ALEXANDER MODULES OF TRIGONAL CURVES

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Melih Üçer
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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

ALEXANDER MODULES OF TRIGONAL CURVES

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We classify the *monodromy Alexander modules* of non-isotrivial trigonal curves.

Keywords: Trigonal curve, Alexander module, braid monodromy, Burau representation, modular group, dessin d'enfant.

ÖZET

ÜÇ KATLI EĞRİLERİN ALEXANDER MODÜLLERİ

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Eşsıfırsal olmayan üç katlı eğrilerin *monodromi Alexander modüllerini* tasnif ettik.

Anahtar sözcükler: Üç katlı eğri, Alexander modülü, örgü monodromisi, Burau temsili, modüler grup, dessin d'enfant.

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Chapter 1

Introduction

Let $C \subset \mathbb{P}^2$ be an algebraic curve. The fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is an important invariant of C. It has been subject of interest since Zariski [1], yet its structure is still not well understood in general. As the singularities of C grow, $\pi_1(\mathbb{P}^2 \smallsetminus C)$ gets more complicated. A precise statement in the direction of this principle is that under certain upper bounds on the singularities, $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is abelian [2]. In this case, $\pi_1(\mathbb{P}^2 \smallsetminus C) = H_1(\mathbb{P}^2 \smallsetminus C) = H^3(\mathbb{P}^2, C)$, hence it is easily described in terms of the degrees of the irreducible components of C. On the other hand, there are many specific curves C for which $\pi_1(\mathbb{P}^2 \smallsetminus C)$ has a known explicit description and is non-abelian. For example, the union of three concurrent lines (the lowest-degree example), the 3-cuspidal quartic (the lowest degree irreducible example) as well as the more interesting cases of the curves of (p, q)-torus type [3] and the branch curves of the generic projections of non-singular hypersurfaces in $\mathbb{P}^3$ [4]. Up to the curves of degree six, $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is almost fully known (see [5] for quintics and [6] for almost all sextics).

Another important invariant is the (conventional) *Alexander module* A^{c}_C. Let d be the degree of C, then there is a canonical epimorphism $\mathrm{lk}\colon \pi_1(\mathbb{P}^2 \smallsetminus C) \twoheadrightarrow \mathbb{Z}_d$ which takes a loop to its linking coefficient with C. Then,

$$A^{\mathrm{c}}_C := K/K', \qquad K := \mathrm{Ker}(\mathrm{lk}).$$

Even though the Alexander module is simpler than the fundamental group, its structure is not fully known in general either. However, there are some general results which show that the Alexander module of a plane curve is significantly restricted, as compared to, e.g. that of knots. To describe these restrictions, observe that, for a generic line $L \subset \mathbb{P}^2$, there is a canonical linking coefficient epimorphism $\underline{\mathrm{lk}}\colon \pi_1(\mathbb{P}^2 \smallsetminus (C \cup L)) \twoheadrightarrow \mathbb{Z}$, and A^{c}_C can be equivalently defined in terms of $\underline{\mathrm{lk}}$ instead of lk, since there is an induced isomorphism $\mathrm{Ker}(\underline{\mathrm{lk}}) = \mathrm{Ker}(\mathrm{lk})$. Secondly, let $\Lambda := \mathbb{Z}[t, t^{-1}]$, then there is a canonical Λ-module structure on A^{c}_C, where t acts as conjugation by an element in $\mathrm{lk}^{-1}(1)$ (alternatively, $\underline{\mathrm{lk}}^{-1}(1)$). Then, $t^d = 1$ on A^{c}_C. This implies that the only isomorphism invariant of the $\mathbb{C}[t, t^{-1}]$-module $A^{\mathrm{c}}_C \otimes \mathbb{C}$ is the order, namely the *Alexander polynomial* $\Delta_C(t)$. Moreover, the roots of $\Delta_C(t)$ are roots of unity of order d. In contrast, the roots of the Alexander polynomial need not be roots of unity for a knot.

The following are a few general facts on the Alexander polynomial. First, for irreducible C, the order of a root of $\Delta_C(t)$ cannot be a prime power [7]. Secondly, for general C, one has an upper bound

$$\Delta_C(t) \mid \prod_{1 \le i \le m} \Delta_{L_i}(t),$$

where $\{L_1, L_2, \ldots, L_n\}$ are the links of the singularities of C [8]. Note that this upper bound also illustrates the principle that these invariants get more complicated as the singularities of C grow. Thirdly, there are formulae which express $\Delta_C(t)$ in terms of the superabundance of certain linear systems, thus $\Delta_C(t)$ can be computed without topological methods (e.g. [9]). The textbook [10] is a good source of information on this subject. In this work, we study the so-called *monodromy Alexander modules*, described in the next few paragraphs.

The invariants described above apply to the curves C on a Hirzebruch surface $\Sigma := \Sigma_m$ as well (note that this setting is, indeed, more general, since $\mathbb{P}^2$ blown up at a point is Σ_1). Instead of $\pi_1(\mathbb{P}^2 \smallsetminus C)$, one considers the fundamental group $\pi_1(\Sigma \smallsetminus (C \cup E))$, where $E \subset \Sigma$ is the exceptional section. Let C be n-gonal, i.e. C intersects a generic fiber of the ruling $\Sigma \to B \cong \mathbb{P}^1$ at n points, and let the *degree* of C refer to $d := nm + [C \cdot E]$. Then, there is a canonical linking coefficient epimorphism $\mathrm{lk}\colon \pi_1(\Sigma \smallsetminus (C \cup E)) \twoheadrightarrow \mathbb{Z}_d$ and the conventional Alexander module A^{c}_C is defined in terms of lk as in the case of plane curves. For a generic fiber F_∞

of the ruling, there is a canonical epimorphism $\underline{\mathrm{lk}}\colon \pi_1(\Sigma \smallsetminus (C \cup E \cup F_\infty)) \twoheadrightarrow \mathbb{Z}$, and A_C^{c} can be equivalently defined in terms of $\underline{\mathrm{lk}}$ instead of lk as in the case of $\mathbb{P}^2$. The Zariski-van Kampen theorem gives a description of the fundamental group $\pi_1(\Sigma \smallsetminus (C \cup E \cup F_\infty))$ (1.1), in fact, this is a standard technique for studying $\pi_1(\mathbb{P}^2 \smallsetminus C)$ (one blows up $\mathbb{P}^2$ at a point of choice and applies the Zariski-van Kampen theorem). Let $b_\infty \in B$ denote the image F_∞, then the restricted projection $\Sigma \smallsetminus (C \cup E \cup F_\infty) \to B \smallsetminus \{b_\infty\}$ is topologically a fiber bundle away from finitely many singular fibers. Let $F_0 \neq F_\infty$ be another generic fiber of the ruling, then $F^\circ := F_0 \smallsetminus (C \cup E)$ is a fiber of this bundle. Clearly, F° is homeomorphic to a disk with n punctures, so that $\pi_1(F^\circ) \cong F_n$, the free group on n generators. Let $\{b_1, b_2, \ldots, b_k\} \subset B$ denote the image of the singular fibers. Then, there is a monodromy action $\pi_1(B \smallsetminus \{b_\infty, b_1, b_2, \ldots, b_k\}) \to \mathrm{Aut}(\pi_1(F^\circ)) \cong \mathrm{Aut}(F_n)$, whose image M_C is called the *braid monodromy group* of C. With some semi-standard choices, one has $M_C \subset \mathbb{B}_n \cdot \mathrm{Inn}(F_n) \subset \mathrm{Aut}(F_n)$ (see Section 2.2). But since the choices are not unique, M_C is well-defined only up to conjugation by $\mathbb{B}_n$. The Zariski-van Kampen theorem states

$$\pi_1(\Sigma \smallsetminus (C \cup E \cup F_\infty)) = F_n / \langle \alpha = m(\alpha) \mid \alpha \in F_n,\ m \in M_C \rangle. \tag{1.1}$$

The composition of $\underline{\mathrm{lk}}\colon \pi_1(\Sigma \smallsetminus (C \cup E \cup F_\infty)) \twoheadrightarrow \mathbb{Z}$ with the quotient epimorphism $F_n \twoheadrightarrow \pi_1(\Sigma \smallsetminus (C \cup E \cup F_\infty))$ is the map $\mathrm{u}\colon F_n \twoheadrightarrow \mathbb{Z}$ sending each generator to 1. Consider the Λ-module A_n defined in terms of u in the same way that A_C^{c} is defined in terms of $\underline{\mathrm{lk}}$:

$$A_n := K_n / K_n', \qquad K_n := \mathrm{Ker}(\mathrm{u}).$$

Here, M_C acts on $A_n \cong \Lambda^{n-1}$ via the *Burau representation*, namely the induced action $\mathbb{B}_n \cdot \mathrm{Inn}(F_n) \to \mathrm{Aut}(A_n) = GL(n-1, \Lambda)$ (see Section 2.2). The Zariski-van Kampen theorem motivates the definition of the *monodromy Alexander module*:

$$A_C := \Lambda^{n-1} / \langle u = m(u) \mid u \in \Lambda^{n-1},\ m \in M_C \rangle.$$

There is a canonical epimorphism $A_C \twoheadrightarrow A_C^{\mathrm{c}}$, which is often (though not always) an isomorphism [11]. Thus, as long as "upper bounds" are concerned, it suffices to classify the monodromy Alexander modules. On the other hand, the latter is easier to compute than the conventional module, because it depends only on the image H_C of $M_C \to GL(n-1, \Lambda)$, which we call the *Burau monodromy group* of C. Note that H_C is well-defined up to conjugacy as a subgroup of the image Bu_n of the Burau representation, which we call the *Burau group*. In fact, for any subgroup $H \subset \mathrm{Bu}_n$, we can define

$$\mathcal{A}(H) := \Lambda^{n-1}/\mathcal{V}(H), \qquad \mathcal{V}(H) := \langle (h-1)\cdot u \mid u \in \Lambda^{n-1}, h \in H \rangle.$$

Consequently, $A_C = \mathcal{A}(H_C)$. Clearly, the ambiguity in H_C does not affect A_C up to isomorphism.

From now on, we consider only the trigonal curves (the case $n = 3$), whose Burau monodromy groups are almost completely characterized in Theorem 1.1 below. This case is a borderline for the description of braid monodromy: the case $n \leq 2$ is quite easy, and the case $n \geq 4$ appears very difficult as of now. We ignore the very special case of *isotrivial* trigonal curves, which have constant j-invariant on all fibers. For the statement of Theorem 1.1, note that there is a canonical isomorphism $\mathbb{B}_3 \cdot \mathrm{Inn}(F_3)/\mathrm{Inn}(F_3) \cong \Gamma := PSL(2, \mathbb{Z})$ which leads to an epimorphism $\mathrm{c}\colon \mathrm{Bu}_3 \twoheadrightarrow \Gamma$ (see Section 2.2). Thus, a finite-index subgroup $H \subset \mathrm{Bu}_3$ is said to be of *genus zero* if $\mathrm{c}(H)$ is of genus zero as a subgroup of the modular group Γ.

Theorem 1.1 (Degtyarev [11])**.** *Let $C \subset \Sigma$ be a non-isotrivial trigonal curve, then H_C is of genus zero. For a partial converse, let $H \subset \mathbb{B}_3 \subset \mathrm{Bu}_3$ be a genus-zero subgroup. Then, there is a non-isotrivial trigonal curve C such that $H = H_C$.*

In view of Theorem 1.1, the problem of classifying the monodromy Alexander modules of non-isotrivial trigonal curves is almost equivalent to the problem of determining $\mathcal{A}(H)$ for all genus-zero $H \subset \mathrm{Bu}_3$. The only difference between the two is that there may be some genus-zero subgroups $H \subset \mathrm{Bu}_3$ which are not Burau monodromy groups of trigonal curves (because Theorem 1.1 has only a partial converse). But once $\mathcal{A}(H)$ is determined for all genus-zero $H \subset \mathrm{Bu}_3$ in the form of an explicit list, the redundant entries can simply be removed.

The Principal Result Given $H \subset \mathrm{Bu}_3$, the module $\mathcal{A}(H)$ is equipped with an epimorphism $\Lambda^2 \twoheadrightarrow \mathcal{A}(H)$, which is always understood but usually omitted from notation. Conversely, given any module A with an epimorphism $\phi\colon \Lambda^2 \twoheadrightarrow A$, we can define the subgroup

$$\mathcal{H}(A) = \mathcal{H}(\phi) := \{h \in \mathrm{Bu}_3 \mid (h-1)\cdot \Lambda^2 \subset \mathrm{Ker}(\phi)\}.$$

Then, for any H and any A, we have

$$\mathcal{A}(H) = \mathcal{A}(\mathcal{H}(\mathcal{A}(H))), \quad H \subset \mathcal{H}(\mathcal{A}(H)), \quad \mathcal{H}(A) = \mathcal{H}(\mathcal{A}(\mathcal{H}(A))).$$

Therefore, the crucial step in determining $\mathcal{A}(H)$ for all genus-zero H is to classify the genus-zero subgroups of the form $\mathcal{H}(A)$, which we call *saturated* subgroups. After this, it remains to determine $\mathcal{A}(H)$ for saturated genus-zero H, which is a straightforward computation. In this work, we determine all saturated genus-zero subgroups in the form of an explicit list. Note that, if H is saturated, then so are all of its conjugates, hence we list the conjugacy classes.

Theorem 1.2 (Main)**.** *All conjugacy classes of saturated genus-zero subgroups* $H \subset \mathrm{Bu}_3$ *are presented in Appendix A.*

In Appendix A, we also give $\mathcal{A}(H)$ for most $H \subset \mathrm{Bu}_3$ in the list, without showing the computation. In fact, the list contains infinitely many entries and we give $\mathcal{A}(H)$ except for finitely many (albeit a large number) of them.

The Structure of the Text In Chapter 2, we cite a few properties of the groups Γ and Bu_3, which are essential for the proof of Theorem 1.2, as well as for reading Appendix A. In Chapter 3, we describe the method of our proof of Theorem 1.2. Finally, in Chapters 4 and 5, we finish the proof of the theorem in two steps.

Chapter 2

Preliminaries

This chapter contains necessary preliminary information on the modular group $\Gamma = PSL(2,\mathbb{Z})$, the braid group $\mathbb{B}_3$, and the Burau representation $\mathbb{B}_3 \to \mathrm{Bu}_3$. The content of this chapter is completely standard; one can consult the classical sources [12, 13, 14].

2.1 The Modular Group

The modular group $\Gamma = PSL(2,\mathbb{Z})$ is often considered together with its left action on the complex upper half plane $\mathbb{H}$ via the inclusion $\Gamma \subset PSL(2,\mathbb{R}) = \mathrm{Aut}(\mathbb{H})$. Explicitly, the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

This Γ-action is discrete and almost free: there are only two orbits for which the stabilizer is nontrivial, but the stabilizer is finite for these two orbits as well. Namely, $\mathrm{Stab}(\omega) = \langle X \rangle$ for $\omega := \frac{1+\sqrt{3}i}{2}$, and $\mathrm{Stab}(i) = \langle Y \rangle$, where

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} : z \mapsto \frac{1}{1-z}, \qquad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \mapsto -\frac{1}{z}.$$

A very classical theorem states

$$\Gamma = \langle X, Y \mid X^3 = Y^2 = 1 \rangle. \tag{2.1}$$

Hence, the abelianization of Γ is isomorphic to $\mathbb{Z}_6$. We fix the abelianization $\mathrm{ab}\colon \Gamma \twoheadrightarrow \mathbb{Z}_6$ such that $\mathrm{ab}(X) = 2$ (note that one necessarily has $\mathrm{ab}(Y) = 3$).

2.1.1 Modular Curves

Since the Γ-action on $\mathbb{H}$ is dicrete and almost free, for any subgroup $K \subset \Gamma$, the quotient $K\backslash\mathbb{H}$ naturally admits a Riemann surface structure (it also admits an orbifold structure, but we do not use this language explicitly). In particular, the quotient $\Gamma\backslash\mathbb{H}$ is isomorphic to $\mathbb{C}$. We adopt Kodaira's normalization which fixes an identification $\Gamma\backslash\mathbb{H} = \mathbb{C}$ by mapping the orbits of $\omega \in \mathbb{H}$ and $i \in \mathbb{H}$ to $0 \in \mathbb{C}$ and $1 \in \mathbb{C}$, respectively.

Let $K \subset \Gamma$ be a finite-index subgroup. The *modular curve* S_K is a standard compactification of the Riemann surface $K\backslash\mathbb{H}$. In particular, $S_\Gamma = \mathbb{P}^1 = \mathbb{C} \cup \infty$. Any inclusion $K_1 \subset K_2$ of subgroups induces a non-constant holomorphic map $S_{K_1} \to S_{K_2}$, i.e. a (possibly ramified) covering. As such, each modular curve S_K comes with a distinguished covering $S_K \to S_\Gamma = \mathbb{P}^1$, which is unramified outside the special points $\{0, 1, \infty\}$. We always consider the modular curves as equipped with this additional structure. For example, an isomorphism between two modular curves S_K and $S_{K'}$ must be understood to commute with the distinguished maps to $\mathbb{P}^1$.

Remark 2.1. The conjugacy class of a finite-index subgroup $K \subset \Gamma$ determines S_K up to isomorphism. Conversely, any connected compact Riemann surface S, equipped with a covering $S \to \mathbb{P}^1$ which is unramified outside $\{0, 1, \infty\}$, is isomorphic to S_K for some K, which is unique up to conjugacy.

Since $K \mapsto K\backslash\Gamma$ establishes a bijection between conjugacy classes of subgroups and isomorphism classes of transitive (right) Γ-sets, one can define the modular curve S_C (up to isomorphism) for a finite transitive Γ-set C as well. This immediately generalizes to an arbitrary (not necessarily transitive) finite Γ-set C: the modular curve S_C is the disjoint union of the modular curves of the orbits in C. Any Γ-equivariant surjection $C_1 \to C_2$ induces a (possibly ramified) covering $S_{C_1} \to S_{C_2}$. We denote the singleton Γ-set by $\{*\}$, thus $S_{\{*\}} = S_\Gamma = \mathbb{P}^1$. In the language of Γ-sets, the converse statement in Remark 2.1 takes the following form.

Remark 2.2. Any compact Riemann surface S, equipped with a covering $S \to \mathbb{P}^1$ which is unramified outside $\{0, 1, \infty\}$, is isomorphic to S_C for some finite right Γ-set C, which is unique up to isomorphism.

In the rest of the text, we primarily use the language of Γ-sets, but everything applies to subgroups as well. Moreover, whenever we speak of a covering, we allow ramification.

The *cusps* of a finite Γ-set C are the points in S_C which map to $\infty \in \mathbb{P}^1$ under the map $S_C \to \mathbb{P}^1$. The *width* of a cusp is the ramification index. The *Euler characteristic* $\chi(C)$ and the *genus* $g(C)$ are defined as those of S_C. Clearly, $\chi(C) = 2 - 2g(C)$. We consider the notion of genus for transitive Γ-sets only.

Remark 2.3. Let C_1, C_2 be finite transitive Γ-sets with a Γ-equivariant surjection $C_1 \to C_2$. If $g(C_1) = 0$, then $g(C_2) = 0$ as well (because there is a covering $S_{C_1} \to S_{C_2}$). This is equvalently stated as follows. Let $K_1 \subset K_2 \subset \Gamma$ be finite-index subgroups. If $g(K_1) = 0$, then $g(K_2) = 0$ as well.

2.1.2 Standard CW-structures on Modular Curves

The terminal bipartite graph ●—○ is canonically embedded in $S_{\{*\}} = \mathbb{P}^1$ as follows: the black vertex goes to 0, the white vertex goes to 1 and the edge goes to the real interval $[0, 1]$. For any finite right Γ-set C, we denote the preimage of this graph under the map $S_C \to S_{\{*\}}$ by $\mathcal{D}_C$. In particular, we denote the terminal bipartite graph itself by $\mathcal{D}_{\{*\}}$. Since the restricted map $S_C \smallsetminus \mathcal{D}_C \to S_{\{*\}} \smallsetminus \mathcal{D}_{\{*\}}$

is unramified outside one point (the point ∞), each component of $S_C \smallsetminus \mathcal{D}_C$ is a 2-cell. Hence, $\mathcal{D}_C$ provides a CW decomposition of S_C. Clearly, each of the 2-cells contains exactly one cusp. Note that these graphs $\mathcal{D}_C$ are ribbon graphs in a natural way, since they are embedded in oriented surfaces. In fact, the ribbon graph $\mathcal{D}_C$ coincides with Grothendieck's *dessins d'enfant* corresponding to the ramified covering $S_C \to \mathbb{P}^1$ (see [15]). The preimage $\mathcal{F}$ of $\mathcal{D}_{\{*\}}$ under the map $\mathbb{H} \to \Gamma\backslash\mathbb{H} = \mathbb{C}$ is a tree (e.g. [16]). Clearly, $\mathcal{F}$ has a black vertex at ω and a white vertex at i. Moreover, ω and i are joined by an edge. The Γ-action on $\mathbb{H}$ restricts to an action on $\mathcal{F}$. Note that the existence of this action immediately shows (2.1), by the Serre theory (see [16]).

The set of edges of $\mathcal{D}_C$ naturally admits the structure of a right Γ-set as follows. Consider the two loops x, y in $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ based at $\frac{1}{2}$, formed as counterclockwise circles of radius $\frac{1}{2}$ centered at 0 and 1, respectively. Then, the lifts of the path x under the covering map $S_C \to \mathbb{P}^1$ define the action of X on the set of edges of $\mathcal{D}_C$, while the lifts of the path y define the action of Y. More explicitly, X takes each edge to the next one among the edges sharing the same black vertex and Y takes each edge to the next one among the edges sharing the same white vertex. Here, "next" refers to the cyclic order coming from the ribbon graph structure. These actions of X and Y uniquely extend to a right Γ-action. (Note that this right Γ-action applies to the tree $\mathcal{F}$ as well).

Lemma 2.1. *For any finite right* Γ*-set* C*, the set of edges of* $\mathcal{D}_C$ *is isomorphic to* C*.*

Proof. Clearly, one can assume that C is transitive. Let K be the stabilizer of any element of C (well-defined up to conjugacy), then $\mathcal{D}_C \cong K\backslash\mathcal{F}$. On the other hand, the right Γ-action on the set of edges of $\mathcal{F}$ equivalently comes from the following identification of this set with Γ: the edge between ω and i is identified with $1 \in \Gamma$, and the induced left Γ-action (which is free and transitive) extends the identification to all the edges. As a result, the set of edges of $K\backslash\mathcal{F}$ is identified with $K\backslash\Gamma \cong C$. □

Remark 2.4. The action of YX on the set of edges of $\mathcal{D}_C$ is described by the lifts of a certain loop formed by joining a clockwise circle around ∞ to $\frac{1}{2}$ along a path lying in the lower half plane, since yx is homotopic to such a loop. Hence, the choice of an isomorphism as in Lemma 2.1 puts the YX-orbits in C in bijection with the cusps, in such a way that the size of a YX-orbit is equal to the width of the corresponding cusp. It is also clear that the X-orbits are put in bijection with the black vertices, and the Y-orbits are put in bijection with the white vertices.

In light of Remark 2.4, we introduce the following terminology for any finite right Γ*-set* C*.* The *black vertices* in C are the X-orbits, the *white vertices* in C are the Y-orbits, the *edges* in C are simply the elements of C and the *regions* in C are the YX-orbits. Continuing to imitate the graph theory language, we say that a vertex in a Γ-set is *monovalent* if it consists of a single element. Furthermore, we refer to a Γ-equivariant surjection $C_1 \to C_2$ as a *covering.* A covering takes vertices to vertices, regions to regions, etc. For the vertices and the regions, we speak of *ramification*, whose meaning must be clear. For example, a vertex which is not monovalent is necessarily unramified. In fact, any term which is well-known in the context of ramified coverings of surfaces must have a clear meaning in this context as well. Examples of such terms are *degree* of a covering, *Galois* (aka *regular*) covering, *abelian* covering, etc.

For any finite right Γ-set C and any element $\gamma \in \Gamma$, we denote the set of γ-orbits in C by C_γ and the set of γ-fixed elements in C by C^γ. In particular, C_{YX} is the set of regions, C_X and C_Y are the sets of black and white vertices, and C^X and C^Y are the sets of monovalent black and monovalent white vertices. We now give a formula for the Euler characteristic of a Γ-set.

Lemma 2.2 (Euler Characteristic Formula)**.** *Let* C *be a finite right* Γ*-set. Then*

$$\begin{aligned}\chi(C) &= |C_X| + |C_Y| - |C| + |C_{YX}| \\ &= -\frac{|C|}{6} + |C_{YX}| + \frac{2}{3} \cdot |C^X| + \frac{1}{2} \cdot |C^Y|\end{aligned}$$

Proof. Note that $\chi(C) = \chi(S_C)$. In the canonical CW-decomposition of S_C, the number of 0-cells (the black and white vertices) is $|C_X| + |C_Y|$, the number of 1-cells (the edges) is $|C|$ and the number of 2-cells is $|C_{YX}|$ since each 2-cell contains exactly one cusp (Remark 2.4). This establishes the formula in the top line. For the bottom line, it is enough to observe that $|C_X| = \frac{|C|}{3} + \frac{2}{3} \cdot |C^X|$ and $|C_Y| = \frac{|C|}{2} + \frac{1}{2} \cdot |C^Y|$. This is because each X-orbit contains 1 or 3 elements and each Y-orbit contains 1 or 2 elements (since $X^3 = Y^2 = 1$). $\square$

Let C be a finite transitive Γ-set, and let K be the stabilizer of any edge (element) in C. By the Kurosh subgroup theorem, one has an isomorphism

$$K \cong \underbrace{\mathbb{Z} * \dots\dots}_{|C|-|C_X|-|C_Y|+1} * \underbrace{\mathbb{Z}_3 * \dots}_{|C^X|} * \underbrace{\mathbb{Z}_2 * \dots}_{|C^Y|}. \tag{2.2}$$

Proof. We use the Serre theory (see [16]) in the proof. Note that K acts on the tree $\mathcal{F}$ such that the action on the edges is free. The *graph of groups* which corresponds to this action consists of the graph $\mathcal{D}_C \cong K\backslash\mathcal{F}$ with the following assignment of groups: each monovalent black vertex is given $\mathbb{Z}_3$, each monovalent white vertex is given $\mathbb{Z}_2$, and each non-monovalent vertex and each edge is given the trivial group. Then, the *total group* of this graph of groups is isomorphic to the free product of the vertex groups and a number of copies of $\mathbb{Z}$, because the edge groups are trivial. The number of copies of $\mathbb{Z}$ is equal to the number of edges minus the number of vertices plus one, i.e. it is given by $|C|-(|C_X|+|C_Y|)+1$. $\square$

We now discuss a few particular Γ-sets which are of special importance in the rest of the text.

The Congruence Subgroups Let $\tilde{\Gamma}$ briefly denote $SL(2, \mathbb{Z})$. For any positive integer n, it is common to denote

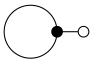
$$\tilde{\Gamma}(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma} \mid a \equiv d \equiv 1 \pmod{n},\ b \equiv c \equiv 0 \pmod{n} \right\},$$

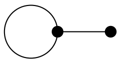
$$\tilde{\Gamma}_1(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma} \mid a \equiv d \equiv 1 \pmod{n},\ c \equiv 0 \pmod{n} \right\},$$

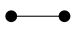
$$\tilde{\Gamma}_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma} \mid c \equiv 0 \pmod{n} \right\}.$$

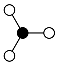
The images of these subgroups under $\tilde{\Gamma} \twoheadrightarrow \Gamma$ are denoted by $\Gamma(n), \Gamma_1(n), \Gamma_0(n)$. A subgroup $K \subset \Gamma$ is called a *congruence* subgroup if $\Gamma(n) \subset K$ for some n. There are finitely many genus-zero congruence subgroups (see [17] for an explicit list of the conjugacy classes of these). In the rest of the text, we consider several genus-zero Γ-sets whose stabilizers are congruence subgroups. In the case that these congruence subgroups do not admit a common notation (e.g. Γ^2 or $\Gamma_1(3)$), we use the notation of [17]. In the following figures, we display the ribbon graph $\mathcal{D}_{K\backslash\Gamma}$ for some genus-zero congruence subgroups K.

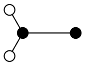
Remark 2.5. For simplicity of the figures, we do not show the non-monovalent white vertices and we instead directly connect the neighboring black vertices. In other words, any edge which connects two black vertices is to be understood to contain a white vertex in the middle. Note that we can display these ribbon graphs as planar graphs because they are of genus zero.

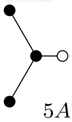

$\Gamma_0(2) = \Gamma_1(2)$

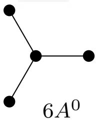

$\Gamma_0(3) = \Gamma_1(3)$

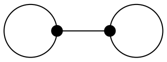

Γ^2

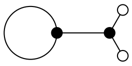

Γ^3

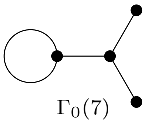

$4A^0$

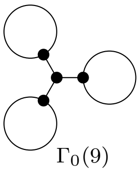

$5A^0$

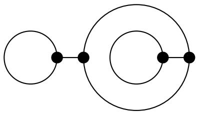

$6A^0$

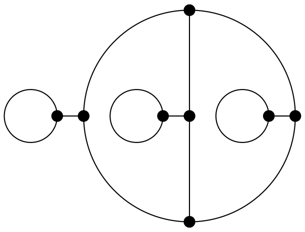

$\Gamma_0(4) = \Gamma_1(4)$

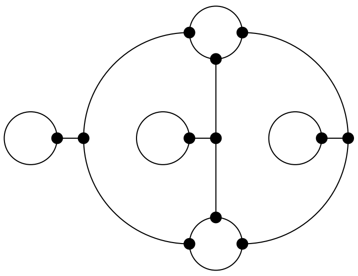

$\Gamma_0(5)$

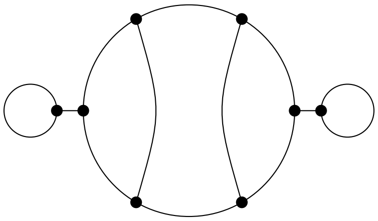

$\Gamma_0(7)$


$\Gamma_0(9)$

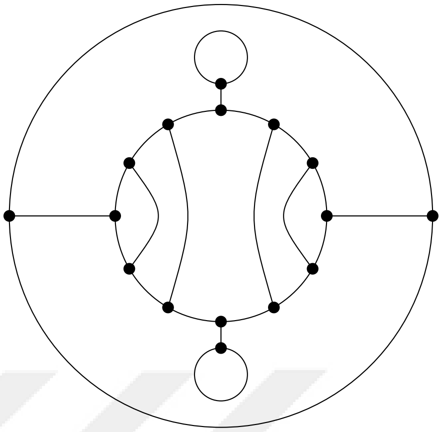

$\Gamma_1(5)$


$\Gamma_1(7)$


$\Gamma_1(9)$



$\Gamma_0(12)$



$\Gamma_1(12)$

The Γ-sets $\Sigma_*(n)$ We first observe the isomorphism $\Gamma(2) \cong F_2$.

Proof. The Γ-set $C := \Gamma(2)\backslash\Gamma$ is equal to $SL(2, \mathbb{F}_2)$ where Γ acts by right translation via $\Gamma \to SL(2, \mathbb{F}_2) = \Gamma/\Gamma(2)$. Thus one has $|C| = 6$, $|C_X| = 2$, $|C_Y| = 3$, and $|C^X| = |C^Y| = 0$. Therefore, $\Gamma(2) \cong F_2$ by (2.2). $\square$

For each positive integer n, consider the subgroup $K := \langle \Gamma(2)', \Gamma(2)^n \rangle$, i.e. the preimage of $n \cdot (\Gamma(2)/\Gamma(2)') \subset (\Gamma(2)/\Gamma(2)')$ under the epimorphism $\Gamma(2) \twoheadrightarrow \Gamma(2)/\Gamma(2)'$. Then, let

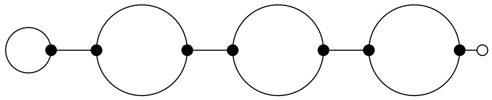
$$\Sigma_1(n) := \langle YX, K \rangle \backslash \Gamma.$$

There is one region of size $2n$, and there is no monovalent black vertex in $\Sigma_1(n)$. There are two regions of size 1 and no monovalent white vertices if n is even, and there is one region of size 1 and one monovalent white vertex if n is odd. All other regions are of size 2, and $|\Sigma_1(n)| = 3n$. Consequently, $\Sigma_1(n)$ is of genus zero. Let n_1, n_2 be coprime, then the only genus-zero orbit in $\Sigma_1(n_1) \times \Sigma_1(n_2)$ is isomorphic to $\Sigma_1(n_1 n_2)$.

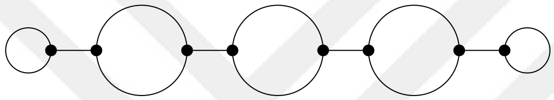
Let $\Sigma_2(n)$ denote the double covering of $\Sigma_1(n)$ ramified at the regions of size 1 and the monovalent white vertices. Equivalently,

$$\Sigma_2(n) = \langle (YX)^2, K \rangle \backslash \Gamma,$$

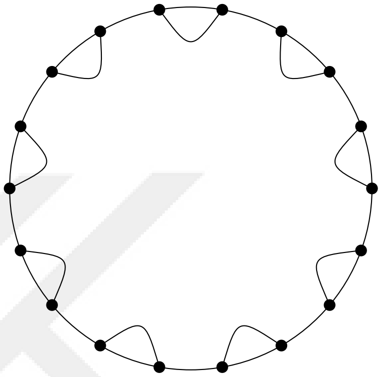
where K is as above. The following figures which display $\mathcal{D}_C$ for a few Γ-sets C of the form $\Sigma_*(n)$ make the general pattern clear (see Remark 2.5 for the convention on the figures).



$\Sigma_1(7)$



$\Sigma_1(8)$



$\Sigma_2(9)$

Note the isomorphisms

$$\Sigma_1(1) \cong \Gamma_1(2)\backslash\Gamma, \qquad \Sigma_2(1) \cong \Gamma(2)\backslash\Gamma, \qquad \Sigma_1(2) \cong \Gamma_1(4)\backslash\Gamma.$$

The Γ-sets $\Pi_*(\iota)$ Let ω denote a third root of unity. Note that the ring $\mathbb{Z}[\omega]$ is a principal ideal domain. We observe that there is an identification $\mathbb{Z}[\omega] = \Gamma'/\Gamma''$.

Proof. The Γ-set $C := \Gamma'\backslash\Gamma$ is equal to $\mathbb{Z}_6$ where Γ acts by translation via the abelianization $\mathrm{ab}\colon \Gamma \twoheadrightarrow \mathbb{Z}_6$ (2.1). Thus one has $|C| = 6$, $|C_X| = 2$, $|C_Y| = 3$, and $|C^X| = |C^Y| = 0$. Therefore, $\Gamma' \cong F_2$ by (2.2). The abelian group $\Gamma'/\Gamma'' \cong \mathbb{Z}^2$ has a distinguished automorphism α, defined by conjugation by any element $\gamma \in \Gamma$ for which $\mathrm{ab}(\gamma) = 1$. Let $\langle\alpha\rangle$ denote the subring of $\mathrm{End}(\Gamma'/\Gamma'')$ generated by α. The abelian group homomorphism $\langle\alpha\rangle \to \Gamma'/\Gamma''$ defined by $\beta \mapsto \beta(XYX^2Y)$ is, in fact, an isomorphism (because Γ' is generated by XYX^2Y and its conjugates). The following equality shows that $\alpha^2 - \alpha + 1 = 0$:

$$\big(X \cdot XYX^2Y \cdot X^{-1}\big) \cdot \big(XYX \cdot (XYX^2Y)^{-1} \cdot (XYX)^{-1}\big) \cdot \big(XYX^2Y\big) = 1.$$

Therefore, there is a ring epimorphism $\mathbb{Z}[\omega] \to \langle\alpha\rangle$ defined by $\omega \mapsto -\alpha$. This epimorphism must be an isomorphism since $\langle\alpha\rangle \cong \mathbb{Z}^2$, thus there is an identification $\mathbb{Z}[\omega] = \Gamma'/\Gamma''$. $\square$

For each non-zero ideal $\iota \subset \mathbb{Z}[\omega]$, consider the preimage K of $\iota \subset \mathbb{Z}[\omega]$ under the epimorphism $\Gamma' \twoheadrightarrow \Gamma/\Gamma'' = \mathbb{Z}[\omega]$. Then, let

$$\Pi_1(\iota) := \langle YX, K \rangle \backslash \Gamma.$$

For each ι, there is exactly one region of size 1 in $\Pi_1(\iota)$. There is one region of size 2 but no monovalent black vertex if $\iota \subset \langle \omega - 1 \rangle$, and there is one monovalent black vertex but no region of size 2 otherwise. There is one region of size 3 but no monovalent white vertex if $\iota \subset \langle 2 \rangle$, and there is one monovalent white vertex but no region of size 3 otherwise. All other regions have size 6. Let α be a generator of ι and let α^* denote the complex conjugate, then $|\Pi_1(\iota)| = \alpha\alpha^*$. Let ι_1 and ι_2 be coprime, then $\Pi_1(\iota_1) \times \Pi_1(\iota_2) \cong \Pi_1(\iota_1\iota_2)$.

Let $\Pi_2(\iota)$ denote the double covering of $\Pi_1(\iota)$ ramified at the region of size 1, and a white vertex or a region of size 3 (whichever is found). Let $\Pi_3(\iota)$ denote the triple covering of $\Pi_1(\iota)$ fully ramified at the region of size 1, and a black vertex or a region of size 2 (whichever is found). Equivalently,

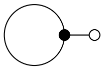
$$\Pi_2(\iota) = \langle (YX)^2, K \rangle \backslash \Gamma, \qquad \Pi_3(\iota) = \langle (YX)^3, K \rangle \backslash \Gamma,$$

where K is as above. Moreover, let

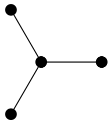
$$\Pi_\bullet(\iota) = \langle X, K \rangle \backslash \Gamma, \qquad \Pi_\circ(\iota) = \langle Y, K \rangle \backslash \Gamma.$$

Then, $\Pi_2(\iota) \cong \Pi_\bullet(\iota)$ if and only if $\iota \not\subset \langle \omega - 1 \rangle$ and $\Pi_3(\iota) \cong \Pi_\circ(\iota)$ if and only if $\iota \not\subset \langle 2 \rangle$. Moreover, $\Pi_\bullet((\omega - 1)\iota)$ is a triple covering of $\Pi_\bullet(\iota)$ and $\Pi_\circ(2\iota)$ is a quadruple covering of $\Pi_\circ(\iota)$. Finally, for $\iota \subset \langle 4 \rangle$, let $\Pi_{3/2}(\iota)$ denote the double covering of $\Pi_3(\iota/2)$ ramified at two of the four regions of size 3 (there are exactly four regions of size 3 in $\Pi_3(\iota/2)$ since $(\iota/2) \subset \langle 2 \rangle$). Note that the ambiguity in the choice of the two regions does not affect $\Pi_{3/2}(\iota)$ up to isomorphism.

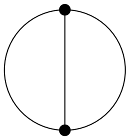
The following figures show $\mathcal{D}_C$ for some Γ-sets of the form $\Pi_*(\iota)$ (see Remark 2.5). Note that $\Pi_*(\iota)$ are of genus zero.



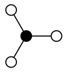
$\Pi_1(\omega-1)$



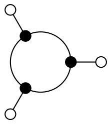
$\Pi_\bullet(\omega-1)$



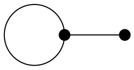
$\Pi_2(\omega-1)$



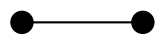
$\Pi_\circ(1)=\Pi_3(1)$



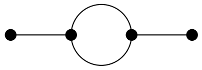
$\Pi_\circ(\omega-1)=\Pi_3(\omega-1)$



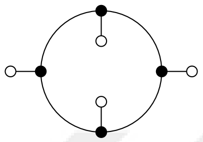
$\Pi_1(2)$



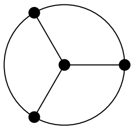
$\Pi_\bullet(1)=\Pi_2(1)$



$\Pi_\bullet(2)=\Pi_2(2)$



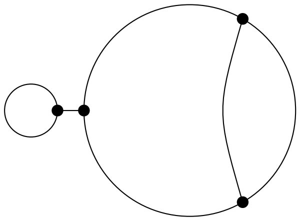
$\Pi_\circ(2)$



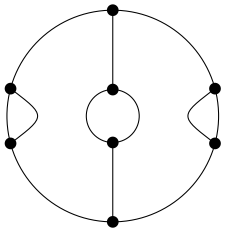
$\Pi_3(2)$

Note the isomorphisms

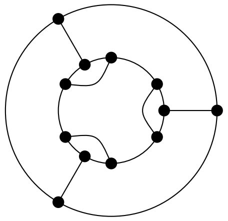
$$\begin{array}{lll}
\Pi_1(2)\cong\Gamma_1(3)\backslash\Gamma, & \Pi_\circ(2)\cong 6E^0\backslash\Gamma, & \Pi_3(2)\cong\Gamma(3)\backslash\Gamma,\\
\Pi_1(\omega-1)\cong\Gamma_1(2)\backslash\Gamma, & \Pi_2(\omega-1)\cong\Gamma(2)\backslash\Gamma, & \Pi_\bullet(\omega-1)\cong 6A^0\backslash\Gamma,\\
 & \Pi_\circ(1)\cong\Gamma^3\backslash\Gamma, & \Pi_\bullet(1)\cong\Gamma^2\backslash\Gamma.
\end{array}$$



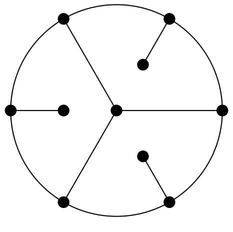
$\Pi_1(2(\omega-1))$



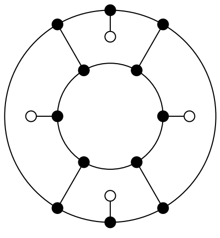
$\Pi_2(2(\omega-1))$



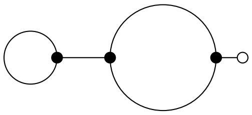
$\Pi_3(2(\omega-1))$



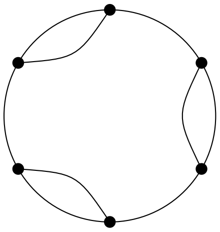
$\Pi_\bullet(2(\omega-1))$



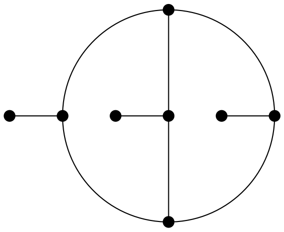
$\Pi_\circ(2(\omega-1))$



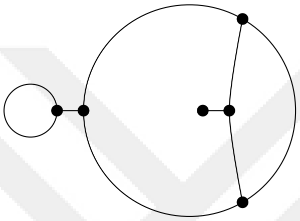
$\Pi_1(3)$



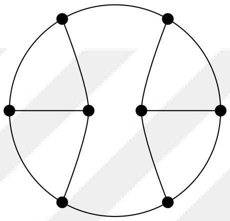
$\Pi_2(3)$



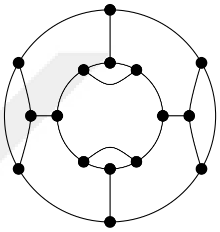
$\Pi_\bullet(3)$



$\Pi_1(4)$

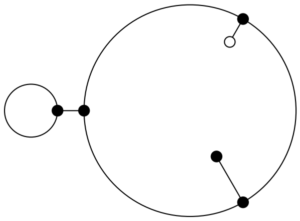


$\Pi_{3/2}(4)$

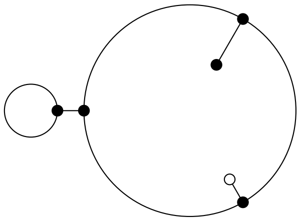


$\Pi_3(4)$

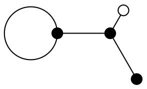
If ι and ι^* are complex conjugate, then $\Pi_i(\iota)$ and $\Pi_i(\iota^*)$ are obtained from each other by composition with an outer automorphism of Γ, such as conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in PGL(2,\mathbb{Z})$ (that is, $X \mapsto X^2$ and $Y \mapsto Y$). Equivalently, the ribbon graphs $\mathcal{D}_{\Pi_i(\iota)}$ and $\mathcal{D}_{\Pi_i(\iota^*)}$ are mirror images. The following figure shows two examples of this.



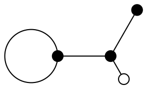
$\Pi_1(\omega - 3)$



$\Pi_1(\omega^2 - 3)$



$\Pi_1(\omega - 2)$



$\Pi_1(\omega^2 - 2)$

It is interesting to note that $\Pi_1(\omega - 1) \cong \Sigma_1(1)$, $\Pi_1(3) \cong \Sigma_1(3)$, $\Pi_2(\omega - 1) \cong \Sigma_2(1)$, and $\Pi_2(3) \cong \Sigma_2(3)$.

2.1.3 Weights on Γ-Sets

We define a *weight* w on Γ-set P as an assignment of real numbers to the vertices and the regions in P. (By a slight abuse of terminology, we call the assigned values as weights as well.) Then, the *Euler characteristic* $\chi(P, w)$ is the sum of weights over the vertices and the regions minus the number of edges. For the *trivial* weight w_0 assigning 1 to every vertex and region, one has $\chi(P, w_0) = \chi(P)$ by Lemma 2.2. A *covering* $(P_1, w_1) \to (P_2, w_2)$ is a covering $\phi\colon P_1 \to P_2$ for which $w_1(\mathfrak{a}) \leq n \cdot w_2(\phi(\mathfrak{a}))$ for any vertex or region $\mathfrak{a}$ where n is the ramification index of $\mathfrak{a} \mapsto \phi(\mathfrak{a})$. Clearly, the existence of a weighted covering $(P_1, w_1) \to (P_2, w_2)$ of degree d requires $\chi(P_1, w_1) \leq d \cdot \chi(P_2, w_2)$.

A Galois covering $C \to P$ of Γ-sets *induce* a weight w on P as follows: for each vertex or region $\mathfrak{a}$ in P, one has $w(\mathfrak{a}) = \frac{1}{n}$ where n is the ramification index of $\mathfrak{a}' \mapsto \mathfrak{a}$ for any $\mathfrak{a}' \subset C$ which maps to $\mathfrak{a}$. Note that n is independent of the choice of $\mathfrak{a}'$ because $C \to P$ is Galois. Clearly, for a monovalent black vertex $\mathfrak{a}_\bullet \subset P$, one has $w(\mathfrak{a}_\bullet) \in \{\frac{1}{3}, 1\}$, and for a monovalent black vertex $\mathfrak{a}_\circ \subset P$, one has $w(\mathfrak{a}_\circ) \in \{\frac{1}{2}, 1\}$. A monovalent vertex $\mathfrak{a} \subset P$ is called *complete* if $w(\mathfrak{a}) = 1$. We denote the set of complete black vertices by $P_\bullet$ and complete white vertices by $P_\circ$. Then, the following formula is straightforward.

Lemma 2.3 (Euler Characteristic Formula). *Let $C \to P$ be a Galois covering, and let w be the induced weight on P. Then,*

$$\chi(P, w) = -\frac{|P|}{6} + \sum_{\mathfrak{a} \in P_{YX}} w(\mathfrak{a}) + \frac{2}{3} \cdot |P_\bullet| + \frac{1}{2} \cdot |P_\circ|.$$

Remark 2.6. Let P be a transitive Γ-set, let $C \to P$ be a Galois covering of degree d, and let w be the induced weight on P. Clearly, $\chi(C) = d \cdot \chi(P, w)$. In particular, $\chi(P, w) > 0$ if and only if any (every) orbit in C is of genus zero. In view of this, we say that a pair (P, w), consisting of a transitive Γ-set and an induced weight, is of *genus zero* if $\chi(P, w) > 0$. Note that this is strictly stronger than P having genus zero.

We finish the section with a lemma on weights induced by abelian coverings.

Lemma 2.4. *Let $C \to P$ be an abelian covering of genus-zero transitive Γ-sets, and let w be the induced weight on P. Consider a vertex or region $\mathfrak{a} \subset P$ (if any) for which $w(\mathfrak{a}) < 1$. Then, $\chi(P, w) \geq w(\mathfrak{a})$.*

Proof. Let Z be the group of deck transformations of $C \to P$, let Z' be the cyclic subgroup of Z which consists of elements which act trivially on the preimage of $\mathfrak{a}$, and let $C' := C/Z'$. The covering $C' \to P$ is unramified at $\mathfrak{a} \subset P$, and the covering $C \to C'$ is fully ramified at all $\mathfrak{a}' \subset C'$ which map to $\mathfrak{a} \subset P$. Since C is of genus zero, there are at most two $\mathfrak{a}' \subset C'$ which map to $\mathfrak{a} \subset P$, because of the full ramification. In other words, the degree of $C' \to P$ is at most 2, thus $|Z| \leq 2 \cdot |Z'|$, i.e. $\frac{1}{|Z'|} \leq \frac{2}{|Z|}$. But $\frac{1}{|Z'|} = w(\mathfrak{a})$ and $\frac{2}{|Z|} = \frac{\chi(C)}{|Z|} = \chi(P, w)$, which concludes the proof. $\square$

2.1.4 Specifications on Γ-sets

Let C be a genus-zero right Γ-set. Consider the following formal abelian group generated by the monovalent vertices and the regions in C:

$$\Big(\bigoplus_{\mathfrak{a}\in C^X} \mathbb{Z}_3 \cdot \mathfrak{a} \oplus \bigoplus_{\mathfrak{a}\in C^Y} \mathbb{Z}_2 \cdot \mathfrak{a} \oplus \bigoplus_{\mathfrak{a}\in C_{YX}} \mathbb{Z} \cdot \mathfrak{a}\Big)/\mathbb{Z} \cdot \sum_{\text{all } \mathfrak{a}} \mathfrak{a}. \tag{2.3}$$

Let $e \in C$ be any edge and let $K \subset \Gamma$ be the stabilizer of e. Then, the abelianization K/K' is canonically identified with the group in (2.3). Note that this is consistent with (2.2), because $|C| - |C_X| - |C_Y| + 1 = |C_{YX}| - 1$ for $g(C) = 0$.

Proof. This is essentially a description of the standard generators of the first homology of a punctured sphere. For each monovalent vertex or region $\mathfrak{a} \subset C$, let $g_{\mathfrak{a}} := X$, $g_{\mathfrak{a}} := Y$ or $g_{\mathfrak{a}} := (YX)^{-|\mathfrak{a}|}$ depending on whether $\mathfrak{a}$ is a black vertex, a white vertex or a region. For each such $\mathfrak{a} \subset C$, choose an element $\gamma_{\mathfrak{a}} \in \Gamma$ such that $e \cdot \gamma_{\mathfrak{a}} \in \mathfrak{a}$. Then, $\gamma_{\mathfrak{a}} g_{\mathfrak{a}} \gamma_{\mathfrak{a}}^{-1} \in K$. Let $1_{\mathfrak{a}} \in K/K'$ denote the image of $\gamma_{\mathfrak{a}} g_{\mathfrak{a}} \gamma_{\mathfrak{a}}^{-1} \in K$ in the abelianization, which is independent of the choice of $\gamma_{\mathfrak{a}}$. It turns out that the set of all $1_{\mathfrak{a}}$ generates K/K' only with the relation $\sum 1_{\mathfrak{a}} = 0$ in addition to the obvious relations $3 \cdot 1_{\mathfrak{a}_\bullet} = 0$ for a black vertex $\mathfrak{a}_\bullet$ and $2 \cdot 1_{\mathfrak{a}_\circ} = 0$ for a white vertex $\mathfrak{a}_\circ$, which proves the statement. $\square$

Definition 2.1 (Specification)**.** Let C be a genus-zero right Γ-set and let Z be an abelian group. A *Z-valued specification* on C is a function s from the set of monovalent vertices and regions in C to Z such that $\sum \mathrm{s}(\mathfrak{a}) = 0$, and $3\,\mathrm{s}(\mathfrak{a}_\bullet) = 0$ for each black vertex $\mathfrak{a}_\bullet$ and $2\,\mathrm{s}(\mathfrak{a}_\circ) = 0$ for each white vertex $\mathfrak{a}_\circ$. In other words, a *Z-valued specification* is a homomorphism from the group in (2.3) to Z.

Let E be a finite set with a transitive right action of $\Gamma \times \mathbb{Z}$. Let C be the set of $(1 \times \mathbb{Z})$-orbits in E, so that C is a Γ-set. Let n be the least positive integer such that $(1, n) \in \Gamma \times \mathbb{Z}$ acts trivially on E. Then, for the stabilizer K of an arbitrary edge $e \in C$, there is a distinguished homomorphism $\mathrm{m}\colon K \to \mathbb{Z}_n$ determined by E as follows: for each $\gamma \in K$, consider some $\tilde{\gamma} \in \Gamma \times \mathbb{Z}$ which projects to γ and which stabilizes some (every) element $\tilde{e} \in E$ which maps to $e \in C$ under the quotient map $E \to C$, then $\mathrm{m}(\gamma)$ is equal to the projection of $\tilde{\gamma}$ to $\mathbb{Z}$ modulo n. (Note that the value of $\mathrm{m}(\gamma)$ is indeed independent of the choice of $\tilde{\gamma}$). In the case that C is of genus zero, the homomorphism m is equivalent to a $\mathbb{Z}_n$-valued specification s on C. It is easy to see that E can be recovered (up to isomorphism) from the pair (C, s). Remark 2.7 below summarizes this paragraph. We say that E is of *genus zero* if and only if C is of genus zero.

Remark 2.7. The isomorphism classes of genus-zero right $(\Gamma \times \mathbb{Z})$-sets are in *bijection* with the isomorphism classes of pairs (C, s) where C is a genus-zero right Γ-set and s is a $\mathbb{Z}_n$-valued specification on C for some n.

Let E_1, E_2 be genus-zero $(\Gamma \times \mathbb{Z})$-sets which correspond to the pairs (C_1, s_1) and (C_2, s_2) where s_1 and s_2 are $\mathbb{Z}_{n_1}$-valued and $\mathbb{Z}_{n_2}$-valued, respectively. There is an equivariant map $E_1 \to E_2$ if and only if $n_2 \mid n_1$ and there is a covering $\phi\colon C_1 \to C_2$ such that $\mathrm{s}_2(\phi(\mathfrak{a})) \equiv \mathrm{s}_1(\mathfrak{a}) \pmod{n_2}$ for all monovalent vertices and regions $\mathfrak{a} \subset C_1$.

Keeping the notation of the previous paragraph, let E_1, E_2 be arbitrary (i.e. we do not assume an equivariant map $E_1 \to E_2$). Let C be the set of $(1 \times \mathbb{Z})$-orbits in the product $(\Gamma \times \mathbb{Z})$-set $E := E_1 \times E_2$. The projection maps $E \to E_1$ and $E \to E_2$ give rise to coverings $C \to C_1$ and $C \to C_2$, hence to a covering $C \to C_1 \times C_2$. Then, the latter covering is cyclic of degree $\gcd(n_1, n_2)$ and induces the following weight w: for each monovalent vertex or region $\mathfrak{a} \subset C_1 \times C_2$:

$$w(\mathfrak{a}) = \frac{\gcd(d_1\, \mathrm{s}_1(\mathfrak{a}_1) - d_2\, \mathrm{s}_2(\mathfrak{a}_2), n_1, n_2)}{\gcd(n_1, n_2)}, \tag{2.4}$$

where $\mathfrak{a}_1 \subset C_1$, $\mathfrak{a}_2 \subset C_2$ are the projections of $\mathfrak{a}$, and d_1, d_2 are the ramification indices of $\mathfrak{a} \mapsto \mathfrak{a}_1$ and $\mathfrak{a} \mapsto \mathfrak{a}_2$. Clearly, $d_i = \frac{|\mathfrak{a}|}{|\mathfrak{a}_i|}$ and $|\mathfrak{a}| = \operatorname{lcm}(|\mathfrak{a}_1|, |\mathfrak{a}_2|)$.

2.2 Braid Groups and Burau Representation

The standard references for this section are [13, 14]. Consider the free group F_n as equipped with a fixed n-tuple $(s_1, s_2, \ldots, s_n)$ of generators. The *braid group* $\mathbb{B}_n$ consists of those elements in the left automorphism group $\operatorname{Aut}(F_n)$ which take each s_i to a conjugate of some s_j and which fix the product $s_1 s_2 \cdots s_n \in F_n$. Consider $\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \in \mathbb{B}_n$ defined as

$$\sigma_i \colon s_i \mapsto s_i s_{i+1} s_i^{-1}, \qquad s_{i+1} \mapsto s_i, \qquad s_j \mapsto s_j \text{ for } j \neq i, i+1.$$

Then, one has

$$\mathbb{B}_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2 \rangle.$$

The action of $\mathbb{B}_n \cdot \operatorname{Inn}(F_n) \subset \operatorname{Aut}(F_n)$ on F_n respects the epimorphism $\mathrm{u} \colon F_n \twoheadrightarrow \mathbb{Z}$ defined by $\mathrm{u}(s_i) = 1$. The *Burau representation* $\mathbb{B}_n \cdot \operatorname{Inn}(F_n) \to GL(n-1, \Lambda)$ is the induced action on $A_n := \operatorname{Ker}(\mathrm{u}) / \operatorname{Ker}(\mathrm{u})' \cong \Lambda^{n-1}$. Here, we identify A_n with Λ^{n-1} by matching the distinguished basis $(s_1 s_2^{-1}, s_2 s_3^{-1}, \ldots, s_{n-1} s_n^{-1})$ of the former with the standard basis of the latter.

In the braid group $\mathbb{B}_3$, let $\mathbb{X} := \sigma_1 \sigma_2$ and let $\mathbb{Y} := \sigma_1 \sigma_2 \sigma_1$. Then, one has $\mathbb{B}_3 = \langle \mathbb{X}, \mathbb{Y} \mid \mathbb{X}^3 = \mathbb{Y}^2 \rangle$. Explicitly written out,

$$\begin{aligned} \mathbb{X} &\colon s_1 \mapsto s_1 s_2 s_1^{-1}, & s_2 &\mapsto s_1 s_3 s_1^{-1}, & s_3 &\mapsto s_1, \\ \mathbb{Y} &\colon s_1 \mapsto s_1 s_2 s_3 s_2^{-1} s_1^{-1}, & s_2 &\mapsto s_1 s_2 s_1^{-1}, & s_3 &\mapsto s_1, \end{aligned}$$

and $\mathbb{X}^3 = \mathbb{Y}^2 \colon s_i \mapsto (s_1 s_2 s_3) \cdot s_i \cdot (s_1 s_2 s_3)^{-1}$. Moreover,

$$\text{the Burau representation } \mathbb{B}_3 \to \mathrm{Bu}_3 \subset GL(2, \Lambda) \text{ is faithful.} \tag{2.5}$$

Hence, we identify $\mathbb{B}_3$ with its image and write

$$\mathbb{X} = \begin{pmatrix} 0 & -t \\ t & -t \end{pmatrix}, \qquad \mathbb{Y} = \begin{pmatrix} 0 & -t \\ -t^2 & 0 \end{pmatrix}.$$

Thus, $\mathbb{X}^3 = \mathbb{Y}^2 = t^3 \cdot 1$. Clearly, Bu_3 is generated by $\mathbb{X}$, $\mathbb{Y}$ and $t \cdot 1$. Then, one has $|\mathrm{Bu}_3 : \mathbb{B}_3| = 3$ since $t \cdot 1 \notin \mathbb{B}_3$.

There is a canonical homomorphism $\mathrm{c} \times \mathrm{d} \colon \mathrm{Bu}_3 \to \Gamma \times \mathbb{Z}$: the first component c is the evaluation of a matrix at $t = -1$ followed by projectivization, and the second component is defined as $\mathrm{d}(b) := \deg(\det(b))$ for any matrix b. Then,

$$\begin{aligned} &\mathrm{c}(\mathbb{X}) = X, && \mathrm{c}(\mathbb{Y}) = Y, && \mathrm{c}(t \cdot 1) = 1, \\ &\mathrm{d}(\mathbb{X}) = 2, && \mathrm{d}(\mathbb{Y}) = 3, && \mathrm{d}(t \cdot 1) = 2. \end{aligned}$$

The only relation between $\tilde{X} := (X, 2)$ and $\tilde{Y} := (Y, 3)$ in $\Gamma \times \mathbb{Z}$ is that $\tilde{X}^3 = \tilde{Y}^2$, because the only relation between X and Y in Γ is $X^3 = Y^2 = 1$. This observation shows (2.5), and that $\mathrm{c} \times \mathrm{d}$ is injective, at once. The image of $\mathrm{c} \times \mathrm{d}$ consists of those pairs (γ, n) for which $\mathrm{ab}(\gamma) \equiv n \pmod 2$, where ab is the abelianization (2.1). Note that the definition of $\mathrm{c} \colon \mathrm{Bu}_3 \to \Gamma$ given here is consistent with that given in the introduction.

Note that $H \mapsto H \backslash \mathrm{Bu}_3$ establishes a bijection between conjugacy classes of subgroups and transitive right Bu_3-sets. In particular, the notion of *genus zero* applies to Bu_3-sets as well. Due to the injection $\mathrm{c} \times \mathrm{d}$, the observations about $(\Gamma \times \mathbb{Z})$-sets found in Section 2.1.4 translate to Bu_3-sets with little modification. For example, Remark 2.7 takes the following form.

Remark 2.8. The isomorphism classes of genus-zero Bu_3-sets are in bijection with the isomorphism classes of pairs (C, s) where C is a genus-zero right Γ-set and s is a $\mathbb{Z}_{2n}$-valued specification on C for some n such that $\mathrm{s}(\mathfrak{a}) \equiv |\mathfrak{a}| \pmod 2$ for any region $\mathfrak{a} \subset C$ and $\mathrm{s}(\mathfrak{a}_\circ) \equiv 1 \pmod 2$ for any monovalent white vertex $\mathfrak{a}_\circ \subset C$.

Therefore, in Appendix A, we list the conjugacy classes in Theorem 1.2 as pairs (C, s) *as in Remark 2.8.* Note that the equation (2.4) applies to the products of genus-zero Bu_3-sets without any change.

Example 2.1. Consider the singleton Γ-set $\{*\}$ with $\mathcal{D}_{\{*\}} = \bullet\!\!-\!\!\!-\!\!\!-\!\!\circ$ and let $\mathbf{1}$ denote the unique region, which is of size 1. Then, for the $\mathbb{Z}_2$-valued specification $\mathrm{s}(\bullet) = 0,\ \mathrm{s}(\circ) = 1,\ \mathrm{s}(\mathbf{1}) = 1$, the pair $(\{*\}, \mathrm{s})$ corresponds to the trivial Bu_3-set. For the $\mathbb{Z}_6$-valued specification $\mathrm{s}(\bullet) = 2,\ \mathrm{s}(\circ) = 3,\ \mathrm{s}(\mathbf{1}) = 1$, the pair $(\{*\}, \mathrm{s})$ corresponds to $\mathbb{B}_3 \backslash \mathrm{Bu}_3$.

Chapter 3

The Method of Proof

In this chapter, we give an overview of how we prove our Theorem 1.2. We begin with the observation that it is enough to consider the subgroups $\mathcal{H}(A)$ for finite modules A.

Lemma 3.1. *Let A be a module equipped with an epimorphism $\Lambda^2 \twoheadrightarrow A$. If $\mathcal{H}(A)$ is of finite index, there is a finite quotient A' of A with $\mathcal{H}(A) = \mathcal{H}(A')$.*

Proof. Since $\mathcal{H}(A)$ has finite index, there is a positive integer n such that $t^n \cdot 1 \in \mathcal{H}(A)$. Thus we have $\begin{bmatrix} t^n - 1 \\ 0 \end{bmatrix} \in \operatorname{Ker}(\Lambda^2 \twoheadrightarrow A)$ and $\begin{bmatrix} 0 \\ t^n - 1 \end{bmatrix} \in \operatorname{Ker}(\Lambda^2 \twoheadrightarrow A)$, hence A is a quotient of $\big(\Lambda/(t^n - 1)\big)^2$. In particular, A is finitely generated over $\mathbb{Z}$. Now, let $\{1, h_1, h_2, \dots, h_k\}$ be a transversal of $\mathcal{H}(A)$ in Bu_3. For each h_i, consider some $u_i \in \Lambda^2$ such that $(h_i - 1)u_i \notin \operatorname{Ker}(\Lambda^2 \twoheadrightarrow A)$ and let $a_i \neq 0 \in A$ be such that $(h_i - 1)u_i \mapsto a_i$. Since A is finitely generated over $\mathbb{Z}$, there is a positive integer ℓ for which $\{a_1, a_2, \dots, a_k\} \cap \ell A = \emptyset$. Consider the finite quotient module $A' := A/\ell A$, and let $a_i' \neq 0 \in A'$ be such that $a_i \mapsto a_i'$. Consequently, $(h_i - 1)u_i \notin \operatorname{Ker}(\Lambda^2 \twoheadrightarrow A \twoheadrightarrow A')$, thus $h_i \notin \mathcal{H}(A')$. Therefore, $\mathcal{H}(A') = \mathcal{H}(A)$. $\square$

From now on, instead of considering a module A equipped with an epimorphism $\Lambda^2 \twoheadrightarrow A$, we consider a module A with no additional structure and consider all possible epimorphisms at the same time. We denote

$$\mathcal{E}(A) := \{\phi \mid \phi\colon \Lambda^2 \twoheadrightarrow A\} = \{(a_1, a_2) \in A^2 \mid \Lambda \cdot a_1 + \Lambda \cdot a_2 = A\}. \tag{3.1}$$

The two definitions of $\mathcal{E}(A)$ given in (3.1) agree, since the two sets are canonically identified by

$$\phi \mapsto (\phi(e_1), \phi(e_2)), \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

There is a canonical right Bu_3-action on $\mathcal{E}(A)$ given by

$$\phi \cdot b = \phi', \qquad \phi'(u) = \phi(b \cdot u), \qquad\qquad \text{for } b \in \mathrm{Bu}_3,\ u \in \Lambda^2. \tag{3.2}$$

Equivalently,

$$(a_1, a_2) \cdot b = (x \cdot a_1 + z \cdot a_2, y \cdot a_1 + w \cdot a_2), \qquad b = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{Bu}_3. \tag{3.3}$$

Lemma 3.2 gives an alternative characterization of $\mathcal{H}(\phi)$.

Lemma 3.2. *Let $\phi\colon \Lambda^2 \twoheadrightarrow A$ be an epimorphism. Then,*

$$\mathcal{H}(\phi) = \mathrm{Stab}(\phi)$$

where the stabilizer is with respect to the right Bu_3*-set* $\mathcal{E}(A)$*.*

Proof. If $b \in \mathcal{H}(\phi)$, then $(b-1) \cdot \Lambda^2 \subset \mathrm{Ker}(\phi)$, thus $\phi(b \cdot u) = \phi(u)$ for all $u \in \Lambda^2$, which shows that $\phi \cdot b = \phi$. All of the implications go both ways, hence $\phi \cdot b = \phi$ implies $b \in \mathcal{H}(\phi)$ as well. □

In view of the above, Theorem 1.2 is, in fact, a classification (up to isomorphism) of the genus-zero Bu_3-orbits in $\mathcal{E}(A)$ for all finite modules A.

Definition 3.1 ($\mathfrak{m}$-local module)**.** Let $\mathfrak{m} \subset \Lambda$ be a maximal ideal. An $\mathfrak{m}$-*local* module is a non-trivial Λ-module annihilated by $\mathfrak{m}^n$ for sufficiently large n.

Note that any maximal ideal $\mathfrak{m} \subset \Lambda$ is in the form $\mathfrak{m} = \langle p, \psi(t) \rangle$ for a prime p and a polynomial $\psi(t)$ irreducible modulo p. Hence, *local* modules are finite.

Remark 3.1. The action of an element $\phi \in \Lambda \smallsetminus \mathfrak{m}$ on an $\mathfrak{m}$-local module A is invertible.

Lemma 3.3. *Let A be a finite module. Then, there is a decompostion $A = \bigoplus A_{\mathfrak{m}}$ into local modules $A_{\mathfrak{m}}$. Moreover, $\mathcal{E}(A) = \prod \mathcal{E}(A_{\mathfrak{m}})$.* □

Lemma 3.3 suggests the following strategy of proof.

Strategy. I. Determine the genus-zero orbits in $\mathcal{E}(A)$ for all local modules A.

II. Determine the genus-zero orbits in the products of the Bu_3-sets found in the first step.

The second step of the proof mainly relies on Equation (2.4) and is completed in Chapter 5. The first step is completed in Chapter 4 through a long casework. In the rest of this chapter, we develop the ideas necessary to complete the first step. We introduce notation and terminology to be constantly used throughout the casework of the next chapter.

3.1 The Γ-set $\mathcal{C}(A)$

For a local module A, we denote the set of t-orbits in $\mathcal{E}(A)$ by $\mathcal{C}(A)$, i.e. $(a_1, a_2) \sim (t^k \cdot a_1, t^k \cdot a_2)$, and we denote the quotient map by $\mathrm{c}\colon \mathcal{E}(A) \to \mathcal{C}(A)$. Since $t \cdot 1 \in \mathrm{Bu}_3$ generates the kernel of $\mathrm{c}\colon \mathrm{Bu}_3 \twoheadrightarrow \Gamma$ (see Section 2.2), the Bu_3-action on $\mathcal{E}(A)$ reduces to a Γ-action on $\mathcal{C}(A)$. Clearly, an epimorphism $A_1 \twoheadrightarrow A_2$ induces a covering $\mathcal{C}(A_1) \to \mathcal{C}(A_2)$. Due to (3.3), the Γ-action on $\mathcal{C}(A)$ can be explicitly described as follows:

$$\begin{array}{rclcl} \mathrm{c}(a_1, a_2) \cdot X & = & \mathrm{c}((a_1, a_2) \cdot (t^{-1}\mathbb{X})) & = & \mathrm{c}(a_2, -a_1 - a_2), \\ \mathrm{c}(a_1, a_2) \cdot Y & = & \mathrm{c}((a_1, a_2) \cdot (t^{-1}\mathbb{Y})) & = & \mathrm{c}(-t \cdot a_2, -a_1), \\ \mathrm{c}(a_1, a_2) \cdot YX & = & \mathrm{c}(-t \cdot a_2, -a_1) \cdot X & = & \mathrm{c}(-a_1, t \cdot a_2 + a_1). \end{array} \tag{3.4}$$

Let $\Omega \subset \mathcal{C}(A)$ be a Γ-orbit, then $S(\Omega) := \mathrm{c}^{-1}(\Omega) \subset \mathcal{E}(A)$ is a Bu_3-orbit. Clearly, $S(\Omega)$ is of genus zero if and only if Ω is of genus zero. *Hence, the first step* *in our strategy is to determine* $S(\Omega)$ *for all genus-zero orbits* $\Omega \subset \mathcal{C}(A)$. In the genus-zero case, $S(\Omega)$ is determined by a $\mathbb{Z}_{2n}$-valued specification s on Ω (Remark 2.8), where n is the least positive integer for which t^n acts trivially on A. The specification s can be explicitly described as follows. For each monovalent vertex or region $\mathfrak{a} \subset \Omega$, consider an integer k for which the appropriate one among $t^k \cdot \mathbb{X}$, $t^k \cdot \mathbb{Y}$, and $t^k \cdot (\mathbb{Y}\mathbb{X})^{|\mathfrak{a}|}$ acts trivially on some (every) element $(a_1, a_2) \in \mathcal{E}(A)$ for which $\mathrm{c}(a_1, a_2) \in \mathfrak{a}$. Note that k is well-defined modulo n. Then, $\mathrm{s}(\mathfrak{a}) = 2k+2$, $2k+3$, or $-(2k+5|\mathfrak{a}|)$, depending on whether $\mathfrak{a}$ is a black vertex, a white vertex or a region.

3.2 Wheels and Rings

In the rest of the text, whenever $\mathfrak{m} = \langle p, \psi(t) \rangle$ refers to a particular maximal ideal, $\Bbbk$ denotes the residue field $\Lambda/\mathfrak{m} = \mathbb{F}_p[t]/\psi(t)$. For an $\mathfrak{m}$-local module A of interest, the vector space $A \otimes \Bbbk = A/\mathfrak{m}A$ must have dimension 1 or 2, since $\mathcal{E}(A)$ is otherwise empty. We treat the two classes of modules separately. If $\dim(A \otimes \Bbbk) = 1$, Nakayama's Lemma implies that $A \cong \Lambda/I$ for an ideal $I \subset \Lambda$. Clearly, $\mathfrak{m}^n \subset I$ for sufficiently large n. Hence, in this class of modules, we only consider these rings Λ/I which we call $\mathfrak{m}$-*rings*. Finally, we call modules A with $\dim(A \otimes \Bbbk) = 2$ *wheels*.

Remark 3.2. By Nakayama's Lemma, for a wheel W, one has $(a_1, a_2) \in \mathcal{E}(W)$ if and only if $a_1, a_2 \in W$ map to linearly independent vectors under an epimorphism $W \twoheadrightarrow \Bbbk^2$.

3.2.1 $\mathfrak{m}$-Rings

Whenever we speak of an $\mathfrak{m}$-ring R, we denote the image of any element $\phi \in \Lambda$ in R by ϕ as well, by a slight abuse of notation. We also denote the image of $\mathfrak{m} \subset \Lambda$ in R by $\mathfrak{m}$. Then, R is a local ring with unique maximal ideal $\mathfrak{m} \subset R$, which is nilpotent. It is easy to see that

$$\mathcal{E}(R) = \{(r_1, r_2) \mid r_1, r_2 \in R \text{ and } \{r_1, r_2\} \not\subset \mathfrak{m}\}.$$

Now, let $R^* := R \smallsetminus \mathfrak{m}$ denote the group of invertible elements, and let $\mathcal{P}(R)$ denote the set of R^*-orbits in $\mathcal{E}(R)$, i.e. $(r_1, r_2) \sim (ur_1, ur_2)$ for all $u \in R^*$. Since the image of $t \in \Lambda \smallsetminus \mathfrak{m}$ is in R^*, the quotient map $\mathrm{pc}\colon \mathcal{E}(R) \to \mathcal{P}(R)$ reduces to a map $\mathrm{p}\colon \mathcal{C}(R) \to \mathcal{P}(R)$. This defines a distinguished Γ-action on $\mathcal{P}(R)$, such that $\mathrm{p}\colon \mathcal{C}(R) \to \mathcal{P}(R)$ is an abelian covering with deck transformation group $R^*/\langle t \rangle$. We always consider $\mathcal{P}(R)$ as equipped with the weights induced by the covering p, even though we do not show it in notation. Clearly, an epimorphism $R_1 \twoheadrightarrow R_2$ induces a covering $\mathcal{P}(R_1) \to \mathcal{P}(R_2)$ compatible with the weights (see Section 2.1.3).

Let $\Omega \subset \mathcal{P}(R)$ be some orbit and let $\tilde{\Omega}$ be any orbit in $\mathrm{p}^{-1}(\Omega) \subset \mathcal{C}(R)$. Then, $S(\tilde{\Omega})$ is independent of the choice of $\tilde{\Omega}$, hence we define $S(\Omega) := S(\tilde{\Omega})$. Therefore, as long as we consider $\mathfrak{m}$-rings, our goal is to determine $S(\Omega)$ for all genus-zero orbits $\Omega \subset \mathcal{P}(R)$, where "genus zero" is in the weighted sense defined in Remark 2.6. In this sense, $\mathcal{P}(R)$ replaces $\mathcal{C}(R)$ in our strategy.

Finally, note that any edge in $\mathcal{P}(R)$ can be expressed in the form of either $\mathrm{pc}(1, r)$ for $r \in R$ or $\mathrm{pc}(m, 1)$ for $m \in \mathfrak{m}$, so that this expression is unique. In particular, $|\mathcal{P}(R)| = |R| + |\mathfrak{m}| = (|\mathbb{k}| + 1) \cdot |\mathfrak{m}|$. We now finish with a lemma on the structure of R^*.

Lemma 3.4. *Let $\mathfrak{m}^*$ denote the kernel of the group epimorphism $R^* \twoheadrightarrow \mathbb{k}^*$. Then, $\mathfrak{m}^*$ is a p-group, hence R^* naturally splits as $R^* = \mathfrak{m}^* \oplus \mathbb{k}^*$.*

Proof. Let n be a sufficiently large positive integer such that $\mathfrak{m}^{p^n} = 0$. Then, one has $(1+m)^{p^{(p^n+n)}} = 1$ for all $m \in \mathfrak{m}$. This concludes the proof since $\mathfrak{m}^*$ is the multiplicative group of the elements which are congruent to 1 modulo $\mathfrak{m}$. □

3.3 Restrictions on $\mathcal{C}(W)$ and $\mathcal{P}(R)$

In this section, we establish formulae about the monovalent vertices and the regions in $\mathcal{C}(W)$ or $\mathcal{P}(R)$ for a wheel W or an $\mathfrak{m}$-ring R. With these formulae, one can compute the Euler characteristic of an orbit by Lemma 2.2 or Lemma 2.3, thus determine whether or not an orbit is of genus zero. We implicitly refer to the Γ-action in (3.4) throughout the section.

For a maximal ideal $\mathfrak{m} = \langle p, \psi(t) \rangle$, *we denote the multiplicative order of the image of* $(-t) \in \Lambda \smallsetminus \mathfrak{m}$ *in* $\mathbb{k}^*$ *by* N, *i.e.* N *is the least positive integer for which* $\psi(t) \mid \Phi_N(-t)$ *in* $\mathbb{F}_p$. *Then the order of* $t \in \mathbb{k}^*$ *is*

$$M := \qquad \underbrace{2N}_{\text{if } 2\nmid N \text{ and } p>2} \qquad \underbrace{N}_{\text{if } 4\mid N \text{ or } p=2} \qquad \underbrace{\frac{N}{2}}_{\text{if } N\equiv 2 \,(\mathrm{mod}\, 4)} \quad .$$

For $(-t) \in R^*$, *one has* $\operatorname{ord}(-t) = Np^k$ *for some* k *due to the decomposition* $R^* = \mathfrak{m}^* \oplus \mathbb{k}^*$ *given in Lemma 3.4. Analogous formulae are obviously valid for* $\operatorname{ord}(t)$ *in* R^* *and for the order of the* t*-action on a wheel* W.

3.3.1 Monovalent Vertices

Lemma 3.5. *Let* W *be a wheel. Then, there is no monovalent vertex in* $\mathcal{C}(W)$.

Proof. There is an epimorphism $W \twoheadrightarrow \mathbb{k}^2$, hence an induced covering $\mathcal{C}(W) \to \mathcal{C}(\mathbb{k}^2)$. Thus, it is enough to show that $\mathcal{C}(\mathbb{k}^2)$ contains no monovalent vertex, i.e. no edge fixed by X or Y. Just by comparing the first coordinates, we see that $(a_1, a_2) \neq t^n(a_2, -a_1 - a_2)$ for any n, thus no edge is fixed by X (because a_1 and a_2 are independent by Remark 3.2). Similarly by comparing the first coordinates, we see that no edge is fixed by Y. $\square$

For the notation and the terminology appearing in the formulation of Lemma 3.6, we refer to Section 2.1.3.

Lemma 3.6. *Let R be an $\mathfrak{m}$-ring. Then,*

- *Any monovalent black vertex in $\mathcal{P}(R)$ consists of an edge $\mathrm{pc}(1,r)$ where $r^2+r+1=0$. The vertex is complete if and only if $r \in \langle t \rangle$.*
- *Any monovalent white vertex in $\mathcal{P}(R)$ consists of an edge $\mathrm{pc}(1,r)$ where $r^2 = \frac{1}{t}$. The vertex is complete if and only if $-r \in \langle t \rangle$.*

Consequently,

- *The number of complete monovalent black vertices is at most 3. In the case $p \neq 3$, the number is 2 if $3 \mid N$ and 0 otherwise (in fact, the total number of monovalent black vertices is 2 or 0).*
- *The number of complete monovalent white vertices is at most 1. In the case $p \neq 2$, the number is 1 if $N \equiv 2 \pmod 4$ and 0 otherwise (the total number of monovalent white vertices is 2 or 0).*

Proof. The monovalent black vertices consist of the edges fixed by X. Since any edge in $\mathcal{P}(R)$ has the form of either $\mathrm{pc}(1,r)$ for $r \in R$ or $\mathrm{pc}(m,1)$ for $m \in \mathfrak{m}$, the edges fixed by X are the solutions of the equations

$$\mathrm{pc}(1,r) = \mathrm{pc}(r,-r-1) \qquad \mathrm{pc}(m,1) = \mathrm{pc}(1,-m-1)$$

The second equation clearly has no solution, while the first equation is satisfied if and only if $r^2+r+1=0$. The vertex is complete if and only if $\mathrm{c}(1,r) = \mathrm{c}(r,-r-1)$, that is, $r \in \langle t \rangle \subset R^*$. The equality $r^2+r+1=0$ implies $r^3=1$, therefore there are at most 3 complete monovalent black vertices (those elements in the cyclic group $\langle t \rangle$ with order dividing 3). Moreover, in the case $p \neq 3$, the equality $r^2+r+1=0$ holds if and only if r is of order 3. Therefore, the total number of monovalent black vertices is 2 if $3 \mid |\Bbbk^*|$ and 0 otherwise. The number of complete vertices is 2 if $3 \mid \mathrm{ord}(t)$, i.e. $3 \mid N$ and 0 otherwise.

Similarly, the monovalent white vertices consist of the edges fixed by Y. Thus, they are the solutions of the equations

$$\mathrm{pc}(1,r) = \mathrm{pc}(-tr,-1) \qquad \mathrm{pc}(m,1) = \mathrm{pc}(-t,-m)$$

As above, the second equation has no solution, while the first equation is satisfied if and only if $r^2 = \frac{1}{t}$. The vertex is complete if and only if $\mathrm{c}(1, r) = \mathrm{c}(-tr, -1)$, that is, $-r \in \langle t \rangle \subset R^*$. If $2 \mid \mathrm{ord}(t)$, then $\langle t^2 \rangle$ is properly contained in $\langle t \rangle$, hence there is no complete monovalent white vertex. If $2 \nmid \mathrm{ord}(t)$, there is a unique square root of $\frac{1}{t}$ in the cyclic group $\langle t \rangle$, hence there is 1 such vertex. Note that, in the case $p \neq 2$, one has $2 \nmid \mathrm{ord}(t)$ if and only if $N \equiv 2 \pmod 4$. In the case $p \neq 2$, the total number of monovalent white vertices (square roots of t) is 2 if $2 \mid \frac{|\Bbbk^*|}{M}$ and 0 otherwise. $\square$

3.3.2 Regions for $N > 1$

The YX-action in (3.4) takes a simpler form under a basis transformation. Namely, for a local module (a wheel or an $\mathfrak{m}$-ring) A with $N > 1$ and $(a_1, a_2) \in \mathcal{E}(A)$, we define

$$\mathrm{c}'(a_1, a_2) := \mathrm{c}(a_1, (t+1)^{-1} \cdot (a_2 - a_1)), \tag{3.5}$$

which is well-defined since the action of $(t + 1) \notin \mathfrak{m}$ on A is invertible by Remark 3.1. Analogously, for an $\mathfrak{m}$-ring R and $(r_1, r_2) \in \mathcal{E}(R)$, we define

$$\mathrm{pc}'(r_1, r_2) := \mathrm{pc}(r_1, (t+1)^{-1}(r_2 - r_1)).$$

In this notation, the YX-action on $\mathcal{C}(W)$ and $\mathcal{P}(R)$ is as follows:

$$\mathrm{c}'(a_1, a_2) \cdot YX = \mathrm{c}'(-a_1, t \cdot a_2), \qquad \mathrm{pc}'(r_1, r_2) \cdot YX = \mathrm{pc}'(r_1, -tr_2) \tag{3.6}$$

This makes it more convenient to describe the sizes of the regions in $\mathcal{C}(W)$ and $\mathcal{P}(R)$.

Lemma 3.7. *Let W be a wheel with $N > 1$. Then, the size of any region in $\mathcal{C}(W)$ is equal to Np^k for some integer $k \geq 0$. Moreover, the following holds for the size of the region which contains the edge $\mathrm{c}'(a_1, a_2) \in \mathcal{C}(W)$:*

(1) Except for the case $p = 2$ and $k = 0$, the size of this region is less than or equal to Np^k if and only if either $(-t)^{Np^k} \cdot a_1 = a_1$ or $(-t)^{Np^k} \cdot a_2 = a_2$.

(2) In the case $p = 2$, the size of this region is equal to N if and only if either $(-t)^N \cdot a_1 = a_1$, or both $t^N \cdot a_2 = a_2$ and $t^{2Ns} \cdot a_1 = -a_1$ for some s.

In particular, the size of any region in $\mathcal{C}(\Bbbk^2)$ is equal to N.

Proof. In fact, it is easy to observe that the size of any region in $\mathcal{C}(\Bbbk^2)$ is equal to N. This implies that the size of any region in $\mathcal{C}(W)$ is divisible by N.

Now suppose that either $p > 2$ or $k > 0$. We will show that the size n of the region which contains the edge $\mathrm{c}'(a_1, a_2) \in \mathcal{C}(W)$ *divides* Np^k if and only if either $(-t)^{Np^k} \cdot a_1 = a_1$ or $(-t)^{Np^k} \cdot a_2 = a_2$. Note that this implies the whole statement but part (2), because these equalities hold for sufficiently large k. We show the "only if" part, then the "if" part becomes clear. Suppose that $n \mid Np^k$, then $((-1)^{Np^k} \cdot a_1, t^{Np^k} \cdot a_2) = t^{d'} \cdot (a_1, a_2)$ for some d'. Let $d_1 = Mp^{k_0}$ be the order of the t-action on W. Then, *the order of the t-action on either a_1 or a_2 is equal to d_1*, because these orders are divisible by M, and their least common multiple is $d_1 = Mp^{k_0}$. Let $d_0 := 0$ if $2 \mid Np^k$, and let $d_0 := \frac{d_1}{2} = Np^{k_0}$ if $2 \nmid Np^k$, so that $(t^{d_0} - (-1)^{Np^k}) \cdot W = 0$. Then, if the order of the t-action on a_1 is equal to d_1, one necessarily has $d' \equiv d_0 \pmod{d_1}$, therefore $t^{Np^k} \cdot a_2 = (-1)^{Np^k} \cdot a_2$, hence $(-t)^{Np^k} \cdot a_2 = a_2$ holds. If the order of the t-action on a_2 is equal to d_1, one necessarily has $d' \equiv Np^k \pmod{d_1}$, therefore $(-1)^{Np^k} \cdot a_1 = t^{Np^k} \cdot a_1$, hence $(-t)^{Np^k} \cdot a_1 = a_1$ holds.

Now it is left to prove part (2). As above, we prove the "only if" part. The proof above fails in this case only in the choice of d_0, so we retain the remaining notation, i.e. d', d_1, k_0. Thus we have $(-a_1, t^N \cdot a_2) = t^{d'} \cdot (a_1, a_2)$. Clearly $N \mid d'$, so let $d' := Ns'$, thus $t^{Ns'} \cdot a_1 = -a_1$. If $s' = 2s$, one must have $t^N \cdot a_2 = a_2$, hence suppose $2 \nmid s'$. As in the proof above, if the order of the t-action on a_2 is equal to d_1, then $d' \equiv N \pmod{d_1}$. If the order of the t-action on a_1 is equal to d_1, then $d_1 \mid 2d' = 2Ns'$, hence $d_1 \mid 2N$, thus $d' \equiv N \pmod{d_1}$. In either case, one has $d' \equiv N \pmod{d_1}$, thus $(-t)^N \cdot a_1 = a_1$. □

For an $\mathfrak{m}$-ring R, we say that a region in $\mathcal{P}(R)$ is a *bulk region* if it contains an edge of the form $\mathrm{pc}'(1, r)$ for $r \in R^*$. Note that all edges in a bulk region are necessarily of this form.

Lemma 3.8. *Let R be a ring with $N > 1$. Then, the size of a bulk region in $\mathcal{P}(R)$ is equal to $\mathrm{ord}(-t) = Np^{k_0}$, and the weight on the region is 1. Similarly, except for the following cases, the size of the region which contains the edge $\mathrm{pc}'(m, 1) \in \mathcal{P}(R)$ or $\mathrm{pc}'(1, m) \in \mathcal{P}(R)$ for some $m \in \mathfrak{m}$ is equal to the least $n = Np^k$ with $((-t)^n - 1) \cdot m = 0$, and the weight on the region is 1.*

(1) The size of the region which contains $\mathrm{pc}'(0, 1) \in \mathcal{P}(R)$ is 1, and the weight on the region is 1.

(2) The size of the region which contains $\mathrm{pc}'(1, 0) \in \mathcal{P}(R)$ is 1, and the weight on the region is 1 if $-1 \in \langle t \rangle \subset R^*$ and $\frac{1}{2}$ otherwise. (Note that, in the case $p > 2$, one has $-1 \in \langle t \rangle \subset R^*$ if and only if $N \not\equiv 2 \pmod 4$.)

(3) If $p = 2$ and $-1 \notin \langle t \rangle \subset R^*$, the weight on a region of size N which contains an edge $\mathrm{pc}'(1, m) \in \mathcal{P}(R)$ is $\frac{1}{2}$.

Proof. Knowing the action in (3.6), the statements about the sizes of the regions are easy to verify. Similarly, the statements about the weights are consequences of the following easy observations. The weight on a region which contains an edge $\mathrm{pc}'(r, 1)$ for some $r \in R$ is 1. The weight on a region of even size is 1. The weight on a region of odd size which contains an edge $\mathrm{pc}'(1, r)$ for some $r \in R$ is 1 or $\frac{1}{2}$ depending on whether or not $-1 \in \langle t \rangle \subset R^*$. □

As it is made clear above, we often use the (c', pc')-notation while discussing the wheels and the $\mathfrak{m}$-rings with $N > 1$ in the next chapter. Thus the following formula, which expresses the action of $X \in \Gamma$ in this notation, is useful:

$$\mathrm{c}'(a, b) \cdot X = \mathrm{c}'(a - b,\ t \cdot b + (t^2 + t + 1) \cdot a). \tag{3.7}$$

In particular, for $m \in \mathfrak{m} \subset R$,

$$\mathrm{pc}'(1, m) \cdot X = \mathrm{pc}'(1,\ (1 - m)^{-1}(tm + t^2 + t + 1)).$$

3.3.3 Regions for $N = 1$

The maximal ideals concerned in this section are of the form $\mathfrak{m} = \langle p, t+1 \rangle$, thus $\Bbbk = \mathbb{F}_p$. We now introduce notation to briefly describe the iterated YX-action on $\mathcal{C}(W)$ and $\mathcal{P}(R)$. For any integer $\ell \geq 0$, let

$$\begin{aligned} \delta_\ell &:= (-t)^{p^\ell \cdot (p-1)} + (-t)^{p^\ell \cdot (p-2)} + \ldots = \sum_{i=0}^{p-1} (-t)^{p^\ell \cdot i}, \\ \omega_\ell &:= (-t)^{p^\ell - 1} + (-t)^{p^\ell - 2} + \ldots = \sum_{i=0}^{p^\ell - 1} (-t)^i = \delta_0 \delta_1 \ldots \delta_{\ell-1}. \end{aligned} \tag{3.8}$$

Note that $\delta_\ell \in \mathfrak{m} \subset \Lambda$. (As usual, whenever an $\mathfrak{m}$-ring R is in question, we consider $\delta_\ell \in \mathfrak{m} \subset R$.) Consequently, for any wheel W, one has $\omega_\ell \cdot W = 0$ for sufficiently large ℓ. Similarly for any $\mathfrak{m}$-ring R, one has $\omega_\ell = 0$ for sufficiently large ℓ. For further convenience, put

$$\lambda := -1 - t, \qquad \text{so that} \qquad (-t)^{p^\ell} = 1 + \omega_\ell \lambda.$$

Note that $\lambda \in \mathfrak{m}$. Now, the iterated YX-action on $\mathrm{c}(a_1, a_2) \in \mathcal{C}(W)$ and $\mathrm{pc}(r_1, r_2) \in \mathcal{P}(R)$ can be expressed as (see 3.4)

$$\begin{aligned} \mathrm{c}(a_1, a_2) \cdot (YX)^{p^\ell} &= (-1)^{p^\ell} \cdot \mathrm{c}(a_1, a_2 + \omega_\ell(\lambda \cdot a_2 - a_1)), \\ \mathrm{pc}(r_1, r_2) \cdot (YX)^{p^\ell} &= \mathrm{pc}(r_1, r_2 + \omega_\ell(\lambda r_2 - r_1)). \end{aligned} \tag{3.9}$$

Note that $(-1)^{p^\ell} \in \langle t \rangle \subset R^*$ possibly except when $p = 2$ and $\ell = 0$. Similarly, there exists an integer d with $(t^d - (-1)^{p^\ell}) \cdot W = 0$ possibly except when $p = 2$ and $\ell = 0$.

Lemma 3.9. *Let W be a wheel with $N = 1$. Then, the size of any region in $\mathcal{C}(W)$ is equal to p^k for some integer $k \geq 1$. Moreover, the size of any region in $\mathcal{C}(\Bbbk^2)$ is p.*

Proof. As a result of (3.9) and the surrounding remarks, $(YX)^{p^\ell}$ acts trivially on $\mathcal{C}(W)$ for sufficiently large ℓ. Similarly, $(YX)^p$ acts trivially on $\mathcal{C}(\Bbbk^2)$ since $\omega_1 \in \mathfrak{m}$ annihilates $\Bbbk^2$. Thus, it is left to note that $\mathrm{c}(a_1, a_2) \neq \mathrm{c}(-a_1, a_1 - a_2)$ for any $\mathrm{c}(a_1, a_2) \in \mathcal{C}(\Bbbk^2)$ (since the vectors a_2, $a_1 - a_2 \in \Bbbk^2$ are linearly independent). $\square$

For an $\mathfrak{m}$-ring R, we say that a region in $\mathcal{P}(R)$ is a *bulk region* if it contains an edge of the form $\mathrm{pc}(1, r)$ for $r \in R$. Note that all edges in a bulk region are necessarily of this form. In addition, we consider the weight on any region $\mathfrak{a} \subset \mathcal{P}(R)$ as equally distributed over the edges in $\mathfrak{a}$. Thus we say that a region $\mathfrak{a}$ of weight $w(\mathfrak{a})$ has *edge-weight* $\frac{w(\mathfrak{a})}{|\mathfrak{a}|}$ (we also say that an edge in $\mathfrak{a}$ has *weight* $\frac{w(\mathfrak{a})}{|\mathfrak{a}|}$). Consequently, the sum of weights over the regions, which appears in Lemma 2.3, can be replaced by the sum of weights over the edges.

Lemma 3.10. *Let R be a ring with $N = 1$. For any region in $\mathcal{P}(R)$, the size is p^k and the edge-weight is $p^{-k'}$ for some integers $k' \geq k \geq 0$, i.e. the weight is $p^{k-k'}$. For a bulk region, $k' = k = k_0$ where k_0 is the least such that $\omega_{k_0} = 0$. Similarly, for the region which contains the edge $\mathrm{pc}(m, 1)$ for some $m \in \mathfrak{m}$, the size is less than or equal to p^k if and only if $\omega_k m(m - \lambda) = 0$ and the edge-weight is greater than or equal to $p^{-k'}$ if and only if $\omega_{k'} m(m - \lambda) = 0$ and $(-1)^{p^{k'}}(1+\omega_{k'}(\lambda - m)) \in \langle t \rangle \subset R^*$. The last condition can be expanded as follows (see Lemma 3.4):*

1. *If $p > 2$, then $1 + \omega_{k'}(\lambda - m) \in \langle 1 + \lambda \rangle \subset \mathfrak{m}^*$.*
2. *If $p = 2$ and $k' > 0$, then $1 + \omega_{k'}(\lambda - m) \in \langle 1 - \omega_1 \rangle \subset \mathfrak{m}^*$.*
3. *If $p = 2$ and $k' = 0$, then $1 - \omega_1 + m \in \langle 1 - \omega_1 \rangle \subset \mathfrak{m}^*$.*

Proof. The first sentence follows from the fact that $(YX)^{p^{k_0}}$ acts trivially on $\mathcal{C}(R)$ (see the proof of Lemma 3.9 and Equation 3.9). Then, for the region which contains the edge $\mathrm{pc}(r_1, r_2)$, the size is less than or equal to p^k if and only if $\mathrm{pc}(r_1, r_2) \cdot (YX)^{p^k} = \mathrm{pc}(r_1, r_2)$, and the edge-weight is greater than or equal to $p^{k'}$ if and only if $\mathrm{c}(r_1, r_2) \cdot (YX)^{p^{k'}} = \mathrm{c}(r_1, r_2)$.

The following observations show the remaining statements:

(a) One has $\mathrm{pc}(1, r) \cdot (YX)^{p^k} = \mathrm{pc}(1, r + \omega_k(\lambda r - 1))$, and one has

$$\mathrm{pc}(1, r) = \mathrm{pc}(1, r + \omega_k(\lambda r - 1))$$

if and only if $\omega_k(\lambda r - 1) = 0$, i.e. $k \geq k_0$ since $\lambda r - 1 \in R^*$.

(b) One has $\mathrm{pc}(m,1)\cdot(YX)^{p^k}=\mathrm{pc}(m,1+\omega_k(\lambda-m))$, and one has

$$\mathrm{pc}(m,1)=\mathrm{pc}(m,1+\omega_k(\lambda-m))$$

if and only if $m(1+\omega_k(\lambda-m))=m$, i.e. $\omega_k m(m-\lambda)=0$.

(c) One has $\mathrm{c}(m,1)\cdot(YX)^{p^{k'}}=(-1)^{p^{k'}}\cdot\mathrm{c}(m,1+\omega_{k'}(\lambda-m))$, and one has

$$\mathrm{c}(m,1)=(-1)^{p^{k'}}\cdot\mathrm{c}(m,1+\omega_{k'}(\lambda-m))$$

if and only if $\mathrm{pc}(m,1)=\mathrm{pc}(m,1+\omega_{k'}(\lambda-m))$ and $(-1)^{p^{k'}}(1+\omega_{k'}(\lambda-m))\in\langle t\rangle$. □

Chapter 4

The Proof: Part I

In this chapter, we begin proving Theorem 1.2, that is, we complete the first step of the proof strategy outlined in Chapter 3. Throughout the chapter, R denotes an $\mathfrak{m}$-ring, W denotes a wheel, and Ω denotes an orbit in $\mathcal{C}(W)$ or $\mathcal{P}(R)$. We check all wheels and $\mathfrak{m}$-rings (simply called *rings* in the rest) case-by-case, and show that, for all genus-zero orbits $\Omega \subset \mathcal{C}(W)$ and $\Omega \subset \mathcal{P}(R)$, the Bu_3-set $S(\Omega)$ appears in the tables in Appendix A. We freely use all the information in Section 3.3 as well as the Lemmas 2.2 and 2.3 in determining $\chi(\Omega)$.

4.1 The Case $N \geq 7$

We first observe that there is no genus-zero orbit in $\mathcal{C}(\mathbb{k}^2)$ (Lemmas 3.5 and 3.7), hence we only consider rings R. The following theorem allows us to restrict attention to a finite set of maximal ideals $\mathfrak{m}$.

Theorem 4.1 (Degtyarev [11]). *Let $\mathfrak{m}$ be a maximal ideal with $N \geq 7$. There is a genus-zero orbit in $\mathcal{P}(\mathbb{k})$ if and only if (p, N) is one of the following pairs:*

$$(2,7),(2,15),(3,8),(5,8),(5,12),(11,10),(13,12),$$
$$(17,8),(19,9),(19,18),(29,7),(37,9),(43,7).$$

Moreover, for any such $\mathfrak{m}$, there is only one orbit in $\mathcal{P}(\Bbbk)$.

Proof. This proof is essentially a re-expression of Degtyarev's original proof in our terminology. However, the initial part of the proof is considerably simplified by our approach. Let Ω be a genus-zero orbit in $\mathcal{P}(\Bbbk)$ and let $\bar{\Omega}$ denote the underlying Γ-set of Ω, i.e. we ignore the weights on the vertices and the regions. Clearly, $\chi(\bar{\Omega}) = 2$. The difference between $\chi(\Omega)$ and $\chi(\bar{\Omega})$ is due to the incomplete monovalent vertices and possibly due to a region of weight $\frac{1}{2}$ (Lemmas 2.2, 2.3, 3.8). In other words, $\chi(\bar{\Omega}) - \chi(\Omega) = \frac{2}{3} \cdot \epsilon_3' + \frac{1}{2} \cdot \epsilon_2'$ where ϵ_3' is the number of incomplete monovalent black vertices in Ω and ϵ_2' is the number of incomplete monovalent white vertices plus the number of regions of weight $\frac{1}{2}$ in Ω. Thus, $\chi(\Omega) = 2 - \frac{2}{3} \cdot \epsilon_3' - \frac{1}{2} \cdot \epsilon_2'$. Now, one can deduce that $\chi(\Omega) \in \{\frac{2}{3}, 1, 2\}$ by using the following three facts: $\epsilon_3', \epsilon_2' \leq 2$ (Lemmas 3.6, 3.8), $\frac{2}{\chi(\Omega)}$ is an integer (Remark 2.6), and $\chi(\Omega) \geq \frac{1}{2}$ if $\epsilon_2' > 0$ (Lemma 2.4). Moreover, $\chi(\Omega) \in \{1, 2\}$ if $3 \mid N$, because $\epsilon_3' = 0$ in this case.

Now consider the case $p > 3$. Let n denote the number of regions of size N, let ϵ_0 denote the number of regions of size 1, let ϵ_3 denote the number of complete monovalent black vertices and let ϵ_2 denote the number of complete monovalent white vertices minus the number of regions of weight $\frac{1}{2}$ in Ω. Then, $\chi(\Omega) = -\frac{nN+\epsilon_0}{6} + n + \epsilon_0 + \frac{2}{3} \cdot \epsilon_3 + \frac{1}{2} \cdot \epsilon_2$ (Lemmas 2.3, 3.8), thus

$$n \cdot \left(\frac{N}{6} - 1\right) = \frac{5\epsilon_0}{6} + \frac{2\epsilon_3}{3} + \frac{\epsilon_2}{2} - \chi(\Omega). \tag{4.1}$$

One can find all of the tuples $(N; n, \epsilon_0, \epsilon_3, \epsilon_2)$ satisfying (4.1) by considering the following restrictions in addition to those on $\chi(\Omega)$ given in the previous paragraph: $\epsilon_0, \epsilon_3 \leq 2$, $-1 \leq \epsilon_2 \leq 1$, $\epsilon_3 = 0$ if $3 \nmid N$ and $\epsilon_2 = 0$ if $N \not\equiv 2 \pmod 4$ (Lemmas 3.6, 3.8). These solutions are as follows (we ignore the solutions $(N; 0, 1, 1, \pm 1)$ since they are meaningless):

$$\begin{array}{lllll} (7;6,2,0,0), & (7;4,2,0,0), & (7;1,1,0,0), & (8;3,2,0,0), & (8;2,2,0,0), \\ (9;2,2,2,0), & (9;4,2,2,0), & (9;1,1,1,0), & (10;1,2,0,0), & (10;1,1,0,1), \\ (12;1,2,2,0), & (12;2,2,2,0), & (18;1,2,2,0). & & \end{array} \tag{4.2}$$

This finding already restricts Ω significantly. However, one needs further consideration to get the full description given in the theorem.

Since $\epsilon_0 \geq 1$ in all of the tuples in (4.2), Ω contains at least one of the edges $\mathrm{pc}'(0,1)$ and $\mathrm{pc}'(1,0) \in \mathcal{P}(\Bbbk)$ (in fact, $\epsilon_0 = 2$ in most cases, which requires Ω to contain both edges). Considering the Γ-action on these edges, one can find many pairs (q_1, q_2) of polynomials such that Ω necessarily contains $\mathrm{pc}'(q_1(t), q_2(t))$: for example, $\mathrm{pc}'(0,1) \cdot Y = \mathrm{pc}'(1,1)$, $\mathrm{pc}'(1,0) \cdot X = \mathrm{pc}'(1, t^2 + t + 1)$ and $\mathrm{pc}'(0,1) \cdot XYXY = \mathrm{pc}'(t(t-1), t^2+1)$ (3.6, 3.7). For each particular tuple $(N; n, \epsilon_0, \epsilon_3, \epsilon_2)$ in (4.2), consider $(n+1)$ pairs $(q_1^1, q_2^1), (q_1^2, q_2^2), \ldots, (q_1^{n+1}, q_2^{n+1})$ such that $\mathrm{pc}'(q_1^i(t), q_2^i(t))$ are necessarily in Ω. Then, there are indices $i \neq j$ such that the edges $\mathrm{pc}'(q_1^i(t), q_2^i(t))$ and $\mathrm{pc}'(q_1^j(t), q_2^j(t))$ are either in the same region or there is an index i such that $\mathrm{pc}'(q_1^i(t), q_2^i(t)) = \mathrm{pc}'(0,1)$ or $\mathrm{pc}'(q_1^i(t), q_2^i(t)) = \mathrm{pc}'(1,0)$. Therefore, there is a number $0 \leq k < N$ such that

$$(-t)^k \cdot q_1^i(t) \cdot q_2^j(t) - q_2^i(t) \cdot q_1^j(t) = 0,$$

or $q_1^i(t) = 0$ or $q_2^i(t) = 0$. One can suitably choose these $(n+1)$ pairs so that the latter equations cannot admit simultaneous solutions with $(-t)^N = 1$ in characteristic p except for finitely many primes p (that is, the resultant of the two polynomials is a nonzero integer). Consequently, one finds finitely many maximal ideals $\mathfrak{m}$ such that $\mathcal{P}(\Bbbk)$ can possibly contain an orbit Ω as above, then checks these candidates to conclude the theorem.

The cases $p = 2, 3$ are easier. We still have Equation (4.1), but the restrictions on the parameters on the right-hand side are slightly different. Then, one observes that $N < 18$ and checks $\mathcal{P}(\Bbbk)$ for all the maximal ideals in this range. □

As a side note, it is interesting to ask if there is exactly one orbit in $\mathcal{P}(\Bbbk)$ for any maximal ideal $\mathfrak{m}$ with $N \geq 7$ (not only those in Theorem 4.1). In order to answer this, one needs to better understand the specializations of the Burau representation at N-th roots of unity.

For the maximal ideals $\mathfrak{m}$ in Theorem 4.1, we denote $S(\mathcal{P}(\Bbbk))$ as follows:

$$\underbrace{H_{\mathrm{II}}(p;a) := S(\mathcal{P}(\Lambda/\langle p, t-a\rangle))}_{\text{for } p\geq 7} \qquad \underbrace{H_{\mathrm{II}}(p;\psi) := S(\mathcal{P}(\Lambda/\langle p, \psi\rangle))}_{\text{for } p\leq 5} \tag{4.3}$$

Remark 4.1. For $\mathfrak{m} = \langle p, \psi\rangle$, one has $|\Bbbk| = p^{\deg(\psi)}$ and $\deg(\psi)$ is the order of p modulo N. Thus, $|\mathcal{P}(\Bbbk)|$ can be found in terms of the pair (p, N) since $|\mathcal{P}(\Bbbk)| = |\Bbbk| + 1$ (Section 3.2.1).

We now consider the relevant rings R case-by-case.

Rings with $p > 3$ Suppose that $\Omega \subset \mathcal{P}(R)$ is of genus zero, and let d denote the degree of the induced covering $\Omega \to \mathcal{P}(\Bbbk)$. The number of regions in Ω is less than $\frac{|\Omega|}{N} + 2$ since all but two regions in $\mathcal{P}(R)$ have size $Np^k \geq N$ (Lemma 3.8). Taking into account the greatest possible numbers of complete monovalent vertices (2 black and 1 white by Lemma 3.6), one finds $-\frac{|\Omega|}{6} + \frac{|\Omega|}{N} + 2 + \frac{1}{2} + \frac{4}{3} > \chi(\Omega) > 0$. Then, a rearrangement of terms yields $d\cdot|\mathcal{P}(\Bbbk)|\cdot\left(\frac{1}{6} - \frac{1}{N}\right) < 2+\frac{1}{2}+\frac{4}{3} < 4$. Now, it is easy to deduce $d < N$ by checking all the relevant pairs (p, N) (see Remark 4.1). But, this implies that Ω contains no edge $\mathrm{pc}'(1, m)$ with $m \neq 0 \in \mathfrak{m}$, since such an edge is contained in a region of size $Np^k \geq N$ which entirely maps to the edge $\mathrm{pc}'(1, 0) \in \mathcal{P}(\Bbbk)$. Therefore, the preimage of $\mathrm{pc}'(1, 0) \in \mathcal{P}(\Bbbk)$ in Ω consists only of $\mathrm{pc}'(1, 0) \in \mathcal{P}(R)$, hence Ω is unique (if it exists) and $d = 1$. Moreover, the covering $\Omega \to \mathcal{P}(\Bbbk)$ is not only an isomorphism of Γ-sets, but weights also agree because of the following. (First, the monovalent vertices in $\mathcal{P}(R)$ and $\mathcal{P}(\Bbbk)$ are in bijection under the induced covering $\mathcal{P}(R) \to \mathcal{P}(\Bbbk)$ so that a complete vertex maps to a complete vertex and an incomplete vertex maps to an incomplete vertex (Lemma 3.6). Secondly, in both $\mathcal{P}(R)$ and $\mathcal{P}(\Bbbk)$, any region but $\{\mathrm{pc}'(1, 0)\}$ has weight 1, while this region has weight 1 or $\frac{1}{2}$ depending on N (Lemma 3.8)). Furthermore, we have $(-t)^N = 1 \in R^*$, thus $t^M = 1 \in R^*$. (Because, the bulk regions in $\mathcal{P}(R)$ are all of size $\mathrm{ord}(-t)$, while Ω contains such regions of size N). Hence, the specifications of $S(\Omega)$ and $S(\mathcal{P}(\Bbbk))$ are both $\mathbb{Z}_{2M}$-valued. All of this implies that $S(\Omega) = S(\mathcal{P}(\Bbbk))$. □

Rings with $p \leq 3$ and $(-t)^N \neq 1$ Let d denote the degree of the covering $\Omega \to \mathcal{P}(\Bbbk)$. Then, $d > 1$ since the bulk regions in Ω have size $\mathrm{ord}(-t) = N \cdot p^{k_0}$ where $k_0 \geq 1$. This implies that Ω contains an edge $\mathrm{pc}'(1, m)$ with $m \neq 0 \in \mathfrak{m}$, therefore $d \geq N$. On the other hand, the number of regions in Ω is less than $\frac{|\Omega|}{Np} + \frac{2d}{N} + 2$ because all but two regions in $\mathcal{P}(R)$ have size $Np^k \geq N$ and all but $2d$ edges are in bulk regions which have size $N \cdot p^{k_0} \geq Np$ (Lemma 3.8). Moreover, the contribution of the complete monovalent vertices to $\chi(\Omega)$ is at most $\frac{1}{2} + \frac{4}{3} < 2$ if $p = 2$ (1 white vertex and 2 black vertices), and it is at most 2 if $p = 3$ (3 black vertices and no white vertex since $N = 8$ according to Theorem 4.1) (Lemma 3.6). Overall, $\chi(\Omega) < -\frac{|\Omega|}{6} + \frac{|\Omega|}{Np} + \frac{2d}{N} + 2 + 2$. By a rearrangement, one gets $\chi(\Omega) < 4 - \frac{d}{N} \cdot \left(|\mathcal{P}(\Bbbk)| \cdot \left(\frac{N}{6} - \frac{1}{p}\right) - 2\right) \leq 0$ where the last inequality is easily verified for the pairs $(2, 7), (2, 15), (3, 8)$ of Theorem 4.1, using $d \geq N$ (see Remark 4.1).

Rings with $p \leq 3$ and $(-t)^N = 1$ Suppose $R \neq \Bbbk$, then $\mathfrak{m} \neq 0$. By replacing R with $R/\mathfrak{m}^2$ if necessary, we assume that $\mathfrak{m}^2 = 0$. Then, $R = \Lambda/\langle \mathfrak{m}^2, (-t)^N - 1\rangle$ since $\mathfrak{m}$ is the only non-trivial ideal in the latter ring. (One can see this as follows. In the ring $\mathbb{F}_p[t]/\psi^2 = \Lambda/\langle \mathfrak{m}^2, p\rangle$, one has $(-t)^N \neq 1$ because $\psi^2 \nmid (-t)^N - 1$ in $\mathbb{F}_p[t]$. Therefore in Λ, the $\Bbbk$-vector space $\mathfrak{m}/\mathfrak{m}^2$ has $\{p, (-t)^N - 1\}$ as a basis. This implies that in the ring $\Lambda/\langle \mathfrak{m}^2, (-t)^N - 1\rangle$, one has $p \neq 0$ and that $\mathfrak{m}$ is generated by any nonzero element.) Then $\mathcal{P}(R)$ contains no monovalent white vertex if $p = 2$ and no monovalent black vertex if $p = 3$ (If $p = 2$, one has $r^2 \in \Bbbk^*$ for any $r \in R^* = \mathfrak{m}^* \oplus \Bbbk^*$ but $\frac{1}{t} \notin \Bbbk^*$ since $-t \in \Bbbk^*$ and $-1 \neq 1 \in \mathfrak{m}^*$. If $p = 3$, one has $r^2 + r + 1 = 3 \neq 0$ for all $r \in \mathfrak{m}^*$.) (Lemmas 3.6, 3.4). Let d be the degree of the covering $\Omega \to \mathcal{P}(\Bbbk)$. Since $\mathcal{P}(\Bbbk)$ does contain a monovalent white vertex if $p = 2$ and a monovalent black vertex if $p = 3$, one has $d > 1$. As in the previous case, this implies $d \geq N$. However, the degree of the covering $\mathcal{P}(R) \to \mathcal{P}(\Bbbk)$ is $|\mathfrak{m}| = |\Bbbk|$ but $|\Bbbk| = N + 1$ for each of the pairs $(2, 7), (2, 15), (3, 8)$ (Remark 4.1). Therefore $\Omega = \mathcal{P}(R)$. Finally, it is easy to verify $\chi(\mathcal{P}(R)) \leq 0$ by direct computation. $\square$

4.2 The Case $N = 6$

In this section, we consider $\mathfrak{m}$ with $t^2 + t + 1 \subset \mathfrak{m}$ and $p > 3$. Note that $\mathfrak{m} = \langle p, t^2 + t + 1\rangle$ if $p \equiv 2 \pmod 3$, and $\mathfrak{m} = \langle p, t - a\rangle$ if $p \equiv 1 \pmod 3$ (here, a is one of the two solutions of $a^2 + a + 1 = 0$ in $\mathbb{F}_p$). For a third root of unity ω, consider the epimorphism $\pi\colon \Lambda \twoheadrightarrow \mathbb{Z}[\omega]$ defined by $t \mapsto \omega$. Then, $\mathfrak{m} \mapsto \pi(\mathfrak{m})$ establishes a bijection between the maximal ideals considered in this section and the primes in $\mathbb{Z}[\omega]$ except $\langle \omega - 1\rangle$ and $\langle 2\rangle$ (note that $\mathbb{Z}[\omega]$ is a principal ideal domain). Now observe that, as in the case $N \geq 7$, there is no genus-zero orbit in $\mathcal{C}(\Bbbk^2)$, hence we only consider rings.

Rings with $t^2 + t + 1 = 0$ First, the edge $\mathrm{pc}'(1, 0) = \mathrm{pc}(1, t) \in \mathcal{P}(R)$ is fixed by Γ (Equations 3.4, 3.6). Then, for the orbit $\Omega_0 := \{\mathrm{pc}'(1, 0)\}$, one has $S(\Omega_0) = H_\Pi(\mathtt{D}; 1) = H_\Sigma(2\mathtt{y}^\bullet; 1)$. Let $\Omega \subset \mathcal{P}(R)$ be any other orbit which maps to the orbit $\Omega_0' := \{\mathrm{pc}'(1, 0)\} \subset \mathcal{P}(\Bbbk)$. Then, all regions in Ω are of size 6 and there are no monovalent vertices in Ω, thus $\chi(\Omega) = 0$ (Lemmas 3.8, 3.6).

Due to the condition $t^2 + t + 1 = 0$, one can write the simple formulae on the left below, thus the corollaries on the right (3.7):

$$\begin{aligned} \mathrm{pc}'(r,1)\cdot Y &= \mathrm{pc}'(1-r,1), & \mathrm{pc}'(r,1)\cdot (YXYX^2) &= \mathrm{pc}'(r+(t+1),1),\\ \mathrm{pc}'(r,1)\cdot X &= \mathrm{pc}'(t^2(r-1),1). & \mathrm{pc}'(r,1)\cdot (YX^2YX) &= \mathrm{pc}'(r+t,1). \end{aligned}$$

This shows that all of the remaining edges, namely those of the form $\mathrm{pc}'(r, 1)$ for $r \in R$, are in the same orbit. Let $\Omega_1 := \{\mathrm{pc}'(r, 1) \mid r \in R\}$ and let n be the least such that $p^n = 0$ in R. Then, $S(\Omega_1) = H_\Pi(1; \pi(\mathfrak{m})^n)$. □

Rings with $t^2 + t + 1 \neq 0$ but $(t^2 + t + 1)^2 = 0$ Let $\Omega \subset \mathcal{P}(R)$ be an orbit which maps to $\Omega_1' := \{\mathrm{pc}'(r, 1) \mid r \in \Bbbk\} \subset \mathcal{P}(\Bbbk)$ and let d be the degree of the covering $\Omega \to \Omega_1'$. Then, only $d = \frac{|\Omega|}{|\Omega_1'|} \leq \frac{|\Omega|}{p}$ edges in Ω are outside the bulk regions which have size $\mathrm{ord}(-t) = 6p^{k_0} \geq 6p$ (Lemma 3.8). Moreover, there is at most 1 monovalent black vertex, at most 1 monovalent white vertex, and at most

1 region of size 1 in Ω (Lemmas 3.8, 3.6). Then, $\chi(\Omega) < -\frac{|\Omega|}{6} + \frac{|\Omega|}{6p} + \frac{d}{6} + \frac{5}{6} + \frac{2}{3} + \frac{1}{2} \leq \left(-\frac{1}{6} + \frac{1}{6p} + \frac{1}{6p}\right) \cdot |\Omega| + 2 \leq -p + 4 < 0$ since $|\Omega| \geq 6p$ and $p \geq 5$.

Now let $\Omega \subset \mathcal{P}(R)$ be an orbit which maps to $\Omega'_0 := \{\mathrm{pc}'(1,0)\} \subset \mathcal{P}(R')$ where $R' := R/(t^2+t+1)$. It turns out there is only one such orbit Ω_0, i.e. $\Omega_0 = \{\mathrm{pc}'(1,(t^2+t+1)r) \mid r \in R\}$. We omit the proof which is analogous to the previous case. Then, $S(\Omega_0) = H_{\mathrm{II}}(\mathtt{D}; \pi(\mathfrak{m})^n)$ where n is the least such that $p^n(t^2+t+1) = 0$ in R. □

Rings with $(t^2+t+1)^2 \neq 0$ Let $R' := R/(t^2+t+1)\mathfrak{m}$. Then R' is as in the previous case, hence it is enough to consider orbits $\Omega \subset \mathcal{P}(R)$ which map to $\Omega'_0 = \{\mathrm{pc}'(1,(t^2+t+1)r) \mid r \in R'\} \subset \mathcal{P}(R')$. Only those edges in Ω which map to $\mathrm{pc}'(1,0) \in \Omega'_0$ can be in regions of size less than or equal to 6. Let d be the degree of the covering $\Omega \to \Omega'_0$, then at most $d = \frac{|\Omega|}{|\Omega'_0|} \leq \frac{|\Omega|}{p}$ edges in Ω are in regions of size 1 or 6. Therefore, $\chi(\Omega) < -\frac{|\Omega|}{6} + \frac{|\Omega|}{6p} + \frac{d}{6} + \frac{2}{6} + \frac{2}{3} \leq \left(-\frac{1}{6} + \frac{1}{6p} + \frac{1}{6p}\right) \cdot |\Omega| + 1 \leq -p + 3 < 0$. □

4.3 The Case $3 \leq N \leq 5$

Rings with $N = 5$ **and** $t^5 + 1 = 0$ The edges $\mathrm{pc}'(0,1) \in \mathcal{P}(R)$ and $\mathrm{pc}'(1,0) \in \mathcal{P}(R)$ are in the same orbit Ω_0. Then, $S(\Omega_0) = H_{\mathrm{I}}(5\mathtt{a}')$ if $R = \Bbbk$ and $p = 2$, and $S(\Omega_0) = H_{\mathrm{I}}(5\mathtt{a})$ otherwise. For $p = 2$, the only orbit in $\mathcal{P}(\Bbbk)$ other than Ω_0 is Ω° with $S(\Omega^\circ) = H_{\mathrm{I}}(5^\circ)$. For $p = 3$, there are two orbits in $\mathcal{P}(\Bbbk)$ other than Ω_0, namely $\Omega^\bullet$ and $\tilde{\Omega}$ with $S(\Omega^\bullet) = H_{\mathrm{I}}(5^\bullet)$ and $S(\tilde{\Omega}) = H_{\mathrm{I}}(5\mathtt{ww})$. Unless $R = \Bbbk$ and $p \leq 3$, there is no complete monovalent vertex in $\mathcal{P}(R)$ and $\mathrm{ord}(t) = 10$. Then, $S(\Omega) = H_{\mathrm{I}}(5\mathtt{ww})$ for any $\Omega \neq \Omega_0$. □

Rings with $N = 5$ **and** $t^5 + 1 \neq 0$ Let $\Omega' \subset \mathcal{P}(\Bbbk)$ be the image of $\Omega \subset \mathcal{P}(R)$. If $\Omega' = \Omega_0$ (defined in the previous case), then $\frac{5}{6} \cdot |\Omega|$ edges in Ω are contained in bulk regions which have size $\mathrm{ord}(-t) = 5p^{k_0} \geq 5p$ (because $|\Omega_0| = 12$). Moreover, Ω contains an edge $\mathrm{pc}'(1,m)$ with $m \neq 0 \in \mathfrak{m}$, thus the degree of the covering $\Omega \to \Omega_0$ is at least 5, implying $|\Omega| \geq 5|\Omega_0| = 60$. Therefore $\chi(\Omega) \leq \big(-\frac{1}{6} + \frac{1}{5p}\cdot\frac{5}{6} + \frac{1}{5}\cdot\frac{1}{6}\big)\cdot|\Omega| + \frac{4}{5}\cdot 2 < \big(-\frac{1}{6} + \frac{1}{10}\cdot\frac{5}{6} + \frac{1}{5}\cdot\frac{1}{6}\big)\cdot 60 + 2 < 0$ since there is no monovalent vertex in Ω_0.

If $\Omega' \neq \Omega_0$, then all regions in Ω are bulk regions. If Ω contains no complete monovalent vertex, then $\chi(\Omega) \leq \big(-\frac{1}{6} + \frac{1}{5p}\big)\cdot|\Omega| < 0$. If Ω contains a complete monovalent vertex, then $p = 3$ and $\Omega' = \Omega^{\bullet}$ (defined in the previous case), because $\mathcal{P}(R)$ contains no complete monovalent white vertex even if $p = 2$, and no complete monovalent black vertex unless $p = 3$ (Lemma 3.6). Note that $|\Omega^{\bullet}| = 10$ and there is exactly 1 complete monovalent black vertex in $\Omega^{\bullet}$. Let ϵ be the number of complete monovalent black vertices in Ω, then the degree of the covering $\Omega \to \Omega^{\bullet}$ is at least ϵ, implying $|\Omega| \geq 10\epsilon$. Consequently, $\chi(\Omega) \leq \big(-\frac{1}{6} + \frac{1}{15}\big)\cdot|\Omega| + \frac{2}{3}\epsilon \leq -\frac{1}{3}\epsilon < 0$. □

Wheels with $N = 5$ and $(t^5 + 1)\cdot W = 0$ For $p = 2$, one has $S(\Omega) = H_{\mathrm{I}}(5\mathtt{ww}')$ for any $\Omega \subset \mathcal{C}(\Bbbk^2)$. Unless $p = 2$ and $W = \Bbbk^2$, one has $S(\Omega) = H_{\mathrm{I}}(5\mathtt{ww})$ for any $\Omega \subset \mathcal{C}(W)$. □

Wheels with $N = 5$ and $(t^5 + 1)\cdot W \neq 0$ First consider the case $p \neq 2$. Then, an edge $\mathrm{c}'(a, b) \in \mathcal{C}(W)$ is contained in a region of size 5 if and only if one of the two equalities $(t^5 + 1)\cdot a = 0$ and $(t^5 + 1)\cdot b = 0$ hold (Lemma 3.7). There is an epimorphism $\pi\colon W \twoheadrightarrow \Bbbk$ whose kernel contains the submodule

$$\{a \in W \mid (t^5 + 1)\cdot a = 0\},$$

because the image of this submodule under the epimorphism $W \twoheadrightarrow \Bbbk^2$ is a subspace of dimension at most 1. Consider the covering $\mathcal{C}(W) \to \mathcal{P}(\Bbbk)$ induced by π. Then, all regions of size 5 in Ω map to the edges $\mathrm{pc}'(0, 1)$ and $\mathrm{pc}'(1, 0) \in \mathcal{P}(\Bbbk)$. Consequently, the number of regions in Ω is bounded by $\frac{1}{5}\cdot\frac{|\Omega|}{6} + \frac{1}{5p}\cdot\frac{5|\Omega|}{6} < \frac{|\Omega|}{6}$ because the edges $\mathrm{pc}'(0, 1)$, $\mathrm{pc}'(1, 0)$ are in the same orbit $\Omega_0 \subset \mathcal{P}(\Bbbk)$ with $|\Omega_0| = 12$.

Now consider the case $p = 2$. The proof in this case is based on the same idea as the case $p \neq 2$, even though formulae might change. First, if an edge $\mathrm{c}'(a, b)$

is contained in a region of size 5, then one of the two equalities $(t^{10}-1)\cdot a = 0$ and $(t^{10}-1)\cdot b = 0$ hold. Thus, if $(t^{10}-1)\cdot W \neq 0$, the arguments in the case $p \neq 2$ literally apply to this case. On the other hand, if $(t^{10}-1)\cdot W = 0$, an edge is $\mathrm{c}'(a,b)$ is contained in a region of size 5 if and only if one of the two conditions $(t^5+1)\cdot a = 0$ and $2a = 0$, $(t^5-1)\cdot b = 0$ hold (Lemma 3.7). Now, either $2W \neq 0$ or $(t^5-1)\cdot W \neq 0$, thus let $\phi \in \{2, t^5-1\}$ such that $\phi \cdot W \neq 0$. As before, consider epimorphisms $\pi\colon W \twoheadrightarrow \Bbbk$ and $\pi'\colon W \twoheadrightarrow \Bbbk$ whose kernels contain the submodules

$$\{a \in W \mid (t^5+1)\cdot a = 0\} \qquad \text{and} \qquad \{a \in W \mid \phi \cdot a = 0\},$$

respectively. Any region of size 5 in Ω either maps to $\mathrm{pc}'(0,1) \in \mathcal{P}(\Bbbk)$ under the covering $\mathcal{C}(W) \to \mathcal{P}(\Bbbk)$ induced by π, or it maps to a particular one of $\mathrm{pc}'(0,1)$ and $\mathrm{pc}'(1,0) \in \mathcal{P}(\Bbbk)$ under the covering $\mathcal{C}(W) \to \mathcal{P}(\Bbbk)$ induced by π'. Finally, the number of regions in Ω is bounded by $\frac{1}{5}\cdot\frac{|\Omega|}{12} + \frac{1}{5}\cdot\frac{|\Omega|}{12} + \frac{1}{10}\cdot\frac{5|\Omega|}{6} < \frac{|\Omega|}{6}$. $\square$

Rings with $N = 4$ and $t^2+1 = 0$ The edges $\mathrm{pc}'(0,1) \in \mathcal{P}(R)$ and $\mathrm{pc}'(1,0) \in \mathcal{P}(R)$ are in the same orbit Ω_0 so that $S(\Omega_0) = H_\Sigma(1\mathtt{aa}; 2)$. For $p = 3$, the only orbit in $\mathcal{P}(\Bbbk)$ other than Ω_0 is $\Omega^\bullet$ with $S(\Omega^\bullet) = H_\Sigma(2\mathtt{d}^\bullet; 2)$. Unless $R = \Bbbk$ and $p = 3$, there is no complete monovalent vertex in $\mathcal{P}(R)$. Then, $S(\Omega) = H_\Sigma(2\mathtt{d}; 2)$ for any $\Omega \neq \Omega_0$. $\square$

Rings with $N = 4$ and $t^2+1 \neq 0$ Let $\Omega' \subset \mathcal{P}(\Bbbk)$ be the image of $\Omega \subset \mathcal{P}(R)$. If $\Omega' = \Omega_0$ (defined in the previous case), then $\frac{2}{3}\cdot|\Omega|$ edges in Ω are contained in bulk regions which are of size at least $4p$ (because $|\Omega_0| = 6$). If $p > 3$, then $\chi(\Omega) \leq \big(-\frac{1}{6} + \frac{1}{4p}\cdot\frac{2}{3} + \frac{1}{4}\cdot\frac{1}{3}\big)\cdot|\Omega| + \frac{3}{4}\cdot 2 < -\frac{p}{2} + \frac{5}{2} \leq 0$ since $|\Omega| \geq 6p$ and there is no monovalent vertex in Ω_0. If $p = 3$, let d be the degree of the covering $\Omega \to \Omega_0$. Then, $3 \mid d$ and $d \equiv 0, 1 \pmod 4$ since Ω contains some edges $\mathrm{pc}'(1,m)$ with $m \neq 0 \in \mathfrak{m}$. Hence one has $d \geq 9$, which implies $|\Omega| \geq 9|\Omega_0| = 54$. Then, $\chi(\Omega) \leq \big(-\frac{1}{6} + \frac{1}{12}\cdot\frac{2}{3} + \frac{1}{4}\cdot\frac{1}{3}\big)\cdot|\Omega| + \frac{3}{4}\cdot 2 \leq 0$.

If $\Omega' \neq \Omega_0$, then all regions in Ω are bulk regions. If Ω contains no complete monovalent vertex, then $\chi(\Omega) \leq \big(-\frac{1}{6} + \frac{1}{4p}\big)\cdot|\Omega| < 0$. If Ω contains complete monovalent vertices, then $p = 3$ and $\Omega' = \Omega^\bullet$ (defined in the previous case).

Note that $|\Omega^\bullet| = 4$, and there is 1 complete monovalent black vertex and 2 incomplete monovalent white vertices in $\Omega^\bullet$. Now if $|\Omega| \geq 24$, then $\chi(\Omega) \leq \left(-\frac{1}{6} + \frac{1}{12}\right) \cdot |\Omega| + \frac{2}{3} \cdot 3 \leq 0$. Hence suppose $|\Omega| < 24$ (necessarily $|\Omega| = 12$) and $\chi(\Omega) \geq 0$.

Since $|\Omega| = 12$, the degree of the covering $\Omega \to \Omega^\bullet$ is 3. This implies that Ω also contains incomplete monovalent white vertices, thus $\chi(\Omega) \geq \frac{1}{2}$ (Lemma 2.4). Let ϵ be the number of complete monovalent black vertices in Ω. Then, $\frac{1}{2} \leq \chi(\Omega) = \left(-\frac{1}{6} + \frac{1}{12}\right) \cdot |\Omega| + \frac{2}{3}\epsilon$ which shows that $\epsilon = 3$. Note that $t^6 + 1 = 0$ since the size of a bulk region in $\Omega \subset \mathcal{P}(R)$ is 12. Therefore, $\epsilon = 3$ requires $3 = 0$ and $t^4 - t^2 + 1 = 0$, thus $R = \mathbb{F}_3[t]/(t^2+1)^2$. In this case, there is a unique genus-zero orbit $\Omega \subset \mathcal{P}(R) = \mathcal{P}(\mathbb{F}_3[t]/(t^2+1)^2)$ for which $S(\Omega) = H_{\mathrm{I}}(12^\bullet)$. $\square$

Wheels with $N = 4$ If $(t^2+1) \cdot W = 0$, one has $S(\Omega) = H_\Sigma(\mathtt{2d}; 2)$ for any $\Omega \subset \mathcal{C}(W)$.

Hence suppose $(t^2+1) \cdot W \neq 0$. Note that an edge $\mathrm{c}'(a, b) \in \mathcal{C}(W)$ is contained in a region of size 4 if and only if one of the two equalities $(t^2+1) \cdot a = 0$ and $(t^2+1) \cdot b = 0$ hold. There is an epimorphism $\pi\colon W \twoheadrightarrow \Bbbk$ whose kernel contains the submodule

$$\{a \in W \mid (t^2+1) \cdot a = 0\},$$

because the image of this submodule under the epimorphism $W \twoheadrightarrow \Bbbk^2$ is a subspace of dimension at most 1. Consider the covering $\mathcal{C}(W) \to \mathcal{P}(\Bbbk)$ induced by π. Then, all regions of size 4 in Ω map to the edges $\mathrm{pc}'(0,1)$ and $\mathrm{pc}'(1,0) \in \mathcal{P}(\Bbbk)$. Consequently, the number of regions in Ω is bounded by $\frac{1}{4} \cdot \frac{|\Omega|}{3} + \frac{1}{4p} \cdot \frac{2|\Omega|}{3} < \frac{|\Omega|}{6}$ because the edges $\mathrm{pc}'(0,1)$, $\mathrm{pc}'(1,0)$ are in the same orbit $\Omega_0 \subset \mathcal{P}(\Bbbk)$ with $|\Omega_0| = 6$. $\square$

Rings with $N = 3$, $p \neq 2$ First suppose $t^3 + 1 = 0$. Then, the edges $\mathrm{pc}'(0,1) \in \mathcal{P}(R)$ and $\mathrm{pc}'(1,0) \in \mathcal{P}(R)$ are in distinct orbits Ω_0, Ω_1 so that $S(\Omega_0) = H_{\Pi}(1^\times; 2)$ and $S(\Omega_1) = H_{\Pi}(\mathtt{1ay}; 2)$. For any other orbit $\Omega \subset \mathcal{P}(R)$, one has $S(\Omega) = H_{\Pi}(\mathtt{3a}; 2)$.

Now suppose $t^3+1 \neq 0$. First note that $\mathcal{P}(R)$ contains no complete monovalent white vertex and it contains exactly 2 complete monovalent black vertices which map to 2 distinct vertices in $\mathcal{P}(\Bbbk)$. Let $\Omega' \subset \mathcal{P}(\Bbbk)$ be the image of $\Omega \subset \mathcal{P}(R)$. If $\Omega' = \Omega_0$ or $\Omega' = \Omega_1$ (defined above), then $\frac{3}{4} \cdot |\Omega|$ edges in Ω are contained in bulk regions which are of size at least $3p$ (because $|\Omega_0| = |\Omega_1| = 4$). If $p > 5$, then $\chi(\Omega) \leq \big(-\frac{1}{6} + \frac{1}{3p} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{1}{4}\big) \cdot |\Omega| + \frac{2}{3} + \frac{2}{3} < -\frac{p}{3} + \frac{7}{3} \leq 0$ since $|\Omega| \geq 4p$. If $p = 5$, let d be the degree of the covering $\Omega \to \Omega'$. Then, $5 \mid d$ and $d \equiv 0, 1 \pmod 3$ since Ω contains some edges $\mathrm{pc}'(1, m)$ or $\mathrm{pc}'(m, 1)$ with $m \neq 0 \in \mathfrak{m}$. Hence one has $d \geq 10$, which implies $|\Omega| \geq 10|\Omega'| = 40$. Then, $\chi(\Omega) \leq \big(-\frac{1}{6} + \frac{1}{15} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{1}{4}\big) \cdot |\Omega| + \frac{2}{3} + \frac{2}{3} \leq 0$. If $\Omega' \neq \Omega_0$ and $\Omega' \neq \Omega_1$, then all regions in Ω are bulk regions and there is no complete monovalent vertex in Ω. Then $\chi(\Omega) \leq \big(-\frac{1}{6} + \frac{1}{3p}\big) \cdot |\Omega| < 0$. $\square$

Wheels with $N = 3$, $p \neq 2$ If $(t^3 + 1) \cdot W = 0$, one has $S(\Omega) = H_{\Pi}(3\mathsf{a}; 2)$ for any $\Omega \subset \mathcal{C}(W)$.

Hence suppose $(t^3+1) \cdot W \neq 0$. Note that an edge $\mathrm{c}'(a, b) \in \mathcal{C}(W)$ is contained in a region of size 3 if and only if one of the two equalities $(t^3 + 1) \cdot a = 0$ and $(t^3 + 1) \cdot b = 0$ hold. There is an epimorphism $\pi \colon W \twoheadrightarrow \Bbbk$ whose kernel contains the submodule

$$\{a \in W \mid (t^3 + 1) \cdot a = 0\},$$

because the image of this submodule under the epimorphism $W \twoheadrightarrow \Bbbk^2$ is a subspace of dimension at most 1. Consider the covering $\mathcal{C}(W) \to \mathcal{P}(\Bbbk)$ induced by π. Then, all regions of size 3 in Ω map to the edges $\mathrm{pc}'(0, 1)$ and $\mathrm{pc}'(1, 0) \in \mathcal{P}(\Bbbk)$. Consequently, the number of regions in Ω is bounded by $\frac{1}{3} \cdot \frac{|\Omega|}{4} + \frac{1}{3p} \cdot \frac{3|\Omega|}{4} < \frac{|\Omega|}{6}$ because the edges $\mathrm{pc}'(0, 1)$, $\mathrm{pc}'(1, 0)$ are in distinct orbits $\Omega_0, \Omega_1 \subset \mathcal{P}(\Bbbk)$ with $|\Omega_0| = |\Omega_1| = 4$. $\square$

4.3.1 The case $N = 3$ and $p = 2$

Note that $\mathfrak{m} = \langle 2, t^2 - t + 1 \rangle$. For any ring R, let $\Omega_0 \subset \mathcal{P}(R)$ and $\Omega_1 \subset \mathcal{P}(R)$ denote the orbits of the edges $\mathrm{pc}'(1, 0)$ and $\mathrm{pc}'(0, 1)$, respectively. The orbits

$\Omega'_0 := \Omega_0 \subset \mathcal{P}(\Bbbk)$ and $\Omega'_1 := \Omega_1 \subset \mathcal{P}(\Bbbk)$ are distinct, and they are the only orbits in $\mathcal{P}(\Bbbk)$. We consider the orbits $\Omega \subset \mathcal{P}(R)$ in two classes according to whether Ω maps to Ω'_0 or Ω'_1.

4.3.1.1 Orbits $\Omega \subset \mathcal{P}(R)$ which map to $\Omega'_1 \subset \mathcal{P}(\Bbbk)$

If $R = \Bbbk$, then $\Omega_1 = \Omega'_1$ is the only orbit and $S(\Omega_1) = H_\Pi(1; 2)$. If $R \neq \Bbbk$ but $(t^2 - t + 1) = 0$, then $S(\Omega_1) = H_\Pi(1^\times; 2)$ and $S(\Omega) = H_\Pi(3\mathsf{a}; 2)$ for any $\Omega \neq \Omega_1$.

The case $(t^2 + t + 1)(t^2 - t + 1) = 0$ **but** $(t^2 - t + 1) \neq 0$ Let n be the least such that $(t^2 - t + 1)^n = 0$, i.e. $2^{n-1}(t^2 - t + 1) = 0$, let $R' := R/(t^2 - t + 1)$ and let $\Omega' \subset \mathcal{P}(R')$ be the image of Ω. We will show, by induction on n, that Ω is the only orbit in $\mathcal{P}(R)$ which maps to Ω'.

Proof. Let $\bar{R} := R/(t^2 - t + 1)^{n-1}$ and let $\bar{\Omega} \subset \mathcal{P}(\bar{R})$ be the image of Ω. Then, $\bar{\Omega}$ is the only orbit in $\mathcal{P}(\bar{R})$ which maps to Ω', which is obvious if $n = 2$ and follows from the induction hypothesis if $n \geq 3$. Thus, it is enough to show that there is only one orbit in $\mathcal{P}(R)$ which maps to $\bar{\Omega}$, which is then necessarily equal to Ω. Note that the covering $\mathcal{P}(R) \to \mathcal{P}(\bar{R})$ is 4-fold.

First suppose $\mathrm{pc}'(0, 1) \in \Omega'$, then $\bar{\Omega}$ contains the edge $\mathrm{pc}'(0, 1) \in \mathcal{P}(\bar{R})$ and the region $\bar{\mathfrak{a}} \subset \mathcal{P}(\bar{R})$ of size 3 which consists of the edges

$$\mathrm{pc}'(\bar{m}_0, 1),\ \ \mathrm{pc}'(-t\bar{m}_0, 1),\ \ \mathrm{pc}'(t^2\bar{m}_0, 1),$$

where $\bar{m}_0 := (t^2 - t + 1)^{n-2}$ (this must be independently verified in the case $n = 2$). Any region in $\mathcal{P}(R)$ which maps to $\bar{\mathfrak{a}} \subset \mathcal{P}(\bar{R})$ has size 6 since $(t^2 - t + 1)\bar{m}_0 \neq 0$ in R. Thus for any orbit $\tilde{\Omega} \subset \mathcal{P}(R)$ which maps to $\bar{\Omega}$, the degree of the covering $\tilde{\Omega} \to \bar{\Omega}$ is even, i.e. 2 or 4. On the other hand, the region $\mathfrak{a} \subset \mathcal{P}(R)$ of size 3 which consists of the edges

$$\mathrm{pc}'(m_0, 1),\ \ \mathrm{pc}'(-tm_0, 1),\ \ \mathrm{pc}'(t^2m_0, 1),$$

where $m_0 := (t^2 - t + 1)^{n-1}$, maps to $\{\mathrm{pc}'(0,1)\} \subset \bar{\Omega}$. Therefore, the orbit which contains the region $\mathfrak{a}$ is the only orbit in $\mathcal{P}(R)$ which maps to $\bar{\Omega}$.

Now suppose $\mathrm{pc}'(0,1) \notin \Omega'$, then Ω contains no region of size 1 and no monovalent vertex. Thus Ω contains exactly four regions of size 3 or no regions of size 3 (the latter possibility will be outruled). On the other hand, consider some $m \in \mathfrak{m}$ such that the edge $\mathrm{pc}'(m,1) \in \mathcal{P}(R)$ maps into Ω'. There is some $r \in R$ such that $(t^2 - t + 1)m = (t^2 - t + 1)^2 r$. Hence, for $\tilde{m} := m - (t^2 - t + 1)r$, the edge $\mathrm{pc}'(\tilde{m},1) \in \mathcal{P}(R)$ also maps into Ω' and is contained in a region of size 3.

Now observe that $\bar{\Omega}$ contains exactly four regions of size 3, which is obvious if $n = 2$ and follows from the above if $n \geq 3$. We will show that all of the regions of size 3 in $\mathcal{P}(R)$ which map into $\bar{\Omega}$ map to the same region of size 3 in $\bar{\Omega}$. (This will imply that there is exactly one orbit in $\mathcal{P}(R)$ which maps to $\bar{\Omega}$.) If $n = 2$, three of the four regions of size 3 in $\bar{\Omega}$ are bulk regions, thus the statement is clear. If $n \geq 3$, the four regions of size 3 in $\bar{\Omega}$ are necessarily of the form

$$\begin{aligned}
&\{\mathrm{pc}'(\bar{m},1), && \mathrm{pc}'(-t\bar{m},1), && \mathrm{pc}'(t^2\bar{m},1)\},\\
&\{\mathrm{pc}'(\bar{m}+\bar{m}_0,1), && \mathrm{pc}'(-t\bar{m}-t\bar{m}_0,1), && \mathrm{pc}'(t^2\bar{m}+t^2\bar{m}_0,1)\},\\
&\{\mathrm{pc}'(\bar{m}-t\bar{m}_0,1), && \mathrm{pc}'(-t\bar{m}+t^2\bar{m}_0,1), && \mathrm{pc}'(t^2\bar{m}+\bar{m}_0,1)\},\\
&\{\mathrm{pc}'(\bar{m}+t^2\bar{m}_0,1), && \mathrm{pc}'(-t\bar{m}+\bar{m}_0,1), && \mathrm{pc}'(t^2\bar{m}-t\bar{m}_0,1)\}.
\end{aligned}$$

for some $\bar{m} \in \bar{R}$, where $\bar{m}_0 := (t^2 - t + 1)^{n-2}$. Thus the statement easily follows from the fact that $(t^2 - t + 1)\bar{m}_0 \neq 0$ in R. □

Consequently, if $(t^2+t+1) = 0$, the only orbit is Ω_1 so that $S(\Omega_1) = H_\Pi(1; 2^n)$. If $(t^2 + t + 1) \neq 0$, one has $S(\Omega_1) = H_\Pi(1^\times; 2^n)$ and $S(\Omega) = H_\Pi(3\mathsf{a}; 2^n)$ for any $\Omega \neq \Omega_1$. □

The case $(t^2 + t + 1)(t^2 - t + 1) \neq 0$ Note that $\frac{3}{4}|\Omega|$ edges in Ω are in bulk regions, which have size at least 12. If Ω contains no region of size 1 and no monovalent black vertex, then $\chi(\Omega) \leq -\frac{|\Omega|}{6} + \frac{1}{12} \cdot \frac{3|\Omega|}{4} + \frac{1}{3} \cdot \frac{|\Omega|}{4} < 0$.

Suppose Ω contains a region of size 1 or a monovalent black vertex. Let

$R' := R/(t^2-t+1)(t^2+t+1)$ and $R'' := R/(t^2-t+1)$, let $\Omega' \subset \mathcal{P}(R')$ and $\Omega'' \subset \mathcal{P}(R'')$ be the images of Ω, and let d, d' be the degrees of the coverings $\Omega \to \Omega'$ and $\Omega' \to \Omega''$. Considering the size of a bulk region in Ω, Ω' and Ω'', one observes that d, d' are even. On the other hand, there is exactly one region of size 1 or exactly one monovalent black vertex in Ω, Ω' and Ω'', thus it impossible that $d = 2$ or $d' = 2$. Therefore, $|\Omega| = d \cdot d' \cdot |\Omega''| \geq 4 \cdot 4 \cdot 4 = 64$. Consequently, $\chi(\Omega) \leq -\frac{|\Omega|}{6} + \frac{1}{12} \cdot \frac{3|\Omega|}{4} + \frac{1}{3} \cdot \frac{|\Omega|}{4} + \frac{2}{3} + \frac{2}{3} = \frac{4}{3} - \frac{|\Omega|}{48} \leq 0$. $\square$

4.3.1.2 Orbits $\Omega \subset \mathcal{P}(R)$ which map to $\Omega'_0 \subset \mathcal{P}(\Bbbk)$

Note that the only region of size 1 is $\{\mathrm{pc}'(1,0)\} \subset \Omega_0$.

The case $(t^2+t+1) = 0$ First note that $\Omega_0 = \{\mathrm{pc}'(1,0)\}$ contains the only monovalent black vertex in $\mathcal{P}(R)$ which maps to $\Omega'_0 \subset \mathcal{P}(\Bbbk)$. Let n be the least such that $2^n = 0$, then $R = \mathbb{Z}_{2^n}[t]/(t^2+t+1)$. If $n = 1$, i.e. $R = \Bbbk$, the only orbit is $\Omega_0 = \Omega'_0$ so that $S(\Omega_0) = H_\Pi(\circ; 1) = H_\Sigma(1^{\times\bullet}; 1)$. If $n > 1$, then $S(\Omega_0) = H_\Pi(\mathtt{D}; 1) = H_\Sigma(2\mathtt{y}^\bullet; 1)$. If $n = 2$, the only orbit other than $\Omega''_0 := \Omega_0$ is Ω° with $S(\Omega^\circ) = H_\Pi(\circ; 2)$.

If $n > 2$, let $R' := R/4 = \mathbb{Z}_4[t]/(t^2+t+1)$. The only complete monovalent white vertex in $\mathcal{P}(R)$ is $\mathrm{pc}(1,-t)$ which maps into $\Omega^\circ \subset \mathcal{P}(R')$. If Ω maps to $\Omega''_0 \subset \mathcal{P}(R')$ and $\Omega \neq \Omega_0$, then $\chi(\Omega) = 0$ (because there is no complete monovalent vertex in Ω, each region in Ω has size 3 or 6, and the regions of size 3 have weight $\frac{1}{2}$ since $-1 \notin \langle t \rangle$). If Ω maps to Ω°, we will show, by induction on n, that Ω is the only orbit in $\mathcal{P}(R)$ which maps to Ω°.

Proof. Let $\bar{R} := R/2^{n-1}$ and let $\bar{\Omega} \subset \mathcal{P}(\bar{R})$ be the image of Ω. Then, $\bar{\Omega}$ is the only orbit in $\mathcal{P}(\bar{R})$ which maps to Ω', which is obvious if $n = 3$ and follows from the induction hypothesis if $n \geq 4$. Note that the covering $\mathcal{P}(R) \to \mathcal{P}(\bar{R})$ is 4-fold. All regions in $\mathcal{P}(R)$ which map to $\Omega^\circ \subset \mathcal{P}(R')$ have size 6, thus Ω contains exactly four monovalent white vertices or no monovalent white vertices (the latter possibility will be outruled). But there are only four monovalent vertices in $\mathcal{P}(R)$

which map to $\Omega^\circ \subset \mathcal{P}(R')$, namely

$$\underbrace{\mathrm{pc}(1,-t)}_{\text{complete}}, \qquad \underbrace{\mathrm{pc}(1,-t+2^{n-1}),\ \ \mathrm{pc}(1,-t+2^{n-1}t),\ \ \mathrm{pc}(1,-t+2^{n-1}t^2)}_{\text{incomplete}}.$$

Hence, these vertices are in the same orbit. But they all map to the same vertex in $\bar{\Omega}$, thus the orbit which contains these vertices is the only orbit in $\mathcal{P}(R)$ which maps to $\bar{\Omega}$, hence it is equal to Ω. $\square$

Consequently, if Ω maps to $\Omega^\circ \subset \mathcal{P}(R')$, one has $S(\Omega) = H_\Pi(\circ; 2^{n-1})$. $\square$

The case $(t^2+t+1)(t^2-t+1) = 0$ **but** $(t^2+t+1) \neq 0$ There is no complete monovalent white vertex in $\mathcal{P}(R)$ since $\mathrm{ord}(t) = 6$ (Lemma 3.6). On the other hand, Ω_0 contains the unique region of size 1 and the unique monovalent black vertex which map to $\Omega_0' \subset \mathcal{P}(\Bbbk)$. If $-1 \notin \langle t \rangle$, the regions of size 1 and 3 have weight $\frac{1}{2}$, thus Ω_0 is the only genus-zero orbit so that $S(\Omega_0) = H_\Pi(\mathtt{D}; 2)$.

Hence suppose that $-1 \in \langle t \rangle$. Since $\mathrm{ord}(t) = 6$, one has either $2 = 0$, i.e. $R = \mathbb{F}_2[t]/(t^2-t+1)^2$, or $(t^3+1) = 0$, i.e. $(t^2-t+1) = 0$. In the former case, Ω_0 is the only orbit so that $S(\Omega_0) = H_\Pi(1\mathtt{b}; 2)$. In the latter case, $S(\Omega_0) = H_\Pi(1\mathtt{ay}; 2)$ and $S(\Omega) = H_\Pi(3\mathtt{a}; 2)$ for any $\Omega \neq \Omega_0$. $\square$

The case $(t^2+t+1)^2(t^2-t+1) = 0$ **but** $(t^2+t+1)(t^2-t+1) \neq 0$ Let $R' := R/(t^2+t+1)(t^2-t+1)$, and let n be the least such that $(t^2+t+1)(t^2-t+1)^n = 0$, i.e. $2^{n-1}(t^2+t+1)(t^2-t+1) = 0$. Note that $t^{6k} = 1 + k(t^6-1)$ for any integer k, hence $\mathrm{ord}(t) = 3 \cdot 2^n$. We will show, by induction on n, that Ω_0 is the only orbit in $\mathcal{P}(R)$ which maps to $\Omega_0'' := \Omega_0 \subset \mathcal{P}(R')$. Consequently, Ω_0 contains the unique region of size 1 and the unique monovalent black vertex which map to $\Omega_0' \subset \mathcal{P}(\Bbbk)$.

Proof. Let $\bar{R} := R/(t^2+t+1)(t^2-t+1)^{n-1}$ and let $\bar{\Omega}_0 := \Omega_0 \subset \mathcal{P}(\bar{R})$. Then, $\bar{\Omega}_0$ is the only orbit in $\mathcal{P}(\bar{R})$ which maps to Ω_0'', which is obvious if $n = 2$ and follows from the induction hypothesis if $n \geq 3$. As a result, $\bar{\Omega}_0$ contains the region $\bar{\mathfrak{a}} \subset \mathcal{P}(\bar{R})$ of size 3 which consists of the edges

$$\mathrm{pc}'(1, \bar{m}_0), \;\; \mathrm{pc}'(1, -t\bar{m}_0), \;\; \mathrm{pc}'(1, t^2\bar{m}_0),$$

where $\bar{m}_0 := (t^2+t+1)(t^2-t+1)^{n-2}$. Any region in $\mathcal{P}(R)$ which maps to $\bar{\mathfrak{a}} \subset \mathcal{P}(\bar{R})$ has size 6 since $(t^2-t+1)\bar{m}_0 \neq 0$ in R. Thus for any orbit $\tilde{\Omega} \subset \mathcal{P}(R)$ which maps to $\bar{\Omega}$, the degree of the covering $\tilde{\Omega} \to \bar{\Omega}$ is even, i.e. 2 or 4 (note that the covering $\mathcal{P}(R) \to \mathcal{P}(\bar{R})$ is 4-fold). On the other hand, the region $\mathfrak{a} \subset \mathcal{P}(R)$ of size 3 which consists of the edges

$$\mathrm{pc}'(1, m_0), \;\; \mathrm{pc}'(1, -tm_0), \;\; \mathrm{pc}'(1, t^2m_0),$$

where $m_0 := (t^2+t+1)(t^2-t+1)^{n-1}$, maps to $\{\mathrm{pc}'(1,0)\} \subset \bar{\Omega}$. Therefore, the orbit which contains the region $\mathfrak{a}$ is the only orbit in $\mathcal{P}(R)$ which maps to $\bar{\Omega}$, thus it is equal to Ω_0. $\square$

As in the previous case, there is no complete monovalent white vertex in $\mathcal{P}(R)$. If $-1 \notin \langle t \rangle$, the regions of size 1 and 3 have weight $\frac{1}{2}$, thus Ω_0 is the only genus-zero orbit so that $S(\Omega_0) = H_\Pi(\mathtt{D}; 2^n)$. If $-1 \in \langle t \rangle$, then

$$R = \mathbb{F}_2[t]/(t^2-t+1)^3 \quad \text{or} \quad R = \mathbb{Z}_4[t]/\langle 2(t^2-t+1), (t^4-t^2+1)\rangle. \tag{4.4}$$

(Because if $-1 \in \langle t \rangle$, one has $2 = 0$ or $t^{3 \cdot 2^{n-1}} + 1 = 0$. Moreover, one must have $n = 2$ in the latter case, because assuming $n \geq 3$, one finds the following contradiction: $t^{3\cdot 2^{n-1}} = 1 + 2^{n-2}(t^6-1)$, thus $2 = (1 + 2^{n-3}(t^6-1))^{-1}(t^{3\cdot 2^{n-1}}+1) = 0$ but $2^{n-2}(t^2+t+1)(t^2-t+1) \neq 0$.) Consequently, Ω_0 is the only orbit. In the former case, $S(\Omega_0) = H_\Pi(1\mathtt{ba}; 4)$ and in the latter case, $S(\Omega_0) = H_\Pi(1\mathtt{bb}; 4)$. $\square$

The case $(t^2+t+1)^2(t^2-t+1) \neq 0$ Let $R' := R/(t^2+t+1)^2(t^2-t+1)$. It is enough to consider the case that Ω maps to $\Omega_0 \subset \mathcal{P}(R')$, since the latter is the only orbit of genus zero in $\mathcal{P}(R')$. Those regions in Ω which map to the regions of size 6 in Ω_0 have size $12 \cdot 2^k$ for some $k \geq 0$. In particular, the degree d of the covering $\Omega \to \Omega_0$ is even. But it is impossible that $d = 2$ because there is at most one region of size 1 in Ω_0, thus $|\Omega| = d \cdot |\Omega_0| \geq 4 \cdot 16 = 64$. Consequently, $\chi(\Omega) \leq -\frac{|\Omega|}{6} + \frac{1}{12} \cdot \frac{3|\Omega|}{4} + \frac{1}{3} \cdot \frac{|\Omega|}{4} + \frac{2}{3} + \frac{2}{3} = \frac{4}{3} - \frac{|\Omega|}{48} \leq 0$. $\square$

4.3.1.3 Wheels

For each positive integer n, let $R_n := \mathbb{Z}_{2^n}[t]/(t^2+t+1)$. Moreover, let

$$\begin{aligned}&R^0 := \Bbbk, \qquad R^{\mathtt{b}} := \mathbb{F}_2[t]/(t^2-t+1)^2, \qquad R^{\mathtt{ba}} := \mathbb{F}_2[t]/(t^2-t+1)^3,\\ &R^{\mathtt{bb}} := \mathbb{Z}_4[t]/\langle 2(t^2-t+1), (t^4-t^2+1)\rangle.\end{aligned}$$

In the rest, $\mathtt{s}$ is a placeholder superscript, i.e. $\mathtt{s} \in \{0, \mathtt{b}, \mathtt{ba}, \mathtt{bb}\}$.

Consider the wheels

$$W_n^{\mathtt{s}} := R^{\mathtt{s}} \cdot e_1 \oplus R_n \cdot e_2.$$

For each pair $(\mathtt{s}, n)$, let $\Omega_* \subset \mathcal{C}(W_n^{\mathtt{s}})$ be the orbit of the edge $\mathrm{c}'(e_1, e_2)$. For any edge $\mathrm{c}'(a, b) \in \Omega_*$, one has $a = \rho_1 \cdot (e_1 + \phi \cdot e_2)$ and $b = \rho_2 \cdot (\mu \cdot e_1 + e_2)$ for some $\phi \in \Lambda$, $\mu \in \mathfrak{m} \subset \Lambda$ and $\rho_1, \rho_2 \in \Lambda \smallsetminus \mathfrak{m}$ (3.7). Hence, $(t^2+t+1)(t^2-t+1) \cdot b = 0$ for any $\mathrm{c}'(a, b) \in \Omega_*$, thus all regions in Ω_* have size 3 or 6. Moreover, the region of $\mathrm{c}'(e_1, e_2) \in \Omega_*$ has size 3, thus Ω_* is of genus zero.

Let $\pi_1 \colon W_n^{\mathtt{s}} \twoheadrightarrow R^{\mathtt{s}}$ and $\pi_2 \colon W_n^{\mathtt{s}} \twoheadrightarrow R_n$ denote the obvious epimorphisms, i.e. those which are defined by $e_1 \mapsto 1, e_2 \mapsto 0$ and $e_1 \mapsto 0, e_2 \mapsto 1$, respectively. Then, the coverings $\mathcal{C}(W_n^{\mathtt{s}}) \to \mathcal{P}(R^{\mathtt{s}})$ and $\mathcal{C}(W_n^{\mathtt{s}}) \to \mathcal{P}(R_n)$ induced by π_1 and π_2 map $\Omega_* \subset \mathcal{C}(W_n^{\mathtt{s}})$ to $\Omega_0 \subset \mathcal{P}(R^{\mathtt{s}})$ and $\Omega_1 \subset \mathcal{P}(R_n)$ (note that Ω_0, Ω_1 have been defined for $\mathfrak{m} = \langle 2, t^2-t+1\rangle$). There may be regions of size greater than 3 in $\Omega_0 \subset \mathcal{P}(R^{\mathtt{s}})$ (this happens for $\mathtt{s} \in \{\mathtt{ba}, \mathtt{bb}\}$). However if $n > 1$, an edge $\mathrm{c}'(a, b) \in \Omega_*$ which maps into a region of size 1 or 3 in $\Omega_1 \subset \mathcal{P}(R_n)$ necessarily maps into a region of size 1 or 3 in $\Omega_0 \subset \mathcal{P}(R^{\mathtt{s}})$.

Consider an edge $\mathrm{c}'(a, b) = \mathrm{c}'(\rho_1(e_1 + \phi \cdot e_2), \rho_2(\mu \cdot e_1 + e_2)) \in \Omega_*$ which maps into regions of size 1 or 3 in $\Omega_0 \subset \mathcal{P}(R^{\mathtt{s}})$ and $\Omega_1 \subset \mathcal{P}(R_n)$, i.e. the edges $\mathrm{pc}'(1, \rho_1^{-1}\rho_2\mu) \in \Omega_0$ and $\mathrm{pc}'(\rho_1\rho_2^{-1}\phi, 1) \in \Omega_1$ are in regions of size 1 or 3. This implies the equations $(t^2-t+1)\mu = 0$ in $R^{\mathtt{s}}$ and $(t^2-t+1)\phi = 0$ in R_n, which further imply that $\mathfrak{m}\mu = 0$ in $R^{\mathtt{s}}$ and $\mathfrak{m}\phi = 0$ in R_n. Consequently, $(t^2+t+1)\cdot b = 0$ and at least one of the two equalities $2\cdot a = 0$, $(t^6+1)\cdot a = 0$ holds (both equalities hold if $\mathtt{s} \in \{0, \mathtt{b}\}$, the former holds if $\mathtt{s} = \mathtt{ba}$, and the latter holds if $\mathtt{s} = \mathtt{bb}$). In any case, the $\mathrm{c}'(a, b) \in \Omega_*$ lies in a region of size 3.

The observations above show that, except the two cases $(\mathtt{s}, n) = (\mathtt{ba}, 1), (\mathtt{bb}, 1)$, the covering $\Omega_* \to \Omega_1$ is ramified only at the unique region $\{\mathrm{pc}'(0,1)\}$ of size 1 and at the unique monovalent black vertex. In particular, the degree of the covering $\Omega_* \to \Omega_1$ is 3. Consequently, $S(\Omega_*)$ is given in the following for $n > 1$.

$$\begin{array}{lclllll} W & = & W_n^0 & W_n^{\mathtt{b}} & W_n^{\mathtt{ba}} & W_n^{\mathtt{bb}} \\ S(\Omega_*) & = & H_\Pi(3; 2^n) & H_\Pi(3\mathtt{b}; 2^n) & H_\Pi(3\mathtt{ba}; 2^n) & H_\Pi(3\mathtt{bb}; 2^n) \end{array}$$

The information given below for the remaining wheels can be directly verified.

$$\begin{array}{lclllll} W & = & W_1^0 & W_1^{\mathtt{b}} & W_1^{\mathtt{ba}} & W_1^{\mathtt{bb}} \\ S(\Omega_*) & = & H_\Pi(3; 2) & H_\Pi(3\mathtt{b}; 2) & H_\Pi(3\mathtt{ba}; 4) & H_\Pi(3\mathtt{bb}; 4) \end{array}$$

From now on, let $\mathtt{s} = 0, \mathtt{b}$. In writing the equalities $a = \rho_1 \cdot (e_1 + \phi \cdot e_2)$ and $b = \rho_2 \cdot (\mu \cdot e_1 + e_2)$ for an edge $\mathrm{c}'(a, b) \in \Omega_* \subset \mathcal{C}(W_n^{\mathtt{s}})$, one can take $\rho_1 \in \{1, t+1\}$ (in fact, one can take $\rho_1 = 1$ if $\mathtt{s} = 0$). Consider three edges in Ω_* which map to the same edge in $\Omega_1 \subset \mathcal{P}(R_n)$, then at least two of them share the same value of ρ_1. For these two edges $\mathrm{c}'(a_1, b_1)$ and $\mathrm{c}'(a_2, b_2)$, let

$$\begin{aligned} a_1 &= \rho_1 \cdot (e_1 + \phi_1 \cdot e_2), & b_1 &= \rho_{21} \cdot (\mu_1 \cdot e_1 + e_2), \\ a_2 &= \rho_1 \cdot (e_1 + \phi_2 \cdot e_2), & b_2 &= \rho_{22} \cdot (\mu_2 \cdot e_1 + e_2). \end{aligned}$$

Since these two edges map to the same edge in $\Omega_1 \subset \mathcal{P}(R_n)$, one has $\rho_1 \rho_{21}^{-1} \phi_1 = \rho_1 \rho_{22}^{-1} \phi_2$ in R_n, hence $\rho_{21}^{-1} \phi_1 = \rho_{22}^{-1} \phi_2$. On the other hand, these edges necessarily map to the same edge in $\Omega_0 \subset \mathcal{P}(R^{\mathtt{s}})$, hence $\rho_{21} \mu_1 = \rho_{22} \mu_2$ in $R^{\mathtt{s}}$. Consider the twisting automorphism $W_n^{\mathtt{s}} \to W_n^{\mathtt{s}}$ defined by $e_1 \mapsto e_1$, $e_2 \mapsto \rho_{22}^{-1} \rho_{21} \cdot e_2$. This automorphism maps $a_2 \mapsto a_1$ and $b_2 \mapsto b_1$, hence induces a non-trivial deck transformation of the 3-fold Galois covering $\Omega_* \to \Omega_1$. This deck transformation must take the edge $\mathrm{c}'(e_1, e_2) \in \Omega_*$ to another edge in the same region, thus $\rho_{22}^{-1} \rho_{21} \in \{t, t^2\}$ in R_n. Consequently, any three edges in Ω_* which map to the same edge in Ω_1 are in the form

$$\begin{aligned} &\mathrm{c}'(\alpha_1 \cdot e_1 + \alpha_2 \cdot e_2, && \beta_1 \cdot e_1 + \beta_2 \cdot e_2), \\ &\mathrm{c}'(\alpha_1 \cdot e_1 + t\alpha_2 \cdot e_2, && \beta_1 \cdot e_1 + t\beta_2 \cdot e_2), \\ &\mathrm{c}'(\alpha_1 \cdot e_1 + t^2\alpha_2 \cdot e_2, && \beta_1 \cdot e_1 + t^2\beta_2 \cdot e_2), \end{aligned} \tag{4.5}$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Lambda$ (in fact, $\alpha_1, \beta_2 \in \Lambda \smallsetminus \mathfrak{m}$ and $\beta_1 \in \mathfrak{m}$). In particular, these three edges map into the unique region of size 3 in Ω_1 if and only if $\{\alpha_2, t\alpha_2, t^2\alpha_2\} = \{2^{n-1}, 2^{n-1}t, 2^{n-1}t^2\}$ in R_n. We now return to the main theorem.

The case $(t^2+t+1)(t^2-t+1) \cdot W = 0$ All regions in $\mathcal{C}(W)$ have size 3 or 6, thus an orbit $\Omega \subset \mathcal{C}(W)$ is of genus zero if and only if Ω contains a region of size 3. An edge $\mathrm{c}'(a_1, a_2) \in \mathcal{C}(W)$ is in a region of size 3 if and only if either $(t^2 - t + 1) \cdot a_1 = 0$, or $2 \cdot a_1 = 0$, $(t^2 + t + 1) \cdot a_2 = 0$ (Lemma 3.7).

Suppose that there is an edge $\mathrm{c}'(a_1, a_2) \in \Omega$ with $2 \cdot a_1 = 0$ and $(t^2 + t + 1) \cdot a_2 = 0$. Let n be the least such that $2^n \cdot a_2 = 0$. Then, consider the epimorphism $W_n^{\mathfrak{b}} \twoheadrightarrow W$ defined by $e_1 \mapsto a_1$, $e_2 \mapsto a_2$. This epimorphism is either an isomorphism or its kernel is generated by $(t^2 - t + 1) \cdot e_1 + 2^{n-1}\sigma \cdot e_2$ for some $\sigma \in \Lambda$. In the former case, $\Omega = \Omega_* \subset \mathcal{C}(W_n^{\mathfrak{b}})$. In the latter case, if $\sigma \in \mathfrak{m}$, then $\Omega = \Omega_* \subset \mathcal{C}(W_n^0)$. But if $\sigma \in \Lambda \smallsetminus \mathfrak{m}$, one necessarily has $n \geq 2$ and one can take $\sigma \in \{1, t, t^2\}$. Hence, there is an isomorphism

$$W_n' := W_n^{\mathfrak{b}}/((t^2 - t + 1) \cdot e_1 + 2^{n-1} \cdot e_2) \to W, \qquad e_1 \mapsto a_1,\ e_2 \mapsto \sigma \cdot a_2$$

which identifies the orbit Ω_*' of the edge $\mathrm{c}'(e_1, e_2) \in \mathcal{C}(W_n')$ with $\Omega \subset \mathcal{C}(W)$.

Consider the epimorphism $W_n' \twoheadrightarrow W_{n-1}^0$, whose kernel is generated by $(t^2 - t + 1) \cdot e_1 = 2^{n-1} \cdot e_2$, then the induced covering $\mathcal{C}(W_n') \to \mathcal{C}(W_{n-1}^0)$ maps Ω_*' to Ω_*. Any edge $\mathrm{c}'(\tilde{e}_1, \tilde{e}_2) \in \mathcal{C}(W_n')$ which maps to $\mathrm{c}'(e_1, e_2) \in \mathcal{C}(W_{n-1}^0)$ satisfies $2 \cdot \tilde{e}_1 = 0$ and $(t^2 + t + 1) \cdot \tilde{e}_2 = 0$, thus any region in $\mathcal{C}(W_n')$ which maps to the region $\mathfrak{a}_0$ of $\mathrm{c}'(e_1, e_2) \in \mathcal{C}(W_{n-1}^0)$ has size 3. There are three other regions $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ of size 3 in Ω_*, which contain three edges as in (4.5). Consider an edge $\mathrm{c}'(a, b) \in \mathcal{C}(W_n')$ which maps to any of these three edges, then $a \in W_n'$ maps to $\alpha \cdot (e_1 + 2^{n-2}\sigma \cdot e_2) \in W_{n-1}^0$ for some $\alpha \in \Lambda \smallsetminus \mathfrak{m}$ and $\sigma \in \{1, t, t^2\}$. Thus one has $2 \cdot a \neq 0$. Moreover, $(t^2 - t + 1) \cdot a = 0$ if and only if $\sigma = t^2$, which holds for exactly one of the three edges, which lies, w.l.o.g., in $\mathfrak{a}_1$. Thus any region in $\mathcal{C}(W_n')$ which maps to $\mathfrak{a}_1$ has size 3 and any region which maps to $\mathfrak{a}_2$ or $\mathfrak{a}_3$ has size 6. Hence, any orbit in $\mathcal{C}(W_n')$ which maps to $\Omega_* \subset \mathcal{C}(W_{n-1}^0)$ is a double covering of Ω_* ramified at $\mathfrak{a}_2$ and $\mathfrak{a}_3$. Consequently, $S(\Omega_*') = H_{\Pi}(3/2; 2^n)$.

Now suppose that there is no edge $\mathrm{c}'(a_1, a_2) \in \Omega$ with $2 \cdot a_1 = 0$ and $(t^2 + t + 1) \cdot a_2 = 0$, so that an edge $\mathrm{c}'(a, b) \in \Omega$ is in a region of size 3 if and only if $(t^2 - t + 1) \cdot a = 0$. For a certain edge $\mathrm{c}'(a, b) \in \Omega$ in a region of size 3, let n be the least such that $(t^2 - t + 1)^n \cdot b = 0$, i.e. $2^{n-1}(t^2 - t + 1) \cdot b = 0$, then $(t^2 - t + 1) \cdot W \cong R_{n-1}$. For a sufficiently large integer k, let

$$R(k) := \mathbb{Z}_{2^k}[t]/(t^2 - t + 1)\langle 2^{n-1}, t^2 + t + 1\rangle,$$
$$W(k) := \big(\mathbb{Z}_{2^k}[t]/(t^2 - t + 1)\big) \cdot e_1 \oplus R(k) \cdot e_2.$$

The epimorphism $W(k) \twoheadrightarrow W$ defined by $e_1 \mapsto a$, $e_2 \mapsto b$ restricts to an isomorphism $(t^2 - t + 1) \cdot W(k) \cong (t^2 - t + 1) \cdot W$. As a result, the induced covering $\mathcal{C}(W(k)) \to \mathcal{C}(W)$ is unramified over $\Omega \subset \mathcal{C}(W)$. (Because if an edge $\mathrm{c}'(\tilde{a}, \tilde{b}) \in \mathcal{C}(W(k))$ maps into a region of size 3 in Ω, then $(t^2 - t + 1) \cdot \tilde{a} \mapsto 0 \in W$, then $(t^2 - t + 1) \cdot \tilde{a} = 0$). Thus, any orbit $\tilde{\Omega} \subset \mathcal{C}(W(k))$ which maps to Ω maps isomorphically onto Ω so that $S(\tilde{\Omega}) = S(\Omega)$. Consider the covering $\mathcal{C}(W(k)) \to \mathcal{P}(R(k))$ induced by the obvious epimorphism $W(k) \twoheadrightarrow R(k)$ defined by $e_1 \mapsto 0$, $e_2 \mapsto 1$, and let Ω' be the image of $\tilde{\Omega}$. Then, $\Omega' \subset \mathcal{P}(R(k))$ is an orbit which maps to $\Omega'_1 \subset \mathcal{P}(\Bbbk)$ (see Section 4.3.1.1). Thus, unless $\Omega' = \Omega_1$, the covering $\tilde{\Omega} \to \Omega'$ is also unramified so that $S(\tilde{\Omega}) = S(\Omega')$. For sufficiently large k, one can guarantee that $\Omega' \neq \Omega_1$ by varying $\tilde{\Omega}$ over all possibilities. Consequently, $S(\Omega) = H_{\Pi}(3\mathsf{a}; 2^n)$. $\square$

The case $(t^2 + t + 1)(t^2 - t + 1) \cdot W \neq 0$ There is an epimorphism $\pi \colon W \to \Bbbk$ whose kernel contains the submodule

$$\{a \in W \mid (t^2 + t + 1)(t^2 - t + 1) \cdot a = 0\},$$

because the image of this submodule under the epimorphism $W \twoheadrightarrow \Bbbk^2$ is a subspace of dimension at most 1. Consider the covering $\mathcal{C}(W) \to \mathcal{P}(\Bbbk)$ induced by π. If $\Omega \subset \mathcal{C}(W)$ maps to $\Omega'_1 \subset \mathcal{P}(\Bbbk)$, at least $\frac{3|\Omega|}{4}$ edges in Ω are in regions of size at least 12 (because if an edge $\mathrm{c}'(a, b) \in \mathcal{C}(W)$ maps into the bulk region in Ω'_1, then $(t^2 + t + 1)(t^2 - t + 1) \cdot a \neq 0$ and $(t^2 + t + 1)(t^2 - t + 1) \cdot b \neq 0$). Thus, $\chi(\Omega) \leq -\frac{|\Omega|}{6} + \frac{1}{3} \cdot \frac{|\Omega|}{4} + \frac{1}{12} \cdot \frac{3|\Omega|}{4} = -\frac{|\Omega|}{48} < 0$. If $\Omega \subset \mathcal{C}(W)$ maps to $\Omega'_0 \subset \mathcal{P}(\Bbbk)$, an edge $\mathrm{c}'(a, b) \in \Omega$ is in a region of size less than or equal to 6 if and only if $(t^2 + t + 1)(t^2 - t + 1) \cdot b = 0$ (because one necessarily has

$(t^2+t+1)(t^2-t+1)\cdot a \neq 0)$. Then, if $(t^2+t+1)^2(t^2-t+1)\cdot W \neq 0$, two edges sharing the same black vertex cannot be both in regions of size less than or equal to 6 (3.7), thus $\chi(\Omega) \leq -\frac{|\Omega|}{6}+\frac{1}{3}\cdot\frac{|\Omega|}{3}+\frac{1}{12}\cdot\frac{2|\Omega|}{3}=0$. If $(t^2+t+1)^2(t^2-t+1)\cdot W=0$, consider an edge $\mathrm{c}'(a_1,a_2)\in\Omega$ in a region of size 3 (note that any genus-zero orbit contains a region of size 3). Since $(t^2-t+1)\cdot a_1\neq 0$, one has $(t^2+t+1)\cdot a_2=0$ and $(t^{6s}+1)\cdot a_1=0$ for some integer s (Lemma 3.7). The second condition, together with $(t^2+t+1)^2(t^2-t+1)\cdot a_1=0$, implies that the submodule $\Lambda\cdot a_1$ is isomorphic to either R^{ba} or R^{bb} (4.4). Let n be the least such that $2^n\cdot a_2=0$. Then, there is an epimorphism $\pi\colon W_n^{\mathsf{ba}}\twoheadrightarrow W$ or $\pi\colon W_n^{\mathsf{bb}}\twoheadrightarrow W$ defined by $e_1\mapsto a_1$, $e_2\mapsto a_2$.

In either case, if π is an isomorphism, then $\Omega=\Omega_*$ (in $\mathcal{C}(W_n^{\mathsf{ba}})$ or $\mathcal{C}(W_n^{\mathsf{bb}})$). If π is not an isomorphism, then $n\geq 2$ and the kernel of π is generated by $(t^2-t+1)^2\cdot e_1+2^{n-1}\sigma\cdot e_2$ for some $\sigma\in\{1,t,t^2\}$. Hence, one has one of the isomorphisms

$$W_n^{\times\mathsf{a}} := W_n^{\mathsf{ba}}/((t^2-t+1)^2\cdot e_1+2^{n-1}\cdot e_2)\to W,$$
$$W_n^{\times\mathsf{b}} := W_n^{\mathsf{bb}}/((t^2-t+1)^2\cdot e_1+2^{n-1}\cdot e_2)\to W,$$

defined by $e_1\mapsto a_1$, $e_2\mapsto\sigma\cdot a_2$, so that the orbit Ω'_* of the edge $\mathrm{c}'(e_1,e_2)$ (in $\mathcal{C}(W_n^{\times\mathsf{a}})$ or $\mathcal{C}(W_n^{\times\mathsf{b}})$) is identified with $\Omega\subset\mathcal{C}(W)$. It turns out that $S(\Omega'_*)=H_{\Pi}(3/2^{\times};2^n)$ in both cases.

Consider the epimorphism $W_n^{\times\mathsf{a}}\twoheadrightarrow W_{n-1}^{\mathsf{b}}$ whose kernel is generated by $(t^2-t+1)^2\cdot e_1=2^{n-1}\cdot e_2$, then the induced covering $\mathcal{C}(W_n^{\times\mathsf{a}})\to\mathcal{C}(W_{n-1}^{\mathsf{b}})$ maps Ω'_* to Ω_*. Any edge $\mathrm{c}'(\tilde{e}_1,\tilde{e}_2)\in\mathcal{C}(W_n^{\times\mathsf{a}})$ which maps to $\mathrm{c}'(e_1,e_2)\in\mathcal{C}(W_{n-1}^{\mathsf{b}})$ satisfies $2\cdot\tilde{e}_1=0$ and $(t^2+t+1)\cdot\tilde{e}_2=0$, thus any region in $\mathcal{C}(W_n^{\times\mathsf{a}})$ which maps to the region $\mathfrak{a}_0$ of $\mathrm{c}'(e_1,e_2)\in\mathcal{C}(W_{n-1}^{\mathsf{b}})$ has size 3. There are three other regions $\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3$ of size 3 in Ω_*, which contain three edges as in (4.5). Consider an edge $\mathrm{c}'(a,b)\in\mathcal{C}(W_n^{\times\mathsf{a}})$ which maps to any of these three edges, then $a\in W_n^{\times\mathsf{a}}$ maps to $\alpha\cdot(e_1+2^{n-2}\sigma\cdot e_2)\in W_{n-1}^{\mathsf{b}}$ for some $\alpha\in\Lambda\smallsetminus\mathfrak{m}$ and $\sigma\in\{1,t,t^2\}$. Thus one has $2\cdot a\neq 0$. Moreover, $(t^6+1)\cdot a=0$ if and only if $\sigma=t^2$, which holds for exactly one of the three edges, which lies, w.l.o.g., in $\mathfrak{a}_1$. Thus any region in $\mathcal{C}(W_n^{\times\mathsf{a}})$ which maps to $\mathfrak{a}_1$ has size 3 and any region which maps to $\mathfrak{a}_2$ or $\mathfrak{a}_3$ has size 6. Hence, any orbit in $\mathcal{C}(W_n^{\times\mathsf{a}})$ which maps to $\Omega_*\subset\mathcal{C}(W_{n-1}^{\mathsf{b}})$ is a double covering of Ω_* ramified at $\mathfrak{a}_2$ and $\mathfrak{a}_3$. Consequently, $S(\Omega'_*) = H_{\Pi}(3/2^{\times}; 2^n)$.

The other case is very similar. Consider the epimorphism $W_n^{\times\mathtt{b}} \twoheadrightarrow W_{n-1}^{\mathtt{b}}$ whose kernel is generated by $(t^2 - t + 1)^2 \cdot e_1 = 2^{n-1} \cdot e_2$, then the induced covering $\mathcal{C}(W_n^{\times\mathtt{b}}) \to \mathcal{C}(W_{n-1}^{\mathtt{b}})$ maps Ω'_* to Ω_*. Any edge $\mathrm{c}'(\tilde{e}_1, \tilde{e}_2) \in \mathcal{C}(W_n^{\times\mathtt{b}})$ which maps to $\mathrm{c}'(e_1, e_2) \in \mathcal{C}(W_{n-1}^{\mathtt{b}})$ satisfies $(t^6 + 1) \cdot \tilde{e}_1 = 0$ and $(t^2 + t + 1) \cdot \tilde{e}_2 = 0$, thus any region in $\mathcal{C}(W_n^{\times\mathtt{b}})$ which maps to the region $\mathfrak{a}_0$ of $\mathrm{c}'(e_1, e_2) \in \mathcal{C}(W_{n-1}^{\mathtt{b}})$ has size 3. There are three other regions $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ of size 3 in Ω_*, which contain three edges as in (4.5). Consider an edge $\mathrm{c}'(a, b) \in \mathcal{C}(W_n^{\times\mathtt{b}})$ which maps to any of these three edges, then $a \in W_n^{\times\mathtt{b}}$ maps to $\alpha \cdot (e_1 + 2^{n-2}\sigma \cdot e_2) \in W_{n-1}^{\mathtt{b}}$ for some $\alpha \in \Lambda \smallsetminus \mathfrak{m}$ and $\sigma \in \{1, t, t^2\}$. Thus one has $(t^6 + 1) \cdot a \neq 0$. Moreover, $2 \cdot a = 0$ if and only if $\sigma = t^2$, which holds for exactly one of the three edges, which lies, w.l.o.g., in $\mathfrak{a}_1$. Thus any region in $\mathcal{C}(W_n^{\times\mathtt{b}})$ which maps to $\mathfrak{a}_1$ has size 3 and any region which maps to $\mathfrak{a}_2$ or $\mathfrak{a}_3$ has size 6. Hence, any orbit in $\mathcal{C}(W_n^{\times\mathtt{b}})$ which maps to $\Omega_* \subset \mathcal{C}(W_{n-1}^{\mathtt{b}})$ is a double covering of Ω_* ramified at $\mathfrak{a}_2$ and $\mathfrak{a}_3$. Consequently, $S(\Omega'_*) = H_{\Pi}(3/2^{\times}; 2^n)$. □

4.4 The Case $N = 2$

In this section, we consider the maximal ideals $\mathfrak{m} = \langle p, t - 1 \rangle$ with $p > 2$.

4.4.1 Rings with $p > 3$

There is no complete monovalent black vertex, and there is exactly 1 complete monovalent white vertex in $\mathcal{P}(R)$ (Lemma 3.6). Let $\Omega_1 \subset \mathcal{P}(R)$ be the orbit of the edge $\mathrm{pc}'(0, 1)$. If $(t - 1) = 0$, then $S(\Omega_1) = H_{\Sigma}(1; 1)$ and $S(\Omega) = H_{\Sigma}(2; 1, 1)$ for any $\Omega \neq \Omega_1$.

The case $(t-1)^2 = 0$ but $(t-1) \neq 0$ Let n be the least such that $p^n(t-1) = 0$. Note that $t^k - 1 = k(t-1)$ for any positive integer k, hence the order $\mathrm{ord}(t) = p^n$.

Therefore, the size of a bulk region in $\mathcal{P}(R)$ is $\operatorname{ord}(-t) = 2p^n$. Let $R' := R/(t-1)$, let $\Omega' \subset \mathcal{P}(R')$ be the image of $\Omega \subset \mathcal{P}(R)$, and let d be the degree of the covering $\Omega \to \Omega'$. Then, $d \geq p^n$ since a bulk region in Ω maps to a region of size 2 in Ω'. On the other hand, the degree of $\mathcal{P}(R) \to \mathcal{P}(R')$ is p^n, thus Ω is the only orbit in $\mathcal{P}(R)$ which maps to Ω', and $d = p^n$. In particular, $S(\Omega_1) = H_\Sigma(1; p^n)$.

Now, if Ω contains no region of size 1 or 2, then Ω cannot contain a monovalent vertex either since $\Omega \neq \Omega_1$. Therefore, $\chi(\Omega) \leq -\frac{|\Omega|}{6} + \frac{|\Omega|}{2p} \leq 0$. Otherwise, if two regions in Ω map to the same region in Ω', their sizes are either both greater than 2 or both less than or equal to 2. (Because, an edge $\mathrm{pc}'(m, 1) \in \Omega$ or $\mathrm{pc}'(1, m) \in \Omega$ is contained in a region of size 1 or 2 if and only if $(t^2 - 1)m = 0$ (Lemma 3.8). If $m, m' \in R$ map to the same element in R', clearly $(t^2 - 1)m = 0$ and $(t^2 - 1)m' = 0$ both hold or both do not.) Consequently, either $\Omega = \Omega_1$ or $S(\Omega) = H_\Sigma(2; p^n, a)$ with $a \in \{-1, 1\}$. ☐

The case $(t-1)^2 \neq 0$ Let $\Omega' \subset \mathcal{P}(R')$ be the image of $\Omega \subset \mathcal{P}(R)$ where $R' := R/(t-1)^2$. Exactly $\frac{|\Omega'|}{3}$ edges in Ω' are contained in regions of size 1 or 2. Then, at most $\frac{|\Omega|}{3p}$ edges in Ω are contained in regions of size 1 or 2. (Because if $m, m' \in R$ are such that $m - m'$ maps to $(t-1)k \in R'$ for an integer k coprime to p, then $(t^2 - 1)m = 0$ and $(t^2 - 1)m' = 0$ cannot both hold.) Consequently, $\chi(\Omega) < -\frac{|\Omega|}{6} + \frac{1}{2} \cdot \frac{1}{3p} \cdot |\Omega| + \frac{1}{2p} \cdot |\Omega| + \frac{1}{2} + \frac{1}{2} = 1 + |\Omega|\left(\frac{2}{3p} - \frac{1}{6}\right) \leq 1 - \frac{|\Omega|}{30} < 0$ since $p \geq 5$ and $|\Omega| \geq p|\Omega'| \geq 3p^2 \geq 75$. ☐

4.4.2 Rings with $p = 3$

There are two orbits Ω'_0, Ω'_1 in $\mathcal{P}(\mathbb{k})$ where Ω'_0 consists of the single edge $\mathrm{pc}'(1, 0) = \mathrm{pc}(1, 1)$. We consider the orbits $\Omega \subset \mathcal{P}(R)$ in two classes according to whether Ω maps to Ω'_0 or Ω'_1.

4.4.2.1 Orbits $\Omega \subset \mathcal{P}(R)$ which map to $\Omega_1' \subset \mathcal{P}(\Bbbk)$

Let $\Omega_1 \subset \mathcal{P}(R)$ be the orbit of the edge $\mathrm{pc}'(0,1)$. If $(t-1)=0$, then $S(\Omega_1) = H_\Sigma(1;1) = H_\Pi(1;\omega-1)$ and $S(\Omega) = H_\Sigma(2;1,1) = H_\Pi(2;\omega-1)$ for any $\Omega \neq \Omega_1$.

The case $(t-1)^2 = 0$ but $(t-1) \neq 0$ The content of the corresponding case in Section 4.4.1 is valid here as well. Hence, $S(\Omega_1) = H_\Sigma(1;3^n)$ and $S(\Omega) = H_\Sigma(2;3^n,1)$ for any genus-zero $\Omega \neq \Omega_1$. Note that $H_\Sigma(2;3^n,-1)$ does not appear since Ω maps to Ω_1'.

The case $(t^3-1) = 0$ but $(t-1) \neq 0$ Let n be the least such that $(t-1)^n = 0$. Let $R' := R/(t-1)$ and let $\Omega' \subset \mathcal{P}(R')$ be the image of $\Omega \subset \mathcal{P}(R)$. We will show, by induction on n, that Ω is the only orbit in $\mathcal{P}(R)$ which maps to Ω'. The statement is known for $n=2$, hence assume $n \geq 3$.

Proof. Let $\bar{R} := R/(t-1)^{n-1}$ and let $\bar{\Omega} \subset \mathcal{P}(\bar{R})$ be the image of Ω. Note that the covering $\mathcal{P}(R) \to \mathcal{P}(\bar{R})$ is 3-fold (because $\mathfrak{m}(t-1)^{n-1} = 0$), and that $\bar{\Omega}$ is the only orbit in $\mathcal{P}(\bar{R})$ which maps to Ω' (by the induction hypothesis). Consider some $m \in \mathfrak{m}$ such that the edge $\mathrm{pc}'(m,1) \in \mathcal{P}(R)$ maps into Ω'. There is some $r \in R$ such that $(t-1)m = (t-1)^2 r$. Hence, for $\tilde{m} := m - (t-1)r$, the edge $\mathrm{pc}'(\tilde{m},1)$ also maps into Ω' and is contained in a region of size 1 or 2. Observe that the edge $\mathrm{pc}'(\tilde{m}+(t-1)^{n-2},1) \in \mathcal{P}(R)$ maps into a region of size 1 or 2 in $\bar{\Omega}$, but is contained in a region of size 6. Hence, the orbit of the edge $\mathrm{pc}'(\tilde{m}+(t-1)^{n-2},1) \in \mathcal{P}(R)$ is the only orbit in $\mathcal{P}(R)$ which maps to Ω', thus it is necessarily equal to Ω. □

Consequently, $S(\Omega_1) = H_\Pi(1;(\omega-1)^n)$ and $S(\Omega) = H_\Pi(2;(\omega-1)^n)$ for any $\Omega \neq \Omega_1$. □

The case $(t-1)^2 \neq 0$ and $(t^3-1) \neq 0$ As in the case $(t-1)^2 \neq 0$ in Section 4.4.1, at most $\frac{|\Omega|}{9}$ edges are in regions of size 1 or 2. On the other hand,

$\frac{2}{3}|\Omega|$ edges are in bulk regions which have size at least 18. In particular, $|\Omega| \geq 27$. As a result of all, $\chi(\Omega) \leq |\Omega| \cdot \big(-\frac{1}{6}+\frac{1}{2}\cdot\frac{1}{9}+\frac{1}{6}\cdot\frac{2}{9}+\frac{1}{18}\cdot\frac{2}{3}\big)+\frac{1}{2}+\frac{1}{2} = 1-\frac{|\Omega|}{27} \leq 0.$ □

4.4.2.2 Orbits $\Omega \subset \mathcal{P}(R)$ which map to $\Omega_0' \subset \mathcal{P}(\Bbbk)$

There is no complete monovalent white vertex in Ω. Thus, if Ω contains no complete monovalent black vertex and no region of size less than or equal to 2, then $\chi(\Omega) \leq 0$. Let Ω_0 denote the orbit which contains the unique region $\{\mathrm{pc}'(1,0)\}$ of size 1, which has weight $\frac{1}{2}$.

The case $t^2+t+1=0$ First note that $\Omega_0 = \{\mathrm{pc}'(1,0)\}$ (Equations 3.6, 3.5, 3.4). Let n be the least such that $(t-1)^n = 0$. If $n=1$, then $R=\Bbbk$, thus the only orbit is $\Omega_0 = \Omega_0'$ so that $S(\Omega_0) = H_\Pi(\bullet;1) = H_\Sigma(2^\bullet;1)$. If $n>1$, then $S(\Omega_0) = H_\Pi(\mathtt{D};1) = H_\Sigma(2\mathtt{y}^\bullet;1)$. If $n=2$, i.e. $R = \mathbb{F}_3[t]/(t-1)^2$, the only orbit other than $\Omega_0'' := \Omega_0$ is $\Omega^\bullet := \{\mathrm{pc}(1,1), \mathrm{pc}(1,t^2)\}$ with $S(\Omega^\bullet) = H_\Pi(\bullet;\omega-1) = H_\Sigma(2\mathtt{x}^\bullet;1)$.

For $n>2$, there is an orbit $\tilde{\Omega} := \{\mathrm{pc}'(1,\pm(t-1)^{n-1})\}$ so that $S(\tilde{\Omega}) = H_\Pi(\mathtt{D};\omega-1) = H_\Sigma(2^\times;1,-1)$. Let $R' := R/(t-1)^2 = \mathbb{F}_3[t]/(t-1)^2$. The only complete monovalent black vertices in $\mathcal{P}(R)$ are $\mathrm{pc}(1,t) \in \Omega_0$ and $\mathrm{pc}(1,t^2)$ which map to $\Omega_0'' \subset \mathcal{P}(R')$ and $\Omega^\bullet \subset \mathcal{P}(R')$ respectively. On the other hand, the only regions of size 1 or 2 are the regions in Ω_0 and $\tilde{\Omega}$, which both map to $\Omega_0'' \subset \mathcal{P}(R')$. If Ω maps to $\Omega_0'' \subset \mathcal{P}(R')$ and Ω is of genus zero, then $\Omega = \Omega_0$ or $\Omega = \tilde{\Omega}$. If Ω maps to $\Omega^\bullet \subset \mathcal{P}(R')$, then we will show, by induction on n, that Ω is the only orbit in $\mathcal{P}(R)$ which maps to $\Omega^\bullet$.

Proof. Let $\bar{R} := R/(t-1)^{n-1}$ and let $\bar{\Omega} \subset \mathcal{P}(\bar{R})$ be the image of Ω. The covering $\mathcal{P}(R) \to \mathcal{P}(\bar{R})$ is 3-fold, since $\mathfrak{m}(t-1)^{n-1} = 0$. Note that all regions which map to $\Omega^\bullet$ have size 6. This finishes the induction step $n=3$ since $\Omega^\bullet$ has a region of size 2. Now observe that there are exactly two incomplete monovalent black vertices which map to $\Omega^\bullet$, namely $\mathrm{pc}(1, t^2 \pm (t-1)^{n-1})$ (Lemma 3.6). In the case $n \geq 4$, the induction hypothesis implies that $\bar{\Omega}$ contains monovalent black vertices $\mathrm{pc}(1, t^2 \pm (t-1)^{n-2}) \in \mathcal{P}(\bar{R})$. But, no monovalent black vertex in $\mathcal{P}(R)$ maps to these. $\square$

Consequently, if Ω maps to $\Omega^\bullet \subset \mathcal{P}(R')$, one has $S(\Omega) = H_{\mathrm{II}}(\bullet; (\omega-1)^{n-1})$. $\square$

The case $(t^3-1) = 0$ **but** $t^2+t+1 \neq 0$ If Ω contains a complete monovalent black vertex, then this vertex consists of $\mathrm{pc}(1,1)$ and one has $3 = 0$, that is, $R = \mathbb{F}_3[t]/(t-1)^3$. Consequently, $S(\Omega) = H_{\mathrm{I}}(6^\bullet)$.

If Ω contains a region of size 1 or 2, then all regions in Ω have size 1 or 2, which can be seen by considering (3.7). Consequently, $S(\Omega) = H_{\mathrm{II}}(\mathtt{D}; \omega-1) = H_{\Sigma}(2^\times; 1, -1)$. Note that this holds even if $\Omega = \Omega_0$ or there is an incomplete monovalent black vertex in Ω, because an orbit in $\mathrm{p}^{-1}(\Omega) \subset \mathcal{C}(R)$ consists of three regions of size 2 in any case. $\square$

The case $(t^2+t+1)(t-1)^2 = 0$ **but** $(t^3-1) \neq 0$ First note that $(t^3-1)^2 = 0$, hence $t^{3k} = 1 + k(t^3-1)$ for any integer k. Let n be the least such that $3^n(t^3-1) = 0$, then $\mathrm{ord}(t) = 3^{n+1}$. If Ω contains a complete monovalent black vertex, then either $3 = 0$ or $t^{2\cdot 3^n} + t^{3^n} + 1 = 0$ (Lemma 3.6). But the latter equality implies $3 = 0$, because $(t^3-1)^2 = 0$ implies $(t^{3^n}-1)^2 = 0$. Thus, $R = \mathbb{F}_3[t]/(t-1)^4$ and Ω contains one of the three edges $\mathrm{pc}(1,1)$, $\mathrm{pc}(1,t^3)$, $\mathrm{pc}(1,t^6)$. Consequently, $S(\Omega)$ equals $H_{\mathrm{I}}(\mathtt{6a}^\bullet)$, $H_{\mathrm{I}}(\mathtt{6b}^\bullet)$ or $H_{\mathrm{I}}(\mathtt{6c}^\bullet)$. Henceforth, suppose Ω contains a region of size 1 or 2.

Let $R' := R/(t^3-1)$, let $\Omega' \subset \mathcal{P}(R')$ be the image of Ω, and let d be the degree of the covering $\Omega \to \Omega'$. Note that the degree of $\mathcal{P}(R) \to \mathcal{P}(R')$ is 3^n. If two regions in Ω map to the same region in Ω', both of them have size less than or equal to 2 or both have size greater than 2. (Because if $m, m' \in R$ map to the same element in R', the equalities $(t-1)m = 0$ and $(t-1)m' = 0$ either both hold or both do not.) Moreover, if two regions in Ω map to distinct regions in Ω', at least one of them have size $2 \cdot 3^n$. This can be seen by considering (3.7) and the equality $(t^2+t+1)(t^{3k}-1) = 3k(t^3-1)$. Thus, there is a region of size

$2 \cdot 3^n$ in Ω which maps to a region of size 2 in Ω'. This shows that $d \geq 3^n$, hence $d = 3^n$ and Ω is the only orbit in $\mathcal{P}(R)$ which maps to $\Omega' \subset \mathcal{P}(R')$. Consequently, $S(\Omega) = H_\Sigma(2^\times; 3^n, -1)$. $\square$

The case $(t^2+t+1)(t^3-1) = 0$ **but** $(t^3-1) \neq 0$ If Ω contains a complete monovalent black vertex, one has $3 = 0$ as in the previous case, because $(t^3-1)^2 = 0$ and $(t^3-1) \neq 0$. Thus, R equals $\mathbb{F}_3[t]/(t-1)^4$ (considered in the previous case) or $\mathbb{F}_3[t]/(t-1)^5$. For $R = \mathbb{F}_3[t]/(t-1)^5$, the complete monovalent black vertices in $\mathcal{P}(R)$ are in orbits which are not of genus zero.

Henceforth, suppose Ω contains a region of size 1 or 2. Then, all regions in Ω have size at most 6, which can be seen by considering (3.7). Let n be the least such that $(t^2+t+1)(t-1)^n = 0$, let $R' := R/(t^3-1)$, and let $\Omega' \subset \mathcal{P}(R')$ be the image of Ω. We will show, by induction on n, that Ω is the only orbit in $\mathcal{P}(R)$ which maps to Ω'. The statement is known for $n = 2$, hence assume $n \geq 3$.

Proof. Let $\bar{R} := R/(t^2+t+1)(t-1)^{n-1}$ and let $\bar{\Omega} \subset \mathcal{P}(\bar{R})$ be the image of Ω. Note that the covering $\mathcal{P}(R) \to \mathcal{P}(\bar{R})$ is 3-fold, and that $\bar{\Omega}$ is the only orbit in $\mathcal{P}(\bar{R})$ which maps to Ω' (by the induction hypothesis). Consider an edge $\mathrm{pc}'(1, m) \in \Omega$ such that $(t-1)m = 0$. Then, observe that the edge $\mathrm{pc}'(1, m + (t^2+t+1)(t-1)^{n-2}) \in \mathcal{P}(R)$ maps into a region of size 1 or 2 in $\bar{\Omega}$, but is contained in a region of size 6. Therefore, the orbit of the edge $\mathrm{pc}'(1, m + (t^2+t+1)(t-1)^{n-2}) \in \mathcal{P}(R)$ is the only orbit in $\mathcal{P}(R)$ which maps to Ω', thus it is necessarily equal to Ω. $\square$

Consequently, $S(\Omega) = H_\Pi(\mathtt{D}; (\omega-1)^n)$. $\square$

The case $(t^2+t+1)(t^3-1) \neq 0$ **and** $(t^2+t+1)(t-1)^2 \neq 0$ If Ω contains a complete monovalent black vertex, consider $R'' := R/(t^3-1)^2$ and the image $\Omega'' \subset \mathcal{P}(R'')$ of Ω. In an analogous way to the previous two cases, one deduces $R'' = \mathbb{F}_3[t]/(t-1)^6$, hence Ω'' is not of genus zero. Henceforth, suppose Ω contains a region of size 1 or 2.

Let $R' := R/(t^2+t+1)(t-1)^2$, let $\Omega' \subset \mathcal{P}(R')$ be the image of Ω, and let n be the least such that $3^n(t^3-1)$ maps to $0 \in R'$. Then, $\frac{1}{3}\cdot|\Omega'|$ edges in Ω' are in regions of size 1 or 2, and $\frac{2}{3}\cdot|\Omega'|$ edges are in regions of size $2\cdot 3^n \geq 6$. Now observe that at most $\frac{1}{9}\cdot|\Omega|$ edges in Ω are in regions of size 1 or 2, because if $m, m' \in R$ are such that $m - m'$ maps to $(t^3-1)k \in R'$ for an integer k coprime to 3, then the equalities $(t-1)m = 0$ and $(t-1)m' = 0$ cannot both hold. Also observe that $\frac{2}{3}\cdot|\Omega|$ edges in Ω (those which map to the regions of size greater than 2 in Ω') are contained in regions of size at least 18. This can be seen by considering (3.7) and the fact that $(t^2+t+1)(t^3-1) \neq 0$. As a result of all, $\chi(\Omega) \leq |\Omega|\cdot\big(-\frac{1}{6}+\frac{1}{2}\cdot\frac{1}{9}+\frac{1}{6}\cdot\frac{2}{9}+\frac{1}{18}\cdot\frac{2}{3}\big) < 0.$ □

4.4.3 Wheels

If $\Omega \subset \mathcal{C}(W)$ contains no region of size 2, then $\chi(\Omega) \leq -\frac{|\Omega|}{6}+\frac{\Omega}{2p} \leq 0$. Hence consider an edge $\mathrm{c}'(a,b) \in \Omega$ in a region of size 2, then either $(t^2-1)\cdot a = 0$ or $(t^2-1)\cdot b = 0$ (Lemma 3.7), thus $(t-1)\cdot a = 0$ or $(t-1)\cdot b = 0$ (because $t+1 \notin \mathfrak{m}$). Thus the submodule $(t-1)\cdot W$ is cyclic, i.e. $(t-1)\cdot W \cong \Lambda/I$ for an ideal $I \subset \Lambda$. For a sufficiently large integer k, let

$$R(k) := \Lambda/\langle p^k,\, (t-1)I\rangle, \qquad W(k) := \big(\mathbb{Z}_{p^k}[t]/(t-1)\big)\cdot e_1 \oplus R(k)\cdot e_2.$$

Thus, there is an epimorphism $W(k) \twoheadrightarrow W$ defined by either $e_1 \mapsto a$, $e_2 \mapsto b$ or $e_1 \mapsto b$, $e_2 \mapsto a$. This epimorphism restricts to an isomorphism $(t-1)\cdot W(k) \cong (t-1)\cdot W$. As a result, the induced covering $\mathcal{C}(W(k)) \to \mathcal{C}(W)$ is unramified. (Because if an edge $\mathrm{c}'(a_1,a_2) \in \mathcal{C}(W(k))$ maps into a region of size $2p^n$ in $\mathcal{C}(W)$, one of the elements $(t^{2p^n}-1)\cdot a_1 \in W(k)$ and $(t^{2p^n}-1)\cdot a_2 \in W(k)$ maps to $0 \in W$, but these elements are in $(t-1)\cdot W(k)$, therefore $(t^{2p^n}-1)\cdot a_1 = 0$ or $(t^{2p^n}-1)\cdot a_2 = 0$, which shows that the edge $\mathrm{c}'(a_1,a_2)$ is contained in a region of size $2p^n$ in $\mathcal{C}(W(k))$.) Thus, any orbit $\tilde{\Omega} \subset \mathcal{C}(W(k))$ which maps to Ω maps isomorphically onto Ω so that $S(\tilde{\Omega}) = S(\Omega)$. Consider the covering $\mathcal{C}(W(k)) \to \mathcal{P}(R(k))$ induced by the obvious epimorphism $W(k) \twoheadrightarrow R(k)$ defined by $e_1 \mapsto 0$, $e_2 \mapsto 1$, and let Ω' be the image of $\tilde{\Omega}$. Then, if Ω' contains no region of size 1 and no monovalent vertex, the covering $\tilde{\Omega} \to \Omega'$ is also unramified so that

$S(\tilde{\Omega}) = S(\Omega')$. For sufficiently large k, one can guarantee that Ω' contains no region of size 1 and no monovalent vertex by varying $\tilde{\Omega}$ over all possibilities. Consequently, $S(\Omega)$ is equal to one of the Bu_3-sets found in the sections on rings. □

4.5 The Case $N = 1$

In this section, we consider the maximal ideals $\mathfrak{m} = \langle p, t+1 \rangle$. We freely refer to the content of Section 3.3.3. In particular, we retain the notation and the terminology defined there.

Remark 4.2. There is only one orbit in $\mathcal{P}(\Bbbk)$ because of the following: $|\mathcal{P}(\Bbbk)| = |\Bbbk| + 1 = p + 1$, there is a bulk region of size p in $\mathcal{P}(\Bbbk)$ (Lemma 3.10), and the unique edge outside the bulk region, namely $\mathrm{pc}(0, 1)$, is not fixed by e.g. $X \in \Gamma$ (Equation 3.4).

The case $p > 7$ There is no genus-zero orbit in this case. Clearly, it is enough to show that $\chi(\mathcal{P}(\Bbbk)) \leq 0$. But, there is no complete monovalent vertex in $\mathcal{P}(\Bbbk)$ (Lemma 3.6), hence $\chi(\mathcal{P}(\Bbbk)) = -\frac{p+1}{6} + 2 \leq 0$ (Remark 4.2). □

The case $p = 7$ There is no genus-zero orbit in $\mathcal{C}(\Bbbk^2)$, because there is no monovalent vertex and all regions are of size 7 (Lemmas 3.5, 3.9). Hence we only consider rings.

One can directly check that there is no genus-zero orbit $\Omega \subset \mathcal{P}(R)$ for

$$R = \Bbbk[\lambda]/\lambda^3, \qquad \underbrace{R = \mathbb{Z}_{49}[\lambda]/(\lambda - 7k)}_{\text{for some } k=0,1,\ldots,6}, \qquad \underbrace{R = \mathbb{Z}_{49}[\lambda]/\langle 7\lambda, \lambda^2 - 7k \rangle}_{\text{for some } k=0,1,\ldots,6}.$$

The only rings which do not admit an epimorphism onto one of these are $R = \Bbbk$ or $R = \Bbbk[\lambda]/\lambda^2$. Then, there are three orbits $\Omega \subset \mathcal{P}(\Bbbk[\lambda]/\lambda^2)$ and $S(\Omega)$ equals $H_{\mathrm{I}}(7\mathtt{a})$, $H_{\mathrm{I}}(7\mathtt{b})$, or $H_{\mathrm{I}}(7\mathtt{c})$. Finally, $S(\mathcal{P}(\Bbbk)) = H_{\mathrm{I}}(7)$. □

4.5.1 The case $p = 5$

Rings with $\omega_1 = 0$ First, $S(\mathcal{P}(\Bbbk)) = H_{\mathrm{I}}(5)$. Unless $R = \Bbbk$, the edges $\mathrm{pc}(0, 1)$ and $\mathrm{pc}(\lambda, 1)$ are distinct edges in the same orbit Ω_0 so that $S(\Omega_0) = H_{\mathrm{I}}(5\mathtt{a})$. If $\lambda^2 = 0$, i.e. $R = \Bbbk[\lambda]/\lambda^2$, there are two other orbits Ω and $S(\Omega)$ equals $H_{\mathrm{I}}(5\mathtt{x})$ or $H_{\mathrm{I}}(5\mathtt{y})$. If $\lambda^2 \neq 0$, any region outside Ω_0 has edge-weight $\frac{1}{5}$. Consequently, $S(\Omega) = H_{\mathrm{I}}(5\mathtt{ww})$ for any $\Omega \neq \Omega_0$. □

Rings with $\omega_1 \neq 0$ First suppose that $\lambda \notin R\omega_1$. Let $R' := R/\mathfrak{m}\omega_1$, $R'' := R/\omega_1$ and let $\Omega' \subset \mathcal{P}(R')$, $\Omega'' \subset \mathcal{P}(R'')$ be the images of Ω. The bulk regions in Ω' have size 25, while those in Ω'' have size 5, thus the covering $\Omega' \to \Omega''$ is 5-fold. Let d be the degree of the covering $\Omega'' \to \mathcal{P}(\Bbbk)$ so that $|\Omega''| = 6d$ and $|\Omega'| = 30d$. Then, there are exactly d bulk regions in Ω' and Ω''. On the other hand, the edges of weight 1 in $\mathcal{P}(R')$ map to distinct edges in $\mathcal{P}(R'')$ (because such an edge $\mathrm{pc}(m, 1)$ satisfies $1 + \lambda - m \in \langle 1 + \lambda \rangle$, but $|\langle 1 + \lambda \rangle| = 5$ in both R' and R''). Then, Ω' contains at most d edges of weight 1. As a result of all, the sum of weights over the edges in Ω' is bounded above by $\frac{4}{5} \cdot d + \frac{1}{5} \cdot 5d + d < 5d = \frac{|\Omega'|}{6}$. Since there is no complete monovalent vertex in $\mathcal{P}(R')$, one has $\chi(\Omega') < 0$.

Now suppose that $\lambda = \omega_1 r$ for some $r \in R$. Note that $\omega_1 = 5 + \lambda\theta$ for $\theta \in \mathfrak{m}$, hence $\lambda = 5r(1 - \theta r)^{-1}$. This requires that $R = \mathbb{Z}_{5^n}$ for some $n \geq 2$ and $\lambda = 5k$. It is easy to see that, in this case, there is only one orbit in $\mathcal{P}(R)$. Then, $\chi(\mathcal{P}(R)) < 0$ when $n = 3$ or when $n = 2$ and $\lambda = 0$. When $n = 2$ and $\lambda = 5k$ for some $k = 1, 2, 3, 4$, one has $\chi(\mathcal{P}(R)) = 1$ and $S(\mathcal{P}(R)) = H_{\mathrm{I}}(25; -5k - 1)$. □

Wheels with $\omega_1 \cdot W = 0$ One has $S(\Omega) = H_{\mathrm{I}}(5\mathtt{w})$ if $W = \Bbbk^2$, and $S(\Omega) = H_{\mathrm{I}}(5\mathtt{ww})$ otherwise. First, observe that all regions in $\mathcal{C}(W)$ are of size 5, hence any orbit is of genus zero and is isomorphic to its image in $\mathcal{C}(\Bbbk^2)$. This shows that $\Omega = \Gamma(5)\backslash\Gamma$. The order of the t-action on W is 2 if $W = \Bbbk^2$, and 10 otherwise. Hence, the specification of $S(\Omega)$ is $\mathbb{Z}_4$-valued and $\mathbb{Z}_{20}$-valued in these two cases, respectively. Moreover, $\mathbb{L}^5$ acts as $-1 = t^5$ on $\mathcal{E}(W)$, hence the specification assigns the value of 5 to each region. □

Wheels with $\omega_1 \cdot W \neq 0$ The size of any region is either 5 or at least 25 (Lemma 3.9). If an edge $\mathrm{c}(a_1, a_2)$ is in a region of size 5, then $\mathrm{c}(a_1, a_2) = \mathrm{c}(a_1, a_2 + \omega_1(\lambda \cdot a_2 - a_1))$ (Equation 3.9). This implies either

$$\omega_1(a_1 - \lambda \cdot a_2) = 0 \qquad \text{or} \qquad \underbrace{((1+\lambda)^k - 1) \cdot a_1 = 0}_{\text{for some } k \text{ for which } ((1+\lambda)^k - 1)\cdot W \neq 0} \quad .$$

Let n be the greatest such that $\omega_n \lambda = (1+\lambda)^{5^n} - 1$ does not annihilate W. The equation $((1+\lambda)^k - 1) \cdot a_1 = 0$ implies $\omega_n \lambda \cdot a_1 = 0$. There are epimorphisms $\pi \colon W \to \Bbbk$ and $\pi' \colon W \to \Bbbk$ whose kernels contain the submodules

$$\{a \in W \mid \omega_1 \cdot a = 0\} \qquad \text{and} \qquad \{a \in W \mid \omega_n \lambda \cdot a = 0\},$$

because the images of these submodules under the epimorphism $W \twoheadrightarrow \Bbbk^2$ are subspaces of dimension at most 1 (note that these equations do not identically hold in W, and W is generated by any two elements which map to linearly independent vectors in $\Bbbk^2$ (Remark 3.2)). Consider the coverings $\mathcal{C}(W) \to \mathcal{P}(\Bbbk)$ induced by π and π'. Then, any region of size 5 in Ω maps to $\mathrm{pc}(0,1) \in \mathcal{P}(\Bbbk)$ under one of these coverings. Therefore, at most $\frac{1}{3} \cdot |\Omega|$ edges are in regions of size 5 (because $|\mathcal{P}(\Bbbk)| = 6$). Consequently, the number of regions is bounded by $\frac{1}{5} \cdot \frac{1}{3} \cdot |\Omega| + \frac{1}{25} \cdot \frac{2}{3} \cdot |\Omega| < \frac{1}{6} \cdot |\Omega|$. Since there is no monovalent vertex in Ω, one has $\chi(\Omega) < 0$. $\square$

4.5.2 The case $p = 3$

Rings with $\omega_1 = 0$ First, $S(\mathcal{P}(\Bbbk)) = H_\Pi(1'; 2)$. Unless $R = \Bbbk$, the edges $\mathrm{pc}(0,1)$ and $\mathrm{pc}(\lambda, 1)$ are in distinct orbits Ω_0 and Ω_1, respectively. Then, $S(\Omega_0) = H_\Pi(1^\times; 2)$ and $S(\Omega_1) = H_\Pi(1\mathtt{ay}; 2)$. If $\lambda^2 = 0$, i.e. $R = \Bbbk[\lambda]/\lambda^2$, there is one orbit $\Omega \subset \mathcal{P}(R)$ other than Ω_0 and Ω_1 so that $S(\Omega) = H_\Pi(1\mathtt{ax}; 2)$. If $\lambda^2 \neq 0$, any region outside Ω_0 and Ω_1 has edge-weight $\frac{1}{3}$. Consequently, $S(\Omega) = H_\Pi(3\mathtt{a}; 2)$ for any $\Omega \neq \Omega_0$. $\square$

Rings with $\omega_2 = 0$ and $\omega_1 \lambda^2 = 0$ but $\omega_1 \neq 0$ First observe that $\delta_1 = 3 + 3\omega_1\lambda + \omega_1^2\lambda^2 = 3(1 + \omega_1\lambda)$, hence $3\omega_1 = \delta_1\omega_1(1+\omega_1\lambda)^{-1} = 0$. This, in turn, implies $\delta_1 = 3 + 3\omega_1\lambda = 3$.

Suppose Ω contains no edge of weight 1 and it contains no complete monovalent vertex. Let d be the degree of the covering $\Omega \to \mathcal{P}(\Bbbk)$, then $|\Omega| = 4d$ and Ω contains exactly $\frac{d}{3}$ bulk regions which are of size 9. Hence, the sum of weights over the edges in Ω is less than or equal to $\frac{d}{3} + \frac{d}{3} = \frac{|\Omega|}{6}$, thus $\chi(\Omega) \leq 0$.

Suppose Ω contains a complete monovalent vertex. Then, $3 = \delta_1 = 0$ since $\delta_0 = \omega_1 \neq 0$ (Lemma 3.6). Thus, $R = \Bbbk[\lambda]/\lambda^3$ or $R = \Bbbk[\lambda]/\lambda^4$. There is one orbit $\Omega \subset \mathcal{P}(\Bbbk[\lambda]/\lambda^3)$ with monovalent vertices so that $S(\Omega) = H_{\mathrm{I}}(9^\bullet)$. There are three orbits $\Omega \subset \mathcal{P}(\Bbbk[\lambda]/\lambda^4)$ with monovalent vertices so that $S(\Omega)$ equals $H_{\mathrm{I}}(9\mathtt{a}^\bullet)$, $H_{\mathrm{I}}(9\mathtt{b}^\bullet)$ or $H_{\mathrm{I}}(9\mathtt{c}^\bullet)$. Hence, suppose Ω contains an edge of weight 1.

If $\lambda = 3r$ for some $r \in R$, i.e. $R = \mathbb{Z}_9[\lambda]/(\lambda - 3k)$ for some $k = 0, 1, 2$, then there is one orbit in $\mathcal{P}(R)$ so that $S(\mathcal{P}(R))$ equals $H_{\mathrm{I}}(9)$ if $k = 0$ and $H_{\mathrm{I}}(9'; -3k-1)$ otherwise. If $\lambda \notin R \cdot 3$, there is an epimorphism $R \twoheadrightarrow \Bbbk[\lambda]/\lambda^2$, hence the edges $\mathrm{pc}(0,1)$ and $\mathrm{pc}(\lambda, 1)$ are in distinct orbits Ω_0 and Ω_1, respectively. Then, $S(\Omega_0) = H_{\mathrm{I}}(9\mathtt{a})$ and $S(\Omega_1) = H_{\mathrm{I}}(9\mathtt{b})$ if $\omega_1\lambda = 0$, and $S(\Omega_0) = H_{\mathrm{I}}(9\mathtt{aa})$ and $S(\Omega_1) = H_{\mathrm{I}}(9\mathtt{bb})$ if $\omega_1\lambda \neq 0$. Note that $\mathrm{pc}(\pm\omega_1\lambda, 1) \in \Omega_0$ and $\mathrm{pc}(\lambda \pm \omega_1\lambda, 1) \in \Omega_1$.

If $\lambda^2 \neq 0$, there is edge of weight 1 outside Ω_0 and Ω_1. If $\lambda^2 = 0$ but $\lambda \notin R \cdot 3$, one has $R = \mathbb{Z}_9[\lambda]/\langle 3\lambda, \lambda^2\rangle$ or $R = \mathbb{Z}_9[\lambda]/\lambda^2$. In the case $R = \mathbb{Z}_9[\lambda]/\langle 3\lambda, \lambda^2\rangle$, there is one orbit $\Omega \subset \mathcal{P}(R)$ other than Ω_0, Ω_1 so that $S(\Omega) = H_{\mathrm{I}}(9\mathtt{x})$. In the case $R = \mathbb{Z}_9[\lambda]/\lambda^2$, there are three orbits $\Omega \subset \mathcal{P}(R)$ other than Ω_0, Ω_1 so that $S(\Omega)$ equals $H_{\mathrm{I}}(9\mathtt{xa})$, $H_{\mathrm{I}}(9\mathtt{xb})$ or $H_{\mathrm{I}}(9\mathtt{xc})$. □

Rings with $\omega_2 = 0$ but $\omega_1\lambda^2 \neq 0$ As in the previous case, we assume that Ω contains either an edge of weight 1 or a complete monovalent vertex. Observe that $\omega_1\lambda^2 \notin R \cdot 3\omega_1$, thus $\omega_1\lambda^2 \neq 0$ in $R' := R/\langle 3\omega_1, \omega_1\lambda^3\rangle$. (Suppose, for contradiction, that $\omega_1\lambda^2 = 3\omega_1 r$ for some $r \in R$. Then, $\delta_1 = 3(1 + \omega_1\lambda + \omega_1^2 r)$, hence $\omega_1\lambda^2 = 3\omega_1 r = \delta_1\omega_1(1 + \omega_1\lambda + \omega_1^2 r)^{-1} r = 0$, since $\delta_1\omega_1 = \omega_2 = 0$.) Let $\Omega' \subset \mathcal{P}(R')$ be the image of Ω. Since $3\omega_1 = 0$ but $\omega_1\lambda^2 \neq 0$ in R', there is an epimorphism $R' \twoheadrightarrow \Bbbk[\lambda]/\lambda^3$. Let $\Omega'' \subset \mathcal{P}(\Bbbk[\lambda]/\lambda^3)$ be the image of Ω', and let d be the degree of the covering $\Omega' \to \Omega''$ so that $|\Omega'| = d \cdot |\Omega''| = 12d$.

If Ω' contains a complete monovalent vertex, then $\Omega'' \subset \mathcal{P}(\Bbbk[\lambda]/\lambda^3)$ is the unique orbit with monovalent vertices (see the previous case). Then, all regions in Ω' have size 9. (Consider an edge $\mathrm{pc}(m,1) \in \Omega'$. Note that m maps to $-\lambda \in \Bbbk[\lambda]/\lambda^2$. Hence, $m(m-\lambda)$ maps to $-\lambda^2 \in \Bbbk[\lambda]/\lambda^3$, thus $\omega_1 \cdot m(m-\lambda) \neq 0$). In particular, one has $d \geq 3$, thus $|\Omega'| \geq 36$. Then, since there are at most 3 complete monovalent vertices in Ω', one has $\chi(\Omega') \leq -\frac{|\Omega'|}{6} + \frac{|\Omega'|}{9} + \frac{2}{3} \cdot 3 \leq 0$.

If Ω' contains no complete monovalent vertex, but it contans an edge of weight 1, then Ω'' equals Ω_0 or Ω_1 (defined in the previous case). We only consider the case $\Omega'' = \Omega_0$ since the other case is completely analogous. There are only two edges of weight 1 in $\mathcal{P}(R')$, namely $\mathrm{pc}(0,1)$ and $\mathrm{pc}(\lambda,1)$. Since $\Omega'' = \Omega_0$, one has only $\mathrm{pc}(0,1) \in \Omega'$. Then, it is easy to see that $d > 1$. If an edge $\mathrm{pc}(m,1) \in \Omega'$ maps to $\mathrm{pc}(\pm\lambda^2,1) \in \Omega_0$, then it has weight $\frac{1}{9}$ (because $1+\omega_1(\lambda-m) = 1+\omega_1\lambda+(\pm\omega_1\lambda^2) \notin \langle 1+\lambda\rangle$.) As a result of all, the sum of weights over the edges in Ω' is bounded by $\frac{2}{3}+\frac{1}{3}\cdot d+\frac{1}{9}\cdot 2d+d < 2d = \frac{|\Omega'|}{6}$, hence $\chi(\Omega') \leq 0$. □

Rings with $\omega_2 \neq 0$ First suppose that $\omega_1\lambda \notin R\omega_2$. Let $R' := R/\mathfrak{m}\omega_2$, $R'' := R/\omega_2$ and let $\Omega' \subset \mathcal{P}(R')$, $\Omega'' \subset \mathcal{P}(R'')$ be the images of Ω. The bulk regions in Ω' have size 27, while those in Ω'' have size 9, thus the covering $\Omega' \to \Omega''$ is 3-fold. Let d be the degree of the covering $\Omega'' \to \mathcal{P}(\Bbbk)$ so that $|\Omega''| = 4d$ and $|\Omega'| = 12d$. Then, there are exactly $\frac{d}{3}$ bulk regions in Ω' and Ω''. If Ω' contains a complete monovalent vertex, then $3 = 0$ in R' because $\omega_2\lambda = 0$ but $\delta_1 \neq 0$ in R' (Lemma 3.6). Then $R' = \Bbbk[\lambda]/\lambda^9$, but there is no genus-zero orbit in $\mathcal{P}(\Bbbk[\lambda]/\lambda^9)$. Hence, suppose Ω' contains no complete monovalent vertex. On the other hand, there are at most d edges of weight 1 in Ω' since these edges map to distinct edges in $\mathcal{P}(R'')$ (because such an edge $\mathrm{pc}(m,1)$ satisfies $1+\lambda-m \in \langle 1+\lambda\rangle$, but $|\langle 1+\lambda\rangle| = 9$ in both R' and R''). As a result of all, the sum of weights over the edges in Ω' is bounded above by $\frac{2}{3}\cdot d+\frac{1}{3}\cdot 3d+\frac{d}{3} = \frac{|\Omega|}{6}$, hence $\chi(\Omega) \leq 0$.

Now suppose that $\omega_1\lambda = \omega_2 r$ for some $r \in R$, but $\lambda \notin R\omega_1$. Note that $\delta_1 = 3+\lambda\theta$ for some $\theta \in \mathfrak{m}$, hence $\omega_1\lambda = \omega_1\delta_1 r$ implies $\omega_1\lambda = 3\omega_1 r(1-\theta r)^{-1}$. This implies that $\omega_1\lambda = 3k\omega_1$ for some integer k. Thus, $\omega_2 = \omega_1\delta_1 = 3\omega_1(1+k\theta)$.

Let $R' := R/\mathfrak{m}\omega_2$ and $R'' := R/\omega_1$. Then, the covering $\mathcal{P}(R') \to \mathcal{P}(R'')$ is 9-fold. Let $\Omega' \subset \mathcal{P}(R')$, $\Omega'' \subset \mathcal{P}(R'')$ be the images of Ω. The bulk regions in Ω' have size 27, while those in Ω'' have size 3, thus the covering $\Omega' \to \Omega''$ is also 9-fold. Let d be the degree of the covering $\Omega'' \to \mathcal{P}(\Bbbk)$ so that $|\Omega''| = 4d$ and $|\Omega'| = 36d$. Then, there are exactly d bulk regions in Ω' and Ω''. As above, there is no complete monovalent vertex in Ω' since $3 \neq 0$, $\delta_1 \neq 0$ and $\delta_0 = \omega_1 \neq 0$ in R'. On the other hand, there are at most $3d$ edges of weight 1 in Ω' because $|\langle 1+\lambda\rangle| = 9$ in R' and $|\langle 1+\lambda\rangle| = 3$ in R''. Therefore, the sum of weights over the edges in Ω' is bounded by $\frac{2}{3}\cdot 3d + \frac{1}{3}\cdot 9d + d = \frac{|\Omega'|}{6}$, hence $\chi(\Omega') \leq 0$.

Finally suppose that $\lambda \in R\omega_1$. This requires that $R = \mathbb{Z}_{3^n}$ for some $n \geq 3$ and $\lambda = 3k$. It is easy to see that, in this case, there is only one orbit in $\mathcal{P}(R)$. Then, $\chi(\mathcal{P}(R)) < 0$ when $n = 4$ or when $n = 3$ and $\lambda = 9k$. When $n = 3$ and $\lambda = 3k$ for some $k = 1, 2, 4, 5, 7, 8$, one has $\chi(\mathcal{P}(R)) = 2$ and $S(\mathcal{P}(R)) = H_{\mathrm{I}}(27; -3k-1)$. □

Wheels with $p = 3$ and $\omega_1 \cdot W = 0$ One has $S(\Omega) = H_{\Pi}(3\mathsf{a}'; 2)$ if $W = \Bbbk^2$, and $S(\Omega) = H_{\Pi}(3\mathsf{a}; 2)$ otherwise. This is proven in a way completely analogous to the case $p = 5$. □

Wheels with $p = 3$ and $\omega_1 \cdot W \neq 0$ The size of any region is either 3 or at least 9 (Lemma 3.9). If an edge $\mathrm{c}(a_1, a_2)$ is in a region of size 3, then $\mathrm{c}(a_1, a_2) = \mathrm{c}(a_1, a_2 + \omega_1(\lambda \cdot a_2 - a_1))$ (Equation 3.9). This implies either

$$\omega_1(a_1 - \lambda \cdot a_2) = 0 \qquad \text{or} \qquad \underbrace{((1+\lambda)^k - 1)\cdot a_1 = 0}_{\text{for some } k \text{ for which } ((1+\lambda)^k-1)\cdot W \neq 0} \quad .$$

Let n be the greatest such that $\omega_n\lambda = (1+\lambda)^{3^n} - 1$ does not annihilate W. The equation $((1+\lambda)^k - 1)\cdot a_1 = 0$ implies $\omega_n\lambda \cdot a_1 = 0$. There are epimorphisms $\pi\colon W \to \Bbbk$ and $\pi'\colon W \to \Bbbk$ whose kernels contain the submodules

$$\{a \in W \mid \omega_1 \cdot a = 0\} \qquad \text{and} \qquad \{a \in W \mid \omega_n\lambda \cdot a = 0\}. \tag{4.6}$$

Consider the coverings $\mathcal{C}(W) \to \mathcal{P}(\Bbbk)$ induced by π and π'. Then, any region of size 3 in Ω maps to $\mathrm{pc}(0,1) \in \mathcal{P}(\Bbbk)$ under one of these coverings. First suppose that one of these coverings maps all regions of size 3 in Ω to $\mathrm{pc}(0,1) \in \mathcal{P}(\Bbbk)$. Then, at most $\frac{|\Omega|}{4}$ edges in Ω are contained in regions of size 3, hence the number of regions is bounded by $\frac{1}{3} \cdot \frac{1}{4} \cdot |\Omega| + \frac{1}{9} \cdot \frac{3}{4} \cdot |\Omega| = \frac{1}{6} \cdot |\Omega|$, thus $\chi(\Omega) \leq 0$.

Henceforth, π and π' have distinct kernels and Ω contains edges $\mathrm{c}(a_1, a_2)$ and $\mathrm{c}(b_1, b_2)$ in regions of size 3 such that $\pi(a_1) = 0$ and $\pi'(b_1) = 0$. Thus one has $n = 0$, otherwise the latter submodule in (4.6) contains the former. Therefore,

$$\omega_1(a_1 - \lambda \cdot a_2) = 0,$$
$$\underbrace{((1+\lambda)^k - 1) \cdot b_1 = 0, \qquad ((1+\lambda)^k - 1) \cdot b_2 = \omega_1(\lambda \cdot b_2 - b_1)}_{\text{for some } k \text{ coprime to } 3}.$$

Note that $(1+\lambda)^k - 1 = \lambda\sigma$ for some $\sigma \in \Lambda \smallsetminus \mathfrak{m}$. Then, $\lambda \cdot b_1 = \sigma^{-1}((1+\lambda)^k - 1) \cdot b_1 = 0$ and $\lambda \cdot b_2 = (\omega_1 - \sigma)^{-1}\omega_1 \cdot b_1$. (Here, σ^{-1} and $(\omega_1 - \sigma)^{-1}$ are meaningful because even though σ, $(\omega_1 - \sigma) \in \Lambda \smallsetminus \mathfrak{m}$ may not be invertible in Λ, their actions on W are invertible (Remark 3.1)). Since $\lambda \cdot b_2 = (\omega_1 - \sigma)^{-1}\omega_1 \cdot b_1$, one has $\lambda^2 \cdot b_2 = (\omega_1 - \sigma)^{-1} \cdot \omega_1 \lambda \cdot b_1 = 0$, hence $\lambda^2 \cdot W = 0$. This allows one to replace $\omega_1 = 3 + 3\lambda + \lambda^2$ by $3(1+\lambda)$ in the equations. Hence, by introducing the brief notation $a_1' := (1+\lambda) \cdot (a_1 - \lambda \cdot a_2)$ and $b_2' := (1+\lambda)^{-1} \cdot (\omega_1 - \sigma) \cdot b_2$, we re-express the equations as follows:

$$3a_1' = 0, \qquad \lambda \cdot b_1 = 0, \qquad \lambda \cdot b_2' = 3b_1.$$

Since a_1' and b_1 generate W, one can write $b_2' = \phi_1 \cdot a_1' + \phi_2 \cdot b_1$ for some $\phi_1, \phi_2 \in \Lambda$. Moreover, $\phi_1 \notin \mathfrak{m}$ since b_1 and b_2' map to linearly independent vectors in $\Bbbk^2$. Thus, $3b_1 = \lambda \cdot b_2' = \lambda\phi_1 \cdot a_1'$. Finally, let $e_1 := b_1$ and $e_2 := \phi_1 \cdot a_1'$. Then,

$$W = \mathbb{Z}_9 \cdot e_1 \oplus \mathbb{Z}_3 \cdot e_2 \quad \text{with} \quad \lambda \cdot e_1 = 0 \text{ and } \lambda \cdot e_2 = 3e_1.$$

For this exceptional wheel W, there are two orbits $\Omega \subset \mathcal{C}(W)$ so that $S(\Omega)$ equals $H_{\mathrm{I}}(9\texttt{wa})$ or $H_{\mathrm{I}}(9\texttt{wb})$. □

4.5.3 The case $p = 2$

Rings with $\omega_2 = 0$ Let $\Omega_1 \subset \mathcal{P}(R)$ be the orbit of the edge $\mathrm{pc}(0,1)$ of weight 1. If $\omega_1 = 0$, then $S(\Omega_1) = H_\Sigma(1;1)$ and $S(\Omega) = H_\Sigma(2;1,1)$ for any $\Omega \neq \Omega_1$.

Henceforth, suppose $\omega_1 \neq 0$. Then, the edge $\mathrm{pc}(\omega_1 + \omega_1\lambda, 1)$ of weight 1 is also in Ω_1.

If $\mathfrak{m}\omega_1 = 0$, then $S(\Omega_1) = H_\Sigma(1;2)$ and $S(\Omega) = H_\Sigma(2;2,1)$ for any $\Omega \neq \Omega_1$. If $2\omega_1 = 0$ but $\omega_1\lambda \neq 0$, then $\omega_1\lambda^2 = 0$ and the distinct edges $\mathrm{pc}(\omega_1, 1)$ and $\mathrm{pc}(\omega_1\lambda, 1)$ of weight 1 are in the same orbit $\Omega_2 \neq \Omega_1$. Then, $S(\Omega_1) = H_\Sigma(1\mathtt{aa};2)$, $S(\Omega_2) = H_\Sigma(1\mathtt{ab};2)$ and $S(\Omega) = H_\Sigma(2\mathtt{a};2,1)$ for any $\Omega \neq \Omega_1$, Ω_2. If $2\omega_1 \neq 0$, then $S(\Omega_1) = H_\Sigma(1\mathtt{aa};2)$ and there is no edge of weight 1 outside Ω_1, thus $S(\Omega)$ equals $H_\Sigma(2\mathtt{a};2,1)$ or $H_\Sigma(2\mathtt{d};2)$ for any $\Omega \neq \Omega_1$. $\square$

Rings with $2\omega_2 = \omega_2\lambda^2 = 0$ but $\omega_2 \neq 0$ Let $\Omega_1 \subset \mathcal{P}(R)$ be the orbit of the edge $\mathrm{pc}(0,1)$ of weight 1. If $\omega_1^2 = 0$, then $S(\Omega_1) = H_\Sigma(1;4)$ and $S(\Omega) = H_\Sigma(2;4,a)$ with $a \in \{-1,1\}$ for any genus-zero $\Omega \neq \Omega_1$. If $\omega_1\lambda = 0$, then $S(\Omega_1) = H_\Sigma(1\mathtt{d}';4)$ and $S(\Omega) = H_\Sigma(2\mathtt{d}';4)$ for any genus-zero $\Omega \neq \Omega_1$. Henceforth, suppose $\omega_1^2 \neq 0$ and $\omega_1\lambda \neq 0$. Then, $S(\Omega_1) = H_\Sigma(1\mathtt{da};4)$ if $\omega_2\lambda = 0$ and $S(\Omega_1) = H_\Sigma(1\mathtt{daa};4)$ if $\omega_2\lambda \neq 0$.

If $2\omega_1 = 0$, then the edge $\mathrm{pc}(\omega_1, 1)$ of weight 1 is in an orbit $\Omega_2 \neq \Omega_1$. Then, $S(\Omega_2) = H_\Sigma(1\mathtt{dx};4)$ if $\omega_2\lambda = 0$ and $S(\Omega_2) = H_\Sigma(1\mathtt{dxx};4)$ if $\omega_2\lambda \neq 0$. If $\omega_1\lambda^2 = 0$, then the edge $\mathrm{pc}(-\omega_1\lambda, 1)$ of weight 1 is in an orbit $\Omega_2 \neq \Omega_1$. Then, $S(\Omega_2) = H_\Sigma(1\mathtt{db};4)$ if $\omega_2\lambda = 0$ and $S(\Omega_2) = H_\Sigma(1\mathtt{dbb};4)$ if $\omega_2\lambda \neq 0$. If $\omega_1\lambda^2 \neq 0$ and $2\omega_1 \neq 0$, there is no edge of weight 1 outside Ω_1, thus simply put $\Omega_2 := \Omega_1$. Consequently, for any genus-zero $\Omega \neq \Omega_1, \Omega_2$, the following holds in all cases: $S(\Omega)$ equals $H_\Sigma(2\mathtt{d};4,1)$ or $H_\Sigma(2\mathtt{d};4,-1)$ if $\omega_2\lambda = 0$ and $S(\Omega)$ equals $H_\Sigma(2\mathtt{da};4,1)$ or $H_\Sigma(2\mathtt{da};4,-1)$ if $\omega_2\lambda \neq 0$. $\square$

4.5.3.1 The Other Rings

Note that there is no complete monovalent vertex since $\omega_1 \neq 0$, hence our primary goal is to put upper bounds on the weights of the edges outside the bulk regions, i.e. the edges $\mathrm{pc}(m, 1)$ for $m \in \mathfrak{m}$. The element $XYXY \in \Gamma$ fixes $\mathrm{pc}(0,1) \in \mathcal{P}(\Bbbk)$, thus it acts on the set of edges outside the bulk regions. Let the size of a bulk region be 2^{k_0}, then each orbit of $XYXY$ in this action consists of 2^{k_0-1} edges. We put upper bounds on the weights of edges in terms of the weights of other edges in the same $(XYXY)$-orbit. Precisely, we write $R \succ (k;n)$ for integers $k, n \geq 1$ if

$$\begin{gathered}\text{at least one of the edges } \operatorname{pc}(m,1),\ \operatorname{pc}(m,1)\cdot (XYXY)^{2^{(n'-1)}\cdot q} \\ \text{has weight at most } 2^{-k}\end{gathered} \tag{4.7}$$

for all positive integers n', q with $n' \leq n$ and $2 \nmid q$, and all $m \in \mathfrak{m}$. In proving statements of the kind $R \succ (k;n)$, we use the following formula:

$$\operatorname{pc}(m,1)\cdot (XYXY)^{2^{(n-1)}\cdot q} = \operatorname{pc}(m + \omega_n \beta_{n,q}(m)\alpha(m), 1) \tag{4.8}$$

where $\alpha(m) := -1 + \omega_1 + \lambda m$ and $\beta_{n,q}(m)$ are recursively defined as follows:

$$\beta_{n,1}(m) := 1, \qquad \beta_{n,q}(m) := \beta_{n,q-2}(m)(1 + \omega_{n+1}\lambda) + \delta_n.$$

Clearly, $\alpha(m), \beta_{n,q}(m) \in R^*$. Moreover, $\beta_{n,q}(m) - 1 \in R\delta_n$.

Proof of (4.8). We prove the formula inductively. For $n = q = 1$, it can be verified directly. For $q = 1$ and $n \geq 2$,

$$\operatorname{pc}(m,1)\cdot (XYXY)^{2^{n-1}} = \operatorname{pc}(m,1)\cdot (XYXY)^{2^{n-2}}\cdot (XYXY)^{2^{n-2}},$$

hence one has the following by induction on n:

$$\begin{gathered}\operatorname{pc}(m,1)\cdot (XYXY)^{2^{n-1}} = \operatorname{pc}(m + \omega_{n-1}\alpha(m), 1)\cdot (XYXY)^{2^{n-2}} = \\ \operatorname{pc}(m + \omega_{n-1}\alpha(m) + \omega_{n-1}\alpha(m + \omega_{n-1}\alpha(m)), 1) = \\ \operatorname{pc}(m + \omega_{n-1}(\alpha(m) + \alpha(m) + \omega_{n-1}\lambda\alpha(m)), 1) = \\ \operatorname{pc}(m + \omega_n\alpha(m), 1).\end{gathered}$$

For $q > 1$ and fixed n,

$$\operatorname{pc}(m,1)\cdot (XYXY)^{2^{(n-1)}\cdot q} = \operatorname{pc}(m,1)\cdot (XYXY)^{2^{(n-1)}\cdot (q-2)}\cdot (XYXY)^{2^n},$$

hence one has the following by induction on q:

$$\begin{gathered}\operatorname{pc}(m,1)\cdot (XYXY)^{2^{(n-1)}\cdot q} = \operatorname{pc}(m + \omega_n\beta_{n,q-2}(m)\alpha(m), 1)\cdot (XYXY)^{2^n} = \\ \operatorname{pc}(m + \omega_n\beta_{n,q-2}(m)\alpha(m) + \omega_{n+1}\alpha(m + \omega_n\beta_{n,q-2}(m)\alpha(m)), 1) = \\ \operatorname{pc}(m + \omega_n\beta_{n,q-2}(m)\alpha(m) + \omega_{n+1}\alpha(m) + \omega_{n+1}\lambda\omega_n\beta_{n,q-2}(m)\alpha(m), 1) = \\ \operatorname{pc}(m + \omega_n(1 + \omega_{n+1}\lambda)\beta_{n,q-2}(m)\alpha(m) + \omega_{n+1}\alpha(m), 1) = \\ \operatorname{pc}(m + \omega_n\beta_{n,q}(m)\alpha(m), 1).\end{gathered}$$

□

We prove many statements of the kind $R \succ (k;n)$ below, and all of these proofs share a common structure. We mention it here to avoid repetition below. In each step of the proof, we choose a particular integer n' in the range $\{1,2,\ldots,n\}$. Then we suppose, for contradiction, that there is some $m \in \mathfrak{m}$ and some positive integer q such that both of the edges $\mathrm{pc}(m,1)$, $\mathrm{pc}(m,1)\cdot(XYXY)^{2^{(n'-1)\cdot q}}$ have weight at least 2^{-k+1} (see 4.7). Since n', m, q never change during the proof step, we briefly write $\alpha := \alpha(m)$ and $\beta := \beta_{n',q}(m)$. (Moreover, we write $\beta = 1+\delta_{n'}\gamma$). The assumption that both edges have weight at least 2^{-k+1} can be formulated as

$$\omega_{k-1}m(m-\lambda)=0, \qquad \omega_{k-1}(m+\omega_{n'}\alpha\beta)(m+\omega_{n'}\alpha\beta-\lambda)=0$$

together with the following:

$$\begin{array}{lll} 1+\omega_{k-1}(\lambda-m)\in\langle 1-\omega_1\rangle, & 1+\omega_{k-1}(\lambda-m-\omega_{n'}\alpha\beta)\in\langle 1-\omega_1\rangle & \text{if } k\geq 2, \\ 1-\omega_1+m\in\langle 1-\omega_1\rangle, & 1-\omega_1+m+\omega_{n'}\alpha\beta\in\langle 1-\omega_1\rangle & \text{if } k=1. \end{array}$$

We now prove some statements of this kind.

If $\omega_3 \neq 0$, then $R \succ (1;2)$. First consider the case $n'=1$. Then, $m(m-\lambda)=0$ and $(m+\omega_1\alpha\beta)(m+\omega_1\alpha\beta-\lambda)=0$. Consequently, $\omega_1\alpha\beta(2m-\lambda+\omega_1\alpha\beta)=0$, thus $\omega_1(2m-\lambda+\omega_1\alpha(1+\delta_1\gamma))=0$. Since $2m-\lambda+\omega_1\alpha=\delta_1(1+m)$, one gets $\omega_1\delta_1(1+m+\omega_1\alpha\gamma)=0$, thus $\omega_2=0$.

Now let $n'=2$. Then, $1-\omega_1+m\in\langle 1-\omega_1\rangle$ and $1-\omega_1+m+\omega_2\alpha\beta\in\langle 1-\omega_1\rangle$ which implies that the order $|\langle 1-\omega_1\rangle|$ is different in the rings $R/\mathfrak{m}\omega_2$ and R/ω_2. But, $(1-\omega_1)^4=1$ in the former ring and $(1-\omega_1)\neq 1$ in the latter ring. As a result, $\omega_1\lambda=(1-\omega_1)^2-1=\omega_2 r$ for some $r\in R^*$. Thus, $\omega_1(\lambda-2r(1-\omega_1 r)^{-1})=0$, implying $\omega_1(\lambda-2s)=0$ for some odd integer s. Therefore, there are integers $s(m), s'$ such that $\omega_1(m-2s(m))=0$ and $\omega_1(\omega_2-4s')=0$. Now, one also has $\omega_2(2m-\lambda+\omega_2\alpha\beta)=0$ as before. Thus, $2\omega_2(2s(m)-s+2s'\alpha\beta)=0$, implying $2\omega_2=0$. Hence, $\omega_3=0$. $\square$

If $\omega_1\omega_2 \neq 0$, then $R \succ (2;1)$. We only have the case $n'=1$. Thus, $\omega_1 m(m-\lambda)=0$ and $\omega_1(m+\omega_1\alpha\beta)(m+\omega_1\alpha\beta-\lambda)=0$, hence $\omega_1^2\alpha\beta(2m-\lambda+\omega_1\alpha\beta)=0$. In a way analogous to the previous proof, one finds $\omega_1^2\delta_1(1+m+\omega_1\alpha\gamma)=0$, thus $\omega_1\omega_2=0$. $\square$

If $\omega_2^2 \neq 0$, *then* $R \succ (3;1)$. We have $n' = 1$. Therefore, $\omega_2 m(m-\lambda) = 0$ and $\omega_2(m+\omega_1\alpha\beta)(m+\omega_1\alpha\beta-\lambda) = 0$, thus $\omega_2\omega_1\alpha\beta(2m-\lambda+\omega_1\alpha\beta) = 0$. In a way analogous to the first proof, one finds $\omega_2\omega_1\delta_1(1+m+\omega_1\alpha\gamma) = 0$, thus $\omega_2^2 = 0$. □

If $\omega_2^2 = 0$ *but* $\mathfrak{m}\omega_1\omega_2 \neq 0$, *then* $R \succ (2;2)$. We already know $R \succ (2;1)$ by the second proof, thus we only consider the case $n' = 2$. Then, $\omega_1 m(m-\lambda) = 0$ and $\omega_1(m+\omega_2\alpha\beta)(m+\omega_2\alpha\beta-\lambda) = 0$. Consequently, $\omega_1\omega_2\alpha\beta(2m-\lambda+\omega_2\alpha\beta) = 0$, thus $\omega_1\omega_2(2m-\lambda) = 0$. Since $\omega_1\omega_2(2+\omega_1\lambda) = 0$, one gets $2\omega_1\omega_2(1+\omega_1 m) = 0$, hence $2\omega_1\omega_2 = 0$. This implies $\omega_1\omega_2\lambda = 2\omega_1\omega_2 m = 0$, hence $\mathfrak{m}\omega_1\omega_2 = 0$. □

Rings with $\omega_4 = 0$ and $\omega_2\lambda^2 \notin R\cdot 2\omega_2$ Let $R' := R/\langle 2\omega_2, \omega_2\lambda^3\rangle$, then $\omega_2\omega_1^2 = \omega_2\lambda^2 \neq 0$ in R'. Let $\Omega' \subset \mathcal{P}(R')$ be the image of Ω, and let d be the degree of the covering $\Omega' \to \mathcal{P}(\Bbbk)$ so that $|\Omega'| = 3d$. There are $\frac{d}{4}$ bulk regions of size 8 in Ω'. We will show that the sum of weights over the edges outside the bulk regions is less than or equal to $\frac{d}{4}$. This will imply that $\chi(\Omega') \leq -\frac{|\Omega'|}{6} + \frac{d}{4} + \frac{d}{4} = 0$. Now, $R' \succ (2;2)$ because $\omega_2^2 = 0$ but $\omega_1 \cdot \omega_1\omega_2 \neq 0$. Moreover,

$R' \succ (3;1)$. We only have the case $n' = 1$. Then, $1+\omega_2(\lambda-m) \in \langle 1-\omega_1\rangle$ and $1+\omega_2(\lambda-m-\omega_1\alpha\beta) \in \langle 1-\omega_1\rangle$. Since $\omega_2^2 = 0$, one has $(1+\omega_2(\lambda-m))^{-1} = 1-\omega_2(\lambda-m)$, hence $1-\omega_2\omega_1\alpha\beta = (1-\omega_2(\lambda-m))\cdot(1+\omega_2(\lambda-m-\omega_1\alpha\beta)) \in \langle 1-\omega_1\rangle$. But, $1-\omega_2\omega_1\alpha\beta = 1-\omega_2\omega_1\alpha(1+\delta_1\gamma) = 1-\omega_2\omega_1\alpha = 1+\omega_2\omega_1(1-\omega_1-\lambda m) = 1+\omega_2\omega_1+\omega_2\omega_1^2$ since $\omega_2\omega_1\lambda m \in \mathfrak{m}\omega_2\lambda^2 = 0$. Note that $(1-\omega_1)^4 = 1+\omega_2\omega_1 \neq 1$ and $(1-\omega_1)^8 = 1$. On the other hand, $(1+\omega_2\omega_1+\omega_2\omega_1^2)^2 = 1$ but $1+\omega_2\omega_1+\omega_2\omega_1^2 \neq 1+\omega_2\omega_1$ and $1+\omega_2\omega_1+\omega_2\omega_1^2 \neq 1$, hence $1+\omega_2\omega_1+\omega_2\omega_1^2 \notin \langle 1-\omega_1\rangle$.

If Ω' contains no edge of weight 1, then the sum of weights over the edges outside the bulk regions is bounded by $\frac{1}{2}\cdot\frac{d}{4}+\frac{1}{4}\cdot\frac{d}{4}+\frac{1}{8}\cdot\frac{d}{2} = \frac{d}{4}$. Suppose Ω' contains an edge $\mathrm{pc}(m,1)$ of weight 1. Let $R'' := R/\omega_2\omega_1$ and let $\Omega'' \subset \mathcal{P}(R'')$ be the image of Ω'. Then $\mathrm{pc}(m+\omega_2\alpha, 1) \in \Omega'$ is in a region of size 4, but maps to a region of size 1 in Ω''. (Because $\omega_1(m+\omega_2\alpha)(m+\omega_2\alpha-\lambda) = \omega_2\omega_1^2 \neq 0$ in R', but $(m+\omega_2\alpha)(m+\omega_2\alpha-\lambda) = 0$ in R''). Therefore, the covering $\Omega' \to \Omega''$ is 4-fold, thus Ω' contains all 16 edges of the form $\mathrm{pc}(m+\omega_2\omega_1(\epsilon_1+\epsilon_2\omega_1), 1)\cdot(XYXY)^k$

for $\epsilon_1, \epsilon_2 \in \{0,1\}$ and $k \in \{0,1,2,3\}$. The region of size 4 which contains the edge $\mathrm{pc}(m + \omega_2\alpha, 1)$ consists of the 4 edges with $k = 2$. This region has weight 1, because $1 + \omega_2(\lambda - m - \omega_2\alpha) = 1 + \omega_2(\lambda - m) \in \langle 1 - \omega_1 \rangle$. Since $R' \succ (3;1)$, the 8 edges with $k = 1, 3$ have weight $\frac{1}{8}$. The 2 edges with $k = 0$ and $\epsilon_1 = 1$ have weight $\frac{1}{4}$, because $1 + \omega_1(\lambda - m - \omega_2\omega_1(1 + \epsilon_2\omega_1)) = 1 + \omega_1(\lambda - m - \omega_2\omega_1) = (1 + \omega_1(\lambda - m))(1 + \omega_2\omega_1^2) \notin \langle 1 - \omega_1 \rangle$ since $1 + \omega_2\omega_1^2 \notin \langle 1 - \omega_1 \rangle$. Finally, the edge $\mathrm{pc}(m + \omega_2\omega_1^2, 1)$ has weight $\frac{1}{2}$, since $1 - \omega_1 + m + \omega_2\omega_1^2 = (1 - \omega_1 + m)(1 + \omega_2\omega_1^2)$. Overall, the sum of weights over these 16 edges is $1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} = 4 = \frac{16}{4}$. □

Rings with $\omega_4 = 0$ and $\omega_2\lambda^2 \in R \cdot 2\omega_2$ Since $\omega_1 = 2 + \lambda$, there is some $r \in R$ such that $\omega_2(\omega_1\lambda - 2r) = 0$. Then, $\omega_3 = \omega_2(2 + \delta_1\omega_1\lambda) = 2\omega_2(1 + \delta_1 r)$, hence $\omega_4 = \omega_3(2 + \delta_2\delta_1\omega_1\lambda) = 2\omega_3(1 + \delta_2\delta_1 r) = 4\omega_2(1 + \delta_1 r)(1 + \delta_2\delta_1 r)$. Since $\omega_4 = 0$, one has $4\omega_2 = 0$ and thus $\omega_2\lambda^2 = 2\omega_2(\epsilon_1 + \epsilon_2\lambda)$ for some $\epsilon_1, \epsilon_2 \in \{0,1\}$.

In particular, $\omega_2^2\lambda = \omega_2\omega_1\delta_1\lambda = \omega_2(2 + \lambda)(2 + \omega_1\lambda)\lambda = \omega_2\lambda^2(\omega_1\lambda + 2\omega_1 + 2) = \omega_2\lambda^2(\lambda^2 + 2\lambda + 2\omega_1 + 2) = 2\omega_2(\epsilon_1 + \epsilon_2\lambda) \cdot 2(\epsilon_1 + \epsilon_2\lambda + \lambda + \omega_1 + 1) = 0$. Hence, $\omega_3 = \omega_2(2 + \omega_2\lambda) = 2\omega_2$. Note that the case $2\omega_2 = \omega_2\lambda^2 = 0$ has already been considered, hence we assume $\omega_3 = 2\omega_2 \neq 0$. In all the subcases below, d denotes the degree of the covering $\Omega \to \mathcal{P}(\Bbbk)$. Thus $|\Omega| = 3d$ and there are $\frac{d}{8}$ bulk regions of size 16 in Ω.

Suppose $\omega_1^2 = 0$ or $\omega_1^2 = 4\omega_1$, then $8\omega_1 = 0$ Let $R' := R/\omega_1$, then the covering $\mathcal{P}(R) \to \mathcal{P}(R')$ is 8-fold. Let $\Omega' \subset \mathcal{P}(R')$ be the image of Ω. Since a bulk region of size 16 in Ω maps to a region of size 2 in Ω', the covering $\Omega \to \Omega'$ is also 8-fold and Ω is the only orbit in $\mathcal{P}(R)$ which maps to Ω'. Let $\Omega_1 \subset \mathcal{P}(R)$ be the orbit of the edge $\mathrm{pc}(0,1)$ of weight 1. Then, $S(\Omega_1) = H_\Sigma(1;8)$ if $\omega_1^2 = 0$ and $S(\Omega_1) = H_\Sigma(1\mathtt{w};8)$ if $\omega_1^2 = 4\omega_1$. For any genus-zero $\Omega \neq \Omega_1$, one has $S(\Omega) = H_\Sigma(2;8,a)$ with $a \in \{1,3,5,7\}$ if $\omega_1^2 = 0$ and $S(\Omega) = H_\Sigma(2\mathtt{w};8,a)$ with $a \in \{-1,1\}$ if $\omega_1^2 = 4\omega_1$. □

Suppose $\omega_2\omega_1 = 0$ but $\omega_1^2 \notin \{0, 4\omega_1\}$ The edges $\mathrm{pc}(0,1)$ and $\mathrm{pc}(2\omega_2, 1)$ of weight 1 are in the same orbit Ω_1 so that $S(\Omega_1) = H_\Sigma(\mathtt{1da}; 8)$. If $\omega_1\lambda^2 \neq 0$, there are no other edges of weight 1. If $\omega_1\lambda^2 = 0$, there are two other edges of weight 1, namely $\mathrm{pc}(-\omega_1\lambda, 1)$ and $\mathrm{pc}(-\omega_1\lambda + 2\omega_2, 1)$. These are in the same orbit $\Omega_2 \neq \Omega_1$ so that $S(\Omega_2) = H_\Sigma(\mathtt{1db}; 8)$. Hence, suppose Ω contains no edge of weight 1. We will show that $R \succ (2;1)$, hence the sum of weights over the edges outside the bulk regions is less than or equal to $\frac{1}{4} \cdot \frac{d}{2} + \frac{1}{2} \cdot \frac{d}{2} = \frac{3d}{8}$. Consequently, $\chi(\Omega) \leq -\frac{|\Omega|}{6} + \frac{d}{8} + \frac{3d}{8} = 0$.

$R \succ (2;1)$. We only have the case $n' = 1$. Then, $1 + \omega_1(\lambda - m) \in \langle 1 - \omega_1 \rangle$ and $1 + \omega_1(\lambda - m - \omega_1\alpha\beta) \in \langle 1 - \omega_1 \rangle$. In fact, odd powers of $1 - \omega_1$ are not possible, hence $1 + \omega_1(\lambda - m)$ and $1 + \omega_1(\lambda - m - \omega_1\alpha\beta)$ are in the set $D := \langle (1-\omega_1)^2 \rangle = \{1, 1+\omega_1\lambda, 1+\omega_2\lambda, 1+\omega_1\lambda+\omega_2\lambda\}$. Moreover, $\omega_1^2\alpha\beta = \omega_1^2\alpha(1+\delta_1\gamma) = \omega_1^2\alpha$. We briefly write $\theta := \omega_1 m$. Then, $\omega_1\alpha = \omega_1(-1+\omega_1+\lambda m) = \omega_1(-1+\omega_1)+\lambda\theta$. Hence, $1+\omega_1(\lambda-m-\omega_1\alpha\beta) = 1+\omega_1\lambda-\theta-\omega_1^2(-1+\omega_1)-\omega_1\lambda\theta = 1-\omega_2-\theta(1+\omega_1\lambda) \in D$. On the other hand, $(1+\omega_1\lambda)(1+\omega_1(\lambda-m)) = 1+2\omega_2-\theta(1+\omega_1\lambda) \in D$. As a result, ω_2 is the difference of two elements in D. One checks that all possibilities come down to the equations $\omega_2 = \omega_1\lambda$ and $\omega_2 = -\omega_1\lambda$. Therefore, $\omega_2 = \omega_1(2+\omega_1\lambda) = \omega_1(2 \pm \omega_2) = 2\omega_1$. Thus $\omega_1\lambda \in \{2\omega_1, -2\omega_1\}$, hence $\omega_1^2 \in \{0, 4\omega_1\}$. □

Suppose $\omega_1\lambda = 4\omega_1$, then $8\omega_1 = 0$ The only edges of weight 1, namely $\mathrm{pc}(0,1)$ and $\mathrm{pc}(4\omega_1, 1)$, are in the same orbit Ω_1 so that $S(\Omega_1) = H_\Sigma(\mathtt{1d'}; 8)$. Hence, suppose Ω contains no edge of weight 1. Note that $R \succ (2;1)$ since $\omega_2\omega_1 \neq 0$. Then, the sum of weights over the edges outside the bulk regions is less than or equal to $\frac{1}{4} \cdot \frac{d}{2} + \frac{1}{2} \cdot \frac{d}{2} = \frac{3d}{8}$. Consequently, $\chi(\Omega) \leq -\frac{|\Omega|}{6} + \frac{d}{8} + \frac{3d}{8} = 0$. □

Suppose $\omega_2\lambda = 0$ but $\omega_1\lambda \neq 4\omega_1$ We will show that $R \succ (1;3)$ and $R \succ (2;2)$, hence the sum of weights over the edges outside the bulk regions in Ω is less than or equal to $\frac{1}{4} \cdot \frac{3d}{4} + \frac{1}{2} \cdot \frac{d}{8} + \frac{d}{8} = \frac{3d}{8}$. Consequently, $\chi(\Omega) \leq -\frac{|\Omega|}{6} + \frac{d}{8} + \frac{3d}{8} = 0$.

$R \succ (1;3)$. We already know $R \succ (1;2)$ since $\omega_3 \neq 0$, thus we only consider the case $n' = 3$. Then, $1 - \omega_1 + m \in \langle 1 - \omega_1 \rangle$ and $1 - \omega_1 + m + \omega_3\alpha\beta \in \langle 1 - \omega_1 \rangle$.

Moreover, $1-\omega_1+m+\omega_3\alpha\beta = 1-\omega_1+m+\omega_3 = (1-\omega_1+m)(1+\omega_3)$, thus $1+\omega_3 \in \langle 1-\omega_1\rangle$. But $(1+\omega_3)^2 = 1$ and $(1+\omega_3) \notin \{1, 1-\omega_1\}$, thus one must have $\omega_3 = \omega_1\lambda$. Then $\omega_1\lambda = 2\omega_2 = \omega_1(4+2\omega_1\lambda) = 4\omega_1(1+\omega_2) = 4\omega_1$. $\square$

$R \succ (2;2)$. We already know $R \succ (2;1)$ since $\omega_2\omega_1 \neq 0$, thus we only consider the case $n' = 2$. Then, $1+\omega_1(\lambda-m) \in \langle 1-\omega_1\rangle$ and $1+\omega_1(\lambda-m-\omega_2\alpha\beta) \in \langle 1-\omega_1\rangle$. Moreover, $1+\omega_1(\lambda-m-\omega_2\alpha\beta) = 1+\omega_1(\lambda-m)+\omega_2\omega_1 = (1+\omega_1(\lambda-m))(1+\omega_2\omega_1)$, thus $1+\omega_2\omega_1 \in \langle 1-\omega_1\rangle$. But $\omega_2\omega_1 = 2\omega_2 = \omega_3$, hence we conclude as in the previous proof. $\square$

Suppose $\omega_2\omega_1 \neq 0$ and $\omega_2\lambda \neq 0$ As in the previous case, we will show that $R \succ (1;3)$ and $R \succ (2;2)$, hence the sum of weights over the edges outside the bulk regions in Ω is less than or equal to $\frac{1}{4}\cdot\frac{3d}{4}+\frac{1}{2}\cdot\frac{d}{8}+\frac{d}{8} = \frac{3d}{8}$. Consequently, $\chi(\Omega) \leq -\frac{|\Omega|}{6}+\frac{d}{8}+\frac{3d}{8} = 0$. In fact, we will show that $R' \succ (1;3)$ and $R' \succ (2;2)$ where $R' := R/2\omega_2\lambda = R/2\omega_2\omega_1$. Note that the inequalities $\omega_2\omega_1 \neq 0$, $\omega_2\lambda \neq 0$ and $\omega_3 = 2\omega_2 \neq 0$ hold in R' as well.

$R' \succ (1;3)$. We already know $R' \succ (1;2)$ since $\omega_3 \neq 0$, thus we only consider the case $n' = 3$. Then, $1-\omega_1+m \in \langle 1-\omega_1\rangle$ and $1-\omega_1+m+\omega_3\alpha\beta \in \langle 1-\omega_1\rangle$. Moreover, $1-\omega_1+m+\omega_3\alpha\beta = 1-\omega_1+m+\omega_3 = (1-\omega_1+m)(1+\omega_3)$, thus $1+\omega_3 \in \langle 1-\omega_1\rangle$. But $(1+\omega_3)^2 = 1$ and $(1+\omega_3) \neq 1$, thus $\omega_3 = \omega_2\lambda$. Then $\omega_2\omega_1 = \omega_2(2+\lambda) = 2\omega_2+\omega_3 = 4\omega_2 = 0$. $\square$

$R' \succ (2;2)$. Note that $\omega_2^2 = \omega_2(2+\lambda)(2+\omega_1\lambda) = 0$. Thus if $\mathfrak{m}\omega_2\omega_1 \neq 0$, we already know $R' \succ (2;2)$, hence suppose $\mathfrak{m}\omega_2\omega_1 = 0$. Even then, we already know $R' \succ (2;1)$ since $\omega_2\omega_1 \neq 0$, thus we only consider the case $n' = 2$. Therefore, $1+\omega_1(\lambda-m) \in \langle 1-\omega_1\rangle$ and $1+\omega_1(\lambda-m-\omega_2\alpha\beta) \in \langle 1-\omega_1\rangle$. Moreover, $1+\omega_1(\lambda-m-\omega_2\alpha\beta) = 1+\omega_1(\lambda-m)+\omega_2\omega_1 = (1+\omega_1(\lambda-m))(1+\omega_2\omega_1)$, thus $1+\omega_2\omega_1 \in \langle 1-\omega_1\rangle$. But $(1+\omega_2\omega_1)^2 = 1$ and $(1+\omega_2\omega_1) \neq 1$, thus $\omega_2\omega_1 = \omega_2\lambda$. As a result, $2\omega_2 = \omega_2(\omega_1-\lambda) = 0$. $\square$

Rings with $\omega_4 \neq 0$, but $\omega_1\omega_2 = 0$ and $\omega_1^2 \in R \cdot 2\omega_1$ Let n be the least such that $2^n\omega_1 = 0$. Since $\omega_1\omega_2 = 0$ and $\omega_1^2 \in R \cdot 2\omega_1$, one has $\omega_1^2 = 0$ or $\omega_1^2 = 2^{n-1}\omega_1$. Then, $\omega_{\ell+1} = \omega_\ell(2 + \omega_\ell\lambda) = 2\omega_\ell$ for any $\ell \geq 1$, thus $\omega_\ell = 2^{\ell-1}\omega_1$. In particular, $\omega_{n+1} = 0$ but $\omega_n \neq 0$, hence the size of a bulk region is 2^{n+1}.

Let $R' := R/\omega_1$, then the degree of the covering $\mathcal{P}(R) \to \mathcal{P}(R')$ is 2^n. Let $\Omega' \subset \mathcal{P}(R')$ be the image of Ω. Since a bulk region of size 2^{n+1} in Ω maps to a region of size 2 in Ω', the covering $\Omega \to \Omega'$ also has degree 2^n and Ω is the only orbit in $\mathcal{P}(R)$ which maps to Ω'. Let $\Omega_1 \subset \mathcal{P}(R)$ be the orbit of the edge pc$(0,1)$ of weight 1. Then, $S(\Omega_1) = H_\Sigma(1; 2^n)$ if $\omega_1^2 = 0$ and $S(\Omega_1) = H_\Sigma(1\mathtt{w}; 2^n)$ if $\omega_1^2 = 2^{n-1}\omega_1$. For any genus-zero $\Omega \neq \Omega_1$, one has $S(\Omega) = H_\Sigma(2; 2^n, a)$ with $a \in \{-1, 1, 2^{n-1}-1, 2^{n-1}+1\}$ if $\omega_1^2 = 0$ and $S(\Omega) = H_\Sigma(2\mathtt{w}; 2^n, a)$ with $a \in \{-1, 1\}$ if $\omega_1^2 = 2^{n-1}\omega_1$. □

Rings with $\omega_4 \neq 0$, and $\omega_1\omega_2 \neq 0$ or $\omega_1^2 \notin R \cdot 2\omega_1$ Let d be the degree of the covering $\Omega \to \mathcal{P}(\Bbbk)$. Then $|\Omega| = 3d$ and the number of the bulk regions in Ω is at most $\frac{2d}{32} = \frac{d}{16}$. We will show that the sum T of weights over the edges outside the bulk regions is less than or equal to $\frac{7d}{16}$, which implies $\chi(\Omega) \leq -\frac{|\Omega|}{6} + \frac{7d}{16} + \frac{d}{16} = 0$. Note that $R \succ (1; 2)$ since $\omega_3 \neq 0$.

If $\omega_2^2 \neq 0$, then $R \succ (3; 1)$. One also has $R \succ (1; 2)$, hence $T \leq \frac{1}{8} \cdot \frac{d}{2} + \frac{1}{2} \cdot \frac{d}{4} + \frac{d}{4} = \frac{7d}{16}$. If $\omega_2^2 = 0$ but $\mathfrak{m}\omega_1\omega_2 \neq 0$, then $R \succ (2; 2)$. Hence, $T \leq \frac{1}{4} \cdot \frac{3d}{4} + \frac{d}{4} = \frac{7d}{16}$. If $\mathfrak{m}\omega_1\omega_2 = 0$, we will show that $R \succ (1; 3)$ and $R \succ (2; 1)$. Hence, $T \leq \frac{1}{4} \cdot \frac{d}{2} + \frac{1}{2} \cdot \frac{3d}{8} + \frac{d}{8} = \frac{7d}{16}$.

$R \succ (1; 3)$. We already know $R \succ (1; 2)$, thus we only have the case $n' = 3$. Then $m(m - \lambda) = 0$ and $(m + \omega_3\alpha\beta)(m + \omega_3\alpha\beta - \lambda) = 0$. Therefore $\omega_3\alpha\beta(2m - \lambda + \omega_3\alpha\beta) = 0$, thus $\omega_3(2m - \lambda) = 0$. But $\omega_3(2 + \lambda) = \omega_3\omega_1 = 0$, thus one gets $2\omega_3(1 + m) = \omega_3(2 + \lambda + 2m - \lambda) = 0$. Hence $2\omega_3 = 0$, thus $\omega_4 = \omega_3(2 + \omega_3\lambda) = 0$. □

$R \succ (2; 1)$. If $\omega_1\omega_2 \neq 0$, we already know $R \succ (2; 1)$. Hence suppose $\omega_2\omega_1 = 0$, thus $\omega_1^2 \notin R \cdot 2\omega_1$. Moreover, $\omega_{\ell+1} = \omega_\ell(2 + \omega_\ell\lambda) = 2\omega_\ell$ for any $\ell \geq 2$, thus

$\omega_\ell = 2^{\ell-2}\omega_2$. In particular, $4\omega_2 = \omega_4 \neq 0$. We will show that $R' \succ (2;1)$ where $R' := R/\langle 2\omega_1^2, \omega_1^2\lambda\rangle$. Note that $\omega_1^2 \notin R' \cdot 2\omega_1$ as well. Moreover, $4\omega_1 \neq 0$ in R' since $4\omega_2 \neq 0$ in R. In the rest, all equalities and inequalities are in R'. Then $\omega_2 = \omega_1(2 + \omega_1\lambda) = 2\omega_1$, thus $\omega_\ell = 2^{\ell-1}\omega_1$ for any ℓ. Let $n \geq 3$ be the least such that $2^n\omega_1 = 0$. Then $|\langle 1 - \omega_1\rangle| = 2^n$ since the following holds for $\ell \geq 2$: $(1-\omega_1)^{2^\ell} = 1 + \omega_\ell\lambda = 1 - 2\omega_\ell = 1 - 2^\ell\omega_1$.

We now prove $R' \succ (2;1)$. We have $n' = 1$, thus $1 + \omega_1(\lambda - m) \in \langle 1 - \omega_1\rangle$ and $1 + \omega_1(\lambda - m - \omega_1\alpha\beta) \in \langle 1 - \omega_1\rangle$. Moreover, $1 + \omega_1(\lambda - m - \omega_1\alpha\beta) = 1 + \omega_1(\lambda - m) + \omega_1^2 = (1 + \omega_1(\lambda - m))(1 + \omega_1^2)$, thus $(1 + \omega_1^2) \in \langle 1 - \omega_1\rangle$. But $(1+\omega_1^2)^2 = 1$ and $(1 + \omega_1^2) \neq 1$, thus $\omega_1^2 = 2^{n-1}\omega_1 \in R' \cdot 2\omega_1$. □

4.5.3.2 Wheels

Wheels with $\omega_1 \cdot W$ cyclic Let $I \subset \Lambda$ be the ideal for which $\omega_1 \cdot W \cong \Lambda/I$. Recall that $\omega_1 = -(t-1)$. For a sufficiently large integer k, let

$$R(k) := \Lambda/\langle 2^k,\, (t-1)I\rangle, \qquad W(k) := \big(\mathbb{Z}_{2^k}[t]/(t-1)\big) \cdot e_1 \oplus R(k) \cdot e_2.$$

Thus, there is an epimorphism $W(k) \twoheadrightarrow W$ which restricts to an isomorphism $\omega_1 \cdot W(k) \cong \omega_1 \cdot W$. As a result, the induced covering $\mathcal{C}(W(k)) \to \mathcal{C}(W)$ is unramified. (Because if an edge $\mathrm{c}(a_1, a_2) \in \mathcal{C}(W(k))$ maps into a region of size 2^n in $\mathcal{C}(W)$, the edges $\mathrm{c}(a_1, a_2) \in \mathcal{C}(W(k))$ and $\mathrm{c}(a_1, a_2) \cdot (YX)^{2^n} = \mathrm{c}(a_1, a_2 + \omega_n(\lambda \cdot a_2 - a_1)) \in \mathcal{C}(W(k))$ map to the same edge in $\mathcal{C}(W)$, i.e. for some integer k', the elements

$$((1-\omega_1)^{k'} - 1) \cdot a_1 \in W(k), \qquad ((1-\omega_1)^{k'} - 1) \cdot a_2 - \omega_n(\lambda \cdot a_2 - a_1) \in W(k)$$

map to $0 \in W$, but these elements are in $\omega_1 \cdot W(k)$, therefore they are equal to 0, which shows that the edge $\mathrm{c}(a_1, a_2)$ is contained in a region of size 2^n in $\mathcal{C}(W(k))$.) Thus, any orbit $\tilde{\Omega} \subset \mathcal{C}(W(k))$ which maps to Ω maps isomorphically onto Ω so that $S(\tilde{\Omega}) = S(\Omega)$. Consider the covering $\mathcal{C}(W(k)) \to \mathcal{P}(R(k))$ induced by the obvious epimorphism $W(k) \twoheadrightarrow R(k)$ defined by $e_1 \mapsto 0$, $e_2 \mapsto 1$, and let Ω' be the image of $\tilde{\Omega}$. Then, if Ω' contains no region of edge-weight 1 and no monovalent vertex, the covering $\tilde{\Omega} \to \Omega'$ is also unramified so that $S(\tilde{\Omega}) = S(\Omega')$. For sufficiently large k, one can guarantee that Ω' contains no region of edge-weight 1 and no monovalent vertex by varying $\tilde{\Omega}$ over all possibilities. Consequently, $S(\Omega)$ is equal to one of the Bu_3-sets found in the section on rings. $\square$

If $\omega_1 \cdot W$ is not cyclic, there is no region of size 2 in $\mathcal{C}(W)$. For contradiction, suppose that an edge $\mathrm{c}(a_1, a_2)$ is in a region of size 2, thus $((1-\omega_1)^k - 1) \cdot a_1 = 0$ and $((1-\omega_1)^k - 1) \cdot a_2 = \omega_1(\lambda \cdot a_2 - a_1)$ for some integer k. If k is odd, the former equation implies $\omega_1 \cdot a_1 = 0$. If k is even, $(1-\omega_1)^k - 1 = \omega_1 \lambda \phi$, hence the latter equation implies $\omega_1(a_1 - \lambda \cdot a_2 + \lambda\phi \cdot a_2) = 0$. In either case, $\omega_1 \cdot W$ is generated by $\omega_1 \cdot a_2$, hence it is cyclic.

Wheels with $\omega_1 \cdot W$ not cyclic and $\omega_2 \cdot W = 0$ One has $S(\Omega) = H_\Sigma(2\mathtt{d}'; 2)$ if $\omega_1 \cdot W \cong \Bbbk^2$ and $S(\Omega) = H_\Sigma(2\mathtt{d}; 2)$ otherwise. $\square$

Wheels with $\omega_1 \cdot W$ not cyclic and $\omega_2 \cdot W \neq 0$ The size of any region is 4 or at least 8. If an edge $\mathrm{c}(a_1, a_2)$ is in a region of size 4, then $\mathrm{c}(a_1, a_2) = \mathrm{c}(a_1, a_2 + \omega_2(\lambda \cdot a_2 - a_1))$ (Equation 3.9). This implies either

$$\omega_2(a_1 - \lambda \cdot a_2) = 0 \qquad \text{or} \qquad \underbrace{((1+\lambda)^k - 1) \cdot a_1 = 0}_{\text{for some } k \text{ for which } ((1+\lambda)^k - 1) \cdot W \neq 0} \quad .$$

Let n be greatest such that $(1-\omega_1)^{2^n} - 1$ does not annihilate W. The equation $((1-\omega_1)^k - 1) \cdot a_1 = 0$ implies $((1-\omega_1)^{2^n} - 1) \cdot a_1 = 0$. There are epimorphisms $\pi \colon W \to \Bbbk$ and $\pi' \colon W \to \Bbbk$ whose kernels contain the submodules

$$\{a \in W \mid \omega_2 \cdot a = 0\} \qquad \text{and} \qquad \{a \in W \mid ((1-\omega_1)^{2^n} - 1) \cdot a = 0\}. \tag{4.9}$$

Consider the coverings $\mathcal{C}(W) \to \mathcal{P}(\Bbbk)$ induced by π and π'. Then, any region of size 4 in Ω maps to $\mathrm{pc}(0,1) \in \mathcal{P}(\Bbbk)$ under one of these coverings. First suppose that one of these coverings maps all regions of size 4 in Ω to $\mathrm{pc}(0,1) \in \mathcal{P}(\Bbbk)$. Then, at most $\frac{|\Omega|}{3}$ edges in Ω are contained in regions of size 4, hence the number of regions is bounded by $\frac{1}{4} \cdot \frac{1}{3} \cdot |\Omega| + \frac{1}{8} \cdot \frac{2}{3} \cdot |\Omega| = \frac{1}{6} \cdot |\Omega|$, thus $\chi(\Omega) \leq 0$.

Henceforth, π and π' have distinct kernels and Ω contains edges $\mathrm{c}(a_1, a_2)$ and $\mathrm{c}(b_1, b_2)$ in regions of size 4 such that $\pi(a_1) = 0$ and $\pi'(b_1) = 0$. Thus one has

$n = 1$ because of the following: if $n = 0$, the former submodule in (4.9) contains the latter and if $n \geq 2$, the latter submodule contains the former. Therefore,

$$\omega_2(a_1 - \lambda \cdot a_2) = 0,$$
$$\underbrace{((1+\omega_1\lambda)^k - 1) \cdot b_1 = 0, \qquad ((1+\omega_1\lambda)^k - 1) \cdot b_2 = \omega_2(\lambda \cdot b_2 - b_1)}_{\text{for some odd } k}.$$

Note that $(1 + \omega_1\lambda)^k - 1 = \omega_1\lambda\sigma$ for some $\sigma \in \Lambda \smallsetminus \mathfrak{m}$. Then, $\omega_1\lambda \cdot b_1 = \sigma^{-1}((1+\omega_1\lambda)^k - 1) \cdot b_1 = 0$ and $\omega_1\lambda \cdot b_2 = (\delta_1 - \sigma)^{-1}\omega_2 \cdot b_1$. (Here, σ^{-1} and $(\delta_1 - \sigma)^{-1}$ are meaningful because even though σ, $(\delta_1 - \sigma) \in \Lambda \smallsetminus \mathfrak{m}$ may not be invertible in Λ, their actions on W are invertible (Remark 3.1)). Since $\omega_1\lambda \cdot b_2 = (\delta_1 - \sigma)^{-1}\omega_2 \cdot b_1$, one has $\omega_1\lambda^2 \cdot b_2 = (\delta_1 - \sigma)^{-1} \cdot \omega_2\lambda \cdot b_1 = 0$, hence $\omega_1\lambda^2 \cdot W = 0$. This allows one to replace $\omega_2 = \omega_1(2 + 2\lambda + \lambda^2)$ by $2\omega_1(1 + \lambda)$ in the equations. Hence, by introducing the brief notation $a_1' = (1 + \lambda)(a_1 - \lambda \cdot a_2)$ and $b_2' = (1 + \lambda)^{-1} \cdot (\delta_1 - \sigma) \cdot b_2$, we re-express the equations as follows:

$$2\omega_1 \cdot a_1' = 0, \qquad \omega_1\lambda \cdot b_1 = 0, \qquad \omega_1\lambda \cdot b_2' = 2\omega_1 \cdot b_1.$$

Since a_1' and b_1 generate W, one can write $b_2' = \phi_1 \cdot a_1' + \phi_2 \cdot b_1$ for some $\phi_1, \phi_2 \in \Lambda$. Moreover, $\phi_1 \notin \mathfrak{m}$ since b_1 and b_2' map to linearly independent vectors in $\Bbbk^2$. Thus, $2\omega_1 \cdot b_1 = \omega_1\lambda \cdot b_2' = \omega_1\lambda\phi_1 \cdot a_1'$. Finally, let $e_1 := \omega_1 \cdot b_1$ and $e_2 := \omega_1\phi_1 \cdot a_1'$. Then,

$$\omega_1 \cdot W = \mathbb{Z}_4 \cdot e_1 \oplus \mathbb{Z}_2 \cdot e_2 \quad \text{with} \quad \lambda \cdot e_1 = 0 \text{ and } \lambda \cdot e_2 = 2e_1.$$

When $\omega_1 \cdot W$ has this exceptional isomorphism type, $S(\Omega) = H_\Sigma(2\mathtt{dw}; 2)$. $\square$

Chapter 5

The Proof: Part II

In this chapter, we finish the proof of Theorem 1.2 which began in the previous chapter, that is, we complete the second step of the proof strategy outlined in Chapter 3. We say that a collection $\mathcal{B}$ of genus-zero Bu_3-sets is *closed* if, for any finite non-empty subcollection $\{E_i\} \subset \mathcal{B}$, the genus-zero orbits in the product Bu_3-set $\prod E_i$ are also in $\mathcal{B}$ (this includes the collections $\{E_i\}$ with repeated members). Thus, the notion of *closure* is defined in an obvious way. We show that the collection $\mathcal{B}_{\mathrm{A}}$ of the Bu_3-sets listed in the tables in Appendix A contains the closure $\overline{\mathcal{B}_\Omega}$ of the collection $\mathcal{B}_\Omega$ of the Bu_3-sets $S(\Omega)$ found in the previous chapter. In fact, the difference $\mathcal{B}_{\mathrm{A}} \smallsetminus \overline{\mathcal{B}_\Omega}$ consists only of the trivial Bu_3-set, which must be contained in $\mathcal{B}_{\mathrm{A}}$.

Consider the following subcollections whose union is equal to $\mathcal{B}_\Omega$:

- $\mathcal{B}'_\Pi$ consists of the Bu_3-sets which appear in the case $N = 6$. Let ω denote a third root of unity and let Π be the set of all primes in the principal ideal domain $\mathbb{Z}[\omega]$ distinct from $(\omega - 1)$ and 2. Then,

$$\mathcal{B}'_\Pi := \{H_\Pi(1;\pi^n) \mid \pi \in \Pi,\ n \geq 1\} \cup \{H_\Pi(\mathtt{D};\pi^n) \mid \pi \in \Pi,\ n \geq 0\}.$$

- $\mathcal{B}^2_\Pi$ consists of the Bu_3-sets which appear in the case $N = 3$, as well as the

Bu_3-sets of the form $H_\Pi(-;-)$ which appear in the case $N=1, p=3$:

$$\begin{aligned}\mathcal{B}^2_\Pi := \{&H_\Pi(1';2), H_\Pi(\texttt{1ax};2), H_\Pi(\texttt{1ay};2), H_\Pi(\texttt{3a}';2),\\ &H_\Pi(\texttt{1b};2), H_\Pi(\texttt{1ba};4), H_\Pi(\texttt{1bb};4),\} \ \cup\\ &\{H_\Pi(\circ;2^n), H_\Pi(\texttt{D};2^n) \mid n \geq 0\} \ \cup\\ &\{H_\Pi(1;2^n), H_\Pi(1^\times;2^n),\\ &\quad H_\Pi(3;2^n), H_\Pi(\texttt{3a};2^n), H_\Pi(\texttt{3b};2^n) \mid n \geq 1\} \ \cup\\ &\{H_\Pi(\texttt{3ba};2^n), H_\Pi(\texttt{3bb};2^n), H_\Pi(3/2,2^n), H_\Pi(3/2^\times,2^n) \mid n \geq 2\}.\end{aligned}$$

- $\mathcal{B}^3_\Pi$ consists of the Bu_3-sets of the form $H_\Pi(-;-)$ which appear in the case $N=2, p=3$:

$$\begin{aligned}\mathcal{B}^3_\Pi :=&\{H_\Pi(\bullet;(\omega-1)^n), H_\Pi(\texttt{D};(\omega-1)^n) \mid n \geq 0\} \ \cup\\ &\quad\{H_\Pi(1;(\omega-1)^n), H_\Pi(2;(\omega-1)^n) \mid n \geq 1\}.\end{aligned}$$

- $\mathcal{B}'_\Sigma$ consists of the Bu_3-sets which appear in the case $N=2, p>3$:

$$\mathcal{B}'_\Sigma := \{H_\Sigma(1;p^n), H_\Sigma(2;p^n,1), H_\Sigma(2;p^n,-1) \mid p>3 \text{ is prime}, \ n \geq 0\}.$$

- $\mathcal{B}^2_\Sigma$ consists of the Bu_3-sets which appear in the case $N=1, p=2$ as well as $H_\Sigma(\texttt{2d}^\bullet;2)$ which appears in the case $N=4$:

$$\begin{aligned}\mathcal{B}^2_\Sigma := \{&H_\Sigma(\texttt{1aa};2), H_\Sigma(\texttt{1ab};2), H_\Sigma(\texttt{2a};2,1),\\ &H_\Sigma(\texttt{1d}';4), H_\Sigma(\texttt{1da};4), H_\Sigma(\texttt{1db};4), H_\Sigma(\texttt{1dx};4),\\ &H_\Sigma(\texttt{1daa};4), H_\Sigma(\texttt{1dbb};4), H_\Sigma(\texttt{1dxx};4),\\ &H_\Sigma(\texttt{1d}';8), H_\Sigma(\texttt{1da};8), H_\Sigma(\texttt{1db};8),\\ &H_\Sigma(\texttt{2d}';2), H_\Sigma(\texttt{2d};2), H_\Sigma(\texttt{2dw};2), H_\Sigma(\texttt{2d}^\bullet;2),\\ &H_\Sigma(\texttt{2d}';4), H_\Sigma(\texttt{2d};4,1), H_\Sigma(\texttt{2d};4,-1),\\ &H_\Sigma(\texttt{2da};4,1), H_\Sigma(\texttt{2da};4,-1)\} \ \cup\\ &\{H_\Sigma(1;2^n), H_\Sigma(2;2^n,a) \mid n \geq 0, \ a^2 \equiv 1 \pmod{2^n}\} \ \cup\\ &\{H_\Sigma(\texttt{1w};2^n), H_\Sigma(\texttt{2w};2^n,1), H_\Sigma(\texttt{2w};2^n,-1) \mid n \geq 3\}.\end{aligned}$$

- $\mathcal{B}^3_\Sigma$ consists of the Bu_3-sets of the form $H_\Sigma(-;-)$ which appear in the case $N=2, p=3$ as well as $H_\Sigma(1^{\times\bullet};1)$ which appears in the case $N=3, p=2$:

$$\begin{aligned}\mathcal{B}^3_\Sigma := \{&H_\Sigma(1^{\times\bullet};1), H_\Sigma(2^\bullet;1), H_\Sigma(\texttt{2x}^\bullet;1), H_\Sigma(\texttt{2y}^\bullet;1)\} \ \cup\\ &\{H_\Sigma(1;3^n), H_\Sigma(2;3^n,1), H_\Sigma(2^\times;3^n,-1) \mid n \geq 0\}.\end{aligned}$$

- $\mathcal{B}^{\mathrm{I}}$ consists of the Bu_3-sets of the form $H_{\mathrm{I}}(-)$ which appear in the cases $N \leq 5$.
- $\mathcal{B}^{\mathrm{II}}$ consists of the Bu_3-sets of the form $H_{\mathrm{II}}(p;-)$ which appear in the case $N \geq 7$.

We compute the closure $\overline{\mathcal{B}_\Omega}$ of $\mathcal{B}_\Omega$ step-by-step, constantly using the observations in the following remarks.

Remark 5.1. Let $\mathcal{B}_1$, $\mathcal{B}_2$ be closed collections of genus-zero Bu_3-sets, and let $\mathcal{B}^\times$ consist of all genus-zero orbits in all products $E_1 \times E_2$ where $E_1 \in \mathcal{B}_1 \smallsetminus \mathcal{B}_2$ and $E_2 \in \mathcal{B}_2 \smallsetminus \mathcal{B}_1$. Then, the closure of $(\mathcal{B}_1 \cup \mathcal{B}_2)$ is clearly $(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}^\times)$.

Remark 5.2. Let $\mathfrak{m} \subset \Lambda$ be a maximal ideal, let A_1, A_2 be $\mathfrak{m}$-local modules, let $E \subset \mathcal{E}(A_1) \times \mathcal{E}(A_2)$ be an orbit, and let A be the image of $\phi\colon \Lambda^2 \twoheadrightarrow A_1 \oplus A_2$ for some $\phi = \phi_1 \times \phi_2$ in E. Clearly, A is also $\mathfrak{m}$-local and E is isomorphic to an orbit in $\mathcal{E}(A)$. Therefore, let $\mathcal{B}_\mathfrak{m}$ be the collection which consists of $S(\Omega)$ for all genus-zero $\Omega \subset \mathcal{C}(A)$ for all $\mathfrak{m}$-local A, together with the trivial Bu_3-set. Then, $\mathcal{B}_\mathfrak{m}$ is closed.

In considering the products $E_1 \times E_2$ of genus-zero Bu_3-sets, we freely use (2.4), which is stated for $(\Gamma \times \mathbb{Z})$-sets but holds for Bu_3-sets without any change. Considering the bijection between the genus-zero Bu_3-sets and certain pairs (C, s) of Γ-sets and specifications (Remark 2.8), we use the following notation throughout the chapter: for a genus-zero Bu_3-set E, the corresponding pair is denoted by $(\mathrm{c}(E), \mathrm{s}_E)$, and for a pair (C, s), the corresponding Bu_3-set is denoted by $S(C, \mathrm{s})$. Then, for given pairs (C_1, s_1) and (C_2, s_2) corresponding to $E_1 := S(C_1, \mathrm{s}_1)$ and $E_2 := S(C_2, \mathrm{s}_2)$, we denote the weight w on $C_1 \times C_2$ given in (2.4) by $w_{(\mathrm{s}_1, \mathrm{s}_2)} = w_{(E_1, E_2)}$.

Remark 5.3. For any two transitive Bu_3-sets E, E' with an equivariant map $E \to E'$, one of the orbits in $E \times E'$ is isomorphic to E. Hence, for another Bu_3-set $\tilde{E}$, all orbits in $E \times \tilde{E}$ also appear (up to isomorphism) in $E \times E' \times \tilde{E}$. Therefore, in the context of Remark 5.1, if all Bu_3-sets in $\mathcal{B}_1 \smallsetminus \mathcal{B}_2$ admit an equivariant map onto some E, one can ignore those $E_2 \in \mathcal{B}_2 \smallsetminus \mathcal{B}_1$ for which all genus-zero orbits in $E \times E_2$ are in $\mathcal{B}_1$.

Remark 5.3 is nothing but the formulation of the following very basic notion of set theory for G-sets: if $A \subset A'$, then $A \cap \tilde{A} = A \cap A' \cap \tilde{A}$ for any $\tilde{A}$.

5.1 The Closure $\overline{\mathcal{B}_\Pi}$ of $\mathcal{B}_\Pi := \mathcal{B}'_\Pi \cup \mathcal{B}^2_\Pi \cup \mathcal{B}^3_\Pi$

Consider the Bu_3-sets $\mathbb{B}_3\backslash\mathrm{Bu}_3$ and $H_\Pi(\mathtt{D};1)$ for which $\mathrm{c}(\mathbb{B}_3\backslash\mathrm{Bu}_3)$ and $\mathrm{s}_{(\mathbb{B}_3\backslash\mathrm{Bu}_3)}$ are given in Example 2.1, and $\mathrm{c}(H_\Pi(\mathtt{D};1))$ and $\mathrm{s}_{H_\Pi(\mathtt{D};1)}$ are given in Table A.1. Consider the weight $w := w_{(\mathbb{B}_3\backslash\mathrm{Bu}_3),\, H_\Pi(\mathtt{D};1)}$ on the Γ-set $P := \mathrm{c}(\mathbb{B}_3\backslash\mathrm{Bu}_3)\times\mathrm{c}(H_\Pi(\mathtt{D};1)) = \{*\}\times\Pi_2(1) = \Pi_2(1)$. Then, (P,w) contains no complete monovalent vertex and contains only one region which is of size 2 and weight $\frac{1}{3}$, hence $\chi(P,w)=0$ (2.4). Therefore,

$$\text{there is no genus-zero orbit in } (\mathbb{B}_3\backslash\mathrm{Bu}_3)\times H_\Pi(\mathtt{D};1). \tag{5.1}$$

The closure $\overline{\mathcal{B}'_\Pi}$ of $\mathcal{B}'_\Pi$ Any Bu_3-set of the form $H_\Pi(1;\pi^n)\in\mathcal{B}'_\Pi$ admits an equivariant map $H_\Pi(1;\pi^n)\to(\mathbb{B}_3\backslash\mathrm{Bu}_3)$. Therefore, if a subcollection $\{E_i\}\subset\mathcal{B}'_\Pi$ is "mixed", i.e. it contains a Bu_3-set of the form $H_\Pi(1;\pi^n)$ and another Bu_3-set of the form $H_\Pi(\mathtt{D};\pi^n)$, then there is no genus-zero orbit in the product $\prod E_i$ (5.1). Therefore, $\overline{\mathcal{B}'_\Pi}$ consists of the genus-zero orbits in the products

$$\prod_{1\le i\le k} H_\Pi(1;\pi_i^{n_i}), \qquad \prod_{1\le i\le k} H_\Pi(\mathtt{D};\pi_i^{n_i}),$$

where $\pi_1,\pi_2,\ldots,\pi_k$ are primes in $\mathbb{Z}[\omega]$ distinct from each other and from $(\omega-1)$ and 2 (Remarks 5.1, 5.2). For coprime $\alpha,\alpha'\in\mathbb{Z}[\omega]$, the weight $w_{H_\Pi(1;\alpha),H_\Pi(1;\alpha')}$ on $\mathrm{c}(H_\Pi(1;\alpha))\times\mathrm{c}(H_\Pi(1;\alpha')) = \Pi_1(\alpha)\times\Pi_1(\alpha') = \Pi_1(\alpha\alpha')$ is trivial. Consequently, any orbit in the product $\prod H_\Pi(1;\pi_i^{n_i})$ is isomorphic to $H_\Pi(1;\prod\pi_i^{n_i})$. Similarly, the weight $w_{H_\Pi(\mathtt{D};\alpha),H_\Pi(\mathtt{D};\alpha')}$ on $\mathrm{c}(H_\Pi(\mathtt{D};\alpha))\times\mathrm{c}(H_\Pi(\mathtt{D};\alpha')) = \Pi_2(\alpha^*)\times\Pi_2((\alpha')^*)$, which is the disjoint union of two copies of $\Pi_2((\alpha\alpha')^*)$, is also trivial. Consequently, any orbit in the product $\prod H_\Pi(\mathtt{D};\pi_i^{n_i})$ is isomorphic to $H_\Pi(\mathtt{D};\prod\pi_i^{n_i})$. Therefore,

$$\begin{aligned}\overline{\mathcal{B}'_\Pi} := \{H_\Pi(1;\iota)\mid \iota\not\subset\langle 2\rangle, \iota\not\subset\langle\omega-1\rangle, \iota\neq\langle 1\rangle\}\ \cup \\ \{H_\Pi(\mathtt{D};\iota)\mid \iota\not\subset\langle 2\rangle, \iota\not\subset\langle\omega-1\rangle\}.\end{aligned}$$

The collections $\mathcal{B}^{\mathtt{D}}$ and $\mathcal{B}_2$ It is easy to verify, by direct computation, that the collections

$$\mathcal{B}_2^{\mathtt{D}} := \{H_\Pi(\mathtt{D}; 2^n) \mid n \geq 0\}, \qquad \mathcal{B}_3^{\mathtt{D}} := \{H_\Pi(\mathtt{D}; (\omega-1)^n) \mid n \geq 0\}$$

are closed, using the fact all genus-zero orbits in a product $\Pi_2(\alpha^{n_1}) \times \Pi_2(\alpha^{n_2})$ are isomorphic to $\Pi_2(\alpha^{\max(n_1,n_2)})$. Hence, the closure of $\mathcal{B}'_\Pi \cup \mathcal{B}_2^{\mathtt{D}} \cup \mathcal{B}_3^{\mathtt{D}}$ is given by

$$\underbrace{\{H_\Pi(1;\iota) \mid \iota \not\subset \langle 2\rangle, \iota \not\subset \langle \omega-1\rangle, \iota \neq \langle 1\rangle\}}_{\mathcal{B}} \cup \underbrace{\{H_\Pi(\mathtt{D};\iota)\}}_{\mathcal{B}^{\mathtt{D}}}.$$

The collection

$$\begin{aligned}\mathcal{B}_2 := \{H_\Pi(\circ; 2^n) \mid n \geq 0\} \cup \{&H_\Pi(1; 2^n), H_\Pi(1^\times; 2^n),\\ &H_\Pi(3; 2^n), H_\Pi(\mathtt{3a}; 2^n), H_\Pi(\mathtt{3b}; 2^n) \mid n \geq 1\} \cup\\ &\{H_\Pi(\mathtt{3ba}; 2^n), H_\Pi(\mathtt{3bb}; 2^n), H_\Pi(3/2, 2^n), H_\Pi(3/2^\times, 2^n) \mid n \geq 2\}\end{aligned}$$

is also closed. Instead of a direct verification, it is easier to observe that the orbits in the product $H_\Pi(\circ;1) \times H_\Pi(\circ;1)$ are isomorphic to $H_\Pi(\circ;1)$, and $\mathcal{B}_2 \smallsetminus \{H_\Pi(\circ;1)\}$ is the subcollection of the closed collection $\mathcal{B}_{\langle 2, t^2+t+1\rangle}$ (Remark 5.2) consisting of the Bu_3-sets which admit an equivariant map onto $\mathbb{B}_3\backslash\mathrm{Bu}_3$. Then, $\mathcal{B}_2 \cup \mathcal{B}^{\mathtt{D}}$ is also closed since the orbits in the product $H_\Pi(\circ;1) \times H_\Pi(\mathtt{D};1)$ are isomorphic to $H_\Pi(\mathtt{D};1) \in \mathcal{B}^{\mathtt{D}}$ (see (5.1) and Remark 5.3). Thus, the closure of $\mathcal{B} \cup \mathcal{B}^{\mathtt{D}} \cup \mathcal{B}_2$ is the union $\mathcal{B}' \cup \mathcal{B}^{\mathtt{D}}$ where

$$\begin{aligned}\mathcal{B}' := \{&H_\Pi(\circ;\iota) \mid \iota \not\subset \langle \omega-1\rangle\} \cup\\ &\{H_\Pi(1;\iota) \mid \iota \not\subset \langle \omega-1\rangle, \iota \neq \langle 1\rangle\} \cup\\ &\{H_\Pi(1^\times;\iota), H_\Pi(3;\iota), H_\Pi(\mathtt{3a};\iota), H_\Pi(\mathtt{3b};\iota) \mid \iota \subset \langle 2\rangle, \iota \not\subset \langle \omega-1\rangle\} \cup\\ &\{H_\Pi(\mathtt{3ba};\iota), H_\Pi(\mathtt{3bb};\iota), H_\Pi(3/2,\iota), H_\Pi(3/2^\times,\iota) \mid \iota \subset \langle 4\rangle, \iota \not\subset \langle \omega-1\rangle\}.\end{aligned}$$

The collections $\mathcal{B}_\bullet$ and $\mathcal{B}_3$ Consider the collection

$$\mathcal{B}_\bullet := \{H_\Pi(\bullet; (\omega-1)^n) \mid n \geq 0\}.$$

Then, $\mathcal{B}_\bullet^{>} := \mathcal{B}_\bullet \smallsetminus \{H_\Pi(\bullet;1), H_\Pi(\bullet;\omega-1)\}$ is closed, as it can be directly verified using the fact that all genus-zero orbits in a product $\Pi_\bullet(\alpha^{n_1}) \times \Pi_\bullet(\alpha^{n_2})$ are isomorphic to $\Pi_\bullet(\alpha^{\max(n_1,n_2)})$. The members of $\mathcal{B}_\bullet^{>}$ admit an equivariant map onto $\mathbb{B}_3\backslash\mathrm{Bu}_3$, hence $\mathcal{B}_\bullet^{>} \cup \mathcal{B}^{\mathtt{D}}$ is closed (5.1). Finally, $\mathcal{B}_\bullet \cup \mathcal{B}^{\mathtt{D}}$ is also closed because the orbits in $H_\Pi(\bullet;1) \times H_\Pi(\bullet;1)$ are isomorphic to $H_\Pi(\bullet;1)$, the orbits in $H_\Pi(\bullet;\omega-1) \times H_\Pi(\bullet;\omega-1)$ are isomorphic to $H_\Pi(\bullet;\omega-1)$ or $H_\Pi(\mathtt{D};\omega-1) \in \mathcal{B}^{\mathtt{D}}$, the orbits in $H_\Pi(\bullet;1) \times H_\Pi(\bullet;\omega-1)$ are isomorphic to $H_\Pi(\bullet;\omega-1)$, the orbits in $H_\Pi(\bullet;1) \times H_\Pi(\mathtt{D};1)$ are isomorphic to $H_\Pi(\mathtt{D};1) \in \mathcal{B}^{\mathtt{D}}$, the orbits in $H_\Pi(\bullet;\omega-1) \times H_\Pi(\mathtt{D};1)$ are isomorphic to $H_\Pi(\mathtt{D};\omega-1) \in \mathcal{B}^{\mathtt{D}}$, and the orbits in $H_\Pi(\bullet;1) \times H_\Pi(\bullet;(\omega-1)^2)$ and $H_\Pi(\bullet;\omega-1) \times H_\Pi(\bullet;(\omega-1)^2)$ are isomorphic to $H_\Pi(\bullet;(\omega-1)^2)$ (see Remark 5.3). Similarly, the collection

$$\mathcal{B}_3 := \{H_\Pi(1;(\omega-1)^n), H_\Pi(2;(\omega-1)^n) \mid n \geq 1\}$$

is closed (as before, one easily considers the exceptional Bu_3-sets $H_\Pi(1;(\omega-1))$, $H_\Pi(2;(\omega-1))$ separately). Then $\mathcal{B}_3 \cup \mathcal{B}^{\mathtt{D}}$ is closed because the members of $\mathcal{B}_3^{>} := \mathcal{B}_3 \smallsetminus \{H_\Pi(1;\omega-1), H_\Pi(2;\omega-1)\}$ admit an equivariant map onto $\mathbb{B}_3\backslash\mathrm{Bu}_3$ (5.1), and the orbits in the products $H_\Pi(1;\omega-1) \times H_\Pi(\mathtt{D};1)$, $H_\Pi(2;\omega-1) \times H_\Pi(\mathtt{D};1)$ are isomorphic to $H_\Pi(\mathtt{D};\omega-1) \in \mathcal{B}^{\mathtt{D}}$ (Remark 5.3). Finally, $\mathcal{B}_\bullet \cup \mathcal{B}_3 \cup \mathcal{B}^{\mathtt{D}}$ is closed because the orbits in $H_\Pi(\bullet;1) \times H_\Pi(1;\omega-1)$ are isomorphic to $H_\Pi(2;\omega-1)$, the orbits in $H_\Pi(\bullet;\omega-1) \times H_\Pi(1;\omega-1)$ are isomorphic to $H_\Pi(\mathtt{D};(\omega-1))$, and there is no genus-zero orbit in $\Pi_\bullet(\omega-1) \times \Pi_1(\omega-1)$. As a result, the closure $\overline{\mathcal{B}^\cup}$ of $\mathcal{B}^\cup := \mathcal{B}' \cup \mathcal{B}^{\mathtt{D}} \cup \mathcal{B}_\bullet \cup \mathcal{B}_3$ is the union $\mathcal{B}'' \cup \mathcal{B}^\cup$ where $\mathcal{B}''$ consists of the genus-zero orbits in the products $E' \times E$ where $E' \in \mathcal{B}'$ and $E \in \mathcal{B}_\bullet \cup \mathcal{B}_3$.

Note that the product Γ-set $\Pi_\circ(1) \times \Pi_\bullet(1)$ contains only one region which has size 6, and it contains no monovalent vertex, hence it is not of genus zero. Moreover, $\Pi_2(1) = \Pi_\bullet(1)$ and $\Pi_3(1) = \Pi_\circ(1)$. Thus $\mathcal{B}''$ consists of the genus-zero orbits in the products

$$\begin{aligned}
&H_\Pi(1;\iota) \times E, && \text{where } \iota \neq \langle 1\rangle, \iota \not\subset \langle \omega-1\rangle,\ E \in \mathcal{B}_\bullet \cup \mathcal{B}_3,\\
&H_\Pi(1^\times;\iota) \times E, && \text{where } \iota \subset \langle 2\rangle, \iota \not\subset \langle \omega-1\rangle,\ E \in \mathcal{B}_\bullet \cup \mathcal{B}_3,\\
&E' \times H_\Pi(1;(\omega-1)^n), && \text{where } E' \in \mathcal{B}',\ n \geq 1,
\end{aligned}$$

and the exceptional products $H_\Pi(\circ;1) \times H_\Pi(\bullet;1)$, $H_\Pi(\circ;1) \times H_\Pi(\bullet;\omega-1)$, $H_\Pi(\circ;1) \times H_\Pi(2;\omega-1)$. The orbits in the exceptional products are isomorphic to $H_\Pi(\mathtt{D};1) \in \mathcal{B}^{\mathsf{D}}$ or $H_\Pi(\mathtt{D};\omega-1) \in \mathcal{B}^{\mathsf{D}}$. Thus

$$\begin{aligned}\overline{\mathcal{B}^{\cup}} := \mathcal{B}^{\mathsf{D}} \cup \{H_\Pi(\circ;\iota), H_\Pi(\bullet;\iota)\} \cup \{H_\Pi(\bullet^{\times};\iota) \mid \iota \subset \langle 2\rangle\} \ \cup \\ \{H_\Pi(1;\iota) \mid \iota \neq \langle 1\rangle\} \ \cup \\ \{H_\Pi(1^{\times};\iota), H_\Pi(3;\iota), H_\Pi(3\mathtt{a};\iota), H_\Pi(3\mathtt{b};\iota) \mid \iota \subset \langle 2\rangle\} \ \cup \\ \{H_\Pi(3\mathtt{ba};\iota), H_\Pi(3\mathtt{bb};\iota), H_\Pi(3/2,\iota), H_\Pi(3/2^{\times},\iota) \mid \iota \subset \langle 4\rangle\} \ \cup \\ \{H_\Pi(2;\iota) \mid \iota \subset \langle \omega-1\rangle\} \cup \{H_\Pi(2^{\times};\iota) \mid \iota \subset \langle 2(\omega-1)\rangle\}.\end{aligned}$$

The Final Step It is left to consider the Bu_3-sets in

$$\begin{aligned}\mathcal{B}^{\mathsf{a}} &:= \{H_\Pi(1';2), H_\Pi(1\mathtt{ax};2), H_\Pi(1\mathtt{ay};2), H_\Pi(3\mathtt{a}';2)\},\\ \mathcal{B}^{\mathsf{b}} &:= \{H_\Pi(1\mathtt{b};2), H_\Pi(1\mathtt{ba};4), H_\Pi(1\mathtt{bb};4)\},\end{aligned}$$

that is, $\overline{\mathcal{B}_\Pi}$ is the closure of $\mathcal{B}^{\mathsf{a}} \cup \mathcal{B}^{\mathsf{b}} \cup \overline{\mathcal{B}^{\cup}}$. The closures of $\mathcal{B}^{\mathsf{a}}$ and $\mathcal{B}^{\mathsf{b}}$ can be easily found:

$$\begin{aligned}\overline{\mathcal{B}^{\mathsf{a}}} &= \mathcal{B}^{\mathsf{a}} \cup \{H_\Pi(3\mathtt{a};2)\},\\ \overline{\mathcal{B}^{\mathsf{b}}} &= \mathcal{B}^{\mathsf{b}} \cup \{H_\Pi(3\mathtt{b};2), H_\Pi(3\mathtt{ba};4), H_\Pi(3\mathtt{bb};4), H_\Pi(\mathtt{D};4)\}.\end{aligned}$$

Thus, the closure $\tilde{\mathcal{B}}$ of $\mathcal{B}^{\mathsf{b}} \cup \overline{\mathcal{B}^{\cup}}$ is the union of $\mathcal{B}^{\mathsf{b}} \cup \overline{\mathcal{B}^{\cup}} = \overline{\mathcal{B}^{\mathsf{b}}} \cup \overline{\mathcal{B}^{\cup}}$ with the collection of the genus-zero orbits in the products $E^{\mathsf{b}} \times E^{\cup}$ where $E^{\mathsf{b}} \in \mathcal{B}^{\mathsf{b}}$ and $E^{\cup} \in \overline{\mathcal{B}^{\cup}}$. But the orbits in $E^{\mathsf{b}} \times (\mathbb{B}_3 \backslash \mathrm{Bu}_3)$ are isomorphic to $H_\Pi(3\mathtt{b};2)$, $H_\Pi(3\mathtt{ba};4)$ or $H_\Pi(3\mathtt{bb};4)$, while the orbits in $E^{\mathsf{b}} \times H_\Pi(\mathtt{D};1)$ are isomorphic to $H_\Pi(\mathtt{D};2)$ or $H_\Pi(\mathtt{D};4)$, thus it is enough to consider those $E^{\cup} \in \overline{\mathcal{B}^{\cup}}$ which do not admit an equivariant map onto either $\mathbb{B}_3 \backslash \mathrm{Bu}_3$ or $H_\Pi(\mathtt{D};1)$ (Remark 5.3). Namely,

$$\begin{gathered}E^{\cup} \in \mathcal{B}^{\mathrm{exc}} := \{H_\Pi(\circ;1), H_\Pi(\circ;\omega-1), H_\Pi(\bullet;1), H_\Pi(\bullet;\omega-1),\\ H_\Pi(1;\omega-1), H_\Pi(2;\omega-1)\}.\end{gathered} \tag{5.2}$$

On the other hand, E^{b} admits an equivariant map onto $H_\Pi(\circ;1)$, hence $E^{\cup}$ can be replaced with the genus-zero orbits in $E^{\cup} \times H_\Pi(\circ;1)$. In other words, the products $E^{\mathsf{b}} \times E^{\cup}$ are replaced with the products $E^{\mathsf{b}} \times E$ where

$$E \in \{H_\Pi(\circ;1), H_\Pi(\circ;\omega-1), H_\Pi(\mathtt{D};1), H_\Pi(\mathtt{D};\omega-1)\}. \tag{5.3}$$

For $E = H_\Pi(\circ;1)$, the orbits in $E^{\mathsf{b}} \times E$ are isomorphic to E^{b}. Moreover, the cases $E = H_\Pi(\mathtt{D};1), H_\Pi(\mathtt{D};\omega-1)$ can be ignored as already mentioned.

Finally, by considering the orbits in the products $H_{\Pi}(\circ;\omega-1)\times H_{\Pi}(\mathtt{1b};2)$, $H_{\Pi}(\circ;\omega-1)\times H_{\Pi}(\mathtt{1ba};4)$, $H_{\Pi}(\circ;\omega-1)\times H_{\Pi}(\mathtt{1bb};4)$, it is seen that

$$\tilde{\mathcal{B}} = \mathcal{B}^{\mathsf{b}} \cup \overline{\mathcal{B}^{\cup}} \cup \{H_{\Pi}(\mathtt{1b};2(\omega-1)), H_{\Pi}(\mathtt{1ba};4(\omega-1)), H_{\Pi}(\mathtt{1bb};4(\omega-1))\}.$$

Similarly, the closure $\overline{\mathcal{B}_{\Pi}}$ of $\mathcal{B}^{\mathsf{a}}\cup\tilde{\mathcal{B}}$ is the union of $\mathcal{B}^{\mathsf{a}}\cup\tilde{\mathcal{B}} = \overline{\mathcal{B}^{\mathsf{a}}}\cup\tilde{\mathcal{B}}$ with the collection of the genus-zero orbits in the products $E^{\mathsf{a}}\times\tilde{E}$ where $E^{\mathsf{a}}\in\mathcal{B}^{\mathsf{a}}$ and $\tilde{E}\in\tilde{\mathcal{B}}$. But the orbits in $E^{\mathsf{a}}\times(\mathbb{B}_3\backslash\mathrm{Bu}_3)$ are isomorphic to $H_{\Pi}(1^{\times};2)$ or $H_{\Pi}(\mathtt{3a};2)$, while the orbits in $E^{\mathsf{a}}\times H_{\Pi}(\mathtt{D};1)$ and $E^{\mathsf{a}}\times H_{\Pi}(\mathtt{1b};2)$ are either not of genus zero or isomorphic to $H_{\Pi}(\mathtt{D};2)$. Thus it is enough to consider the case that $\tilde{E}\in\tilde{\mathcal{B}}$ does not admit an equivariant map onto any of $\{\mathbb{B}_3\backslash\mathrm{Bu}_3, H_{\Pi}(\mathtt{D};1), H_{\Pi}(\mathtt{1b};2)\}$, namely $\tilde{E}\in\mathcal{B}^{\mathrm{exc}}$ (5.2). The products $H_{\Pi}(1';2)\times\tilde{E}$ for $\tilde{E}\in\mathcal{B}^{\mathrm{exc}}$ are isomorphic to

$$\begin{gathered}H_{\Pi}(\mathtt{1ay};2), H_{\Pi}(\mathtt{1ay};2(\omega-1)), H_{\Pi}(\bullet';2), H_{\Pi}(\bullet';2(\omega-1)),\\ H_{\Pi}(1';2(\omega-1)), H_{\Pi}(2';2(\omega-1)).\end{gathered}$$

The products $H_{\Pi}(\mathtt{1ax};2)\times\tilde{E}$ for $\tilde{E}\in\mathcal{B}^{\mathrm{exc}}$ are isomorphic to

$$\begin{gathered}H_{\Pi}(\mathtt{3a};2), H_{\Pi}(\mathtt{3a};2(\omega-1)), H_{\Pi}(\bullet\mathtt{ax};2), H_{\Pi}(\bullet\mathtt{ax};2(\omega-1)),\\ H_{\Pi}(\mathtt{1ax};2(\omega-1)), H_{\Pi}(\mathtt{2ax};2(\omega-1)).\end{gathered}$$

The products $H_{\Pi}(\mathtt{3a}';2)\times H_{\Pi}(\circ;1)$, $H_{\Pi}(\mathtt{3a}';2)\times H_{\Pi}(\circ;\omega-1)$, $H_{\Pi}(\mathtt{3a}';2)\times H_{\Pi}(1;\omega-1)$ are isomorphic to

$$H_{\Pi}(\mathtt{3a};2), H_{\Pi}(\mathtt{3a};2(\omega-1)), H_{\Pi}(\mathtt{3a}';2(\omega-1)),$$

and the products $H_{\Pi}(\mathtt{3a}';2)\times H_{\Pi}(\bullet;1)$, $H_{\Pi}(\mathtt{3a}';2)\times H_{\Pi}(\bullet;\omega-1)$, $H_{\Pi}(\mathtt{3a}';2)\times H_{\Pi}(2;\omega-1)$ are not of genus zero. Finally, the products $H_{\Pi}(\mathtt{1ay};2)\times\tilde{E}$ for $\tilde{E}\in\mathcal{B}^{\mathrm{exc}}$ are not all transitive, but the orbits in them are isomorphic to

$$H_{\Pi}(\mathtt{1ay};2), H_{\Pi}(\mathtt{1ay};2(\omega-1)), H_{\Pi}(\mathtt{D};2), H_{\Pi}(\mathtt{D};2(\omega-1)).$$

Table A.1 shows the closed collection $\overline{\mathcal{B}_{\Pi}}$ found in this section together with the trivial Bu_3-set $H_{\Pi}(1;1)$.

5.2 The Closure $\overline{\mathcal{B}_{\Sigma}}$ of $\mathcal{B}_{\Sigma} := \mathcal{B}'_{\Sigma}\cup\mathcal{B}^{2}_{\Sigma}\cup\mathcal{B}^{3}_{\Sigma}$

Remark 5.4. In a product of the form $\Sigma_{*}(n)\times\Sigma_{*}(n')$, all genus-zero orbits are of the form $\Sigma_{*}(\mathrm{lcm}(n,n'))$ unless $n=n'=2$, in which case there are also orbits isomorphich to $\Gamma(4)\backslash\Gamma$, which is the double covering of $\Sigma_2(2)$ ramified at the regions of size 2.

The closure $\overline{\mathcal{B}'_\Sigma}$ of $\mathcal{B}'_\Sigma$ For each $n \equiv \pm 1 \,(\mathrm{mod}\, 6)$, consider the collection

$$\mathcal{B}(n) := \{H_\Sigma(1; n), H_\Sigma(2; n, a) \mid a^2 \equiv 1 \,(\mathrm{mod}\, n)\}.$$

For coprime n, n' and for $E(n) \in \mathcal{B}(n)$, $E(n') \in \mathcal{B}(n')$, the weight $w_{E(n),E(n')}$ on $\mathrm{c}(E(n)) \times \mathrm{c}(E(n'))$ is trivial (2.4). Hence, as $E(n), E(n')$ vary, the genus-zero orbits in the products $E(n) \times E(n')$ vary over the members of $\mathcal{B}(nn')$ (see Remark 5.4). Now, $\overline{\mathcal{B}'_\Sigma}$ consists of the genus-zero orbits in the products $\prod_{i=1}^{k} E(p_i^{m_i})$ where $p_1, p_2, \ldots, p_k > 3$ are distinct primes (Remarks 5.1, 5.2). Therefore,

$$\overline{\mathcal{B}'_\Sigma} = \{H_\Sigma(1; n), H_\Sigma(2; n, a) \mid n \equiv \pm 1 \,(\mathrm{mod}\, 6),\ a^2 \equiv 1 \,(\mathrm{mod}\, n)\}.$$

The closure $\overline{\mathcal{B}^3_\Sigma}$ of $\mathcal{B}^3_\Sigma$ The collections $\mathcal{B}_3^- := \{H_\Sigma(2^\times; 3^n, -1) \mid n \geq 0\}$ and $\mathcal{B}_3^+ := \{H_\Sigma(1; 3^n), H_\Sigma(2; 3^n, 1) \mid n \geq 0\}$ are clearly closed (Remark 5.4). Then $\mathcal{B}_3 := \mathcal{B}_3^- \cup \mathcal{B}_3^+$ is also closed, because the orbits in $H_\Sigma(2^\times; 1, -1) \times H_\Sigma(1; 1)$ and $H_\Sigma(2^\times; 1, -1) \times H_\Sigma(2; 1, 1)$ are isomorphic to $H_\Sigma(2^\times; 1, -1)$ (Remark 5.3), while there is no genus-zero orbit in $H_\Sigma(2^\times; 1, -1) \times H_\Sigma(1; 3)$. Consider the collection

$$\mathcal{B}_\bullet := \{H_\Sigma(1^{\times\bullet}; 1), H_\Sigma(2^\bullet; 1), H_\Sigma(2\mathrm{x}^\bullet; 1), H_\Sigma(2\mathrm{y}^\bullet; 1)\}.$$

Then $\mathcal{B}_\bullet \cup \mathcal{B}_3^-$ is closed, because $\mathcal{B}_\bullet \cup \{H_\Sigma(2^\times; 1, -1)\}$ is closed, and the orbits in the products $E \times H_\Sigma(2^\times; 1, -1)$ are isomorphic to $H_\Sigma(2^\times; 1, -1)$ for all $E \in \mathcal{B}_\bullet$. Thus, the closure of $\mathcal{B}^3_\Sigma = \mathcal{B}_\bullet \cup \mathcal{B}_3$ is the union of $\mathcal{B}^3_\Sigma$ with the collection of the genus-zero orbits in the products $E_\bullet \times E$ where $E_\bullet \in \mathcal{B}_\bullet$ and $E \in \mathcal{B}_3^+$.

There is an equivariant map $E \to H_\Sigma(1; 1)$ and the product $H_\Sigma(2^\bullet; 1) \times H_\Sigma(1; 1)$ is isomorphic to $H_\Sigma(2; 1, 1) \in \mathcal{B}_3^+$, hence the case $E_\bullet = H_\Sigma(2^\bullet; 1)$ is removed. Thus $E_\bullet = H_\Sigma(1^{\times\bullet}; 1)$, $E_\bullet = H_\Sigma(2\mathrm{x}^\bullet; 1)$, or there is an equivariant map $E_\bullet \to H_\Sigma(2\mathrm{y}^\bullet; 1)$. But there is no genus-zero orbit in the products $H_\Sigma(2\mathrm{x}^\bullet; 1) \times H_\Sigma(1; 3)$, $H_\Sigma(2\mathrm{y}^\bullet; 1) \times H_\Sigma(1; 3)$, $H_\Sigma(1^{\times\bullet}; 1) \times H_\Sigma(1; 9)$, and $H_\Sigma(1^{\times\bullet}; 1) \times H_\Sigma(2; 3, 1)$, hence it is left to consider the case $E \in \{H_\Sigma(1; 1), H_\Sigma(2; 1, 1)\}$ as well as the exceptional product $H_\Sigma(1^{\times\bullet}; 1) \times H_\Sigma(1; 3)$ which is isomorphic to $H_\Sigma(1^\times; 3)$. For

$E \in \{H_\Sigma(1;1), H_\Sigma(2;1,1)\}$, the orbits in the products $H_\Sigma(2^\times;1,-1) \times E$, $H_\Sigma(2\mathtt{x}^\bullet;1) \times E$, $H_\Sigma(2\mathtt{y}^\bullet;1) \times E$ are isomorphic to $H_\Sigma(2^\times;1,-1)$. Finally, $H_\Sigma(1^{\times\bullet};1) \times H_\Sigma(2;1,1) \cong H_\Sigma(2^\times;1,-1)$ and $H_\Sigma(1^{\times\bullet};1) \times H_\Sigma(1;1) \cong H_\Sigma(1^\times;1)$. In summary, the closure of $\mathcal{B}^3_\Sigma$ is given by

$$\overline{\mathcal{B}^3_\Sigma} = \mathcal{B}^3_\Sigma \cup \{H_\Sigma(1^\times;1), H_\Sigma(1^\times;3)\}.$$

The closure $\tilde{\mathcal{B}}$ of $\mathcal{B}'_\Sigma \cup \mathcal{B}^3_\Sigma$ The closure $\mathcal{B}$ of $\overline{\mathcal{B}'_\Sigma} \cup \mathcal{B}_3$ is the union of $\overline{\mathcal{B}'_\Sigma} \cup \mathcal{B}_3$ with the collection of the genus-zero orbits in the products $E' \times E$, where $E' \in \overline{\mathcal{B}'_\Sigma}$ and $E \in \mathcal{B}_3$, hence it is clearly given by

$$\begin{aligned}\mathcal{B} := \{H_\Sigma(2^\times;n,a) \mid n \equiv 1 \,(\mathrm{mod}\, 2),\ a^2 \equiv 1 \,(\mathrm{mod}\, 3n), a \equiv -1 \,(\mathrm{mod}\, 3)\} \ \cup \\ \{H_\Sigma(1;n), H_\Sigma(2;n,a) \mid n \equiv 1 \,(\mathrm{mod}\, 2),\ a^2 \equiv 1 \,(\mathrm{mod}\, n), a \equiv 1 \,(\mathrm{mod}\, 3)\}.\end{aligned}$$

Therefore, the closure $\tilde{\mathcal{B}}$ of $\overline{\mathcal{B}'_\Sigma} \cup \overline{\mathcal{B}^3_\Sigma}$ is the union of

$$\mathcal{B} \cup \underbrace{\mathcal{B}_\bullet \cup \{H_\Sigma(1^\times;1), H_\Sigma(1^\times;3)\}}_{\mathcal{B}'_\bullet}$$

with the collection of the genus-zero orbits in the products $E \times E_\bullet$ where $E \in \mathcal{B}$ and $E_\bullet \in \mathcal{B}'_\bullet$. There is an equivariant map $E \to H_\Sigma(1,1)$, and the products $H_\Sigma(1,1) \times H_\Sigma(2^\bullet;1)$, $H_\Sigma(1,1) \times H_\Sigma(2\mathtt{x}^\bullet;1)$ and $H_\Sigma(1,1) \times H_\Sigma(2\mathtt{y}^\bullet;1)$ are isomorphic to $H_\Sigma(2;1,1) \in \mathcal{B}$ or $H_\Sigma(2^\times;1,-1) \in \mathcal{B}$, hence it is enough to consider the case $E_\bullet \in \{H_\Sigma(1^{\times\bullet};1), H_\Sigma(1^\times;1), H_\Sigma(1^\times;3)\}$. The orbits in $H_\Sigma(2;1,1) \times H_\Sigma(1^{\times\bullet};1)$ and $H_\Sigma(2;1,1) \times H_\Sigma(1^\times;1)$ are isomorphic to $H_\Sigma(2^\times;1,-1) \in \mathcal{B}$, while the orbits in $H_\Sigma(2;1,1) \times H_\Sigma(1^\times;3)$ are not of genus zero, hence $E = H_\Sigma(1;n)$. Moreover, it can be assumed that $n > 3$, because $E \in \overline{\mathcal{B}^3_\Sigma}$ otherwise. Then, there is no genus-zero orbit in $\Sigma_1(n) \times \mathrm{c}(H_\Sigma(1^\times;3))$. If $3 \mid n$, the weight $w_{E,H_\Sigma(1^{\times\bullet};1)}$ on $\mathrm{c}(E) \times \mathrm{c}(H_\Sigma(1^{\times\bullet};1)) = \Sigma_1(n) \times \{*\} = \Sigma_1(n)$ assigns the weight of $\frac{1}{3}$ to all regions of size 1 and 2, hence there is no genus-zero orbit in $E \times E_\bullet$. Finally, if $3 \nmid n$, the orbits in $H_\Sigma(1;n) \times H_\Sigma(1^{\times\bullet};1)$ and $H_\Sigma(1;n) \times H_\Sigma(1^\times;1)$ are isomorphic to $H_\Sigma(1^\times;n)$ or $H_\Sigma(2^\times;n,a) \in \mathcal{B}$ for some $a \equiv 1 \,(\mathrm{mod}\, n)$. Consequently,

$$\tilde{\mathcal{B}} = \underbrace{\mathcal{B} \cup \{H_\Sigma(1^\times;n) \mid n \equiv \pm 1 \,(\mathrm{mod}\, 6)\}}_{\mathcal{B}'} \ \cup\ \mathcal{B}'_\bullet.$$

The Final Step The collection $\mathcal{B}_\Sigma^2$ is closed because $\mathcal{B}_\Sigma^2 = \mathcal{B}_{\langle 2,t-1\rangle} \cup \{H_\Sigma(2\mathtt{d}^\bullet;2)\}$ where $\mathcal{B}_{\langle 2,t-1\rangle}$ is closed (Remark 5.2), the orbits in the product $H_\Sigma(2\mathtt{d}^\bullet;2) \times H_\Sigma(2\mathtt{d}^\bullet;2)$ are isomorphic to $H_\Sigma(2\mathtt{d}^\bullet;2)$ or $H_\Sigma(2\mathtt{d};2)$, and the product $H_\Sigma(2\mathtt{d}^\bullet;2) \times H_\Sigma(1;1)$ is isomorphic to $H_\Sigma(2\mathtt{d};2) \in \mathcal{B}_{\langle 2,t-1\rangle}$ (Remark 5.3). Let

$$\begin{aligned}\mathcal{B}_2 := {} & \{H_\Sigma(1\mathtt{aa};2), H_\Sigma(1\mathtt{ab};2), H_\Sigma(2\mathtt{a};2,1)\} \;\cup \\ & \{H_\Sigma(1;2^n), H_\Sigma(2;2^n,a) \mid n \geq 0,\ a^2 \equiv 1 \pmod{2^n}\} \;\cup \\ & \{H_\Sigma(1\mathtt{w};2^n), H_\Sigma(2\mathtt{w};2^n,1), H_\Sigma(2\mathtt{w};2^n,-1) \mid n \geq 3\},\end{aligned}$$

and let $\mathcal{B}^{\mathrm{d}} := \mathcal{B}_\Sigma^2 \smallsetminus \mathcal{B}_2$. The union of $\mathcal{B}' \cup \mathcal{B}_2$ with the collection of the genus-zero orbits in the products $E' \times E_2$, where $E' \in \mathcal{B}'$ and $E_2 \in \mathcal{B}_2$, can be easily found:

$$\begin{aligned}\mathcal{B}'' := {} & \{H_\Sigma(1;n), H_\Sigma(2;n,a) \mid a^2 \equiv 1 \pmod{n},\ a \equiv 1 \pmod{3}\} \;\cup \\ & \{H_\Sigma(1^\times;n) \mid n \equiv \pm 1 \pmod{3}\} \;\cup \\ & \{H_\Sigma(2^\times;n,a) \mid a^2 \equiv 1 \pmod{3n},\ a \equiv -1 \pmod{3}\} \;\cup \\ & \{H_\Sigma(1\mathtt{w};n), H_\Sigma(2\mathtt{w};n,a) \mid \\ & \qquad n \equiv 0 \pmod{8},\ a^2 \equiv 1 \pmod{n},\ a \equiv 1 \pmod{3}\} \;\cup \\ & \{H_\Sigma(1\mathtt{w}^\times;n) \mid n \equiv \pm 8 \pmod{24}\} \;\cup \\ & \{H_\Sigma(2\mathtt{w}^\times;n,a) \mid n \equiv 0 \pmod{8},\ a^2 \equiv 1 \pmod{3n},\ a \equiv -1 \pmod{3}\} \;\cup \\ & \{H_\Sigma(1\mathtt{aa};n), H_\Sigma(1\mathtt{ab};n), H_\Sigma(2\mathtt{a};n,a) \mid \\ & \qquad n \equiv 2 \pmod{4},\ a^2 \equiv 1 \pmod{n},\ a \equiv 1 \pmod{3}\} \;\cup \\ & \{H_\Sigma(1\mathtt{aa}^\times;n), H_\Sigma(1\mathtt{ab}^\times;n) \mid n \equiv \pm 2 \pmod{12}\} \;\cup \\ & \{H_\Sigma(2\mathtt{a}^\times;n,a) \mid n \equiv 2 \pmod{4},\ a^2 \equiv 1 \pmod{3n},\ a \equiv -1 \pmod{3}\}.\end{aligned}$$

Hence, the closure $\overline{\mathcal{B}_\Sigma}$ of $\tilde{\mathcal{B}} \cup \mathcal{B}_\Sigma^2$ is the union of $\mathcal{B}'' \cup \mathcal{B}'_\bullet \cup \mathcal{B}^{\mathrm{d}}$ with the collection of the genus-zero orbits in the products

$$\begin{aligned} & E_\bullet \times E, && \text{where} && E_\bullet \in \tilde{\mathcal{B}} \smallsetminus \mathcal{B}' = \mathcal{B}_\bullet \cup \{H_\Sigma(1^\times;3)\}, \\ & && && E \in \mathcal{B}_\Sigma^2 \smallsetminus \{H_\Sigma(1;1), H_\Sigma(2;1,1)\}, \\ & E' \times E^{\mathrm{d}}, && \text{where} && E' \in \mathcal{B}' \smallsetminus \{H_\Sigma(1;1), H_\Sigma(2;1,1)\}, \\ & && && E^{\mathrm{d}} \in \mathcal{B}^{\mathrm{d}}. \end{aligned}$$

First consider the products $E_\bullet \times E$. The case $E_\bullet = H_\Sigma(1^\times;3)$ can be removed, because there is no genus-zero orbit in $\mathrm{c}(H_\Sigma(1^\times;3)) \times \Sigma_1(2)$. The case $E \in$

$\mathcal{B}_2$ can also be removed, because then $E_\bullet$ can be replaced with $E_\bullet \times H_\Sigma(1;1)$ (Remark 5.3), which is isomorphic to

$$H_\Sigma(1^\times;1), H_\Sigma(2;1,1), H_\Sigma(2^\times;1,-1) \in \mathcal{B}'.$$

It is left to consider the case $E_\bullet \in \mathcal{B}_\bullet$ and $E \in \mathcal{B}^{\mathrm{d}}$. Then, the genus-zero orbits are isomorphic to

$$\begin{aligned}
&H_\Sigma(\mathtt{1d'}^\times;4), H_\Sigma(\mathtt{1da}^\times;4), H_\Sigma(\mathtt{1db}^\times;4), H_\Sigma(\mathtt{1dx}^\times;4),\\
&H_\Sigma(\mathtt{1daa}^\times;4), H_\Sigma(\mathtt{1dbb}^\times;4), H_\Sigma(\mathtt{1dxx}^\times;4),\\
&H_\Sigma(\mathtt{1d'}^\times;8), H_\Sigma(\mathtt{1da}^\times;8), H_\Sigma(\mathtt{1db}^\times;8),\\
&H_\Sigma(\mathtt{2d'}^\times;2), H_\Sigma(\mathtt{2d}^\times;2), H_\Sigma(\mathtt{2dw}^\times;2), H_\Sigma(\mathtt{2dx}^\bullet;2), H_\Sigma(\mathtt{2dy}^\bullet;2),\\
&H_\Sigma(\mathtt{2d'}^\times;4), H_\Sigma(\mathtt{2d}^\times;4,1), H_\Sigma(\mathtt{2d}^\times;4,-1),\\
&H_\Sigma(\mathtt{2da}^\times;4,1), H_\Sigma(\mathtt{2da}^\times;4,-1).
\end{aligned} \tag{5.4}$$

Now consider the products $E' \times E^{\mathrm{d}}$. If $E' \in \{H_\Sigma(1^\times;1), H_\Sigma(2^\times;1,-1)\}$, the genus-zero orbits are among those given in (5.4). Otherwise, $\mathrm{c}(E')$ is of the form $\Sigma_*(n)$ where $n \equiv 1 \,(\mathrm{mod}\, 2)$ and $n > 1$. But there is no genus-zero orbit in $\Sigma_1(n) \times \mathrm{c}(E^{\mathrm{d}})$. (It is enough to check this claim for $E^{\mathrm{d}} \in \{H_\Sigma(\mathtt{1d'};4), H_\Sigma(\mathtt{2d}^\bullet;2)\}$ since there is a covering from $\mathrm{c}(E^{\mathrm{d}})$ to either $\mathrm{c}(H_\Sigma(\mathtt{1d'};4))$ or $\mathrm{c}(H_\Sigma(\mathtt{2d}^\bullet;2))$.)

Table A.2 shows the closed collection $\overline{\mathcal{B}_\Sigma}$ found in this section together with the trivial Bu$_3$-set $H_\Sigma(1^\bullet;1)$.

5.3 The Closure $\overline{\mathcal{B}_\Omega}$ of $\overline{\mathcal{B}'_\Pi} \cup \overline{\mathcal{B}'_\Sigma} \cup \mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}}$

The collection $\mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}}$ is closed Consider the following disjoint, closed collections whose union is equal to $\mathcal{B}^{\mathrm{I}}$:

- $\mathcal{B}^{\mathrm{I}}_3$ consists of the Bu$_3$-sets of the form $H_{\mathrm{I}}(-)$ which appear in the case $N = 1, p = 3$ (see Chapter 4). In other words,

  $$\mathcal{B}^{\mathrm{I}}_3 := \{E \in \mathcal{B}_{\langle 3,t+1\rangle} \mid \mathrm{c}(E) \text{ contains a region of size at least } 9\}.$$

  Then $\mathcal{B}^{\mathrm{I}}_3$ is closed, because $\mathcal{B}_{\langle 3,t+1\rangle}$ is closed (Remark 5.2).

- $\mathcal{B}_5^{\mathrm{I}}$ consists of the Bu_3-sets which appear in the case $N = 5$ and the case $N = 1, p = 5$. Then, $\mathcal{B}_5^{\mathrm{I}}$ is the union of the closed collections

  $$\begin{aligned} \mathcal{B}_{\langle 2,\Phi_5(-t)\rangle} &= \{H_{\mathrm{I}}(5^{\circ}), H_{\mathrm{I}}(5\mathtt{a}'), H_{\mathrm{I}}(5\mathtt{a}), H_{\mathrm{I}}(5\mathtt{ww}'), H_{\mathrm{I}}(5\mathtt{ww})\}, \\ \mathcal{B}_{\langle 3,\Phi_5(-t)\rangle} &= \{H_{\mathrm{I}}(5^{\bullet}), H_{\mathrm{I}}(5\mathtt{a}), H_{\mathrm{I}}(5\mathtt{ww})\}, \\ \mathcal{B}_{\langle 5,t+1\rangle} &= \{H_{\mathrm{I}}(5), H_{\mathrm{I}}(5\mathtt{a}), H_{\mathrm{I}}(5\mathtt{x}), H_{\mathrm{I}}(5\mathtt{y}), H_{\mathrm{I}}(5\mathtt{w}), H_{\mathrm{I}}(5\mathtt{ww})\} \cup \\ &\qquad \{H_{\mathrm{I}}(25; a) \mid a = 4, 9, 14, 19\}, \end{aligned}$$

  where $\Phi_5(-t) = (t^4 - t^3 + t^2 - t + 1)$. Then, $\mathcal{B}_{\langle 3,\Phi_5(-t)\rangle} \cup \mathcal{B}_{\langle 5,t+1\rangle}$ is closed because the orbits in $H_{\mathrm{I}}(5^{\bullet}) \times H_{\mathrm{I}}(5)$ are isomorphic to $H_{\mathrm{I}}(5\mathtt{ww})$. Similarly, $\mathcal{B}_{\langle 2,\Phi_5(-t)\rangle} \cup \mathcal{B}_{\langle 5,t+1\rangle}$ is closed because the orbits in $H_{\mathrm{I}}(5^{\circ}) \times H_{\mathrm{I}}(5)$ and $H_{\mathrm{I}}(5\mathtt{ww}') \times H_{\mathrm{I}}(5)$ are isomorphic to $H_{\mathrm{I}}(5\mathtt{ww})$, while the orbits in $H_{\mathrm{I}}(5\mathtt{a}') \times H_{\mathrm{I}}(5)$ are isomorphic to $H_{\mathrm{I}}(5\mathtt{a})$ or $H_{\mathrm{I}}(5\mathtt{ww})$. Finally, $\mathcal{B}_5^{\mathrm{I}}$ is closed because the orbits in $H_{\mathrm{I}}(5^{\bullet}) \times H_{\mathrm{I}}(5^{\circ})$ and $H_{\mathrm{I}}(5^{\bullet}) \times H_{\mathrm{I}}(\mathtt{5a'})$ are isomorphic to $H_{\mathrm{I}}(5\mathtt{ww})$.

- $\mathcal{B}_7^{\mathrm{I}}$ is the closed collection $\mathcal{B}_{\langle 7,t+1\rangle}$.

- $\mathcal{B}_6^{\mathrm{I}} := \{H_{\mathrm{I}}(6^{\bullet}), H_{\mathrm{I}}(6\mathtt{a}^{\bullet}), H_{\mathrm{I}}(6\mathtt{b}^{\bullet}), H_{\mathrm{I}}(6\mathtt{c}^{\bullet}), H_{\mathrm{I}}(12^{\bullet})\}$ is easily verified to be closed by straightforward computation.

Then $\mathcal{B}^{\mathrm{I}}$ is closed because, for any $E_1, E_2 \in \mathcal{B}^{\mathrm{I}}$ so that E_1 and E_2 are in distinct subcollections listed above, there is no genus-zero orbit in $\mathrm{c}(E_1) \times \mathrm{c}(E_2)$. In the case $E_2 \in \mathcal{B}_6^{\mathrm{I}}$, the claim is obvious if there is no monovalent black vertex in $\mathrm{c}(E_1)$. In order to cover the remaining cases $\mathrm{c}(E_1) = \mathrm{c}(H_{\mathrm{I}}(9^{\bullet}))$ and $\mathrm{c}(E_1) = \mathrm{c}(H_{\mathrm{I}}(5^{\bullet}))$, it is enough to check the products

$$\begin{aligned} &(9C^0\backslash\Gamma) \times (4A^0\backslash\Gamma), \qquad (9C^0\backslash\Gamma) \times (6A^0\backslash\Gamma), \\ &(5A^0\backslash\Gamma) \times (4A^0\backslash\Gamma), \qquad (5A^0\backslash\Gamma) \times (6A^0\backslash\Gamma), \end{aligned} \tag{5.5}$$

where $4A^0$, $5A^0$, $6A^0$, $9C^0$ are congruence subgroups given in the notation of [17]. In the case $E_2 \in \mathcal{B}_7^{\mathrm{I}}$, it is enough to check the products

$$(\Gamma_0(3)\backslash\Gamma) \times (\Gamma_0(7)\backslash\Gamma), \quad (\Gamma_0(5)\backslash\Gamma) \times (\Gamma_0(7)\backslash\Gamma), \quad (5A^0\backslash\Gamma) \times (\Gamma_0(7)\backslash\Gamma). \tag{5.6}$$

Finally, for the remaining case $E_1 \in \mathcal{B}_3^{\mathrm{I}}$ and $E_2 \in \mathcal{B}_5^{\mathrm{I}}$, it is enough to check the products

$$(\Gamma_0(3)\backslash\Gamma) \times (\Gamma_0(5)\backslash\Gamma), \qquad (\Gamma_0(3)\backslash\Gamma) \times (5A^0\backslash\Gamma). \tag{5.7}$$

The collection $\mathcal{B}^{\mathrm{II}}$ is also closed. In fact, for any two distinct members $E_1, E_2 \in \mathcal{B}^{\mathrm{II}}$, there is no genus-zero orbit in $E_1 \times E_2$ [11, p.198]. (Note that $\mathcal{B}^{\mathrm{II}} = \bigcup \mathcal{B}_{\mathfrak{m}}$ where the union is taken over the maximal ideals $\mathfrak{m} \subset \Lambda$ given in Theorem 4.1. Here, there are 52 maximal ideals and each $\mathcal{B}_{\mathfrak{m}}$ consists of a single member).

Moreover, for any $E^{\mathrm{I}} \in \mathcal{B}^{\mathrm{I}}$ and $E^{\mathrm{II}} \in \mathcal{B}^{\mathrm{II}}$, there is no genus-zero orbit in $\mathrm{c}(E^{\mathrm{I}}) \times \mathrm{c}(E^{\mathrm{II}})$. It is enough to observe that, for

$$C^{\mathrm{I}} \in \{\Gamma_0(3)\backslash\Gamma,\ \Gamma_0(5)\backslash\Gamma,\ 5A^0\backslash\Gamma,\ \Gamma_0(7)\backslash\Gamma,\ 4A^0\backslash\Gamma,\ 6A^0\backslash\Gamma\}, \tag{5.8}$$

and for $\mathfrak{m} \subset \Lambda$ as in Theorem 4.1, the product $C^{\mathrm{I}} \times \mathcal{P}(\Bbbk)$ is transitive of genus non-zero (here, we consider $\mathcal{P}(\Bbbk)$ simply as a Γ-set, ignoring the weights).

Proof. Exactly two regions in $\mathcal{P}(\Bbbk)$ have size 1 and other regions have size N. Then, in the case $C^{\mathrm{I}} \in \{4A^0\backslash\Gamma,\ 5A^0\backslash\Gamma,\ 6A^0\backslash\Gamma\}$, the transitivity immediately follows because there is only one region in C^{I}. In the remaining case $C^{\mathrm{I}} = (\Gamma_0(k)\backslash\Gamma)$, there are two regions in C^{I} such that one region has size 1 and one region has size k. Then, the transitivity follows from the observation that there is no covering $\mathcal{P}(\Bbbk) \to (\Gamma_0(k)\backslash\Gamma)$. Indeed, suppose that there is such a covering, then $N = nk$ for an integer n so that n divides the degree d of the covering. On the other hand, $d \equiv 2 \pmod N$, thus $n \in \{1, 2\}$. This leaves the possibilities $k = N = 7$ and $k = 5, N = 10$. In the former case, one has $p \in \{2, 29, 43\}$ and $|\mathcal{P}(\Bbbk)| \in \{9, 30, 44\}$, but then $|\mathcal{P}(\Bbbk)|$ is not divisible by 8. In the latter case, $p = 11$ and $|\mathcal{P}(\Bbbk)| = 12$, hence the covering $\mathcal{P}(\Bbbk) \to (\Gamma_0(5)\backslash\Gamma)$ must be the double covering ramified at the region of size 5 and at one of the monovalent white vertices. Consequently, the two monovalent white vertices in $\mathcal{P}(\Bbbk)$ must be related as

$$\mathrm{pc}'(1, t^2(t+1)+1) \cdot (YX)^5 = \mathrm{pc}'(1, -t^2(t+1)+1),$$

i.e. $-t^5(t^2(t+1)+1) = -t^2(t+1)+1$, which simplifies to the absurd equality $2 = 0$.

Once the transitivity of $C^{\mathrm{I}} \times \mathcal{P}(\Bbbk)$ is established, the genus can be shown to be non-zero by applying the Riemann-Hurwitz formula to the covering $C^{\mathrm{I}} \times \mathcal{P}(\Bbbk) \to$

$\mathcal{P}(\Bbbk)$. The following observation is sufficient for the conclusion: there is at least one monovalent vertex in $\mathcal{P}(\Bbbk)$, and if $3 \mid N$ or $5 \mid N$, there are at least two monovalent vertices. $\square$

Table A.4 shows the closed collection $\mathcal{B}^{\mathrm{I}}$ and Table A.5 shows the closed collection $\mathcal{B}^{\mathrm{II}}$.

The closure of $\overline{\mathcal{B}_\Sigma} \cup \mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}}$ We wish to determine the genus-zero orbits in the products $E_\Sigma \times E$, where $E_\Sigma \in \overline{\mathcal{B}_\Sigma}$ and $E \in \mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}}$. In the case $E_\Sigma = H_\Sigma(1^{\times\bullet}; 1)$, the genus-zero orbits which are not already in $\mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}}$ are listed in Table A.6. In the case

$$E_\Sigma \in \{H_\Sigma(2^\bullet; 1), H_\Sigma(2\texttt{x}^\bullet; 1), H_\Sigma(2\texttt{y}^\bullet; 1), H_\Sigma(2\texttt{d}^\bullet; 1), H_\Sigma(2\texttt{dx}^\bullet; 1), H_\Sigma(2\texttt{dy}^\bullet; 1)\},$$

there is a covering $\mathrm{c}(E_\Sigma) \to (\Gamma^2 \backslash \Gamma)$. But if the total number of regions of odd size and monovalent white vertices in $\mathrm{c}(E)$ is greater than 2, the product $(\Gamma^2\backslash\Gamma) \times \mathrm{c}(E)$ is transitive of genus non-zero. This leaves the possibilities

$$E \in \mathcal{B}^{\mathrm{I}}_6, \qquad E = H_{\mathrm{II}}(3; \psi), \qquad E = H_{\mathrm{II}}(13; a). \tag{5.9}$$

If $E \in \{H_{\mathrm{I}}(6^\bullet), H_{\mathrm{I}}(6\texttt{a}^\bullet), H_{\mathrm{I}}(6\texttt{b}^\bullet), H_{\mathrm{I}}(6\texttt{c}^\bullet)\}$,

$$\text{the genus-zero orbits in } H_\Sigma(2^\bullet; 1) \times E \text{ and } H_\Sigma(2\texttt{x}^\bullet; 1) \times E \text{ are isomorphic to } E, \tag{5.10}$$

and there is no genus-zero orbit in $E_\Sigma \times E$ otherwise. If $E = H_{\mathrm{I}}(12^\bullet)$, the genus-zero orbits in $H_\Sigma(2^\bullet; 1) \times E$ and $H_\Sigma(2\texttt{d}^\bullet; 1) \times E$ are isomorphic to E, and there is no genus-zero orbit in $E_\Sigma \times E$ otherwise. If $E = H_{\mathrm{II}}(3; \psi)$ or $E = H_{\mathrm{II}}(13; a)$, the product $(4A^0\backslash\Gamma) \times \mathrm{c}(E)$ is transitive of genus non-zero, hence $E_\Sigma \in \{H_\Sigma(2^\bullet; 1), H_\Sigma(2\texttt{x}^\bullet; 1), H_\Sigma(2\texttt{y}^\bullet; 1)\}$. Then the product Bu_3-set $E_\Sigma \times E$ is transitive, and those of genus zero among these products are listed in (A.1).

If E_Σ is not as above, there is a covering $\mathrm{c}(E_\Sigma) \to \Sigma_1(1)$. Note that the product $\Sigma_1(1) \times \mathrm{c}(E)$ is transitive for all E, because there is no covering $\mathrm{c}(E) \to \Sigma_1(1)$.

Proof. Suppose that there is a covering $\mathrm{c}(E) \to \Sigma_1(1)$. Then $\mathrm{c}(E)$ contains regions of even size, and it contains no monovalent black vertex. This leaves the possibilities:

$$E = H_{\mathrm{II}}(5; t^2 \pm 2), \qquad E = H_{\mathrm{II}}(11; a), \qquad E = H_{\mathrm{II}}(17; a).$$

If $E = H_{\mathrm{II}}(5; t^2 \pm 2)$, the regions in $\mathrm{c}(E)$ have size 1 or 8, hence the degree of the covering $\mathrm{c}(E) \to \Sigma_1(1)$ should be divisible by 4. However $|\,\mathrm{c}(E)| = 78$, thus the degree should be 26. If $E = H_{\mathrm{II}}(11; a)$, one region in $\mathrm{c}(E)$ has size 2 and other regions have size 1 or 10, hence the degree d of the covering $\mathrm{c}(E) \to \Sigma_1(1)$ should satisfy $d \equiv 0, 1 \pmod 5$. However $|\,\mathrm{c}(E)| = 24$, thus the degree should be 8.

If $E = H_{\mathrm{II}}(17; a)$, we do not see a way to outrule the covering $\mathrm{c}(E) \to \Sigma_1(1)$ simply on the basis of numerical invariants, thus we resort to a deeper method. Depending on the value of a, define $\gamma(a) \in \Gamma$ as follows:

$$\begin{aligned} &\gamma(2) := (XYXYX^2YX^2YX^2), && \gamma(9) := (YX^2YX^2YXYXYXY), \\ &\gamma(8) := (YX^2YX^2YX^2YXYX^2YXY), && \gamma(15) := (XYXYXYX^2YXYX^2). \end{aligned}$$

For $\Bbbk := \mathbb{F}_{17}[t]/(t-a)$, it can be verified, by direct computation, that $\mathrm{pc}'(0,1) \cdot \gamma(a) = \mathrm{pc}'(1,0)$ in $\mathcal{P}(\Bbbk)$ (3.7). Since $\mathrm{c}(E)$ is the double covering of $\mathcal{P}(\Bbbk)$ ramified at the two incomplete monovalent white vertices, there are edges $e, e' \in \mathrm{c}(E)$ satisfying $e \cdot \gamma(a) = e'$ so that the covering $\mathrm{c}(E) \to \Sigma_1(1)$ maps e, e' to the unique edge $e_0 \in \Sigma_1(1)$ which lies in the region of size 1. But, it can be directly verified that $e_0 \cdot \gamma(a) \neq e_0$. □

Then, $\Sigma_1(1) \times \mathrm{c}(E)$ is of genus zero if and only if

$$\begin{aligned} E \in \mathcal{B} := \{&H_{\mathrm{I}}(9'; a), H_{\mathrm{I}}(5), H_{\mathrm{I}}(5\mathtt{a}'), H_{\mathrm{I}}(5\mathtt{a}), H_{\mathrm{I}}(5\mathtt{x}), H_{\mathrm{I}}(5\mathtt{y})\} \cup \\ &\{H_{\mathrm{II}}(2; t^3+t+1), H_{\mathrm{II}}(2; t^3+t^2+1), \\ &\quad H_{\mathrm{II}}(3; \psi), H_{\mathrm{II}}(11; a), H_{\mathrm{II}}(17; a)\}. \end{aligned} \tag{5.11}$$

In any case, there are at least two regions of size 1 in $\mathrm{c}(E)$, thus let $\mathfrak{a}_1, \mathfrak{a}_2 \subset \Sigma_1(1) \times \mathrm{c}(E)$ be two regions of size 2 projecting to the region of size 2 in $\Sigma_1(1)$. Then, for any $n > 1$, the covering $\pi_n \colon \Sigma_1(n) \times \mathrm{c}(E) \to \Sigma_1(1) \times \mathrm{c}(E)$ is fully ramified at $\mathfrak{a}_1, \mathfrak{a}_2$, thus $\Sigma_1(n) \times \mathrm{c}(E)$ is of genus zero if and only if π_n is not ramified elsewhere. This happens only if $n = 2$, there is no monovalent white vertex in $\mathrm{c}(E)$, and all but two regions in $\mathrm{c}(E)$ have size divisible by 4, i.e.

$E = H_{\mathrm{II}}(3;\psi)$ (for example, if $n > 2$, the covering π_n is ramified at the regions of size 1).

In view of the above, we wish to determine the genus-zero orbits in the products

$$\begin{aligned}
&E_\Sigma \times E, &&\text{where}\quad E_\Sigma \in \{H_\Sigma(1;1), H_\Sigma(1^\times;1),\\
& && \qquad H_\Sigma(2;1,1), H_\Sigma(2^\times;1,-1)\},\ \ E \in \mathcal{B},\\
&E_\Sigma \times H_{\mathrm{II}}(3;\psi), &&\text{where}\quad E_\Sigma \in \{H_\Sigma(1;2), H_\Sigma(1^\times;2),\\
& && \qquad H_\Sigma(1\mathtt{aa};2), H_\Sigma(1\mathtt{ab};2),\\
& && \qquad H_\Sigma(1\mathtt{aa}^\times;2), H_\Sigma(1\mathtt{ab}^\times;2),\\
& && \qquad H_\Sigma(2;2,1), H_\Sigma(2^\times;2,-1),\\
& && \qquad H_\Sigma(2\mathtt{a};2,1), H_\Sigma(2\mathtt{a}^\times;2,-1)\\
& && \qquad H_\Sigma(2\mathtt{d}';2), H_\Sigma(2\mathtt{d};2), H_\Sigma(2\mathtt{dw};2),\\
& && \qquad H_\Sigma(2\mathtt{d}'^\times;2), H_\Sigma(2\mathtt{d}^\times;2), H_\Sigma(2\mathtt{dw}^\times;2).
\end{aligned}$$

Considering (5.9) as well as the fact that $\Sigma_2(2) \times \mathrm{c}(H_{\mathrm{II}}(3;\psi))$ is not of genus zero, it is enough to check the products

$$\begin{aligned}
&E_\Sigma \times E, &&\text{where}\quad E_\Sigma \in \{H_\Sigma(1;1), H_\Sigma(1^\times;1)\},\ \ E \in \mathcal{B},\\
&E_\Sigma \times H_{\mathrm{II}}(3;\psi), &&\text{where}\quad E_\Sigma \in \{H_\Sigma(1;2), H_\Sigma(1^\times;2),\\
& && \qquad H_\Sigma(1\mathtt{aa};2), H_\Sigma(1\mathtt{ab};2),\\
& && \qquad H_\Sigma(1\mathtt{aa}^\times;2), H_\Sigma(1\mathtt{ab}^\times;2),\\
& && \qquad H_\Sigma(2;1,1), H_\Sigma(2^\times;1,-1)\}.
\end{aligned}$$

The products of the first kind are all of genus zero except the products $H_\Sigma(1^\times;1) \times H_{\mathrm{I}}(9';a)$, and they are listed in Table A.7. For the products of the second kind, observe that if $E_\Sigma \in \{H_\Sigma(1\mathtt{aa};2), H_\Sigma(1\mathtt{ab};2)\}$, the weight $w := w_{E_\Sigma, H_{\mathrm{II}}(3;\psi)}$ on $P := \Sigma_1(2) \times \mathrm{c}(H_{\mathrm{II}}(3;\psi))$ assigns the value of $\frac{1}{4}$ to two regions of size 1 and the value of $\frac{1}{2}$ to one region of size 1, hence $\chi(P,w) = 0$. Therefore, $E_\Sigma \in \{H_\Sigma(1;2), H_\Sigma(1^\times;2), H_\Sigma(2;1,1), H_\Sigma(2^\times;1,-1)\}$. In this case $E_\Sigma \times H_{\mathrm{II}}(3;\psi)$ is transitive of genus zero, and these are listed in (A.2) and (A.3).

In the rest, let $\mathcal{B}^{\mathrm{III}}$ denote the collection of the Bu_3-sets found in this section:

$$\mathcal{B}^{\mathrm{III}} := \overline{(\mathcal{B}_\Sigma \cup \mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}})} \smallsetminus (\overline{\mathcal{B}_\Sigma} \cup \mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}}).$$

Then, $\overline{\mathcal{B}_\Omega}$ is the union of $\overline{\mathcal{B}_\Pi} \cup \overline{\mathcal{B}_\Sigma} \cup \mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}} \cup \mathcal{B}^{\mathrm{III}}$ with the collection of the genus-zero orbits in the products

$$\begin{aligned} &E_\Pi \times E_\Sigma, && \text{where} && E_\Pi \in \overline{\mathcal{B}_\Pi} \smallsetminus \overline{\mathcal{B}_\Sigma} \\ &&&&& E_\Sigma \in \overline{\mathcal{B}_\Sigma} \smallsetminus \overline{\mathcal{B}_\Pi}, \\ &E_\Pi \times E', && \text{where} && E_\Pi \in \overline{\mathcal{B}_\Pi} \smallsetminus \overline{\mathcal{B}_\Sigma} \\ &&&&& E' \in \mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}} \cup \mathcal{B}^{\mathrm{III}}. \end{aligned} \tag{5.12}$$

The Products $E_\Pi \times E_\Sigma$ Since $E_\Pi \in \overline{\mathcal{B}_\Pi} \smallsetminus \overline{\mathcal{B}_\Sigma}$, there is a covering $\mathrm{c}(E_\Pi) \to C_\Pi$ for some Γ-set C_Π such that

$$C_\Pi \in \{\Pi_1(\iota) \mid \iota \neq \langle 1\rangle, \langle \omega - 1\rangle, \langle 3\rangle\} \cup \{\Pi_\bullet(3), \Pi_\circ(2)\}. \tag{5.13}$$

Moreover, since $E_\Sigma \in \overline{\mathcal{B}_\Sigma} \smallsetminus \overline{\mathcal{B}_\Pi}$, there is a covering $\mathrm{c}(E_\Sigma) \to \Sigma_1(n)$ for some $n \neq 1, 3$, or there is a covering $\mathrm{c}(E_\Sigma) \to (4D^0\backslash\Gamma)$. But there is no genus-zero orbit in $C_\Pi \times (4D^0\backslash\Gamma)$, while there is no genus-zero orbit in $C_\Pi \times \Sigma_1(n)$ unless $C_\Pi \in \{\Pi_1(2), \Pi_1(2(\omega-1))\}$ and $n = 2$. Moreover, the products $\Pi_2(2) \times \Sigma_1(2)$, $\Pi_3(2) \times \Sigma_1(2)$ and $\Pi_1(2) \times \Sigma_2(2)$ are of genus non-zero.

Proof. First consider the products $C_\Pi \times (4D^0\backslash\Gamma)$. The case $C_\Pi = \Pi_\circ(2)$ is clear since all regions have size at least 12 and there is no monovalent vertex. For the case $C_\Pi = \Pi_\bullet(3)$, one can directly verify that $\Pi_\bullet(\omega - 1) \times (4A^0\backslash\Gamma)$ is of genus non-zero. Finally in the case $C_\Pi = \Pi_1(\iota)$, one can conclude that $\Pi_1(\iota) \times (4A^0\backslash\Gamma)$ is of genus non-zero, by applying the Riemann-Hurwitz formula to the covering $\Pi_1(\iota) \times (4A^0\backslash\Gamma) \to \Pi_1(\iota)$.

Now consider the products $C_\Pi \times \Sigma_1(n)$. If $C_\Pi = \Pi_\bullet(3)$, all regions have size at least 6 and there is no monovalent vertex, hence the statement follows. If $C_\Pi = \Pi_\circ(2)$ and n is even, similarly all regions have size at least 6 and there is no monovalent vertex. If $C_\Pi = \Pi_\circ(2)$ and n is odd, consider an orbit $C \subset \Pi_\circ(2) \times \Sigma_1(n)$. Note that all regions in $\Pi_\circ(2)$ have size 6, while there is a region of size 1 in $\Sigma_1(n)$, hence let the degree of the covering $C \to \Sigma_1(n)$ be $6d$. Thus, d regions in C map to the region of size 1 in $\Sigma_1(n)$, while $2d$ regions map to each region of size 2. Since $n > 3$, there are at least two regions of size 2 in $\Sigma_1(n)$, hence the Riemann-Hurwitz formula gives $\chi(C) \leq 2 \cdot 6d - (5d + 4d + 4d) < 0$.

From now on, $C_\Pi = \Pi_1(\iota)$. Suppose $\iota \not\subset \langle \omega - 1 \rangle$ and let $3m + 1 := |\Pi_1(\iota)|$. If $3 \mid n$, the covering $\Pi_1(\iota) \times \Sigma_1(n) \to \Pi_1(\iota) \times \Sigma_1(3) = \Pi_1(\iota) \times \Pi_1(3) = \Pi_1(3\iota)$ of degree $\frac{n}{3} > 1$ is fully ramified over $3m + 1 \geq 4$ regions of size 6 (those which project to the region of size 6 in $\Sigma_1(3) = \Pi_1(3)$), hence $\Pi_1(\iota) \times \Sigma_1(n)$ is of genus non-zero. If $3 \nmid n$, the covering $\Pi_1(\iota) \times \Sigma_1(n) \to \Pi_1(\iota) \times \Sigma_1(1) = \Pi_1((\omega - 1)\iota)$ of degree $n > 1$ is similarly fully ramified over $(m+1)$ regions, hence $\Pi_1(\iota) \times \Sigma_1(n)$ is of genus non-zero unless $m = 1$ and there is no other ramification, which implies $\iota = \langle 2 \rangle$ and $n = 2$.

Considering the prime factorization of ι, it is left to prove that there is no genus-zero orbit in $\Pi_1((\omega - 1)^3) \times \Sigma_1(n)$ and $\Pi_1(6) \times \Sigma_1(2)$. For the latter, it is enough to observe that $\Pi_1(6) \times \Sigma_1(2) = \Pi_1(2) \times \Sigma_1(3) \times \Sigma_1(2)$, and that the genus-zero orbits in $\Sigma_1(3) \times \Sigma_1(2)$ are of the form $\Sigma_*(6)$ (Remark 5.4). As for the former, let $n' := \gcd(3, n)$. Then, the covering $\Pi_1((\omega - 1)^3) \times \Sigma_1(n) \to \Pi_1((\omega - 1)^3) \times \Sigma_1(n') = \Pi_1((\omega - 1)^3) \times \Pi_1(*)$ of degree $\frac{n}{n'} > 1$ is fully ramified over 3 regions of size 6 in the orbit which is isomorphic to $\Pi_1((\omega - 1)^3)$, and over 6 regions of size 6 in the orbit which is isomorphic to $\Pi_2((\omega - 1)^3)$. □

Therefore, for the products $E_\Pi \times E_\Sigma$ in (5.12), it is enough to consider the case

$$\begin{aligned} E_\Pi \in \; & \{H_\Pi(1, 2), H_\Pi(1^\times, 2), H_\Pi(1', 2), H_\Pi(1\mathtt{ax}, 2), H_\Pi(1\mathtt{ay}, 2), H_\Pi(1\mathtt{b}, 2), \\ & \quad H_\Pi(1, 2(\omega - 1)), H_\Pi(1^\times, 2(\omega - 1)), H_\Pi(1', 2(\omega - 1)), \\ & \quad\quad H_\Pi(1\mathtt{ax}, 2(\omega - 1)), H_\Pi(1\mathtt{ay}, 2(\omega - 1)), H_\Pi(1\mathtt{b}, 2(\omega - 1))\}, \\ E_\Sigma \in \; & \{H_\Sigma(1; 2), H_\Sigma(1\mathtt{aa}; 2), H_\Sigma(1\mathtt{ab}; 2), \\ & \quad H_\Sigma(1^\times; 2), H_\Sigma(1\mathtt{aa}^\times; 2), H_\Sigma(1\mathtt{ab}^\times; 2)\}. \end{aligned}$$

It can be assumed that $E_\Sigma \in \mathcal{B}_2 := \{H_\Sigma(1; 2), H_\Sigma(1\mathtt{aa}; 2), H_\Sigma(1\mathtt{ab}; 2)\}$, because otherwise $E_\Sigma = H_\Pi(\circ; 1) \times E'_\Sigma$ for some $E'_\Sigma \in \mathcal{B}_2$, thus $E_\Pi \times E_\Sigma$ is equal to $\tilde{E}_\Pi \times E'_\Sigma$ where $\tilde{E}_\Pi := E_\Pi \times H_\Pi(\circ; 1)$ (note that $\tilde{E}_\Pi \in \overline{\mathcal{B}_\Pi} \smallsetminus \overline{\mathcal{B}_\Sigma}$ as well). Moreover, the only genus-zero orbit in $\Pi_1(2(\omega - 1)) \times \Sigma_1(2)$ is isomorphic to $\Pi_1(2) \times \Sigma_1(2)$, hence the genus-zero orbits in $H_\Pi(*; 2(\omega - 1)) \times E_\Sigma$ are isomorphic to those in

$H_\Pi(*;2) \times E_\Sigma$. Consequently, we restrict attention to

$$\begin{aligned} E_\Pi \in \ & \{H_\Pi(1,2), H_\Pi(1^\times,2), H_\Pi(1',2), H_\Pi(1\texttt{ax},2), H_\Pi(1\texttt{ay},2), H_\Pi(1\texttt{b},2)\}, \\ E_\Sigma \in \ & \{H_\Sigma(1;2), H_\Sigma(1\texttt{aa};2), H_\Sigma(1\texttt{ab};2)\}. \end{aligned}$$

In this case, $E_\Pi \times E_\Sigma$ is transitive and of genus zero. These Bu_3-sets are listed in Table A.8.

The Products $E_\Pi \times E'$ Recall that $E_\Pi \in \overline{\mathcal{B}_\Pi} \smallsetminus \overline{\mathcal{B}_\Sigma}$ and $E' \in \mathcal{B}^{\mathrm{I}} \cup \mathcal{B}^{\mathrm{II}} \cup \mathcal{B}^{\mathrm{III}}$ (5.12). First suppose

$$\begin{aligned} E' \notin \mathcal{B}_\bullet := \ & \{H_{\mathrm{I}}(6^\bullet), H_{\mathrm{I}}(6\texttt{a}^\bullet), H_{\mathrm{I}}(6\texttt{b}^\bullet), H_{\mathrm{I}}(6\texttt{c}^\bullet), \\ & H_{\mathrm{I}}(9^\bullet), H_{\mathrm{I}}(9\texttt{a}^\bullet), H_{\mathrm{I}}(9\texttt{b}^\bullet), H_{\mathrm{I}}(9\texttt{c}^\bullet)\}, \end{aligned}$$

Then, there is either a covering $\mathrm{c}(E') \to \mathcal{P}(\mathbb{k})$ where $\mathbb{k}$ is as in Theorem 4.1, or a covering $\mathrm{c}(E') \to C'$ where

$$C' \in \{\Gamma_0(9)\backslash\Gamma,\ 9H^0\backslash\Gamma,\ \Gamma_1(5)\backslash\Gamma,\ 5A^0\backslash\Gamma,\ \Gamma_0(7)\backslash\Gamma,\ 4D^0\backslash\Gamma\}.$$

On the other hand, there is a covering $\mathrm{c}(E_\Pi) \to C_\Pi$ where C_Π is as in (5.13). But among the products $C_\Pi \times \mathcal{P}(\mathbb{k})$ and $C_\Pi \times C'$, only the following contain genus-zero orbits:

$$\Pi_1(2) \times (\Gamma_0(9)\backslash\Gamma), \quad \Pi_1(2(\omega-1)) \times (\Gamma_0(9)\backslash\Gamma), \quad \Pi_1(2) \times (9H^0\backslash\Gamma). \tag{5.14}$$

Proof. For the case $C_\Pi = \Pi_\circ(2)$, first note that the products $\Pi_\circ(1) \times \mathcal{P}(\mathbb{k})$, hence $\Pi_\circ(2) \times \mathcal{P}(\mathbb{k})$, are of genus non-zero as it can be verified by applying the Riemann-Hurwitz formula to the covering $\Pi_\circ(1) \times \mathcal{P}(\mathbb{k}) \to \mathcal{P}(\mathbb{k})$ of degree 3 (it helps to observe that there is at least one monovalent black vertex in $\mathcal{P}(\mathbb{k})$ if $3 \mid N$). As for the products $\Pi_\circ(2) \times C'$, all regions have size at least 6 and there is no monovalent black vertex, hence it is enough to check those C' which contain monovalent white vertices, i.e. $C' = 5A^0\backslash\Gamma$. But even $\Pi_\circ(1) \times (5A^0\backslash\Gamma)$ is of genus non-zero. Moreover, the case $C_\Pi = \Pi_\bullet(3)$ is already complete, because $\Pi_\bullet(\omega - 1) = (6A^0\backslash\Gamma)$ and there is no genus-zero orbit in $(6A^0\backslash\Gamma) \times C'$ (5.5) and $(6A^0\backslash\Gamma) \times \mathcal{P}(\mathbb{k})$ (5.8).

In the remaining case $C_\Pi = \Pi_1(\iota)$, first consider the products $\Pi_1(\iota) \times \mathcal{P}(\Bbbk)$. Since $\Pi_1(3) = \Sigma_1(3)$, there is no genus-zero orbit in $\Pi_1(3) \times \mathcal{P}(\Bbbk)$ (5.11), hence one can take $\iota \not\subset \langle \omega - 1 \rangle$. Moreover, since $\Pi_1(2) = (\Gamma_0(3)\backslash\Gamma)$, there is no genus-zero orbit in $\Pi_1(2) \times \mathcal{P}(\Bbbk)$ (5.8), hence one can take $\iota \not\subset \langle 2 \rangle$ as well. Thus, one region in $\Pi_1(\iota)$ has size 1, and other regions have size 6.

Suppose that there is a covering $\mathcal{P}(\Bbbk) \to \Pi_1(\iota)$. Then $6 \mid N$, thus the pair (p, N) equals $(5, 12)$, $(13, 12)$ or $(19, 18)$, hence $|\mathcal{P}(\Bbbk)| \in \{26, 14, 20\}$. Since $|\Pi_1(\iota)| = 6k + 1$ must divide $|\mathcal{P}(\Bbbk)|$, one has $(p, N) = (5, 12)$ or $(p, N) = (13, 12)$, and $\mathcal{P}(\Bbbk)$ is a double covering of $\Pi_1(\iota)$. Consequently, the two black vertices in $\mathcal{P}(\Bbbk)$ must be related as

$$\mathrm{pc}'(1, t^4(t + 1) + 1) \cdot (YX)^6 = \mathrm{pc}'(1, t^8(t + 1) + 1),$$

i.e. $t^6(t^4(t + 1) + 1) = t^8(t + 1) + 1$, which simplifies to $t^4 - t^3 + 1 = 0$. This can be easily checked to not hold for the two maximal ideals with $(p, N) = (5, 12)$ and the four maximal ideals with $(p, N) = (13, 12)$.

Let $C \subset \Pi_1(\iota) \times \mathcal{P}(\Bbbk)$ be an orbit and let the degree of the covering $C \to \mathcal{P}(\Bbbk)$ be $6m + \epsilon$ where $\epsilon \in \{0, 1\}$. Since there is no covering $\mathcal{P}(\Bbbk) \to \Pi_1(\iota)$, one has $m \geq 1$. The number of regions in C which map to a region of size 1 in $\mathcal{P}(\Bbbk)$ is $m + \epsilon$, the number of vertices which map to a monovalent black vertex is $2m + \epsilon$, and the number of vertices which map to a monovalent white vertex is $3m + \epsilon$. There are two regions of size 1 and at least one monovalent vertex in $\mathcal{P}(\Bbbk)$, hence the Riemann-Hurwitz formula gives

$$\chi(C) \leq (6m + \epsilon) \cdot 2 - (5m + 5m + 3m) = 2\epsilon - m < 2.$$

Now consider the products $\Pi_1(\iota) \times C'$ except the cases $C' \in \{\Gamma_0(9)\backslash\Gamma,\ 9H^0\backslash\Gamma\}$. As before, there is no genus-zero orbit in $\Pi_1(3) \times C'$ (5.11) and in $\Pi_1(2) \times C'$ (5.5, 5.6, 5.7). Moreover, the products $\Pi_1(\iota) \times C''$ with $\iota \not\subset \langle \omega - 1 \rangle$, $\iota \not\subset \langle 2 \rangle$, and

$$C'' \in \{\Gamma_0(5)\backslash\Gamma,\ 5A^0\backslash\Gamma,\ \Gamma_0(7)\backslash\Gamma,\ 4A^0\backslash\Gamma\},$$

are all of genus non-zero. (This can be seen by considering the covering $\Pi_1(\iota) \times C' \to \Pi_1(\iota)$ and using the Riemann-Hurwitz formula.)

It is left to consider the products $\Pi_1(\iota) \times C'$ where $C' \in \{\Gamma_0(9)\backslash\Gamma,\, 9H^0\backslash\Gamma\}$. If $\iota \not\subset \langle\omega - 1\rangle$ and $\iota \not\subset \langle 2\rangle$, let $6k+1 := |\Pi_1(\iota)|$ and observe that the 3-fold coverings

$$\begin{array}{rcccl} \Pi_1(\iota) \times (\Gamma_0(9)\backslash\Gamma) & \to & \Pi_1(\iota) \times (\Gamma_0(3)\backslash\Gamma) & = & \Pi_1(2\iota), \\ \Pi_1(\iota) \times (9H^0\backslash\Gamma) & \to & \Pi_1(\iota) \times (\Gamma(3)\backslash\Gamma) & = & \Pi_3(2\iota) \end{array}$$

are fully ramified at $3k+1$ and $2(3k+1)$ regions, respectively (this can be seen by considering the ramification of the coverings $\Gamma_0(9)\backslash\Gamma \to \Gamma_0(3)\backslash\Gamma$ and $9H^0\backslash\Gamma \to \Gamma(3)\backslash\Gamma$). Similarly, the 3-fold coverings

$$\begin{array}{rcl} \Pi_1(4) \times (\Gamma_0(9)\backslash\Gamma) & \to & \Pi_1(4) \times \Pi_1(2), \\ \Pi_1(4) \times (9H^0\backslash\Gamma) & \to & \Pi_1(4) \times \Pi_3(2) \end{array}$$

are fully ramified at 2 regions and 1 monovalent black vertex in an orbit which is isomorphic to $\Pi_1(4)$, and at 4 or 6 regions in an orbit which is isomorphic to $\Pi_3(4)$. Moreover, there is no genus-zero orbit in $\Pi_1(3) \times (\Gamma_0(9)\backslash\Gamma)$ and $\Pi_1(\omega - 1) \times (9H^0\backslash\Gamma)$ (5.11). This concludes the statement. □

In addition to (5.14), observe that the products

$$\begin{array}{lll} \Pi_1(\omega-1) \times (\Gamma_1(9)\backslash\Gamma), & \Pi_2(1) \times (\Gamma_0(9)\backslash\Gamma), & \Pi_3(1) \times (\Gamma_0(9)\backslash\Gamma) \\ \Pi_1(\omega-1) \times (27A^0\backslash\Gamma), & \Pi_2(1) \times (9H^0\backslash\Gamma), & \end{array}$$

are of genus non-zero (5.9, 5.11). Moreover, there is no genus-zero orbit in the product Bu_3-sets $H_{\mathrm{II}}(3;2) \times H_{\mathrm{I}}(9\mathtt{wa})$ and $H_{\mathrm{II}}(3;2) \times H_{\mathrm{I}}(9\mathtt{wb})$ because the specification $\mathrm{s}_{H_{\mathrm{II}}(3;2)}$ assigns values divisible by 3 to all regions, while the specifications $\mathrm{s}_{H_{\mathrm{I}}(9\mathtt{wa})}$ and $\mathrm{s}_{H_{\mathrm{I}}(9\mathtt{wb})}$ assign values not divisible by 3 to three regions. As a result, for the products $E_{\mathrm{II}} \times E'$ with $E' \notin \mathcal{B}_\bullet$, it is enough to consider

- the case

$$\begin{array}{rcl} E_{\mathrm{II}} & \in & \{H_{\mathrm{II}}(1,2), H_{\mathrm{II}}(1^\times,2), H_{\mathrm{II}}(1',2), \\ & & \quad H_{\mathrm{II}}(1\mathtt{ax},2), H_{\mathrm{II}}(1\mathtt{ay},2), H_{\mathrm{II}}(1\mathtt{b},2)\}, \\ E' & \in & \{H_{\mathrm{I}}(9), H_{\mathrm{I}}(9\mathtt{a}), H_{\mathrm{I}}(9\mathtt{b}), H_{\mathrm{I}}(9\mathtt{x}), H_{\mathrm{I}}(9\mathtt{aa}), H_{\mathrm{I}}(9\mathtt{bb}), \\ & & \quad H_{\mathrm{I}}(9\mathtt{xa}), H_{\mathrm{I}}(9\mathtt{xb}), H_{\mathrm{I}}(9\mathtt{xc}), H_{\mathrm{I}}(9\mathtt{wa}), H_{\mathrm{I}}(9\mathtt{wb}), \\ & & \quad\quad H_{\mathrm{I}}(9';a), H_{\mathrm{I}}(9';a) \times H_{\mathrm{II}}(1;\omega-1), H_{\mathrm{I}}(27;a)\}, \end{array}$$

- the case

$$\begin{aligned} E_{\Pi} &\in \{H_{\Pi}(1, 2(\omega-1)), H_{\Pi}(1^{\times}, 2(\omega-1)), H_{\Pi}(1', 2(\omega-1)),\\ &\qquad H_{\Pi}(1\mathtt{ax}, 2(\omega-1)), H_{\Pi}(1\mathtt{ay}, 2(\omega-1)), H_{\Pi}(1\mathtt{b}, 2(\omega-1))\},\\ E' &\in \{H_{\mathrm{I}}(9'; a), H_{\mathrm{I}}(9'; a) \times H_{\Pi}(1; \omega-1)\}, \end{aligned}$$

- and the exceptional products $H_{\Pi}(3\mathtt{a}'; 2) \times H_{\mathrm{I}}(9\mathtt{wa})$ and $H_{\Pi}(3\mathtt{a}'; 2) \times H_{\mathrm{I}}(9\mathtt{wb})$ (the orbits in the exceptional products are isomorphic to $H_{\mathrm{I}}(9\mathtt{wa})$ and $H_{\mathrm{I}}(9\mathtt{wb})$).

Note that the second case is implicit in the first because $H_{\Pi}(*; 2(\omega-1)) = H_{\Pi}(*; 2) \times H_{\Pi}(1; \omega-1)$, while the orbit in $H_{\Pi}(1; \omega-1) \times H_{\Pi}(1; \omega-1)$ which is isomorphic to $H_{\Pi}(2; \omega-1)$ must be ignored because $\Pi_2(1) \times (\Gamma_0(9)\backslash\Gamma)$ is of genus non-zero. As for the first case, it is helpful to observe that the genus-zero orbits in $\mathrm{c}(E_{\Pi}) \times \mathrm{c}(E') = \Pi_1(2) \times \mathrm{c}(E')$ are isomorphic to $\mathrm{c}(E')$. Then, the following hold for the products $E_{\Pi} \times E'$ in the first case:

- There is no genus-zero orbit if $E_{\Pi} = H_{\Pi}(1\mathtt{b}, 2)$.
- The genus-zero orbits in $H_{\Pi}(1'; 2) \times E'$ are isomorphic to E'.
- The orbits in $H_{\Pi}(1; 2) \times E'$ are isomorphic to those in $H_{\Pi}(1^{\times}; 2) \times E'$.
- There is no genus-zero orbit if

  $$E' \in \{H_{\mathrm{I}}(9\mathtt{wa}), H_{\mathrm{I}}(9\mathtt{wb}), H_{\mathrm{I}}(27; a), H_{\mathrm{I}}(9'; a) \times H_{\Pi}(1; \omega-1)\}$$

  unless $E_{\Pi} = H_{\Pi}(1'; 2)$.
- If $E' \in \{H_{\mathrm{I}}(9'; a), H_{\mathrm{I}}(9)\}$, the genus-zero orbits in $H_{\Pi}(1^{\times}; 2) \times E'$ are isomorphic to $H_{\mathrm{I}}(9\mathtt{a})$, the genus-zero orbits in $H_{\Pi}(1\mathtt{ay}; 2) \times E'$ are isomorphic to $H_{\mathrm{I}}(9\mathtt{b})$, and the genus-zero orbits in $H_{\Pi}(1\mathtt{ax}; 2) \times E'$ are isomorphic to $H_{\mathrm{I}}(9\mathtt{x})$.
- Let

  $$\begin{aligned} \mathcal{B}^{\mathsf{a}} &:= \{H_{\mathrm{I}}(9\mathtt{a}), H_{\mathrm{I}}(9\mathtt{aa})\}\\ \mathcal{B}^{\mathsf{b}} &:= \{H_{\mathrm{I}}(9\mathtt{b}), H_{\mathrm{I}}(9\mathtt{bb})\}\\ \mathcal{B}^{\mathsf{x}} &:= \{H_{\mathrm{I}}(9\mathtt{x}), H_{\mathrm{I}}(9\mathtt{xa}), H_{\mathrm{I}}(9\mathtt{xb}), H_{\mathrm{I}}(9\mathtt{xc})\}. \end{aligned}$$

Then, the genus-zero orbits in $H_{\Pi}(1^{\times};2)\times E'$ are isomorphic to E' if $E'\in\mathcal{B}^{\mathsf{a}}$, and there is no genus-zero orbit if $E'\notin\mathcal{B}^{\mathsf{a}}$. The genus-zero orbits in $H_{\Pi}(1\mathtt{ay};2)\times E'$ are isomorphic to E' if $E'\in\mathcal{B}^{\mathsf{b}}$, and there is no genus-zero orbit if $E'\notin\mathcal{B}^{\mathsf{b}}$. The genus-zero orbits in $H_{\Pi}(1\mathtt{ax};2)\times E'$ are isomorphic to E' if $E'\in\mathcal{B}^{\mathsf{x}}$, and there is no genus-zero orbit if $E'\notin\mathcal{B}^{\mathsf{x}}$.

Finally consider the products $E_{\Pi}\times E'$ with $E'\in\mathcal{B}_{\bullet}$. If the specification $\mathrm{s}:=\mathrm{s}_{E_{\Pi}}$ on $\mathrm{c}(E_{\Pi})$ is $\mathbb{Z}_{6k}$-valued but there is no monovalent black vertex $\mathfrak{a}_{\bullet}\in\mathrm{c}(E_{\Pi})$ with $\mathrm{s}(\mathfrak{a}_{\bullet})\equiv 0\,(\mathrm{mod}\,3)$, the weight $w:=w_{E_{\Pi},E'}$ on $P:=\mathrm{c}(E_{\Pi})\times\mathrm{c}(E')$ assigns the value of $\frac{1}{3}$ to all monovalent black vertices (if any). In this case, or if there is no monovalent black vertex in $\mathrm{c}(E_{\Pi})$, there is no complete monovalent black vertex in (P,w). Hence, there is no genus-zero orbit in (P,w), because there is a covering $P\to\mathrm{c}(E')$ and there are three monovalent black vertices in $\mathrm{c}(E')$. Consequently, it is enough to consider the case

$$\begin{aligned}E_{\Pi}\in\ &\{H_{\Pi}(1';2),H_{\Pi}(1\mathtt{ax};2),H_{\Pi}(\bullet';2),H_{\Pi}(\bullet\mathtt{ax};2),\\&\qquad H_{\Pi}(\bullet';2(\omega-1)),H_{\Pi}(\bullet\mathtt{ax};2(\omega-1))\}.\end{aligned}$$

In other words, for some $E'_{\Pi}\in\{H_{\Pi}(1';2),H_{\Pi}(1\mathtt{ax};2)\}$, one has $E_{\Pi}=E'_{\Pi}$ or $E_{\Pi}=E'_{\Pi}\times E_{\bullet}$ where $E_{\bullet}\in\{H_{\Pi}(\bullet;1),H_{\Pi}(\bullet;\omega-1)\}$. But either there is no genus-zero orbit in $E_{\bullet}\times E'$, or the genus-zero orbits are isomorphic to E' (5.10). Hence, the genus-zero orbits in $E_{\Pi}\times E'$ are isomorphic to those in $E'_{\Pi}\times E'$. Moreover, $\mathrm{c}(H_{\Pi}(1';2))=\mathrm{c}(H_{\Pi}(1\mathtt{ax};2))=\Pi_1(2)$ and the weights $w':=w_{H_{\Pi}(1';2),E'}$ and $w^{\mathsf{ax}}:=w_{H_{\Pi}(1\mathtt{ax};2),E'}$ are both trivial, thus the orbits in $H_{\Pi}(1';2)\times E'$ and $H_{\Pi}(1\mathtt{ax};2)\times E'$ are isomorphic. Finally, in the case $E'\in\{H_{\mathrm{I}}(9^{\bullet}),H_{\mathrm{I}}(9\mathtt{a}^{\bullet}),H_{\mathrm{I}}(9\mathtt{b}^{\bullet}),H_{\mathrm{I}}(9\mathtt{c}^{\bullet})\}$, the only genus-zero orbit in $H_{\Pi}(1';2)\times E'$ is isomorphic to E', while in the case $E'\in\{H_{\mathrm{I}}(6^{\bullet}),H_{\mathrm{I}}(6\mathtt{a}^{\bullet}),H_{\mathrm{I}}(6\mathtt{b}^{\bullet}),H_{\mathrm{I}}(6\mathtt{c}^{\bullet})\}$, the products $H_{\Pi}(1';2)\times E'$ are of genus zero, and these are listed in (A.4).

Appendix A

The List of the Main Theorem

In this chapter, we present the conjugacy classes of saturated genus-zero subgroups (Theorem 1.2). In fact, since $H \mapsto H\backslash\mathrm{Bu}_3$ establishes a bijection between the conjugacy classes of subgroups and transitive right Bu_3-sets, we present the latter (note that the proof of Theorem 1.2 is entirely expressed in the language of Bu_3-sets). The saturated genus-zero Bu_3-sets come in the form of two infinite families and a finite number of sporadic ones. The sporadic Bu_3-sets are presented in Section A.1, while the members of the infinite families are listed in Tables A.1 and A.2 in terms of the corresponding pairs of Γ-sets and specifications (Remark 2.8). In each table, the first column gives the "name" of the Bu_3-set. The second column shows the corresponding Γ-set (for the notation of the second column, see Section 2.1.4). The third column describes the corresponding specification. The following is a guide for reading this column (note that all variables in the guide are replaced by constants in the tables):

$s(\bullet) = \{a_1, a_2, \ldots, a_k\}$ $s(\circ) = \{b_1, b_2, \ldots, b_m\}$	There are k monovalent black vertices and m monovalent white vertices so that the specification assigns them the values $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_m$.
$s(\mathbf{n}) = \{a_1, a_2, \ldots, a_k\}$	There are k regions of size n and the specification assigns them the values $a_1, a_2, \ldots, a_k$.
$s(\mathbf{big}) = a_1$ or $s(\mathbf{big}) = \{a_1, a_2\}$	This is found in Table A.2 and **big** refers to the regions of size $2n$ in the Γ-sets $\Sigma_1(n)$ or $\Sigma_2(n)$.
$s(\bullet) = \{a, \ldots\}$ $s(\circ) = \{b, \ldots\}$ $s(\mathbf{n}) = \{c, \ldots\}$	The specification assigns the value a to all monovalent black vertices, the value b to all monovalent white vertices and the value c to all regions of size n.

Finally, the fourth column shows the module for most (except for finitely many) entries. More explicitly, for a Bu_3-set E in the list, we show the module $\mathcal{A}(E) := \mathcal{A}(H)$ where $H \subset \mathrm{Bu}_3$ is the stabilizer of any element in E, i.e. $E \cong H \backslash \mathrm{Bu}_3$ (since H is well-defined up to conjugacy, the module $\mathcal{A}(E)$ is well-defined up to isomorphism).

In Table A.1, the parameter ι refers to a nonzero ideal of the principal ideal domain $\mathbb{Z}[\omega]$, where ω is a third root of unity.

Table A.1: The Π-family

The saturated Bu_3-set	The Γ-set	The specification s	The module
$H_\Pi(1;1)$: Trivial	$\Pi_1(1)$	$\mathbb{Z}_2$-valued.	0
$H_\Pi(1;\omega-1)$	$\Pi_1(\omega-1)$	$\mathbb{Z}_2$-valued.	(A.5)
$H_\Pi(1;\iota)$ where $\iota \neq \langle 1\rangle, \langle \omega-1\rangle$	$\Pi_1(\iota)$	$\mathbb{Z}_6$-valued. $s(\mathbf{1}) = 1$, $s(\mathbf{6}) = \{0, \ldots\}$. $s(\mathbf{2}) = 2$ if $\iota \subset \langle \omega-1\rangle$. $s(\bullet) = 2$ if $\iota \not\subset \langle \omega-1\rangle$. $s(\mathbf{3}) = 3$ if $\iota \subset \langle 2\rangle$. $s(\circ) = 3$ if $\iota \not\subset \langle 2\rangle$.	(A.5)

$H_{\Pi}(1^{\times};\iota)$ where $\iota \subset \langle 2\rangle$	$\Pi_1(\iota)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = 1$, $\mathrm{s}(\mathbf{3}) = 3$, $\mathrm{s}(\mathbf{6}) = \{6,\ldots\}$. $\mathrm{s}(\mathbf{2}) = 2$ if $\iota \subset \langle \omega - 1\rangle$. $\mathrm{s}(\bullet) = 8$ if $\iota \not\subset \langle \omega - 1\rangle$.	(A.6)
$H_{\Pi}(1';2)$	$\Pi_1(2)$	$\mathbb{Z}_{4}$-valued. $\mathrm{s}(\mathbf{1}) = 1$, $\mathrm{s}(\mathbf{3}) = 3$, $\mathrm{s}(\bullet) = 0$.	
$H_{\Pi}(1'; 2(\omega-1))$	$\Pi_1(2(\omega-1))$	$\mathbb{Z}_{4}$-valued. $\mathrm{s}(\mathbf{1}) = 1$, $\mathrm{s}(\mathbf{2}) = 2$, $\mathrm{s}(\mathbf{3}) = 3$, $\mathrm{s}(\mathbf{6}) = 2$.	
$H_{\Pi}(1\mathtt{ax};2)$	$\Pi_1(2)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = 9$, $\mathrm{s}(\mathbf{3}) = 3$, $\mathrm{s}(\bullet) = 0$.	
$H_{\Pi}(1\mathtt{ax}; 2(\omega-1))$	$\Pi_1(2(\omega-1))$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = 9$, $\mathrm{s}(\mathbf{2}) = 6$, $\mathrm{s}(\mathbf{3}) = 3$, $\mathrm{s}(\mathbf{6}) = 6$.	
$H_{\Pi}(1\mathtt{ay};2)$	$\Pi_1(2)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = 5$, $\mathrm{s}(\mathbf{3}) = 3$, $\mathrm{s}(\bullet) = 4$.	
$H_{\Pi}(1\mathtt{ay}; 2(\omega-1))$	$\Pi_1(2(\omega-1))$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = 5$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{3}) = 3$, $\mathrm{s}(\mathbf{6}) = 6$.	
$H_{\Pi}(1\mathtt{b};2)$	$\Pi_1(2)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = 11$, $\mathrm{s}(\mathbf{3}) = 9$, $\mathrm{s}(\bullet) = 4$.	
$H_{\Pi}(1\mathtt{b}; 2(\omega-1))$	$\Pi_1(2(\omega-1))$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = 11$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{3}) = 9$, $\mathrm{s}(\mathbf{6}) = 6$.	
$H_{\Pi}(1\mathtt{ba};4)$	$\Pi_1(4)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = 23$, $\mathrm{s}(\mathbf{3}) = 21$, $\mathrm{s}(\mathbf{6}) = \{18,18\}$, $\mathrm{s}(\bullet) = 16$.	
$H_{\Pi}(1\mathtt{ba}; 4(\omega-1))$	$\Pi_1(4(\omega-1))$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = 23$, $\mathrm{s}(\mathbf{2}) = 22$, $\mathrm{s}(\mathbf{3}) = 21$, $\mathrm{s}(\mathbf{6}) = \{18,\ldots\}$.	

$H_\Pi(\mathtt{1bb};4)$	$\Pi_1(4)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = 11$, $\mathrm{s}(\mathbf{3}) = 9$, $\mathrm{s}(\mathbf{6}) = \{18,18\}$, $\mathrm{s}(\bullet) = 16$.	
$H_\Pi(\mathtt{1bb};$ $4(\omega-1))$	$\Pi_1(4(\omega-1))$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = 11$, $\mathrm{s}(\mathbf{2}) = 22$, $\mathrm{s}(\mathbf{3}) = 9$, $\mathrm{s}(\mathbf{6}) = \{18,\ldots\}$.	
$H_\Pi(\bullet;1)$	$\Pi_\bullet(1)$	$\mathbb{Z}_2$-valued.	(A.7)
$H_\Pi(\bullet;\omega-1)$	$\Pi_\bullet(1)$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\bullet) = \{0,2\}$, $\mathrm{s}(\mathbf{2}) = 4$.	(A.7)
$H_\Pi(\bullet;\iota)$ where $\iota \neq \langle 1\rangle, \langle \omega-1\rangle$	$\Pi_\bullet(\iota)$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{6}) = \{0,\ldots\}$. $\mathrm{s}(\bullet) = \{2,\ldots\}$. $\mathrm{s}(\mathbf{2}) = 2$ if $\iota \not\subset \langle \omega-1\rangle$.	(A.7)
$H_\Pi(\bullet^\times;\iota)$ where $\iota \subset \langle 2\rangle$	$\Pi_\bullet(\iota)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{6}) = \{6,\ldots\}$, $\mathrm{s}(\bullet) = \{8,\ldots\}$. $\mathrm{s}(\mathbf{2}) = 2$ if $\iota \not\subset \langle \omega-1\rangle$.	(A.8)
$H_\Pi(\bullet';2)$	$\Pi_\bullet(2)$	$\mathbb{Z}_4$-valued. $\mathrm{s}(\mathbf{2}) = 2$, $\mathrm{s}(\mathbf{6}) = 2$, $\mathrm{s}(\bullet) = \{0,0\}$.	
$H_\Pi(\bullet';$ $2(\omega-1))$	$\Pi_\bullet(2)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{6}) = 6$, $\mathrm{s}(\bullet) = \{0,8\}$.	
$H_\Pi(\bullet\mathtt{ax};2)$	$\Pi_\bullet(2)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{2}) = 6$, $\mathrm{s}(\mathbf{6}) = 6$, $\mathrm{s}(\bullet) = \{0,0\}$.	
$H_\Pi(\bullet\mathtt{ax};$ $2(\omega-1))$	$\Pi_\bullet(2(\omega-1))$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{6}) = \{6,\ldots\}$, $\mathrm{s}(\bullet) = \{0,\ldots\}$.	
$H_\Pi(2;\omega-1)$	$\Pi_2(\omega-1)$	$\mathbb{Z}_2$-valued.	(A.9)

$H_\Pi(2;\iota)$ where $\iota \subset \langle \omega-1\rangle$ and $\iota \neq \langle \omega-1\rangle$	$\Pi_2(\iota)$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{2}) = \{2,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{0,\ldots\}$.	(A.9)
$H_\Pi(2^\times;\iota)$ where $\iota \subset \langle 2(\omega-1)\rangle$	$\Pi_2(\iota)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{2}) = \{2,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{6,\ldots\}$.	(A.10)
$H_\Pi(2';$ $2(\omega-1))$	$\Pi_2(2(\omega-1))$	$\mathbb{Z}_4$-valued. $\mathrm{s}(\mathbf{2}) = \{2,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{2,\ldots\}$.	
$H_\Pi(2\texttt{ax};$ $2(\omega-1))$	$\Pi_2(2(\omega-1))$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{2}) = \{6,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{6,\ldots\}$.	
$H_\Pi(\circ;1)$	$\Pi_1(1)$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{1}) = 5$, $\mathrm{s}(\bullet) = 4$, $\mathrm{s}(\circ) = 3$.	(A.11)
$H_\Pi(\circ;\omega-1)$	$\Pi_1(\omega-1)$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{1}) = 5$, $\mathrm{s}(\mathbf{2}) = 4$, $\mathrm{s}(\circ) = \{3,\ldots\}$.	(A.11)
$H_\Pi(\circ;\iota)$ where $\iota \neq \langle 1\rangle, \langle \omega-1\rangle$	$\Pi_\circ(\iota)$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{6}) = \{0,\ldots\}$, $\mathrm{s}(\circ) = \{3,\ldots\}$. $\mathrm{s}(\mathbf{3}) = 3$ if $\iota \not\subset \langle 2\rangle$.	(A.11)
$H_\Pi(3;\iota)$ where $\iota \subset \langle 2\rangle$	$\Pi_3(\iota)$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{3}) = \{3,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{0,\ldots\}$.	(A.12)
$H_\Pi(3\texttt{a};\iota)$ where $\iota \subset \langle 2\rangle$	$\Pi_3(\iota)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{3}) = \{3,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{6,\ldots\}$.	(A.13)
$H_\Pi(3\texttt{a}';2)$	$\Pi_3(2)$	$\mathbb{Z}_4$-valued. $\mathrm{s}(\mathbf{3}) = \{3,\ldots\}$.	
$H_\Pi(3\texttt{a}';$ $2(\omega-1))$	$\Pi_3(2(\omega-1))$	$\mathbb{Z}_4$-valued. $\mathrm{s}(\mathbf{3}) = \{3,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{2,\ldots\}$.	

$H_{\Pi}(3\texttt{b};\iota)$ where $\iota \subset \langle 2\rangle$	$\Pi_3(\iota)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{3}) = \{9,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{6,\ldots\}$.	(A.14)
$H_{\Pi}(3\texttt{ba};\iota)$ where $\iota \subset \langle 4\rangle$	$\Pi_3(\iota)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{3}) = \{21,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{18,\ldots\}$.	(A.15)
$H_{\Pi}(3\texttt{bb};\iota)$ where $\iota \subset \langle 4\rangle$	$\Pi_3(\iota)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{3}) = \{9,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{18,\ldots\}$.	(A.16)
$H_{\Pi}(3/2;\iota)$ where $\iota \subset \langle 4\rangle$	$\Pi_{3/2}(\iota)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{3}) = \{3,3,9,9\}$, $\mathrm{s}(\mathbf{6}) = \{6,\ldots\}$. Two regions of size 3 take the same value under s if and only if they map to the same region under the double covering $\Pi_{3/2}(\iota) \to \Pi_3(\iota/2)$.	(A.17)
$H_{\Pi}(3/2^{\times};\iota)$ where $\iota \subset \langle 4\rangle$	$\Pi_{3/2}(\iota)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{3}) = \{9,9,21,21\}$, $\mathrm{s}(\mathbf{6}) = \{18,\ldots\}$. Two regions of size 3 take the same value under s if and only if they map to the same region under the double covering $\Pi_{3/2}(\iota) \to \Pi_3(\iota/2)$.	(A.18)
$H_{\Pi}(\texttt{D};\iota)$	$\Pi_2(\iota^*)$ where ι^* is the complex conjugate.	$\mathbb{Z}_{2n}$-valued where n is the least positive integer such that $n \in (\omega-1)\iota$. $\mathrm{s}(\mathbf{2}) = \{-2,\ldots\}$, $\mathrm{s}(\mathbf{6}) = \{-6,\ldots\}$. $\mathrm{s}(\bullet) = \{2k, 2k\}$ if $\iota \not\subset \langle \omega - 1\rangle$ where $\lvert k\rvert$ is the least positive integer such that $\lvert k\rvert \in \iota$ and $k \equiv 2 \pmod 3$.	(A.19)

In Table A.2, the parameter n refers to a positive integer and the parameter a takes values in the indicated cyclic group which depends on n. In the cases where there is an ambiguity in the description of the specification, we clarify it using figures (see Remark 2.5).

Table A.2: The Σ-family

The saturated Bu_3-set	The Γ-set	The specification s	The module
$H_\Sigma(1;n)$	$\Sigma_1(n)$	$\mathbb{Z}_{2n}$-valued. $\mathrm{s}(\mathbf{2}) = \{2,\ldots\}$, $\mathrm{s}(\textbf{big}) = 0$. $\mathrm{s}(\mathbf{1}) = \{1, 1+n\}$ if $2 \mid n$. $\mathrm{s}(\mathbf{1}) = 1$ if $2 \nmid n$. $\mathrm{s}(\circ) = n$ if $2 \nmid n$.	(A.20)
$H_\Sigma(1^\bullet;1)$: Trivial	$\{*\}$	$\mathbb{Z}_2$-valued.	0
$H_\Sigma(1^\times;n)$ where $3 \nmid n$	$\Sigma_1(n)$	$\mathbb{Z}_{6n}$-valued. $\mathrm{s}(\textbf{big}) = 4n$. $\mathrm{s}(\mathbf{2}) = \{2+2n,\ldots\}$ if $n \equiv 1 \pmod 3$. $\mathrm{s}(\mathbf{2}) = \{2-2n,\ldots\}$ if $n \equiv 2 \pmod 3$. $\mathrm{s}(\mathbf{1}) = \{1+2n, 1-n\}$ if $n \equiv 2 \pmod 6$. $\mathrm{s}(\mathbf{1}) = \{1-2n, 1+n\}$ if $n \equiv 4 \pmod 6$. $\mathrm{s}(\mathbf{1}) = 1+2n$ if $n \equiv 5 \pmod 6$. $\mathrm{s}(\mathbf{1}) = 1-2n$ if $n \equiv 1 \pmod 6$. $\mathrm{s}(\circ) = 3n$ if $2 \nmid n$.	(A.21)
$H_\Sigma(1^{\times\bullet};1)$	$\{*\}$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{1}) = 5$, $\mathrm{s}(\bullet) = 4$, $\mathrm{s}(\circ) = 3$.	(A.11)
$H_\Sigma(1^\times;3)$	The triple covering of $\Sigma_1(3)$ fully ramified at the region of size 1 and the region of size 2.	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{3}) = 3$, $\mathrm{s}(\mathbf{6}) = \{0,\ldots\}$, $\mathrm{s}(\circ) = \{3,\ldots\}$.	(A.21)
$H_\Sigma(1\mathtt{w};n)$ where $8 \mid n$	$\Sigma_1(n)$	$\mathbb{Z}_{2n}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1+n\}$, $\mathrm{s}(\textbf{big}) = 0$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2, 2+n\}$, starting and ending with $2+n$. (Figure A.1)	(A.22)

$H_{\Sigma}(1\mathtt{w}^{\times}; n)$ where $3 \nmid n$ and $8 \mid n$	$\Sigma_1(n)$	$\mathbb{Z}_{6n}$-valued. $\mathrm{s}(\mathbf{big}) = 4n$. $\mathrm{s}(\mathbf{1}) = \{1, 1+n\}$ if $n \equiv 1 \pmod{3}$. $\mathrm{s}(\mathbf{1}) = \{1, 1-n\}$ if $n \equiv 2 \pmod{3}$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2+2n, 2-n\}$, starting and ending with $2-n$, if $n \equiv 1 \pmod{3}$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2-2n, 2+n\}$, starting and ending with $2+n$, if $n \equiv 2 \pmod{3}$.	(A.23)
$H_{\Sigma}(1\mathtt{aa}; n)$ where $n \equiv 2 \pmod{4}$	$\Sigma_1(n)$	$\mathbb{Z}_{4n}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1+\frac{n^2}{2}\}$, $\mathrm{s}(\mathbf{big}) = 2n$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2, 2+2n\}$ (Figure A.2).	(A.24)
$H_{\Sigma}(1\mathtt{ab}; n)$ where $n \equiv 2 \pmod{4}$	$\Sigma_1(n)$	$\mathbb{Z}_{4n}$-valued. $\mathrm{s}(\mathbf{1}) = \{1+2n, 1+\frac{n^2}{2}+2n\}$, $\mathrm{s}(\mathbf{big}) = 2n$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2, 2+2n\}$ (Figure A.2).	(A.25)
$H_{\Sigma}(1\mathtt{aa}^{\times}; n)$ where $3 \nmid n$ and $n \equiv 2 \pmod{4}$	$\Sigma_1(n)$	$\mathbb{Z}_{12n}$-valued. $\mathrm{s}(\mathbf{big}) = 10n$. $\mathrm{s}(\mathbf{1}) = \{1+4n, 1+\frac{n^2}{2}-4n\}$ if $n \equiv 1 \pmod{3}$. $\mathrm{s}(\mathbf{1}) = \{1-4n, 1+\frac{n^2}{2}+4n\}$ if $n \equiv 2 \pmod{3}$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2-4n, 2+2n\}$ if $n \equiv 1 \pmod{3}$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2+4n, 2-2n\}$ if $n \equiv 2 \pmod{3}$. The alternating pattern is similar to $H_{\Sigma}(1\mathtt{aa}; n)$.	(A.26)

$H_\Sigma(\mathtt{1ab}^\times; n)$ where $3 \nmid n$ and $n \equiv 2 \pmod 4$	$\Sigma_1(n)$	$\mathbb{Z}_{12n}$-valued. s(**big**) $= 10n$. s(**1**) $= \{1 - 2n, 1 + \frac{n^2}{2} + 2n\}$ if $n \equiv 1 \pmod 3$. s(**1**) $= \{1 + 2n, 1 + \frac{n^2}{2} - 2n\}$ if $n \equiv 2 \pmod 3$. s(**2**) alternates between $\{2 - 4n, 2 + 2n\}$ if $n \equiv 1 \pmod 3$. s(**2**) alternates between $\{2 + 4n, 2 - 2n\}$ if $n \equiv 2 \pmod 3$. The alternating pattern is similar to $H_\Sigma(\mathtt{1ab}; n)$.	(A.27)
$H_\Sigma(\mathtt{1d}'; 4)$	The double covering of $\Sigma_1(4)$ ramified at the region of size 2 and at one of the regions of size 1: $\Gamma_1(8)\backslash\Gamma$	$\mathbb{Z}_4$-valued. s(**1**) $= \{1, 1\}$, s(**2**) $= 2$, s(**4**) $= 0$, s(**8**) $= \{0, 0\}$.	
$H_\Sigma(\mathtt{1da}; 4)$	The same as $H_\Sigma(\mathtt{1d}'; 4)$	$\mathbb{Z}_8$-valued. s(**1**) $= \{1, 1\}$, s(**2**) $= 2$, s(**4**) $= 4$, s(**8**) $= \{0, 0\}$.	
$H_\Sigma(\mathtt{1db}; 4)$	The same as $H_\Sigma(\mathtt{1d}'; 4)$	$\mathbb{Z}_8$-valued. s(**1**) $= \{5, 5\}$, s(**2**) $= 2$, s(**4**) $= 4$, s(**8**) $= \{0, 0\}$.	
$H_\Sigma(\mathtt{1dx}; 4)$	The same as $H_\Sigma(\mathtt{1d}'; 4)$	$\mathbb{Z}_8$-valued. s(**1**) $= \{7, 7\}$, s(**2**) $= 6$, s(**4**) $= 4$, s(**8**) $= \{0, 0\}$.	
$H_\Sigma(\mathtt{1daa}; 4)$	The same as $H_\Sigma(\mathtt{1d}'; 4)$	$\mathbb{Z}_{16}$-valued. s(**1**) $= \{1, 9\}$, s(**2**) $= 10$, s(**4**) $= 12$, s(**8**) $= \{8, 8\}$.	

$H_\Sigma(1\texttt{dbb};4)$	The same as $H_\Sigma(1\texttt{d}';4)$	$\mathbb{Z}_{16}$-valued. $\mathrm{s}(\mathbf{1}) = \{5, 13\}$, $\mathrm{s}(\mathbf{2}) = 2$, $\mathrm{s}(\mathbf{4}) = 12$, $\mathrm{s}(\mathbf{8}) = \{8, 8\}$.	
$H_\Sigma(1\texttt{dxx};4)$	The same as $H_\Sigma(1\texttt{d}';4)$	$\mathbb{Z}_{16}$-valued. $\mathrm{s}(\mathbf{1}) = \{7, 15\}$, $\mathrm{s}(\mathbf{2}) = 6$, $\mathrm{s}(\mathbf{4}) = 4$, $\mathrm{s}(\mathbf{8}) = \{8, 8\}$.	
$H_\Sigma(1\texttt{d}';8)$	The double covering of $\Sigma_1(8)$ ramified at the regions of size 2 which are not in the middle.	$\mathbb{Z}_8$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1, 5, 5\}$, $\mathrm{s}(\mathbf{2}) = \{6, 6\}$, $\mathrm{s}(\mathbf{4}) = \{0, 0\}$, $\mathrm{s}(\mathbf{16}) = \{0, 0\}$. Two regions of size 1 take the same value under s if and only if they map to the same region under the double covering onto $\Sigma_1(8)$. (Figure A.3)	
$H_\Sigma(1\texttt{da};8)$	The same as $H_\Sigma(1\texttt{d}';8)$	$\mathbb{Z}_{16}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1, 9, 9\}$, $\mathrm{s}(\mathbf{2}) = \{2, 2\}$, $\mathrm{s}(\mathbf{4}) = \{4, 4\}$, $\mathrm{s}(\mathbf{16}) = \{0, 0\}$. Two regions of size 1 take the same value under s if and only if they map to the same region under the double covering onto $\Sigma_1(8)$.	
$H_\Sigma(1\texttt{db};8)$	The same as $H_\Sigma(1\texttt{d}';8)$	$\mathbb{Z}_{16}$-valued. $\mathrm{s}(\mathbf{1}) = \{5, 5, 13, 13\}$, $\mathrm{s}(\mathbf{2}) = \{10, 10\}$, $\mathrm{s}(\mathbf{4}) = \{4, 4\}$, $\mathrm{s}(\mathbf{16}) = \{0, 0\}$. Two regions of size 1 take the same value under s if and only if they map to the same region under the double covering onto $\Sigma_1(8)$.	
$H_\Sigma(1\texttt{d}'^{\times};4)$	The same as $H_\Sigma(1\texttt{d}';4)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = \{5, 5\}$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{4}) = 8$, $\mathrm{s}(\mathbf{8}) = \{4, 4\}$.	

$H_\Sigma(1\mathtt{da}^\times;4)$	The same as $H_\Sigma(1\mathtt{d}';4)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{17,17\}$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{4}) = 20$, $\mathrm{s}(\mathbf{8}) = \{16,16\}$.	
$H_\Sigma(1\mathtt{db}^\times;4)$	The same as $H_\Sigma(1\mathtt{d}';4)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{5,5\}$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{4}) = 20$, $\mathrm{s}(\mathbf{8}) = \{16,16\}$.	
$H_\Sigma(1\mathtt{dx}^\times;4)$	The same as $H_\Sigma(1\mathtt{d}';4)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{23,23\}$, $\mathrm{s}(\mathbf{2}) = 22$, $\mathrm{s}(\mathbf{4}) = 20$, $\mathrm{s}(\mathbf{8}) = \{16,16\}$.	
$H_\Sigma(1\mathtt{daa}^\times;4)$	The same as $H_\Sigma(1\mathtt{d}';4)$	$\mathbb{Z}_{48}$-valued. $\mathrm{s}(\mathbf{1}) = \{17,41\}$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{4}) = 44$, $\mathrm{s}(\mathbf{8}) = \{40,40\}$.	
$H_\Sigma(1\mathtt{dbb}^\times;4)$	The same as $H_\Sigma(1\mathtt{d}';4)$	$\mathbb{Z}_{48}$-valued. $\mathrm{s}(\mathbf{1}) = \{5,29\}$, $\mathrm{s}(\mathbf{2}) = 34$, $\mathrm{s}(\mathbf{4}) = 44$, $\mathrm{s}(\mathbf{8}) = \{40,40\}$.	
$H_\Sigma(1\mathtt{dxx}^\times;4)$	The same as $H_\Sigma(1\mathtt{d}';4)$	$\mathbb{Z}_{48}$-valued. $\mathrm{s}(\mathbf{1}) = \{23,47\}$, $\mathrm{s}(\mathbf{2}) = 22$, $\mathrm{s}(\mathbf{4}) = 20$, $\mathrm{s}(\mathbf{8}) = \{40,40\}$.	
$H_\Sigma(1\mathtt{d}'^\times;8)$	The same as $H_\Sigma(1\mathtt{d}';8)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{17,17,5,5\}$, $\mathrm{s}(\mathbf{2}) = \{22,22\}$, $\mathrm{s}(\mathbf{4}) = \{8,8\}$, $\mathrm{s}(\mathbf{16}) = \{8,8\}$. Two regions of size 1 take the same value under s if and only if they map to the same region under the double covering onto $\Sigma_1(8)$.	

$H_\Sigma(1\mathtt{da}^\times; 8)$	The same as $H_\Sigma(1\mathtt{d}'; 8)$	$\mathbb{Z}_{48}$-valued. $\mathrm{s}(\mathbf{1}) = \{17, 17, 41, 41\}$, $\mathrm{s}(\mathbf{2}) = \{34, 34\}$, $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\mathbf{16}) = \{32, 32\}$. Two regions of size 1 take the same value under s if and only if they map to the same region under the double covering onto $\Sigma_1(8)$.	
$H_\Sigma(1\mathtt{db}^\times; 8)$	The same as $H_\Sigma(1\mathtt{d}'; 8)$	$\mathbb{Z}_{48}$-valued. $\mathrm{s}(\mathbf{1}) = \{5, 5, 29, 29\}$, $\mathrm{s}(\mathbf{2}) = \{10, 10\}$, $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\mathbf{16}) = \{32, 32\}$. Two regions of size 1 take the same value under s if and only if they map to the same region under the double covering onto $\Sigma_1(8)$.	
$H_\Sigma(2; n, a)$ where $a \in \mathbb{Z}_n$ such that $a^2 \equiv 1 \pmod{n}$, and $a \equiv 1 \pmod{3}$ in the case $3 \mid n$.	$\Sigma_2(n)$	$\mathbb{Z}_{2n}$-valued. $\mathrm{s}(\mathbf{2}) = \{2a, \ldots\}$, $\mathrm{s}(\mathbf{big}) = \{0, 0\}$.	(A.28)
$H_\Sigma(2^\bullet; 1)$	$\Gamma^2\backslash\Gamma$	$\mathbb{Z}_2$-valued.	(A.7)
$H_\Sigma(2^\times; n, a)$ where $a \in \mathbb{Z}_{3n}$ such that $a^2 \equiv 1 \pmod{3n}$, and $a \equiv -1 \pmod{3}$.	$\Sigma_2(n)$	$\mathbb{Z}_{6n}$-valued. $\mathrm{s}(\mathbf{2}) = \{2a, \ldots\}$, $\mathrm{s}(\mathbf{big}) = \{4n, 4n\}$.	(A.29)
$H_\Sigma(2\mathtt{x}^\bullet; 1)$	$\Gamma^2\backslash\Gamma$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{2}) = 4$, $\mathrm{s}(\bullet) = \{0, 2\}$.	(A.7)

$H_\Sigma(2\mathtt{y}^\bullet; 1)$	$\Gamma^2 \backslash \Gamma$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{2}) = 4$, $\mathrm{s}(\bullet) = \{4, 4\}$.	(A.19)
$H_\Sigma(2\mathtt{w}; n, a)$ where $8 \mid n$ and $a \in \mathbb{Z}_{n/2}$ such that $a^2 \equiv 1 \pmod{n}$, and $a \equiv 1 \pmod{3}$ in the case $3 \mid n$.	$\Sigma_2(n)$	$\mathbb{Z}_{2n}$-valued. $\mathrm{s}(\mathbf{big}) = \{0, 0\}$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2a, 2a + n\}$.	(A.30)
$H_\Sigma(2\mathtt{w}^\times; n, a)$ where $8 \mid n$ and $a \in \mathbb{Z}_{3n/2}$ such that $a^2 \equiv 1 \pmod{3n}$, and $a \equiv -1 \pmod{3}$.	$\Sigma_2(n)$	$\mathbb{Z}_{6n}$-valued. $\mathrm{s}(\mathbf{big}) = \{4n, 4n\}$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2a, 2a + 3n\}$.	(A.31)
$H_\Sigma(2\mathtt{a}; n, a)$ where $n \equiv 2 \pmod{4}$ and $a \in \mathbb{Z}_n$ such that $a^2 \equiv 1 \pmod{n}$, and $a \equiv 1 \pmod{3}$ in the case $3 \mid n$.	$\Sigma_2(n)$	$\mathbb{Z}_{4n}$-valued. $\mathrm{s}(\mathbf{big}) = \{2n, 2n\}$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2a, 2a + 2n\}$.	(A.32)

$H_\Sigma(2\mathtt{a}^\times; n, a)$ where $n \equiv 2 \pmod 4$ and $a \in \mathbb{Z}_{3n}$ such that $a^2 \equiv 1 \pmod{3n}$, and $a \equiv -1 \pmod 3$.	$\Sigma_2(n)$	$\mathbb{Z}_{12n}$-valued. $\mathrm{s}(\mathbf{big}) = \{10n, 10n\}$. $\mathrm{s}(\mathbf{2})$ alternates between $\{2a, 2a + 6n\}$.	(A.33)
$H_\Sigma(2\mathtt{d}'; 2)$	The double covering of $\Sigma_2(2)$ ramified at the regions of size 2: $\Gamma(4)\backslash\Gamma$	$\mathbb{Z}_4$-valued. $\mathrm{s}(\mathbf{4}) = \{0, \ldots\}$.	
$H_\Sigma(2\mathtt{d}; 2)$	The same as $H_\Sigma(2\mathtt{d}'; 2)$	$\mathbb{Z}_8$-valued. $\mathrm{s}(\mathbf{4}) = \{4, \ldots\}$.	
$H_\Sigma(2\mathtt{dw}; 2)$	The 4-fold covering of $\Sigma_2(2)$ fully ramified at the regions of size 2: $8N^0\backslash\Gamma$ ([17])	$\mathbb{Z}_8$-valued. $\mathrm{s}(\mathbf{8}) = \{0, 0\}$ and $\mathrm{s}(\mathbf{4})$ alternates between $\{0, 4\}$. (Figure A.4)	
$H_\Sigma(2\mathtt{d}^\bullet; 2)$	The quotient of $\Gamma(4)\backslash\Gamma$ by an automorphism of order 3: $4D^0\backslash\Gamma$ ([17]) (Figure A.6)	$\mathbb{Z}_8$-valued. $\mathrm{s}(\mathbf{4}) = \{4, 4\}$, $\mathrm{s}(\bullet) = \{0, 0\}$.	

$H_\Sigma(\texttt{2d}';4)$	The double covering of $\Sigma_2(4)$ ramified at two non-adjacent regions of size 2	$\mathbb{Z}_4$-valued. $\mathrm{s}(\mathbf{2}) = \{2,\ldots\}$, $\mathrm{s}(\mathbf{4}) = \{0,0\}$, $\mathrm{s}(\mathbf{8}) = \{0,\ldots\}$.	
$H_\Sigma(\texttt{2d};4,1)$	The same as $H_\Sigma(\texttt{2d}';4)$	$\mathbb{Z}_8$-valued. $\mathrm{s}(\mathbf{2}) = \{2,\ldots\}$, $\mathrm{s}(\mathbf{4}) = \{4,4\}$, $\mathrm{s}(\mathbf{8}) = \{0,\ldots\}$.	
$H_\Sigma(\texttt{2d};4,-1)$	The same as $H_\Sigma(\texttt{2d}';4)$	$\mathbb{Z}_8$-valued. $\mathrm{s}(\mathbf{2}) = \{6,\ldots\}$, $\mathrm{s}(\mathbf{4}) = \{4,4\}$, $\mathrm{s}(\mathbf{8}) = \{0,\ldots\}$.	
$H_\Sigma(\texttt{2da};4,1)$	The same as $H_\Sigma(\texttt{2d}';4)$	$\mathbb{Z}_{16}$-valued. $\mathrm{s}(\mathbf{2})$ alternates between $\{2,10\}$ (Figure A.4). $\mathrm{s}(\mathbf{4}) = \{12,12\}$, $\mathrm{s}(\mathbf{8}) = \{8,\ldots\}$.	
$H_\Sigma(\texttt{2da};4,-1)$	The same as $H_\Sigma(\texttt{2d}';4)$	$\mathbb{Z}_{16}$-valued. $\mathrm{s}(\mathbf{2})$ alternates between $\{6,14\}$ (Figure A.4). $\mathrm{s}(\mathbf{4}) = \{12,12\}$, $\mathrm{s}(\mathbf{8}) = \{8,\ldots\}$.	
$H_\Sigma(\texttt{2d}'^{\times};2)$	The same as $H_\Sigma(\texttt{2d}';2)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{4}) = \{8,\ldots\}$.	
$H_\Sigma(\texttt{2d}^{\times};2)$	The same as $H_\Sigma(\texttt{2d}';2)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{4}) = \{20,\ldots\}$.	
$H_\Sigma(\texttt{2dw}^{\times};2)$	The same as $H_\Sigma(\texttt{2dw};2)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{8}) = \{16,16\}$ and $\mathrm{s}(\mathbf{4})$ alternates between $\{8,20\}$. The alternating pattern is similar to $H_\Sigma(\texttt{2dw};2)$.	

$H_\Sigma(\mathtt{2dx}^\bullet; 2)$	The same as $H_\Sigma(\mathtt{2d}^\bullet; 2)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\bullet) = \{0, 8\}$.	
$H_\Sigma(\mathtt{2dy}^\bullet; 2)$	The same as $H_\Sigma(\mathtt{2d}^\bullet; 2)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\bullet) = \{16, 16\}$.	
$H_\Sigma(\mathtt{2d}'^\times; 4)$	The same as $H_\Sigma(\mathtt{2d}'; 4)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{2}) = \{10, \ldots\}$, $\mathrm{s}(\mathbf{4}) = \{8, 8\}$, $\mathrm{s}(\mathbf{8}) = \{4, \ldots\}$.	
$H_\Sigma(\mathtt{2d}^\times; 4, 1)$	The same as $H_\Sigma(\mathtt{2d}'; 4)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{2}) = \{10, \ldots\}$, $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\mathbf{8}) = \{16, \ldots\}$.	
$H_\Sigma(\mathtt{2d}^\times; 4, -1)$	The same as $H_\Sigma(\mathtt{2d}'; 4)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{2}) = \{22, \ldots\}$, $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\mathbf{8}) = \{16, \ldots\}$.	
$H_\Sigma(\mathtt{2da}^\times; 4, 1)$	The same as $H_\Sigma(\mathtt{2d}'; 4)$	$\mathbb{Z}_{48}$-valued. $\mathrm{s}(\mathbf{2})$ alternates between $\{10, 34\}$. (The alternating pattern is similar to $H_\Sigma(\mathtt{2da}; 4, 1)$.) $\mathrm{s}(\mathbf{4}) = \{44, 44\}$, $\mathrm{s}(\mathbf{8}) = \{40, \ldots\}$.	
$H_\Sigma(\mathtt{2da}^\times; 4, -1)$	The same as $H_\Sigma(\mathtt{2d}'; 4)$	$\mathbb{Z}_{48}$-valued. $\mathrm{s}(\mathbf{2})$ alternates between $\{22, 46\}$. (The alternating pattern is similar to $H_\Sigma(\mathtt{2da}; 4, -1)$.) $\mathrm{s}(\mathbf{4}) = \{44, 44\}$, $\mathrm{s}(\mathbf{8}) = \{40, \ldots\}$.	

There are some common members in the Π-family and the Σ-family. The following is a complete list of them:

Table A.3: Common Members of the Infinite Families

$H_\Sigma(1^\bullet; 1) = H_\Pi(1; 1)$
$H_\Sigma(1^{\times\bullet}; 1) = H_\Pi(\circ; 1)$
$H_\Sigma(1; 1) = H_\Pi(1; \omega - 1)$
$H_\Sigma(1^\times; 1) = H_\Pi(\circ; \omega - 1)$
$H_\Sigma(2; 1, 1) = H_\Pi(2; \omega - 1)$
$H_\Sigma(1; 3) = H_\Pi(1; 3)$
$H_\Sigma(1^\times; 3) = H_\Pi(\circ; 3)$
$H_\Sigma(2; 3, 1) = H_\Pi(2; 3)$

$H_\Sigma(2^\bullet;1) = H_\Pi(\bullet;1)$ $H_\Sigma(2\mathtt{x}^\bullet;1) = H_\Pi(\bullet;\omega-1)$

$H_\Sigma(2\mathtt{y}^\bullet;1) = H_\Pi(\mathtt{D};1)$ $H_\Sigma(2^\times;1,-1) = H_\Pi(\mathtt{D};\omega-1)$ $H_\Sigma(2^\times;3,-1) = H_\Pi(\mathtt{D};3)$

A.1 Sporadic Classes

In this section, we present the sporadic Bu_3-sets in various forms. Table A.4 shows several Bu_3-sets in terms of Γ-sets and specifications, as in Tables A.1 and A.2. For most entries in the table, we write the stabilizer instead of the Γ-set, only for brevity. Note that most of these subgroups of Γ are congruence subgroups.

Table A.4: Basic Sporadic Classes of the First Kind

The saturated Bu_3-set	The Γ-set or its stabilizer	The specification s
$H_\mathrm{I}(9';a)$ where $a \in \{2,5\}$	$\Gamma_0(9)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = \{1,5,9\}$, $\mathrm{s}(\mathbf{9}) = 9$. The distribution of $\{1,5,9\}$ to the regions of size 1 depends on a (see Figure A.5).
$H_\mathrm{I}(9)$	$\Gamma_1(9)$	$\mathbb{Z}_4$-valued. $\mathrm{s}(\mathbf{1}) = \{1,1,1\}$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{9}) = \{1,1,1\}$.
$H_\mathrm{I}(9\mathtt{a})$	$\Gamma_1(9)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = \{1,1,1\}$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{9}) = \{9,9,9\}$.
$H_\mathrm{I}(9\mathtt{b})$	$\Gamma_1(9)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = \{5,5,5\}$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{9}) = \{9,9,9\}$.
$H_\mathrm{I}(9\mathtt{x})$	$\Gamma_1(9)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = \{9,9,9\}$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{9}) = \{9,9,9\}$.
$H_\mathrm{I}(9\mathtt{aa})$	$\Gamma_1(9)$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{1}) = \{1,13,25\}$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{9}) = \{9,9,9\}$.

$H_{\mathrm{I}}(9\mathtt{bb})$	$\Gamma_1(9)$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{1}) = \{5, 17, 29\}$, $\mathrm{s}(\mathbf{3}) = \{15, 15\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9, 9\}$.
$H_{\mathrm{I}}(9\mathtt{xa})$	$\Gamma_1(9)$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{1}) = \{9, 9, 9\}$, $\mathrm{s}(\mathbf{3}) = \{27, 27\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9, 9\}$.
$H_{\mathrm{I}}(9\mathtt{xb})$	$\Gamma_1(9)$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{1}) = \{21, 21, 21\}$, $\mathrm{s}(\mathbf{3}) = \{27, 27\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9, 9\}$.
$H_{\mathrm{I}}(9\mathtt{xc})$	$\Gamma_1(9)$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{1}) = \{33, 33, 33\}$, $\mathrm{s}(\mathbf{3}) = \{27, 27\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9, 9\}$.
$H_{\mathrm{I}}(9^{\bullet})$	The composition of the triple covering of $\Gamma_1(3)\backslash\Gamma$ fully ramified at the regions of size 1 and size 3 with the triple covering fully ramified at two of the three monovalent black vertices: $9J^0\backslash\Gamma$ ([17]) (Figure A.6)	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{3}) = \{3, 3, 3\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9, 9\}$, $\mathrm{s}(\bullet) = \{0, 0, 0\}$.
$H_{\mathrm{I}}(9\mathtt{a}^{\bullet})$	The same as $H_{\mathrm{I}}(9^{\bullet})$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{3}) = \{27, 27, 27\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9, 9\}$, $\mathrm{s}(\bullet) = \{0, 0, 0\}$.
$H_{\mathrm{I}}(9\mathtt{b}^{\bullet})$	The same as $H_{\mathrm{I}}(9^{\bullet})$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{3}) = \{27, 27, 27\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9, 9\}$, $\mathrm{s}(\bullet) = \{12, 12, 12\}$.

$H_{\mathrm{I}}(9\texttt{c}^{\bullet})$	The same as $H_{\mathrm{I}}(9^{\bullet})$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{3}) = \{27, 27, 27\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9, 9\}$, $\mathrm{s}(\bullet) = \{24, 24, 24\}$.
$H_{\mathrm{I}}(9\texttt{wa})$	The triple covering of $\Gamma(3)\backslash\Gamma$ fully ramified at two regions: $9H^{0}\backslash\Gamma$ ([17])	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{3}) = \{3, 3, 3, 7, 7, 7\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9\}$ Two regions of size 3 take the same value under s if and only if they map to the same region under the triple covering onto $\Gamma(3)\backslash\Gamma$.
$H_{\mathrm{I}}(9\texttt{wb})$	The same as $H_{\mathrm{I}}(9\texttt{wa})$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{3}) = \{3, 3, 3, 11, 11, 11\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9\}$ Two regions of size 3 take the same value under s if and only if they map to the same region under the triple covering onto $\Gamma(3)\backslash\Gamma$.
$H_{\mathrm{I}}(27; a)$ where $a \in \{2, 5, 11, 14, 20, 23\}$	The triple covering of $\Gamma_0(9)\backslash\Gamma$ fully ramified at the region of size 9 and at one of the regions of size 1: $27A^{0}\backslash\Gamma$ ([17])	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 13, 25, 5, 17, 19\}$, $\mathrm{s}(\mathbf{3}) = 27$, $\mathrm{s}(\mathbf{27}) = 27$. The distribution of $\{1, 13, 25, 5, 17, 19\}$ to the regions of size 1 depends on a. This is analogous to $H_{\mathrm{I}}(9'; a)$.
$H_{\mathrm{I}}(5)$	$\Gamma_1(5)$	$\mathbb{Z}_4$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1\}$, $\mathrm{s}(\mathbf{5}) = \{1, 1\}$.
$H_{\mathrm{I}}(5\texttt{a}')$	$\Gamma_1(5)$	$\mathbb{Z}_{10}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 9\}$, $\mathrm{s}(\mathbf{5}) = \{5, 5\}$.
$H_{\mathrm{I}}(5\texttt{a})$	$\Gamma_1(5)$	$\mathbb{Z}_{20}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 9\}$, $\mathrm{s}(\mathbf{5}) = \{5, 5\}$.
$H_{\mathrm{I}}(5\texttt{x})$	$\Gamma_1(5)$	$\mathbb{Z}_{20}$-valued. $\mathrm{s}(\mathbf{1}) = \{5, 5\}$, $\mathrm{s}(\mathbf{5}) = \{5, 5\}$.

$H_{\mathrm{I}}(5\mathtt{y})$	$\Gamma_1(5)$	$\mathbb{Z}_{20}$-valued. $\mathrm{s}(\mathbf{1}) = \{13, 17\}$, $\mathrm{s}(\mathbf{5}) = \{5, 5\}$.
$H_{\mathrm{I}}(5\mathtt{w})$	$\Gamma(5)$	$\mathbb{Z}_{4}$-valued. $\mathrm{s}(\mathbf{5}) = \{1, \ldots\}$.
$H_{\mathrm{I}}(5\mathtt{ww}')$	$\Gamma(5)$	$\mathbb{Z}_{10}$-valued. $\mathrm{s}(\mathbf{5}) = \{5, \ldots\}$.
$H_{\mathrm{I}}(5\mathtt{ww})$	$\Gamma(5)$	$\mathbb{Z}_{20}$-valued. $\mathrm{s}(\mathbf{5}) = \{5, \ldots\}$.
$H_{\mathrm{I}}(25; a)$ where $a \in \{4, 9, 14, 19\}$	The 5-fold covering of $\Gamma_1(5)\backslash\Gamma$ fully ramified at the regions of size 5: $25B^0\backslash\Gamma$	$\mathbb{Z}_{20}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 5, 9, 13, 17, 1, 5, 9, 13, 17\}$, $\mathrm{s}(\mathbf{25}) = \{5, 5\}$. The distribution of $\{1, 5, 9, 13, 17, 1, 5, 9, 13, 17\}$ to the regions of size 1 depends on a. This is analogous to $H_{\mathrm{I}}(9'; a)$.
$H_{\mathrm{I}}(5^{\circ})$	$5E^0$ (see [17])	$\mathbb{Z}_{10}$-valued. $\mathrm{s}(\mathbf{5}) = \{5, 5, 5\}$, $\mathrm{s}(\circ) = \{5, 5, 5\}$.
$H_{\mathrm{I}}(5^{\bullet})$	$5F^0$ (see [17])	$\mathbb{Z}_{20}$-valued. $\mathrm{s}(\mathbf{5}) = \{5, 5, 5, 5\}$, $\mathrm{s}(\bullet) = \{0, 0\}$.
$H_{\mathrm{I}}(7)$	$\Gamma_1(7)$	$\mathbb{Z}_{4}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1, 1\}$, $\mathrm{s}(\mathbf{7}) = \{3, 3, 3\}$.
$H_{\mathrm{I}}(7\mathtt{a})$	$\Gamma_1(7)$	$\mathbb{Z}_{28}$-valued. $\mathrm{s}(\mathbf{1}) = \{21, 21, 21\}$, $\mathrm{s}(\mathbf{7}) = \{7, 7, 7\}$.
$H_{\mathrm{I}}(7\mathtt{b})$	$\Gamma_1(7)$	$\mathbb{Z}_{28}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 9, 25\}$, $\mathrm{s}(\mathbf{7}) = \{7, 7, 7\}$.
$H_{\mathrm{I}}(7\mathtt{c})$	$\Gamma_1(7)$	$\mathbb{Z}_{28}$-valued. $\mathrm{s}(\mathbf{1}) = \{5, 13, 17\}$, $\mathrm{s}(\mathbf{7}) = \{7, 7, 7\}$.

$H_{\mathrm{I}}(6^\bullet)$	$\Pi_\bullet(3)$	$\mathbb{Z}_6$-valued. $\mathrm{s}(\mathbf{6}) = \{0, \ldots\}$, $\mathrm{s}(\bullet) = \{0, \ldots\}$.
$H_{\mathrm{I}}(6\mathtt{a}^\bullet)$	The same as $H_{\mathrm{I}}(6^\bullet)$	$\mathbb{Z}_{18}$-valued. $\mathrm{s}(\mathbf{6}) = \{12, \ldots\}$, $\mathrm{s}(\bullet) = \{0, \ldots\}$.
$H_{\mathrm{I}}(6\mathtt{b}^\bullet)$	The same as $H_{\mathrm{I}}(6^\bullet)$	$\mathbb{Z}_{18}$-valued. $\mathrm{s}(\mathbf{6}) = \{12, \ldots\}$, $\mathrm{s}(\bullet) = \{6, \ldots\}$.
$H_{\mathrm{I}}(6\mathtt{c}^\bullet)$	The same as $H_{\mathrm{I}}(6^\bullet)$	$\mathbb{Z}_{18}$-valued. $\mathrm{s}(\mathbf{6}) = \{12, \ldots\}$, $\mathrm{s}(\bullet) = \{12, \ldots\}$.
$H_{\mathrm{I}}(12^\bullet)$	The triple covering of $4D^0\backslash\Gamma$ ([17]) fully ramified at the regions of size 4. (Figure A.6)	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{12}) = \{12, 12\}$, $\mathrm{s}(\bullet) = \{0, 0, 8, 8, 16, 16\}$. The monovalent black vertices which map to the same vertex under the triple covering onto $4D^0\backslash\Gamma$ take different values under s.

Table A.5 gives partial information about several Bu_3-sets, by describing the specification without showing the Γ-set. The specification information implicitly shows the sizes of the regions and the numbers of the monovalent vertices in the Γ-set, however this information does not determine the Γ-set up to isomorphism. In fact, the Γ-set depends on the parameters ψ and a, which is why we omit it from the table. Note, however, that the Bu_3-sets given in this table are uniquely characterized in (4.3).

Table A.5: Basic Sporadic Classes of the Second Kind

The saturated Bu_3-set	The specification s
$H_{\mathrm{II}}(2;\psi)$ where $\psi \in \{t^3 + t + 1, t^3 + t^2 + 1\}$	$\mathbb{Z}_{14}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 13\}$, $\mathrm{s}(\mathbf{7}) = 7$, $\mathrm{s}(\circ) = 7$.

$H_{\mathrm{II}}(2;\psi)$ where $\psi \in \{t^4+t+1, t^4+t^3+1\}$	$\mathbb{Z}_{30}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 29\}$, $\mathrm{s}(\mathbf{15}) = 15$, $\mathrm{s}(\bullet) = \{10, 20\}$, $\mathrm{s}(\circ) = 15$.
$H_{\mathrm{II}}(3;\psi)$ where $\psi \in \{t^2+t+2, t^2+2t+2\}$	$\mathbb{Z}_{16}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 7\}$, $\mathrm{s}(\mathbf{8}) = 8$, $\mathrm{s}(\bullet) = 0$.
$H_{\mathrm{II}}(5;\psi)$ where $\psi \in \{t^2+2, t^2+3\}$	$\mathbb{Z}_{16}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1, 1, 7, 7, 7\}$, $\mathrm{s}(\mathbf{8}) = \{8, \ldots\}$.
$H_{\mathrm{II}}(5;\psi)$ where $\psi \in \{t^2+2t+4, t^2+3t+4\}$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1, 11, 11\}$, $\mathrm{s}(\mathbf{12}) = \{12, \ldots\}$, $\mathrm{s}(\bullet) = \{8, 8, 16, 16\}$.
$H_{\mathrm{II}}(11;a)$ where $a \in \{3, 4, 5, 9\}$.	$\mathbb{Z}_{10}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1\}$, $\mathrm{s}(\mathbf{2}) = 8$, $\mathrm{s}(\mathbf{10}) = \{0, 0\}$, $\mathrm{s}(\circ) = \{5, 5\}$.
$H_{\mathrm{II}}(13;a)$ where $a \in \{2, 6, 7, 11\}$.	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 11\}$, $\mathrm{s}(\mathbf{12}) = 12$, $\mathrm{s}(\bullet) = \{8, 16\}$.
$H_{\mathrm{II}}(17;a)$ where $a \in \{2, 8, 9, 15\}$.	$\mathbb{Z}_{16}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1, 7, 7\}$, $\mathrm{s}(\mathbf{8}) = \{8, \ldots\}$.
$H_{\mathrm{II}}(19;a)$ where $a \in \{2, 3, 10, 13, 14, 15\}$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 17\}$, $\mathrm{s}(\mathbf{9}) = \{9, 9\}$, $\mathrm{s}(\bullet) = \{12, 24\}$.
$H_{\mathrm{II}}(19;a)$ where $a \in \{4, 5, 6, 9, 16, 17\}$	$\mathbb{Z}_{18}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1\}$, $\mathrm{s}(\mathbf{2}) = 16$, $\mathrm{s}(\mathbf{18}) = \{0, 0\}$, $\mathrm{s}(\bullet) = \{6, 6, 12, 12\}$, $\mathrm{s}(\circ) = \{9, 9\}$.
$H_{\mathrm{II}}(29;a)$ where $a \in \{4, 5, 6, 9, 13, 22\}$	$\mathbb{Z}_{28}$-valued. $\mathrm{s}(\mathbf{1}) = \{1, 1, 13, 13\}$, $\mathrm{s}(\mathbf{7}) = \{7, \ldots\}$.

$H_{\mathrm{II}}(37;a)$ where $a \in \{3,4,21,25,28,30\}$	$\mathbb{Z}_{36}$-valued. $\mathrm{s}(\mathbf{1}) = \{1,1,17,17\}$, $\mathrm{s}(\mathbf{9}) = \{9,\ldots\}$, $\mathrm{s}(\bullet) = \{12,12,24,24\}$.
$H_{\mathrm{II}}(43;a)$ where $a \in \{2,8,22,27,32,39\}$	$\mathbb{Z}_{28}$-valued. $\mathrm{s}(\mathbf{1}) = \{1,1,1,13,13,13\}$, $\mathrm{s}(\mathbf{7}) = \{7,\ldots\}$.

We now list the remaining Bu_3-sets as products of those already given (note that the products are transitive).

$$H_{\mathrm{II}}(13;a)\times H_{\Pi}(\bullet;1),$$

$$H_{\mathrm{II}}(3;\psi)\times H_{\Pi}(\bullet;1),\qquad H_{\mathrm{II}}(3;\psi)\times H_{\Pi}(\bullet;\omega-1),\qquad H_{\mathrm{II}}(3;\psi)\times H_{\Pi}(\mathtt{D};1), \tag{A.1}$$

$$H_{\mathrm{II}}(3;\psi)\times H_{\Pi}(2;\omega-1),\qquad H_{\mathrm{II}}(3;\psi)\times H_{\Pi}(\mathtt{D};\omega-1), \tag{A.2}$$

$$H_{\mathrm{II}}(3;\psi)\times H_{\Sigma}(1;2),\qquad H_{\mathrm{II}}(3;\psi)\times H_{\Sigma}(1^{\times};2), \tag{A.3}$$

$$H_{\mathrm{I}}(6^{\bullet})\times H_{\Pi}(1';2),$$

$$H_{\mathrm{I}}(6\mathtt{a}^{\bullet})\times H_{\Pi}(1';2),\qquad H_{\mathrm{I}}(6\mathtt{b}^{\bullet})\times H_{\Pi}(1';2),\qquad H_{\mathrm{I}}(6\mathtt{c}^{\bullet})\times H_{\Pi}(1';2) \tag{A.4}$$

Table A.6:

$H_{\mathrm{I}}(5)\times H_{\Pi}(\circ;1)$	$H_{\mathrm{I}}(5\mathtt{a}')\times H_{\Pi}(\circ;1)$	$H_{\mathrm{I}}(5\mathtt{a})\times H_{\Pi}(\circ;1)$
$H_{\mathrm{I}}(5\mathtt{x})\times H_{\Pi}(\circ;1)$	$H_{\mathrm{I}}(5\mathtt{y})\times H_{\Pi}(\circ;1)$	
$H_{\mathrm{I}}(5\mathtt{w})\times H_{\Pi}(\circ;1)$	$H_{\mathrm{I}}(5\mathtt{ww}')\times H_{\Pi}(\circ;1)$	$H_{\mathrm{I}}(5\mathtt{ww})\times H_{\Pi}(\circ;1)$
$H_{\mathrm{I}}(25;a)\times H_{\Pi}(\circ;1)$	$H_{\mathrm{I}}(5^{\circ})\times H_{\Pi}(\circ;1)$	$H_{\mathrm{I}}(5^{\bullet})\times H_{\Pi}(\circ;1)$
$H_{\mathrm{I}}(7)\times H_{\Pi}(\circ;1)$		
$H_{\mathrm{I}}(7\mathtt{a})\times H_{\Pi}(\circ;1)$	$H_{\mathrm{I}}(7\mathtt{b})\times H_{\Pi}(\circ;1)$	$H_{\mathrm{I}}(7\mathtt{c})\times H_{\Pi}(\circ;1)$
$H_{\mathrm{II}}(2;\psi)\times H_{\Pi}(\circ;1)$	$H_{\mathrm{II}}(3;\psi)\times H_{\Pi}(\circ;1)$	
$H_{\mathrm{II}}(5;t^2+2)\times H_{\Pi}(\circ;1)$	$H_{\mathrm{II}}(5;t^2+3)\times H_{\Pi}(\circ;1)$	
$H_{\mathrm{II}}(11;a)\times H_{\Pi}(\circ;1)$	$H_{\mathrm{II}}(13;a)\times H_{\Pi}(\circ;1)$	$H_{\mathrm{II}}(17;a)\times H_{\Pi}(\circ;1)$
$H_{\mathrm{II}}(29;a)\times H_{\Pi}(\circ;1)$	$H_{\mathrm{II}}(43;a)\times H_{\Pi}(\circ;1)$	

Table A.7:

$H_{\mathrm{I}}(9';a)\times H_{\Pi}(1;\omega-1)$	
$H_{\mathrm{I}}(5)\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{I}}(5)\times H_{\Pi}(\circ;\omega-1)$
$H_{\mathrm{I}}(5\mathtt{a}')\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{I}}(5\mathtt{a}')\times H_{\Pi}(\circ;\omega-1)$
$H_{\mathrm{I}}(5\mathtt{a})\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{I}}(5\mathtt{a})\times H_{\Pi}(\circ;\omega-1)$
$H_{\mathrm{I}}(5\mathtt{x})\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{I}}(5\mathtt{x})\times H_{\Pi}(\circ;\omega-1)$
$H_{\mathrm{I}}(5\mathtt{y})\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{I}}(5\mathtt{y})\times H_{\Pi}(\circ;\omega-1)$
$H_{\mathrm{II}}(2;t^3+t+1)\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{II}}(2;t^3+t+1)\times H_{\Pi}(\circ;\omega-1)$
$H_{\mathrm{II}}(2;t^3+t^2+1)\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{II}}(2;t^3+t^2+1)\times H_{\Pi}(\circ;\omega-1)$
$H_{\mathrm{II}}(3;\psi)\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{II}}(3;\psi)\times H_{\Pi}(\circ;\omega-1)$
$H_{\mathrm{II}}(11;a)\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{II}}(11;a)\times H_{\Pi}(\circ;\omega-1)$
$H_{\mathrm{II}}(17;a)\times H_{\Pi}(1;\omega-1)$	$H_{\mathrm{II}}(17;a)\times H_{\Pi}(\circ;\omega-1)$

Table A.8 shows the sporadic Bu_3-sets which are products of the members of the infinite families. Specifically, $H_{\Pi\Sigma}(*,!;2)=H_{\Pi}(*;2)\times H_{\Sigma}(!;2)$.

Table A.8: Blends of the Infinite Families

The saturated Bu_3-set	The stabilizer of the Γ-set	The specification s
$H_{\Pi\Sigma}(1,1;2)$	$\Gamma_0(12)$	$\mathbb{Z}_{12}$-valued. s($\mathbf{1}$) = $\{1,7\}$, s($\mathbf{3}$) = $\{3,9\}$, s($\mathbf{4}$) = 4, s($\mathbf{12}$) = 0.
$H_{\Pi\Sigma}(1,1\mathtt{aa};2)$	$\Gamma_0(12)$	$\mathbb{Z}_{24}$-valued. s($\mathbf{1}$) = $\{1,19\}$, s($\mathbf{3}$) = $\{3,9\}$, s($\mathbf{4}$) = 4, s($\mathbf{12}$) = 12.
$H_{\Pi\Sigma}(1,1\mathtt{ab};2)$	$\Gamma_0(12)$	$\mathbb{Z}_{24}$-valued. s($\mathbf{1}$) = $\{13,7\}$, s($\mathbf{3}$) = $\{15,21\}$, s($\mathbf{4}$) = 4, s($\mathbf{12}$) = 12.
$H_{\Pi\Sigma}(1',1;2)$	$\Gamma_1(12)$	$\mathbb{Z}_4$-valued. s($\mathbf{1}$) = $\{1,1\}$, s($\mathbf{2}$) = 2, s($\mathbf{3}$) = $\{3,3\}$, s($\mathbf{4}$) = $\{0,0\}$, s($\mathbf{6}$) = 2, s($\mathbf{12}$) = $\{0,0\}$.

$H_{\Pi\Sigma}(1', \texttt{1aa}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_8$-valued. $\mathrm{s}(\mathbf{1}) = \{1,1\}$, $\mathrm{s}(\mathbf{2}) = 6$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{4}) = \{4,4\}$, $\mathrm{s}(\mathbf{6}) = 2$, $\mathrm{s}(\mathbf{12}) = \{4,4\}$.
$H_{\Pi\Sigma}(1', \texttt{1ab}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_8$-valued. $\mathrm{s}(\mathbf{1}) = \{5,5\}$, $\mathrm{s}(\mathbf{2}) = 6$, $\mathrm{s}(\mathbf{3}) = \{7,7\}$, $\mathrm{s}(\mathbf{4}) = \{4,4\}$, $\mathrm{s}(\mathbf{6}) = 2$, $\mathrm{s}(\mathbf{12}) = \{4,4\}$.
$H_{\Pi\Sigma}(1^{\times}, 1; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = \{1,1\}$, $\mathrm{s}(\mathbf{2}) = 2$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{4}) = \{4,4\}$, $\mathrm{s}(\mathbf{6}) = 6$, $\mathrm{s}(\mathbf{12}) = \{0,0\}$.
$H_{\Pi\Sigma}(1^{\times}, \texttt{1aa}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{1,1\}$, $\mathrm{s}(\mathbf{2}) = 14$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{4}) = \{4,4\}$, $\mathrm{s}(\mathbf{6}) = 18$, $\mathrm{s}(\mathbf{12}) = \{12,12\}$.
$H_{\Pi\Sigma}(1^{\times}, \texttt{1ab}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{13,13\}$, $\mathrm{s}(\mathbf{2}) = 14$, $\mathrm{s}(\mathbf{3}) = \{15,15\}$, $\mathrm{s}(\mathbf{4}) = \{4,4\}$, $\mathrm{s}(\mathbf{6}) = 18$, $\mathrm{s}(\mathbf{12}) = \{12,12\}$.
$H_{\Pi\Sigma}(\texttt{1ax}, 1; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = \{9,9\}$, $\mathrm{s}(\mathbf{2}) = 6$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{4}) = \{0,0\}$, $\mathrm{s}(\mathbf{6}) = 6$, $\mathrm{s}(\mathbf{12}) = \{0,0\}$.
$H_{\Pi\Sigma}(\texttt{1ax}, \texttt{1aa}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{9,9\}$, $\mathrm{s}(\mathbf{2}) = 6$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{4}) = \{12,12\}$, $\mathrm{s}(\mathbf{6}) = 18$, $\mathrm{s}(\mathbf{12}) = \{12,12\}$.
$H_{\Pi\Sigma}(\texttt{1ax}, \texttt{1ab}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{21,21\}$, $\mathrm{s}(\mathbf{2}) = 6$, $\mathrm{s}(\mathbf{3}) = \{15,15\}$, $\mathrm{s}(\mathbf{4}) = \{12,12\}$, $\mathrm{s}(\mathbf{6}) = 18$, $\mathrm{s}(\mathbf{12}) = \{12,12\}$.
$H_{\Pi\Sigma}(\texttt{1ay}, 1; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = \{5,5\}$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{3}) = \{3,3\}$, $\mathrm{s}(\mathbf{4}) = \{8,8\}$, $\mathrm{s}(\mathbf{6}) = 6$, $\mathrm{s}(\mathbf{12}) = \{0,0\}$.

$H_{\Pi\Sigma}(1\mathtt{ay}, 1\mathtt{aa}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{17, 17\}$, $\mathrm{s}(\mathbf{2}) = 22$, $\mathrm{s}(\mathbf{3}) = \{3, 3\}$, $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\mathbf{6}) = 18$, $\mathrm{s}(\mathbf{12}) = \{12, 12\}$.
$H_{\Pi\Sigma}(1\mathtt{ay}, 1\mathtt{ab}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{5, 5\}$, $\mathrm{s}(\mathbf{2}) = 22$, $\mathrm{s}(\mathbf{3}) = \{15, 15\}$, $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\mathbf{6}) = 18$, $\mathrm{s}(\mathbf{12}) = \{12, 12\}$.
$H_{\Pi\Sigma}(1\mathtt{b}, 1; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{12}$-valued. $\mathrm{s}(\mathbf{1}) = \{11, 11\}$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{3}) = \{9, 9\}$, $\mathrm{s}(\mathbf{4}) = \{8, 8\}$, $\mathrm{s}(\mathbf{6}) = 6$, $\mathrm{s}(\mathbf{12}) = \{0, 0\}$.
$H_{\Pi\Sigma}(1\mathtt{b}, 1\mathtt{aa}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{11, 11\}$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{3}) = \{9, 9\}$, $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\mathbf{6}) = 6$, $\mathrm{s}(\mathbf{12}) = \{12, 12\}$.
$H_{\Pi\Sigma}(1\mathtt{b}, 1\mathtt{ab}; 2)$	$\Gamma_1(12)$	$\mathbb{Z}_{24}$-valued. $\mathrm{s}(\mathbf{1}) = \{23, 23\}$, $\mathrm{s}(\mathbf{2}) = 10$, $\mathrm{s}(\mathbf{3}) = \{21, 21\}$, $\mathrm{s}(\mathbf{4}) = \{20, 20\}$, $\mathrm{s}(\mathbf{6}) = 6$, $\mathrm{s}(\mathbf{12}) = \{12, 12\}$.

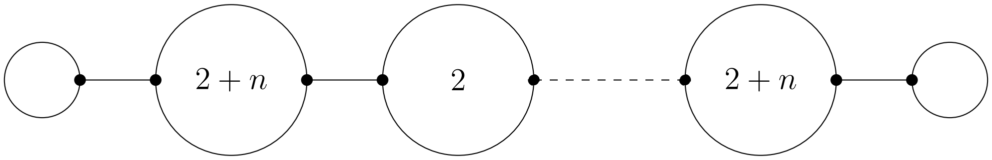


Figure A.1: The $\mathbb{Z}_{2n}$-valued specification of $H_{\Sigma}(1\mathtt{w}; n)$ assigns alternating values to the regions of size 2 in $\Sigma_1(n)$.

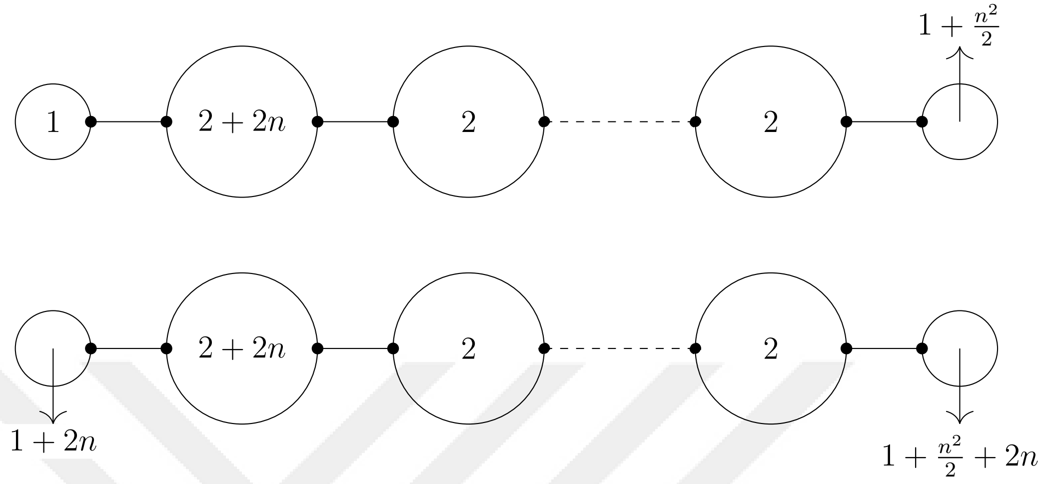


Figure A.2: The $\mathbb{Z}_{4n}$-valued specifications of $H_\Sigma(1\mathtt{aa};n)$ and $H_\Sigma(1\mathtt{ab};n)$ assign alternating values to the regions of size 2 in $\Sigma_1(n)$. If the values assigned to the regions of size 1 are doubled, the alternating pattern extends to them as well.

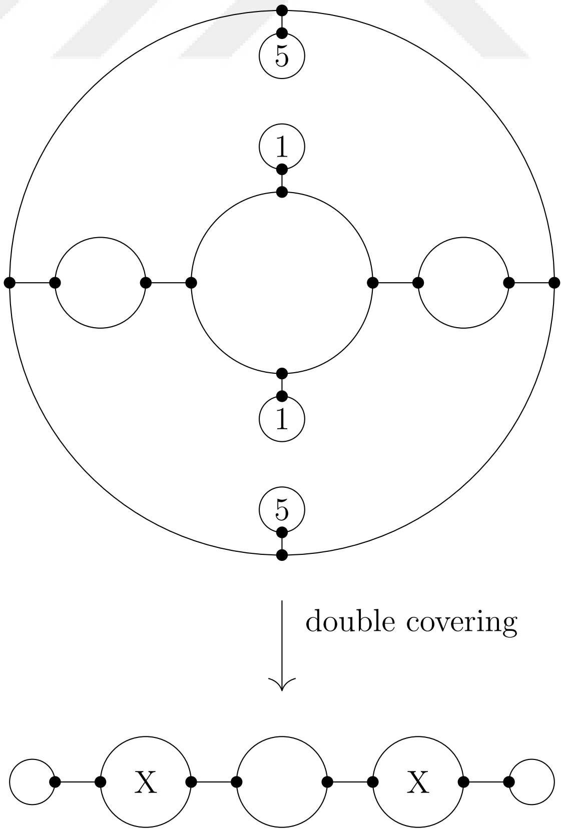
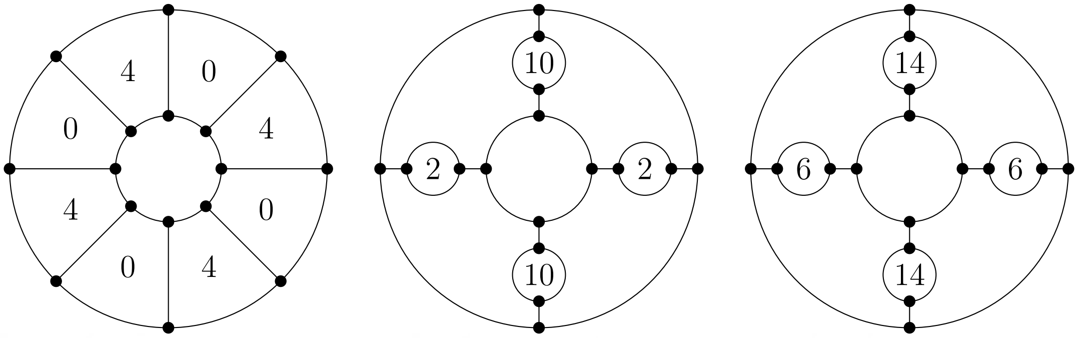
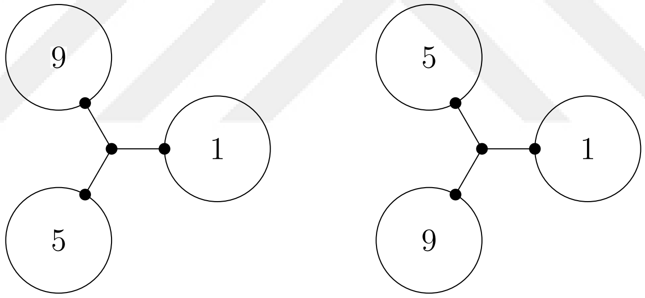


Figure A.3: The $\mathbb{Z}_8$-valued specification of $H_\Sigma(1\mathtt{d}';8)$ assigns the same value to two regions if they map to the same region in $\Sigma_1(8)$ under the double covering ramified at the regions marked by X.



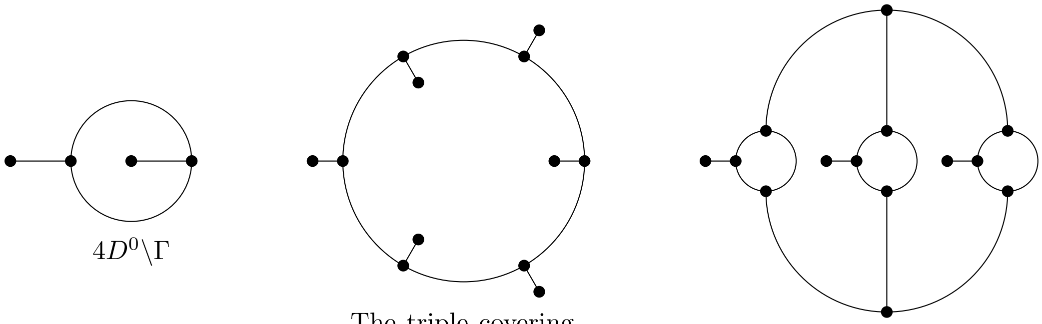
$H_\Sigma(2\texttt{dw};2)$ $H_\Sigma(2\texttt{da};4,1)$ $H_\Sigma(2\texttt{da};4,-1)$

Figure A.4: The alternating patterns in the specifications of some members of the Σ-family.



$H_{\mathrm{I}}(9';2)$ $H_{\mathrm{I}}(9';5)$

Figure A.5: The Bu_3-sets $H_{\mathrm{I}}(9';a)$.



$4D^0\backslash\Gamma$

The triple covering of $4D^0\backslash\Gamma$ ramified at the regions.

$9J^0\backslash\Gamma$

Figure A.6: A few particular Γ-sets.

A.2 The Modules

In this section, we give the modules $\mathcal{A}(E)$ for some Bu_3-sets E in the Tables A.1 and A.2. We first give the modules $\mathcal{A}(H_\Pi(-;-))$. For any nonzero ideal $\iota \subset \mathbb{Z}[\omega]$, let $I(\iota) \subset \Lambda$ denote the preimage of ι under the epimorphism $\Lambda \twoheadrightarrow \mathbb{Z}[\omega]$ defined by $t \mapsto \omega$. Note that $(t^2+t+1) \subset I$.

$$\begin{aligned} \mathcal{A}(H_\Pi(1;\iota)) &= \Lambda/I(\iota) && \text{if } \iota \not\subset \langle \omega - 1\rangle, \\ & \quad \Lambda/(t-1)I(\iota/(\omega-1)) && \text{if } \iota \subset \langle \omega - 1\rangle. \end{aligned} \tag{A.5}$$

$$\begin{aligned} \mathcal{A}(H_\Pi(1^\times;\iota)) &= \Lambda/(t^2-t+1)I(\iota/2) && \text{if } \iota \not\subset \langle \omega - 1\rangle, \\ & \quad \Lambda/(t^2-t+1)(t-1)I(\iota/2(\omega-1)) && \text{if } \iota \subset \langle \omega - 1\rangle. \end{aligned} \tag{A.6}$$

$$\mathcal{A}(H_\Pi(\bullet;\iota)) = \Lambda/I((\omega-1)\iota). \tag{A.7}$$

$$\mathcal{A}(H_\Pi(\bullet^\times;\iota)) = \Lambda/(t^2-t+1)I((\omega-1)\iota/2). \tag{A.8}$$

$$\mathcal{A}(H_\Pi(2;\iota)) = \Lambda/(t-1) \oplus \Lambda/(t-1)I(\iota/(\omega-1)). \tag{A.9}$$

$$\begin{aligned} \mathcal{A}(H_\Pi(2^\times;\iota)) &= \Lambda/(t-1) \ \oplus \\ & \qquad \Lambda/(t^2-t+1)(t-1)I(\iota/2(\omega-1)). \end{aligned} \tag{A.10}$$

$$\begin{aligned} \mathcal{A}(H_\Pi(\circ;\iota)) &= \Lambda/I(2\iota) && \text{if } \iota \not\subset \langle \omega - 1\rangle, \\ & \quad \Lambda/(t-1)I(2\iota/(\omega-1)) && \text{if } \iota \subset \langle \omega - 1\rangle. \end{aligned} \tag{A.11}$$

$$\begin{array}{rcll} \mathcal{A}(H_\Pi(3;\iota)) & = & \mathbb{F}_2[t]/(t^2-t+1)\oplus\Lambda/I(\iota) & \text{if } \iota\not\subset\langle\omega-1\rangle, \\ & & \mathbb{F}_2[t]/(t^2-t+1)\oplus\Lambda/(t-1)I(\iota/(\omega-1)) & \text{if } \iota\subset\langle\omega-1\rangle. \end{array} \tag{A.12}$$

$$\begin{array}{rcll} \mathcal{A}(H_\Pi(3\texttt{a};\iota)) & = & \Lambda/(t^2-t+1)\ \oplus & \\ & & \quad\Lambda/(t^2-t+1)I(\iota/2) & \text{if } \iota\not\subset\langle\omega-1\rangle, \\ & & & \\ & & \Lambda/(t^2-t+1)\ \oplus & \\ & & \quad\Lambda/(t^2-t+1)(t-1)I(\iota/2(\omega-1)) & \text{if } \iota\subset\langle\omega-1\rangle. \end{array} \tag{A.13}$$

$$\begin{array}{rcll} \mathcal{A}(H_\Pi(3\texttt{b};\iota)) & = & \mathbb{F}_2[t]/(t^2-t+1)^2\oplus\Lambda/I(\iota) & \text{if } \iota\not\subset\langle\omega-1\rangle, \\ & & \mathbb{F}_2[t]/(t^2-t+1)^2\oplus\Lambda/(t-1)I(\iota/(\omega-1)) & \text{if } \iota\subset\langle\omega-1\rangle. \end{array} \tag{A.14}$$

$$\begin{array}{rcll} \mathcal{A}(H_\Pi(3\texttt{ba};\iota)) & = & \mathbb{F}_2[t]/(t^2-t+1)^3\oplus\Lambda/I(\iota) & \text{if } \iota\not\subset\langle\omega-1\rangle, \\ & & \mathbb{F}_2[t]/(t^2-t+1)^3\oplus\Lambda/(t-1)I(\iota/(\omega-1)) & \text{if } \iota\subset\langle\omega-1\rangle. \end{array} \tag{A.15}$$

$$\begin{array}{rcll} \mathcal{A}(H_\Pi(3\texttt{bb};\iota)) & = & \mathbb{Z}_4[t]/\langle 2(t^2-t+1),(t^4-t^2+1)\rangle\oplus\Lambda/I(\iota) & \text{if } \iota\not\subset\langle\omega-1\rangle, \\ & & & \\ & & \mathbb{Z}_4[t]/\langle 2(t^2-t+1),(t^4-t^2+1)\rangle\ \oplus & \\ & & \quad\Lambda/(t-1)I(\iota/(\omega-1)) & \text{if } \iota\subset\langle\omega-1\rangle. \end{array} \tag{A.16}$$

We describe the modules $\mathcal{A}(H_\Pi(3/2;\iota))$ and $\mathcal{A}(H_\Pi(3/2^\times;\iota))$ in terms of a polynomial $\phi\in\Lambda$ such that $\phi\notin I(\iota)$ but $2\phi\in I(\iota)$ (such a $\phi\in\Lambda$ can be found since $\iota\subset\langle 4\rangle\subset\langle 2\rangle$). Then,

$$
\begin{array}{lcll}
\mathcal{A}(H_{\Pi}(3/2;\iota)) & = & \big(\mathbb{F}_2[t]/(t^2-t+1)^2\big)\cdot e_1 \oplus \big(\Lambda/I(\iota)\big)\cdot e_2 & \\
 & & \quad \text{modulo the relation} & \\
 & & \quad (t^2-t+1)\cdot e_1 = \phi\cdot e_2 & \text{if } \iota\not\subset\langle\omega-1\rangle, \\
 & & & \\
 & & \big(\mathbb{F}_2[t]/(t^2-t+1)^2\big)\cdot e_1 \ \oplus & \\
 & & \quad \big(\Lambda/(t-1)I(\iota/(\omega-1))\big)\cdot e_2 & \\
 & & \quad \text{modulo the relation} & \\
 & & \quad (t^2-t+1)\cdot e_1 = (t-1)\phi\cdot e_2 & \text{if } \iota\subset\langle\omega-1\rangle.
\end{array}
\tag{A.17}
$$

$$
\begin{array}{lcll}
\mathcal{A}(H_{\Pi}(3/2^{\times};\iota)) & = & \big(\mathbb{F}_2[t]/(t^2-t+1)^3\big)\cdot e_1 \oplus \big(\Lambda/I(\iota)\big)\cdot e_2 & \\
 & & \quad \text{modulo the relation} & \\
 & & \quad (t^2-t+1)^2\cdot e_1 = \phi\cdot e_2 & \text{if } \iota\not\subset\langle\omega-1\rangle, \\
 & & & \\
 & & \big(\mathbb{F}_2[t]/(t^2-t+1)^3\big)\cdot e_1 \ \oplus & \\
 & & \quad \big(\Lambda/(t-1)I(\iota/(\omega-1))\big)\cdot e_2 & \\
 & & \quad \text{modulo the relation} & \\
 & & \quad (t^2-t+1)^2\cdot e_1 = (t-1)\phi\cdot e_2 & \text{if } \iota\subset\langle\omega-1\rangle.
\end{array}
\tag{A.18}
$$

Note that even though $\phi\in\Lambda$ is not unique, the formulae given in terms of ϕ are valid for any choice (the resulting modules are isomorphic). Finally,

$$
\begin{array}{lcll}
\mathcal{A}(H_{\Pi}(\mathtt{D};\iota)) & = & \Lambda/(t^2+t+1)I(\iota) & \text{if } \iota\not\subset\langle\omega-1\rangle, \iota\not\subset\langle 2\rangle, \\
 & & \Lambda/(t^2+t+1)(t^2-t+1)I(\iota/2) & \text{if } \iota\not\subset\langle\omega-1\rangle, \iota\subset\langle 2\rangle, \\
 & & \Lambda/(t-1)\oplus\Lambda/(t^3-1)I(\iota/(\omega-1)) & \text{if } \iota\subset\langle\omega-1\rangle, \iota\not\subset\langle 2\rangle, \\
 & & \Lambda/(t-1)\ \oplus & \\
 & & \quad \Lambda/(t^3-1)(t^2-t+1)I(\iota/2(\omega-1)) & \text{if } \iota\subset\langle\omega-1\rangle, \iota\subset\langle 2\rangle.
\end{array}
\tag{A.19}
$$

We now give the modules $\mathcal{A}(H_\Sigma(-;-))$.

$$\mathcal{A}(H_\Sigma(1;n)) = \Lambda/(t-1)\langle n,\, t-1\rangle. \tag{A.20}$$

$$\mathcal{A}(H_\Sigma(1^\times;n)) = \Lambda/(t-1)\langle n,\, t-1\rangle \oplus \mathbb{F}_2[t]/(t^2-t+1). \tag{A.21}$$

$$\mathcal{A}(H_\Sigma(1\mathtt{w};n)) = \Lambda/(t-1)\langle n,\, t-1-n/2\rangle. \tag{A.22}$$

$$\mathcal{A}(H_\Sigma(1\mathtt{w}^\times;n)) = \Lambda/(t-1)\langle n,\, t-1-n/2\rangle \oplus \mathbb{F}_2[t]/(t^2-t+1). \tag{A.23}$$

$$\mathcal{A}(H_\Sigma(1\mathtt{aa};n)) = \Lambda/(t-1)(t^2+1)\langle n/2,\, t-1\rangle. \tag{A.24}$$

$$\mathcal{A}(H_\Sigma(1\mathtt{ab};n)) = \Lambda/(t-1)\langle n,\, 2(t-1),\, (t-1)^2\rangle. \tag{A.25}$$

$$\mathcal{A}(H_\Sigma(1\mathtt{aa}^\times;n)) = \Lambda/(t-1)(t^2+1)\langle n/2,\, t-1\rangle \oplus \mathbb{F}_2[t]/(t^2-t+1). \tag{A.26}$$

$$\mathcal{A}(H_\Sigma(1\mathtt{ab}^\times;n)) = \Lambda/(t-1)\langle n,\, 2(t-1),\, (t-1)^2\rangle \oplus \mathbb{F}_2[t]/(t^2-t+1). \tag{A.27}$$

$$\mathcal{A}(H_\Sigma(2;n,a)) = \Lambda/(t-1) \oplus \Lambda/(t-1)\langle n,\, t-1\rangle. \tag{A.28}$$

$$\mathcal{A}(H_\Sigma(2^\times;n,a)) = \Lambda/(t-1) \oplus \Lambda/(t^3-1)\langle n,\, t-1\rangle. \tag{A.29}$$

$$\mathcal{A}(H_\Sigma(2\mathtt{w};n,a)) = \Lambda/(t-1) \oplus \Lambda/(t-1)\langle n,\, t-1-n/2\rangle. \tag{A.30}$$

$$\mathcal{A}(H_\Sigma(2\mathtt{w}^\times;n,a)) = \Lambda/(t-1) \oplus \Lambda/(t^3-1)\langle n,\, t-1-n/2\rangle. \tag{A.31}$$

$$\mathcal{A}(H_\Sigma(2\mathtt{a};n)) = \Lambda/(t-1) \oplus \Lambda/(t-1)(t^2+1)\langle n/2,\, t-1\rangle. \tag{A.32}$$

$$\mathcal{A}(H_\Sigma(2\mathtt{aa}^\times;n)) = \Lambda/(t-1) \oplus \Lambda/(t^3-1)(t^2+1)\langle n/2,\, t-1\rangle. \tag{A.33}$$

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