S}^*$ *for all* $g \in \mathcal{S}^*$.

These results are not hold for the case of harmonic. If $f, g \in \mathcal{C}_{\mathcal{H}}$, then $f * g$ may not in $\mathcal{C}_{\mathcal{H}}$ and may not be univalent. Even though the studies on the convolution of harmonic functions have some progress in literature, but still very little is known on this topic. For more details of" convolution of analytic functions, we refer to the book by Ruscheweyh [48].

1.3 Scope of Thesis

This thesis deals with applications of $q-$calculus in harmonic mappings. We extend some important definitions of the harmonic univalent functions to the $q-$harmonic univalent functions by using $q-$derivative operator. Later, we define two new subclasses of the $q-$harmonic univalent functions and examine their properties. Our thesis consists of two original parts.

In Chapter 1, we discuss on some basic definitions, theorems and conjectures of the analytic and harmonic univalent functions, respectively.

In Chapter 2, we first give the definitions and properties of $q-$derivative and $q-$integral. By using $q-$derivative operator, we define the $q-$Jacobian of a function $f = h + \overline{g} \in \mathcal{H}_q$ as

$$J_{q,f}(z) = |D_q h(z)|^2 - |D_q g(z)|^2.$$

We first prove that the function $f = h + \overline{g}$ is locally univalent in $\mathbb{D}$ if its $q-$Jacobian is nonvanishing for each $z \in \mathbb{D}$. Thus, we observe that for locally univalent and sense-preserving $q-$harmonic mappings, the $q-$second dilatation satisfies the condition

$$|w_q(z)| = \left|\frac{D_q g(z)}{D_q h(z)}\right| < 1.$$

In view of these results, we define the class $\mathcal{S}_{\mathcal{HC}_q(b)}$ which contains a function f that is the solution of the non-linear partial differential equation $w_q(z) = \frac{D_q g(z)}{D_q h(z)}$ with $|w_q(z)| < 1$, $w_q(z) \prec b_1 \frac{1+z}{1-qz}$ and the function h is in the class $\mathcal{C}_q(b)$. We also prove the $q-$Jack's Lemma due to its usefulness in our theorems. By using subordination technique and the $q-$Jack's Lemma, we obtain distortion bounds for the functions in the class $\mathcal{S}_{\mathcal{HC}_q(b)}$.

In Chapter 3, we define the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ of harmonic univalent convex functions of complex order and type α given by

$$Re\left[1 + \frac{1}{b}\left(\frac{zD_q(zD_q f(z))}{zD_q f(z)} - 1\right)\right] \geq \alpha,$$

where $b \in \mathbb{C} \setminus \{0\}$, $q \in (0,1)$, $\alpha \in [0,1)$ and $z \in \mathbb{D}$. We first obtain a sufficient coefficient condition that guarentees that a function f in $\mathcal{H}_q$ is sense-preserving harmonic univalent in $\mathbb{D}$ and belongs to the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. Using this coefficient condition, we then examine four ratios of partial sums of the functions f given by

$$Re\left\{\frac{f(z)}{f_m(z)}\right\},\quad Re\left\{\frac{f_m(z)}{f(z)}\right\},\quad Re\left\{\frac{f(z)}{f_s(z)}\right\},\quad Re\left\{\frac{f_s(z)}{f(z)}\right\},$$

for the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. We finally introduce distortion bounds, covering theorem and convolution conditions for the functions in the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$.

The original results presented in this thesis are contained in the following papers:

1. A. Çetinkaya, Y. Polatoğlu, *$q-$Harmonic mappings for which analytic part is $q-$convex functions of complex order*, Hacettepe University Bulletin of Natural Sciences and Engineering - Series B: Mathematics and Statistics, 47 (4) (2018), 813-820.

2. A. Çetinkaya, O. Mert, *A certain class of harmonic mappings related to functions of bounded boundary rotation*, In Proc. of 12^{th} International Symposium on Geometric Function Theory and Applications, (2016), 67-76.

3. A. Çetinkaya, Y. Polatoğlu, *Partial sums of harmonic univalent convex functions by using quantum calculus*, submitted.

A part of the original results, proved in the thesis, were presented at the following international conferences:

1. A. Çetinkaya, Y. Polatoğlu, *$q-$Harmonic mappings for which analytic part is $q-$convex functions of complex order*, XVIII th Conference on Analytic Functions and Related Topics (AF&RT 2016), June 26-29, 2016, Chelm, POLAND.
2. A. Çetinkaya, O. Mert, *A certain class of harmonic mappings related to functions of bounded boundary rotation,* 12^{th} International Symposium on Geometric Function Theory and Applications (GFTA 2016), August 25-28, 2016, Alba Iulia, ROMANIA.

Chapter 2

QUANTUM CALCULUS IN HARMONIC FUNCTIONS

The quantum calculus (or $q-$calculus) dates back to the studies of Leonhard Euler (1707-1783), Carl Friedrich Gauss (1777-1855), Eduard Heine (1821-1881) and Carl Gustav Jacobi (1804-1851). The systematic development of $q-$calculus begins with Jackson, who reconsidered the Euler-Jackson $q-$difference operator in 1908, and who introduced definite $q-$integral in 1910 (see [25, 26, 27]). Later, researchers used the quantum calculus in several areas of physics and mathematics because of its applications. The quantum calculus is especially used in the theory of basic hypergeometric functions, $q-$difference equations, $q-$integral equations, orthogonal polynomials, combinatorics, ordinary fractional calculus, analytic number theory, optimal control problems, the theory of relativity, and more recently in the Geometric Function Theory.

Harmonic univalent functions have been studied since 1984, but the impact of $q-$calculus in this branch is extremely new. In this part of thesis, motivated by some recent papers (see for example [14]), we make an attempt to apply $q-$calculus and harmonic univalent functions, and introduce some elementary definitions for $q-$harmonic univalent functions by using $q-$derivative operator. We also prove the $q-$Jack's Lemma. We next define a new class $\mathcal{S}_{\mathcal{HC}_q(b)}$ of $q-$harmonic mappings for which analytic part is $q-$convex functions of complex order. Finally, we get distortion bounds for the functions in the class $\mathcal{S}_{\mathcal{HC}_q(b)}$.

2.1 $q-$Derivative and $q-$Integral

Let $n \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$. Also, q can be taken as real and $q \in (0, 1)$. The *quantum number* or *$q-$number* of n is given by

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + ... + q^{n-1},$$

which is a polynomial in q possesing the degree $n-1$. In the limiting case $q \to 1^-$, $q-$numbers reduce to the ordinary numbers.

Earlier, Euler and Jackson considered a kind of $q-$derivative, but the first systematic study of $q-$derivative was initiated by Jackson in 1908. By using the $q-$number, Jackson [25] defined the $q-$derivative as follows.

Definition 2.1.1. ([25]) Let f be an arbitrary function. The $q-$differential of f is

$$d_q f(x) = f(qx) - f(x).$$

Thus, the *$q-$derivative operator* (or *$q-$difference operator*) of a function f is defined by

$$(D_q f)(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \; (x \neq 0).$$

If f is differentiable function, then it turn into the ordinary derivative when $q \to 1^-$ as

$$\lim_{q \to 1^-} (D_q f)(x) = \lim_{q \to 1^-} \frac{f(qx) - f(x)}{(q-1)x} = f'(x).$$

The $q-$derivative operator has the following rules [30]:

1) The $q-$derivative of a function has the linearity property

$$D_q(af(x) \pm bg(x)) = aD_q f(x) \pm bD_q g(x),$$

where a, b are real or complex constants.

2) The $q-$derivative operator has the following two product rule of arbitrary functions f and g

$$D_q(f(x).g(x)) = f(qx)D_q g(x) + g(x)D_q f(x),$$

$$D_q(f(x).g(x)) = f(x)D_q g(x) + g(qx)D_q f(x).$$

3) The $q-$derivative operator has two division rules

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)},$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(x)g(qx)},$$

where $g(x) \neq 0$ and $g(qx) \neq 0$.

For example; the $q-$derivative of the function $f(x) = x^n$ yields

$$D_qx^n = \frac{1-q^n}{1-q}x^{n-1} = [n]_qx^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$ is the $q-$number of n.

The $q-$derivative and $q-$antiderivative are closely related. The function F is a $q-$antiderivative of f if $D_qF = f$. It is denoted by

$$F(x) = \int f(x)d_qx.$$

In ordinary calculus, the uniqueness is up to adding a constant, since the derivative of a function vanishes if and only if it is constant. The situation in quantum calculus is more subtle.

Theorem 2.1.2. ([*30*]) *"Let $q \in (0,1)$. Then, up to adding a constant, any function $f(x)$ has at most one $q-$antiderivative that is continuous at $x = 0$."*

The *$q-$integral* of any function f was defined by Jackson [26] as

$$\int f(x)d_qx = x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n), \tag{2.1}$$

provided that the series on right hand side converges absolutely. This integral is also called *Jackson integral*. From the formula in (2.1), one easily get a more general formula given as

$$\int f(x)d_qg(x) = \sum_{n=0}^{\infty} f(xq^n)(g(xq^n) - g(xq^{n+1})).$$

By applying the Jackson integral formula given in (2.1), the definite $q-$integral is given as follows.

Definition 2.1.3. ([30]) Suppose $0 < a < b$. The *definite $q-$integral* defined as

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

where

$$\int_0^a f(x)d_qx = a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n).$$

If f is continuous on $[0, a]$, then it is easily seen that

$$\lim_{q\to 1^-}\int_0^a f(x)d_qx = \int_0^a f(x)dx.$$

The improper $q-$integral of $f(x)$ on $[0,\infty)$ is defined as

$$\int_0^\infty f(x)d_qx = \sum_{n=-\infty}^{\infty} \int_{q^{n+1}}^{q^n} f(x)d_qx, \ (0<q<1)$$

and

$$\int_0^\infty f(x)d_qx = \sum_{n=-\infty}^{\infty} \int_{q^n}^{q^{n+1}} f(x)d_qx, \ (q>1).$$

The quantum calculus can be considered as an extension of ordinary calculus explored by Gottfried Wilhelm Leibniz (1646-1716) and Issac Newton (1643-1727). Newton-Leibniz formula (also called the fundamental theorem of calculus) shows the relation between derivative and definite integral in ordinary calculus. In view of a $q-$antiderivative, the relation between the $q-$derivative and definite $q-$integral is clear. Analogues to the ordinary case, we present the following fundamental theorem for quantum calculus.

Theorem 2.1.4. ([*30*]) *(a) "If F is a $q-$antiderivative of f, namely $D_qF = f$, and F is continuous at $x = 0$, then*

$$\int_a^b f(x)d_qx = F(b) - F(a),$$

where $0 \le a < b \le \infty$.

(b) If f' exists in a neighbourhood of $x = 0$ and is continuous at $x = 0$, where f' is the ordinary derivative of f, then

$$\int_a^b D_qf(x)d_qx = f(b) - f(a),$$

where $0 \le a < b \le \infty$*."*

Comprehensive details of the quantum calculus can be found in the books by Kac and Cheung [30], and Gasper and Rahman [17]. Moreover, one may refer to the book by Fine [15].

2.2 $q-$Harmonic Univalent Functions

In this section, we generalize the harmonic univalent functions to the $q-$harmonic univalent functions, and introduce some new definitions for $q-$harmonic univalent functions by using $q-$derivative operator. Such functions will be denoted by $\mathcal{H}_q$. In the limiting case $q \to 1^-$, $\mathcal{H}_q$ reduces to the traditional harmonic univalent functions.

Let $q \in (0,1)$ and let $f = h + \overline{g}$ given by (1.7) be element of the class $\mathcal{H}_q$ in $\mathbb{D}$. The power series converges for $|z| < 1$ when $q \in (0,1)$ and this guarentees the analyticity of the power series in the open unit disc ([17]). Thus, the functions h and g defined in (1.7) have $q-$derivatives given by

$$(D_q h)(z) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} \quad \text{and} \quad (D_q g)(z) = \sum_{n=1}^{\infty} [n]_q b_n z^{n-1}.$$

In the limiting case $q \to 1^-$, we conclude that $D_q h(z)$ and $D_q g(z)$ reduces to

$$h'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \quad \text{and} \quad g'(z) = \sum_{n=1}^{\infty} n b_n z^{n-1}.$$

For defining $q-$harmonic univalent functions, we need to examine situation of locally univalent and sense-preserving property. For our purpose, we first define the q case of Jacobian for $q-$harmonic functions.

Definition 2.2.1. Let the function $w = f(z) = h(z) + \overline{g(z)},\ z = (x,y)$ be in the class $\mathcal{H}_q$. Let $J_{q,f}$ denote the *$q-$Jacobian* of the function $f = u + iv$. From Theorem 1.2.1 and the relation (1.4), we know that

$$f(z) = u(z) + iv(z) = Re\{h(z) + g(z)\} + iIm\{h(z) - g(z)\}.$$

We will conveniently use the notation ∂_q instead of D_q, and show the first-order partial $q-$derivatives of $f = u + iv$ with respect to x and y as in the following statements:

$$\frac{\partial_q u}{\partial_q x} = \partial_q u_x, \quad \frac{\partial_q u}{\partial_q y} = \partial_q u_y, \quad \frac{\partial_q v}{\partial_q x} = \partial_q v_x, \quad \frac{\partial_q v}{\partial_q y} = \partial_q v_y.$$

Hence, using the Jacobian formula given by (1.1), we get

$$\begin{aligned} J_{q,f}(z) = \frac{\partial_q(u,v)}{\partial_q(x,y)} &= \det \begin{vmatrix} \partial_q u_x & \partial_q u_y \\ \partial_q v_x & \partial_q v_y \end{vmatrix} \\ &= \det \begin{vmatrix} \partial_q((Reh)_x + (Reg)_x) & \partial_q((Reh)_y + (Reg)_y) \\ \partial_q((Imh)_x - (Img)_x) & \partial_q((Imh)_y - (Img)_y) \end{vmatrix}. \end{aligned}$$

Since f is analytic, it satisfies Cauchy-Riemann equations $\partial_q(Reh)_x = \partial_q(Imh)_y$, $\partial_q(Reg)_x = \partial_q(Img)_y$, $\partial_q(Reh)_y = -\partial_q(Imh)_x$ and $\partial_q(Reg)_y = -\partial_q(Img)_x$. Therefore, we get

$$\begin{aligned} J_{q,f}(z) &= \det \begin{vmatrix} \partial_q((Reh)_x + (Reg)_x) & \partial_q(-(Imh)_x - (Img)_x) \\ \partial_q((Imh)_x - (Img)_x) & \partial_q((Reh)_x - (Reg)_x) \end{vmatrix} \\ &= (\partial_q(Reh)_x)^2 - (\partial_q(Reg)_x))^2 + (\partial_q(Imh)_x)^2 - (\partial_q(Img)_x))^2 \\ &= |\partial_q h|^2 - |\partial_q g|^2. \end{aligned}$$

Finally, we conclude that the $q-$Jakobian of f is obtained as

$$J_{q,f}(z) = |D_q h(z)|^2 - |D_q g(z)|^2.$$

We note that in the limiting sense $q \to 1^-$, the $q-$Jacobian $J_{q,f}$ reduces to the familiar Jacobian J_f. We next prove that the $q-$Jacobian of locally univalent $q-$harmonic functions is nonvanishing.

Lemma 2.2.2. *"If a function $f(z) = h(z) + \overline{g(z)}$ given in the class $\mathcal{H}_q$ is locally univalent in $\mathbb{D}$, then its $J_{q,f}(z_0) \neq 0$ for all $z \in \mathbb{D}$."*

Proof. Asumme that the $q-$Jacobian of a nonconstant function $f = u + iv$ is $J_{q,f}(z_0) = 0$ for some point $z_0 \in \mathbb{D}$. Thus, the matrix

$$\begin{pmatrix} \partial_q u_x & \partial_q v_x \\ \partial_q u_y & \partial_q v_y \end{pmatrix}$$

has a vanishing determinat at z_0, thus the homogeneous system of linear equations

$$a\partial_q u_x + b\partial_q v_x = 0$$

$$a\partial_q u_y + b\partial_q v_y = 0$$

has nontrivial solution $(a, b) \neq (0, 0)$. (For $q-$case, a critical point of a real-valued function $u(x, y)$ is a point z_0, where $\partial_q u_x$ and $\partial_q u_y$ vanish at that point.) In other words, the real-valued harmonic function $\varphi = au + bv$ has a critical point at z_0. Assume that $f(z_0) = 0$, and consider the level-set $\varphi = au + bv = 0$ near the point z_0. This level-set of nonconstant harmonic function through a critical point z_0 consists locally of two or more analytic arcs intersecting with equal angles at z_0. On the other hand, f maps $\varphi = au + bv = 0$ into a line. Due to f is locally univalent at z_0, it can not maps the level-set φ consisting of several intersecting arcs onto a line. Thus, this is a contradiction. □

Since $J_{q,f}(z_0) \neq 0$ on $\mathbb{D}$, if $|D_q h(z)| > |D_q g(z)|$, then f is sense-preserving; and if $|D_q g(z)| > |D_q h(z)|$, then f is sense-reversing. The $q-$*second dilatation* is defined by $w_q(z) = D_q g(z)/D_q h(z)$, and for sense-preserving and locally univalent $q-$harmonic functions, the $q-$second dilatation satisfies the following condition.

Definition 2.2.3. "A function $f = h + \overline{g}$ given in the class $\mathcal{H}_q$ is sense-preserving and locally univalent in $\mathbb{D}$ if and only if the $q-$second dilatation w_q holds the condition

$$|w_q(z)| = \left| \frac{D_q g(z)}{D_q h(z)} \right| < 1, \tag{2.2}$$

where $q \in (0, 1)$ and $z \in \mathbb{D}$."

We note that the $q-$second dilatation reduces to the second dilatation when $q \to 1^-$.

We now give some lemmas and theorems due to their usefulness in our main results. We first start with the $q-$Jack's Lemma. By using $q-$derivative operator, the $q-$Jack's Lemma can be presented as below.

Lemma 2.2.4. *"Let ϕ be a nonconstant and analytic function in $\mathbb{D}$ with the condition $\phi(0) = 0$. If $|\phi(z)|$ takes its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{D}$, then*

$$z_0 D_q \phi(z_0) = m\phi(z_0)$$

for some $m \geq 1 \in \mathbb{R}$ and $q \in (0,1)$."

Proof. Using the $q-$derivative operator, we get

$$D_q \phi(z) = \frac{\phi(z) - \phi(qz)}{z - qz} = \frac{\phi(z) - \phi(z_0)}{z - z_0}, \quad qz = z_0.$$

Taking limit for $z \to z_0$, we observe that

$$\lim_{z \to z_0} D_q \phi(z) = D_q \phi(z_0) = \lim_{z \to z_0} \frac{\phi(z) - \phi(z_0)}{z - z_0} = \phi'(z_0).$$

Therefore, in view of Jack's Lemma we obtain

$$z_0 D_q \phi(z_0) = m\phi(z_0).$$

We note that one easily obtain the familiar Jack's Lemma when $q \to 1^-$. □

The following lemma was given in [43].

Lemma 2.2.5. ([*43*]) *If h is an element of $\mathcal{C}_q(b)$, then*

$$F_2(|b|, Reb, q, r) \leq |D_q h(z)| \leq F_1(|b|, Reb, q, r),$$

where

$$F_1(|b|, Reb, q, r) = \left[(1 - qr)^{Reb+|b|}.(1 + qr)^{Reb-|b|}\right]^{-\frac{1-q^2}{2q^2 logq^{-1}}},$$

$$F_2(|b|, Reb, q, r) = \left[(1 - qr)^{Reb-|b|}.(1 + qr)^{Reb+|b|}\right]^{-\frac{1-q^2}{2q^2 logq^{-1}}}.$$

We present a result for $q-$version of the Carathédory class. Denote by $\mathcal{P}_q$ the $q-$Carathédory class holds the following lemma. This lemma leads to prove the next theorem.

Lemma 2.2.6. ([54]) *"Denote by $\mathcal{P}_q$ the class of analytic functions p of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ in $\mathbb{D}$ holds the condition*

$$\left| p(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad (z \in \mathbb{D})$$

where $q \in (0,1)$ is a fixed real number."

Theorem 2.2.7. *p is an element of $\mathcal{P}_q$ if and only if $p(z) \prec \dfrac{1+z}{1-qz}$. This result is sharp for $p(z) = \dfrac{1+\phi(z)}{1-q\phi(z)}$, where ϕ is a Schwarz function.*

Proof. "Let $p \in \mathcal{P}_q$, then

$$\left| p(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q} \Leftrightarrow |p(z) - m| \leq m,$$

where $m = \dfrac{1}{1-q} > 1$. Therefore, we observe that

$$\left| \frac{1}{m} p(z) - 1 \right| \leq 1.$$

Thus, the function $\psi(z) = \frac{1}{m} p(z) - 1$ is analytic and has modulus at most 1 in $\mathbb{D}$; so

$$\phi(z) = \frac{\psi(z) - \psi(0)}{1 - \overline{\psi(0)}\psi(z)} = \frac{(\frac{1}{m}p(z) - 1) - (\frac{1}{m} - 1)}{1 - (\frac{1}{m} - 1)(\frac{1}{m}p(z) - 1)}$$

holds the conditions of Schwarz Lemma. This shows that

$$p(z) = \frac{1+\phi(z)}{1 - (1 - \frac{1}{m})\phi(z)} \Rightarrow p(z) \prec \frac{1+z}{1-qz}.$$

Conversely, assume that the function p in $\mathbb{D}$ holds the condition $p(0) = 1$ and

$$p(z) \prec \frac{1+z}{1-qz} \Rightarrow p(z) = \frac{1+\phi(z)}{1-(1-\frac{1}{m})\phi(z)}$$

$$p(z) - m = m \frac{\frac{1-m}{m} + \phi(z)}{1 + \frac{1-m}{m}\phi(z)}.$$

The function $\left(\frac{\frac{1-m}{m} + \phi(z)}{1 + \frac{1-m}{m}\phi(z)} \right)$ maps the disc onto itself, then we write

$$|p(z) - m| \leq m \Leftrightarrow$$

$$\left| p(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}.$$

This shows that $p \in \mathcal{P}_q$." □

We must note that the linear tranformation $\frac{1+z}{1-qz}$ maps the disc $\mathbb{D}$ onto the disc with radius $\rho(r) = \frac{(1+q)r}{1-q^2r^2}$ and centre $C(r) = \frac{1+qr^2}{1-q^2r^2}$.

2.3 The Class $\mathcal{S}_{\mathcal{HC}_q(b)}$

By using the concept of subordination given in Theorem 2.2.7 and the $q-$derivative operator, the class $\mathcal{C}_q(b)$ was defined in [43]. In view of the class $\mathcal{C}_q(b)$, we define the class $\mathcal{S}_{\mathcal{HC}_q(b)}$. The classes $\mathcal{C}_q(b)$ and $\mathcal{S}_{\mathcal{HC}_q(b)}$ are, respectively, given as follows.

Definition 2.3.1. ([43]) "Let $q \in (0,1)$ and $b \in \mathbb{C}\backslash\{0\}$. A function f given by (1.2) is in $\mathcal{C}_q(b)$ if it holds the condition

$$1 + \frac{1}{b}\left(qz\frac{D_q(D_q f(z))}{D_q f(z)}\right) \prec \frac{1+z}{1-qz}.$$

Such functions are called $q-$convex function of complex order b."

Definition 2.3.2. "Let $q \in (0,1)$ and let $f = h + \overline{g}$ given by (1.7) be in the class $\mathcal{H}_q$. If f is the solution of the non-linear partial differential equation $w_q(z) = \frac{D_q g(z)}{D_q h(z)}$ with $|w_q(z)| < 1$, $w_q(z) \prec b_1 \frac{1+z}{1-qz}$ and the function h is in $\mathcal{C}_q(b)$, then such functions will be called $q-$harmonic mappings for which analytic part is $q-$convex functions of complex order denoted by $\mathcal{S}_{\mathcal{HC}_q(b)}$."

2.4 Distortion Bounds

We first present a theorem, which includes subordination techniques for the functions in the class $\mathcal{S}_{\mathcal{HC}_q(b)}$ as the following.

Theorem 2.4.1. *If $f = h + \overline{g}$ given by (1.7) is in $\mathcal{S}_{\mathcal{HC}_q(b)}$, then*

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1+z}{1-qz}. \tag{2.3}$$

Proof. "Since $f = h + \overline{g} \in \mathcal{S}_{\mathcal{HC}_q(b)}$, then

$$\frac{D_q g(z)}{D_q h(z)} \prec b_1 \frac{1+z}{1-qz}.$$

The linear transformation $w = b_1 \frac{1+z}{1-qz}$ maps $|z| = r$ onto the disc with the centre $C(r) = \left(\frac{\alpha_1(1+qr^2)}{1-q^2r^2}, \frac{\alpha_2(1+qr^2)}{1-q^2r^2}\right)$ and the radius $\rho(r) = \frac{|b_1|(1+q)r}{1-q^2r^2}$ where $b_1 = \alpha_1 + i\alpha_2$. Hence, by using the definition of the class $\mathcal{S}_{\mathcal{HC}_q(b)}$ and the concept of subordination, we get

$$w_q(\mathbb{D}_r) = \left\{\frac{D_q g(z)}{D_q h(z)} : \left|\frac{D_q g(z)}{D_q h(z)} - \frac{b_1(1+qr^2)}{1-q^2r^2}\right| \leq \frac{|b_1|(1+q)r}{1-q^2r^2}, q \in (0,1)\right\}. \tag{2.4}$$

For verifying the conditions of Schwarz function, we consider the analytic function ϕ by

$$\frac{g(z)}{h(z)} = b_1 \frac{1+\phi(z)}{1-q\phi(z)}. \tag{2.5}$$

The function ϕ is a well-defined function and

$$\frac{g(z)}{h(z)}=\frac{b_1z+b_2z^2+...}{z+a_2z^2+...}=b_1\frac{1+\phi(z)}{1-q\phi(z)}\Rightarrow$$

$$\left.\frac{g(z)}{h(z)}\right|_{z=0}=b_1=b_1\frac{1+\phi(0)}{1-q\phi(0)}.$$

This proves that $\phi(0)=0$. Now, we must show that $|\phi(z)|<1$ for each $z\in\mathbb{D}$. If we take $q-$derivative on both sides of (2.5) and simplify, we get

$$\begin{aligned}D_q\left(\frac{g(z)}{h(z)}\right)&=D_q\left(b_1\frac{1+\phi(z)}{1-q\phi(z)}\right)\\ \frac{h(qz)D_qg(z)-g(qz)D_qh(z)}{h(z)h(qz)}&=b_1\frac{D_q(1+\phi(z))(1-q\phi(qz))-D_q(1-q\phi(z))(1+\phi(qz))}{(1-q\phi(z))(1-q\phi(qz))}\\ \frac{D_qg(z)}{h(z)}-\frac{g(qz)D_qh(z)}{h(z)h(qz)}&=b_1\frac{D_q\phi(z)-q\phi(qz)D_q\phi(z)+qD_q\phi(z)+q\phi(qz)D_q\phi(z)}{(1-\phi(z))(1-\phi(qz))}.\end{aligned}$$

If we multiply both sides of the last expression by $h(z)/D_qh(z)$ and simplify, we obtain

$$\begin{aligned}\frac{D_qg(z)}{D_qh(z)}-\frac{g(qz)}{h(qz)}&=b_1\frac{(1+q)D_q\phi(z)}{(1-q\phi(z))(1-q\phi(qz))}\frac{h(z)}{D_qh(z)}\\ \frac{D_qg(z)}{D_qh(z)}&=\frac{g(qz)}{h(qz)}+b_1\frac{(1+q)D_q\phi(z)}{(1-q\phi(z))(1-q\phi(qz))}\frac{h(z)}{D_qh(z)}\\ \frac{D_qg(z)}{D_qh(z)}&=b_1\left(\frac{1+\phi(qz)}{1-q\phi(qz)}+\frac{(1+q)zD_q\phi(z)}{(1-q\phi(z))(1-q\phi(qz))}.\frac{h(z)}{zD_qh(z)}\right).\end{aligned}\tag{2.6}$$

Contrary, suppose that there is a point $z_0\in\mathbb{D}_r$ such that $|\phi(z_0)|=1$. By using Lemma 2.2.4, equation (2.6) gives

$$\frac{D_qg(z_0)}{D_qh(z_0)}=b_1\left(\frac{1+\phi(qz_0)}{1-q\phi(qz_0)}+\frac{(1+q)m\phi(z_0)}{(1-q\phi(z_0))(1-q\phi(qz_0))}.\frac{h(z)}{zD_qh(z)}\right)\notin w_q(\mathbb{D}_r).$$

This is contradiction with (2.4) and thus, $|\phi(z)|<1$ for each $z\in\mathbb{D}$. ” □

We can derive some consequences of this theorem. Using Theorem 2.4.1 and Lemma 2.2.5, distortion bounds are given as below.

Corollary 2.4.2. *If $f=h+\overline{g}\in\mathcal{S}_{\mathcal{HC}_q(b)}$, then we have*

$$F_2(|b|,Reb,|b_1|,q,r)\leq|D_qg(z)|\leq F_1(|b|,Reb,|b_1|,q,r),\tag{2.7}$$

where

$$F_1(|b|,Reb,|b_1|,q,r)=\left[\left(1-qr\right)^{Reb+|b|}\left(1+qr\right)^{Reb-|b|}\right]^{-\frac{1-q^2}{2q^2logq^{-1}}}\frac{|b_1|(1+r)}{1-qr},$$

$$F_2(|b|,Reb,|b_1|,q,r)=\left[\left(1-qr\right)^{Reb-|b|}\left(1+qr\right)^{Reb+|b|}\right]^{-\frac{1-q^2}{2q^2logq^{-1}}}\frac{|b_1|(1-r)}{1+qr}.$$

Proof. Since $f = h + \overline{g} \in \mathcal{S}_{\mathcal{HC}_q(b)}$, from Theorem 2.4.1 we write $\frac{D_q g(z)}{D_q h(z)} \prec b_1 \frac{1+z}{1-qz}$, where $h \in \mathcal{C}_q(b)$. Therefore we have

$$\left| \frac{D_q g(z)}{D_q h(z)} - \frac{b_1(1+qr^2)}{1-q^2r^2} \right| \leq \frac{|b_1|(1+q)r}{1-q^2r^2}.$$

This inequality yields

$$|D_q g(z)| \leq |D_q h(z)| \frac{|b_1|(1+r)}{1-qr}.$$

If we use Lemma 2.2.5, we get the right side of (2.7).

Similarly, we get

$$|D_q g(z)| \geq |D_q h(z)| \frac{|b_1|(1-r)}{1+qr}.$$

Using Lemma 2.2.5, we obtain left side of (2.7). □

Corollary 2.4.3. *If $f = h + \overline{g} \in \mathcal{S}_{\mathcal{HC}_q(b)}$, then*

$$f = h(z) + \overline{h(z) b_1 \frac{1+\phi(z)}{1-q\phi(z)}}, \tag{2.8}$$

where ϕ is a Schwarz function.

Proof. By using Theorem 2.4.1, we get

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1+z}{1-qz} \Rightarrow \frac{g(z)}{h(z)} = b_1 \frac{1+\phi(z)}{1-q\phi(z)}.$$

Thus, we obtain

$$g(z) = h(z) b_1 \frac{1+\phi(z)}{1-q\phi(z)},$$

which gives (2.8). □

Chapter 3

QUANTUM CALCULUS IN HARMONIC UNIVALENT CONVEX FUNCTIONS

After the paper of Clunie and Sheil-Small [6], harmonic univalent functions rapidly developed and many familiar results of the univalent functions were generalized to the harmonic univalent functions. The class of univalent convex functions was also generalized to harmonic univalent functions. In literature, there are considerable results on these functions. In this chapter, we define the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ of $q-$harmonic univalent convex functions of complex order and type α by using $q-$difference operator. This definition is a generalization of the harmonic univalent convex functions.

Let the function $f = h + \overline{g}$ defined by (1.7) be in the class $\mathcal{H}_q$ with $b_1 = 0$. Then, the sequences of partial sums of functions f are given by

$$f_m(z) = z + \sum_{n=2}^{m} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n},$$

$$f_s(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{s} \overline{b_n z^n}.$$

The studies of partial sums of starlike functions and convex functions were first initiated by Silverman [50] and Silvia [51]. Later, Porwal [45], Porwal and Dixit [46] derived some results of partial sums for harmonic univalent functions.

This part of thesis includes some ratios of partial sums of the functions in the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. We first obtain a sufficient coefficient condition that guarentees that a function f in $\mathcal{H}_q$ is sense-preserving harmonic univalent in $\mathbb{D}$ and belongs to the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. By using this coefficient condition, we then examine four ratios of partial sums of f in $\mathcal{H}_q$ given by

$$Re\left\{\frac{f(z)}{f_m(z)}\right\},\quad Re\left\{\frac{f_m(z)}{f(z)}\right\},\quad Re\left\{\frac{f(z)}{f_s(z)}\right\},\quad Re\left\{\frac{f_s(z)}{f(z)}\right\}.$$

We finally introduce distortion theorems, covering theorem and convolution conditions for the functions in the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$.

3.1 The Class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$

By using the $q-$difference operator, definition of the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ is given as follows.

Definition 3.1.1. "A function $f = h + \overline{g}$ given by (1.7) is in the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ if

$$Re\left[1+\frac{1}{b}\left(\frac{zD_q(zD_qf(z))}{zD_qf(z)}-1\right)\right] \geq \alpha, \tag{3.1}$$

where $b \in \mathbb{C}\setminus\{0\}$, $q \in (0,1)$, $\alpha \in [0,1)$ and $z \in \mathbb{D}$. Such functions are called $q-$harmonic univalent convex functions of complex order and type α."

For various parameters, we obtain some known subclasses as special cases:

1) When $b = 1$ and $q \to 1^-$, we get the class $\lim_{q\to 1^-}\mathcal{C}_{\mathcal{H}_q}(1,\alpha) \equiv \mathcal{C}_{\mathcal{H}}(\alpha)$ of harmonic univalent convex functions of order α defined in [29].

2) When $b = 1$, $\alpha = 0$ and $q \to 1^-$, we get the class $\lim_{q\to 1^-}\mathcal{C}_{\mathcal{H}_q}(1,0) \equiv \mathcal{C}_{\mathcal{H}}$ of harmonic univalent convex functions defined in [6].

3) When $b = 1$, $q \to 1^-$ and $g(z) \equiv 0$, we get the class $\lim_{q\to 1^-}\mathcal{C}_{\mathcal{H}_q}(1,\alpha) \equiv \mathcal{C}(\alpha)$ of univalent convex functions of order α defined in [47].

4) When $\alpha = 0$, $q \to 1^-$ and $g(z) \equiv 0$, we get the class $\lim_{q\to 1^-}\mathcal{C}_{\mathcal{H}_q}(b,0) \equiv \mathcal{C}(b)$ of univalent convex functions of complex order defined in [56].

5) When $b = 1$, $\alpha = 0$, $q \to 1^-$ and $g(z) \equiv 0$, we get the class $\lim_{q\to 1^-}\mathcal{C}_{\mathcal{H}_q}(1,0) \equiv \mathcal{C}$ of univalent convex functions defined in [53].

3.2 Coefficient Bounds and Univalence Criteria

In the following theorem, we prove sufficient coefficient condition and show univalence criteria for functions belonging to the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$.

Theorem 3.2.1. *Let $b \in \mathbb{C} \setminus \{0\}$, $q \in (0,1)$, $\alpha \in [0,1)$, $z \in \mathbb{D}$, and let $f = h + \overline{g}$ defined by (1.7) be in $\mathcal{H}_q$. If*

$$\sum_{n=2}^{\infty} \lambda_n |a_n| + \sum_{n=1}^{\infty} \nu_n |b_n| \leq |b|(1-\alpha), \tag{3.2}$$

where

$$\lambda_n = [n]_q([n]_q - 1 + |b|(1-\alpha)), \quad (n \geq 2), \tag{3.3}$$

$$\nu_n = [n]_q([n]_q + 1 - |b|(1-\alpha)), \quad (n \geq 1), \tag{3.4}$$

then f is sense-preserving and univalent function in $\mathbb{D}$; thus, $f \in \mathcal{C}_{\mathcal{H}_q}(b,\alpha)$.

Proof. "In order to show that the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ is univalent, we will show that $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$. Then

$$\begin{aligned}
\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| \\
&= 1 - \left|\frac{\sum\limits_{n=1}^{\infty} b_n(\overline{z}_1^n - \overline{z}_2^n)}{(z_1 - z_2) + \sum\limits_{n=2}^{\infty} a_n(z_1^n - z_2^n)}\right| \\
&> 1 - \frac{\sum\limits_{n=1}^{\infty} n|b_n|}{1 - \sum\limits_{n=2}^{\infty} n|a_n|} \\
&\geq 1 - \frac{\sum\limits_{n=1}^{\infty} \frac{[n]_q\left([n]_q + 1 - |b|(1-\alpha)\right)}{|b|(1-\alpha)}|b_n|}{1 - \sum\limits_{n=2}^{\infty} \frac{[n]_q\left([n]_q - 1 + |b|(1-\alpha)\right)}{|b|(1-\alpha)}|a_n|} \geq 0.
\end{aligned}$$

This proves univalence of f. Also, f is sense-preserving in $\mathbb{D}$ because

$$\begin{aligned}
|D_q h(z)| \geq 1 - \sum_{n=2}^{\infty} [n]_q |a_n||z|^{n-1} &> 1 - \sum_{n=2}^{\infty} \frac{[n]_q\big([n]_q - 1 + |b|(1-\alpha)\big)}{|b|(1-\alpha)}|a_n| \\
&\geq \sum_{n=1}^{\infty} \frac{[n]_q\big([n]_q + 1 - |b|(1-\alpha)\big)}{|b|(1-\alpha)}|b_n| \\
&> \sum_{n=1}^{\infty} [n]_q |b_n||z|^{n-1} \geq |D_q g(z)|.
\end{aligned}$$

We know that $Re(\omega) > \alpha$ if and only if $|1-\alpha+\omega| > |1+\alpha-\omega|$. In view of the relation given in (3.1), we will show that

$$\begin{aligned} &\left|1-\alpha+\frac{(b-1)zD_qf(z)+zD_q(zD_qf(z))}{bzD_qf(z)}\right| \\ &-\left|1+\alpha-\frac{(b-1)zD_qf(z)+zD_q(zD_qf(z))}{bzD_qf(z)}\right| \geq 0. \end{aligned} \tag{3.5}$$

The series expansions of $zD_qf(z)$ and $zD_q(zD_qf(z))$ can be written as

$$\begin{aligned} zD_qf(z) &= zD_qh(z) - zD_qg(z) \\ &= z + \sum_{n=2}^{\infty}[n]_q a_n z^n - \sum_{n=1}^{\infty}[n]_q b_n \overline{z^n} \end{aligned}$$

and

$$\begin{aligned} zD_q(zD_qf(z)) &= zD_q(zD_qh(z)) + zD_q(zD_qg(z)) \\ &= z + \sum_{n=2}^{\infty}[n]_q^2 a_n z^n + \sum_{n=1}^{\infty}[n]_q^2 b_n \overline{z^n}. \end{aligned}$$

Substituting these series expansions into the left side of (3.5), we observe that

$$\begin{aligned} &= \left| b(2-\alpha)z + \sum_{n=2}^{\infty}[n]_q([n]_q - 1 + b(2-\alpha))a_n z^n \right. \\ &\quad \left. + \sum_{n=1}^{\infty}[n]_q([n]_q + 1 - b(2-\alpha))b_n \overline{z^n} \right| \\ &\quad - \left| b\alpha z - \sum_{n=2}^{\infty}[n]_q([n]_q - 1 - b\alpha)a_n z^n - \sum_{n=1}^{\infty}[n]_q([n]_q + 1 + b\alpha))b_n \overline{z^n} \right| \\ &\geq |b|(2-\alpha)|z| - \sum_{n=2}^{\infty}[n]_q([n]_q - 1 + |b|(2-\alpha))|a_n||z|^n \\ &\quad - \sum_{n=1}^{\infty}[n]_q([n]_q + 1 - |b|(2-\alpha))|b_n||z|^n \\ &\quad - |b|\alpha|z| - \sum_{n=2}^{\infty}[n]_q([n]_q - 1 - |b|\alpha)|a_n||z|^n - \sum_{n=1}^{\infty}[n]_q([n]_q + 1 + |b|\alpha))|b_n||z|^n \\ &\geq 2|b|(1-\alpha)|z| - 2\sum_{n=2}^{\infty}[n]_q([n]_q - 1 + |b|(1-\alpha))|a_n||z|^n \\ &\quad - 2\sum_{n=1}^{\infty}[n]_q([n]_q + 1 - |b|(1-\alpha))|b_n||z|^n \\ &\geq |b|(1-\alpha)|z|\left(1 - \sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n||z|^{n-1} - \sum_{n=1}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n||z|^{n-1}\right) \geq 0, \end{aligned}$$

by (3.2). This proves that $f \in \mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. The function defined by

$$f(z) = z + \sum_{n=2}^{\infty}\frac{|b|(1-\alpha)}{\lambda_n}x_n z^n + \sum_{n=1}^{\infty}\frac{|b|(1-\alpha)}{\nu_n}\overline{y_n z^n},$$

where $\sum\limits_{n=2}^{\infty}|x_n|+\sum\limits_{n=1}^{\infty}|y_n|=1$, show that the estimate in (3.2) is sharp." □

In order to establish that (3.2) is also necessary condition, we need to define a class $\mathcal{TH}_q$ containing the functions $f=h+\overline{g}$, where

$$h(z)=z-\sum_{n=2}^{\infty}|a_n|z^n \quad \text{and} \quad g(z)=-\sum_{n=1}^{\infty}|b_n|z^n,\ (z\in\mathbb{D}). \tag{3.6}$$

The bound in (3.2) is necessary condition for the class $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ defined by $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha):=\mathcal{TH}_q\cap\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. We prove the next coefficient characterization.

Theorem 3.2.2. *If $f=h+\overline{g}$ defined by* (3.6) *belongs to $\mathcal{TH}_q$, then $f\in\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}\lambda_n|a_n|+\sum_{n=1}^{\infty}\nu_n|b_n|\le|b|(1-\alpha), \tag{3.7}$$

where λ_n and ν_n given, respectively, by (3.3) *and* (3.4)*, and where $b\in\mathbb{C}\setminus\{0\}$, $q\in(0,1)$ and $\alpha\in[0,1)$.*

Proof. "Due to $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)\subset\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$, we need to show that only if part of this theorem. Using the functions $f=h+\overline{g}$ defined by (3.6), we observe that the condition (3.1) can be written as

$$Re\left\{\frac{(b-1)zD_qf(z)+zD_q(zD_qf(z))}{bzD_qf(z)}-\alpha\right\}\ge 0,$$

which is equivalent to

$$Re\left\{\frac{b(1-\alpha)z-\sum\limits_{n=2}^{\infty}\lambda_n|a_n|z^n-\sum\limits_{n=1}^{\infty}\nu_n|b_n|\overline{z}^n}{z-\sum\limits_{n=2}^{\infty}[n]_q|a_n|z^n+\sum\limits_{n=1}^{\infty}[n]_q|b_n|\overline{z}^n}\right\}\ge 0. \tag{3.8}$$

The expression given in (3.8) must satisfy for all $z\in\mathbb{D}$, ($|z|=r<1$). If we choose the values of z on the positive real axis, we must have

$$\frac{|b|(1-\alpha)-\sum\limits_{n=2}^{\infty}\lambda_n|a_n|r^{n-1}-\sum\limits_{n=1}^{\infty}\nu_n|b_n|r^{n-1}}{1-\sum\limits_{n=2}^{\infty}[n]_q|a_n|r^{n-1}+\sum\limits_{n=1}^{\infty}[n]_q|b_n|r^{n-1}}\ge 0. \tag{3.9}$$

The numerator in (3.9) is negative for r sufficiently close to 1 if the bound in (3.7) does not satisfy. Hence, there is a point $z_0=r_0$ between $(0,1)$ for which the quotient in (3.9) is negative. Due to contradiction, we say that $f\in\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$."

□

3.3 Partial Sums

We now establish some ratios of partial sums of the functions in the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$.

Theorem 3.3.1. *Let* $f = h + \overline{g}$ *defined by* (1.7) *be in* $\mathcal{H}_q$ *with* $b_1 = 0$. *If* f *holds the condition given by* (3.2) *and*

$$\lambda_n \geq \begin{cases} |b|(1-\alpha), & n=2,3,\ldots,m \\ \\ \lambda_{m+1}, & n=m+1,m+2,\ldots, \end{cases}$$

then

$$i) \quad Re\left(\frac{f(z)}{f_m(z)}\right) \geq 1 - \frac{|b|(1-\alpha)}{\lambda_{m+1}}, \tag{3.10}$$

$$ii) \quad Re\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{\lambda_{m+1}}{\lambda_{m+1} + |b|(1-\alpha)}. \tag{3.11}$$

These estimates are sharp for

$$f(z) = z + \frac{|b|(1-\alpha)}{\lambda_{m+1}} z^{m+1}. \tag{3.12}$$

Proof. i) "Since $f \in \mathcal{C}_{\mathcal{H}_q}(b,\alpha)$, by (3.2) we have

$$\sum_{n=2}^{\infty} \frac{\lambda_n}{|b|(1-\alpha)}|a_n| + \sum_{n=1}^{\infty} \frac{\nu_n}{|b|(1-\alpha)}|b_n| \leq 1,$$

where λ_n and ν_n given, respectively, by (3.3) and (3.4). For proving (3.10), we consider

$$\begin{aligned}
\omega_1(z) &= \frac{\lambda_{m+1}}{|b|(1-\alpha)}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{|b|(1-\alpha)}{\lambda_{m+1}}\right)\right\} \\
&= 1 + \frac{\lambda_{m+1}}{|b|(1-\alpha)}\left(\frac{f(z) - f_m(z)}{f_m(z)}\right) \\
&= 1 + \frac{\lambda_{m+1}}{|b|(1-\alpha)}\left(\frac{z + \sum\limits_{n=2}^{\infty} a_n z^n + \sum\limits_{n=2}^{\infty} \overline{b_n z^n} - \left(z + \sum\limits_{n=2}^{m} a_n z^n + \sum\limits_{n=2}^{\infty} \overline{b_n z^n}\right)}{z + \sum\limits_{n=2}^{m} a_n z^n + \sum\limits_{n=2}^{\infty} \overline{b_n z^n}}\right) \\
&= 1 + \frac{\frac{\lambda_{m+1}}{|b|(1-\alpha)} \sum\limits_{n=m+1}^{\infty} a_n z^n}{z + \sum\limits_{n=2}^{m} a_n z^n + \sum\limits_{n=2}^{\infty} \overline{b_n z^n}},
\end{aligned} \tag{3.13}$$

which is analytic in $\mathbb{D}$ with $\omega_1(0) = 1$. It is sufficent to show that $Re\omega_1(z) > 0$ or equivalently

$$\left|\frac{\omega_1(z) - 1}{\omega_1(z) + 1}\right| \leq 1.$$

Substituting (3.13) into this inequality, we get

$$\left|\frac{\omega_1(z)-1}{\omega_1(z)+1}\right| \le \frac{\frac{\lambda_{m+1}}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2(\sum\limits_{n=2}^{m}|a_n|+\sum\limits_{n=2}^{\infty}|b_n|)-\frac{\lambda_{m+1}}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|a_n|}.$$

The last inequality is bounded by 1 if and only if

$$\sum_{n=2}^{m}|a_n|+\sum_{n=2}^{\infty}|b_n|+\frac{\lambda_{m+1}}{|b|(1-\alpha)}\sum_{n=m+1}^{\infty}|a_n| \le 1. \tag{3.14}$$

It suffices to show that left side of (3.14) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n|,$$

which is equaivalent to

$$\sum_{n=2}^{m}\frac{\lambda_n-|b|(1-\alpha)}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{\infty}\frac{\nu_n-|\tau|(1-\alpha)}{|b|(1-\alpha)}|b_n|+\sum_{n=m+1}^{\infty}\frac{\lambda_n-\lambda_{m+1}}{|b|(1-\alpha)}|a_n| \ge 0.$$

If we take $f(z)=z+\frac{|b|(1-\alpha)}{\lambda_{m+1}}z^{m+1}$ with $z=re^{i\pi/m}$ and r approaches to 1 from left, then we get

$$\frac{f(z)}{f_m(z)}=1+\frac{|b|(1-\alpha)}{\lambda_{m+1}}z^m \to 1-\frac{|b|(1-\alpha)}{\lambda_{m+1}}.$$

ii) Similarly, for proving (3.11) we consider

$$\begin{aligned}
\omega_2(z) &= \frac{\lambda_{m+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\left\{\frac{f_m(z)}{f(z)}-\left(1-\frac{|b|(1-\alpha)}{\lambda_{m+1}+|b|(1-\alpha)}\right)\right\} \\
&= 1+\frac{\lambda_{m+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\left(\frac{f_m(z)-f(z)}{f(z)}\right) \\
&= 1+\frac{\lambda_{m+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\left(\frac{z+\sum\limits_{n=2}^{m}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}-\left(z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}\right)}{z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}}\right) \\
&= 1-\frac{\frac{\lambda_{m+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}a_nz^n}{z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}},
\end{aligned}$$

which is analytic in $\mathbb{D}$ with $\omega_2(0)=1$. Therefore

$$\left|\frac{\omega_2(z)-1}{\omega_2(z)+1}\right| \le \frac{\frac{\lambda_{m+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2(\sum\limits_{n=2}^{m}|a_n|+\sum\limits_{n=2}^{\infty}|b_n|)-\frac{\lambda_{m+1}-|b|(1-\alpha)}{|b|(1-\alpha)}\sum\limits_{n=m+1}^{\infty}|a_n|} \le 1$$

if and only if

$$\sum_{n=2}^{m}|a_n|+\sum_{n=2}^{\infty}|b_n|+\frac{\lambda_{m+1}}{|b|(1-\alpha)}\sum_{n=m+1}^{\infty}|a_n|\le 1. \tag{3.15}$$

Since left side of (3.15) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n|,$$

this completes the proof." □

Theorem 3.3.2. *Let $f=h+\overline{g}$ defined by* (1.7) *be in $\mathcal{H}_q$ with $b_1=0$. If f holds the condition given by* (3.2) *and*

$$\nu_n \ge \begin{cases} |b|(1-\alpha), & n=2,3,\ldots,s \\ \\ \nu_{s+1}, & n=s+1,s+2,\ldots, \end{cases}$$

then

$$i)\quad Re\left(\frac{f(z)}{f_s(z)}\right)\ge 1-\frac{|b|(1-\alpha)}{\nu_{s+1}}, \tag{3.16}$$

$$ii)\quad Re\left(\frac{f_s(z)}{f(z)}\right)\ge \frac{\nu_{s+1}}{\nu_{s+1}+|b|(1-\alpha)}. \tag{3.17}$$

These estimates are sharp for

$$f(z)=z+\frac{|b|(1-\alpha)}{\nu_{s+1}}\overline{z}^{s+1}. \tag{3.18}$$

Proof. i) "Since $f\in\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$, by (3.2) we have

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n|+\sum_{n=1}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n|\le 1,$$

where λ_n and ν_n given, respectively, by (3.3) and (3.4). For proving (3.16), we consider

$$\begin{aligned}\omega_3(z)&=\frac{\nu_{s+1}}{|b|(1-\alpha)}\left\{\frac{f(z)}{f_s(z)}-\left(1-\frac{|b|(1-\alpha)}{\nu_{s+1}}\right)\right\}\\ &=1+\frac{\nu_{s+1}}{|b|(1-\alpha)}\left(\frac{f(z)-f_s(z)}{f_s(z)}\right)\\ &=1+\frac{\nu_{s+1}}{|b|(1-\alpha)}\left(\frac{z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}-\left(z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{s}\overline{b_nz^n}\right)}{z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{s}\overline{b_nz^n}}\right)\\ &=1+\frac{\frac{\nu_{s+1}}{|b|(1-\alpha)}\sum\limits_{n=s+1}^{\infty}b_n\overline{z^n}}{z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{s}\overline{b_nz^n}}.\end{aligned}$$

which is analytic in $\mathbb{D}$ with $\omega_3(0) = 1$. It is sufficent to show that $Re\omega_3(z) > 0$, or

$$\left|\frac{\omega_3(z)-1}{\omega_3(z)+1}\right| \leq \frac{\frac{\nu_{s+1}}{|b|(1-\alpha)}\sum\limits_{n=s+1}^{\infty}|b_n|}{2-2(\sum\limits_{n=2}^{\infty}|a_n|+\sum\limits_{n=2}^{s}|b_n|)-\frac{\nu_{s+1}}{|b|(1-\alpha)}\sum\limits_{n=s+1}^{\infty}|b_n|} \leq 1$$

if and only if

$$\sum_{n=2}^{\infty}|a_n|+\sum_{n=2}^{s}|b_n|+\frac{\nu_{s+1}}{|b|(1-\alpha)}\sum_{n=s+1}^{\infty}|b_n| \leq 1. \tag{3.19}$$

It suffices to show that left side of (3.19) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n|$$

which is equaivalent to

$$\sum_{n=2}^{\infty}\frac{\lambda_n-|b|(1-\alpha)}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{s}\frac{\nu_n-|b|(1-\alpha)}{|b|(1-\alpha)}|b_n|+\sum_{n=s+1}^{\infty}\frac{\nu_n-\nu_{s+1}}{|b|(1-\alpha)}|a_n| \geq 0.$$

To prove that $f(z) = z + \frac{|b|(1-\alpha)}{\nu_{s+1}}\overline{z}^{s+1}$ gives the sharp result, we observe that for $z = re^{i\pi/s+2}$ we have

$$\frac{f(z)}{f_s(z)} = 1 + \frac{|b|(1-\alpha)}{\nu_{s+1}}r^s e^{-i(s+2)\frac{\pi}{s+2}} \to 1 - \frac{|b|(1-\alpha)}{\nu_{s+1}},$$

when $r \to 1^-$.

ii) Similarly, we obtain

$$\begin{aligned}
\omega_4(z) &= \frac{\nu_{s+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\left\{\frac{f_s(z)}{f(z)}-\left(1-\frac{|b|(1-\alpha)}{\nu_{s+1}+|b|(1-\alpha)}\right)\right\}\\
&= 1+\frac{\nu_{s+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\left(\frac{f_s(z)-f(z)}{f(z)}\right)\\
&= 1+\frac{\nu_{s+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\left(\frac{z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{s}\overline{b_nz^n}-\left(z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}\right)}{z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}}\right)\\
&= 1-\frac{\frac{\nu_{s+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\sum\limits_{n=s+1}^{\infty}b_n\overline{z^n}}{z+\sum\limits_{n=2}^{\infty}a_nz^n+\sum\limits_{n=2}^{\infty}\overline{b_nz^n}}.
\end{aligned}$$

which is analytic in $\mathbb{D}$ with $\omega_4(0) = 1$. Therefore

$$\left|\frac{\omega_4(z)-1}{\omega_4(z)+1}\right| \leq \frac{\frac{\nu_{s+1}+|b|(1-\alpha)}{|b|(1-\alpha)}\sum\limits_{n=s+1}^{\infty}|b_n|}{2-2(\sum\limits_{n=2}^{\infty}|a_n|+\sum\limits_{n=2}^{s}|b_n|)-\frac{\nu_{s+1}-|b|(1-\alpha)}{|b|(1-\alpha)}\sum\limits_{n=s+1}^{\infty}|b_n|} \leq 1$$

if and only if

$$\sum_{n=2}^{\infty}|a_n|+\sum_{n=2}^{s}|b_n|+\frac{\nu_{s+1}}{|b|(1-\alpha)}\sum_{n=s+1}^{\infty}|b_n|\leq 1. \tag{3.20}$$

Since left side of (3.20) is bounded above by

$$\sum_{n=2}^{\infty}\frac{\lambda_n}{|b|(1-\alpha)}|a_n|+\sum_{n=2}^{\infty}\frac{\nu_n}{|b|(1-\alpha)}|b_n|,$$

the proof is completed." □

3.4 Distortion Bounds and Covering Theorem

Distortion bounds for the functions belonging to the class $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ can be proven as below. This result also yields the covering theorem.

Theorem 3.4.1. *If $f\in\mathcal{TC}_{\mathcal{H}_q}(\alpha)$, then for $|z|=r<1$ we have*

$$|f(z)|\leq(1+|b_1|)r+\frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\left(1-\frac{2-|b|(1-\alpha)}{|b|(1-\alpha)}|b_1|\right)r^2, \tag{3.21}$$

and

$$|f(z)|\geq(1-|b_1|)r-\frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\left(1-\frac{2-|b|(1-\alpha)}{|b|(1-\alpha)}|b_1|\right)r^2. \tag{3.22}$$

Proof. "Let $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$. Then

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n\\
&\leq (1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2\\
&\leq (1+|b_1|)r+\frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\\
&\quad\sum_{n=2}^{\infty}\frac{[2]_q([2]_q-1+|b|(1-\alpha))}{|b|(1-\alpha)}\big(|a_n|+|b_n|\big)r^2\\
&\leq (1+|b_1|)r+\frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\\
&\quad\sum_{n=2}^{\infty}\left(\frac{[n]_q([n]_q-1+|b|(1-\alpha))}{|b|(1-\alpha)}|a_n|+\frac{[n]_q([n]_q+1-|b|(1-\alpha))}{|b|(1-\alpha)}|b_n|\right)r^2\\
&\leq (1+|b_1|)r+\frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\left(1-\frac{2-|b|(1-\alpha)}{|b|(1-\alpha)}|b_1|\right)r^2.
\end{aligned}$$

This proves (3.21).

Similarly, we get

$$\begin{aligned}|f(z)| &\le (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n\\
&\le (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2\\
&\le (1-|b_1|)r - \frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\\
&\quad \sum_{n=2}^{\infty}\frac{[2]_q([2]_q-1+|b|(1-\alpha))}{|b|(1-\alpha)}\big(|a_n|+|b_n|\big)r^2\\
&\le (1-|b_1|)r - \frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\\
&\quad \sum_{n=2}^{\infty}\left(\frac{[n]_q([n]_q-1+|b|(1-\alpha))}{|b|(1-\alpha)}|a_n| + \frac{[n]_q([n]_q+1-|b|(1-\alpha))}{|b|(1-\alpha)}|b_n|\right)r^2\\
&\le (1-|b_1|)r - \frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\left(1-\frac{2-|b|(1-\alpha)}{|b|(1-\alpha)}|b_1|\right)r^2.\end{aligned}$$

This proves (3.22). Thus, the proof is completed." □

From inequality (3.22), we obtain the covering result as follows:

Corollary 3.4.2. *If $f \in \mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$, then*

$$\left\{w : |w| < \left(1-\frac{|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}\right) + \left(\frac{2-|b|(1-\alpha)}{[2]_q([2]_q-1+|b|(1-\alpha))}-1\right)|b_1|\right\} \subset f(\mathbb{D}).$$

For $q \to 1^-$, $\alpha = b_1 = 0$ and $b = 1$, Corollary 3.4.2 yields the following result given in [29].

Remark 3.4.3. *If $f \in \mathcal{T}^0\mathcal{C}_{\mathcal{H}}(0)$, then $\{w : |w| < 3/4\} \subset f(\mathbb{D})$.*

3.5 Convolution Conditions

By using the definition of convolution, we show that $\mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ is closed under convolution.

Theorem 3.5.1. *For $0 \le \beta \le \alpha < 1$, suppose $f \in \mathcal{TC}_{\mathcal{H}_q}(b,\alpha)$ and $F \in \mathcal{TC}_{\mathcal{H}_q}(b,\beta)$, then $f * F \in \mathcal{TC}_{\mathcal{H}_q}(b,\alpha) \subset \mathcal{TC}_{\mathcal{H}_q}(b,\beta)$.*

Proof. "By using (3.6), the harmonic functions f and F given by (1.10) can be written as

$$f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n - \sum_{n=1}^{\infty}|b_n|\overline{z}^n$$

and

$$F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n - \sum_{n=1}^{\infty} |B_n| \overline{z}^n.$$

Due to the definition of convolution of two harmonic functions, we write

$$(f * F)(z) = z + \sum_{n=2}^{\infty} |a_n||A_n| z^n + \sum_{n=1}^{\infty} |b_n||B_n| \overline{z}^n.$$

Since $F \in \mathcal{TC}_{\mathcal{H}_q}(\beta)$, from Theorem 3.2.2 we observe that $|A_n| \leq 1$ and $|B_n| \leq 1$. Thus, we obtain

$$\sum_{n=2}^{\infty} \frac{[n]_q([n]_q - 1 + |b|(1-\alpha))}{|b|(1-\alpha)} |a_n||A_n| + \sum_{n=1}^{\infty} \frac{[n]_q([n]_q + 1 - |b|(1-\alpha))}{|b|(1-\alpha)} |b_n||B_n|$$
$$\leq \sum_{n=2}^{\infty} \frac{[n]_q([n]_q - 1 + |b|(1-\alpha))}{|b|(1-\alpha)} |a_n| + \sum_{n=1}^{\infty} \frac{[n]_q([n]_q + 1 - |b|(1-\alpha))}{|b|(1-\alpha)} |b_n| \leq 1.$$

Due to Theorem 3.2.2, we prove that $f * F \in \mathcal{TC}_{\mathcal{H}_q}(b, \alpha) \subset \mathcal{TC}_{\mathcal{H}_q}(b, \beta)$." □

Chapter 4

CONCLUSION AND FUTURE SCOPE

Harmonic univalent functions and their subclasses are well-studied topics in Geometric Function Theory, and there are considerable results on this area in literature. But, the relation between harmonic univalent functions and quantum calculus is quite new and very little is known about this topic. After the paper by Ismail, Merkes and Styer [23], quantum calculus was used in the analytic univalent functions, and researchers published a great number of papers on these functions by using $q-$calculus. Motivated by these recent developments, we attempted to introduce some elementary definitions for $q-$harmonic univalent functions by using $q-$difference operator, and defined two new subclasses of these functions and investigated their properties.

In Chapter 2, we defined the $q-$Jacobian and concluded that the $q-$Jacobian is nonvanishing for locally univalent $q-$harmonic functions. We further defined the $q-$second dilatation, and observed that for locally univalent, sense preserving $q-$harmonic functions, the $q-$second dilatation satisfies the condition $|w_q(z)| < 1$. We also proved the $q-$Jack's Lemma. In the light of these new terms, we defined the class $\mathcal{S}_{\mathcal{HC}_q(b)}$ of "q-harmonic mappings for which analytic part is $q-$convex functions of complex order." Finally, making use of subordination technique and the $q-$Jack's Lemma, we obtained distortion bounds for the functions in the class $\mathcal{S}_{\mathcal{HC}_q(b)}$.

In Chapter 3, we defined the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$ of "$q-$harmonic univalent convex functions of complex order and type α" by using $q-$difference operator. We first investigated sufficient coefficient estimates and univalence criteria for the functions in this class. By using these coefficient estimates, we explored four ratios of partial sums of the functions in the class $\mathcal{C}_{\mathcal{H}_q}(b,\alpha)$. Finally, we proved distortion bounds, covering theorem and convolution conditions for the functions in this class.

We concluded that our definitions and classes are improvement of the familiar harmonic univalent functions. In the limiting case, we obtained their special cases.

In this thesis, we attemtped to study on $q-$harmonic univalent convex functions. But, $q-$calculus can be used in the other classes of harmonic univalent functions such as harmonic univalent starlike functions, harmonic univalent close-to-convex functions, and so on. Recently, some important differential and integral operators was defined by using $q-$calculus. We can also say that classes of the $q-$harmonic univalent functions can be defined by these new operators.

References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. Math.* **17** (1915/1916), 12-22.

[2] Y. Avcı, E. Zlotkiewicz, On harmonic univalent mappings, *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, **44** (1990), 1-7.

[3] L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, *S.-B. Preuss. Akad. Wiss.* **138** (1916), 940-955.

[4] L. Bieberbach, Über einige Extremalprobleme im Geibete der konformen Abbildung, *Math. Ann.* **77** (1916), 153-172.

[5] C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annehmen, *Math. Ann.* **64** (1907), 95-115.

[6] J. Clunie, T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **9** (1984), 3-25.

[7] A. Çetinkaya, Yaşar Polatoğlu, $q-$harmonic mappings for which analytic part is $q-$convex functions of complex order, *Hacettepe University Bulletin of Natural Sciences and Engineering - Series B: Mathematics and Statistics,* **47** (2018), no. 4, 813-820.

[8] A. Çetinkaya, O. Mert, A certain class of harmonic mappings related to functions of bounded boundary rotation, *In Proc. of* 12^{th} *International Symposium on Geometric Function Theory and Applications*, (2016), 67-76.

[9] A. Çetinkaya, Y. Polatoğlu, Partial sums of harmonic univalent convex functions by using quantum calculus, *communicated to journal.*

[10] M. Dorff, J. Rolf, *Anamorphosis, mapping problems, and harmonic univalent functions*, In book: Explorations in Complex Analysis, 197-269, Math. Assoc. of America, Washington, 2012.

[11] L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.* **154** (1985), 137-152.

[12] P. L. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.

[13] P. L. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Math. (156), Cambridge University Press, 2004.

[14] T. Ernst, A method for $q-$calculus, *J. Nonlinear Math. Phys.* **10** (2003), no. 4, 487-525.

[15] N. J. Fine, *Basic hypergeometric series and applications*, Math. Surveys Monogr. 1988.

[16] P. R. Garabedian, M. Schiffer, A proof of the Bieberbach conjecture of fourth coefficient, *J. Rational Mech. Anal.* **4** (1955), 428-465.

[17] G. Gasper, M. Rahman, *Basic hypergeometric series,* Cambridge University Press, 2004.

[18] A. W. Goodman, *Univalent functions*, Mariner Pub. Co. Inc., Vol. I, II, Tampa, Florida, 1983.

[19] I. Graham, G. Kohr, *Geometric Function Theory in one and higher dimensions*, Marcel Dekker Inc., New York-Basel, 2003.

[20] T. H. Gronwall, Some remarks on conformal representation, *Ann. Math.* **16** (1914/15), 72-76.

[21] W. Hengartner, G. Schober, Univalent harmonic functions, *Trans. Amer. Math. Soc.* **299** (1987), no. 1, 1-31.

[22] W. Hengartner, G. Schober, On the boundary behavior of orientation-preserving harmonic mappings, *Complex Variables Theory Appl.* **5** (1986), no. 2-4, 197-208.

[23] M. E. H. Ismail, E. Merkes, D. Styer, A generalization of starlike functions, *Complex Variables Theory Appl.* **14** (1990), no. 1-4, 77-84.

[24] I. S. Jack, Functions starlike and convex of order α, *J. London Math. Soc.* **32** (1971), 469-474.

[25] F. H. Jackson, On $q-$functions and a certain difference operator, *Trans. Royal Soc. Edinburgh*, **46** (1909), no. 2, 253-281.

[26] F. H. Jackson, On $q-$definite integrals, *Quart. J. Pure Appl. Math.* **41** (1910), 193-203.

[27] F. H. Jackson, $q-$difference equations, *Amer. J. Math.* **32** (1910), 305-314.

[28] J. M. Jahangiri, Harmonic functions starlike in the unit disk, *J. Math. Anal. Appl.* **235** (1999), 470-477.

[29] J. M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, *Ann. Univ. Mariae Curie-Skłodowska Sect. A*, **52** (1998), 57-66.

[30] V. Kac, P. Cheung, *Quantum calculus*, Springer-Verlag, New York, 2002.

[31] P. Koebe, Über die Uniformisierung beliebiger analytischer kurven, *Nachr. Kgl. Ges. Wiss. Göttingen, Math. -Phys. KI*, (1907), 191-210.

[32] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.* **42** (1936), 689-692.

[33] J. E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.* **23** (1925), 481-519.

[34] C. Loewner, Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises $|z| < 1$, die durch Funktionen mit nichtverschwindender Ableitung geliefertwerden, *S.-B. Sächs. Akad. Wiss. Leipzig*, **69** (1917), 89-106.

[35] C. Loewner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, *Math. Ann.* **89** (1923), 103-121.

[36] S. S. Miller, P. T. Mocanu, *Differential subordinations theory and applications*, Marcel Dekker Inc., New York, 2000.

[37] R. Nevanlinna, Über die schlichten Abbildungen des Einheitskreises, *Översikt av Finska Vetenskaps-Soc. Förh.* **62A** (1920), 1-14.

[38] R. Nevanlinna, Über die konforme Abbildung von Sterngebieten, *Översikt av Finska Vetenskaps-Soc. Förh.* **63A** (1920/21), 1-21.

[39] M. Ozawa, On the Bieberbach conjecture for the sixth coefficient, *Kodai Math. Sem. Rep.* **21** (1969), 97-128.

[40] M. Ozawa, Y. Kubato, On the eight coefficient of univalent functions, *J. Analyse Math.* **23** (1970), 323-352.

[41] R. N. Pedersen, A proof of the Bieberbach conjecture for the sixth coefficient, *Arch. Rational Mech. Anal.* **31** (1968), 331-351.

[42] R. N. Pedersen, M. Schiffer, A proof of the Bieberbach conjecture for the fifth coefficient, *Arch. Rational Mech. Anal.* **45** (1972), 161-193.

[43] Y. Polatoğlu, M. Aydoğan, O. Mert, Some properties of $q-$convex functions of complex order, *Sarajevo J. Math.* **13** (2017), 189-196.

[44] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht in Göttingen, 1975.

[45] S. Porwal, Partial sums of certain harmonic univalent function, *Lobachevskii J. Math.* **32** (2011), 366-375.

[46] S. Porwal, K. K. Dixit, Partial sums of starlike harmonic univalent function, *Kyungpook Math. J.* **50** (2010), 433-445.

[47] M. S. Robertson, On the theory of univalent functions, *Ann. Math.* **37** (1936), 374-408.

[48] S. Ruscheweyh, *Convolutions in geometric function theory*, Montreal: Les Presses de lUniversite de Montreal, QC. 1982.

[49] S. Ruscheweyh, T. Sheil-Small, Hadamard products of schlicht functions and the Polya-Schoenberg conjecture, *Comment Math. Helv.* **48** (1973), 119-135.

[50] H. Silverman, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.* **209** (1997), 221-227.

[51] E. M. Silvia, On partial sums of convex functions of order alpha, *Houston J. Math.* **11** (1985), 397-404.

[52] H. M. Srivastava, *Univalent functions, fractional calculus, and associated generalized hypergeometric functions, in: "Univalent functions, fractional calculus, and their applications"*, (H. M. Srivastava and S. Owa Editors), Halsted Press (Ellis Horwood Limited, Chichester), pp. 329-354, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[53] E. Study, Vorlesungen über ausgewählte Gegenstände der Geometrie, 2.Heft, Leipzig und Berlin, 1913.

[54] E. Ö. Uçar, Coefficient inequality for $q-$starlike functions, *Appl. Math. Comput.* **276** (2016), 122-126.

[55] J. Thomae, Beiträge zur Theorie der durch die Heinesche Reihe ..., *J. Reine Angew. Math.* **70** (1869), 258-281.

[56] P. Wiatrowski, The coefficients of a certain family of holomorphic functions, *Zeszyty Nauk. Univ. Lodz. Nauki. Math. Przyrod. Ser. II, Zeszyt,* **39** (1971), 75-85.

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