

Product Ranking, Pricing, and Recommendation for E-commerce Retailers and Platforms

by

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**Product Ranking, Pricing, and Recommendation for E-commerce
Retailers and Platforms**

Koç University

Graduate School of Business

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ABSTRACT

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Product ranking and pricing in an online store or marketplace plays a crucial role in customer engagement and subsequently in sales and total revenue. An online retailer or marketplace operator must decide on the set of the products and their positions and prices at various points, such as campaigns, first page, landing pages, side-bar recommendations, or search query lists. In this dissertation, we investigate the optimal ranking and pricing in centralized and decentralized environments at online e-commerce platforms.

In the first part, we propose a new personalized content-based method specially designed for online retailers. The focus of the study is on customers' activities in an online store, such as clicks, and purchases, more than the feedback such as ratings or comments. This property is useful for online retailers where there is limited rating feedback, but there are several search sessions including clicks and purchases. We test the method using data provided by an apparel retailer. Our method outperforms benchmark methods (Collaborative filtering, Popular products, and MNL choice model) and it has a strictly better performance in recommending new products. Also, our method outperforms benchmark methods in recommending products to customers who are generally interested in less popular (fringe) products. The main practical contribution of this study for online retailers is a novel personalized recommending system that is build-up based on panel data (instead of rating data) that causes better use of limited detail provided in the dataset and improved the quality of recommendation.

In the second part, we consider a decentralized pricing and ranking problem in an online marketplace where sellers decide on their prices simultaneously and ranking

takes place based on a pre-defined rule by the marketplace operator (MO). Empirical studies show that the ranking of sellers in an online marketplace is highly correlated to prices. We model the price competition as a full information game where all players (sellers) know the price-based ranking rule defined by the marketplace operator and the demand function of each seller. We obtain the Nash equilibrium of each game under binary and probabilistic ranking policies and show how ranking policy and commission rate affect the pricing strategies and equilibrium. Then, we investigate the profit of MO and third-party sellers dependent on the ranking rules and market structure. From a practical and implementation point of view, this study sheds light on the optimal pricing and ranking strategies in an online marketplace with three main contributions. First, we find the optimal ranking rule for MO is dependent on the sale margin and cost of the products. Further, we show that the MO may deviate from the optimal ranking to either increase the profit of each supply-side party by implementing a less competitive ranking rule and higher commission rate or enhance the consumers' experience quality such as demand satisfaction rate and demand surplus by implementing a competitive ranking rule and lower commission rate. Finally, in cases that MO participates as a seller, s/he may rank himself higher than other sellers. This strategy increases the profit of MO in any case, however, surprisingly it increases both total supply-side profit and customers demand surplus if the demand functions are asymmetric.

In the third part, we study product ranking and pricing at e-commerce platforms, where customers are generally window shoppers. Window shoppers are a type of customer that browse online stores as a form of leisure or external search behavior without a clear intent to purchase a specific product. We develop an optimization model with deterministic demand and vertically differentiated positions (ranking) where products that are placed in earlier positions are more likely to be considered and, therefore, purchased. We show that revenue-ordered ranking is optimal when there is infinite inventory for products. However, when there is a finite inventory case, despite the deterministic and invertible structure of demand, we prove that the optimal price would result in product stock-out during the campaign in some cases. Also, we show that solving the problem with a set of predefined prices (and not joint pricing ranking) would result in insignificant profit loss if the diversity of the product's demand parameters is limited. The precise optimization model in the infinite inventory case is intractable due to the existence of integer variables and

the complexity of the constraints. Thus, we propose two heuristic methods to solve the problem and provide an upper bound by relaxing the integer variables. We show that the performance of the heuristic methods is dependent on the customer's search behavior. Selecting the appropriate heuristic method, we provide a near-optimal solution in all cases. Finally, numerical analyses show that trivial solutions such as sorting by price or sorting by inventory would result in significant profit loss.



ÖZETÇE

Çevrimiçi Perakendeciler ve E-Ticaret Platformları için Ürün Sıralama ve Fiyatlandırması

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Bir çevrimiçi mağazada veya pazaryerinde ürün sıralama ve fiyatlandırması, satışlar ve toplam gelir üzerinde çok önemli bir rol oynar. Bir çevrimiçi perakendeci veya pazaryeri operatörü, kampanyalar, ilk sayfa, açılış sayfaları, kenar çubuğu önerisi veya arama sorgusu listeleri gibi çeşitli noktalarda ürün grubuna ve konumlarına ve fiyatlarına karar vermelidir. Bu tez çalışmasında, çevrimiçi perakendeciler e-ticaret platformlarında merkezi ve merkezi olmayan ortamlarda en uygun sıralama ve fiyatlandırmayı araştırıyoruz.

İlk bölümde, çevrimiçi perakendeciler için özel olarak tasarlanmış yeni, kişiselleştirilmiş içerik tabanlı bir yöntem öneriyoruz. Çalışmanın odak noktası, puanlama veya yorumlar gibi geri bildirimlerden çok, müşterilerin bir çevrimiçi mağazadaki tıklamalar ve satın almalar gibi faaliyetleridir. Bu özellik, puanlama ve yorum olarak geri bildirimlerden daha çok tıklamalar ve satın almalarından oluşan arama oturumu bulunan çevrimiçi perakendeciler için kullanışlıdır. Bir giyim perakendecisi tarafından sağlanan verileri kullanarak yöntemi test ediyoruz. Yöntemimiz, kıyaslama yöntemlerinden (İşbirlikçi filtreleme, Popüler ürünler ve MNL seçim modeli) daha iyi performans gösterir ve yeni ürünlerin tavsiye edilmesinde kesinlikle daha iyi bir performansa sahiptir. Ayrıca, yöntemimiz, genellikle daha az popüler ürünlerle ilgilenen müşterilere ürün öneren kıyaslama yöntemlerinden daha iyi performans gösterir. Bu çalışmanın çevrimiçi perakendeciler için temel pratik katkısı, veri setinde sağlanan sınırlı ayrıntıların daha iyi kullanılmasına neden olan ve öneri kalitesini iyileştiren panel verilere (derecelendirme verileri yerine) dayalı olarak oluşturulan yeni bir kişiselleştirilmiş tavsiye sistemidir.

İkinci bölümde, satıcıların aynı anda fiyatlarına karar verdiği ve sıralamanın piyasa operatörü (PO) tarafından önceden tanımlanmış bir kurala göre gerçekleştiği

bir çevrimiçi pazaryerinde merkezi olmayan bir fiyatlandırma ve sıralama problemini ele alıyoruz. Ampirik araştırmalar, çevrimiçi bir pazaryerindeki satıcıların sıralamasının fiyatlar ile yüksek oranda ilişkili olduğunu göstermektedir. Fiyat rekabetini, tüm oyuncuların (satıcıların) piyasa operatörü tarafından tanımlanan fiyata dayalı sıralama kuralını ve her satıcının talep fonksiyonunu bildiği tam bir bilgi oyunu olarak modelliyoruz. İkili ve olasılıklı sıralama politikaları altında her oyunun Nash dengesini elde ediyoruz ve sıralama politikasının ve komisyon oranının fiyatlandırma stratejilerini ve dengeyi nasıl etkilediğini gösteriyoruz. Ardından, sıralama kurallarına ve piyasa yapısına bağlı olarak PO ve üçüncü taraf satıcıların kârını araştırırız. Pratik ve uygulama açısından, bu çalışma, üç ana katkı ile çevrimiçi bir pazardaki en uygun fiyatlandırma ve sıralama stratejilerine ışık tutuyor. İlk olarak, PO için en uygun sıralama kuralının, ürünlerin satış marjına ve maliyetine bağlı olduğunu buluyoruz. Ayrıca, PO'nun daha az rekabetçi bir sıralama kuralı ve daha yüksek komisyon oranı uygulayarak her bir arz tarafının karını artırmak veya talep tatmin oranı ve talep fazlası gibi tüketicilerin deneyim kalitesini artırmak için optimal sıralamadan sapabileceğini gösteriyoruz. rekabetçi bir sıralama kuralı ve daha düşük komisyon oranı uygulayarak. Son olarak, PO'nun satıcı olarak katıldığı durumlarda, kendisini diğer satıcılardan daha üst sıralarda yer alabilir. Bu strateji her durumda PO'nun karını arttırır, ancak şaşırtıcı bir şekilde hem toplam arz yönlü karı hem de talep fonksiyonları asimetric ise müşterilerin talep fazlasını arttırır.

Üçüncü bölümde, müşterilerin genellikle vitrin müşterisi olduğu e-ticaret platformlarında ürün sıralamasını ve fiyatlandırmasını inceliyoruz. Vitrin müşterileri, belirli bir ürünü satın almak için net bir niyet olmaksızın, boş zaman veya harici arama davranışı olarak çevrimiçi mağazalara göz atan bir müşteri türüdür. Deterministik talep ve dikey olarak farklılaştırılmış konumlara (sıralama) sahip bir optimizasyon modeli geliştiriyoruz, burada daha önceki konumlara yerleştirilen ürünlerin dikkate alınması ve dolayısıyla satın alınması daha olasıdır. Ürünler için sonsuz envanter olduğunda gelir sıralı sıralamanın optimal olduğunu gösteriyoruz. Ancak, sınırlı bir stok durumu söz konusu olduğunda, talebin deterministik ve tersinir yapısına rağmen, bazı durumlarda optimal fiyatın seans boyunca ürün stoğu ile sonuçlanacağını kanıtıyoruz. Ayrıca, ürünün talep parametrelerinin çeşitliliği sınırlıysa, sorunu önceden tanımlanmış bir dizi fiyatla (ve ortak fiyatlandırma sıralamasıyla değil) çözmenin önemsiz kar kaybıyla sonuçlanacağını gösteriyoruz. Sonsuz envanter durumunda kesin optimizasyon modeli, tamsayı değişkenlerinin varlığı ve

kısıtlamaların karmaşıklığı nedeniyle zorludur. Bu nedenle, problemi çözmek ve tamsayı değişkenlerini gevşeterek bir üst sınır sağlamak için iki sezgisel yöntem öneriyoruz. Sezgisel yöntemlerin performansının müşterinin arama davranışına bağlı olduğunu gösteriyoruz. Uygun buluşsal yöntemi seçerek, her durumda optimuma yakın bir çözüm sunuyoruz. Son olarak, sayısal analizler, fiyata göre sıralama veya envantere göre sıralama gibi önemsiz çözümlerin sonuç vereceğini göstermektedir.



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Chapter 1

INTRODUCTION

Product ranking and positioning in an online store or marketplace plays a crucial role in customer engagement and subsequently in sales and total revenue. A product with a more prominent position on the page has a higher chance of being visited by a customer. An online retailer or marketplace operator must decide on the set of the products and their positions at various points, such as first page, landing pages, side-bar recommendations, or search query lists. Surveys have shown that 38% of people stop engaging with a website if the content or layout is unattractive (Adobe 2015). This result emphasizes the importance of the assortment of products on the first few pages and their ranking.

The ranking decision has a higher impact on the market when products are offered by third-party sellers as it can affect the customer's preference for the seller. For example, in an online marketplace, multiple competing sellers may offer products that are identically the same (e.g. electronic devices) or near substitutes. In such a case, the ranking may affect a customer's preferences of the sellers if he/she only considers the seller in the most prominent position. For instance, Amazon uses the "buy box" option where one of the sellers has a very dominant position. Customers can directly add a product to the basket from the buy box and it is fulfilled from the seller at the dominant position. 82% of Amazon sales go through the buy box, and the percentage is even higher for mobile purchases (Chen et al. 2016). This motivates third-party sellers to compete for higher ranks.

1.1 Recommendation Systems in E-retailers

Recommendation systems (RS) are practical solutions for e-commerce businesses to manage the assortment of the products and the order in which they are presented on the web pages. RS are commonly used where there are several items to present, but there is a limited displaying capacity on each page. RS consist of software tools and techniques, suggesting items such as products, music, and videos to users. RS help both the user and the retailer by recommending related items (that are estimated to be interesting for the user or beneficial for the retailer) and filtering unrelated ones. Industry reports frequently claim that recommendation systems account for significant traffic, contributing anywhere from 10% to 30% of total revenue. On Amazon.com, nearly 30% of traffic is via click-through on recommendations (MacKenzie and Noble 2013). In another case, Best Buy is reported to boost its online sales, in 2016's second quarter by 23.7%, thanks in part to their recommendation system (Gustafson 2016). RS mainly use the history of all users to recognize the interests of an individual user and recommend what he/she might prefer.

There are two well-known types of recommendation systems: Collaborative Filtering and Content-based. Collaborative Filtering (CF) uses the past interaction of items and users to detect either users that have similar preferences (called neighbors) or similar items based on the common item preferences of customers in history. Generally, user-based CF recommends products preferred by the neighbors to the main user and item-based CF recommends similar items. On the other hand, content-based systems (CB) use pre-tagged attributes of items to find the similarities among items to recommend similar items (Ricci et al. 2015, Balabanović and Shoham 1997). Even though collaborative filtering and content-based RS have a positive impact on website traffic and sales, they have some limitations. Basic CB doesn't provide personalized content. On the other hand, CF faces the so-called "cold start" problem, which means that recommendation is challenging when an item or user is new (Mashtoff 2015). Cold start is a challenge for CF because CF provides recommendations

based on the history of preferences and data is limited for new items and users.

Flash sales apparel retailers such as Rue La La and Privalia face cold start problem more often because many products are introduced every week, and they are available in limited times and quantities. The short period of product offers leads to limited history for each product. In such an environment, we are interested in the following research questions. How can we learn customers' preferences and their choices? What is the optimal set of products to offer? What are the challenges and limitations of existing methods? Would personalization help when there is limited information as mentioned?

1.2 Online Marketplaces and Decentralized Ranking-Pricing Decisions

Product ranking decision is more complicated in an online marketplace, where sellers are ranked as well as the products. An online marketplace is an e-commerce website where product or service information is provided by multiple third parties, whereas transactions are processed by the marketplace operator such as Amazon, Hepsiburada, and Gittigidiyor. Generally, the ranking of the third-party sellers and their products is decided by the marketplace operator (MO) based on some properties of the products and sellers, such as price and fulfillment strategy. This means that ranking takes place automatically based on some parameters of the sellers and their products. For example, MO may choose a price-sorted ranking rule (where products with lower prices would be placed at higher positions). Thus in marketplace platforms, ranking and pricing decisions are decentralized. In such a setting, sellers set their prices taking the ranking rule and its impact on sales into account.

The decentralized pricing and ranking problem mentioned above can be modeled as a two-phase price competition game. In the first phase, MO decides on the ranking rule. In the second phase, sellers decide on their prices simultaneously and ranking takes place based on the ranking rule defined in phase one. In such a game, we are interested in the following research questions. What do the dynamics of this game imply for sellers' pricing? Does there exist a Nash equilibrium? How does the total profit of the MO get affected by the ranking rule and other properties of the market?

1.3 Centralized Ranking and Pricing in an Online Retailer in Existence of Window Shoppers

A significant segment of the customers at online stores spend a considerable amount of time without any clear intention to buy, referred to as “window shoppers.” New research from Bazaarvoice finds that not only 61% of consumers are globally more likely to browse for new products online compared to in-store, but that they also find it easier (64%) and more enjoyable (54%) (Release 2022). Furthermore, window-shopping is not only a form of leisure but also results in impulsive sales and subsequently, higher profit for retailers. Reports state that 67% of users admit to window shopping for fun online, 77% of these online window shoppers make impulse purchases, and 70% of them will come back and make a purchase from their device within the first hour of seeing the product (Borland et al. 2018, Ouellette 2022). Similarly, Forbes declares that 90% of mobile searches lead to action and over 50% of them lead to sales (Conner 2014). These statistics show that there is a high opportunity for impulsive sales provided by window shoppers for retailers. Behavioral studies state that window shoppers behave distinctly from other customers in three main aspects: i) Window shoppers may engage thorough impulsive purchases after a shorter search time in comparison with other customers. ii) A window shopper customer may purchase more than one product at any session. iii) Finally, the customer may browse products from completely different categories.

While window shopping has been studied extensively from behavioral and psychological perspectives, there are very limited analytical studies about the topic. One common practice by online retailers to entice users to visit the e-store is planning short time campaigns. In a campaign, the retailer decides on a single-shot ranking and prices of each product for a specific amount of time by answering the following research questions. What is the optimal ranking and pricing strategy? Is ranking and pricing problem independent of each other or decomposable? How do inventory limits affect the ranking and pricing?

The remainder of the article is structured as follows. In Chapter 2, we focus

on the recommendation systems for flash sales apparel retailers. We propose a personalized content-based recommendation system and measure the performance of the method by its accuracy in estimating a customer's choices compared with well-known methods as benchmarks. In Chapter 3, we mainly discuss decentralized pricing and ranking decisions at a marketplace, for substitute products. We search for the Nash equilibrium of the game and the corresponding profit of sellers and marketplace operator. In Chapter 4, we study product ranking and pricing at e-commerce platforms, where customers are generally window shoppers and there is finite inventory for each product. Chapter 5 concludes with a discussion and suggestions for future work.

Chapter 2

PERSONALIZED CONTENT-BASED RANKING IN AN ONLINE APPAREL RETAILER

Our study in this chapter is motivated by flash sales apparel retailers such as Rue La La and Privalia. A flash sales retailer every week introduces many products which are available in limited times and quantities. We are interested in a personalized recommendation system that can boost sales and raise the conversion rate where the short period of product offers leads to limited history for each product and the problem so-called “cold start”. We introduce a personalized content-based RS, using products’ attributes to provide a preference profile for each customer. We shortly denote this method as PCB in the rest of the paper. This method uses the customers’ activity history (search and purchase behavior) to recognize the customers’ preference for the attributes. We represent each product with an attribute vector. For each product, the corresponding attribute vector consists of all explicitly known attributes of a product, e.g. color, price level, and brand. For each customer, a score of interest is assigned to each attribute vector, representing his/her interest in that combination of attributes. Scores are a weighted sum of the “subscores” where subscores keep track of customer’s past behavior about each attribute vector. Each time a product is clicked or purchased by the customer, the subscores of relating attributes are increased. In other words, the number of times that a product is clicked or purchased is a positive feedback of customer’s preferences in that products’ attributes. PCB offers products with higher scores to the customer.

We measure the performance of the method by its accuracy in estimating a customer’s next action. Given “available products set” and customers’ history up to a specified time, the method tries to anticipate which products would be clicked or purchased by the customer after that time. We test the method in secondary data

provided by an apparel retailer and compare it with benchmarks such as Collaborative Filtering (CF), a heuristic method recommending popular products (PP), and another heuristic method based on the Multinomial Logit choice model (MNL). In general, our method outperforms other methods in anticipating clicks and purchases. We also take a deeper look at the performance of the methods in different situations and for different segments. As mentioned, a flash sale website introduces many new products every day that cause the cold issue for the CF. We define products that are introduced in the last 24 hours as “new products” and measure the performance of the methods in recommending them. PCB has strictly better performance in recommending new products. We notice that the performance of the benchmark methods has very high variance in different customer segments. As mentioned, CF recommends products based on the neighbors’ preferences, assuming that neighbors have similar preferences. We segment the customers based on the number of neighbors in their neighborhood. As predicted, CF performance is worse than other methods when the number of neighbors is small. In another test, we divide the customers based on the popularity of products that they generally prefer. The first group consists of customers who usually prefer popular products. In contrast, the second group consists of customers who are interested in other products. PCB outperforms benchmarks in the second segment.

The remainder of this chapter is structured as follows. In Section 2.1, we review the literature and related studies. In Section 2.2, We mainly discuss our data and model, respectively. In Section 2.3, the implementation details and test methods are presented. Section 2.4 contains the results of tests and Section 2.5 concludes with a discussion and suggestions for future work.

2.1 Literature review

Our work is related to two different streams of literature: recommendation systems and assortment planning.

2.1.1 Recommendation System

Studies about recommendation systems mainly start in the mid-90s (Goldberg and Terry. 1992). In this structure, the main objective is to find the best set of items to be offered to a customer, based on the previously provided data. There are two well-known types of recommendation systems: Content-based and collaborative filtering (Burke 2007).

Content-based recommendation (CB) is based on recommending items similar to the items which are previously preferred by the user. In this approach, the similarity of items is estimated based on the similarity of predefined attributes of the items (Mladenec 1999). Two main limitations of CB recommendation is overspecialization and new user problem (Pasquale Lops and Semeraro 2015). By overspecialization, we mean that the recommender system is limited by recommending just similar things and cannot recommend novel content. On the other hand, the new user problem is common issue for all personalized recommending systems, because RS cannot recognize the user's preferences due to the lack of sufficient data about the user (Ricci et al. 2015).

Collaborative filtering recommendation systems (CF) use the idea “similar users” called neighbors. Let R represents the activity matrix of all customers with the dimension of $U \times I$ (Users $\times$ Items). Each column in this matrix represents an item and each row represents a customer and each array R_{ij} represents relation of related item i and related user j . The relation between item and user can be a rating provided by the user (e.g. IMDB) or it can be a click or a purchase decision in a retailing environment (Ricci et al. 2015, Ning et al. 2015).

The first step of the user-based CF approach is finding similar users called neighbors (Hill and Furnas. 1995, Konstan and Riedl. 1997). Neighbors could be the top N users with the highest similarity. In the next step, CF recommends the items that are preferred more by the neighbors.

CF approach has a difficulty dealing with new users and new items. For a new user (new item), there is insufficient and limited data. The huge number of available items and users in the matrix R cause the matrix sparsity. Sparsity in the matrix

means a high number of unknown values in the estimation process which affects the estimation performance of CF (Masthoff 2015).

Later studies in this area, develop the main idea presented above in various aspects. As a contribution to the main idea, some studies are assuming a vector of user properties instead of a single user and a vector of item properties instead of a single item. The other example of new methods can be multicriteria ranking (e.g. considering story rating and visual effects rating of a movie separately instead of a single rating). Gediminas Adomavicius and Kwon (2015) provide a full review on multicriteria RS.

As a summary, recommendation systems literature is very rich, and existing techniques are well-developed. Most of the techniques and studies have focused on a rating-based structure, and it has a strong performance in some categories. The categories such as movies, books, articles are some of the well-suited categories. In contrast, the other categories such as apparel have some challenges in adopting this method. Most of the apparel retailers don't have any rating data for their products. In such cases, "click or not" and "purchasing or not" decisions are imperfect proxies for ratings.

2.1.2 Assortment Planning

Assortment planning (AP), helps the retailer to decide on the amount and assortment of items carried in each store at each period due to store capacities. Recent studies of AP are provided for online retailers where physical limits change to presenting and display limits. In an online AP, assortment can be both dynamic and personalized (Kök et al. 2015).

In contrast with classic AP, in dynamic assortment planning a retailer can update its assortment during the season. This research is motivated by fast fashion companies such as Zara. These type of companies can observe the demand at the start of the season and optimize their assortment for the rest of the season. This means that company faces an exploration-exploitation trade-off. In this trade-off, a retailer can learn more about the demand of each assortment by presenting different

assortment or can choose the best assortments due to the information gathered up to that time and start exploiting. Dynamic AP studies start with Caro and Galien (2007). In this study, they provide a multi-armed bandit (MAB) perspective for the AP problem to learn the best assortment. MAB form of AP problem assumes each available assortment as a bandit and the prize is revenue collected by that assortment. They solve this problem by using Lagrangian relaxation. In the next contributions to the AP problem, Sauré and Zeevi (2013) study the DAP in an online retailer. They define a MAB problem, too and measure the “regret” (the magnitude of the loss, relative to the best possible performance) of any policy in terms of its dependence on the length of the selling horizon, the number of products, and the capacity constraint. They have proposed 3 different heuristics to solve this MAB problem. The policies provided in this study identify the optimal assortment of products with high probability with limited exploration. Also they show that some products can be easily and quickly identified as suboptimal.

An online retailer has an opportunity to offer an special and more appealing assortment for each customer to boost the conversion rate and optimize its profit. Relatively, the next stream of AP studies considers personalized and dynamic assortments. Bernstein et al. (2015) add a personalized perspective to DAP problem to provide a better solution under limited inventory assumption. They show it is optimal to reserve some products for some of the customers if the inventory of that product is too limited. This phenomenon happens because some of the customers may prefer to purchase substitute products while others may leave without purchasing any products. Golrezaei and Rusmevichientong (2014) define a more general model for personalized dynamic AP. They have developed a model to maximize the revenue by considering a limited inventory assumption and personalizing assortment due to the location of arriving customers. They demonstrate that this personalization approach can lead to over 10% improvements in revenue compared to a policy that treats all customers the same. They provide a worse case performance guarantee relative to the benchmark methods.

The importance of personalized dynamic assortment planning increases with the

extensive focus on e-commerce. However, there are very limited studies in this field. In contrast with recommendation systems literature where personalization methods are well developed, there is few study with personalization considerations in the AP literature.

2.2 Data and Model

2.2.1 Data

Our data is provided by online flash seller apparel retailer “Privalia”. Data contains a 3-year history of customers’ behavior including clicks and purchased products. There is products’ information including color, brand, price, and gender, and campaign information including their start and end date. Privalia is an online apparel retailer selling the products in the campaigns (campaigns last 9 days on average) and usually, products are new (not repeated in campaigns). Thus RS is typically subject to the cold start problem related to new products. Even though content-based methods solve the item-based cold start problem, they require detailed data about product similarity. Unfortunately, the detail about each product is limited in our data set. Further, CB methods typically does not differentiate between users.

2.2.2 Product Attributes, Features, and Attribute Vector

We represent each product with an attribute vector. Each attribute vector consists of the product’s attributes and sales features (The vector includes attributes such as color, price, brand, and features such as the number of sales and introduction time of the product.) For instance, product code 689712 is represented by an attribute vector $[Shoe, Red, Nike, Expensive, bestseller, new]^T$.

Consider that each product is described with K attributes and k^{th} attribute can get L_k exclusive values, for example, color can get values such as red, blue and black. Then, product i can be represented by the corresponding attribute vector $a(i) = [a_1^i, a_1^i, \dots, a_K^i]^T$ where a_k^i is the value of k^{th} attribute of product i out of L_k different values.

2.2.3 Attribute-Vector-based Score and Subscores

To investigate the users' interest in the attributes, we count the number of times they visit, click, or purchase products with a specific attribute vector. More formally, assume all products with the corresponding attribute vector $a(i)$, then $C_{jt}^{a(i)}$ and $p_{jt}^{a(i)}$ are the number of times that these products are clicked and purchased (respectively) by the customer j up to time t . We call these values subscores in the rest of the paper. In other words, we capture the number of times that an attribute vector $a(i)$ is clicked and purchased as feedback from the customer's behavior.

PCB recommendation system assigns a score to each attribute vector to identify customer's interest in that attribute-feature combination. The score of customer j for attribute vector $a(i)$ at time t denoted by $S_{jt}^{a(i)}$ is a function of related attribute vector and subscores; more formally:

$$S_{jt}^{a(i)} = f\left(a(i), C_{jt}^{a(i)}, p_{jt}^{a(i)}\right) \quad (2.1)$$

Table 2.1 present a simple example with five products, their corresponding attribute vectors, and customers' activity history. Note that in Table 2.1 there are two different products (codes: 4356978 and 1669733) with the same attribute vector a^1 because their all captured attributes are the same. It means that all subscores related to this attribute vector i.e. $C_1^{a^1t}$, $p_1^{a^1t}$, $C_2^{a^1t}$, and $p_2^{a^1t}$ are affected by both products. Table 2.2 includes the customers' scores based on the values provided in Table 2.1.

2.2.4 Linear Scoring Function

We use a linear function for scoring function $f(\cdot)$ at the first step. Different $f(\cdot)$ functions could be used as well. We discuss possible alternatives in the next section. With a linear function, we assign the scores as follows (note that b_0 , b_1 , and b_2 are coefficients for calculating the total score):

$$S_{jt}^{a(i)} = b_0 + b_1 \cdot C_{jt}^{a(i)} + b_2 \cdot p_{jt}^{a(i)} \quad (2.2)$$

Table 2.1: Examples of customers' activity history

Product Code	4356978	5794612	8000456	1669733	1223409
Color	Blue	White	Red	Blue	White
Price Level	Expensive	Expensive	Moderate	Expensive	Moderate
Brand	Nike	Adidas	Adidas	Nike	Adidas
Attribute vector	$a(1)$	$a(2)$	$a(3)$	$a(1)$	$a(4)$
Customer 1	<i>Clicked</i>	<i>Clicked</i> <i>Purchased</i>	—	<i>Clicked</i>	—
Customer 2	<i>Clicked</i> <i>Purchased</i>	—	<i>Clicked</i>	—	—

Table 2.2: Examples of Scores provided by customers' activity history

Customers	$a(1)$	$a(2)$	$a(3)$	$a(4)$
1	$S_{1t}^{a(1)} = f(a(1), 2, 0)$	$S_{1t}^{a(2)} = f(a(2), 1, 1)$	$S_{1t}^{a(3)} = f(a(3), 0, 0)$	$S_{1t}^{a(4)} = f(a(4), 0, 0)$
2	$S_{2t}^{a(1)} = f(a(1), 1, 1)$	$S_{2t}^{a(2)} = f(a(2), 0, 0)$	$S_{2t}^{a(3)} = f(a(3), 1, 0)$	$S_{2t}^{a(4)} = f(a(4), 0, 0)$

Accuracy, Efficiency Trade-off

A customer's preference for an attribute may be dependent on the other attributes of the product. For instance, a customer may prefer white Nike shoes but prefers red Adidas ones, which means that his/her preference for color is dependent on the product's brand. Attribute vector helps us evaluating customer's interest in an exclusive combination of attributes. However, there are many possible combinations. Remember that each products is defined with K attributes and each attribute can get L_k different values (e.g. red, blue), then set of all possible attribute vectors, A , would have a size of $\prod_K L_k$. Each member of the set (attribute vector) $a(i)$,

is an exclusive combination of attributes (e.g. $[Shoe, Red, Nike, Expensive, \dots]^T$). Some of these combinations may not correspond to any existing product. Remember Table 2.1, where there are $K = 3$ attributes (i.e. color, price level, brand), where color can get three values of white, red, and blue, the price has three levels of cheap moderate and expensive, and finally, there are two brands. Thus, L_1, L_2, L_3 are 3, 3, and 2 respectively. In this simple example, A consists of 18 attribute vectors while some of the attribute vectors may not correspond to an existing product. For example, maybe there is no red cheap Adidas product, however the attribute vector can be assumed, and it is a part of A .

The high number of possible vectors implies enormous interaction of customer and products to calculate the score for each vector. As presented above, by Equation 2.1, the score is a function of subscores, however, if the number of attribute vectors is too high, subscores for a specific attribute vector would be zero frequently. Zero subscores provide no information about the customer's preference. To reduce the sparsity, we can decrease number of the specific values extracted from the data set. We do this by measuring each customer's interaction with each specific attribute separately instead of a specific vector (combination of attributes). It means that the subscores for each K attribute of a product are calculated separately and the final score is calculated by these subscores. More formally, instead of Equation 2.1, we use the equation below where $\hat{C}_{jt}^{a(i)}$ and $\hat{p}_{jt}^{a(i)}$ are approximations for $C_{jt}^{a(i)}$ and $p_{jt}^{a(i)}$ respectively:

$$S_{jt}^{a(i)} = f\left(a(i), \hat{C}_{jt}^{a(i)}, \hat{p}_{jt}^{a(i)}\right) \quad (2.3)$$

Let's assume that we have K attributes for describing each product, and $C_{jt}^{a_k(i)}$ are $p_{jt}^{a_k(i)}$ are the number of times that k^{th} attribute is clicked and purchased (respectively) by the customer j up to time t . Then, we approximate the subscore values with linear functions as below where $b_{10}, b_{11}, b_{20}, b_{21}, \dots$ are coefficient values for each attribute:

$$\hat{C}_{jt}^{a(i)} = b_{10} + b_{11}C_{jt}^{a_1(i)} + b_{12}C_{jt}^{a_2(i)} + \dots + b_{1K}C_{jt}^{a_K(i)} \quad (2.4)$$

$$\hat{p}_{jt}^{a(i)} = b_{20} + b_{21}p_{jt}^{a_1(i)} + b_{22}p_{jt}^{a_2(i)} + \dots + b_{2K}p_{jt}^{a_K(i)} \quad (2.5)$$

Table 2.3: Example 2: Attribute Vectors and Corresponding Purchase Subscores

Product Code	1457277	7618843	2003871	3890366
Attribute Vector	$a(1)$	$a(2)$	$a(3)$	$a(4)$
$p_{1t}^{a(i)}$ Value	3	1	1	0
Color	Blue	White	Red	Blue
Brand	Nike	Adidas	Adidas	Adidas

This may cause some misleading estimation of customers' interest (for example, a customer could be interested in the expensive Adidas products but not expensive Nike ones, meaning her/his interest in expensive products is dependant and not the same in all brands). This is a trade-off of accuracy in preference estimation and efficiency in data usage.

Tables 2.3, 2.4, and 2.5 represent a group of examples of how approximated subscores are calculated out of a customer's interaction history. Tables 2.3 represents the preferences of the customer in the level of attribute vectors (the combination of two attributes of color and brand). Then, in Table 2.4, we represent preferences of the same customer in the level of attributes. Finally, in Table 2.5, we approximate a value for each subscore based on the values represented in Table 2.4. For example, in Table 2.4, number of purchases by the customer is aggregated for color blue (attribute vectors $a(1)$ and $a(4)$) and brand Adidas (attribute vectors $a(2)$, $a(3)$ and $a(4)$). Also, you can observe that attribute vector $a(4)$ is not purchased by the customer, however, we know that customer has purchased some products with blue color and some products with Adidas brand that causes approximated purchase value for product 3890366 be higher than zero in Table 2.5.

2.2.5 Random Forest Implementation

In the previous section, we proposed a linear $f()$ function to assign the score. As an alternative scoring function, we implement a random forest (RF) as well. Each

Table 2.4: Example 2: Attribute-level Purchase Values

Attribute Type	Color			Brand	
Attribute Values	Blue	Red	White	Adidas	Nike
Attribute Code	$a_1(1)$	$a_1(2)$	$a_1(3)$	$a_2(1)$	$a_2(2)$
$p_{1t}^{a_k(i)}$ Values	3	2	1	2	3

Table 2.5: Example 2: Linear Approximation of Purchase Subscores Using Attribute-level Purchase Values

Product Code	1457277		7618843		2003871		3890366	
Attributes	Blue	Nike	White	Adidas	Red	Adidas	Blue	Adidas
$p_{1t}^{a_k(i)}$	2	3	1	2	2	2	3	2
$\hat{p}_{1t}^{a(i)}$	$b_{20} + 2.b_{21} + 3.b_{22}$		$b_{20} + 1.b_{21} + 2.b_{22}$		$b_{20} + 2.b_{21} + 2.b_{22}$		$b_{20} + 3.b_{21} + 2.b_{22}$	

entry (observation) in RF represents a customer-product interaction. In that sense, for each observation in the data, products' attribute vector and corresponding customer's subscores are given as observed values and customer's preference is the final decision. In our data, we only have the clicks and purchases information, however, to train the RF we need the information of the products not preferred as well as the products preferred by the customer. In other words, we need both types of observations that lead to positive and negative decisions. It is possible to label all products which are not clicked or purchased as "not preferred," however, the number of products is too high and we are not sure that the customer has seen them or not. We narrow down this set of products as follows. Let's denote a campaign as "considered" if the customer clicked or purchased at least one product of that campaign. We assume that all products in the considered campaigns are visited by the customer and if they are not clicked or purchased, we label them as "not

Table 2.6: RF observation Example

Observation Index		Customer-based Subscores (Preference Percentage)			Product-based features		Lable
Customer ID	Product ID	Color	Brand	Price-level	Age*	Pop.-level	Action
234912	9813428	12%	10%	25%	New	2	Preferred
156729	9813428	11%	7%	25%	New	2	Not-Preferred
234912	1634728	10%	5%	22%	Old	3	Not-Preferred

preferred”.

Each entry in RF represents a customer-product interaction. In that sense, for each observation in the data, products’ attribute vector and corresponding customer’s preference percentage of that attributes are given as independent variables and customer’s preference is the dependent variable. Table 2.6 represents some examples of observation used in RF implementation.

2.2.6 Benchmarks

Collaborative Filtering (CF)

CF is a well-known method in the recommendation system literature. This method mainly uses the similarity of customers’ behavior. The similarity between customers is calculated by the number of common activities they have e.g. purchasing the same product. A neighborhood is formed for each customer which contains the customers who have similar behaviors with him/her. CF recommends the most preferred items by neighbors that are not yet viewed by the focal customer.

Popular Products

Popular items recommendation is used by most of the e-retailers. For example, Amazon recommends popular products with the title of “best sellers.” This method simply shows the most visited/purchased products at the current time.

Logit-based Choice Models

The multinomial logit choice is commonly used in assortment planning and customer choice modeling. In this model, each customer has a utility function consists of two parts of the average utility of each product, and the personal preference of the customer for each product which are i.i.d. Gumble distribution. Based on the average utility of each product, the probability of choosing a product out of a set of products can be calculated (Talluri and Ryzin. 2004).

2.3 Implementation and Test

2.3.1 Implementation of Linear PCB and Random Forest

In the current dataset, there are four attributes for each product: Color, Price, Brand, and category. We implement two types of linear PCB methods. The attribute vector in the first one includes only predefined product attributes such as color and price. In the second implementation, attribute vector includes sales features such as sales amount and the time passed after product introduction in addition to predefined attributes. This separation helps us realize the effect of the sales features on the estimation performance. We denote the later version as PCBf. Similar to the linear PCB, we implement two types of random forest denoted by RF and RFf.

2.3.2 Implementation of Benchmarks

To implement CF (/CFf), we calculate the similarity between any pair of customers by measuring the number of times they have purchased or clicked on the same product (/feature). Formally, 's $D_{jj'}$ denote the similarity between customers j and j' , and I_j^C and I_j^P denote the set of products clicked and purchased by customer j ,

respectively. Then we can calculate similarity between any customer pair based on clicks or purchases as follows:

$$D_{jj'} = \frac{|I_j^C \cap I_{j'}^C|}{|I_j^C| + |I_{j'}^C|} \quad (2.6)$$

$$D_{jj'} = \frac{|I_j^P \cap I_{j'}^P|}{|I_j^P| + |I_{j'}^P|} \quad (2.7)$$

On the other side, to implement the popular products recommendation, we define popularity as follows: The top products with the number of clicks or purchases. We denote the methods with click as PPC and with purchase as PPP. Finally, using the following linear regression, we provide MNL utilities as below:

$$U_i = \gamma_0 + \gamma_1 \cdot a(i) \quad (2.8)$$

where γ_1 is the vector of coefficients and $a(i)$ is attribute vector of product i . U_i is estimated using the purchase quantity of product i in the train.

2.3.3 Performance Measurement

All methods are tested where a method's performance is measured by the accuracy of anticipating the customers' choices. We define test period of each customer as a period that a customer has at least a website visit session. Products that are available at the website in the test period of customer j are considered as available products set denoted by I_j^α , and products that are selected (clicked or purchased) by the customer are considered as chosen products set denoted by I_j^β . All activities before the test period are included in the train data. The train data and available product set are information available to all methods.

Type 1 Measurement

We measure the performance of methods by two different techniques. In the first technique, with implementing the likelihood method, each method gives a probability distribution to the set of available products, in a way that the summation of the probabilities would be equal to 1. Formally, method Q gives probability $Prob_{ij}^Q$ that

product i to be selected by customer j . Additionally, let's N_{ij} denote the the number of times product i is selected by customer j up to the test time. In all methods, we use a linear normalization for available product set that make the summation summation of probabilities equal to 1. Then following are the probability-distributing policy of each method.

PCB

In the all of our methods (i.e. PCB, PCBf, RF, and RFf), we assign a probability based on the score of each product and its corresponding attribute vector.

$$Prob_{ij}^{PCB} = \frac{S_j^{a(i)}}{\sum_{I_j^\alpha} S_j^{a(i)}} \quad (2.9)$$

CF

Due to the literature, one of commonly used CF versions is weighted rating prediction method. We implement similar method where a product receives a higher probability when it is preferred by a customer with higher nearness parameter, $D_{jj'}$.

$$Prob_{ij}^{CF} = \frac{\sum_{j' \in J} D_{jj'} \cdot N_{ij'}}{\sum_{I_j^\alpha} \sum_{j' \in J} D_{jj'} \cdot N_{ij'}} \quad (2.10)$$

Popular Products

We use the products' purchase or click number to assign probabilities in popular products method.

$$Prob_{ij}^{PP} = \frac{\sum_{j' \in J} N_{ij'}}{\sum_{I_j^\alpha} \sum_{j' \in J} N_{ij'}} \quad (2.11)$$

MNL

We exactly use the basic form of MNL probability distribution due to the literature using the utility values we estimate from the training data set.

$$Prob_{ij}^{MNL} = \frac{e^{U_i}}{\sum_{I_j^\alpha} e^{U_i}} \quad (2.12)$$

Table 2.7: Probability distribution of each method; Example 1

Product code	998734	121356	200567
CF: Prob. Provided	0.5	0.4	0.1
PCB: Prob. Provided	0.2	0.6	0.2
Customer's Choice	No	Yes	No

Table 2.8: Probability distribution of each method; Example 2

Product code	998734	121356	200567
CF: Prob. Provided	0.2	0.4	0
PCB: Prob. Provided	0.5	0.3	0.2
Customer's Choice	No	No	Yes

The performance of the method is the sum of the probabilities assigned to the chosen products set. More formally, assume $Prob_{ij}^Q$ is the probability of choosing product i by customer j , given by method Q , the performance of method Q is measured as below:

$$Performance(Q) = \frac{\sum_j (\sum_{i \in I_j^\beta} Prob_{ij}^Q)}{|J|} \quad (2.13)$$

$$s.t. \sum_{i \in I_j^\alpha} Prob_{ij}^Q \leq 1 \quad \forall j \in J \quad (2.14)$$

For instance, consider there are three available products and two customer. Table 2.7 and Table 2.8 represent the choice of these customers. CF and PCB provide exclusive probability distribution for each customer. In this example, CF has average performance of 0.2 ($(0.4 + 0) / 2$) and PCB has average performance equal to 0.4 ($(0.6 + 0.2) / 2$).

Type 2 Measurement

In the second technique, we measure the performance of the methods in offering a limited number of products. In that sense, each method nominates “Top N” products. Each method gains a positive value for each product match from the “Top N” list and chosen products set. Let’s I_j^Q denote the set of top N products selected by method Q to be recommended to customer j . Then, performance of each method is measured as follows:

$$Performance2(Q) = \frac{\sum_j (|I_j^\beta \cap I_j^Q|)}{|J|} \quad (2.15)$$

Not that I_j^Q is the set of the products with the highest probability assigned by method Q .

2.4 Results

2.4.1 General Performance

As mentioned, in the tests, the available products set is a set of all products available at the website during the visit period of the customer. Each method estimates the set of all clicked products (denoted by the visit to click) or purchased products (denoted by the visit to purchase) in the test period. Table 2.9 represents the performance of each method.

1. RFf is the best method predicting clicks. On the other hand, PCB outperforms benchmarks predicting purchases.
2. There is no significant difference between PCB and PCBf, while, RFf outperforms all methods predicting clicks. Without the features the performance level is below the CF and linear PCB. This shows that features are more effective in the RF than linear PCB. Also, note that considering features in CF has a negligible effect. In the rest of the paper, for ease of read and elimination of redundancy, we represent the slightly better type of each method that is RFf,

Table 2.9: General Performance

Method	Visit to Click			Visit to Purchase		
	Probability	Top 5	Top 20	Probability	Top 5	Top 20
RF	13.3	68.9	243.7	3.3	20.7	93.1
RFf	16.3	95	300.5	3.9	26.9	123.5
PCB	14.8	78.1	267.8	4.0	27.8	122.3
PCBf	15.6	82.2	285.8	3.9	26.6	118.5
CF	15.6	82.2	285.8	3.2	19.7	75.3
CFf	16.3	85.5	291.3	2.9	19.7	71.4
PPC	12.0	278.9	857.5	1.0	23.9	62.5
PPP	9.2	150.3	520.3	2.1	45.9	138.7
MNL	2.9	7.7	39.7	0.6	1.0	6.9

* Results are multiplied by 1000 for ease of reading.

PCB, and CF as we observe the difference is insignificant in the remaining results.

- For the top 5/20 products measurements both PPC and PPP outperform other methods by a significant margin. Recall that we don't have the data of visited products, we are assuming all available products as visited. However, visited products are limited and the website is recommending popular products regularly, which may boost the performance of PPC and PPP. We refer to this effect as "unfair effect of visited products."
- Let's call products' information represented in the data explicitly such as color, brand, and price as explicit information. On the other hand, other attributes that are not included in the data are hidden information such as design elements. While PPC/P methods may catch hidden information to find the most preferred products, MNL uses explicit information to do so. According to the results, we see a big difference in their performance levels. We conclude that

the explicit information (i.e. number of explicitly predefined attributes) is too limited for the MNL model to perform well. Note that CF is able to discover the hidden information while PCB uses only explicit information. Despite limited explicit information, personalization causes PCB and RF methods to outperform CF.

5. As expected, PPC performs better than PPP in Visit to Click test and PPP performs better in Visit to Click test.

2.4.2 Product Visit Bias

Recall that in Table 2.9 PPC and PPP have the best performance selecting the top 5/20 products. Recall that we don't have the data of visited products, we are. Although products visited by a customer are limited we do not have the list of that products and assume that all available products are visited. However we know that the website is recommending popular products regularly, which enhance the probability of visiting these products. We design another test to reduce the bias described above. We select a set of top 50 products which are clicked and purchased the most in the test periods. This set have received about 50% of clicks and purchases although the set includes only about 5% of all available products. This information shows that this set of the products is offered to and visited by the customers. Thus, we test each method estimating clicks and purchases in this limited set, instead of all available products set. This reduces the probability that a product selected by a method has not been visited by the customers. Subsequently, mentioned bias is reduced. Results of the test is provided in Table 2.10.

As predicted, by reducing the the bias, PPC/P methods lose their advantage and gets outperformed by other methods even when we measure top5/20 performance. Additionally, results shows an effect of the personalized recommendation in the limited sets. Note that on average, a product in this set have received 10 times more click and purchase than a product in available products set. This is the main reason of higher performance values in Table 2.10 in compare with previous section.

Table 2.10: Performance of the methods recommending top 50 popular products

Method	Visit to Click			Visit to Purchase		
	Probability	Top 5	Top 20	Probability	Top 5	Top 20
RF	64	306	1043	15.1	75	287
RFf	66	401	1104	16.9	79	276
PCB	79	453	1192	18.2	84	301
PCBf	77	420	1133	18.0	83	304
CF	60	318	1074	14.7	66	310
PPC	38	283	889	5.7	23.9	90
PPP	31	170	561	12.1	48	151

* Results are multiplied by 1000 for ease of reading.

However, methods cannot easily enhance their performance. When we measure the performance over all available products, methods such as PPP/C, CF and RF easily recognise the popular products and distribute higher probability to these products. It means that even without limiting the set, they recommend these products more (i.e. Each method set a higher probability for a product in this set). When we measure the performance inside the popular set, there is no more advantage of recognising the popular products, which makes predicting the clicks and purchases more challenging. Thus, methods without any personalization cannot enhance their performance anymore such as PPC and PPP. In contrast, a completely personalized method such as PCB have the highest performance increase. On the other hand, CF loses part of its advantage even its performance is increased due to the personalized structure of the method. Due to a similar reason, RFf method lose the advantage over RF method. Therefore, as presented in the table, PCB has the best performance predicting both clicks and purchases in all measurement types in this set.

Table 2.11: New Product Recommendation Performance

Method	Visit to Click			Visit to Purchase		
	Probability	Top 5	Top 20	Probability	Top 5	Top 20
RF	10.3	55.7	193.6	6.6	35.7	107.0
RFf	12.1	65	211.5	7.1	40.5	120.4
PCB	11.8	67.7	203.4	7.8	44.3	139.8
PCBf	11.6	64.8	198.4	7.2	41.1	131.1
CF	3.6	20.1	86.9	1.7	11.9	47.9
PPC	0.4	0.0	0.0	0.1	0.0	0.0
PPP	2.2	39.4	108.1	1.0	28.2	63.7
MNL	1.3	7.7	39.7	0.4	1.0	6.9

* Results are multiplied by 1000 for ease of reading.

2.4.3 New Products

As mentioned, new product recommendation is a challenge for some of the methods. We have designed a test of recommending new products (new product is defined as the product introduced in the last 24 hours) and have measured the performance of each method. Table 2.11 represents the results of the new product recommendation performance of each method.

- As we expect, PCB and RFf are the best methods to recommend new products with a meaningful difference.

2.4.4 Customer-based Segmentations

Neighborhood Segmentation

We have observed that the performance of CF has a very high variance in the customers. To find out the customers' attributes which are effective on the CF performance, we measure the strength of customers' neighborhood by some parameters such as the number of neighbors (n), the average nearness of the neighborhood,

and the number of the neighbors with a higher nearness level than a threshold α ($\alpha = 0.8, 0.7, 0.5$). We use linear regression and assign performance as dependent variable and each of the parameters above as independent variable to realize if the effect of these attributes is significant or not. As a result, we find that the number of neighbors (n) and the number of neighbors that have at least two same activity (m) have significant effects on the CF performance, while the level of significance for other methods is too low in comparison with CF.

To illustrate the performance of methods, performance is measured in the first and the last decile of customers as two extreme region of the range. We denote the first decile as “Bottom” and last decile as “Top”, compared to grand average denoted as “Average.”. Results are represented at Table 2.12 and Table 2.13.

- The variance of CF’s performance is noticeable in these tables. While CF outperforms all methods in the “top” segment in all cases by a high margin, it has a weak performance in the “bottom” segment. For example, in Table 2.12 CF is outperformed by all methods predicting purchases, and in Table 2.13 it has lower performance in predicting clicks than RFf, PCB, PPC, and PPP methods.
- CF performance of predicting purchases for the bottom segment is almost zero (rounded to tenths). We investigate the reason behind that and we found that the bottom segment has no neighbors and subsequently CF sets equal probabilities on all available products which are so close to zero.
- RFf and PCB are the best methods to predict clicks and purchases respectively for the bottom segment in both tables.
- We observe that all methods have a lower performance level at the “bottom” segment. Recall that customers in the bottom segment have fewer neighbors than other segments. This means that either these customers are new or their preferences are incompatible with the majority of the customers. These reasons cause all methods to have a challenge in predicting their preferences.

Table 2.12: Effect of the number of neighbors (n) on the performance

Method	Visit to Click			Visit to Purchase		
	Bottom	Average	Top	Bottom	Average	Top
RFf	12.0	16.3	16.8	1.9	3.9	4.2
PCB	9.9	14.8	14.8	2.1	4.0	4.6
CF	9.8	15.6	18.5	0.0	3.2	5.2
PPC	9.6	12.0	14.7	0.8	1.0	1.2
PPP	6.6	9.2	12.7	1.3	2.1	3.2
MNL	2.7	2.9	3.0	0.6	0.6	0.6

* Results are multiplied by 1000 for ease of reading.

2.4.5 Preference Segmentation

We define two new parameters: i) Product i 's Popularity PI_i : The popularity of the product in the train data is equal to the number of times it is purchased. ii) Customer j 's average interest in popular product $\overline{PI}_j$: The parameter measure average popularity of product selected by the customer (equal to the average popularities of products that the customer has clicked or purchased). We use linear regression and assign performance as dependent variable and each of the parameters above as independent variable to realize if the effect of these attributes is significant or not. We found that $\overline{PI}_j$ has a significant effects on the performance. To illustrate the performance of methods, performance is measured in the first and the last decile of customers denoted by "Bottom" and "Top" respectively, compared to grand average denoted by "Average". Results are provided in Table 2.14.

- In contrast with other methods, our content-based methods especially PCB have better performance at the bottom segment (customers who don't prefer popular products generally). We can conclude our methods are able to capture idiosyncratic preferences better than basic methods as it utilizes information on customers-attribute level interaction.

Table 2.13: Effect of size of neighborhood (m) on the performance

Method	Visit to Click			Visit to Purchase		
	Bottom	Average	Top	Bottom	Average	Top
RFf	11.9	16.3	16.8	1.8	3.9	4.7
PCB	11.6	14.8	15.6	2.0	4.0	4.8
CF	6.1	15.6	22.2	1.8	3.2	7.0
PPC	10.0	12.0	13.8	1.0	1.0	1.2
PPP	7.6	9.2	13.2	1.7	2.1	3.6
MNL	2.9	2.9	2.9	0.6	0.6	0.8

* Results are multiplied by 1000 for ease of reading.

- The effect of $\overline{PI}_j$ on the performance of CF is even higher than PPC and PPP methods. We observe that CF has a challenge in predicting both clicks and purchases of the bottom segment.

2.5 Extensions

2.5.1 Hybrid Method

As mentioned in Section 2.4, the performance of the methods is dependent on parameters such as the number of the customer's neighbors, and we implement an RF to select the best recommendation method for each customer based on these parameters. Remember that we show that three parameters have a significant impact on the performance i.e. i) customer's reference popularity $\overline{PI}$, ii) the number of the neighbors, n and iii) the number of neighbors with more than one common preferred product, m . Given these three parameters, RF decides on the method to provide the recommendation. We denote this RF hybrid method as "smart selection."

Table 2.15 represents the result of Smart selection.

1. As predicted, smart selection outperforms all methods in all tests using the best one in each segment.

Table 2.14: Effect of customers' average interest in popular product ($\overline{PI}_j$) on the performance

Method	Visit to Click			Visit to Purchase		
	Bottom	Average	Top	Bottom	Average	Top
RFf	16.9	16.3	16.0	4.0	3.9	3.9
PCB	16.6	14.8	14.8	4.3	4.0	4.0
CF	8.8	15.6	15.5	0.6	3.2	4.0
PPC	12.0	9.6	10.4	0.8	1.0	1.1
PPP	6.6	9.2	9.7	1.2	2.1	2.5
MNL	2.9	2.9	2.9	0.6	0.6	0.6

* Results are multiplied by 1000 for ease of reading.

- Table 2.16 represents the percentage of using each method for prediction in smart selection. These results show that the parameters defined previously such as n , and $\overline{PI}$ are practically usable to have better predictions. We observe that CF is used in more than half of the cases. After CF, our content-based methods (i.e. PCB and RF) are used around one third of times. This is aligned with our results that show CF has high performance when neighborhood is constructed well, and PCB and RF can perform in other segments.

2.6 Conclusions and Future Research

In this article, we proposed a personalized content-based recommendation method. The content-based recommendation helps us address so called item-based cold start issue. On the other hand, products in our data are identified with a few attributes. Remember that we called details clearly captured by data, as explicit information, e.g. color price, etc. and we called other information which is not clearly defined, as hidden information e.g. design elements. As mentioned, explicit information (i.e. the number of attributes explicitly mentioned for each product in the dataset)

Table 2.15: General Performance

Method	Visit to Click			Visit to Purchase		
	Probability	Top 5	Top 20	Probability	Top 5	Top 20
Smart	19.8	114.8	354.8	6.0	40.9	156.6
RF	13.3	68.9	243.7	3.3	20.7	93.1
RFf	16.3	95	300.5	3.9	26.9	123.5
PCB	14.8	78.1	267.8	4.0	27.8	122.3
PCBf	15.6	82.2	285.8	3.2	19.7	75.3
CF	15.6	82.2	285.8	3.2	19.7	75.3
CFf	16.3	85.5	291.3	2.9	19.7	71.4
PPC	12.0	278.9	857.5	1.0	23.9	62.5
PPP	9.2	150.3	520.3	2.1	45.9	138.7
MNL	2.9	7.7	39.7	0.6	1.0	6.9

* Results are multiplied by 1000 for ease of reading.

is strictly limited. Personalization causes better use of these limited detail and improved the quality of recommendation.

Results of segmentation show that PCB is able to address the new product problem. Also, we find three customer-specific parameters which affect the performance of benchmark methods named 1. customer's reference popularity $\overline{PI}$, 2. the number of the neighbors, n and 3. the number of neighbors with more than one common preferred product, m . PCB outperforms benchmarks when $\overline{PI}$, n or m is low. Finally, we implement a hybrid method that selects the best method for the recommendation using the parameters mentioned above. The hybrid method outperforms all other methods in general.

2.6.1 Product Views Without Click or Purchase

One hypothesis is that if a product has been offered many times to a particular customer, but the customer has not clicked on the product or purchased it, this

Table 2.16: Smart selection of the methods

	PCB and RF	CF	PPP and PPC
V to C	34%	55%	11%
V to P	36%	55%	9%

could be counted as a negative sign of interest. In other words, in contrast with clicks and purchasing which are positive feedbacks of interest, seeing but not clicking or purchasing can be assumed to indicate negative feedback. It is important to account for these negative feedbacks, to differentiate between products which have been offered sufficiently many times and which have been not offered. This would give a higher chance to the rarely offered products to be offered more. To test this hypothesis, we need click-stream data including the set of products offered to the customers. Then, we can define $V_{jt}^{a(i)}$ as the number of times product with corresponding attribute vector $a(i)$ are offered to the customer j up to time t . Then, the score function would be updated as follows:

$$S_{jt}^{a(i)} = f\left(a(i), V_{jt}^{a(i)}, C_{jt}^{a(i)}, p_{jt}^{a(i)}\right) \quad (2.16)$$

2.6.2 Data Pooling from Different Categories

We are also interested in finding relations across-categories. We try to find the best way to use the history of the customer in a category to enrich other categories' data. For instance, consider a customer who had purchased several items in the t-shirt, shoe, and blouse categories. However, we don't have any information about his/her preferences in the jacket category. To find some appealing products in the jacket category for the customer, we need to answer two questions. First, is the customer's preference in brand, price level, etc. in the jacket category, similar to the other categories? In other words, is a customer's interest in an attribute in some categories generalizable for all categories? For example, the customer may be generally interested in expensive products in all categories, however, his/her color preference

may be different in various categories. Data of an attribute from any category is more valuable in all other categories if the generality of that attribute has a higher value (e.g. price level). Second, customer's preferences in which category is more relevant to the jacket category? One can explore the correlation between categories to measure the similarity of the two categories. The correlation can be measured by product similarity or by analyzing the customer choice. Recall that scores are the weighted sum of subscores, and we can use "generality of attributes" and "category correlation" to assign better weights for each subscore from each category. We hypothesize that using "generality of attributes" and "category correlation" would improve the recommendation performance.

Chapter 3

DECENTRALIZED RANKING AND PRICING IN AN ONLINE MARKETPLACE

Online Marketplace platforms play an essential role in e-commerce. An online marketplace is an e-commerce website where product or service information is provided by multiple third parties, whereas transactions are processed by the marketplace operator such as Amazon.com, Trendyol, and Hepsiburada. Due to the operator's strategy, for each product, there may exist more than one seller with different prices. When an identical product is offered by more than one seller, operational attributes of sellers such as price and rank of the seller on the web-page can affect the customers' preference of the seller. For example, a customer may search for the seller with the lowest price or the seller with the best ratings. Amazon.com uses the "buy box" option to differentiate one of the sellers with a dominant position. Customers can directly add a product to the basket from the buy box and it is fulfilled by that seller. 82% of Amazon sales go through the buy box, and the rate is even higher for mobile purchases (Chen et al. 2016). Also, the ranking of sellers and the probability of winning the buy box is highly correlated by prices (Chen et al. 2016).

We are interested in joint pricing and ranking decisions at a marketplace when products are identical or close substitutes. In this setting, the marketplace operator (MO) decides on the ranking rule (e.g. price-sorted, or based on bidding). On the other hand, the ranking rule is known to all sellers and they decide on their prices. Note that MO may or may not participate as a seller. We are concerned about optimal competitive pricing schemes and optimal ranking rule to maximize the revenue of each party.

We study the decentralized pricing and ranking problem mentioned above with

a game theoretical approach and search for a Nash equilibrium of the game. We assume the game is played in full information, which means that all players (sellers) know the ranking rule and demand function of themselves. They decide on their prices simultaneously and ranking takes place based on the pre-defined rule. After characterising the Nash equilibrium of the game, we investigate the total profit of MO dependent on the ranking rule as well as sellers.

The remainder of this chapter is structured as follows. In Section 3.1, we review the literature and related studies. In Section 3.2, we mainly discuss our model including price competition model and ranking rules. In Section 3.3, we define four types of the game with two players and Section 3.4 contains analyses of symmetric games. In Section 3.5, we provide numerical analyses and investigate the symmetric game with more details. Section 3.6 contains analyses of asymmetric games. In Section 3.7, we extend our model with the multiple sellers games. Finally, Section 3.8 concludes with a discussion and suggestions for future work.

3.1 Literature Review

Our work is related to the different streams of literature: product ranking and positioning, price competition, and seller competition in online platforms.

3.1.1 Product Ranking

Product ranking has been well-studied in the academic literature. Most of these studies imply that the products on the top of the list of an online retailer's web pages get more attention from the customers. There are several studies on the effect of the ranking on demand. One of the remarkable studies of ranking effect is provided by Ursu (2018). She does an experimental study at Expedia on the effect of ranking on the click-through rate (CTR), and subsequently on the demand. This experiment uses a randomized ranking and roll-outs any correlated effects. Results show that the isolated and causal effect of ranking on CTR and demand is significant. Similar results are provided in the sponsored search ranking in a study provided by Animesh et al. (2011), indicating the significance of the ranking effect

on CTR. Ghose et al. (2012). In this study, a hotel's utility is measured based on user-generated content. Subsequently, a utility-based ranking system is proposed in which a hotel that provides a comparably higher average utility would appear at the top of the list. The paper is followed by another study where this ranking method is tested competing with the benchmark ranking methods (Ghose et al. 2014). They show that a utility-based ranking mechanism can cause an increase in search engine revenue. Also, they found that ranking may affect customers' hotel ratings regardless of hotels' utilities.

To investigate how product ranking and positioning affect the customer's choice, most of the empirical studies use a sequential search model (Chen and Yao 2017, Kim et al. 2010, 2017). In the sequential search model, a customer usually decides on the consideration set first and then purchases the best alternative within the set (Weitzman 1979). In this model, the customer starts forming the consideration set from the first position and adding up the products up to a stopping point. The optimal stopping point is a function of the intrinsic utilities of the products and the customer's search cost. The sequential search model assumes that customers buy at most one product and the stopping point is dependent on the ranking of products. In contrast, the stopping point of a customer can be independent of ranking decision and defined by her/his search cost, if one of the following assumptions holds. One, if all products are substitutes that means their intrinsic utilities are the same. Two, customers are window shoppers and browse products looking for any they might like (denoted as screening activity that is independent of any purchase needs (Bloch et al. 1989)). This means each customer considers only the first r product/s with a certain probability. This assumption is used to study the effect of ranking on vertically differentiated positions by Aouad and Segev (2021). A similar method is used by Gallego et al. (2020) where they show that the ranking problem under the MNL choice model is NP-hard when prices are given. They propose heuristic algorithms with guaranteed performance relative to an optimal algorithm under reasonable assumptions. However, they show that when the firm can decide on the prices, optimal ranking and pricing strategy is descending product's utility

sorting followed by ascending order of markups. The study by Lei et al. (2021) complements these works by considering the problem with a dynamic ranking configuration in the setting with inventory constraints and cardinality constraints. In this study, the stochastic structure of the problem makes the optimization problem intractable and the deterministic relaxation of the problem needs a randomization policy to ensure the feasibility of the solution. The study provides a near-optimal heuristic randomization solution for the problem that chooses what to show to each customer by sampling from a joint probability distribution of prices and ranking sample space. In contrast with all mentioned studies that assume the demand is known to the retailer prior to the ranking decision, a bunch of studies considers the ranking problem for the cases that demand is unknown and retailer needs to learn the demand of each product during the season to maximize the profit. One of the remarkable studies in this area belongs to Ferreira et al. (2019). They propose online learning algorithms for product ranking that maximize the engagement of the customers with the site. They show that the algorithms converge to the best possible ranking with complete information. Also, they show that the algorithm yields a significant increase in the number of customers that engage with the site in practice. This study is followed by other studies provided by Bastani et al. (2021) and Golrezaei et al. (2021) considering a personalized ranking of products and existence of fake users, respectively.

Our work differs from all mentioned studies, as we are interested in ranking decision in a decentralized market, where items are provided by third-party firms and ranking is decided by the marketplace operator. In that sense, firms may take some actions (e.g. pricing strategies) to be presented in a more prominent position. Further, we consider pricing as a factor that affects the ranking of the sellers and subsequently customer choice. However, the underlying ranking effect in our study is similar to that in Aouad and Segev (2021) and Gallego et al. (2020).

3.1.2 Price Competition

Our model bears strong similarities to the well-studied price competition and oligopoly problems, as our problem is similar to a price competition in multiple markets for a given ranking rule in the platform. One of the commonly used choice models to study the pricing effect is multinomial logit (MNL). This model, independently proposed by Luce (1959) and Plackett (1975). However, the popularity of this model was notably driven by its efficient estimation procedures (Talluri and Ryzin. 2004). Assuming an MNL model, optimal pricing of a firm (centralized pricing decision) is constant markup (equal price minus cost) for all the products in the set (Anderson and De Palma 1992, Hopp and Xu 2005, Gallego and Stefanescu 2009). On the other hand, for price competition game (decentralized pricing) supermodularity and subsequently the existence of a pure Nash equilibrium of the game under the MNL model is provided by Milgrom and Roberts (1990). This result is extended to a more general form of demand functions, referred to as the attraction model by Bernstein and Federgruen (2004) and Federgruen and Yang (2009). This stream is followed by Cachon and Kök (2007) and Kök and Xu (2011) that consider the nested logit model with identical price sensitivities for the products of the same firm and characterize the Nash equilibrium. Gallego and Wang (2014) extend the result for nested logit models with product-differentiated price sensitivities and show that the adjusted markup, which is defined as price minus cost minus the reciprocal of the price sensitivity, is constant across all the products in each nest at optimal prices.

Another well-established price competition method is Bertrand (1883) model. Bertrand model is useful when products are identical or near substitute. The existence, uniqueness, and stability of the Nash equilibrium in the Bertrand model even with non-linear cost function is provided by Gallego et al. (2006). Farahat and Perakis (2010) and Federgruen and Hu (2015) extend previous studies for cases that each seller may offer more than one product considering an affine demand function. Federgruen and Hu (2015) provide a full characterization of the equilibrium behavior even arbitrary cost vectors. We refer readers for a vast review of price competition and oligopoly models such as Cournot, Bertrand, Edgeworth, Chamberlin,

and Robinson to Vives (1999). In the current paper, we use differentiated Bertrand competition to model the price competition between sellers.

Asymmetry in Price Competition

We study the asymmetric competition between firms as well as symmetric cases. Several studies consider an asymmetry of firms in price competition with/without product differentiation. Most of these studies assume that sellers differ due to a cost asymmetry and analyze its effect on the equilibrium (Ledvina and Sircar 2011, Dugar and Mitra 2011, 2016, Zanchettin 2006). For example, Boone et al. (2012) provide an experimental study of a Bertrand competition with asymmetric costs. The results show that a firm with a lower cost will set the price at the marginal cost of a less efficient competitor. In contrast with cost asymmetry consideration, few studies consider asymmetric demand under Bertrand competition to model differentiated firms (Liang et al. 2006). In our research, we study both cost and demand asymmetries.

3.1.3 Seller Competition on Online Platforms

Online marketplace platforms bring sellers together at the same platform which causes a higher level of competition. In this study, we are interested in the competition of sellers on an online marketplace platform such as Amazon.com. Chen et al. (2016) empirically study the price competition at Amazon.com and how the “buy box” and ranking of the sellers are affected by prices. The results of this study show that the winning of the buy box is highly correlated with the ranking of the sellers and their prices. For example, results show that the winner of the buy box is changed almost as many times as the seller with the lowest price changes. Also, they observe that with a high probability, the winner of the buy box is the seller with the lowest price. Finally, the study finds that many sellers prefer to implement algorithmic pricing at the platform to set a price below the competitors. In a similar study, Sikdar et al. (2019) use a random forest to identify the key predictors of price change. This study shows that Amazon.com as a seller has a more complicated

pricing strategy in comparison with other sellers. They found that Amazon adapts its products' price considering parameters such as the existence of used items at the market and third-party sellers' price, rating, and shipment methods. In our research, we build our model setup utilizing these observations from the market.

Besides empirical studies, the decentralized ranking has been well-studied theoretically, although there are key differences in modeling assumptions and analysis between this stream of work and ours. Most of these studies consider a bidding or auction-based positioning. In that sense, prominent positions are sold to the sellers based on their bids. One of the remarkable studies in this field belongs to Xu et al. (2011). In this study, a joint ranking-pricing decision of two cost-asymmetric firms is considered. The competition proceeds in two steps. In the first step, each firm bids for the prominent position, and the seller with the highest bid earns position 1. In the second step, two firms compete by pricing. Due to the ranking, just a portion of customers checks both products, while others just visit the product in the first position. The competition is modeled as a basic Bertrand Model where a seller with a lower price earns the total demand. Subsequently, there is no pure Nash equilibrium of the game and the mixed equilibrium is asymmetric. They show that position 1 may or may not be more valuable, even if it is cost-free. They find that the equilibrium prices are higher when the weaker competitor wins the prominent position which may cause higher profit than a prominent advertising position. Also, they find that due to the structure of the market, the expected price in position 1 might not be higher, even firms had paid more to be at position 1. In a more general setting, Chan and Park (2015) and Choi and Mela (2019) study the platform's ad. placement strategy based on a joint model of browsing (impressions), clicking, and purchase, on the demand side; and sellers' valuations and advertising competition on the supply side. Choi and Mela (2019) find that sorting items below the fifth position by expected sales revenue while conducting an auction in the top 5 positions yields the greatest improvement in profit by balancing the value gathered from advertising in the top positions and the transaction revenues in the lower positions. In contrast, Chu et al. (2020) show that, in the complete information case, rank-

ing with decreasing order of “net value added to the marketplace by the seller” is optimal if the platform considers a multi-objective optimization that includes the supply-side surplus, customers’ surplus, and sales revenue. They propose that an auction can be used in the incomplete information case to reveal the private information of sellers. The ranking provided by the auction is called “surplus-ordered ranking mechanism” which is near-optimal and outperforms benchmarks. In contrast with these studies, we focus on the competition intensity and subsequently pricing strategy of the seller based on the ranking policy of the marketplace. In that sense, one of the related studies belongs to Chen et al. (2021). They investigate the platform’s optimal assortment display policy, which is characterized by the partitioning of products (/sellers) into the subsets and assigning a specific amount of traffic directed to each partition. They find that it is optimal for the platform to display the entire assortment to all the customers when demand is sufficiently high. However, when demand is less, the platform may divide sellers and customers into the partitions to decrease the competition intensity, and subsequently increase the prices and her profit. Additionally, the fairness of such a policy is considered based on the closeness of the attractiveness of each partition and the traffic allocated to each partition. Results show that increasing the fairness of policy may result in a decrease in the platform’s profit.

Our study is motivated by online marketplace platforms such as Amazon.com, where one of the significant factors to define the ranking of sellers is the price of the products. marketplace operators allow the sellers to compete for the more prominent positions by pricing to motivate them to set lower prices. Generally, the ranking rule is predefined and once sellers set their prices (or reprice their product) ranking will take place automatically. In contrast with other studies, there is no bidding or auction for higher positions, and a seller can compete for a higher position with her/his price. This means that pricing has an impact on both ranking and demand. Through this mechanism, we investigate the effect of some predefined ranking rules on the market, pricing, and operator’s profit.

3.2 Model

3.2.1 Environment, Notations and Assumptions

Let $i \in I$, and $p_i \in R$ denote the sellers and their prices, respectively. We denote the demand and profit of seller i by $d_i(\mathbf{p})$ and $\pi_i(\mathbf{p})$, respectively, where $\mathbf{p}$ is vector of prices including prices of all sellers. Let index $m \in I$ denote the marketplace operator and p_m denote the marketplace operator's price (if it exists as a seller). We assume that platform earns the total benefit of the transaction when it is the seller and receives commission from other sellers' transactions. Commissions are based on the revenue (not the profit) by rate of β ($\beta < 1$). Then, we have following equation:

$$E[\pi_m(\mathbf{p})] = E[(p_m - s_m) \cdot d_m(\mathbf{p})] + E\left[\sum_{I \setminus \{m\}} \beta p_i \cdot d_i(\mathbf{p})\right] \quad (3.1)$$

$$E[\pi_i(\mathbf{p})] = E[((1 - \beta)p_i - s_i) \cdot d_i(\mathbf{p})] \quad (3.2)$$

where s_i is the constant marginal cost of the product i .

3.2.2 Differentiated Bertrand Ranking Competition

We use differentiated Bertrand competition model for the price competition in our study. However, the basic form of differentiated Bertrand does not add the ranking effect to the account and we modify the model as follows. We assume that each position is vertically differentiated from another position and each customer considers only first r product/s with probability α_r . As mentioned earlier, this assumption is supported by the literature (Aouad and Segev 2015). It is also compatible with all studies which are using a sequential ranking model (Chen and Yao 2017, Ursu 2018, Kim et al. 2017, 2010). We assume that all products are substitutes. In that sense, the intrinsic utilities of products are the same. This means that the stopping point of a customer is defined by the search cost of the customer and it is independent of ranking decisions. Subsequently, with probability α_r the stopping point of each customer is the position r ; it means that the customer only considers the first r products. Also, we assume that when the first r products are considered, demand

occurs due to differentiated Bertrand model denoted as $d_i^r(\mathbf{p})$. More formally:

$$d_i^r(\mathbf{p}) = a_r - b_r p_i + c_r \sum_{j=1}^r p_j \quad (3.3)$$

In Equation 3.3, a_r is the market size and b_r and c_r are related price elasticity parameters for related group of customers who consider the first r products.

Without loss of generality, we assume that $a_r = 1 \forall r \in R$. In that sense, our market size is 1 and $d_i^r(\mathbf{p})$ can be considered as probability of purchasing product i by a customer when exactly first r products are considered. Let $\theta(i)$ denote the position of product i . Then, we can formulate the expected demand as below:

$$E[d_i(\mathbf{p})] = \sum_r \alpha_r \cdot 1[\theta(i) \leq r] \cdot d_i^r(\mathbf{p}) \quad (3.4)$$

3.2.3 Two-position Competition with Two Players

Let us assume that there are just two differentiated positions on the page. We refer to the predominant position as “buy box” or the “first position”, which has only space for one product/seller. We call the other position as “consider all” or the “second position” which includes all sellers. In that sense, each customer only consider the first position/seller with probability α (i.e. $r = 1$) and consider both sellers with probability $1 - \alpha$ (i.e. $r = 2$).

We assume that all players are rational. The first result of rationality is that prices are always higher than cost i.e. $p_j \geq \frac{s}{(1-\beta)}$. Also, we assume that cost is not too high. It means that there is a positive demand for some prices higher than cost i.e. $\frac{s}{(1-\beta)} < \frac{1}{b_1}$, and $\frac{s}{(1-\beta)} < \frac{1}{b_2 - c_2}$. Finally, a technical assumption is that $b_2 - c_2 \geq b_1$. This implies that if market sizes are equal for $r = 1$ and $r = 2$, then $d_i^1(p, p) \geq d_i^2(p, p)$. This assumption is not restricting as products are substitutes to each other, we expect that existence of a competitor affects demand in negative way.

3.2.4 Price-sorted Binary and Probabilistic Ranking

We denote the marketplace operator’s ranking rule as binary, if the ranking is a deterministic function of prices. In an ascending price-sorted binary ranking, the

seller with the lowest price wins the first position until the next change in the prices. We assume that MO considers the prices of two sellers as the same if there is less than ϵ difference between their prices and allocate the first position to each player an equal fraction of time. In other words, a seller need to set a price at least ϵ below all competitors to earn the first position.

We denote the marketplace operator's ranking rule as probabilistic, if the marketplace operator assigns each seller to the first position with some probability due to its price. In that sense, the first position is shared among the sellers. More formally, when a customer arrives, seller i is at first position by probability $\gamma_i(\mathbf{p})$.

3.3 Games

3.3.1 Two-product Symmetric Bertrand Ranking

First, we study the pricing game between two third-party sellers with symmetric profit function (i.e. cost and demand parameters are equal for the two players) denoted as Game BR and Game PR. Note that MO does not compete as a seller. We assume that players play with full information (i.e. all players know the ranking rule and demand function of themselves). The demand functions in Bertrand two-position ranking models are as follows:

$$d_i(\mathbf{p}) = \alpha \cdot \gamma_i(p_i, p_j) \cdot (1 - b_1 p_i) + (1 - \alpha) (1 - b_2 p_i + c_2 p_j) \quad (3.5)$$

Ranking rule in Game BR is binary. This means that a seller win the first position if its price is lower than the competitor (i.e. $\gamma_i(\mathbf{p}) = 1$ if $p_i \leq p_j - \epsilon$, $\gamma_i(\mathbf{p}) = 0.5$ if $|p_i - p_j| \leq \epsilon$, $i \neq j$). However, in Game PR, ranking rule is probabilistic and probability of winning first position is dependent on the ratio of the prices: $\gamma_i(\mathbf{p}) = \frac{p_j}{p_i + p_j}$, $i \neq j$.

3.3.2 Two-product Asymmetric Bertrand Ranking

In a more general form of the Games BR and PR we assume that the profit function of the sellers are asymmetric. We study two cases of asymmetric demand and cost.

In the games with asymmetric cost, we assume that the asymmetry is endogenous and caused by cost. In that sense, the marginal cost of selling a product is different for each seller (i.e. $s_i \neq s_j, i \neq j$). Without loss of generality, we assume that $s_1 > s_2$. Let denote the games with probabilistic and binary ranking as Games PR-AS and BR-AS, respectively. In the games with asymmetric demand, we assume that each customer has a different valuation of each seller. This results in different demand levels for the products even if the sellers set prices the same. Without loss of generality let assume that seller one has higher demand. more formally:

$$d_1^1(\mathbf{p}) > d_2^1(\mathbf{p}) \quad (3.6)$$

$$d_1^2(\mathbf{p}) > d_2^2(\mathbf{p}) \quad (3.7)$$

To implement an exogenous demand asymmetry in Bertrand model, we assume that market size of two sellers are not identical. Then, we can modify demand function as follows, where δ is a positive value such that $0 < \delta < 1$.

$$d_1(\mathbf{p}) = \alpha \cdot \gamma_1(p_1, p_2) \cdot (1 - b_1 p_1) + (1 - \alpha) (1 - b_2 p_1 + c_2 p_2) \quad (3.8)$$

$$d_2(\mathbf{p}) = \alpha \cdot \gamma_2(p_2, p_1) \cdot (\delta - b_1 p_2) + (1 - \alpha) (\delta - b_2 p_2 + c_2 p_1) \quad (3.9)$$

Let Game PR-AD denotes the game with probabilistic ranking rule and Game BR-AD denotes the binary one.

3.4 Analyses

3.4.1 Structure of the Profit and Best Response of the sellers at the Games with Binary Ranking Rule

For a specific ranking rule, games get simplified to a price competition. Each seller would decide on the optimal price of her/his product. We call the optimal price of a seller when price of the other seller is known, as “best response.” To study the best response strategy, we define the total profit of a seller $\pi_i(p_i, p_j)$, as summation of two partial profit $\pi_i^1(p_i, p_j)$ and $\pi_i^2(p_i, p_j)$, where $r = 1$ and $r = 2$, respectively. In other words, the total profit of a seller can be defined as summation of profits

received from customers who consider the first position ($r = 1$) and customers who considers both positions ($r = 2$). More formally;

$$\pi_i(p_i, p_j) = \pi_i^1(p_i, p_j) + \pi_i^2(p_i, p_j) \quad (3.10)$$

$$\pi_i^1(p_i, p_j) = ((1 - \beta)p_i - s) d_i^1(\mathbf{p}) = ((1 - \beta)p_i - s) \cdot \alpha \cdot \gamma_i(p_i, p_j) \cdot (1 - b_1 p_i) \quad (3.11)$$

$$\pi_i^2(p_i, p_j) = ((1 - \beta)p_i - s) d_i^2(\mathbf{p}) = ((1 - \beta)p_i - s) (1 - \alpha) (1 - b_2 p_i + c_2 p_j) \quad (3.12)$$

In Equation 3.10, $\gamma_i(p_i, p_j)$ represents the fraction of time that seller i is in the first position. In a game with the binary ranking rule, $\gamma_i(p_i, p_j)$ is a discontinuous function with value equal to 0 or 1. Note that when $\gamma_i(p_i, p_j)$ is equal to zero, $\pi_i^1(p_i, p_j)$ is equal to zero, and subsequently, $\pi_i(p_i, p_j)$ is equal to $\pi_i^2(p_i, p_j)$. On the other hand, we define an auxiliary function denoted as $\pi_i^0(p_i, p_j)$, where $\pi_i^0(p_i, p_j) = \pi_i(p_i, p_j)$ if $\gamma_i(p_i, p_j) = 1$. More formally:

$$\pi_i^0(p_i, p_j) = ((1 - \beta)p_i - s) \cdot (\alpha \cdot (1 - b_1 p_i) + (1 - \alpha) (1 - b_2 p_i + c_2 p_j)) \quad (3.13)$$

Figure 3.1 illustrates an example of profit functions $\pi_i(p_i, p_j)$, $\pi_i^0(p_i, p_j)$, $\pi_i^1(p_i, p_j)$, and $\pi_i^2(p_i, p_j)$. Note that, we can interpret $\pi_i^0(p_i, p_j)$ as total profit of the seller, if s/he is placed in the first position permanently. Similarly, $\pi_i^2(p_i, p_j)$ can be interpreted as total profit of the seller, if s/he is placed in the second position permanently.

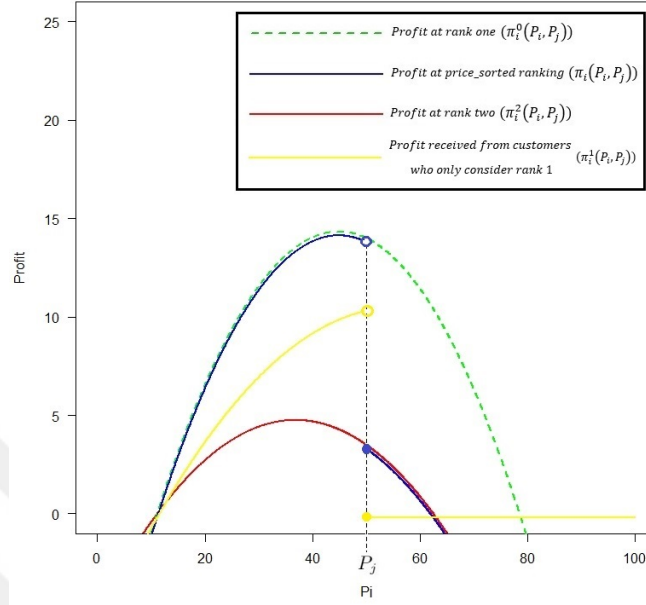
We search for the best response of seller i , denoted as $p_i^*(p_j)$ and defined as follows:

$$p_i^*(p_j) = \arg \max_{P_i} (\pi_i(p_i, p_j)) \quad (3.14)$$

As $\pi_i(p_i, p_j)$ is not a continuous function that jumps downward at $p_i = p_j$, to find the optimal price $p_i^*(p_j)$, we define auxiliary and constrained optimal prices as follows:

p_i^{2*} is the optimal price of seller i when, s/he is placed in the second position permanently.

p_i^{0*} is the optimal price of seller i when, s/he is placed in the first position permanently.



(a)

Figure 3.1: Profit functions examples where $p_j = 50$

p_i^{1*} is the optimal price of seller i that keeps her/him in the first position when binary ranking rule is implemented.

More formally:

$$p_i^{2*}(p_j) = \arg \max_{p_i} (\pi_i^2(p_i, p_j)) \quad (3.15)$$

$$p_i^{0*}(p_j) = \arg \max_{p_i} (\pi_i^0(p_i, p_j)) \quad (3.16)$$

and,

$$p_i^{1*}(p_j) = \arg \max_{p_i} (\pi_i(p_i, p_j)) \quad s.t. \quad \gamma_i(p_i, p_j) = 1 \quad (3.17)$$

To find the $p_i^{2*}(p_j)$ we can take derivative of $\pi_i^2(p_i, p_j)$:

$$\frac{\partial \pi_i^2(p_i, p_j)}{\partial p_i} = (1 - \alpha)(1 - \beta) \left((1 - b_2 p_i + c_2 p_j) - b_2 (p_i - s') \right) \quad (3.18)$$

$$\frac{\partial^2 \pi_i^2(p_i, p_j)}{\partial p_i^2} = -2(1 - \alpha)(1 - \beta)(b_2) < 0 \quad (3.19)$$

$$p_i^{2*}(p_j) = \frac{1 + c_2 p_j + b_2 s'}{2b_2} \quad (3.20)$$

Similarly, we can find $p_i^{0*}(p_j)$ by taking derivative of $\pi_i^0(p_i, p_j)$:

$$\begin{aligned} \frac{\partial \pi_i^0(p_i, p_j)}{\partial p_i} = & (1 - \beta) \left(\alpha \cdot (1 - b_1 p_i) + (1 - \alpha) (1 - b_2 p_i + c_2 p_j) - (\alpha \cdot b_1 + (1 - \alpha) b_2) (p_i - s') \right) \end{aligned} \quad (3.21)$$

$$\frac{\partial^2 \pi_i^0(p_i, p_j)}{\partial p_i^2} = -2(1 - \beta)(\alpha \cdot b_1 + (1 - \alpha) b_2) < 0 \quad (3.22)$$

$$p_i^{0*}(p_j) = \frac{1 + (1 - \alpha) c_2 p_j}{2(\alpha \cdot b_1 + (1 - \alpha) b_2)} + \frac{s'}{2} \quad (3.23)$$

Also, let $[\underline{p}_i(p_j), \overline{p}_i(p_j)]$ denote the price range for product i that keeps seller i in the first position (i.e. $\gamma_i(p_i, p_j) = 1$). We know that in an ascending-price-sorted binary ranking, $\underline{p}_i(p_j) = 0$ and $\overline{p}_i(p_j) = p_j - \epsilon$.

Figure 3.2 represents examples of $\pi_i(p_i, p_j)$, $\pi_i^2(p_i, p_j)$, and $\pi_i^0(p_i, p_j)$ at four different levels of competitor's price p_j . This figure shows how p_j affects $\pi_i(p_i, p_j)$, and the form of $p_i^*(p_j)$. In graph (a) $p_j = 50$ and $p_i^*(p_j) = p_i^{1*}(p_j) = p_i^{0*}(p_j)$. In the graph (b) and (c) $p_i^*(p_j) = p_i^{1*}(p_j) = p_j - \epsilon$, where $p_j = 30$ and $p_j = 25$, respectively. However, in graph (d), $p_i^*(p_j) = p_i^{2*}(p_j)$, where $p_j = 16$. As represented in the figure, we can categorize all cases into the three general groups. In the first group, competitor's price p_j is too high (e.g. case (a)), hence, the price that maximize the seller i 's profit is lower than competitor's price. This allows seller i to win the first position. In the second group, competitor's price p_j is too low (e.g. case (d)), thus, it is not optimal for seller i to set a lower price to win the first position. In such a case, seller i prefers to charge a higher price that maximizes his/her profit at the second position. In the third group, p_j is neither too low, nor too high (e.g.

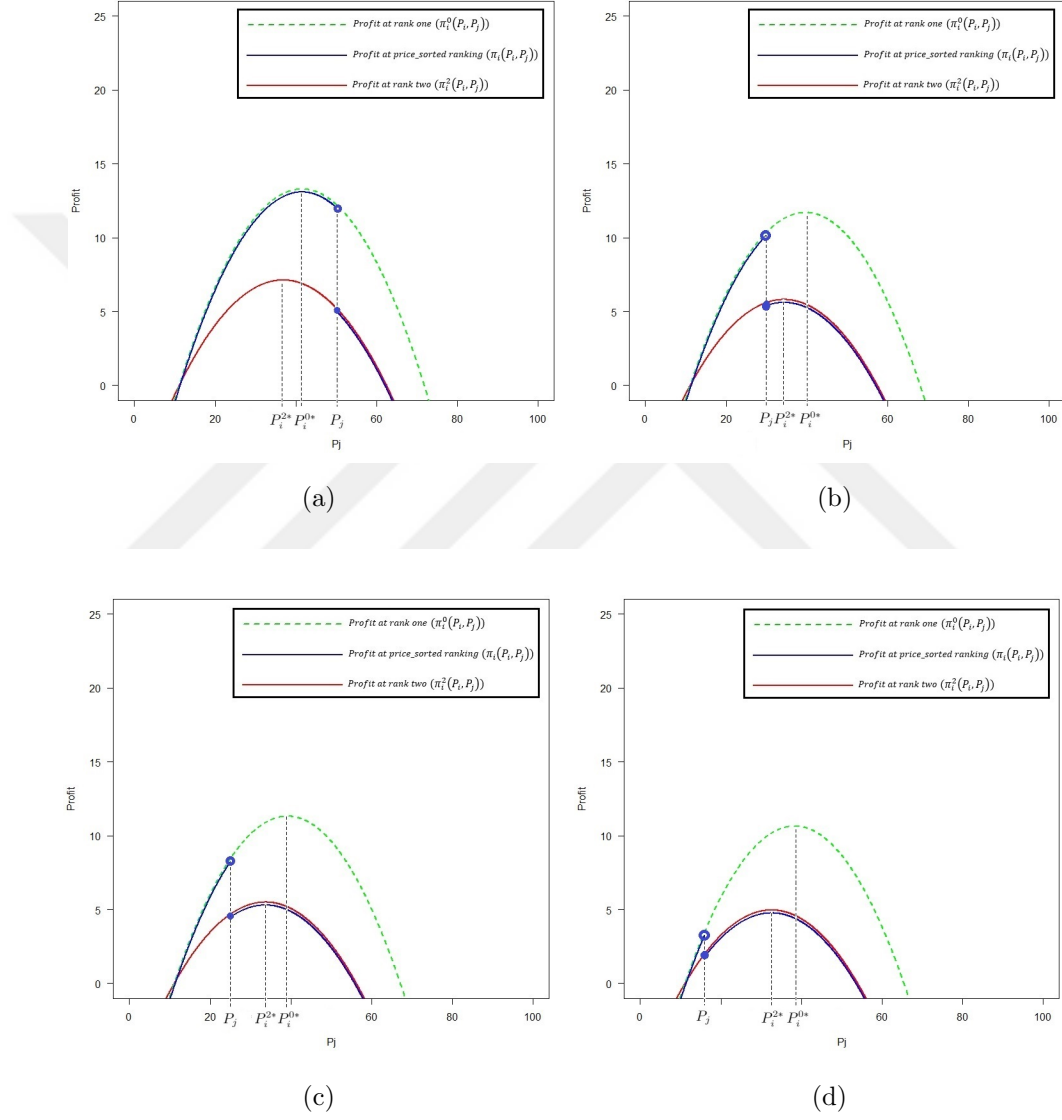


Figure 3.2: Profit functions examples where (a) $p_j = 50$, (b) $p_j = 30$, (c) $p_j = 25$, and (d) $p_j = 15$

cases b and c), thus, winning the first position is optimal for seller i and $p_j - \epsilon$ is best response. Proposition 3.1 characterizes the best response of a seller ($p_i^*(p_j)$) as discussed.

Proposition 3.1 *Best response in the Games with Binary Ranking Rule*

1. If $p_i^{0*}(p_j) \geq \bar{p}_i(p_j) = p_j - \epsilon$, then $p_i^{1*}(p_j) = \bar{p}_i(p_j)$, and $p_i^*(p_j) = p_i^{2*}(p_j)$ or $p_i^*(p_j) = \bar{p}_i(p_j) = p_j - \epsilon$.
2. If $p_i^{0*}(p_j) < \bar{p}_i(p_j) = p_j - \epsilon$, then $p_i^*(p_j) = p_i^{1*}(p_j) = p_i^{0*}(p_j)$.

Due to Proposition 3.1, when $\bar{p}_i(p_j) > p_i^{0*}(p_j)$, the optimal price of seller i is $p_i^{0*}(p_j)$. However, when $\bar{p}_i(p_j) \leq p_i^{0*}(p_j)$, the optimal price for seller i can be either $p_i^{2*}(p_j)$ or $\bar{p}_i(p_j) = p_j - \epsilon$. We characterize this decision using a threshold point defined below.

$$y_j^t = \min\{p_j | p_i^*(p_j) = p_i^{1*}(p_j)\} \quad (3.24)$$

$$T_i^1 = p_i^{1*}(y_j^t) \quad (3.25)$$

y_j^t is the minimum p_j value such that is optimal for seller i to compete for winning the first position by setting a lower price. In other words, $y_j^t = \min\{p_j | \pi_i^2(p_i^{2*}(p_j), p_j) < \pi_i(p_i^{1*}(p_j), p_j)\}$, where $\pi_i(p_i^{1*}(p_j), p_j)$ is the maximum profit that seller i can earn when it is in the first position. Also, $\pi_i^2(p_i^{2*}(p_j), p_j)$ is the maximum profit that seller i can earn when it is in the second position. Then, we define threshold 1, T_i^1 , as best response to y_j^t . This means that T_i^1 is the lowest price that winning the first position is optimal. In reaction to a price less than or equal to T_i^1 , staying in the second position with a price equal to $p_i^{2*}(p_j)$ maximize the profit of seller i . Formally, for any $p_j \leq T_i^1$ the best response of seller i is $p_i^{2*}(p_j)$.

We denote the optimal price when the competitor sets his/her price equal to $T_j^1 = T_i^1$ as Threshold 2 $T_i^2 = p_i^{2*}(T_j^1)$. Proposition 3.2 defines the relation between these two thresholds and their role in the equilibrium.

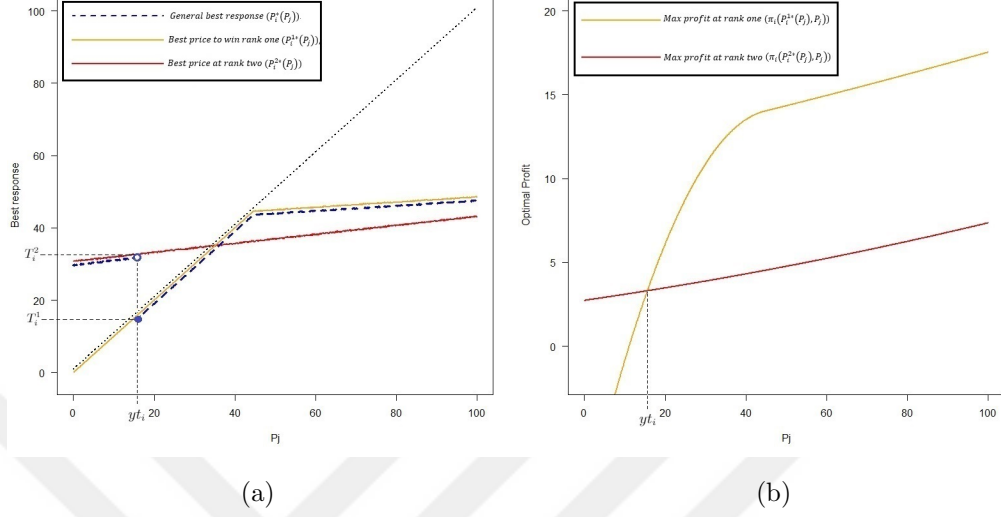


Figure 3.3: (a) Best responses and (b) corresponding profit in a game with binary ranking

Proposition 3.2 *Role of Thresholds T_i^1, T_i^2 , in Game BR*

- $T_i^1 \leq T_i^2$,
- In Game BR, only price range of $T_i^1 \leq p_i \leq T_i^2$ survive iterated elimination of strictly dominated strategy.

Figure 3.3 illustrates how we define y_j^t , T_i^1 , and T_i^2 at Game BR. Note that x-axis in all graphs is the competitor's price (p_j). Due to Proposition 3.1, we know that optimal price of seller i is either p_i^{1*} or p_i^{2*} . Figure 3.3(a) represents an example of p_i^{1*} , p_i^{2*} , and Figure 3.3(b) represents their corresponding profits. As represented in Figure 3.3(b), y_j^t is the price where two profit functions intersect. Then, we can define T_i^1 and T_i^2 in Figure 3.3(a) at price y_j^t . T_i^1 is value of p_i^{1*} (yellow graph) at price y_j^t and T_i^2 is value of p_i^{1*} (red graph) at price $y_j^t - \epsilon$.

3.4.2 Game BR: Two-player Symmetric Competition with Binary Ranking Rule

In this section, we investigate the existence of Nash equilibrium in a symmetric competition with the binary ranking rule. Due to Proposition 3.2, we find that there exists no pure strategy Nash equilibrium in Game BR. The following statements are a sketch of the proof:

1. Due to Proposition 3.2, any p_i such that $p_i < T_i^1$ or $p_i > T_i^2$ doesn't survive iterated elimination of strictly dominated strategies. Subsequently, cannot be the Nash equilibrium.
2. For any $T_i^1 < p_i \leq T_i^2$, $p_i - \epsilon$, cause higher profit and T_i^2 is best response to by T_i^1 . By definition of pure strategy Nash equilibrium, in an equilibrium no player has a beneficial deviation, mean that no other price has a higher profit. In that sense, pure strategy Nash equilibrium happen at a price set $(p_i^*(p_j), p_j)$ such that $p_j^*(p_i^*(p_j)) = p_j$, which is not possible at this case.

Although the Game BR has no pure strategy Nash equilibrium, any game with finite players and strategies has a mixed strategy Nash equilibrium. We denote the mixed strategy Nash equilibrium of Game BR by $e^{BR} = (\mathbf{p}^{BR}, \Sigma^{BR})$, where strategies (prices) denoted by $\mathbf{p}^{BR}$ are played with probability distributions denoted by Σ^{BR} . Recall that $\mathbf{p}^{BR}$ includes all possible prices between (and including) T_i^1 and T_i^2 . Figure 3.4(b) represents examples of the best response correspondence of Game PR. We provide more details about mixed strategy Nash equilibrium of Game BR in Section 3.5.1.

3.4.3 Game PR: Two-player Symmetric Competition with Probabilistic Ranking Rule

We define the Game PR as a two-product symmetric competition with the probabilistic ranking rule. Recall that, when ranking rule is probabilistic, profit function of sellers are continuous and differentiable. We show that the profit function of sellers is concave (see appendix A.4 for proof). Due to the literature, any game with a

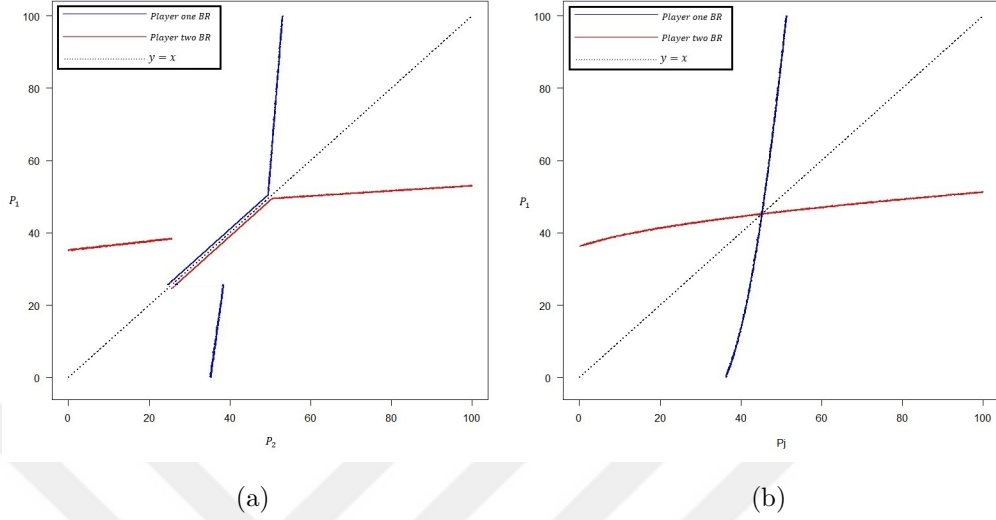


Figure 3.4: Best response correspondence of symmetric games with (a) binary and (b) probabilistic ranking rule

concave payoff function has at least one pure strategy Nash equilibrium. Also, any symmetric game has at least one symmetric Nash equilibrium.

Proposition 3.3 *Nash Equilibrium of Game PR*

There exists a unique symmetric pure strategy Nash Equilibrium of Game PR.

We denote the symmetric Nash equilibrium of Game PR as e^{PR} . Also, in all numeric examples, we observe that pure strategy Nash equilibrium of Game PR is unique. Figure 3.4(b) represents examples of the best response correspondence of Game PR. We provide more details about pure strategy Nash equilibrium of Game PR in Section 3.5.1.

3.4.4 Marketplace Operator's Profit in Symmetric Games

Recall that profit function of a seller is continuous when ranking rule is probabilistic. It means that any slight change in the price results in proportionally slight change in the profit received. However, profit function of a seller has a down-ward jump when ranking rule is binary. By this ranking rule, a seller can earn whole profit

received from the first position if he set a price at least ϵ below the competitor and nothing otherwise. It makes the competition more intense when ranking rule is binary. Thus, sellers set lower prices in Game BR in comparison with Game PR. Knowing that pricing strategy of the sellers is different in Games BR and PR, we are interested in optimal ranking rule that maximize the MO's profit.

In both games, marketplace operator receives commission from sales. We investigate the MO's profit, when sellers set prices the same (i.e. $p_i = p_j = p$). This is useful to investigate the optimal price as we can simplify the profit function of MO $\pi_m(p_i, p_j)$ to a single-parameter function $\pi_m(p)$ as follows:

$$\pi_m(p) = (\beta \cdot p) \cdot (\alpha \cdot (1 - b_1 p) + 2(1 - \alpha)(1 - b_2 p + c_2 p)) \quad (3.26)$$

Let $p = p^{mo*}$ be the price that maximizes the $\pi_m(p)$. Note that, $\pi_m(p)$ is a concave function and MO earn more profit as sellers' prices get closer to $p_i = p_j = p^{mo*}$. As presented in Equation 3.26, $\pi_m(p)$ is not a function of cost and ranking rule. However, both cost and ranking rule affect the pricing strategies of the sellers and subsequently, the MO profit. Proposition 3.4 states the effect of ranking rule on the profit of MO. Inequality 3.27 is a sufficient condition for Proposition 3.4. This inequality implies if both sellers set the prices the same, that demand is larger when $r = 2$ in comparison with $r = 1$. In other words, if both sellers set their prices the same, a customer who considers only position one ($r = 1$) has a higher probability to purchase the product rather than a customer who considers both products $r = 2$. It is reasonable assumption as competitor's product is substitute. Inequality 3.27 guarantees that $T_i^2 \leq e^{PR}$ in Proposition 3.4.

$$3b_1 < 2b_2 - 2c_2 \quad (3.27)$$

In Proposition 3.4, $\pi_m(e^{BR})$ denotes profit of MO when ranking rule is binary and sellers set the prices based on the mixed strategy Nash equilibrium of Game BR, e^{BR} . Similarly, $\pi_m(e^{PR})$ denotes profit of MO when ranking rule is probabilistic and sellers set the prices equal to pure strategy Nash equilibrium of Game PR, e^{PR} . Finally, $\pi_m(p^{mo*})$ denotes maximum profit of MO which is achieved if both sellers set the prices equal to p^{mo*} .

Proposition 3.4 *Marketplace Operator's Optimal Ranking Rule*

For any game that Condition 3.27 holds, there are cost values $\underline{s}$ and $\bar{s}$ such that:

- If $s \leq \underline{s}$:

$$T_i^1 \leq T_i^2 \leq e^{PR} \leq p^{mo*}$$

$$\pi_m(e^{BR}) < \pi_m(e^{PR}) < \pi_m(p^{mo*})$$

- If $s \geq \bar{s}$:

$$p^{mo*} \leq T_i^1 \leq T_i^2 \leq e^{PR}$$

$$\pi_m(p^{mo*}) > \pi_m(e^{BR}) > \pi_m(e^{PR})$$

In Proposition 3.4, we define $\underline{s}$ such that $\frac{\partial \pi_i^{PR}(p,p)}{\partial p} = 0$, if $p = p^{mo*}$. It guarantees that for any $s \leq \underline{s}$, $e^{PR} \leq p^{mo*}$. Thus:

$$\underline{s} = \frac{(2-\alpha)(\alpha b_1 + 4(1-\alpha)((1-\alpha)(2-\alpha)b_2 - \alpha b_1 + 2(1-\alpha)(b_2 - c)))}{2(\alpha b_1 + 2(1-\alpha)(b_2 - c))(2\alpha b_1 + 8(1-\alpha)b_2 + \alpha^2 b_1 - 4\alpha(1-\alpha)c)} \quad (3.28)$$

On the other hand, we define $\bar{s} = \max\{\bar{s}^1, \bar{s}^2\}$, where:

$$\bar{s}^1 = p^{mo*} \quad (3.29)$$

$$\bar{s}^2 = \frac{2b_2 - (1-\alpha)(b_2 - 2c_2)}{(\alpha b_1(3b_2 + c_2) + (1-\alpha)(3b_2 + 2c_2))} \quad (3.30)$$

$\bar{s}^1$ guarantees that all prices (including T_i^1 , T_i^2 , and e^{PR}) are higher than p^{mo*} . $\bar{s}^2$ guarantees that for any mixed strategy Nash equilibrium of Game BR, $\pi_m(T_i^2, T_j^2) > \pi_m(e^{BR}, e^{BR})$.

Proposition 3.4 shows that a single ranking rule cannot be optimal for MO in every case. In that sense, part 1 states that when the cost is low, it is optimal for MO to implement a probabilistic ranking. On the other hand, part 2 states that when the cost is high, a more competitive ranking rule (binary ranking) is optimal for MO because sellers set lower prices. Figure 3.5 represents examples of $\underline{s}$ and $\bar{s}$ based on different price elasticity parameters and values of α . Note that we assume that costs are not high such that there is positive demand for some prices higher than

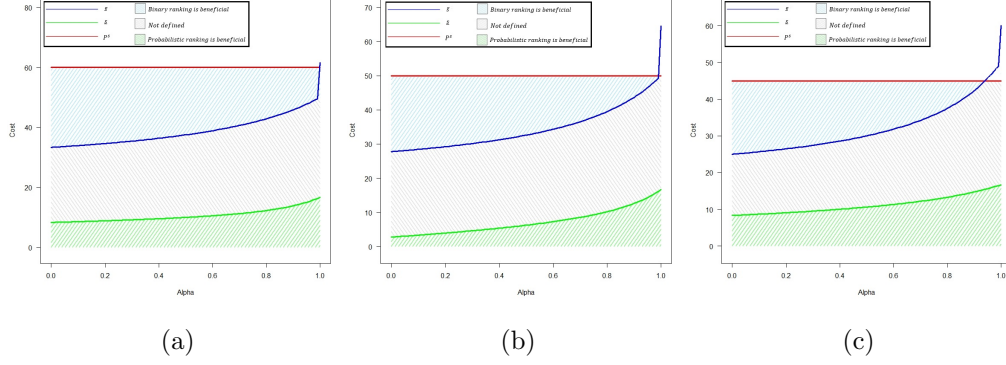


Figure 3.5: Examples of $\underline{s}$ and $\bar{s}$ where (a) $b_1 = 0.01, b_2 = 0.02, c_2 = 0.005$, (b) $b_1 = 0.01, b_2 = 0.02, c_2 = 0.002$, and (c) $b_1 = 0.01, b_2 = 0.03, c_2 = 0.01$

cost. In that sense, we denote the shut down price of the second position as p^s (i.e. $d_i^2(p^s, p^s) = 0$). p^s is represented by red lines in the figures. Due to Proposition 3.4, we know that in the green region of the graph, binary ranking rule earns a higher profit for MO, while in the blue region, probabilistic ranking rule does. In the grey region, we cannot characterize the comparison of the ranking rules. The result is this region is dependent on the exact value of e^{BR} . We discuss Proposition 3.4 and MO's profit numerically in Section 3.5.2.

3.5 Numerical Analyses and Results

In this section, we represent some numerical examples of the games. We numerically analyze the effect of parameters such as cost s , ranking α , commission rate β , and price elasticity parameters b_1, b_2 , and c_2 on optimal prices, thresholds and profits.

3.5.1 Symmetric Games, Effect of Parameters on the Equilibrium

Figures 3.6, 3.7, and 3.8 represent examples of T_i^1 and T_i^2 in the Game BR and e^{PR} in the Game PR different market values. Recall that in mixed strategy Nash equilibrium of Game BR, only prices between T_i^1 and T_i^2 have a positive probability to be played by the sellers.

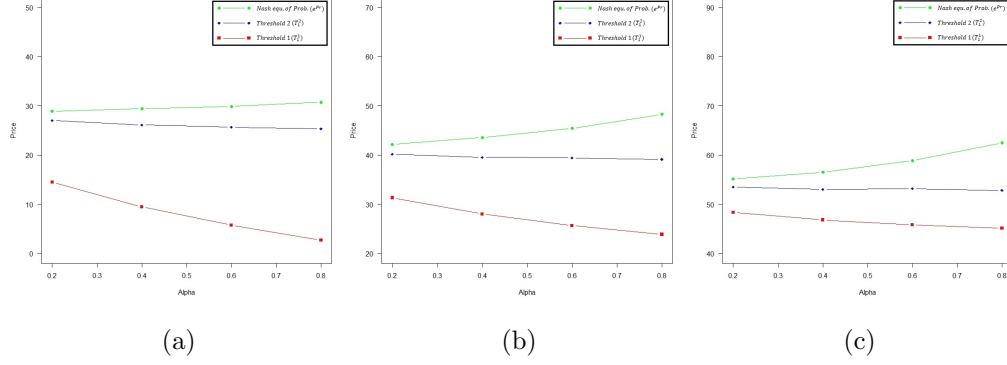


Figure 3.6: Effect of cost and alpha on the equilibrium price and thresholds where (a) $s = 0$, (b) $s = 20$, and (c) $s = 40$

In Figure 3.6 $b_1 = 0.01$, $b_2 = 0.02$, $c_2 = 0.05$, and $\beta = 0.1$. The value of s is 0, 20, and 40 in the Figures (a), (b), and (c), respectively. Each graph represents four examples with four levels of the alpha (i.e. $\alpha = 0.2, 0.4, 0.6, 0.8$). Note that, when α has a higher values, larger fraction of the customers only consider the first position. Subsequently, sellers receive more demand from the first position and less from the second position.

- As predicted, T_i^1, T_i^2 and e^{PR} are increasing in cost in all cases.
- We observe that T_i^1 is decreasing in α . This is intuitive, because by increasing α , the competition for the first position becomes more intense.
- There is no significant change in values of T_i^2 in the graphs. By the definition, T_i^2 is a specific point on the $p_i^{2*}(p_j)$ function and this function is not affected by α .
- In contrast with T_i^1, e^{PR} is increasing in α . We know that sellers prefer to charge a higher price when they are in the first position in comparison with the second position. This causes that Nash equilibrium be at a higher price when α is higher.

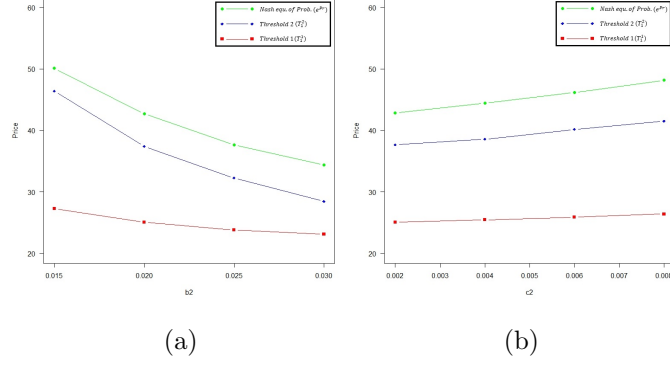


Figure 3.7: Effect of price elasticity parameters (a) b_2 and (b) c_2 on thresholds and equilibrium where $s = 20$

Figure 3.7 represents the effect of price elasticity parameters on Games BR and PR. Figures 3.7(a) consist of three groups of examples with different c_2 values (i.e. $c_2 = 0.002, 0.005, 0.010$, while $b_1 = 0.010$ and $b_2 = 0.020$). We know that the equilibrium price is increasing in c in the general form of differentiated Bertrand price competition. Our results confirm the same in the modified model of the Bertrand ranking competition. Figure 3.7(b) represents the effect of b_2 on the thresholds and equilibrium. As predicted, an increase in price elasticity b_2 results in a decrease in equilibrium prices. It is intuitive, because demand is decreasing in b_2 . Also, we observe the gap between T_i^1 and T_i^2 is decreasing in b_2 . When b_2 is higher, the demand in the second position is lower, which results in a lower T_i^2 .

We investigate more than 50 numerical examples of Game BR with different values of α , β , cost, and price elasticity parameters. We find all possible symmetric mixed strategy Nash equilibria of these cases. As proposed in Proposition 3.2, all non-dominated strategies for mixed strategies are between T_i^1 and T_i^2 . We define ϵ value such that there exist 10 to 30 strategies between these thresholds (Note that, if value of ϵ get too high, the number of strategies between two thresholds narrows down to zero, which results in nonrepresentational pure strategy equilibrium. On the other hand, if value of ϵ get too high, the high number of strategies between two

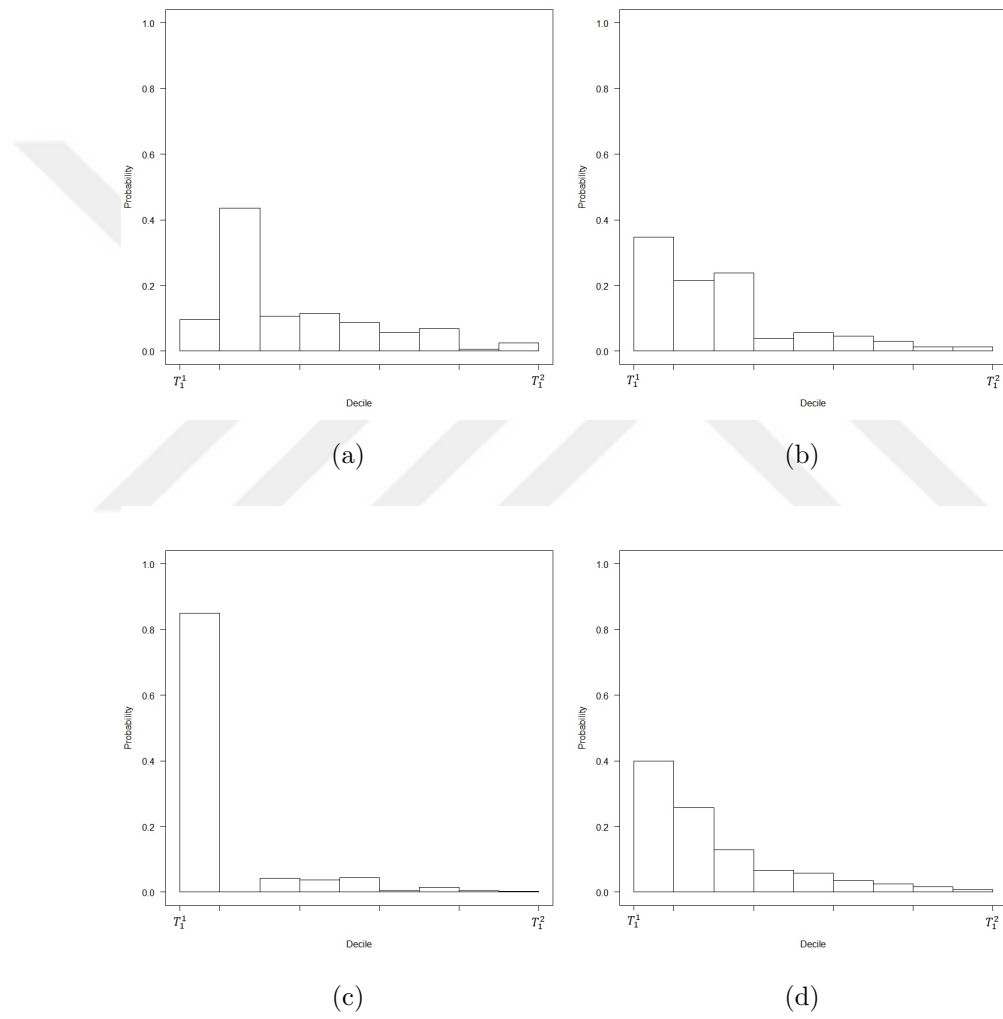


Figure 3.8: Mixed strategy Nash equilibrium probability distribution

thresholds makes probability distribution calculation Untraceable). In any mixed strategy Nash equilibrium, there can exist some strategies with zero probability. In that sense, we consider any combination of strategies with positive probabilities in that region. Then, for each combination we check the possibility of a mixed strategy (In a mix strategy equilibrium, all strategies which are played with a positive probability must be best response. It implies that these strategies must have equal payoffs that is higher than payoff of any strategy with positive probability). Finally, to represent the probability distribution over the strategies, we divide all strategies into ten sorted deciles from T_i^1 to T_i^{10} . Figure 3.8 represents the probability distribution of the equilibria in each decile for three examples (Figures 10(a), 10(b), and 10(c)) and the average of all cases (Figure 3.8(d)). As shown in the figure, players are more likely to set the prices in the first deciles (closer to T_i^1) rather than last deciles (closer to T_i^{10}). Also, we observe that in all cases the first three deciles cover 78% (/63%) and the first five deciles cover 91% (/84%) of mix strategies probability distribution on average (/at least) in all 50 cases.

3.5.2 Marketplace Operator and Sellers' Profit in Symmetric Games

In this section, we investigate profit of sellers and MO in different cases. We compare effects of two different ranking rules on the profits at different costs and α values numerically and some examples are presented in Figures 3.10 and 3.9.

Figure 3.9 represents sellers' profit at different symmetric cases. Note that sellers are motivated to set lower prices when there price-based competition for ranking in compare with the case where rankings are independent of prices. As mentioned, competition at binary ranking is more intense that cause the sellers even set lower prices in comparison with probabilistic ranking. Subsequently, sellers earn less profits at all of the binary ranking cases. Figure 3.10 represents MO's profit at different symmetric cases. The results of Figure 3.10 are aligned with Proposition 3.4. The binary ranking rule causes a higher profit for MO in the higher cost values and the probabilistic rule causes a higher profit for MO in the lower cost values. The difference in profits is bigger for all parties when cost is low.

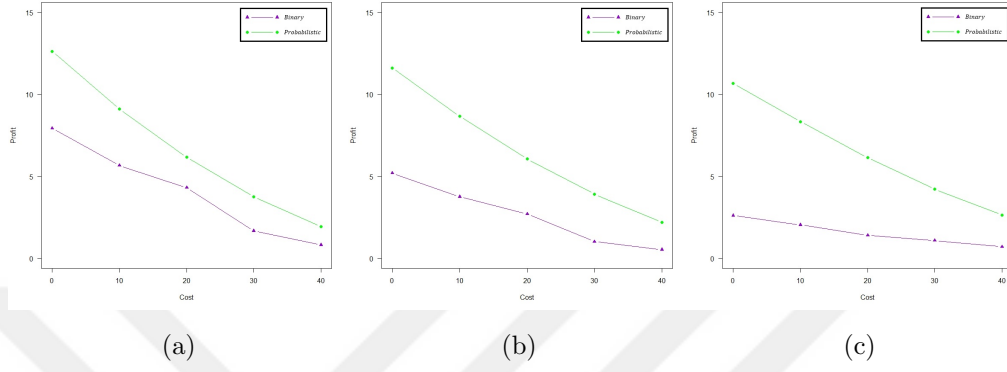


Figure 3.9: Sellers' profit at symmetric games where (a) ($\alpha = 0.4$), (b) ($\alpha = 0.6$), and (c) ($\alpha = 0.8$)

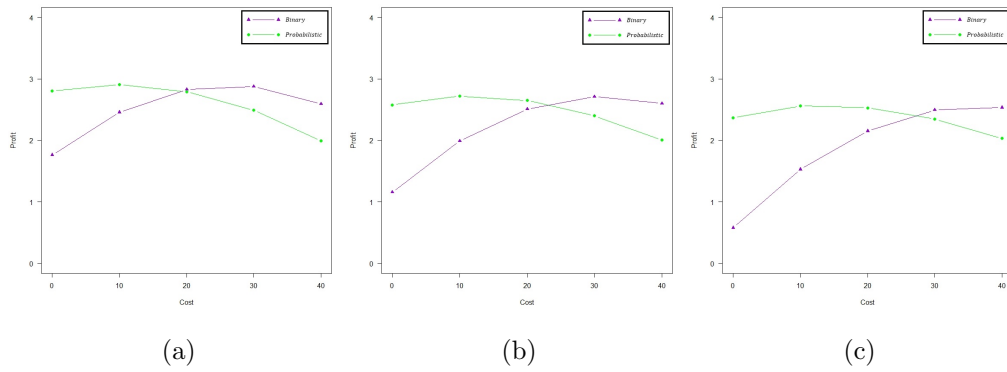


Figure 3.10: MO's expected profit at symmetric games (α is equal to 0.4 (a), 0.6 (b), and 0.8(c))

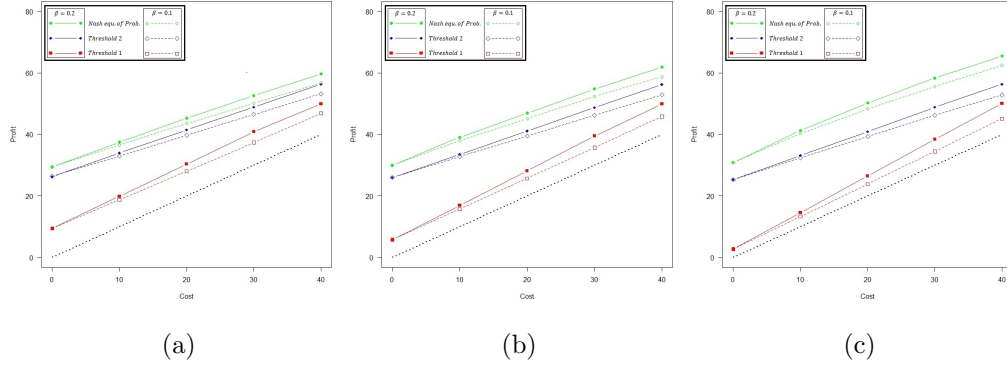


Figure 3.11: Effect of β on the prices, thresholds and equilibrium, where (a) ($\alpha = 0.4$, (b) ($\alpha = 0.6$, and (c) ($\alpha = 0.8$

3.5.3 Effect of Beta on the Profits and Customers' Surplus

β is another decision variable for MO after the ranking rule. β directly affects the profit of the MO and the sellers. It also affects the pricing strategies of the sellers which subsequently affect customers' experience of marketplace. We observe that the thresholds and equilibria change when we use different β values. To illustrate the effect of β , we present examples where β is increased from 0.1 to 0.2. Figure 3.11 shows that how prices are inflated when β is doubled. Figure 3.12 illustrates the effect of β value on the market metrics such as MO's and sellers' profit and demand satisfaction rate and consumer surplus. We also calculate the total surplus of each policy by adding up MO's and sellers' profit and customer's surplus. Figure 3.13 represents the total surplus of the market for different α and β values.

- We observe that higher commission rate β acts similar as cost multiplier. While T_i^1, T_i^2 and e^{PR} don't change when cost is zero, for any positive cost value all T_i^1, T_i^2 and e^{PR} are increasing in β .
- As we can see in Figure 3.12(a), the increased value of the β almost doubled the MO's Profit. On the other hand, as expected, it causes a decrease in sellers' profit.

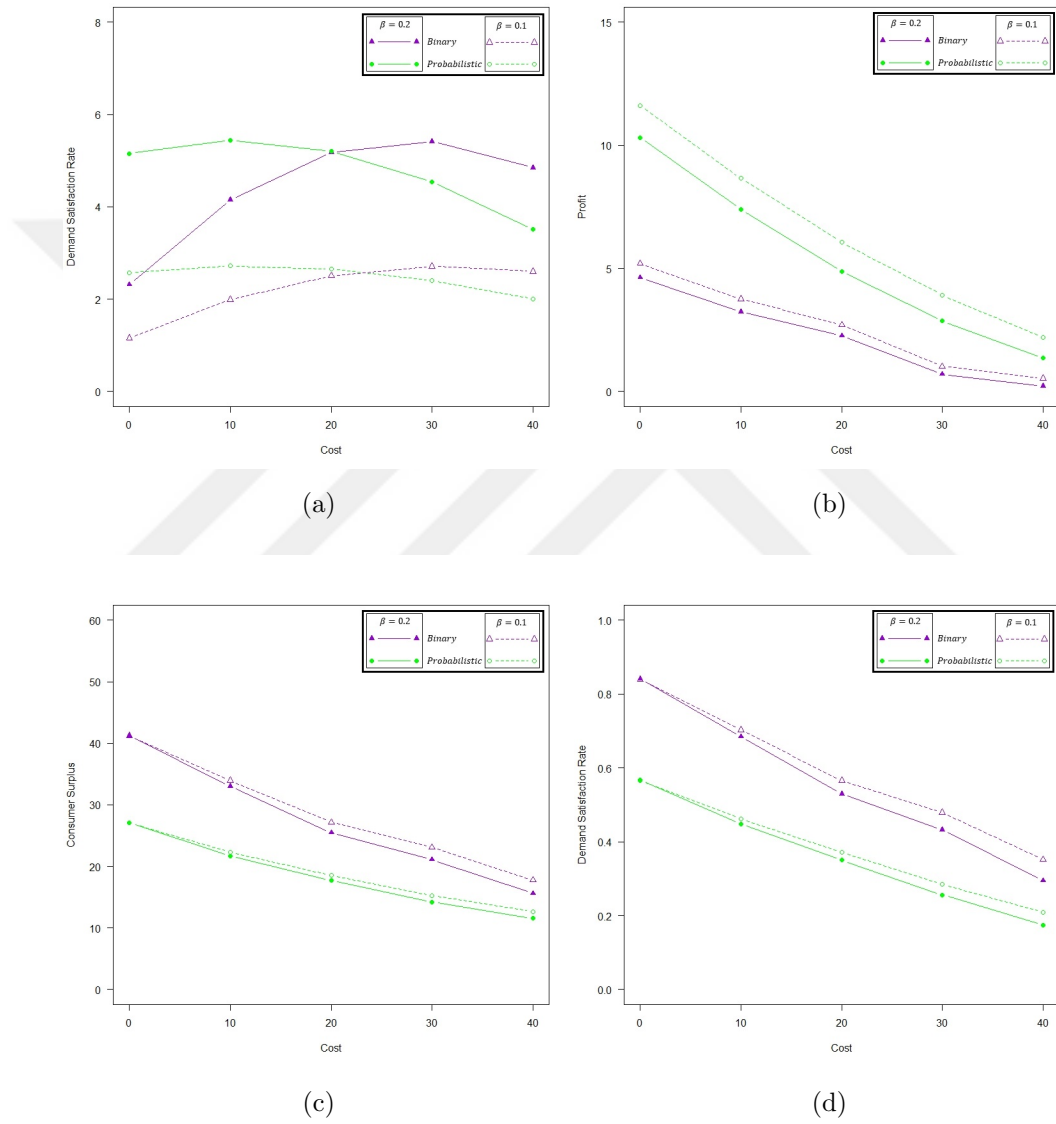


Figure 3.12: Effect of β on the MO's profit (a), seller's profit (b), demand satisfaction rate (c), and demand surplus (d))

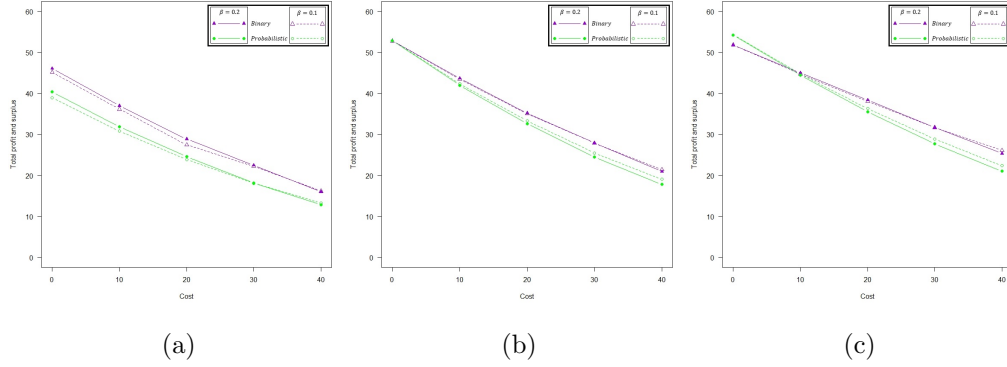


Figure 3.13: Effect of β on the total surplus of the market where (a) $\alpha = 0.4$, (b) $\alpha = 0.6$, and (c) $\alpha = 0.8$

- The customers are affected by β value through prices. In that sense, consumer surplus and demand satisfaction rate is decreased in any case that prices are increased.
- Total surplus is calculated as equally weighted summation of the MO and sellers' profit and consumer surplus. We observe that the effect of β on total surplus is insignificant, specially at lower cost levels. However, the total surplus is effected by ranking rules and α values. For higher α values total surplus is aligned with MO's profit of each ranking. It means that the total surplus of the market is higher at probabilistic ranking in lower cost values and binary ranking cause a higher total surplus at higher cost levels. In contrast, for lower α values, binary ranking rule has a higher total surplus.

3.6 Asymmetric Games

3.6.1 Analyses

The structure of competition in asymmetric games is similar to symmetric games. In asymmetric Games AS and AD with the probabilistic ranking rule, sellers' profit functions are concave, similar to Game PR. It means that there is a pure strategy

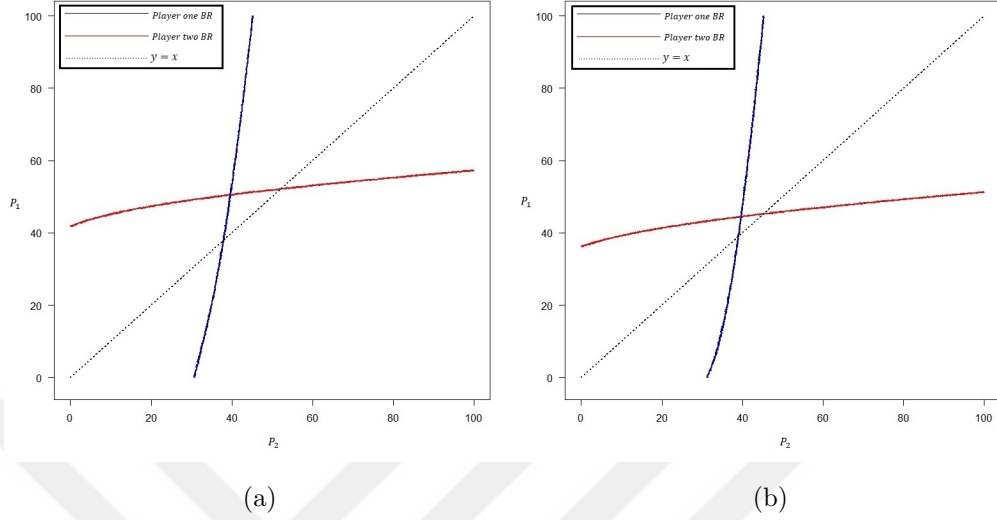


Figure 3.14: Best response correspondence of (a) asymmetric cost and (b) asymmetric demand game with probabilistic ranking

Nash equilibrium of Games PR-AS and PR-AD denoted by e^{PR-AS} and e^{PR-AD} . Figure 3.14 represents two examples of best response correspondence graphs of Games PR-AS and PR-AD. As is observed in the graphs, the prices of the sellers are not the same in the Nash equilibrium of asymmetric games.

In asymmetric games with the binary ranking rule, we observe a pure strategy in some limited conditions (When the difference in demand or cost is too high). We refer to these conditions as extreme asymmetry in the rest of the paper. In all asymmetric cases, we can define T_i^1 and T_i^2 as well as symmetric games. However, in asymmetric games, $T_2^1 \neq T_1^1$ and $T_2^2 \neq T_1^2$. Proposition 3.5 characterizes the asymmetry of thresholds and mixed strategy of asymmetric games.

Proposition 3.5 *Nash Equilibrium of Games BR-AS and BR-AD*

For Games BR-AS and BR-AD:

1. $T_1^1 > T_2^1$
2. If $p_2^{0*}(T_1^1) < T_1^1$, there exists a pure Nash strategy Nash equilibrium.

3. If $p_2^{0*}(T_1^1) > T_1^1$, there exists no pure Nash strategy Nash equilibrium and price range of $T_1^1 \leq p_i \leq T_1^2$ survive iterated elimination of strictly dominated strategy.

Proposition 3.5 sheds light on the conditions of the Nash equilibrium of Games DR-AS and DR-AD. As mentioned, in the extreme asymmetry cases, there exists a pure strategy Nash equilibrium of the game, when equations $p_1^*(p_2^*) = p_1^{2*}(p_2^*)$, and $p_2^*(p_1^*) = p_2^{0*}(p_1^*)$ hold. This means that player 1 prefers to stay in the second position and optimize his/her profit at a price higher than player 2. Meanwhile, the price that optimizes player 2's profit keeps player 2 in the first position. In other words, there is a cross point in best response correspondence of the games, where player 1 is in the second position and player 2 is in the first position. When the asymmetry is not extreme (i.e. $p_2^{0*}(T_1^1) > T_1^1$), there exists no pure strategy Nash equilibrium and players would behave similar to the Game BR. In that sense, rational players would end up playing between two thresholds. Remember that we assume that player 1 has a higher cost in Game BR-AS (/more demand in Game BR-AD) which causes the optimal price of player 1 be higher than player 2. Formally, $p_1^{0*}(p) > p_2^{0*}(p)$ and $T_1^1 > T_2^1$. It means that, for any competing price $p_2 < T_1^1$, player 1 would prefer to charge a higher price staying in the second position and don't compete for the first position. Vice versa, seller 2 would compete for the first position at any price that seller 1 would. This shows that T_1^1 is lower bound of pricing for rational sellers and price range of $T_1^1 \leq p_i \leq T_1^2$ survive iterated elimination of strictly dominated strategy. Figure 3.15 illustrates two examples of best response correspondence of asymmetric games. Figure 3.15(a) represents a case with a pure strategy Nash equilibrium, while Figure 3.15(b) represents the case without a Nash equilibrium.

3.6.2 Numerical Analyses, Effect of Parameters on the Equilibrium

As mentioned in Section 3.3, we study two types of asymmetry of cost and demand. Also, we assumed that seller 1 has a higher cost per demand (/has higher demand) than seller two when the cost (/demand) is asymmetric. By Proposition 3.5, we know that $T_1^1 > T_2^1$. In Game BR-AS, we know that T_1^1 is independent of cost

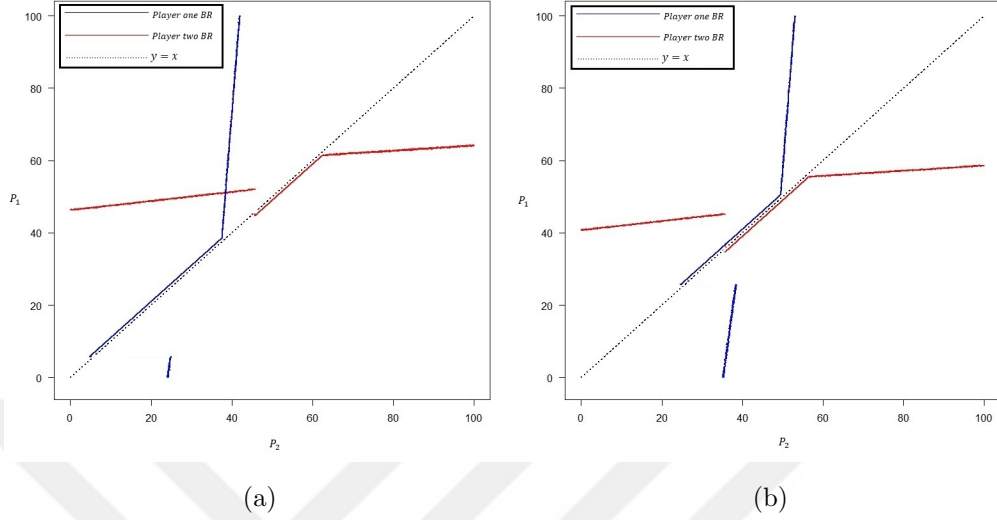


Figure 3.15: Best response correspondence of (a) asymmetric cost and (b) asymmetric demand game with binary ranking

of seller 2 s_2 . For example, T_1^1 is the same either cost of seller 2 be 0, 10 or 20. Table 3.1 represents examples of T_1^1 in different cases of Game BR-AS. Game BR-AD, follows the similar situation where T_1^1 is independent of demand function of seller 2 d_2 . Table 3.2 represents examples of T_1^1 in different cases of Game BR-AD. As mentioned, in Table 3.1, T_1^1 is the same for cases 1 and 2 and cases 3, 4 and 5, because the cost s_1 is the same. Similarly, in Table 3.2, cases 1 and 11, cases 2 and 12 and so on has the same T_1^1 value because a_1 is the same. Also, we can observe that in Games PR-AS and PR-AD asymmetry in cost or demand results in asymmetric Nash prices of the sellers.

3.6.3 Marketplace Operator's Profit in Asymmetric Games

We illustrate some examples of asymmetric cases and related profit of MO. We compare effect of two different ranking rule on the MO's profit for different α values. Figure 3.16 represents examples of games with asymmetric cost and Figure 3.17 represents examples of games with asymmetric demand. These results are similar to

Table 3.1: Asymmetric cost game (Game 3) examples

Case	α	a_1	a_2	s_1	s_2	T_1^1	T_1^2	e_1^2	e_2^2
1	0.4	1	1	20	0	28	39.8	41.9	31.4
2	0.4	1	1	20	10	28	39.5	42.7	37.6
3	0.4	1	1	40	0	46.8	53.1	53.6	32.95
4	0.4	1	1	40	20	46.8	53.2	55.1	45.2
5	0.4	1	1	40	30	46.8	53.3	55.9	50.9
6	0.8	1	1	20	0	23.8	39.3	46.1	33.8
7	0.8	1	1	20	10	23.8	39.2	47.2	41.2
8	0.8	1	1	40	0	45.1	53	59.2	35.7
9	0.8	1	1	40	20	45.1	53	61	49.7
10	0.8	1	1	40	30	45.1	52.9	61.7	56.2

symmetric cases and aligned with Proposition 3.4, when we compare of MO's profit by two ranking rules. However, we can see that binary ranking generally performs better in asymmetric cases in comparison with symmetric cases.

3.7 Existence of MO as a Seller

Several marketplace operators provide services as a seller at their marketplace such as Amazon. When a MO exists as a seller at the competition, it may prefer to rank itself higher than other sellers regardless of any ranking rule. This may help MO to earn more profit in short term. However, it can lead to dissatisfaction of third-party sellers and customers in long term. The observations by Chen et al. (2016) show that MO favors himself in the ranking, albeit third-party sellers have a chance to win the buy box.

To study the case, we define a two-player game where MO exist as a seller (denoted by $i = m$) in competition with one third-party seller (denoted by $i = 1$). We still investigate ascending price-sorted ranking rules while MO can favor himself

Table 3.2: Asymmetric demand game (Game 4) examples

Case	α	a_1	a_2	s_1	s_2	T_1^1	T_1^2	e_1^2	e_2^2
1	0.4	1	0.8	0	0	9.4	26.4	28.6	24.4
2	0.4	1	0.8	10	10	18.7	33.1	36	31.6
3	0.4	1	0.8	20	20	28	39.5	42.6	38.4
4	0.4	1	0.8	30	30	37.3	46.4	49.5	44.8
5	0.4	1	0.8	40	40	46.8	52.9	56	51.3
6	0.6	1	0.8	0	0	5.7	25.8	28.8	25.4
7	0.6	1	0.8	10	10	15.7	32.9	37.1	32.7
8	0.6	1	0.8	20	20	25.6	39.5	44.6	39.9
9	0.6	1	0.8	30	30	35.7	46.1	44.6	39.9
10	0.6	1	0.8	40	40	45.8	52.9	58.1	53.3
11	0.4	1	0.6	0	0	9.4	26.2	27.6	19.3
12	0.4	1	0.6	10	10	18.7	32.9	35.1	26.4
13	0.4	1	0.6	20	20	28	39.5	42.1	32.8
14	0.4	1	0.6	30	30	37.3	46.4	48.7	39.4
15	0.6	1	0.6	0	0	5.7	25.8	27.8	19.8
16	0.6	1	0.6	10	10	15.7	32.5	36.3	27.3
17	0.6	1	0.6	20	20	25.6	39.2	43.6	33.8
18	0.6	1	0.6	30	30	35.7	46.1	50.6	40.5

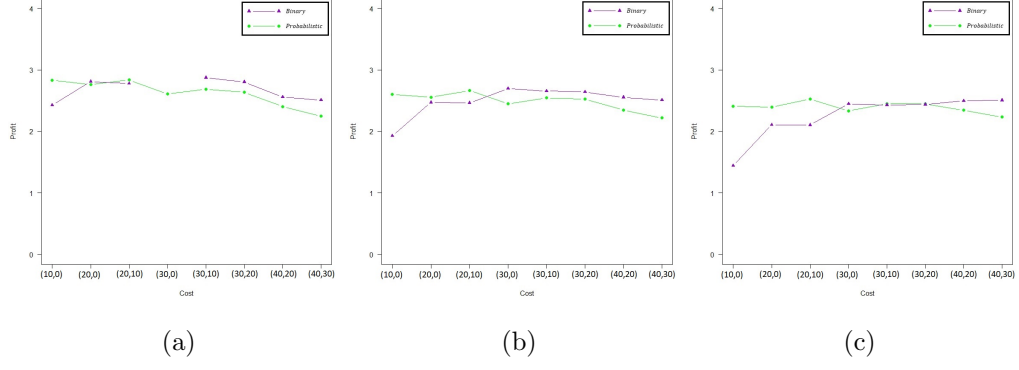


Figure 3.16: MO's profit at asymmetric cost games where (a) ($\alpha = 0.4$), (b) ($\alpha = 0.6$), (c) ($\alpha = 0.8$)

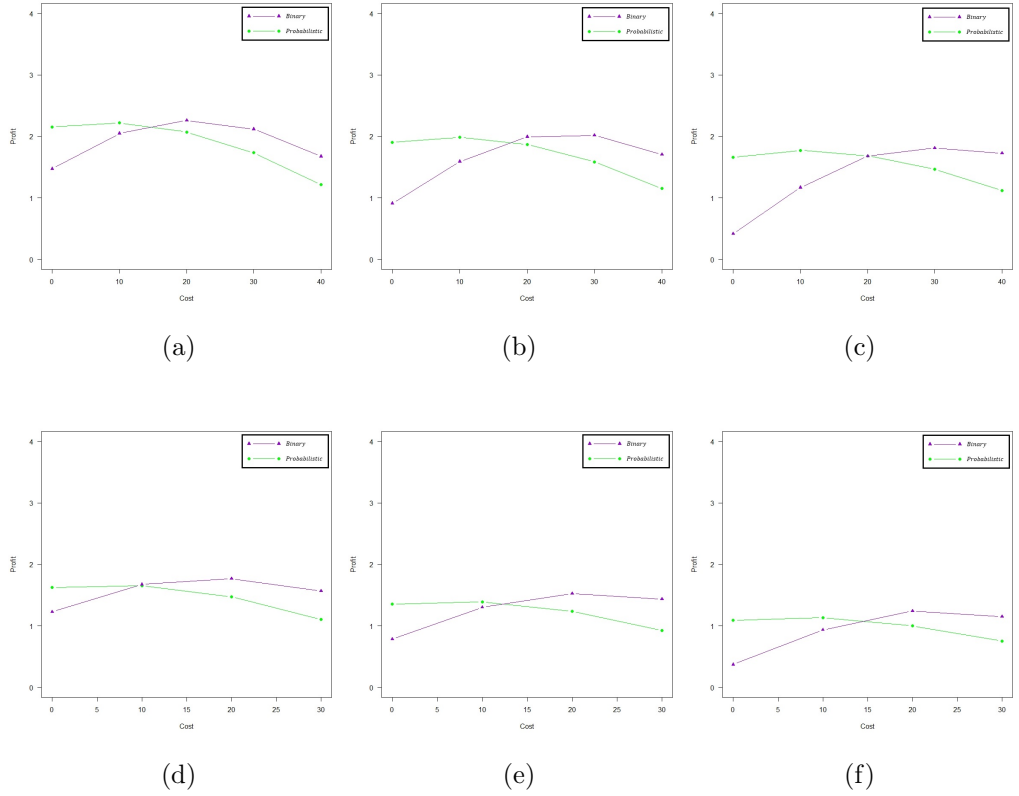


Figure 3.17: MO's profit at asymmetric demand games where (a) ($\alpha = 0.4$), (b) ($\alpha = 0.6$), (c) ($\alpha = 0.8$)

in the ranking by defining a margin ν , which is subtracted from MO's price for ranking consideration. In other words ranking take place as if MO has set $p_m - \nu$. Formally, when ranking rule is deterministic third party seller win the first position if its price is more than ν lower than the MOs' price (i.e. $\gamma_1(\mathbf{p}) = 1$ if $p_1 \leq p_m - \nu - \epsilon$, $\gamma_1(\mathbf{p}) = 0.5$ if $|p_1 - (p_m - \nu)| \leq \epsilon$). Similarly, when ranking rule is probabilistic, $\gamma_1(\mathbf{p}) = \frac{p_m - \nu}{p_m + p_1}$.

We provide numerical examples of the game with different cost and margin levels (i.e. $\nu = 0, 5, 10, 20, 40$). Note that when $\nu = 0$, the ranking rules take place as they do in Sections 4, 5, and 6. On the other hand, when the margin is too high such as $\nu = 40$, the deterministic ranking rule becomes a static ranking rule where MO is always in the first position and third-party seller is always in the second position. This static ranking strategy leads to a pure strategy Nash equilibrium in the game. Figures 3.18 and 3.19 represents the profit of MO, third-party seller and total supply-side in two cost levels $s = 0$, and $s = 30$. Followings are some conclusions of these results. First, profit of MO is increasing in margin in all cases and both ranking rules. Recall that in Sections 5 and 6 we show that sellers always prefer the probabilistic ranking while MO prefers the deterministic one when cost is high. We observe that when MO participate as seller, he prefers probabilistic ranking rule at any cost unless there is high margin and static ranking. Second, A third-party seller always earn the maximum profit when the ranking rule is probabilistic and there is no margin. Finally, total profit of supply-side is not affected by margin when ranking rule is neither probabilistic nor static. It means that, margin and ranking rule just define how would parties share the total profits. However, as expected, deterministic ranking rule makes competition more intense which decrease the total profit of supply-side.

Due to the branding and advertisement strategies, customers may prefer MO over third-party sellers. In this case, one can assume that demand is asymmetric and MO receive higher demand. We provide numerical examples where market size of MO and third-party seller is different in their demand function similar to asymmetric Games DR-AD and PR-AD (i.e. $a_m = 1$ and $a_1 = 0.6$). Figures 3.20

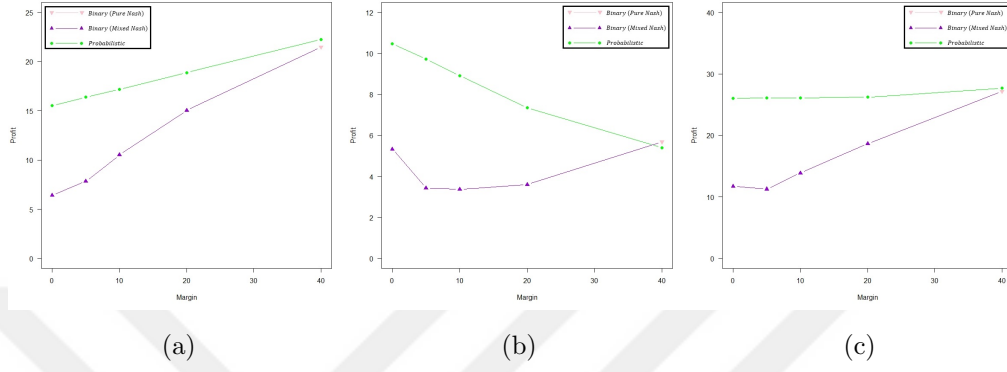


Figure 3.18: Profit of (a) MO, (b) third-party seller, and (c) total supply-side profit where $s = 0$

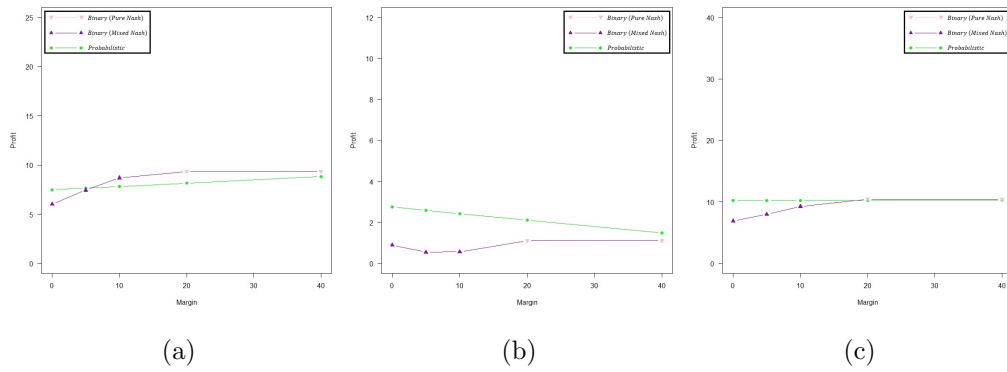


Figure 3.19: Profit of (a) MO, (b) third-party seller, and (c) total supply-side profit where $s = 30$

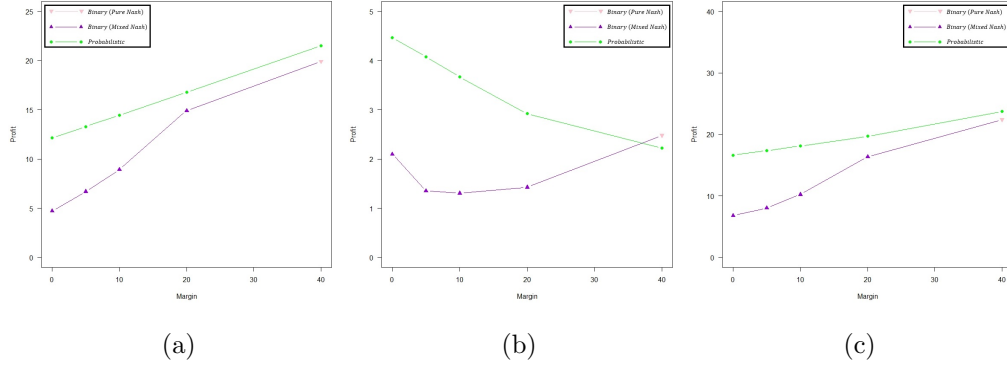


Figure 3.20: Profit of (a) MO, (b) third-party seller, (c) total supply-side when demand is asymmetric and $\alpha = 0.6$, and $s = 0$

and 3.21 represents profit of MO, third-party seller, total profit of supply-side parties, demand surplus, and total market surplus in different margin levels where cost is zero ($s = 0$) and demand is asymmetric.

Note that a higher margin value causes a higher probability of winning the first position for MO who would receive larger demand. As a result MO exploit first position better, that means MO satisfy more demand and earn higher profit at the first position in comparison with third-party seller. Hence, high margin value leads to the following results when demand is asymmetric. First, in contrast with the symmetric case, total supply-side profit is increasing in margin even when the ranking rule is probabilistic. Second, similar to the symmetric case, margin cause higher profits for MO and lower profit of the third-party seller. However, in asymmetric case, MO can exploit the first position better because he has relatively larger demand. Third, even though prices are increasing in the margin, demand surplus and rate are increasing in the margin when the ranking rule is deterministic. Finally, the total surplus of the market is increasing in the margin when demand is asymmetric with both deterministic and probabilistic ranking rules. In other words, the overall market benefits from a higher margin value and existence of MO at the first position.

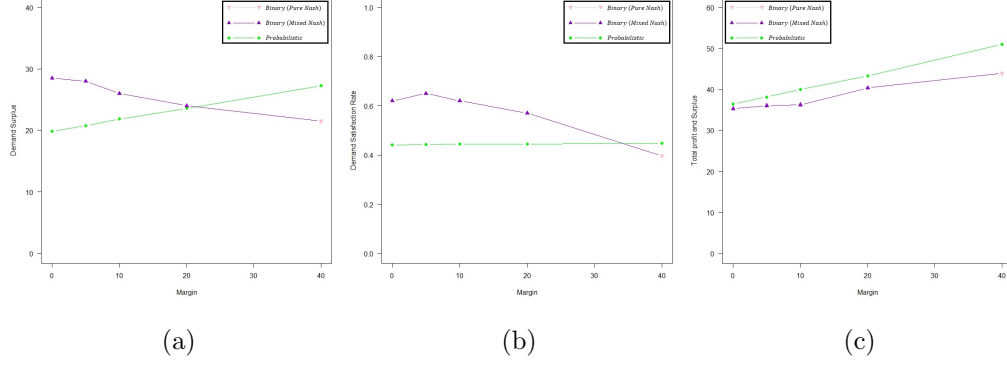


Figure 3.21: (a) Demand surplus, (b) demand satisfaction rate, and (c) total market surplus when demand is asymmetric and $\alpha = 0.6$, and $s = 0$

3.8 Conclusion

In this chapter, we study a decentralized ranking and pricing decision in an online marketplace. We consider the case where more than one seller offer some products that are identically the same or near substitutes. In such an environment, empirical studies show that marketplace operators such as Amazon tend to rank sellers with lower prices, higher (Chen et al. 2016). We model the problem as a game, where the marketplace operator decides on the ranking rule and based on the ranking role sellers play price competition in the market which affects their demand directly (by price) and indirectly (by ranking). We propose two ascending price-sorted ranking rules named deterministic and probabilistic. In the deterministic ranking rule, any seller with the lowest price is displayed at the first position permanently. In probabilistic ranking all seller has a chance to win the first position, however, a seller with lower price has a higher probability to be displayed at the first position. We show that there exist a Nash equilibrium of the games with probabilistic ranking while games with the deterministic ranking rule don't have any pure strategy Nash equilibrium in most of the cases (unless there is an extreme asymmetry between sellers).

Results show that sellers always prefer the probabilistic ranking and customers

prefer deterministic ranking. However, MO's preference of ranking rule is dependent on the cost of products. In that sense, MO earns more profit by probabilistic ranking when the cost is low and earns more by deterministic ranking when the cost is high. Also, we observe that the total surplus of the market is aligned with MO's profit.

We investigate the commission rate as another decision variable of MO. It is obvious that the commission rate directly affects the MO's profit. However, the combination of the ranking rule and the commission rate is a strategic decision for MO. MO can favor himself and sellers by a less competitive ranking rule (probabilistic) and a high commission rate. On the other hand, MO can pursue a strategy to attract more customers with lower profit margins by adjusting a more intense ranking rule (deterministic) and a lower commission rate. Finally, we extend our study with games in which MO participates as a seller. Practically, MO intends to place his product at the first position regardless of any ranking rule. This strategy increases the profit of MO in any case, however, this strategy also increases both total supply-side profit and customers demand surplus if the demand functions are asymmetric and MO has a larger demand than a third-party seller. In other words, in an asymmetric competition entire market benefits from such a strategy.

3.9 Extensions and Future Research

We provide a vast analyses of different cases where two seller compete in an market, however, in an online marketplace such as amazon there can exist more than two competing sellers. Note that propose the appropriate price competition and ranking model in Section 3.2 in a general form that is suitable for multiple sellers (more than two sellers). However, our study and analyses can be generalize and extend for multiple sellers. Also, in this study, we just propose only two price-based ranking rules. In that sense, our study can be extended by a more flexible price-based ranking rule or other ranking rules. For instance, Amazon already declares the factors other than price such as fulfillment method as effective parameters in the ranking.

Chapter 4

RANKING AND PRICING IN AN ONLINE RETAILER WITH LIMITED INVENTORY

In this chapter, we study product ranking and pricing at e-commerce platforms, where customers are generally window shoppers. Window shoppers are a type of customer that browse online stores as a form of leisure or external search behavior without a clear intent to purchase a specific product. New research from Bazaarvoice finds that not only 61% of consumers are globally more likely to browse for new products online compared to in-store, but that they also find it easier (64%) and more enjoyable (54%) (Release 2022). Furthermore, window-shopping is not only a form of leisure but also results in impulsive sales and subsequently higher profit for retailers. Reports state that 67% of users admit to window shopping for fun online, 77% of these online window shoppers make impulse purchases, and 70% of them will come back and make a purchase from their device within the first hour of seeing the product (Borland et al. 2018, Ouellette 2022). Similarly, Forbes declares that 90% of mobile searches lead to action and over 50% of them lead to sales (Conner 2014). These statistics show that there is a high opportunity for impulsive sales provided by window shoppers for retailers.

One common practice by online retailers to entice users to visit the e-store is designing short time campaigns including products from different categories. Surveys show that discounts and sales campaigns are the top inspirations for online window shoppers, and they are three times more effective than TV ads (CouponChief 2021). For example, in 2018, 174 million Americans shopped on Black Friday and Cyber Monday, spending an average of \$335 per person (Ouellette 2022). The majority of shopping happens online (58 million) rather than offline (51 million) during these special days.

While window shopping has been studied extensively from a behavioral and psychological perspective, there are very limited analytical studies about the topic. We are interested in joint pricing and ranking decisions in an online retailer when there is a finite inventory for each product and customers are window shoppers. Our research is distinct from other studies of ranking and pricing in three main aspects: i) A customer may purchase more than one product. ii) Offered products can be selected from completely different categories. iii) Finally, as a result of two previous properties, the demand for each product is independent of other products in the offered set. We assume that demand for a specific product at a specific rank (position) occurs at a constant rate during the session. In this setting, the retailer decides on a single-shot ranking and prices of each product to maximize the profit for a specific amount of time referred to as a campaign.

We develop an optimization model where demand for each product is deterministic and invertible. We model the effect of ranking on the demand as follows. Products are organized into vertically differentiated positions (ranking) and a consumer examines only the first r products, where r is a random variable. Thus, products that are placed in earlier positions are more likely to be considered and, therefore, purchased than those that are placed in later positions. We can refer to three key findings of the study as follows. First, we show that optimal pricing decision is independent of the ranking when there is an infinite inventory for each product. In this case, we can define the optimal price for a product based on its intrinsic properties. Also, a descending-profit-ordered product ranking would provide the maximum total profit. Second, when there is a finite inventory for each product, optimal ranking and pricing decisions are not decomposable and the optimization problem is more complicated. In this case, despite the deterministic and invertible structure of demand, we prove that the optimal price would result in product stock-out during the campaign on some occasions. As mentioned, products at higher positions receive more attention and subsequently, higher demand. Therefore, product stock-outs at the higher positions cause a higher number of products to be presented at these positions, which leads to higher profit. Third, we prove that solving the problem

with a set of predefined prices (and not joint pricing ranking) would result in insignificant profit loss if the diversity of the product's demand parameters is limited. In other words, the effect of the joint pricing (with ranking) on the profit is limited and dependent on products variety.

The precise optimization model is intractable due to the existence of integer variables and the complexity of the constraints. Thus, we propose two heuristic methods to solve the problem and provide an upper bound by relaxing the integer variables. We show that the performance of the heuristic methods is dependent on the customer's search behavior. Our approximation-based heuristic method, OVP, has a better performance when customers have a higher tendency to search longer and consider more products. On the other hand, our heuristic method designed based on the upper-bound solution, UB_Alg, has a better performance when customers consider a few products at the top positions. Selecting the appropriate heuristic method, we provide a near-optimal solution in all cases. Also, numerical analyses show that trivial solutions such as sorting by price or sorting by inventory would result in significant profit loss.

Our research contributes to the research on ranking and pricing (Gallego et al. 2020, Lei et al. 2021, Derakhshan et al. 2022) in the following aspects. This is the first study to address the problem of pricing and ranking when there is a finite inventory for each product. We also provide an analytical model to study i) the window shopper customers' behavior, and ii) the effect of product stock-out on ranking and presentation of the products. Finally, our results provide insight for online retailers to set the optimal ranking and pricing decisions.

The remainder of this chapter is structured as follows. In Section 4.1, we review the literature and related studies. In Section 4.2, we mainly discuss our model including the general optimization model, upper bound, and heuristics. In Section 4.3, we analyze the problem where the prices of the products are predefined. In Section 4.4, we consider the general case where prices are decision variables. In Section 4.5, we provide numerical analyses. Finally, Section 4.6 concludes with a discussion and suggestions for future work.

4.1 Literature Review

In this section, to position our contributions, we review research streams that are relevant for the key aspects of our model, i.e. product ranking and pricing when there is a limited inventory, generalized assignment problem, scheduling problem, and hedonic shopping and behavior of window shoppers.

4.1.1 Joint Ranking and Pricing with Limited Inventory

The ranking problem has some basic similarities with the dynamic assortment planning problem. Dynamic assortment planning considers the problem of selecting a set of products to be offered to the customers at online stores without any specific ranking of products in the set. One of the first studies in dynamic assortment planning with limited inventories belongs to Bernstein et al. (2015). They consider an assortment planning problem where products are substitutes and each of them has a limited inventory. Prices of the products are predefined and identical. On the other hand, customers are heterogeneous with respect to their product preferences. The retailer must decide on the set of products to be offered depending on the customer's preferences, the inventory levels, and the remaining time in the season. The study proves that it is optimal to reserve some products with low inventory levels for potential future customers. In that sense, the retail does not offer these products to some of the customers even though there is stock for that products. They show that the potential revenue impact of such a policy can be significant, especially when customer heterogeneity is high and when the products' inventory-to-demand ratios are asymmetric.

In a similar study, Golrezaei and Rusmevichientong (2014) propose a more general model where the retailer decides on a personalized and dynamic assortment of products to be offered to each customer. They propose a real-time assortment optimization to balance the inventory level of the products using a penalty function. Also, the optimization problem selects a personalized set of products based on the customer's location. They show that such a policy can lead to over 10% improve-

ments in revenue compared to a policy that treats all customers the same. Fata et al. (2019) improve the bound for the special case where all of the assortments have size one and the problem transforms to the online stochastic matching with timeouts problem. They provide a polynomial-time approximation algorithm that earns at least 0.51 of the optimum. This improves upon the previous-best approximation ratio of 0.46. Ma and Simchi-Levi (2020) extend the problem where the retailer decides on the prices as well. In this study, they extend all of the results to allow for items having multiple feasible prices. Their algorithms achieve the best-possible weight-dependent competitive ratios, which depend on the sets of feasible prices given in advance.

In contrast with previous studies, Rusmevichientong and Topaloglu (2012) consider the assortment optimization problems under the multinomial logit choice model where true parameters of the logit model are assumed to be unknown. They provide a robust assortment that maximizes the worst-case expected revenue over all parameter values in the uncertainty set. Numerical experiments in the paper show that the robust approach is especially beneficial when there is significant uncertainty in the parameter values, resulting in over 10% improvement in the worst-case performance, but it can also maintain comparable average revenue if average revenue is the performance measure of interest.

Generally, demands for new products are unknown to the retailer at the start of the season. In that sense, learning the demand add a new aspect to the problem. Ferreira et al. (2016) study the pricing problem where products are new and the retailer must predict demand. They propose a machine learning technique to estimate historical lost sales and predict the future demand for new products. They solve an optimization problem with a non-parametric demand prediction model under price competition. They develop an algorithm to efficiently solve the subsequent price optimization that incorporates reference price effects. The implementation of the method in an apparel retailer shows that sales do not decrease due to the implementation of the new pricing method. One of the retailer's main challenges is pricing and predicting demand for products that it has never sold before, which account for

the majority of sales and revenue. price increases. However, the revenue of the test group is increased by approximately 9.7%.

In contrast with the practical demand learning method proposed in the previous study, Ferreira et al. (2018) develop a theoretical demand learning method. Without knowledge of the demands, the retailer faces a trade-off commonly referred to as the exploration-exploitation trade-off. Through exploration, the retailer offered different products to learn more about the products' attractiveness. By the exploitation, the retailer maximizes the revenue using the information gathered up to that moment. In this study, they propose "Thompson sampling" to address the challenge of balancing the exploration-exploitation trade-off under the presence of inventory constraints. They prove a theoretical performance guarantee for the algorithm and show that it outperforms the benchmark in numerical cases.

Most of the studies in this area consider an MNL choice model to investigate customers' behavior. However, Feldman and Topaloglu (2017) study the problem under a substitution model. In this choice model, a customer arrives into the system to purchase a particular product. If this product is available for purchase, then the customer purchases it. Otherwise, the customer transitions to another product or to the no purchase option, until she reaches an available product or the no purchase option. They solve assortment planning problems, as well as single resource revenue management problems, and network revenue management problems. They define a linear program to solve the problem by approximation. Then, an algorithm is proposed to recover the optimal solution to the original problem from the linear program.

While assortment planning problems assume that all products in the set are considered by the customers, empirical studies show that customers only consider a limited number of the products at the upper positions (Ghose and Li. 2012, Ursu 2018). In the rest of this section, We review a few studies where a product ranking model is used instead of assortment planning. Gallego et al. (2020) propose one of the first models of product ranking and pricing where there is infinite inventory for each product. They assume that products are assigned to some virtual web pages

and consumers consider only products on the top pages with different consumers willing to see different numbers of pages. Within the set of the considered products, consumers select a product, if any, due to a general choice model. They prove that the product ranking with predefined prices problem is NP-hard even under the MNL choice model. However, they propose an algorithms with guaranteed performance to solve the problem. Interestingly, this study shows that the retailer can solve the product ranking and pricing problem optimally with a straightforward and simple algorithm. They show that sorting products by descending order of their value gap and setting the same price markup for all products on the same page is optimal. In a similar study, Wang and Sahin (2018) study the ranking and pricing where, a consumer forms her consideration set first, then, she evaluates all products in her consideration set and chooses the one with the highest net utility. Similar to the previous study, they show that assortment problems with search cost are generally NP-hard. They show that the revenue-ordered assortment fails to be optimal, although it can obtain at least half the optimal revenue. However, they state that for the joint assortment planning and pricing problem with homogeneous consumers, the intrinsic-utility ordered assortment and the quasi-same-price policy, which charges the same price for all products except at most one, are optimal.

In another remarkable study with a focus on the fulfillment methods, Lei et al. (2021) consider a joint product ranking, pricing, and order fulfillment problem with limited inventory. They show that the original problem is NP-hard and solve it using techniques such as randomized algorithms and graph-based algorithms. They prove that the policy is asymptotically optimal and show that has a near-optimal performance in numerical cases.

To the best of our knowledge, there is only one paper that models a ranking and dynamic pricing problem with limited inventories. In contrast with previous studies that consider the MNL choice model, Najafi et al. (2019) consider a cascade choice model, where customers consider the products on the list one by one and decide either to purchase the product, continue searching by considering the next products or leave. They state some characterization of the optimal ranking and

pricing policy. They solve the problem by adapting two heuristic policies to retain their strong performance guarantees.

As mentioned, our study is motivated by short-term campaigns where many customers visit the online store simultaneously. Thus, we consider that the products' ranking and prices are fixed during the campaign. Also, we assume that visitors are window-shopping customers who may purchase more than one product. We show a limited inventory level, and subsequently, product stock-outs affect the ranking and pricing at the store.

4.1.2 Generalized Assignment Problem and Parallel Machines scheduling Problem

The product ranking problem has similarities to the well-developed job assignment problem as each product must be assigned to exactly one position. Generally, in the job assignment problem, each job is assigned to an agent at a cost that may vary depending on the agent-task assignment and the objective function of the problem is the minimization of the total cost. Similarly, in the ranking problem, the objective function is the maximization of cost (that can be considered as the minimization of some negative costs) where the profit of assigning a product to each rank may vary depending on the product-rank assignment. While a classic assignment problem can be solved by a simplex solver, a generalized assignment problem (GAP) where there are capacity constraints for each agent is NP-hard. One of the remarkable studies in this area belongs to Savelsbergh (1997). They use a branch-and-price algorithm to solve the problem. Generally, the cost of assigning a job to an agent is fixed in the GAP problem, however, a variation of this problem named multilevel generalized assignment problem considers the case in which agents can execute tasks at different efficiency levels with different costs. Ceselli and Righini (2006) propose an exact solution for this problem based on a decomposition into a set-partitioning constraints problem and a multiple-choice knapsack problem. Recently, many studies consider a fuzzy structure for assignment costs. We can refer to Kar et al. (2014) as a remarkable study on generalized fuzzy assignment problem.

We assume that the demand for products at higher positions is higher. Thus,

when there is a limited inventory for each product, products at a higher position would be consumed and sold out faster. In that sense, the ranking problem is similar to a specific type of job scheduling problem named uniform parallel machine scheduling. In the uniform parallel machine scheduling problem, multiple jobs of varying processing times must need to be scheduled on multiple uniform machines (i.e. the machines are identical except for speed). The objective of the problem is to minimize the makespan (i.e. the total time required to execute the schedule). We refer readers for a vast review on the topic to Cheng and Sin (1990). Generally, scheduling problems are solved using the branch and bound method. Chen and Powell (1999) propose one of the effective methods to solve the problem using column generation to solve the linear problem in each node of branch and bound tree. Additional constraints and generalization of the problem have been studied in more recent studies. In one of the noteworthy studies, Yeh et al. (2015) considers the problem where there is a resource consumption constraint. Bulhoes et al. (2020) study the parallel machine scheduling problems where there are constraints such as deadlines and sequence dependency.

Despite the similarities in the problem structures, our problem differs from uniform machine scheduling in three main aspects: First, we assume that makespan (campaign time) is fixed and our objective is to maximize the profit. Note that we consider that the demand rate is a function of the price which is a decision variable in our problem, thus, the time that a specific product would be presented at a specific position is not fixed. Second, in both GAP and scheduling problem, the goal is to finish jobs up to a specific time, however, in ranking problem, there is no requirement on selling out all units of a product. The seller may receive a positive profit even though the product is not sold out and there is left over inventory. Third, in ranking problems there is a vertical relevance among different positions while machines in scheduling problems and agents in GAP are not correlated with each other. We discuss how the vertical relevance of the positions affects the ranking during the campaign in Section 4.2.3.

4.1.3 Window Shopping and Hedonic Search

Studies show that hedonic shopping motivations are strong drivers of visiting online stores for users (Kim and Eastin 2011, Childers et al. 2001, To and Sung 2014). To and Sung (2014) investigate the hedonic motivations that exist in the online shopping environment in comparison with traditional hedonic values. They state two hedonic motivations unique to internet shopping; privacy and online shopping achievement. It appears that the most important hedonic value to online shoppers is having the choice to interact or not interact with others while shopping on the Internet. In a more general investigation of online window shoppers' behavior, Kim and Eastin (2011) indicate that hedonic shopping motivation is a robust predictor of exploratory information seeking and impulse buying. Results demonstrate that online shopping behaviors affected by hedonic shopping motivation affect pre-purchase browsing time. Also, pre-purchase browsing time has a positive relationship with online buying frequency.

Although window shoppers do not have any clear intent to purchase a product, studies show a significant effect of traffic and store visits on sales. In that sense, attracting shoppers to stores and converting the incoming traffic into sales profitably is vital for the financial health of retailers. (Perdikaki et al. 2012) use proprietary data of an brick-and-mortar apparel retailer to study the relationship between store traffic, labor, and sales performance. They find that store sales volume exhibits diminishing returns to scale with respect to traffic, and labor moderates the impact of traffic on sales. For example, for values of traffic and labor corresponding to the mean, increasing average traffic per hour by one unit increases average sales volume per hour by \$9.97. However, at online stores, there is no labor capacity constraint. Accordingly, Moe and Fader (2004) develop a model to study the determining factor in conversion behavior (i.e., converting store visits into purchases). Their model is able to accommodate visits motivated by different purposes (e.g. planned purchases visits, hedonic browsing, and window shopping). The results indicate that visits have a positive effect on sales. They also show that purchasing probability increases as a function of previous purchasing experiences. Moe (2003) theoretically

develop and empirically test a typology of store visits in which visits vary according to the shoppers' underlying objectives. By using page-to-page clickstream data from a given online store, visits are categorized as a buying, browsing, searching, or knowledge-building visit, based on observed in-store navigational patterns, including the general content of the pages viewed. They show that about 2% of window shopping sessions end up with purchases. Similarly, several studies confirm that visits and window shopping has a significant effect on sales. Venkatesh and Agarwal (2006) state that purchase probabilities are highly correlated with visits. Park et al. (2012) state that two types of web browsing occur: utilitarian and hedonic. In an estimated structural model, the variety of selection has a positive effect on utilitarian web browsing, whereas price has a positive effect on hedonic web browsing. Additionally, utilitarian web browsing has a negative effect on impulse buying, whereas hedonic web browsing has a positive effect on impulse buying for apparel on shopping websites.

Window shopping is well studied in behavioral and psychological papers as well (Ali et al. 2020, Li et al. 2020, Floh and Madlberger 2013). However, to our knowledge, there is only one operational paper that considers window shopping. Ferreira et al. (2019) design online learning algorithms for product ranking where customers are window shoppers and their product preferences and attention spans are unknown to the retailer. They develop an online learning-then-earning algorithms that balance product popularity with diversity. They show that the learning algorithm has a striking performance both theoretically and practically. We extend the window shopping literature with an analytical model to study ranking and pricing where there is a finite inventory for each product.

4.2 Model

We consider a short-term campaign designed by an e-retailer that is visited by a large number of window shopper customers. We assume that products are offered in limited inventories and leftover inventories at the end of the campaign have no value. We assume that the retailer decides on the price and ranking of each product

at the start of the campaign. It means that the prices of all products are fixed and the same for all customers during the campaign. However, the list of the offered product may change due to the stock-outs. We would discuss the stock-out and its effect on ranking in Section 4.2.3.

On the other hand, window shopper customers visit the store without any need for specific products. They dedicate a limited time to product search which can be resulted in some impulsive purchases based on the products' attributes and price. Thus, we assume that the search time of the customer is predefined and independent of products' ranking.

4.2.1 Environment, Notations and Assumptions

Let $i \in I$, $p_i \in R$, and $q_i \in R+$ denote the products, their prices, and their inventories, respectively (I is the set of all products). We assume that each product is assigned to a position (ranking) and each position is vertically differentiated from another position. Each customer considers only first r products with probability α_r . It means that customers consider all products starting from the first position up to a stopping point of the customer is the position r . As a result, a specific product at position r would be considered by all customers that have a stopping point equal to or higher than r . As mentioned earlier, this assumption is supported by the literature (Aouad and Segev 2015). As mentioned, the stopping point of a window shopper is defined by the dedicated search time and is independent of the products' ranking. It means that our setting is also compatible with all studies which are using a sequential ranking model (Chen and Yao 2017, Ursu 2018, Kim et al. 2017, 2010).

Let binary matrix X denote the ranking decision of the retailer where $x_{ir} = 1$, if product i is assigned to position r at the start of the campaign and $x_{ir} = 0$, otherwise. Let θ_r denote the set of all products assigned to the first r positions (i.e. $\{i \in I \mid \sum_{k=1}^r x_{ik} \geq 1\}$) and their prices (i.e. $\{p_i \mid i \in I \ \& \ \sum_{k=1}^r x_{ik} \geq 1\}$).

4.2.2 Individual, Potential, and Actual Demand

As we mentioned, we assume that each customer considers all products starting from the first position to his/her stopping point. For each product in this consideration set, the customer decides on purchasing the product individually based on its prices and independent of other considered products. Let $d_i(p_i)$ denote individual demand for product i when it is considered by a customer and its price is equal to p_i . We define $d_i(p_i) = a_i - b_i(p_i)$ where $a_i \leq 1$ and b_i are positive parameters that identifies products' attractiveness and price elasticity, respectively. Then, the total demand for each product is dependent on the number of customers that consider a product. We define the potential demand of product i as the total demand for product i during the campaign regardless of its inventory if it is assigned to rank r at the start of the campaign. We denote the potential demand as $\bar{d}_{ir}(\theta_r)$ which is equal to the multiplication of the expected number of the customers that consider the product and individual demand of that product. Formally, if the number of the customers that consider the product be equal to n , potential demand would be $\bar{d}_{ir}(\theta_r) = n \cdot d_i(p_i)$. We discuss the calculation of n in Section 4.2.5.

Due to the limited inventory of each product, the actual demand of a product is limited by its inventory. Thus, we define $d_{ir}(\theta_r) = \min(\bar{d}_{ir}(\theta_r), q_i)$ as actual demand (sales) of product i at rank r .

4.2.3 Effect of Products' Stock-out on Ranking

We assume that demand is deterministic and occurs at a constant rate at a specific price and position. We also assume that if during the campaign a product stocked out, it would be deleted from the list of the products in the remaining time of the campaign and all products with lower ranks (with a position below the stocked out product) would shift up one position. More formally, let $t \in [0, T]$ denote the time during the campaign and q_{it} denote remaining inventory of product i at time t . Then, we define t_i^s as the time that product i gets stocked out (i.e. $t_i^s = \min\{t | q_{it} = 0\}$). Subsequently, after t_i^s all products below product i shift up one position.

Based on the number of the products that are stocked out, a product starting at

position $r > 1$ can shift up one or more positions. Remember that products would receive a higher potential demand at a higher position. To identify the products' position and their potential demand during the campaign, it is essential to keep track of the number of the stocked out products and their timing. Thus, first, we introduce $\gamma_r(\theta_r) \in [0, 1]$ as the ratio of total campaign time that product started in position r is stocked out and it is not available at store (i.e. $\gamma_r = 1 - (t_i^s/T)$). Then, we calculate $\eta_{mr}(\theta_r)$ as portion of the time in which, at least m products from rank 1 to rank $r-1$ are stocked out. In other words, a product at the position r at $t = 0$ would be shifted up to the position $r - m$ at time $T - \eta_{mr}(\theta_r)$. Note that $\eta_{mr}(\theta_r)$ can be defined for $r \geq 2$ and $m < r$ and by the definition, $\eta_{1r}(\theta_r) \geq \eta_{2r}(\theta_r) \geq \dots \geq \eta_{r-1r}(\theta_r)$. The value of $\eta_{mr}(\theta_r)$ is dependent on the stock-out timing of the products in upper positions (i.e. values of $\gamma_1(\theta_1), \gamma_1(\theta_1), \dots, \gamma_{r-1}(\theta_{r-1})$). Starting with $\eta_{12}(\theta_2) = \gamma_1(\theta_1)$, for $r > 2$ and $1 \leq m < r$, we define $\eta_{mr}(\theta_r)$ recursively, as follows. Note that, to be consistent with the definition, we define $\eta_{0r}(\theta_r) = 1$.

$$\eta_{mr}(\theta_r) = \begin{cases} \eta_{m-1r-1}(\theta_{r-1}) & \gamma_{r-1}(\theta_{r-1}) > \eta_{m-1r-1}(\theta_{r-1}) \\ \gamma_{r-1}(\theta_{r-1}) & \eta_{mr-1}(\theta_{r-1}) < \gamma_{r-1}(\theta_{r-1}) \leq \eta_{m-1r-1}(\theta_{r-1}) \\ \eta_{mr-1}(\theta_{r-1}) & \gamma_{r-1}(\theta_{r-1}) \leq \eta_{mr-1}(\theta_{r-1}) \end{cases} \quad (4.1)$$

The Equation 4.1 shows how $\eta_{mr}(\theta_r)$ is dependent on the $\eta_{mr-1}(\theta_{r-1})$ and $\eta_{m-1r-1}(\theta_{r-1})$. Intuitively, to have m stocked out products down to the position r , at least we need to have $m - 1$ stocked out products down to position $r - 1$. Then, following cases identify the value of the $\eta_{mr}(\theta_r)$.

1. If the product at rank $r - 1$ stocks out faster than $m - 1$ products at higher positions (i.e. $\gamma_{r-1}(\theta_{r-1}) > \eta_{m-1r-1}(\theta_{r-1})$), then, $\eta_{mr}(\theta_r) = \eta_{m-1r-1}(\theta_{r-1})$.
2. If the product at rank $r - 1$ stocks out later than $m - 1$ products at higher positions, however faster than m th product (i.e. $\eta_{mr-1}(\theta_{r-1}) < \gamma_{r-1}(\theta_{r-1}) \leq \eta_{m-1r-1}(\theta_{r-1})$), then, $\eta_{mr}(\theta_r) = \gamma_{r-1}(\theta_{r-1})$.

Table 4.1: Example of $\gamma_r(\theta_r)$

	r				
	1	2	3	4	5
$\gamma_r(\theta_r)$	0.6	0.3	0.8	0.4	0.5

Table 4.2: Example of $\eta_{mr}(\theta_r)$

Rank	m				
	$\eta_{1r}(\theta_r)$	$\eta_{2r}(\theta_r)$	$\eta_{3r}(\theta_r)$	$\eta_{4r}(\theta_r)$	$\eta_{5r}(\theta_r)$
5	0.8	0.6	0.4	0.3	—
6	0.8	0.6	0.5	0.4	0.3

3. Finally, if the product at rank $r - 1$ stocks out later than m products at higher positions (i.e. $\gamma_{r-1}(\theta_{r-1}) \leq \eta_{mr-1}(\theta_{r-1})$), then, $\eta_{mr}(\theta_r) = \eta_{mr-1}(\theta_{r-1})$.

As an example we refer to Table 4.1 and Table 4.2. Considering γ_r values for $r = 1, 2, \dots, 5$ represented in Table 4.1, we calculate $\eta_{m6}(\theta_6) \forall m \in \{1, 2, \dots, 5\}$ based on $\eta_{m5}(\theta_5)$ values in Table 4.2. Note that all values in Table 4.2 refer to one of γ_r values in Table 4.1.

- Product 3 stocks out faster than all other products. Thus, $\eta_{16}(\theta_6) = \eta_{15}(\theta_5) = \gamma_3 = 0.8$
- Product 3 is followed by product 1. Thus, $\eta_{26}(\theta_6) = \eta_{25}(\theta_5) = \gamma_1 = 0.6$
- Product 5 stocks out faster than remaining products. Thus, distinctly from two previous cases, $\eta_{36}(\theta_6) = \gamma_5 = 0.5$
- Finally, products 4 and 2 remains at the store longer than all products. Thus, $\eta_{46}(\theta_6) = \eta_{35}(\theta_5) = \gamma_4 = 0.4$ and $\eta_{56}(\theta_6) = \eta_{45}(\theta_5) = \gamma_2 = 0.3$.

4.2.4 Optimal Pricing

Let define p_i^* as the price that maximize the $p_i d_i(p_i)$ (i.e. $p_i^* = \operatorname{argmax}\{p_i d_i(p_i)\}$). When there is limited inventory, the total potential demand at p_i^* can be higher than the inventory of the product (i.e. $\bar{d}_{ir}(\theta_r) > q_i$). In other words, there would be an excess demand for the product at such a price. Thus, in such a case, retailer may prefer to set a higher price to earn higher profit per inventory unit. Therefore, let $p_{ir}^e(\theta_r)$ be the price for product i at position r such that the potential demand of the product be equal to its inventory (i.e. $\bar{d}_{ir}(\theta_r) = q_i$). It means that product i at price $p_{ir}^e(\theta_r)$ would neither stock out nor have any leftover. Note that $p_{ir}^e(\theta_r)$ always exists due to deterministic and invertible structure of the demand function. We can define optimal pricing region for any product using these two definitions. Proposition 4.1 identifies the optimal pricing strategies.

Proposition 4.1 *Optimal Pricing*

For any product i at position r :

1. *If $p_i^* \geq p_{ir}^e(\theta_r)$, optimal price would be equal to p_i^* .*
2. *If $p_i^* < p_{ir}^e(\theta_r)$, optimal price would be in $[p_i^*, p_{ir}^e(\theta_r)]$*

Proposition 4.1 have some important implications. First, due to Part 1, if there exist infinite inventories for all products, p_i^* would be the optimal price for all products. Thus, in this case, optimal pricing is independent of the ranking decision. It means that we can solve the ranking and pricing problem separately when there is an infinite inventory for each product. Second, we know that for a single product pricing problem, it is not optimal to have any excess demand and the optimal price would be $\max(p_{ir}^e(\theta_r), p_i^*)$. However, due to Proposition 4.1 Part 2, for a multiple-product ranking setting such as we propose, it may be optimal to have prices that result in stock-outs during the campaign, as any price $p_i < p_{ir}^e(\theta_r)$ would result in product i 's stock-out. It may seem counterintuitive as we propose to set a price to stock out the product under deterministic demand; however, in the existing ranking setting, excess demand is not lost and is shifted to the products at lower positions.

We know that in an online store the first page and positions receive the highest portion of customers' attention. Thus a pricing strategy that results in stock-outs would result in a higher number of the products presented at first positions. Thus, the high attention for the first positions can be shared among more products. We provide a simple example that shows the optimality of stock-outs as follows (We refer readers for a general case that proves the optimality of the stock-outs to Appendix B.1)

For example, consider a simple case with two products. Let's assume that during the campaign, 100 customers visits the store (all checking the product at the first position) and only 10 of them check the product at the second position. Let's assume that the demand function and inventory of both products are identical, i.e. $d_i(p_i) = 1 - 0.01p_i$ and $q_i = 20$ and . It means that $p_i^* = 50$ and $p_{i1}^e = 80$. Now consider two following pricing strategies:

1. Set $p_{i1}^e = 80$ and for product at rank 1 which result in no stock out, and $p_i^* = 50$ for product two: Profit provided by product at the first and the second positions is equal to 1600 and 250, respectively.
2. Optimal solution when product at the first position is stocked out at price $p_i = 63$ and price for product at second position is $p_i = 61$: Profit provided by product at the first and the second positions is equal to 1260 and 1221, respectively.

In this case, the second strategy provides a higher profit (i.e. 631 more than the first strategy). Note that in the second strategy $\gamma_1 = 0.46$. It means that 54 customers arrive at the store when both products are offered at the store, however, the last 46 customers only see one product. As products share the demand at the first position, they provide a more balanced profit and the retailer receives a higher total profit.

4.2.5 General Optimization Model

Let's A denotes the number of customers during the campaign. For ease of use, we can define $\beta_r = A \cdot \sum_{s=r}^R \alpha_s$ as the expected number of the customer who consider

the rank r . Then, we can form the optimization model denoted as OPT to solve the joint product pricing and ranking problem mentioned as follows.

$$\begin{aligned}
\mathbf{OPT:} \max_{P, X} \pi &= \sum_I \sum_R x_{ir} \cdot p_i \cdot d_{ir}(\theta_r) \\
d_{ir}(\theta_r) &\leq \bar{d}_{ir}(\theta_r) && \forall i \in I, \forall r \in R \\
d_{ir}(\theta_r) &\leq q_i && \forall i \in I, \forall r \in R \\
\bar{d}_{ir}(\theta_r) &= A. \left(\sum_{m=r}^R \alpha_m + \sum_{m=1}^{r-1} \alpha_{r-m} \cdot \eta_{mr}(\theta_r) \right) d_i(p_i) && \forall i \in I, \forall r \in R \\
\sum_I x_{ir} &= 1 && \forall r \in R \\
\sum_R x_{ir} &= 1 && \forall i \in I \\
x_{ir} &\in \{0, 1\} && \forall i \in I, \forall r \in R \\
p_i &\in R+ && \forall i \in I
\end{aligned} \tag{4.2}$$

There are main similarities between Model OPT and the classic job assignment problem, where $p_i \cdot d_{ir}(\theta_r)$ is the value of assigning product i to position r and constraints 4 and 5 guarantee that each position includes exactly one product and each product is assigned to exactly one position, respectively. However, the main difference is provided by $d_{ir}(\theta_r)$ function. Remember that actual demand $d_{ir}(\theta_r)$ is the minimum of the potential demand and inventory (guaranteed by constraints 1 and 2, respectively). Constraints 1 and 2 are followed by constraint 3 which represent the potential demand of the products. In constraint 3, $A.(\sum_{m=r}^R \alpha_m + \sum_{m=1}^{r-1} \alpha_{r-m} \cdot \eta_{mr}(\theta_r))$ identifies the expected number of the customers who would consider the product i which consists of two part as follows. First, all customers who have a stopping point equal to or higher than r would consider the product i ($A.(\sum_{m=r}^R \alpha_m)$). Second, any customer with stopping point equal to $r - m$ would consider the product i after at least m products get stock out and product i shift up m positions ($A.(\alpha_{r-m} \cdot \eta_{mr}(\theta_r))$). Although $d_i(p_i)$ is a linear function, $d_{ir}(\theta_r)$ is non-linear due to the structure of η function (i.e. $d_{ir}(\theta_r)$ is piece-wise linear with price-dependent boundaries). The non-linear form of $d_{ir}(\theta_r)$ and existence of inte-

ger variables x_{ir} make the problem OPT intractable when the number of products increases. Thus, in this chapter, we will not attempt to exactly solve OPT, and in the rest of this article, we provide heuristic methods and an upper-bounds to solve the problem.

4.2.6 Upper-bound Model, UB

One straightforward approach to solving the problem OPT is relaxation of the integer variable x_{ir} , representing the portion of the time that product i is assigned to position r . It means that we allow a position to be shared among products, and one product can be assigned to more than one position. However, a product cannot be assigned to more than one position at the same time during the campaign. Thus, the summation of the assignment of a product cannot exceed 1. In such a setting, we can assume that if a product gets stocked out, it can be easily not assigned to any position. Subsequently, stock-outs and positions of products would be handled by the solver and there is no more need to $\eta_{mr}(\theta_r)$. Using these assumptions, we relax x_{ir} and define problem UB as follows:

$$\begin{aligned}
 \text{UB: } \max_{P,X} \pi &= \sum_I p_i \cdot d_i(p_i) \cdot \sum_R x_{ir} \cdot \beta_r \\
 \sum_R x_{ir} \cdot d_i(p_i) \cdot \beta_r &\leq q_i & \forall i \in I \\
 \sum_I x_{ir} &\leq 1 & \forall r \in R \\
 \sum_R x_{ir} &\leq 1 & \forall i \in I \\
 x_{ir} &\in [0, 1] & \forall i \in I, \forall r \in R \\
 p_i &\in R+ & \forall i \in I
 \end{aligned} \tag{4.3}$$

The solution of UB guarantees an upper bound for the problem OPT as any feasible solution in OPT has a corresponding feasible solution in UB. However, we investigate whether any solution of UB is feasible to implement at the store theoretically and practically. Let us define “rotational ranking” policy as a ranking

policy where the retailer decides at the start of the season to change the ranking of products at some specific times $t^1, t^2, t^3, \dots, t^n$. Then, the solution of UB can be interpreted as a rotational ranking policy where product i is assigned to position r for $x_{ir}T$ amount of time. In that sense, the solution of UB is implementable. However, note that a retailer may need to change the ranking $|I|!$ times, where time intervals can be irrational values (e.g. 0.01s). Therefore, such a solution would not be practically implementable in most cases. In Section 4.3.2, we propose two heuristic methods to provide an integer solution based on the solution of UB.

4.2.7 Heuristic Model, OVP

As we mentioned, $\eta_{mr}(\theta_r)$ is a complicated non-linear function that makes problem OPT intractable. Thus, we provide a heuristic method, replacing $\eta_{mr}(\theta_r)$ with an approximation. To approximate and replace the η function with a less complicated function, we need to investigate i) how η affects the final solution (total profit), and ii) the effect of decision variables on η . By definition, $\bar{d}_{ir}(p_i, r)$ is increasing in $\eta_{mr}(\theta_r)$. Subsequently, profit earned by each product is non-decreasing in $\eta_{mr}(\theta_r)$. On the other hand, $\eta_{mr}(\theta_r)$ is defined based on stock outs of products at higher position. In that sense, the portion of the time that a product at position r is stocked out provides a positive opportunity value for products at lower positions. Therefore, we approximate the value provided by η , using the portion of the time that each product is stocked out, γ_r . More formally, let define $c_i^q(p_i) = \frac{q_i}{p_i d_i(p_i)}$ as consumption time of product i . Then, we define the opportunity value, denoted by o_{ir} , as the utility provided by portion of time that product i at position r is stocked out, as follows:

$$o_{ir} = f(\beta_r - c_i^q(p_i)) = v \cdot (\beta_r - c_i^q(p_i)) \quad (4.4)$$

In Equation 4.4, v represents the opportunity value provided by an expected customer considering a product (i.e. a unit of β_r). The expected value provided by a unit of β_r is dependent on the product at the rank r and it is equal to $p_i d_i(p_i)$ for all the time that product i is at rank r . In that sense, maximum value for v is

$\max_i\{p_i d_i\}$ and minimum value for v would $\min_i\{p_i d_i\}$. Thus, the rational value for v must be in $[\min_i\{p_i d_i\}, \max_i\{p_i d_i\}]$. Based on this setting, we define a heuristic method denoted as OVP to solve the problem OPT as follows:

$$\begin{aligned}
\text{OVP: } \max_x \pi &= \sum_I \sum_R x_{ir} \cdot (p_i \cdot d_{ir}(p_i, r) + o_{ir}) \\
d_{ir}(p_i, r) &\leq \bar{d}_{ir}(p_i, r) && \forall i \in I, \forall r \in R \\
d_{ir}(p_i, r) &\leq q_i && \forall i \in I, \forall r \in R \\
\bar{d}_{ir}(p_i, r) &= A \cdot \left(\sum_{m=r}^R \alpha_m \right) d_i(p_i) && \forall i \in I, \forall r \in R \\
\sum_I x_{ir} &= 1 && \forall r \in R \\
\sum_R x_{ir} &= 1 && \forall i \in I \\
x_{ir} &\in \{0, 1\} && \forall i \in I, \forall r \in R \\
p_i &\in R^+ && \forall i \in I
\end{aligned} \tag{4.5}$$

In comparison with the general problem OPT, OVP has a simpler structure due to the elimination of the η function from constraint 3. Note that x_{ir} is still integer and objective function is non linear even when x_{ir} is relaxed (i.e. $(p_i \cdot d_{ir}(p_i, r) + o_{ir})$ is concave in p_i). However, for a given set of prices, OVP is a classic job assignment problem that can be solved as a linear optimization model. We discuss the solution of OVP for a given set of prices in Section 4.3 and solving strategy when prices are decision variable in Section 4.4. Finally, in Section 4.5, we solve problem OVP numerically under different v values and investigate its effect on the solution.

4.3 Ranking with Predefined Prices

In this section, we investigate the ranking problem when prices are predefined before the campaign and the retailer only must decide on the ranking of the products. Note that, when we have a predefined price for each product, individual demand and potential demand are constant as well. Subsequently, Problem OPT simplifies

to a complete integer problem. Although the feasible region is finite and countable for integer problems, the ranking problem with predefined prices has $|I|!$ feasible solutions, where η should be defined for each feasible solution, separately. Due to the complexity of the η function, the problem is still intractable when the number of products gets too high. However, for a given set of prices, we propose some optimal ranking rules that limit the solution region of the problem. We discuss these rules in the next section.

4.3.1 Optimal Ranking Rules

As mentioned in Section 4.2.3, the structure of OPT is recursive and the optimal decision for each position r is dependent on θ_{r-1} . Thus, we state the optimal decision based on the decisions for upper positions, θ_{r-1} . Let S_r^a denote set of all available products for position r (i.e. all product that are assigned to non of positions 1 to $r - 1$). Then, we partition S_r^a into two product sets $S_r^d(\theta_{r-1})$ and $S_r^q(\theta_{r-1})$ as set of the products such that $\forall i \in S_r^d : \bar{d}_{ir}(\theta_r) \leq q_i$ and $\forall i \in S_r^q : \bar{d}_{ir}(\theta_r) > q_i$, respectively. While set $S_r^q(\theta_{r-1})$ identifies the set of product that would be sold out completely if be assigned to position r , set $S_r^d(\theta_{r-1})$ identifies the set of the product that would have left over inventories. Also, let's define $i_r^* = \operatorname{argmax}_i \{p_i \cdot d_i(p_i) | i \in S_r^d\}$. As we state later in Proposition 4.2, optimal product in $S_r^d(\theta_{r-1})$ to be assigned to position r is product i_r^* . However, to find the optimal product in $S_r^q(\theta_{r-1})$, and generally in S_r^a , to be assigned to rank r , we need further definitions.

Let's define product j and j' in S_r^q , interchangeably q-type if both following conditions holds: i) If we assign j to the rank r , then, $j' \in S_{r+1}^q$; ii) If we assign j' to the rank r , then, $j \in S_{r+1}^q$.

Using the definitions provided above, we state three ranking rules in Proposition 4.2:

Proposition 4.2 *Optimal ranking of products with predefined prices*

1. In the optimal raking, any product $i \neq i_r^* \in S_r^d$ cannot be assigned to position r .

2. In the optimal ranking, any product $j \in S_r^q$, cannot be assigned to position r , if $p_j \cdot d_j(p_j) < p_{i_r^*} \cdot d_{i_r^*}(p_{i_r^*})$.
3. In the optimal ranking, any interchangeably q -type products j and j' followed by empty set of S_{r+2}^q should be ranked by increasing $c_i^q(p_i)$ values.

Note that conditioned on θ_{r-1} , expected number of the customers that consider position r is independent on product i and is equal to $A \cdot (\sum_{m=r}^R \alpha_m + \sum_{m=1}^{r-1} \alpha_{r-m} \cdot \eta_{mr}(\theta_r))$. It means that potential profit for product i in S_{r+2}^a is fixed and equal to $p_i d_i(p_i) \cdot (A \cdot (\sum_{m=r}^R \alpha_m + \sum_{m=1}^{r-1} \alpha_{r-m} \cdot \eta_{mr}(\theta_r)))$. Proposition 4.2 Part 1 and Part 2 are concluded from this property.

Corollary 1 *Optimal ranking and pricing - infinite inventory.*

If $q_i > A \cdot d_i(p_i^*) \forall i \in I$, then optimal price is $p_i^* \forall i \in I$ and optimal ranking is sorting product by $p_i^* \cdot d_i(p_i^*)$.

Corollary 1 is resulted by Proposition 4.1 Part 1 and Proposition 4.2 Part 1. Due to the condition of Corollary 1, all products satisfy the condition of Proposition 4.1 Part 1. Thus optimal prices are independent of the ranking and is equal to $p_i^* \forall i \in I$. Also, the condition of Corollary 1 indicates that $S_1^d = I$. Then, due to Proposition 4.2 Part 1, i_r^* should be assigned to position r for all $r \in R$. Equivalently, optimal ranking is sorting product by $p_i^* \cdot d_i(p_i^*)$.

Counterintuitively, Proposition 4.2 Part 3 proposes to rank products with an ascending inventory order. By the definition, regardless of the ranking decision, both interchangeably q -type products would be sold out and the total profit provided by them would be $p_j q_j + p_{j'} q_{j'}$ in both ranking scenarios. However, the ranking of these two products has an effect on the demand for products at lower positions. A ranking due to Proposition 4.2 Part 3 would result in a larger demand for the product at position $r + 2$, and a higher total profit.

Let's define a threshold position r^d such that $\bar{d}_{i^d}(\theta_{r^d}) > q_i \forall i \in S_r^a$. Also, We know that all product in S_r^d are in S_{r+1}^d if not assigned to the rank r . It means that

considering Proposition 4.2 Part 1, the optimal ranking strategy from rank r^d to R is to sort the products by decreasing $p_i d_i$ (p_i value).

The structure of the problem and the optimal solution is highly dependent on customers' search behavior. Let us define $\omega_r = \frac{\beta_{r+1}}{\beta_r} = \frac{\sum_{s=r+1}^R \alpha_s}{\sum_{s=r}^R \alpha_s}$ that is the portion of customers that would continue their search by considering position $r+1$ conditional on considering position r . In that sense, ω is a determining factors for optimal solution. For example if $\omega_r \leq 0.6$, $\forall r \in R$, at most 10% of customers would visit rank 7 and only 1% customers would visit rank 11. Lower ω values would result in long list of position with a minor demand for each position (a long tail effect). Thus, corollary result of part 1 is useful to reduce the optimal solution possibilities for such positions.

4.3.2 Upper Bound and Heuristic Methods

We can use the model UB to provide the upper bound for the ranking problem with the set of predefined prices. When prices are predefined, optimization problem UB transforms into a linear maximization problem. Therefore, a simplex solver would guarantee the general optimal solution. We denote the UB model with predefined prices as LUB.

We propose two heuristic methods to solve the problem. Form of heuristic method OVP is easier under predefined prices scenario as well. For a set of given prices, problem OVP is equivalent to the classic job assignment problem. Due to the nature of the assignment problem, even with the relaxation of the integer decision variables, the optimization ends up with an integer solution. Thus, we can guarantee a polynomial-time and optimal solution for problem OVP. Quality of approximation in heuristic model OVP is dependent on o_{ir} function. We analyze the performance of the OVP in numerical cases in Section 4.5.1.

We propose the second heuristic method denoted as UB_Alg, specifically to solve the ranking problem with predefined prices. UB_Alg is an algorithmic solution to provide a feasible integer solution based on the non-integer result provided by LUB. The UB_Alg algorithm uses optimal ranking rules stated in Proposition 4.2 that

limit the feasible solution of the problem OPT.

Algorithm 1 Algorithm UB_Alg

```

Set  $c_i^{UB} = \sum_R x_{ir} \cdot \beta_r \quad \forall i \in I$ 
Sort products by  $c_i^{UB}$ 
 $r = 2$ 
while  $r < r^d$  do
    if Products in positions  $r$  and  $r - 1$  are interchangeably q-type then
        Sort the products in positions  $r$  and  $r - 1$  by increasing  $c_i^q(p_i)$ 
    end if
     $r \leftarrow r + 1$ 
end while
Sort all remaining products by decreasing  $p_i d_i(p_i)$  value.
  
```

Algorithm UB_Alg simply transforms the non-integer solution provided by UB to a feasible solution where optimal ranking rules in Proposition 4.2 are applied. We discuss the performance of the method numerically in Section 4.5.1.

4.4 Joint Pricing and Ranking

In this section, we investigate the joint product ranking and pricing problem with limited inventories. Remember that by Proposition 4.1, p_i^* is the lowest price that can be optimal for product i . However, the retailer might set a higher price if potential demand at this price is higher than the existing inventory. Therefore, we solve the problem as predefined price problem with $p_i = p_i^*$, $\forall i \in I$. Then, we investigate whether setting a higher price for some products would result in higher total profit.

Let c_i^M be expected number of customers that would consider the product i under solution provided by model M (i.e. $d_{ir}(\theta_r) = c_i^M \cdot d_i(p_i)$). By the definition, in any solution $\sum_I c_i^M \leq \sum_R \beta_r$. Also, consider that the total number of the customers are fixed and constant. Subsequently, $\sum_R \beta_r$ and $\sum_I c_i^M$ are limited as well. Remember

that we defined $c_i^q(p_i) = \frac{q_i}{p_i d_i(p_i)}$ as expected number of customers that must consider product i to sold out the product under price p_i and due to Proposition 4.1, for any $c_i^M < c_i^q(p_i^*)$, p_i^* is the optimal price. However, for products that are sold out (i.e. $c_i^M \geq c_i^q(p_i^*)$) optimal price is equal to $p_i^{M*} = d_i^{-1}(\frac{q_i}{c_i^M}) = \frac{a_i}{b_i} - \frac{q_i}{c_i^M b_i}$. However, setting a price higher than p_i^* would result in a higher c_i^M value (product gets sold out later), which could be resulted in less sold inventories from some other products. In that sense, objective of OPT problem is equivalent to maximizing the summation profit provided per unit of c_i^M over all $i \in I$. We know that expected profit provided by a unit of c_i^M is equal to $p_i d_i(p_i)$ and it is decreasing in price for $p_i > p_i^*$. More formally, under optimal pricing, if $c_i^M < c_i^q(p_i^*)$, expected profit provided by a unit of c_i^M is $p_i^* d_i(p_i^*)$. However, if $c_i^M < c_i^q(p_i^*)$ expected profit provided by a unit of c_i^M is $p_i^{M*} d_i(p_i^{M*})$ which is decreasing in c_i^M . Thus, we can propose a threshold where assigning higher c_i^M to a product would not be optimal. Let $\pi^{min} = \min_i \{p_i d_i(p_i) | c_i^M > 0\}$ be the minimum profit provided by a unit of c_i^M such that the product has a positive demand under solution of method M. Then, maximum value for c_i^M can be defined such that marginal profit provided by a unit of c_i^M be greater than π^{min} . Thus, effect of pricing on profit is highly dependent on diversity of $p_i d_i(p_i)$ and subsequently, demand parameters a_i and b_i . To measure the diversity of the products we define $\sigma = \max_i (\frac{p_i^* d_i(p_i^*)}{\pi^{min}})$.

Let $\Delta\pi$ be the maximum difference between optimal objective value (profit) of problem OPT and the objective value of problem OPT with predefined prices $p_i = p_i^*$, $\forall i \in I$. Proposition 4.3 states the magnitude of $\Delta\pi$:

Proposition 4.3 *Effect of pricing on the optimal profit value. The $\Delta\pi$ is bounded and dependent on σ :*

$$\frac{\Delta\pi}{\pi} < \left(\frac{\sigma - 1}{\sigma + 1}\right)^2 \quad (4.6)$$

We discuss the value of $\Delta\pi$ dependent on σ in numerical cases in Section 4.5.2.

4.4.1 Algorithmic Upper Bound

Model UB has a nonlinear multidimensional concave objective function when prices are decision variables. Additionally, constraint 1 is nonlinear and convex. Many convex optimization problem solvers can provide a polynomial-time and efficient solution. However, they do not guarantee the general optimal solution. Thus, we cannot rely on the solution of UB as an upper bound. Therefore, we define an algorithmic solution denoted as UB.Price to provide an upper bound for problem OPT.

To define UB.Price algorithm we need some definitions as follows. As mentioned, $p_i^{M*} d_i p_i^{M*}$ is decreasing in c_i^M . Thus, with abuse of the notation, we define the derivative of $p_i^{M*} d_i p_i^{M*}$ in c_i^M as follows:

$$\pi'_i = \frac{\partial \pi_i}{\partial c_i^M} = \frac{\partial \left(\frac{a_i \cdot q_i}{b_i} - \frac{q_i^2}{b_i c_i^M} \right)}{\partial c_i^M} = \frac{q_i^2}{b_i c_i^{M2}} \quad (4.7)$$

Then, we can state Algorithm UB.Price as follows.

Using algorithm UB.Price we start with $(p_i = p_i^*, \forall i \in I)$ and increase the price of the products as much as it is profitable for the retailer. During this process, we always consider to keeping $\sum_I c_i^M$ fixed as it is fixed and limited in the problem. In that sense, we must set ϵ such that two following conditions hold: 1) $\epsilon < c_j^{LUB}$ 2) $\pi'_k > \pi^{min}$ after updating the prices. Note that some feasibility conditions may be ignored during the procedure (i.e. constraints 2 and 3 in Model UB). We provide numerical analyses of the algorithm's performance in Section 4.5.2.

4.4.2 Adaptive OVP Heuristics

We show that OVP model is a job assignment problem even if prices are decision variables. Note that in this model the profit of assigning product i to rank r ($p_i \cdot d_{ir}(p_i, r) + o_{ir}$) is a concave function of the price p_i . Thus, we can define the optimal price of product i for rank r and for a given v_r value as follow:

$$p_i = \max \left\{ p_i^*, \frac{a_i - \sqrt{b_i \cdot v_r}}{b_i} \right\} \quad (4.8)$$

Algorithm 2 Algorithm UB_Price

Solve the LUB with $(p_i = p_i^*, \forall i \in I)$ Set $c_i^{LUB} \quad \forall i \in I$ Set $z = 1$ **while** z **do** Update π^{min} Set $j = \operatorname{argmin}_i \{p_i \cdot d_i(p_i) | c_i^M > 0\}$ Set $k = \operatorname{argmax}_i (\pi'_i)$ **if** $\pi'_k > \pi^{min}$ **then**

Set new prices

 Set $c_j^{LUB} = c_j^{LUB} - \epsilon$ Set $c_k^{LUB} = c_k^{LUB} + \epsilon$ Update p_k^{LUB*} due to c_k^{LUB} **else** Set $z = 0$ **end if****end while**

Note that when we set the price $\frac{a_i - \sqrt{b_i \cdot v_r}}{b_i}$ in Equation 4.8 is the highest price that we can assign to product i at position r such that marginal profit provided by a unit of c_i^M is equal to v_r . By setting the optimal prices due to the optimality condition in Equation 4.8, the OVP problem simplifies to job assignment problem. However, in this case, v value has an effect both on ranking and pricing. In that sense, the effect of v on the total profit is higher. Thus, we propose a recursive and adaptive method denoted as OVP_Price to improve the approximation of v value. With abuse of notation, let $\pi(OVP)$ denotes the profit provided by the solution of OVP, and l_r denote the $p_i d_i(p_i)$ of the product presented at position r at time $t = T$.

Algorithm 3 Algorithm OVP_Price

```

Set  $z^0 \leftarrow 0$ 
Set  $v \leftarrow \min_i \{p_i d_i(p_i)\}$ 
Set  $p_{ir} = p_{ir}^{OVP*}, \forall i \in I$ 
Solve the OVP
Set  $z = \pi(OVP)$ 
Set  $\delta z = z$ 
Set  $z^0 = z$ 
while  $\delta z > \epsilon$  do
    Update  $v_r \leftarrow l_r$ 
    Update  $p_{ir} = p_{ir}^{OVP*}, \forall i \in I$ 
    Solve the OVP
    Set  $z = \pi(OVP)$ 
    Set  $\delta z = z - z^0$ 
end while
  
```

In each step of OVP_Price, the value of v_r is updated due to the value of the lost sales in the previous step. We analyze the performance of OVP_Price numerically in Section 4.5.2.

Table 4.3: Mean and variance of parameters' distribution functions

Parameter	Mean	Boundaries		
		Level. 1	Level. 2	Level. 3
p_i	$p_i^* = \frac{a_i}{2b_i}$	$+/- 0.1 p_i^*$	$+/- 0.2 p_i^*$	$+/- 0.5 p_i^*$
q_i	$\sum_R \beta_r / [I]$	$+/- 5$	$+/- 10$	$+/- 20$
a_i	0.9	$+/- 0.01$	$+/- 0.05$	$+/- 0.10$
b_i	0.3	$+/- 0.05$	$+/- 0.1$	$+/- 0.2$

4.5 Numerical Solution

In this section, we represent some numerical examples of the problem. First, we investigate the problem when prices are predefined and show how different parameters affect the performance of each heuristic method. Then, we solve the general problem where prices are decision variables as well as product ranking.

4.5.1 Ranking with Predefined Prices

We numerically solve the product ranking problem with predefined prices for more than 500 cases. We test six different values for ω (i.e. $\omega = 0.95, 0.9, 0.8, 0.7, 0.6$, and 0.5) and random values for demand parameters a_i , and b_i , inventory level q_i , and price p_i . We analyze the effect of parameters' variance on the performance by deriving random variables from uniform distribution with three different variance levels. Table 4.3 represents the mean and boundaries of uniform distribution for each random parameter.

We solve each numerical case, using LUB, heuristics, and benchmark methods. Followings are list of these method and variations:

1. OVP_av: Heuristic OVP method where v value is constant for all positions (rs) and equal to $mean_i\{p_i d_i(p_i)\}$
2. OVP_min: Heuristic OVP method where v value is constant for all positions

- (rs) and equal to $\min_i \{p_i d_i(p_i)\}$
3. OVP_max: Heuristic OVP method where v value is constant for all positions (rs) and equal to $\max_i \{p_i d_i(p_i)\}$
 4. OVP_ord1: Heuristic OVP method where v_r values are rank-dependent and ranging from $\max_i \{p_i d_i(p_i)\}$ (assigned to v_1) to $\min_i \{p_i d_i(p_i)\}$ (assigned to v_R)
 5. OVP_ord2: Heuristic OVP method where v_r values are rank-dependent and ranging from $\max_i \{p_i d_i(p_i)\}$ (assigned to v_1) to $\min_i \{p_i d_i(p_i)\}$ (assigned to v_R)
 6. OVP_zero (benchmark): Benchmark based on OVP method, where v value is zero for all rs, to investigate the effect of v value on the performance.
 7. UB_Alg: Heuristic method UB_Alg defined in Section 4.3.2
 8. UB_Int (benchmark): UB_Int in a benchmark method for UB_Alg where products are sorted by c_i^{LUB} but no optimal ranking rule is implemented.
 9. p-sort (benchmark): Benchmark method where products are sorted by decreasing prices.
 10. q-sort (benchmark): Benchmark method where products are sorted by decreasing inventories.
 11. c-sort (benchmark): Benchmark method where products are sorted by decreasing $c_i^q(p_i)$ values.

We analyze the effect of the variant parameters on the solution of LUB using a regression model, where, we set the performance of LUB as the dependent variable and parameters' variance as independent variables. Table 4.4 represents the result of the regression.

Table 4.4: Effect of the parameters on profit upper bound

Method	Parameters				
	Intercept.	ω	p_var	q_var	d_var
LUB	6709**	-1965**	-2992**	-4.78	16297**

* and ** denote statistical significance at the 5%, and 1% levels, respectively.

As mentioned earlier, ω is determining factor in the structure of the problem and has a significant effect on the profit. Table 4.4 shows that variance of the price and inventory have a significant and negative effect on the profit as well.

We represent the performance of all heuristic and benchmark methods as a percentage of the upper bound (i.e. profit provided by LUB). To find the effective factors, we use a regression model where we set the performance of each method as the dependent variable (separately in an exclusive regression) and parameters' variance as independent variables. Table 4.5 represents the result of these regressions.

As expected, ω has a significant effect on all methods except benchmarks c-sort and p-sort. One noteworthy result of the regression is the effect of the ω on the performance. Performance of all versions of OVP heuristics is increasing in ω . In contrast, the performance of UB_Alg is decreasing in ω . This property is helpful to select an appropriate method to solve the problem in different cases. We would discuss this issue in more detail later.

Strikingly, UB methods (UB_Alg and UB_Int) have a robust performance and are not affected by variances (i.e. variance of p , q , and d). On the other hand, all versions of OVP and benchmarks have lower performance when prices have a higher variance level. However, this effect is more severe for the benchmark method p-sort. Note that, when variance levels are lower, all products are similar to each other and price is a suitable parameter to sort the products. However, for the higher levels of price variance, the problem is too complicated to be solved by trivial sorting methods.

Due to the high effect of ω on the performance, we present and compare the

Table 4.5: Effect of the parameters on performance

Method	Parameters				
	Intercept.	ω	p_var	q_var	d_var
OVP_av	0.67**	0.27**	-0.03**	0.00	0.05*
OVP_min	0.68**	0.26**	-0.03**	0.00	0.05*
OVP_max	0.68**	0.26**	-0.04**	0.00	0.05*
OVP_ord1	0.70**	0.23**	-0.05**	0.00	0.05
OVP_ord2	0.69**	0.24**	-0.06**	0.00	0.05*
OVP_zero	0.66**	0.15**	-0.04	0.00	0.14**
UB_Alg	1.01**	-0.05**	0.00	0.00	0.00
UB_int	0.57**	0.42**	0.02*	0.00	0.00
q-sort	0.61**	0.03*	-0.05**	0.00	0.20**
c-sort	0.48**	0.03	-0.12**	0.00	0.34**
p-sort	0.97**	0.00	-0.22**	0.00	0.05**

* and ** denote statistical significance at the 5%, and 1% levels, respectively.

performance of the method separately by each value of ω . Table 4.6 represents the mean performance of each method over 81 cases for any identical value of ω .

Note that when ω has a lower value, a few first positions receive higher attention from customers, followed by a long tail of positions with minor considerations. Remember that UB_Alg solves the problem for long-tail optimally using the optimal ranking rules. We can observe that for all $\omega \leq 0.8$, UB_Alg is the best method to solve the problem.

On the other hand, for higher values of ω , total customer attention is shared among the positions. In these cases, the approximation provided O_{ir} function has a better performance. Subsequently, OVP methods have better performance in the higher values of ω . Similarly, the solution provided by LUB is closer to integer solutions in these cases. Thus, we observe that the benchmark method UB_Int outperforms all methods in these cases.

Table 4.6: Average performance of each method over different ω values

Method	ω					
	0.95	0.9	0.8	0.7	0.6	0.5
OVP_av	96%	95%	93%	89%	87%	84%
OVP_min	96%	95%	92%	89%	87%	84%
OVP_max	96%	96%	92%	89%	87%	85%
OVP_ord1	95%	93%	89%	87%	86%	84%
OVP_ord2	95%	94%	90%	87%	86%	85%
OVP_zero	95%	94%	89%	87%	89%	89%
UB_Alg	96%	98%	99%	99%	99%	99%
UB_int	99%	98%	92%	87%	83%	81%
q-sort	82%	80%	82%	80%	80%	80%
c-sort	81%	78%	79%	79%	78%	78%
p-sort	95%	95%	95%	95%	95%	95%

Performance of each method is presented as the percentage of upper bound achieved by the method.

Table 4.7: Effect of the parameters on the upper bound

Method	Parameters				
	Intercept.	ω	q_mean	q_var	σ
UB	41242**	-3925*	265**	204	-2819**

* and ** denote statistical significance at the 5%, and 1% levels, respectively.

Finally, the performance of the benchmark method p-sort seems robust and near-optimal. However, remember that as presented in Table 4.5, the variance of the price has a significant effect on the performance of this method. While p-sort on overage provides 99% performance over cases with the lowest price variance level, its performance drops to 89% over cases with the highest price variance level.

4.5.2 Joint Pricing and Ranking

We numerically solve the joint ranking and pricing problem for more than 500 cases. As we mentioned, the retailer may set a higher price for a product when its potential demand is higher than its inventory. In that sense, the inventory level of products has a determining effect on the optimal pricing. Thus, We test three different inventory levels. Also, due to Proposition 4.3, σ is another determining parameter that limits the effect of the joint pricing on the profit. We investigate numerical cases with three different values of $\sigma = 2, 4, \text{ and } 8$. Also, these cases vary in the value of ω (i.e. $\omega = 0.95, 0.8, \text{ and } 0.6$).

Effect of the Pricing on the Profit Upper Bound

We provide the upper bound for each case by UB_Price algorithm. We measure the effect of each parameter on the upper bound using a regression. Table 4.7 represents the result of the regression.

Table 4.7 shows that ω , mean inventory level (q_mean), and ω have significant effects. As expected, profit is increasing in mean inventory level as the retailer can sell a larger number of high-profit products.

Table 4.8: Effect of the parameters on $\Delta\pi$

	Parameters				
	Intercept.	ω	q_mean	q_var	σ
$\Delta\pi$	0.023*	0.007	-0.001**	0.001	0.011**

* and ** denote statistical significance at the 5%, and 1% levels, respectively.

Table 4.9: Effect of the parameters on the performance of OVP_Price

Method	Parameters				
	Intercept.	ω	q_mean	q_var	σ
OVP_Price	0.866**	0.145**	-0.001**	0.004	-0.004*

* and ** denote statistical significance at the 5%, and 1% levels, respectively.

As mentioned in Section 4.4, instead of solving the joint ranking and pricing problem, the retailer may solve the ranking problem with $p_i = p_i^*$, $\forall i \in I$. We denote the difference between upper bound provided by UB_Alg and profit provided by LUB (with $p_i = p_i^*$, $\forall i \in I$) as $\Delta\pi$. Theoretically, we state that $\Delta\pi$ is bounded in Proposition 4.3. However, difference is even lower in numerical cases. The maximum value of $\Delta\pi$ is 14% (/9%, and 3%) over all cases (/where σ is 4, and 2). Using a regression, we investigate the effect of each parameter on the $\Delta\pi$. Table 4.8 represents the result of the regression. As expected, regression shows that average inventory level (q_mean) and σ have significant effects.

OVP_Price

We present the performance of our heuristic algorithm OVP_Price in comparison with the upper bound provided by UB_Alg. We measure the effect of the variant parameters using a regressions model. Table 4.9 represents the result of the regression. Similar, to other OVP methods, OVP_Price has a significantly increasing performance in ω . Average inventory level and σ have significant effects as well.

We solve each numerical case, using OVP_Price and compare it to the upper

bound provided by UB_Alg. We also solve each case with some benchmark methods. Followings are the list of these methods and variations:

1. UB: Even though optimization model UB does not guarantee the upper bound, we can solve the model efficiently with a convex problem solver.
2. UB_PP: We solve the LUB model with the set of predefined prices $p_i = p_i^*$, $\forall i \in I$ to measure the effect of joint pricing and ranking on the profit.
3. OVP_av/OVP_min/OVP_max: We solve the problem with OVP optimization problem where v is constant (i.e. not adaptive as we propose in OVP_Price) and is equal to $mean_i\{p_i d_i(p_i)\} / \min_i\{p_i d_i(p_i)\} / \max_i\{p_i d_i(p_i)\}$.
4. UB_Alg: We solve the UB_Alg model with set of predefined prices $p_i = p_i^*$, $\forall i \in I$.

We present the average performance of OVP_Price and other methods in Tables 4.10, 4.11, and 4.12.

Through all observations, the performance OVP_Price is higher than all benchmark OVP methods (i.e. OVP_av, OVP_min, and OVP_max). These results indicate that our adaptive algorithm to update the v_r values improve the performance of OVP methods significantly. Through all cases, we observe that solution of UB is so close to the upper bound. It means that the upper bound provided by UB_Price is as tight as UB.

As mentioned, $\Delta\pi$ is decreasing in inventory levels. Table 4.10 confirms this statement as we can observe that UB_PP and UB_Alg have lower performance when inventory levels are lower. Interestingly, OVP_Price method has a higher performance either when inventory levels are low or high. That is caused by adaptive v values which can be set to the minimum(/minimum) value for high(/low) inventories. Table 4.11 confirms our regression results that inventory variance (q-var) has no significant effect on the performance of the methods. Remember that both proposition 4.3 and Table 4.12 state the significance of σ on the pricing. Table 4.11 are

Table 4.10: Average performance of each method over different inventory levels, q_mean

Method	q_mean		
	$\sum_R \beta_r / 2[I]$	$\sum_R \beta_r / [I]$	$\sum_R \beta_r / 0.5[I]$
UB	98%	99%	99.9%
UB_PP	92%	96%	98%
OVP_Price	98%	92%	93%
OVP_av	90%	86%	88%
OVP_min	91%	88%	88%
OVP_max	90%	91%	86%
UB_Alg	91%	94%	97%

Performance of each method is presented as the percentage of upper bound achieved by the method.

Table 4.11: Average performance of each method over different variance levels of q_i , q_var

Method	q_var		
	0.2	0.5	0.8
UB	99%	99%	99%
UB_PP	95%	96%	95%
OVP_Price	95%	94%	95%
OVP_av	87%	89%	90%
OVP_min	89%	87%	88%
OVP_max	89%	89%	89%
UB_Alg	94%	95%	95%

Performance of each method is presented as the percentage of upper bound achieved by the method.

Table 4.12: Average performance of each method over different σ values

Method	σ		
	2	4	8
UB	99.9%	99%	98%
UB_PP	99%	96%	92%
OVP_Price	96%	94%	94%
OVP_av	94%	88%	84%
OVP_min	94%	87%	83%
OVP_max	94%	88%	87%
UB_Alg	98%	94%	91%

Performance of each method is presented as the percentage of upper bound achieved by the method.

aligned with this statement and shows that all methods based on predefined prices (i.e. UB_PP, UB_Alg, OVP_av, OVP_min, and OVP_max) have lower performance when σ has a higher value. However, OVP_Price has an outstanding performance adapting to higher σ values. As a result, while UB_Alg outperforms all methods when $\sigma = 2$, OVP_Price outperforms all methods for higher σ values

4.6 Conclusion

In this chapter, the research is motivated by short-term campaigns including products from different categories, designed by an online retailer to entice customers to visit the e-store. Thus, we assume that customers are window shoppers. It means that the customer browses online stores as a form of leisure or external search behavior without a current intent to purchase a specific product. Depending on the deals, a window-shopping customer may purchase some of the products. In such an environment, we analyze the joint pricing and ranking decisions of the online retailer when there is a finite inventory for each product.

Our research provides some key results as follows. We show that optimal pricing

and ranking decision are decomposable and straightforward when the inventory for each product is infinite. In contrast, the problem is more complicated when inventory is limited. Despite the deterministic and invertible structure of demand, we prove that the optimal price would result in product stock-out during the campaign in some cases. Also, we show that solving the problem with a set of predefined prices would result in a negligible profit loss if the diversity of the product's demand parameters is limited. In that sense, the effect of joint pricing on profit is dependent on and bounded by products' variety.

The precise optimization model for finite inventory case is intractable due to the integer and complex structure of the problem. Thus, we propose two heuristic methods to solve the problem and provide an upper bound by relaxing the integer variables. We show that the performance of the heuristic methods is dependent on the customer's search behavior. Our approximation heuristic method, OVP, has a better performance when customers have a higher tendency to search longer and consider more products. On the other hand, our heuristic method designed based on the upper bound solution, UB_Alg, has a better performance when customers only consider a few products at the top positions. Selecting the appropriate heuristic method we provide a near-optimal solution in all cases. Also, numerical analyses show that trivial solutions such as sorting by price or sorting by inventory would result in a significant profit loss.

As a summary, this is one of the first studies to address the problem of joint pricing and ranking when there is a finite inventory for each product. We also provide an analytical model to study i) the window shopper customers' behavior, and ii) the effect of product stock-out on ranking and presentation of the products. Finally, our results provide insight for online retailers to set the optimal ranking and pricing decisions.

4.6.1 Future Research

Multiple campaigns and Existence of Strategic Customers

We consider a single-session campaign where prices are fixed during the campaign and all leftover inventory units remain unsold. However, the retailer may use the leftover inventories in consecutive campaigns. Thus our study can be extended in three directions. First, the retailer might be interested in optimizing overall profit over multiple sessions. Second, the retailer may prefer dynamic pricing over sessions. Third, the existence of dynamic pricing may result in the strategic behavior of the customers. It means that customers may postpone their purchasing decision to receive better deals.

Existence of Substitute Products

Remember that we assume that demand is independent due to the structure of the campaigns. However, the study can be extended for the cases where some products are substitutes. In such a case, a set dependent demand should be implemented (e.g. substitution matrix, Bertrand price competition, or MNL model).

Chapter 5

CONCLUSION

In this dissertation, we consider the product ranking and pricing in online e-commerce platforms. We show the importance of product ranking in different centralized platforms such as e-retailers and decentralized platforms such as marketplaces. We propose theoretical and practical ranking and pricing strategies in each case.

In the first part, we focus on personalizing recommendation systems. We propose a personalized content-based method specially designed for online retailers using panel data. We use data on customers' activities at the store, such as clicks, and purchases. This property is useful for online retailers where there are fewer rating feedbacks, but there are several search sessions including clicks and purchases. We test the method using data provided by an apparel retailer. Our method outperforms benchmark methods (Collaborative filtering, Popular products, and MNL choice model) and it has a strictly better performance in recommending new products. Also, our method outperforms benchmark methods in recommending products to the customers who are generally interested in less popular (fringe) products. The main practical contribution of this study for online retailers is a novel personalized recommending system that is build-up based on panel data (instead of rating data) that causes better use of limited detail provided in the dataset and improves the quality of recommendation.

In the second part, we consider a decentralized pricing and ranking problem in an online marketplace where sellers decide on their prices simultaneously and ranking takes place based on a predefined rule by the marketplace operator (MO). Empirical studies show that the ranking of sellers is highly correlated with their prices. We investigate the effect of different ranking policies on the pricing competition between

the sellers as well as the profitability of the sellers and the marketplace. MO can implement one of two ranking policies: Under a binary ranking policy (BR), the seller with the lowest price always gets the highest ranking. Under a probabilistic ranking policy (PR), however, a seller increases the probability of receiving a higher ranking by lowering its price. We model the price competition as a full information game where all players (sellers) know the ranking policy and the demand function of themselves.

We characterize the unique symmetric Nash equilibrium of the PR game and show that no such equilibrium exists under BR. We characterize the range of prices in the mixed-strategy equilibrium under BR and show that BR leads to lower prices and lower seller profits than PR. Unexpectedly, even MO profit and total seller side surplus can be higher under PR when the profit margin of products is on the higher side.

Our study has three main contributions. First, we find that the optimal ranking rule for MO is dependent on the profit margin and cost of the products. Further, we show that the MO can increase the profit of all supply-side parties by implementing a less competitive ranking rule and higher commission rate. Surprisingly, in cases where MO participates as a seller with an inherently high demand for its product, assigning a higher rank to the MO product, even when it may not have the lowest price, increases both the total supply-side profit and demand surplus, hurting the seller profits.

In the third part, we study product ranking and pricing at e-commerce platforms, where customers are generally window shoppers. Window shoppers are a type of customer that browse online stores as a form of leisure or external search behavior without a clear intent to purchase a specific product. We develop an optimization model with deterministic demand and vertically differentiated positions (ranking) where products that are placed in earlier positions are more likely to be considered and, therefore, purchased. We show that revenue-ordered ranking is optimal when there is infinite inventory for products. However, when there is a finite inventory case, despite the deterministic and invertible structure of demand, we prove that

the optimal price would result in product stock-out during the campaign in some cases. Also, we show that solving the problem with a set of predefined prices (and not joint pricing ranking) would result in insignificant profit loss if the diversity of the product's demand parameters is limited.

The precise optimization model in the infinite inventory case is intractable due to the existence of integer variables and the complexity of the constraints. Thus, we propose two heuristic methods to solve the problem and provide an upper bound by relaxing the integer variables. We show that the performance of the heuristic methods is dependent on the customer's search behavior. Selecting the appropriate heuristic method, we provide a near-optimal solution in all cases. Finally, numerical analyses show that trivial solutions such as sorting by price or sorting by inventory would result in significant profit loss. To the best of our knowledge, this is the first study to address the problem of joint pricing and ranking where customers are window shoppers. We also provide an analytical model to study the effect of product stock-out on the ranking and presentation of the products. Finally, our results provide insight for online retailers to set the optimal ranking and pricing decisions when there is a limited inventory for each product.

We propose demand learning algorithms as a general direction to extend the study. Generally, the attractiveness of products is unknown to the sellers at the start of the season. Thus, the retailer may prefer to offer different sets of products to learn more about the demand for each product. Our research in all three chapters can be extended by implementing a learning algorithm. The problem is more complicated in the decentralized setting as we use in the second part. While the marketplace operator knows less about the products' attributes, third-party sellers are less informed about the customers' preferences. Thus, beyond the classic procedure, a learning method must address the information sharing between the operator and the sellers.

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Appendix A

A.1 Preliminary Lemmas and Structure of Profit Functions

Let us define $s' = \frac{s}{(1-\beta)}$ for ease of use in whole appendices.

A.1.1 Profit Functions Inequality

Lemma 1 For any price $p_i > s'$:

$$\pi_i^0(p_i, p_j) \geq \pi_i(p_i, p_j) \geq \pi_i^2(p_i, p_j) \quad (\text{A.1})$$

Proof of Lemma 1 For any price $p_i > s'$:

$$(1 - \beta)p_i - s \geq 0 \quad (\text{A.2})$$

then:

$$\pi_i^1(p_i, p_j) \geq 0 \quad (\text{A.3})$$

By definition of $\pi_i(p_i, p_j)$ represented in Equation 3.10:

$$\pi_i(p_i, p_j) = \pi_i^1(p_i, p_j) + \pi_i^2(p_i, p_j) \geq \pi_i^2(p_i, p_j) \quad (\text{A.4})$$

Also, we know that $0 \leq \gamma_i(p_i, p_j) \leq 1$. By definition of $\pi_i^1(p_i, p_j)$ represented in Equation 3.11:

$$\pi_i^0(p_i, p_j) \geq \pi_i(p_i, p_j) \geq \pi_i^2(p_i, p_j) \quad (\text{A.5})$$

A.1.2 Optimal Prices

To find the $p_i^{2*}(p_j)$ we can take derivative of $\pi_i^2(p_i, p_j)$:

$$\frac{\partial \pi_i^2(p_i, p_j)}{\partial p_i} = (1 - \alpha)(1 - \beta) \left((1 - b_2 p_i + c_2 p_j) - b_2 (p_i - s') \right) \quad (\text{A.6})$$

$$\frac{\partial^2 \pi_i^2(p_i, p_j)}{\partial p_i^2} = -2(1 - \alpha)(1 - \beta)(b_2) < 0 \quad (\text{A.7})$$

$$p_i^{2*}(p_j) = \frac{1 + c_2 p_j + b_2 s'}{2b_2} \quad (\text{A.8})$$

Similarly, we can find $p_i^{0*}(p_j)$ by taking derivative of $\pi_i^0(p_i, p_j)$:

$$\frac{\partial \pi_i^0(p_i, p_j)}{\partial p_i} = (1 - \beta) \left(\alpha \cdot (1 - b_1 p_i) + (1 - \alpha)(1 - b_2 p_i + c_2 p_j) - (\alpha \cdot b_1 + (1 - \alpha)b_2) (p_i - s') \right) \quad (\text{A.9})$$

$$\frac{\partial^2 \pi_i^0(p_i, p_j)}{\partial p_i^2} = -2(1 - \beta)(\alpha \cdot b_1 + (1 - \alpha)b_2) < 0 \quad (\text{A.10})$$

$$p_i^{0*}(p_j) = \frac{1 + (1 - \alpha)c_2 p_j}{2(\alpha \cdot b_1 + (1 - \alpha)b_2)} + \frac{s'}{2} \quad (\text{A.11})$$

As mentioned, we are interested in the profit of marketplace operator when prices are equal i.e. $p_i = p_j = p$:

$$\pi_m(p, p) = \beta \cdot p \cdot (\alpha \cdot (1 - b_1 p) + 2(1 - \alpha)(1 - b_2 p + c_2 p)) \quad (\text{A.12})$$

$$\frac{\partial \pi_m(p, p)}{\partial p} = \beta (\alpha \cdot (1 - b_1 p) + 2(1 - \alpha)(1 - b_2 p + c_2 p) + p(-\alpha b_1 - 2(1 - \alpha)(b_2 - c_2))) \quad (\text{A.13})$$

$$\frac{\partial^2 \pi_m(p, p)}{\partial p^2} = \beta (2\alpha \cdot (-b_1) + 4(1 - \alpha)(-b_2 + c_2)) \quad (\text{A.14})$$

Then,

$$p^{mo*} = \frac{2 - \alpha}{2(\alpha b_1 + 2(1 - \alpha)(b_2 - c_2))} \quad (\text{A.15})$$

A.2 Proof of Proposition 3.1

A.2.1 Proposition 3.1

Optimal Price in the Games with Deterministic Ranking Rule

1. If $p_i^{0*}(p_j) \geq \bar{p}_i(p_j) = p_j - \epsilon$, then $p_i^{1*}(p_j) = \bar{p}_i(p_j)$ and the best response of seller i ($p_i^*(p_j)$) is equal to either $p_i^{2*}(p_j)$ or $\bar{p}_i(p_j)$.
2. If $p_i^{0*}(p_j) < \bar{p}_i(p_j) = p_j - \epsilon$, then $p_i^*(p_j) = p_i^{1*}(p_j) = p_i^{0*}(p_j)$.

A.2.2 A Lemma for Proof of Proposition 3.1 Part 1

Lemma 2 For any pair of prices p^1, p^2 , such that $p^1 < p^2 < p_i^{0*}(p_j)$:

$$\pi_i^0(p^1, p_j) < \pi_i^0(p^2, p_j) < \pi_i^0(p_i^{0*}(p_j), p_j) \quad (\text{A.16})$$

Proof of Lemma 2 Due to the definition, $\pi_i^0(p_i, p_j)$ is a linear combination of two differentiated Bertrand profit function (that is a convex function) and subsequently, it is concave. Also, by the definition, $p_i^{0*}(p_j)$ is the optimal point of the function that returns the maximum value of the function. Then, Lemma 2 is a property of concave functions.

A.2.3 Proof of Proposition 3.1 Part 1

We evaluate the Proposition 3.1 part 1 in two price regions as follows:

1. For any price $p' \leq \bar{p}_i(p_j) = p_j - \epsilon$:

$$\pi_i(p', p_j) = \pi_i^0(p', p_j) \quad (\text{A.17})$$

And by lemma 1:

$$\pi_i^0(p', p_j) \leq \pi_i^0(\bar{p}_i(p_j), p_j) \quad (\text{A.18})$$

$$\pi_i^0(p', p_j) = \pi_i(p', p_j) \leq \pi_i(\bar{p}_i(p_j), p_j) = \pi_i^0(\bar{p}_i(p_j), p_j) \quad (\text{A.19})$$

2. For any price $p'' > \bar{p}_i(p_j)$:

$$\pi_i(p'', p_j) = \pi_i^2(p'', p_j) \quad (\text{A.20})$$

By the definition, $p_i^{2*}(p_j)$ is the optimal point of $\pi_i^2(p_i, p_j)$ (that returns the maximum value of the function). Then:

$$\pi_i^2(p'', p_j) \leq \max_{P_i} \pi_i^2(p_i, p_j) = \pi_i^2(p_i^{2*}(p_j), p_j) \quad (\text{A.21})$$

Also, by Lemma 1, for any price $p_i > s'$:

$$\pi_i(p_i, p_j) \geq \pi_i^2(p_i, p_j) \quad (\text{A.22})$$

Then:

$$\pi_i^2(p'', p_j) = \pi_i(p'', p_j) \leq \pi_i^2(p_i^{2*}(p_j), p_j) \leq \pi_i(p_i^{2*}(p_j), p_j) \quad (\text{A.23})$$

Equation A.19 shows that $\bar{p}_i(p_j)$ is the optimal price of seller i in the price range $p' \leq \bar{p}_i(p_j)$. On the other hand, Equation A.23 shows that $p_i^{2*}(p_j)$ is the optimal price of seller i in the price range $p'' > \bar{p}_i(p_j)$. Considering both equations, we conclude that if $\bar{p}_i(p_j) \leq p_i^{0*}(p_j)$, best response of seller i ($p_i^*(p_j)$) is either $p_i^{2*}(p_j)$ or $\bar{p}_i(p_j)$.

A.2.4 Proof of Proposition 3.1 Part 2

If $\bar{p}_i(p_j) > p_i^{0*}(p_j)$, then:

$$\pi_i(p_i^{0*}(p_j), p_j) = \pi_i^0(p_i^{0*}(p_j), p_j) \quad (\text{A.24})$$

By the definition, $p_i^{0*}(p_j)$ is the optimal point that return the maximum value of $\pi_i^0(p_i, p_j)$, It means that:

$$\pi_i^0(p_i^{0*}(p_j), p_j) > \pi_i^0(P, p_j) \quad \forall P \neq (p_i^{0*}(p_j)) \quad (\text{A.25})$$

By Lemma 1, for any price $p_i > s'$:

$$\pi_i(p_i, p_j) \leq \pi_i^0(p_i, p_j) \quad (\text{A.26})$$

then:

$$\pi_i(p_i^{0*}(p_j), p_j) = \pi_i^0(p_i^{0*}(p_j), p_j) \geq \pi_i^0(P, p_j) \geq \pi_i(P, p_j) \quad \forall P \neq (p_i^{0*}(p_j)) \quad (\text{A.27})$$

Thus, we conclude that if $p_i^{0*}(p_j) < \bar{p}_i(p_j)$, then, the best response of seller i ($p_i^*(p_j)$) is $p_i^{0*}(p_j)$.

A.3 Proof of Proposition 3.2

A.3.1 Proposition 3.2

Role of Thresholds T_i^1, T_i^2 , in Game DR.

- $T_i^1 \leq T_i^2$,
- In Game DR, only price range of $T_i^1 \leq p_i \leq T_i^2$ survive iterated elimination of strictly dominated strategy.

A.3.2 Proof of Proposition 3.2 Part 1

Remember that y_j^t is the minimum p_j value such that is optimal for seller i to compete for winning the first position by setting a lower price. Subsequently, T_i^1 is best response to y_j^t , defined as follows:

$$y_j^t = \min\{p_j | p_i^*(p_j) = p_i^{1*}(p_j)\} \quad (\text{A.28})$$

$$T_i^1 = p_i^*(y_j^t) = p_i^{1*}(y_j^t) \quad (\text{A.29})$$

Equation A.29 implies that:

$$T_i^1 < y_j^t \quad (\text{A.30})$$

Note that in symmetric games $T_i^1 = T_j^1$ and $T_i^2 = T_j^2$.

We prove Proposition 3.2 Part 1 by contradiction. Let assume that $T_i^1 > T_i^2$. Then, in a game with deterministic ranking rule, we have following equation :

$$\gamma_i(T_i^2, T_j^1) = 1 \quad (\text{A.31})$$

By its definition, $T_i^2 = p_i^*(T_j^1)$. Considering Equation A.22, we conclude that:

$$p_i^{1*}(T_j^1) = p_i^*(T_j^1) = T_i^2 \quad (\text{A.32})$$

Equation A.32 shows that T_i^1 satisfies the condition of Equation A.28. It means that:

$$y_j^t \leq T_i^1 \quad (\text{A.33})$$

Equation A.33 contradicts with Equation A.30, caused by our starting assumption $T_i^1 > T_i^2$.

A.3.3 Lemmas for Proof of Proposition 3.2 Part 2

Lemma 3 .

1. $\pi_i(p_j - \epsilon, p_j)$ is concave in p_j .

2. $\pi_i^2(p_i^{2*}(p_j), p_j)$ is convex in p_j .

3. $\pi_i(p_j - \epsilon, p_j) - \pi_i^2(p_i^{2*}(p_j), p_j)$ is concave in p_j .

Proof of Lemma 3 Let's form the $\pi_i(p_j - \epsilon, p_j)$ (the profit of the price that guarantees the first position for seller p_j) and $\pi_i^2(p_i^{2*}(p_j), p_j)$ (Maximum profit when seller i is placed in the second position) as follows:

$$\pi_i(p_j - \epsilon, p_j) = (1 - \beta)(p_j - \epsilon - s')(1 - (\alpha b_1 + (1 - \alpha)b_2)(p_j - \epsilon) + (1 - \alpha)c_2 p_j) \quad (\text{A.34})$$

By Equation A.8, we know that:

$$p_i^{2*}(p_j) = \frac{1 + c_2 p_j + b_2 s'}{2b_2} \quad (\text{A.35})$$

Then:

$$\begin{aligned} \pi_i^2(p_i^{2*}(p_j), p_j) &= (1 - \beta)(p_i^{2*}(p_j) - s')((1 - \alpha)(1 - b_2 p_i^{2*}(p_j) + c_2 p_j) \\ &= (1 - \beta)(1 - \alpha) \frac{(1 + c_2 p_j - b_2 s')^2}{4b_2} \end{aligned} \quad (\text{A.36})$$

Then, we can take derivative of each function as follows:

$$\begin{aligned} \frac{\partial \pi_i(p_j - \epsilon, p_j)}{\partial p_j} &= \\ (1 - \beta)(1 - (\alpha b_1 + (1 - \alpha)b_2)(p_j - \epsilon) + (1 - \alpha)c_2 p_j) \\ &+ (p_j - \epsilon - s')(-(\alpha b_1 + (1 - \alpha)b_2 - (1 - \alpha)c_2)) \end{aligned} \quad (\text{A.37})$$

$$\frac{\partial^2 \pi_i(p_j - \epsilon, p_j)}{\partial p_j^2} = (1 - \beta)(-2(\alpha b_1 + (1 - \alpha)(b_2 - c_2))) \quad (\text{A.38})$$

Remember that, we assume $b_1 < b_2 - c_2$. It implies that $b_2 > c_2$. It means that the value of Equation A.38 is negative. Then, we conclude that $\pi_i(p_j - \epsilon, p_j)$ is concave in p_j .

$$\frac{\partial \pi_i^2(p_i^{2*}(p_j), p_j)}{\partial p_j} = \frac{(1 - \beta)(1 - \alpha)}{2b_2} (c_2 + (c_2)^2 p_j - b_2 c_2 s') \quad (\text{A.39})$$

$$\frac{\partial^2 \pi_i^2(p_i^{2*}(p_j), p_j)}{\partial p_j^2} = \frac{(1 - \beta)(1 - \alpha)}{2b_2} ((c_2)^2) > 0 \quad (\text{A.40})$$

Equation A.40 implies that $\pi_i^2(p_i^{2*}(p_j), p_j)$ is convex in p_j . Subsequently, $\pi_i(p_j - \epsilon, p_j) + (-\pi_i^2(p_i^{2*}(p_j), p_j))$ is sum of two concave function and it is concave in p_j .

$$\frac{\partial^2 \pi_i(p_j - \epsilon, p_j)}{\partial p_j^2} - \frac{\partial^2 \pi_i^2(p_i^{2*}(p_j), p_j)}{\partial p_j^2} = -4(\alpha b_1 + (1-\alpha)(b_2 - c_2))b_2(1-\alpha)(c_2)^2 \quad (\text{A.41})$$

Bertrand model implies that $b_2 > c_2$. It means that the value of Equation A.38 is negative. Then, we conclude that $\pi_i(p_j - \epsilon, p_j) - \pi_i^2(p_i^{2*}(p_j), p_j)$ is concave.

Let us define T_j^3 as follows:

$$T_j^3 = p_i^{0*}(T_j^3) \quad (\text{A.42})$$

Lemma 4 .

$$\pi_i(p_i^{1*}(T_j^3), T_j^3) > \pi_i^2(p_i^{2*}(T_j^3), T_j^3) \quad (\text{A.43})$$

Proof of Lemma 4

Recall that:

$$p_i^{2*}(p_j) = \frac{1 + c_2 p_j + b_2 s'}{2b_2} \quad (\text{A.44})$$

and,

$$p_i^{0*}(p_j) = \frac{1 + (1-\alpha)c_2 p_j}{2(\alpha b_1 + (1-\alpha)b_2)} + \frac{s'}{2} \quad (\text{A.45})$$

It shows that for any $p_i < \frac{1}{b_1}$, $p_i^{2*}(p_j) \leq p_i^{0*}(p_j) - \epsilon < p_i^{0*}(p_j)$. We know that $T_j^3 < \frac{1}{b_1}$. Then by Lemma 1 and definition of $p_i^{0*}(p_j)$:

$$\pi_i(p_i^{1*}(T_j^3), T_j^3) \geq \pi_i(p_i^{2*}(T_j^3), T_j^3) \geq \pi_i^2(p_i^{2*}(T_j^3), T_j^3) \quad (\text{A.46})$$

Lemma 5 .

1. For any $p_j > T_i^1$, $p_i^*(p_j) < p_j$
2. For any $p_j < T_i^1$, $p_i^*(p_j) = p_i^{2*}(p_j) \leq T_i^2$

Proof of Lemma 5

1. Part 1:

We investigate the first part of Lemma 5 in two price regions:

(a) $p_j > T_j^3$:

By definition of T_j^3 , in this price region, $p_i^{0*}(p_j) \leq \bar{p}_i(p_j) = p_j - \epsilon$. Then, by Proposition 3.1 Part 2, $p_i^*(p_j) = p_i^{0*}(p_j) < p_j$.

(b) $p_j \leq T_j^3$:

By definition of T_j^3 , in this price region, $p_i^{0*}(p_j) \geq \bar{p}_i(p_j)$, and by Proposition 3.1 Part 1, $p_i^*(p_j) = p_i^{2*}(p_j)$ or $p_i^*(p_j) = \bar{p}_i(p_j) = p_j - \epsilon$. We show that for $p_j > T_i^1$, $p_i^*(p_j) = \bar{p}_i(p_j) = p_j - \epsilon$ in following steps:

i. By Lemma 3, we know that $\pi_i(p_j - \epsilon, p_j) - \pi_i^2(p_i^{2*}(p_j), p_j)$ is concave.

Also, by Proposition 3.1 Part 1, in this price region, $p_i^{1*}(p_j) = p_j - \epsilon$.

It means that if we have a pair of price $(p^1 < p^2)$ such that:

$$\pi_i(p_i^{1*}(p^1), p^1) - \pi_i^2(p_i^{2*}(p^1), p^1) > 0 \quad (\text{A.47})$$

$$\pi_i(p_i^{1*}(p^2), p^2) - \pi_i^2(p_i^{2*}(p^2), p^2) > 0 \quad (\text{A.48})$$

Then, for any price, p , such that $p^1 \leq p \leq p^2$, we have following inequality:

$$\pi_i(p_i^{1*}(p), p) - \pi_i^2(p_i^{2*}(p), p) > 0 \quad (\text{A.49})$$

ii. Remember that by definition of y_j^t :

$$p_i^*(y_j^t) = p_i^{1*}(y_j^t) = T_i^1 = y_j^t - \epsilon \quad (\text{A.50})$$

It means that inequality A.47 holds for $T_i^1 + \epsilon = y_j^t$:

$$\pi_i(p_i^{1*}(T_i^1 + \epsilon), T_i^1 + \epsilon) > \pi_i^2(p_i^{2*}(T_i^1 + \epsilon), T_i^1 + \epsilon) \quad (\text{A.51})$$

Also, by Lemma 4, we know that inequality A.48 holds for T_j^3 :

$$\pi_i(p_i^{1*}(T_j^3), T_j^3) > \pi_i^2(p_i^{2*}(T_j^3), T_j^3) \quad (\text{A.52})$$

- iii. We conclude that for any p such that $T_i^1 < p \leq T_j^3$, we have following inequality:

$$\pi_i(p_i^{1*}(p_j), p_j) > \pi_i(p_i^{2*}(p_j), p_j) \quad (\text{A.53})$$

Subsequently, $p_i^*(p_j) = \bar{p}_i(p_j) = p_j - \epsilon$

2. Part 2

By the definition of T_i^1 , we know that for any $p_j < T_i^1$ we have $p_i^*(p_j) = p_i^{2*}(p_j)$. Also, $p_i^{2*}(p_j)$ is increasing in p_j ($\frac{\partial p_i^{2*}}{\partial p_j} = \frac{c}{2b_2} > 0$). It means that for any $p_j < T_i^1$, $p_i^{2*}(p_j) \leq p_i^{2*}(T_i^1) = T_i^2$. Then, we conclude that for any $p_j < T_i^1$, $p_i^*(p_j) \leq T_i^2$

A.3.4 Proof of Proposition 3.2 Part 2

Let's define p^{max} such that $d_i(p^{max}, p^{max}) = 0$). Recall that we assumed that there are some prices such that profit is positive for sellers. Hence any rational player do not set any price $p \geq p^{max}$. Then, we define some steps of the iterated elimination of strictly dominated strategies (i.e. never best responses):

1. Steps type n - Starting with $[0, p_i^n]$ if $p_i^n > T_i^2$, the strategies narrow down to $[0, p_i^n - \epsilon]$:

By Lemma 5, always $p_i^*(p_j) < p_j$, hence, p_i^n is never best response.

2. Step types m: At any iteration starting with $[p_i^m, T_i^2]$ if $p_i^m < T_i^1$, the strategies narrow down to $[p_i^{2*}(p_i^m), T_i^2]$.

Iterated iterated elimination of strictly dominated strategies (i.e. never best responses) by steps types n and m ends up with strategy region $[T_i^1, T_i^2]$.

A.4 Proof of Proposition 3.3

A.4.1 Proposition 3.3

There exist a unique symmetric pure strategy Nash Equilibrium of Game PR.

A.4.2 Proof of Proposition 3.3

We take derivative of $\pi_i^{PR}(p_i, p_j)$ to find the Nash equilibrium of Game PR.

$$\begin{aligned} \frac{\partial \pi_i^{PR}(p_i, p_j)}{\partial p_i} = \\ (1 - \beta) \left(\alpha \frac{p_j \cdot (1 - b_1 p_i)}{p_i + p_j} + (1 - \alpha) (1 - b_2 p_i + c_2 p_j) - (p_i - s') \left(\frac{\alpha \cdot p_j \cdot (1 + b_1 p_j)}{(p_i + p_j)^2} + (1 - \alpha) b_2 \right) \right) \end{aligned} \quad (\text{A.54})$$

We set the prices equal i.e. $p_i = p_j = p$ to investigate symmetric Nash equilibrium.

$$\frac{\partial \pi_i^{PR}(p, p)}{\partial p} = \frac{(1 - \beta)}{4} ((-3\alpha b_1 - 4(1 - \alpha)(2b_2 - c_2))p + (4 - 3\alpha + s'(\alpha b_1 + 4(1 - \alpha)b_2)) + \frac{s'\alpha}{p}) \quad (\text{A.55})$$

For ease of use, let's define $f'(p)$ as follows:

$$f'(p) = \frac{\partial \pi_i^{PR}(p, p)}{\partial p} \cdot \frac{4}{(1 - \beta)} \quad (\text{A.56})$$

$$f'(p) = (-3\alpha b_1 - 4(1 - \alpha)(2b_2 - c_2))p + (4 - 3\alpha + s'(\alpha b_1 + 4(1 - \alpha)b_2)) + \frac{s'\alpha}{p} \quad (\text{A.57})$$

Further, we use the $f'(p)$ to compare T_i^1 , T_i^2 , e_i^2 and p^{mo*} . To find the symmetric Nash equilibrium of Game PR with check the first order condition $f'(p) = 0$. This equation is equivalent to following quadratic equation.

$$(-3\alpha b_1 - 4(1 - \alpha)(2b_2 - c_2))p^2 + (4 - 3\alpha + s'(\alpha b_1 + 4(1 - \alpha)b_2))p + s'\alpha = 0 \quad (\text{A.58})$$

The inequality below shows that the roots of this quadratic equation have different signs. It means that there is only one positive solution for the equation.

$$(-3\alpha b_1 - 4(1 - \alpha)(2b_2 - c_2)) \cdot s'\alpha < 0 \quad (\text{A.59})$$

Then, we check the second order condition.

$$\begin{aligned}
& \frac{\partial^2 \pi_i^{PR}(p_i, p_j)}{\partial p_i^2} = \\
& (1 - \beta) \left(\frac{-\alpha \cdot p_j \cdot (1 + b_1 p_j)}{(p_i + p_j)^2} - (1 - \alpha) b_2 - \left(\frac{\alpha \cdot p_j \cdot (1 + b_1 p_i)}{(p_i + p_j)^2} + (1 - \alpha) b_2 \right) + \right. \\
& \quad \left. (p_i - s') \left(\frac{-2 \cdot \alpha \cdot p_j \cdot (1 + b_1 p_j)}{(p_i + p_j)^3} \right) = \right. \\
& -2(1 - \beta) \left(\frac{\alpha \cdot p_j \cdot (1 + b_1 p_j)}{(p_i + p_j)^2} + (1 - \alpha) b_2 + (p_i - s') \left(\frac{\alpha \cdot p_j \cdot (1 + b_1 p_j)}{(p_i + p_j)^3} \right) \right) = \\
& \quad \left. -2(1 - \beta) \left(\frac{\alpha \cdot p_j \cdot (1 + b_1 p_j)}{(p_i + p_j)^2} \left(1 - \frac{(p_i - s')}{(p_i + p_j)} \right) + (1 - \alpha) b_2 \right) \right) \quad (\text{A.60})
\end{aligned}$$

Equation A.60 show that $\frac{\partial^2 \pi_i^{PR}(p_i, p_j)}{\partial p_i^2}$ is negative and payoff function $\pi_i^{PR}(p_i, p_j)$ is concave. We conclude that the positive root of Equation A.58 is unique symmetric Nash equilibrium of Game PR.

A.5 Proof of Proposition 3.4

A.5.1 Proposition 3.4

marketplace operator's Optimal Ranking Rule

For any game that Condition 3.27 holds, there are cost values $\underline{s}$ and $\bar{s}$ such that:

- If $s \leq \underline{s}$:

$$T_i^1 \leq T_i^2 \leq e^{PR} \leq p^{mo*}$$

$$\pi_m(e^{DR}) < \pi_m(e^{PR}) < \pi_m(p^{mo*})$$

- If $s \geq \bar{s}$:

$$p^{mo*} \leq T_i^1 \leq T_i^2 \leq e^{PR}$$

$$\pi_m(p^{mo*}) > \pi_m(e^{DR}) > \pi_m(e^{PR})$$

A.5.2 Definition of $\underline{s}$ and $\bar{s}$

Recall that, we define Let $f'(P) = \frac{\partial \pi_i^{PR}(p,p)}{\partial p} \cdot \frac{4}{(1-\beta)}$ in equation A.56. Also we proved that $\pi_i^{PR}(p,p)$ is concave. It mean that we have following properties of $f'(P)$ function:

- $f'(p) = 0 \iff p = e^{PR}$,
- $f'(p) < 0 \iff p > e^{PR}$
- $f'(p) > 0 \iff p < e^{PR}$

Then, to have the inequality $e^{PR} \leq p^{mo*}$, we define $\underline{s}$ such that $f'(p^{mo*}) = 0$:

$$f'(p^{mo*}) = \frac{-\alpha b_1 - 4(1-\alpha)((1-\alpha)(2-\alpha)b_2 - \alpha b_1 + 2(1-\alpha)(b_2 - c))}{2(\alpha b_1 + 2(1-\alpha)(b_2 - c))} + \frac{2\alpha b_1 + 8(1-\alpha)b_2 + \alpha^2 b_1 - 4\alpha(1-\alpha)c}{2-\alpha} \quad (A.61)$$

Then:

$$\underline{s} = \frac{(2-\alpha)(\alpha b_1 + 4(1-\alpha)((1-\alpha)(2-\alpha)b_2 - \alpha b_1 + 2(1-\alpha)(b_2 - c)))}{2(\alpha b_1 + 2(1-\alpha)(b_2 - c))(2\alpha b_1 + 8(1-\alpha)b_2 + \alpha^2 b_1 - 4\alpha(1-\alpha)c)} \quad (A.62)$$

On other hand ,we define $\bar{s} = \max\{\bar{s}^1, \bar{s}^2\}$, where:

$$\bar{s}^1 = p^{mo*} \quad (A.63)$$

$$\bar{s}^2 = \frac{2b_2 - (1-\alpha)(b_2 - 2c_2)}{(\alpha b_1(3b_2 + c_2) + (1-\alpha)(3b_2 + 2c_2))} \quad (A.64)$$

$\bar{s}^1$ guarantees that all prices (including T_i^1 , T_i^2 , and e^{PR}) are higher than p^{mo*} . $\bar{s}^2$ guarantees that for any mixed strategy Nash equilibrium of Game DR, $\pi_m(T_i^2, T_j^2) > \pi_m(e^{DR}, e^{DR})$.

A.5.3 Proof of Proposition 3.4

Part 1: Order of p^{mo*} By the definition of $\underline{s}$, for $s' \leq \underline{s}$, $f'(p^{mo*}) \leq 0$. By convexity of $f'(\cdot)$, if $f'(P) \leq 0$, then $p \geq e^{PR}$. So, we conclude that for $s' \leq \underline{s}$, $e^{PR} \leq p^{mo*}$.

On the other hand, we know that for $p < s'$ profit of a seller is negative. Also, we have assume that there some prices such that profit of the sellers are positive. Then $\leq T_i^1 \geq s' \geq p^{mo*}$. Part 2: $T_i^1 \leq e^{PR}$

We show that $T_i^1 \leq e^{PR}$ by contradiction: Let assume that $T_i^1 > e^{PR}$. By the definition of T_j^1 , for $p_j \leq T_j^1$:

$$\pi_i(p_i, p_j) < \pi_i^2(p_i^{2*}(p_j), p_j) \quad \forall p_i \leq p_j \quad (\text{A.65})$$

By Proposition 3.2, $T_i^1 < T_i^2$. It means that $e^{PR} < T_i^2$. Also, by definition of profit functions, in both ranking rules, if two sellers set prices the same, they would be represented at rank 1, half of the time. It means that:

$$\pi_i^{PR}(e^{PR}, e^{PR}) = \pi_i(e^{PR}, e^{PR}) < \pi_i^2(p_i^{2*}(p_j), p_j) \quad (\text{A.66})$$

Equation A.66 shows that seller i has a profitable deviation from e^{PR} which contradict with the definition of Nash equilibrium caused by our starting assumption $T_i^1 > e^{PR}$.

Part 3: $T_i^2 \leq e^{PR}$

Let define p^x as the price such that $p_i^{2*}(p^x) = p^x = \frac{1+c_2p_j+b_2s'}{2b_2}$.

Lemma 6 .

For any $p_j < p^x$, $p_j < p_i^{2*}(p_j) < p^x$

Proof of Lemma 6 The derivative of p_i^{2*} to p_j is positive and below 1. Also, $p_i^{2*}(0)$ is a strictly positive value. and by the definition $p_i^{2*}(p_j)$ intersect the $y = x$ at point p^x . Then, for any $p_j < p^x$, $p_j < p_i^{2*}(p_j) < p^x$

Lemma 7 .

For any $T_i^2 \leq p^x$

Proof of Lemma 7 By Proposition 3.2, we know that $T_i^1 < T_i^2 = p_i^{2*}(T_i^1)$. By the definition of p^x , if For any , $p_j < p_i^{2*}(p_j)$, $p_j < p^x$. It implies that $T_i^2 \leq p^x$.

We evaluate $f'(p^x)$:

$$f'(p^x) = \frac{-3\alpha b_1(s'b_2 + 1)}{2b_2 - c_2} - 4(1 - \alpha)(s'b_2 + 1) + 4 - 3\alpha + s'(\alpha b_1 - 4(1 - \alpha)b_2) - \frac{s'\alpha(2b_2 - c_2)}{(s'b_2 + 1)}$$

(A.67)

$$f'(p^x) = \frac{\alpha}{(2b_2 - c_2)(s'b_2 + 1)} \left((s'b_2 + 1) \left(-b_1b_2s' - 3b_1 + 2b_2 - c_2 - s'b_1c_2 \right) + s'(2b_2 - c_2)^2 \right) \quad (\text{A.68})$$

$$f'(p^x) = \frac{\alpha}{(2b_2 - c_2)(s'b_2 + 1)} \left(-(s')^2b_2(b_1b_2 + b_1c_2) + s'(-4b_1b_2 + 6b_2^2 - 5b_2c_2 + c_2^2 - b_1c_2) + 2b_2 - 3b_1 - c_2 \right) \quad (\text{A.69})$$

Remember that, we assume that cost is not too high such that there are some prices which result in positive profits. It implies that $s' < \frac{1}{b_1}$. Then:

$$f'(p^x) > \frac{\alpha}{(2b_2 - c_2)(s'b_2 + 1)} \left(s'(-4b_1b_2 + 5b_2^2 - 6b_2c_2 + c_2^2 - b_1c_2) + 2b_2 - 3b_1 - c_2 \right) \quad (\text{A.70})$$

Now, we can use condition 3.27:

$$f'(p^x) > \frac{\alpha}{(2b_2 - c_2)(s'b_2 + 1)} \left(s'(-4b_1b_2 + 2(3b_1 + 2c_2)b_2 + (b_2^2 - 2b_2c_2 + c_2^2) - 4b_2c_2 - b_1c_2) + c_2 \right) \quad (\text{A.71})$$

By condition of Bertrand model, $c_2 < b_2$. Thus, we conclude that:

$$f'(p^x) > \frac{\alpha}{(2b_2 - c_2)(s'b_2 + 1)} \left(s'(b_1(2b_2 - c_2 + (b_2^- c_2)^2) + c_2) \right) > 0 \quad (\text{A.72})$$

It proves that $p^x < e^{PR}$. Also, due to Lemma 7, we know that $T_i^2 < p^x$. We conclude that $T_i^2 < p^x < e^x$.

Part 4: Order of profits We proved that the profit function $\pi_m(p, p)$ is concave in p and p^{mo*} is the optimal point of the function.

$$\pi_m(p, p) = \beta \cdot p \cdot (\alpha \cdot (1 - b_1p) + 2(1 - \alpha)(1 - b_2p + c_2p)) \quad (\text{A.73})$$

$$\frac{\partial \pi_m(p, p)}{\partial p} = \beta (\alpha \cdot (1 - b_1p) + 2(1 - \alpha)(1 - b_2p + c_2p) + p(-\alpha b_1 - 2(1 - \alpha)(b_2 - c_2))) \quad (\text{A.74})$$

$$\frac{\partial^2 \pi_m(p, p)}{\partial p^2} = \beta (2\alpha \cdot (-b_1) + 4(1 - \alpha)(-b_2 + c_2)) \quad (\text{A.75})$$

Then,

$$p^{mo*} = \frac{2 - \alpha}{2(\alpha b_1 + 2(1 - \alpha)(b_2 - c_2))} \quad (\text{A.76})$$

For any pair of prices p^1, p^2 , such that $p^1 < p^2 < p_i^{0*}(p_j)$:

$$\pi_i^0(p^1, p_j) < \pi_i^0(p^2, p_j) < \pi_i^0(p_i^{0*}(p_j), p_j) \quad (\text{A.77})$$

Part 5 Profit of mix strategies For ease of use, let $\pi_m^{DR}(p_j, p_j)$ denotes profit of MO when ranking rule is deterministic. Then we have following equations:

$$\begin{aligned} \pi_m^{DR}(p_j, p_j) = & \beta (1[p_i \leq p_j] \cdot p_i \cdot \alpha \cdot (1 - b_1 p_i) + 1[p_i < p_j] \cdot p_i \cdot \alpha \cdot (1 - b_1 p_i)) + \\ & (1 - \alpha) p_i \cdot (1 - b_2 p_i + c_2 p_j) + (1 - \alpha) p_j \cdot (1 - b_2 p_j + c_2 p_i) \end{aligned} \quad (\text{A.78})$$

$$\begin{aligned} \pi_m^{DR}(T_i^2, T_i^2) = & \beta (T_i^2 \cdot \alpha \cdot (1 - b_1 T_i^2) + \\ & 2(1 - \alpha) T_i^2 \cdot (1 - (b_2 - c_2) T_i^2)) \end{aligned} \quad (\text{A.79})$$

$$\begin{aligned} \pi_m^{DR}(T_i^2 - \delta, T_i^2) = & \beta ((T_i^2 - \delta) \alpha \cdot (1 - b_1 (T_i^2 - \delta)) + \\ & (1 - \alpha) (T_i^2 - \delta) \cdot (1 - b_2 (T_i^2 - \delta) + c_2 T_j^2) + (1 - \alpha) T_j^2 \cdot (1 - b_2 T_j^2 + c_2 (T_i^2 - \delta))) \end{aligned} \quad (\text{A.80})$$

Let's define $\Delta \pi_m$ as follows:

$$\begin{aligned} \Delta \pi_m = & \pi_m^{DR}(T_i^2 - \delta, T_i^2) - \pi_m^{DR}(T_i^2, T_i^2) = \\ & \beta (\alpha \cdot (-\delta + 2b_1 T_i^2 \delta - b_1 \delta^2)) + (1 - \alpha) (-\delta + 2b_2 T_i^2 \delta - b_2 \delta^2 - 2c_2 T_j^2 \delta) = \\ & \beta \delta (\alpha \cdot (-1 + 2b_1 T_i^2 - b_1 \delta)) + (1 - \alpha) (-1 + 2b_2 T_i^2 - b_2 \delta - 2c_2 T_j^2) = \\ & \beta \delta (-1 - \alpha b_1 \delta - (1 - \alpha) b_2 \delta + 2(\alpha b_1 + (1 - \alpha) b_2 - c_2) T_i^2) \end{aligned} \quad (\text{A.81})$$

We already showed that for $s' \leq \underline{s}$, $T_i^2 < p^{mo*}$. Then:

$$\begin{aligned} \Delta \pi_m & < \beta \delta (-1 - \alpha b_1 \delta - (1 - \alpha) b_2 \delta + 2(\alpha b_1 + (1 - \alpha) b_2 - c_2) \frac{(2 - \alpha)}{2(\alpha b_1 + 2(1 - \alpha)(b_2 - c_2))}) = \\ & \beta \delta (-\alpha b_1 \delta - (1 - \alpha) b_2 \delta + \frac{(2 - \alpha)(\alpha b_1 + (1 - \alpha) b_2 - c_2) - (\alpha b_1 + 2(1 - \alpha)(b_2 - c_2))}{\alpha b_1 + 2(1 - \alpha)(b_2 - c_2)}) = \\ & \beta \delta (-\alpha b_1 \delta - (1 - \alpha) b_2 \delta + \frac{(1 - \alpha) \alpha b_1 - \alpha (1 - \alpha) (b_2 - c_2)}{\alpha b_1 + 2(1 - \alpha)(b_2 - c_2)}) = \\ & \beta \delta (-\alpha b_1 \delta - (1 - \alpha) b_2 \delta + \frac{\alpha (1 - \alpha) (b_1 - b_2 + c_2)}{\alpha b_1 + 2(1 - \alpha)(b_2 - c_2)}) < 0 \end{aligned}$$

$$(A.82)$$

On the other hand, when $s' \leq \bar{s}$, we show that $\Delta\pi_m > 0$. To do that we use following preliminary results for evaluating the value of $\Delta\pi_m$:

$$T_i^1 > s' \quad (A.83)$$

Then:

$$T_i^2 = p_i^{2*}(T_i^1) > p_i^{2*}(s') = \frac{1 + (b_2 + c_2)s'}{2b_2} \quad (A.84)$$

$$\max\delta = T_i^2 - T_i^1 = p_i^{2*}(T_i^1) - T_i^1 = \frac{1 + (-2b_2 + c_2)T_i^1 + b_2s'}{2b_2} < \frac{1 + (-b_2 + c_2)s'}{2b_2} \quad (A.85)$$

Then:

$$\begin{aligned} \Delta\pi_m > \beta\delta(-1 + \frac{-(\alpha b_1 + (1 - \alpha)b_2)(1 - (b_2 - c_2)s') + 2(\alpha b_1 + 2(1 - \alpha)(b_2 - c_2))(1 + (b_2 + c_2))s'}{2b_2}) = \\ \beta\delta(-1 + \frac{(1 - \alpha)(b_2 - 2c_2) + s'((\alpha b_1(3b_2 + c_2) + (1 - \alpha)(b_2 - c_2)(3b_2 + 2c_2))}{2b_2}) \end{aligned} \quad (A.86)$$

Then, $\Delta\pi_m > 0$, if following inequality holds:

$$s' > \frac{2b_2 - (1 - \alpha)(b_2 - 2c_2)}{(\alpha b_1(3b_2 + c_2) + (1 - \alpha)(b_2 - c_2)(3b_2 + 2c_2))} = \bar{s}^2 \quad (A.87)$$

Recall that we define $\bar{s} = \max\{\bar{s}^1 = p^{mo*}, \bar{s}^2\}$. This guarantees that any mix strategy played in the equilibrium has a higher profit for MO than $\pi_m(T_i^2, T_j^2)$ and subsequently, $\pi_m(p^{mo*}, p^{mo*}) > \pi_m(e^{PR}, e^{PR}) > \pi_m(e^{DR}, e^{DR})$

A.6 Proof of Proposition 3.5

A.6.1 Proposition 3.5

Nash Equilibrium of Games DR-AS and DR-AD

For Games DR-AS and DR-AD:

1. $T_1^1 > T_2^1$
2. If $p_2^{0*}(T_1^2) < T_1^1$, there exists a pure Nash strategy Nash equilibrium.
3. If $p_2^{0*}(T_1^2) > T_1^1$, there exists no pure Nash strategy Nash equilibrium and price range of $T_1^1 \leq p_i \leq T_1^2$ survive iterated elimination of strictly dominated strategy.

A.6.2 Proof Part 1

Game DR-AS

We know that price $p_i = p_j - \epsilon$ guarantees the first position for seller i , where profit function of the seller is as follows:

$$\pi_i(p_j - \epsilon, p_j) = (1 - \beta)(p_j - \epsilon - s') (1 - (\alpha b_1 + (1 - \alpha) b_2)(p_j - \epsilon) + (1 - \alpha) c_2 p_j) \quad (\text{A.88})$$

On the other hand, the best price that seller i can set to maximize his profit at the second position is $p_i^{2*}(p_j)$, and By Equation A.8, we know that:

$$p_i^{2*}(p_j) = \frac{1 + c_2 p_j + b_2 s'}{2b_2} \quad (\text{A.89})$$

Then, the maximum profit that a seller i at the second position is as follows:

$$\pi_i^2(p_i^{2*}(p_j), p_j) = (1 - \beta)(p_i^{2*}(p_j) - s') ((1 - \alpha)(1 - b_2 p_i^{2*}(p_j) + c_2 p_j)) = (1 - \beta)(1 - \alpha) \frac{(1 + c_2 p_j - b_2 s')^2}{4b_2} \quad (\text{A.90})$$

Remember that T_i^1 is the minimum price that seller i would set to win the first position. Then, By the definition of T_1^1 we have following inequality:

$$\pi_1^{DR}(T_1^1, T_1^1 + \epsilon) > \pi_1^{DR}(p_1^{2*}(T_1^1 + \epsilon), T_1^1 + \epsilon) \quad (\text{A.91})$$

And explicitly:

$$(1 - \beta)(T_1^1 - s') (1 - (\alpha b_1 + (1 - \alpha) b_2)(T_1^1) + (1 - \alpha) c_2 (T_1^1 + \epsilon)) > (1 - \beta)(1 - \alpha) \frac{(1 + c_2 (T_1^1 + \epsilon) - b_2 s')^2}{4b_2} \quad (\text{A.92})$$

To prove the first part of Proposition 3.5, we must show following inequality:

$$\pi_2^{DR}(T_1^1, T_1^1 + \epsilon) > \pi_2^{DR}(p_2^{2*}(T_1^1 + \epsilon), T_1^1 + \epsilon) \quad (\text{A.93})$$

or explicitly:

$$(1-\beta)(T_1^1 - s_2')(1 - (\alpha b_1 + (1-\alpha)b_2)(T_1^1) + (1-\alpha)c_2(T_1^1 + \epsilon)) > (1-\beta)(1-\alpha) \frac{(1 + c_2(T_1^1 + \epsilon) - b_2 s_2')^2}{4b_2} \quad (\text{A.94})$$

Inequality A.93 shows that T_1^1 is profitable for seller 2 to set and win the first position. Hence T_2^1 (minimum price that seller 2 would set to win the first position) is less or at most equal to T_1^1 .

First alternative: By derivative:

Note that inequality A.91 and inequality A.93 differ only in value of s' and $s_1' > s_2'$. Therefore, we take derivative of each profit function (profit functions A.88 and A.90) at point $T_1^1 + \epsilon$ to s' as follows, then we compare them to each other:

$$\frac{\partial \pi_i(T_1^1, T_1^1 + \epsilon)}{\partial s'} = (1-\beta)(-1)(1 - (\alpha b_1 + (1-\alpha)b_2)(T_1^1) + (1-\alpha)c_2(T_1^1 + \epsilon)) \quad (\text{A.95})$$

$$\frac{\partial \pi_i^2(p_i^{2*}(T_1^1 + \epsilon), (T_1^1 + \epsilon))}{\partial s'} = \frac{(1-\beta)((1-\alpha))}{4b_2}(-2b_2)(1 + c_2(T_1^1 + \epsilon) - b_2 s') \quad (\text{A.96})$$

We know that both functions are decreasing in price. Hence, to conclude the inequality A.93 by inequality A.91, we must show that the derivative of function A.88 is less than function A.88. Formally:

$$(1-\beta)(-1)(1 - (\alpha b_1 + (1-\alpha)b_2)(T_1^1) + (1-\alpha)c_2(T_1^1 + \epsilon)) < \frac{(1-\beta)((1-\alpha))}{4b_2}(-2b_2)(1 + c_2(T_1^1 + \epsilon) - b_2 s') \quad (\text{A.97})$$

Which is equivalent to the following inequality

$$(1 - (\alpha b_1 + (1-\alpha)b_2)(T_1^1) + (1-\alpha)c_2(T_1^1 + \epsilon)) > \frac{(1-\alpha)}{2}(1 + c_2(T_1^1 + \epsilon) - b_2 s') \quad (\text{A.98})$$

$$\left(\frac{(1+\alpha)}{2} - (\alpha b_1 + (1-\alpha)b_2)(T_1^1) + \frac{(1-\alpha)}{2}c_2(T_1^1 + \epsilon)\right) > \frac{(1-\alpha)}{2}(-b_2 s') \quad (\text{A.99})$$

We know that $T_1^1 < p_i^{0*}(T_1^1 + \epsilon)$. Also, by Equation A.11, we know that:

$$p_i^{0*}(p_j) = \frac{1 + (1-\alpha)c_2 p_j}{2(\alpha b_1 + (1-\alpha)b_2)} + \frac{s'}{2} \quad (\text{A.100})$$

By implying $T_1^1 < p_i^{0*}(T_1^1 + \epsilon)$ to inequality A.99, we can reform the inequality as follows:

$$\left(\frac{(1+\alpha)}{2} - (\alpha b_1 + (1-\alpha)b_2)\right) \left(\frac{1 + (1-\alpha)c_2(T_1^1 + \epsilon)}{2(\alpha b_1 + (1-\alpha)b_2)} + \frac{s'}{2}\right) + \frac{(1-\alpha)}{2} c_2(T_1^1 + \epsilon) > \frac{(1-\alpha)}{2} (-b_2 s') \quad (\text{A.101})$$

$$\left(\frac{(1+\alpha)}{2} - \left(\frac{1 + (1-\alpha)c_2(T_1^1 + \epsilon)}{2} + (\alpha b_1 + (1-\alpha)b_2)\frac{s'}{2}\right) + \frac{(1-\alpha)}{2} c_2(T_1^1 + \epsilon)\right) > \frac{(1-\alpha)}{2} (-b_2 s') \quad (\text{A.102})$$

$$\left(\frac{\alpha}{2} - (\alpha b_1 + (1-\alpha)b_2)\frac{s'}{2}\right) > \frac{(1-\alpha)}{2} (-b_2 s') \quad (\text{A.103})$$

$$\left(\frac{\alpha}{2} - (\alpha b_1)\frac{s'}{2}\right) > 0 \quad (\text{A.104})$$

Remember that we assumed that cost is not too high such that there are some profitable prices to set (i.e. $s' < \frac{1}{b_1}$) which show that inequality A.104 and subsequently inequality A.93 holds.

Game DR-AD

Similar to Game DR-AS we start with the definition of T_1^1 :

$$\pi_1^{DR}(T_1^1, T_1^1 + \epsilon) > \pi_1^{DR}(p_1^{2*}(T_1^1 + \epsilon), T_1^1 + \epsilon) \quad (\text{A.105})$$

Explicitly:

$$(1-\beta)(T_1^1 - s')(a_1 - (\alpha b_1 + (1-\alpha)b_2)(T_1^1) + (1-\alpha)c_2(T_1^1 + \epsilon)) > (1-\beta)(1-\alpha) \frac{(a_1 + c_2(T_1^1 + \epsilon) - b_2 s')^2}{4b_2} \quad (\text{A.106})$$

Let's assume that $a_2 = a_1 - \delta$ where $0 < \delta < a_1$, then we must show following inequality:

$$\pi_2^{DR}(T_1^1, T_1^1 + \epsilon) > \pi_2^{DR}(p_2^{2*}(T_1^1 + \epsilon), T_1^1 + \epsilon) \quad (\text{A.107})$$

or explicitly:

$$(1-\beta)(T_1^1 - s')(a_2 - (\alpha b_1 + (1-\alpha)b_2)(T_1^1) + (1-\alpha)c_2(T_1^1 + \epsilon)) > (1-\beta)(1-\alpha) \frac{(a_2 + c_2(T_1^1 + \epsilon) - b_2 s')^2}{4b_2}$$

$$(A.108)$$

Inequality A.107 shows that T_1^1 is profitable for seller 2 as to set and win the first position as well. Hence, T_2^1 (minimum price that seller 2 would set to win the first position) is less or at most equal to T_1^1 . We can replace a_2 with $a_1 - \delta$ in inequality A.108 as follows:

$$(1-\beta)(T_1^1 - s')((a_1 - \delta) - (\alpha b_1 + (1-\alpha)b_2)(T_1^1) + (1-\alpha)c_2(T_1^1 + \epsilon)) > (1-\beta)(1-\alpha) \frac{((a_1 - \delta) + c_2(T_1^1 + \epsilon))}{4b_2} \quad (A.109)$$

Which is equivalent to the following inequality:

$$-(T_1^1 - s')\delta > (1-\alpha) \frac{-2a_1\delta + \delta^2 - 2\delta(c_2(T_1^1 + \epsilon) - b_2s')}{4b_2} \quad (A.110)$$

$$-(T_1^1 - s') > (1-\alpha) \frac{-2a_1 + \delta - 2(c_2(T_1^1 + \epsilon) - b_2s')}{4b_2} \quad (A.111)$$

We can divide $-2a_1 + \delta - 2(c_2(T_1^1 + \epsilon) - b_2s')$ into two parts.

$$-(T_1^1 - s') > (1-\alpha) \frac{-(a_1 + c_2(T_1^1 + \epsilon) - b_2s') - (a_2 + 2c_2(T_1^1 + \epsilon) - b_2s')}{4b_2} \quad (A.112)$$

Note that each part can form a best response $p_1^{2*}(T_1^1 + \epsilon)$ as follows:

$$(T_1^1 - s') < (1-\alpha) \frac{(p_1^{2*}(T_1^1 + \epsilon)) + (p_2^{2*}(T_1^1 + \epsilon))}{2} \quad (A.113)$$

To prove the inequality A.113 we use proof by contradiction. Let's assume that $T_2^1 > T_1^1$, then:

$$\pi_2^{DR}(T_1^1, T_1^1 + \epsilon) < \pi_2^{DR}(p_1^{2*}(T_1^1 + \epsilon), T_1^1 + \epsilon) \quad (A.114)$$

Then:

$$(T_1^1 - s')(a_2 - (\alpha b_1 + (1-\alpha)b_2)(T_1^1) + (1-\alpha)c_2(T_1^1 + \epsilon)) < (1-\alpha) \frac{(a_2 + c_2(T_1^1 + \epsilon) - b_2s')^2}{4b_2} \quad (A.115)$$

Let's evaluate inequality in two condition:

- if $p_1^{2*}(T_1^1 + \epsilon) < T_1^1$: We know that for any $p < T_i^1$, $p_i^{2*}(p) > T_i^1$. Then this condition contradicts with our starting assumption.

- if $p_1^{2*}(T_1^1 + \epsilon) \geq T_1^1$:

$$d_2^1(T_1^1, T_1^1 + \epsilon) < d_2^2(T_1^1, T_1^1 + \epsilon) < d_2^2(p_1^{2*}(T_1^1 + \epsilon), T_1^1 + \epsilon) \quad (\text{A.116})$$

then, considering inequalities A.115 and A.116, we conclude that:

$$(T_1^1 - s') < (1 - \alpha)(p_2^{2*}(T_1^1 + \epsilon)) \quad (\text{A.117})$$

which proves that inequality A.113 and subsequently Proposition 3.5 Part 1 which contradicts with our starting assumption.

A.6.3 Part 2

sketch: Consider the best response correspondence of two player in region $T_2^1, T_2^1 + \epsilon$ to T_1^1, T_1^2 :

Recall that best response of seller 1 to any price p such that $p < T_1^1$ is $p_1^{2*}(p)$. Also we know that $p_1^{2*}(p)$ is a continuous and increasing function. In this region, its values start with $p_1^{2*}(T_2^1)$ and ends with T_1^2 , where $T_2^1 < p_1^{2*}(T_2^1) < T_1^2$. On the other hand, best response of seller 2 to any price p such that $p > T_1^2$ is $p_1^{1*}(p)$. Also we know that $p_1^{1*}(p)$ is a continuous and increasing function. In this region, its values start with $T_2^1 - \epsilon$ and ends with $p_2^{0*}(T_1^2)$, where $p_2^{0*}(T_1^2) < T_1^2$. Statements above guarantees that there is an intersection of two best response function in this region that show there exist a pure strategy Nash equilibrium.

A.6.4 Part 3

The best response correspondence of sellers in this case is similar to Game DR. Thus, we refer reader to Proof of Proposition 3.2 Part 2.

Appendix B

B.1 Proof of Proposition 4.1

B.1.1 Proposition 4.1

Optimal Pricing

For any product i at position r :

1. If $p_i^* \geq p_{ir}^e(\theta_r)$, optimal price would be equal to p_i^* .
2. If $p_i^* < p_{ir}^e(\theta_r)$, optimal price would be in $[p_i^*, p_{ir}^e(\theta_r)]$

B.1.2 Proof of Part 1

Let c_i^{OPT} be expected number of customers that would consider the product i under the optimal solution (i.e. $d_{ir}(\theta_r) = c_i^{OPT} \cdot d_i(p_i)$). The condition of Part 1 ($p_i^* \geq p_{ir}^e(\theta_r)$) indicates that $c_i^M < c_i^q(p_i^*)$. With abuse of notation, let $\pi_i(p_i)$ denote the profit provided by product i when its price is equal to p_i . Then, we state the optimality of total profit in two following price regions:

1. For any price $p_i \geq p_{ir}^e(\theta_r)$:

Actual demand (sales) of the product i is equal to the potential demand.

Therefore:

$$\pi_i(p_i) = p_i d_{ir}(\theta_r) = c_i^{OPT} \cdot p_i \cdot d_i(p_i) \quad (\text{B.1})$$

As c_i^{OPT} is fixed, optimizing $\pi_i(p_i)$ is equivalent to optimizing $p_i d_i(p_i)$. By the definition, p_i^* is the optimal price to maximize $p_i d_i(p_i)$. Thus, optimal price would be equal to p_i^* .

2. For any price $p_i \leq p_{ir}^e(\theta_r)$:

Actual demand (sales) of the product i is equal to inventory of product q_i .
Therefore:

$$\pi_i(p_i) = p_i d_{ir}(\theta_r) = c_i^{OPT} \cdot p_i \cdot q_i \quad (\text{B.2})$$

As $c_i^{OPT} \cdot q_i$ is fixed, $\pi_i(p_i)$ is increasing in p_i . Thus, $\pi_i(p_i)$ is less than $\pi_i(p_{ir}^e(\theta_r))$.
Also, from previous price region we know that $\pi_i(p_i^*)$ results in a higher profit than $\pi_i(p_{ir}^e(\theta_r))$

Adding both regions to the account, p_i^* is the optimal price.

B.1.3 Proof of Part 2

First, we prove than any $p_i < p_i^*$ or $p_i > p_{ir}^e(\theta_r)$ cannot be optimal. Then, we investigate the case where $p_i^* \leq p_i \leq p_{ir}^e(\theta_r)$.

1. For $p_i \leq p_i^*$:

Actual demand (sales) of the product i is equal to inventory of product q_i .
Therefore:

$$\pi_i(p_i) = p_i d_{ir}(\theta_r) = c_i^{OPT} \cdot p_i \cdot q_i \quad (\text{B.3})$$

As $c_i^{OPT} \cdot q_i$ is fixed, $\pi_i(p_i)$ is increasing in p_i . Thus, $\pi_i(p_i)$ is less than $\pi_i(p_i^*)$

2. For $p_i > p_{ir}^e(\theta_r)$:

Actual demand (sales) of the product i is equal to the potential demand.

$$\pi_i(p_i) = p_i d_{ir}(\theta_r) = c_i^{OPT} \cdot p_i \cdot d_i(p_i) \quad (\text{B.4})$$

As c_i^{OPT} is fixed, optimizing $\pi_i(p_i)$ is equivalent to optimizing $p_i d_i(p_i)$. Note that $p_i d_i(p_i)$ is a concave function in p_i . Thus, $\pi_i(p_i)$ is decreasing in p_i , $\forall p_i \geq p_i^*$. Therefore, $\pi_i(p_i)$ is less than $\pi_i(p_{ir}^e(\theta_r))$

3. For $p_i^* \leq p_i \leq p_{ir}^e(\theta_r)$: The optimal price is $p_i = d_i^{-1}(c_i^{OPT})$, However, due to c_i^{OPT} , optimal price could be $p_i^* \leq p_i \leq p_{ir}^e(\theta_r)$. We present general cases where both p_i^* and $p_{ir}^e(\theta_r)$ would be suboptimal (i.e. optimal price can lay anywhere inside the region). Assume two products i and i' that are assigned to two consecutive positions r and $r+1$. Let β_r and β_{r+1} denote the expected number of customers that consider position r and $r+1$, respectively. We assume that $c_i^q(p_i^*) < \beta_r$ and $c_{i'}^q(p_{i'}^*) < \beta_{r+1}$. We compare the total profit provided in two following cases: 1) No product stocks out 2) Product i stock out at time t_i^s .

(a) No stock-out: As $c_i^q(p_i^*) < \beta_r$, p_i must be equal to $p_{ir}^e(\theta_r)$ to avoid the stock-out. On the other hand, $c_{i'}^q(p_{i'}^*) < \beta_{r+1}$, $p_{i'}^*$ is the optimal price for product i' . Then, total profit provided by two product is as follows:

$$p_{ir}^e(\theta_r) \cdot q_i + \beta_{r+1} \cdot p_{i'}^* d_{i'}(p_{i'}^*) \quad (\text{B.5})$$

(b) Product i stocks out at time t_i^s : Note that in this case product i' shift one position up at time t_i^s , due to stock-out of product i . Then, potential demand of product i' would be as follows:

$$\bar{d}_{i'r+1}(\theta_{r+1}) = (\beta_{r+1} + (1 - \frac{t_i^s}{T})(\beta_r - \beta_{r+1})) \cdot d_{i'}(p_{i'}) \quad (\text{B.6})$$

Let's set $p_i = p_i^*$ and assume that $q_{i'}$ is large enough such that $\bar{d}_{i'r+1}(\theta_{r+1}) = (\beta_{r+1} + (1 - \frac{t_i^s}{T})(\beta_r - \beta_{r+1})) \cdot d_{i'}(p_{i'}^*) < q_{i'}$. Then, optimal price for product i' would be $p_{i'}^*$. Then, total profit provided by two product is as follows:

$$p_i^* \cdot q_i + (\beta_{r+1} + (1 - \frac{t_i^s}{T})(\beta_r - \beta_{r+1})) \cdot p_{i'}^* d_{i'}(p_{i'}^*) \quad (\text{B.7})$$

Note that the total profit provided by all products assigned to positions 1 to $r-1$ is the same in both cases is not affected by the pricing decision for products r and $r+1$. On the other hand, all products assigned to positions below $r+1$ shift up one position up at time t_i^s and receive a higher potential demand, due to stock-out of product i . As profit is non-decreasing in potential demand, total profit provided by products assigned to positions $r+2$ to R in

case 2 is grater or equal to case 1. However, if following inequality hold we can guarantee that case 2 provide a higher total profit:

$$p_i^* \cdot q_i + (\beta_{r+1} + (1 - \frac{t_i^s}{T})(\beta_r - \beta_{r+1})) \cdot p_{i'}^* d_{i'}(p_{i'}^*) - p_{ir}^e(\theta_r) \cdot q_i - \beta_{r+1} \cdot p_{i'}^* d_{i'}(p_{i'}^*) > 0 \quad (\text{B.8})$$

Inequality B.8 can be simplified as follows:

$$(p_i^* - p_{ir}^e(\theta_r)) \cdot q_i + (1 - \frac{t_i^s}{T})(\beta_r - \beta_{r+1}) \cdot p_{i'}^* d_{i'}(p_{i'}^*) > 0 \quad (\text{B.9})$$

By implementing value of each definition, Inequality B.8 would be equivalent to following inequality:

$$(\frac{a_i}{2b_i} - (\frac{a_i}{b_i} - \frac{q_i}{\beta_r b_i})) \cdot q_i + (1 - \frac{2q_i}{\beta_r a_i})(\beta_r - \beta_{r+1}) \cdot \frac{a_{i'}^2}{4b_{i'}} > 0 \quad (\text{B.10})$$

That is equivalent to:

$$(1 - \frac{2q_i}{\beta_r a_i})(\beta_r - \beta_{r+1}) \cdot \frac{a_{i'}^2}{4b_{i'}} > (\frac{a_i}{2b_i} - \frac{q_i}{\beta_r b_i}) \cdot q_i \quad (\text{B.11})$$

Inequality B.11 is equivalent to following inequality:

$$(\beta_r - \beta_{r+1}) \cdot \frac{a_{i'}^2}{4b_{i'}} > \frac{a_i}{2b_i} \cdot q_i \quad (\text{B.12})$$

In summary, if Inequality B.12 hold for any two products i and i' that are assigned to two consecutive positions r and $r + 1$, it is optimal to stock out the product i . Note that this example guarantee the optimality of stock out even though we do not add to the account the significant excess profit provided by lower positions.

B.2 Proof of Proposition 4.2

B.2.1 Proposition 4.2

Optimal ranking of products with predefined prices

1. In the optimal raking, any product $i \neq i_r^* \in S_r^d$ cannot be assigned to position r .

2. In the optimal raking, any product $j \in S_r^q$, cannot be assigned to position r , if $p_j \cdot d_j(p_j) < p_{i_r^*} \cdot d_{i_r^*}(p_{i_r^*})$.
3. In the optimal ranking, any interchangeably q-type products j and j' followed by empty set of S_{r+2}^q should be ranked by increasing $c_i^q(p_i)$ values.

With abuse of notation, let $\pi_i(p_i)$ denote the profit provided by product i .

B.2.2 Proof of Part 1

We prove Part 1 with contradiction. Let's assume that product $i^1 \neq i_r^*$ is assigned to position r at the optimal solution. Subsequently, product i_r^* is assigned to position $r' > r$. Thus, total profit provided by two product is:

$$\pi_{i^1} + \pi_i = c_{i^1}^{OPT} \cdot p_{i^1} \cdot d_{i^1}(p_{i^1}) + c_{i_r^*}^{OPT} \cdot p_{i_r^*} \cdot d_{i_r^*}(p_{i_r^*}) \quad (\text{B.13})$$

By the definition of i_r^* , $p_{i_r^*} \cdot d_{i_r^*}(p_{i_r^*}) > p_{i^1} \cdot d_{i^1}(p_{i^1})$. Thus, swapping two products without changing ranking of other products would result in a higher profit as follows:

$$c_{i^1}^{OPT} \cdot p_{i_r^*} \cdot d_{i_r^*}(p_{i_r^*}) + c_{i_r^*}^{OPT} \cdot p_{i^1} \cdot d_{i^1}(p_{i^1}) > \pi_{i^1} + \pi_i = c_{i^1}^{OPT} \cdot p_{i^1} \cdot d_{i^1}(p_{i^1}) + c_{i_r^*}^{OPT} \cdot p_{i_r^*} \cdot d_{i_r^*}(p_{i_r^*}) \quad (\text{B.14})$$

Therefore, assigning product i_r^* to position r and i^1 position r' would result in higher profit. This contradict with our starting optimality assumption.

B.2.3 Proof of Part 2

Similar to part 1, we prove Part 2 with contradiction. Let's assume that product j is assigned to position r at optimal solution. Subsequently, product i_r^* is assigned to position $r' > r$. Let's j_k denote the product that is assigned to the position k for all positions r to $r' - 1$. Due to optimality condition in Part 1, $\{j_k | r \leq k \leq r' - 1\}$ must be in S_r^q . Then, we investigate the optimality in two following cases:

1. For all products j_k such that $r \leq k \leq r' - 1$, $p_{j_k} \cdot d_{j_k}(p_{j_k}) < p_{i_r^*} \cdot d_{i_r^*}(p_{i_r^*})$: Replacing product i_r^* with product product j_r would result in higher profit.

2. There is at least one product $j_{k'}$ such that $r \leq k' \leq r' - 1$ & $p_{j_{k'}}.d_{j_{k'}}(p_{j_{k'}}) > p_{i_r^*}.d_{i_r^*}(p_{i_r^*})$: Let's replace product $j_{k'}$ with j_r , then
 - (a) if j_r is in $S_{k'}^d$, then due to the Part 1 of the proposition swap product j with i_r^* would result in higher profit.
 - (b) if j_r is in $S_{k'}^q$, due to Part 2 item 1, swap with i_r^* would result in higher profit.

This contradicts with our starting optimal ranking assumption.

B.2.4 Proof of Part 3

let's assume that, two consecutive position r and $r + 1$ are considered by expected number of customers equal to β_r and β_{r+1} , respectively. Note that our main assumption about ranking implies that $\beta_r > \beta_{r+1}$ for any r . Also, let product j and j' be two interchangeably q-type assigned to position r and $r + 1$. Without loss of generality, we assume that $c_j^q(p_j) < c_{j'}^q(p_{j'})$. We investigate two cases where either j is assigned to r and j' is assigned to $r + 1$ or vice versa. Without loss of generality, we assume $T = 1$. Then, we present how β_r and β_{r+1} is allocated among products j , j' , and products in lower positions $r + 2$ and $r + 3$, as follows:

1. Case 1, j is assigned to r , j' to $r + 1$:

- Up to time $c_j^q(p_j)/\beta_r$:

In position r , $c_j^q(p_j)$ out of β_r is consumed by product j .

In position $r + 1$, $c_{j'}^q(p_{j'}) \cdot \beta_{r+1}/\beta_r$ out of β_{r+1} is consumed by product j' .

- : Time $c_j^q(p_j)/\beta_r$ to $(c_{j'}^q(p_{j'}) - c_j^q(p_j) \cdot (\beta_{r+1}/\beta_r))/\beta_r$:

In position r , $c_{j'}^q(p_{j'}) - c_j^q(p_j) \cdot (\beta_{r+1}/\beta_r)$ out of β_r is consumed by product j' .

In position $r + 1$, $(c_{j'}^q(p_{j'}) - c_j^q(p_j) \cdot (\beta_{r+1}/\beta_r))$ out of β_{r+1} is allocated to rank $r + 2$.

- time $c_{j'}^q(p_{j'}) - c_j^q(p_j) \cdot \beta_{r+1} / (\beta_r \cdot \beta_r)$ to T:

In position r , $\beta_r - c_{j'}^q(p_{j'}) - c_j^q(p_j) + c_j^q(p_j)(\beta_{r+1}/\beta_r)$ out of β_r is allocated to rank $r + 2$.

In position $r + 1$, $\beta_{r+1}(-c_{j'}^q(p_{j'}) - c_j^q(p_j) + c_j^q(p_j)(\beta_{r+1}/\beta_r))(\beta_{r+1}/\beta_r)$ out of β_{r+1} is allocated to rank $r + 3$.

2. Case 2, j' is assigned to r , j to $r + 1$:

(a) Case 2.1, j stock outs faster:

- up to time $c_j^q(p_j)/\beta_r$:

In position r , $c_j^q(p_j)(\beta_r/\beta_{r+1})$ out of β_r is consumed by product j' .

In position $r + 1$, $c_j^q(p_j)$ out of β_{r+1} is consumed by product j .

- time $c_j^q(p_j)/\beta_r$ to $(c_{j'}^q(p_{j'}) - c_j^q(p_j) \cdot (\beta_r/\beta_{r+1}))/\beta_r$:

In position r , $c_{j'}^q(p_{j'}) - c_j^q(p_j) \cdot (\beta_r/\beta_{r+1})$ out of β_r is consumed by product j' .

In position $r + 1$, $(c_{j'}^q(p_{j'}) - c_j^q(p_j) \cdot (\beta_r/\beta_{r+1}))(\beta_{r+1}/\beta_r)$ out of β_{r+1} is allocated to rank $r + 2$

- time $(c_{j'}^q(p_{j'}) - c_j^q(p_j) \cdot (\beta_r/\beta_{r+1}))/\beta_r$ to 1:

In position r , $\beta_r - c_{j'}^q(p_{j'})$ out of β_r is allocated to rank $r + 2$.

In position $r + 1$, $\beta_{r+1} - c_{j'}^q(p_{j'})(\beta_{r+1}/\beta_r)$ out of β_{r+1} is allocated to rank $r + 3$.

(b) Case 2.2, j' stock outs faster:

- up to time $c_{j'}^q(p_{j'})/\beta_r$:

In position r , $c_{j'}^q(p_{j'})$ out of β_r is consumed by (product j')

In position $r + 1$, $c_{j'}^q(p_{j'})(\beta_{r+1}/\beta_r)$ out of β_{r+1} (product j)

- time $c_{j'}^q(p_{j'})/\beta_r$ to $(c_j^q(p_j) - c_{j'}^q(p_{j'}) \cdot (\beta_{r+1}/\beta_r))/\beta_r$:

In position r , $c_j^q(p_j) - c_{j'}^q(p_{j'}) \cdot (\beta_{r+1}/\beta_r)$ out of β_r is consumed by (product j)

In position $r + 1$, $(c_j^q(p_j) - c_{j'}^q(p_{j'}) \cdot (\beta_{r+1}/\beta_r))(\beta_{r+1}/\beta_r)$ out of β_{r+1} (allocated to rank $r + 2$)

- time $(c_j^q(p_j) - c_{j'}^q(p_{j'}) \cdot (\beta_{r+1}/\beta_r)) / \beta_r$ to 1:

In position r , $\beta_r - c_{j'}^q(p_{j'}) - c_j^q(p_j) + c_{j'}^q(p_{j'}) \cdot (\beta_{r+1}/\beta_r)$ out of β_r is consumed by (allocated to rank $r + 2$)

In position $r+1$, $\beta_{r+1} + (-c_{j'}^q(p_{j'}) - c_j^q(p_j) + c_{j'}^q(p_{j'}) \cdot (\beta_{r+1}/\beta_r))(\beta_{r+1}/\beta_r)$ out of β_{r+1} is consumed by (allocated to rank $r + 3$)

Due to the stock-outs at positions r and $r + 1$, product at position $r + 2$ shift to upper positions. Followings are the excess consideration allocated to position $r + 2$ out of out of $\beta_r + \beta_{r+1}$, due to stock-outs:

1. Case 1: $\beta_r - c_{j'}^q(p_{j'}) - c_j^q(p_j) + (c_j^q(p_j) + c_{j'}^q(p_{j'}) - c_j^q(p_j)(\beta_{r+1}/\beta_r))(\beta_{r+1}/\beta_r)$
2. Case 2.1: $\beta_r - c_{j'}^q(p_{j'}) - c_j^q(p_j) + c_{j'}^q(p_{j'}) (\beta_{r+1}/\beta_r)$
3. Case 2.2: $\beta_r - c_{j'}^q(p_{j'}) - c_j^q(p_j) + (c_j^q(p_j) + c_{j'}^q(p_{j'}) - c_{j'}^q(p_{j'}) (\beta_{r+1}/\beta_r))(\beta_{r+1}/\beta_r)$

As $c_{j'}^q(p_{j'}) > c_j^q(p_j)$ and $\beta_r > \beta_{r+1}$, excess consideration allocated to position $r + 2$ out of $\beta_r + \beta_{r+1}$ is higher in case 1 rather than cases 2.1 and 2.2.

Note that, in part 3, we state that $s_{r+2}^q = \emptyset$. It means that all remaining products are in s_{r+2}^d . Thus, due to part 1, all remaining products should be sorted by $p_i d_i(p_i)$. It means that product i_{r+2}^* must be assigned to position $r + 2$. On the other hand, the retailer can sell more units of product at position $r + 2$ under case 1 rather than two other cases. As $p_{i_{r+2}^*} d_{i_{r+2}^*}(p_{i_{r+2}^*})$ is higher than $p_i d_i(p_i)$ for all products in the lower positions. Therefore, the retailer provides a higher profit under case 1.

B.3 Proof of Proposition 4.3

B.3.1 Proposition 4.3

Effect of pricing on the optimal profit value.

$$\frac{\Delta\pi}{\pi} < \left(\frac{\sigma - 1}{\sigma + 1}\right)^2 \quad (\text{B.15})$$

For any product i such that $\bar{d}_{ir}(\theta_r) \geq q_i$, $c_i^M \cdot d_i(p_i) = q_i$. Thus, we can write p_i as function of c_i^M as follows:

$$a_i - b_i p_i = \frac{q_i}{c_i^M} \quad (\text{B.16})$$

$$p_i = \frac{a_i}{b_i} - \frac{q_i}{c_i^M \cdot b_i} \quad (\text{B.17})$$

Then, we can define the profit provided by product i , $\pi_i(p_i) = p_i \cdot q_i$, and its derivative in $c_i^q(p_i)$, respectively, as follows:

$$p_i \cdot q_i = \left(\frac{a_i}{b_i} - \frac{q_i}{c_i^q(p_i) \cdot b_i} \right) \cdot q_i = \frac{a_i \cdot q_i}{b_i} - \frac{q_i^2}{c_i^q(p_i) b_i} \quad (\text{B.18})$$

Then:

$$\frac{\partial p_i \cdot q_i}{\partial c_i^q(p_i)} = \frac{q_i^2}{b_i c_i^q(p_i)^2} = \frac{q_i^2}{b_i (c_i^q(p_i^*) + \delta)^2} \quad (\text{B.19})$$

$$\frac{\partial p_i \cdot q_i}{\partial c_i^q(p_i)} = \frac{q_i^2}{b_i \left(\frac{2q_i}{a_i} + \delta \right)^2} = \frac{q_i^2}{b_i \left(\frac{(2q_i)^2}{(a_i)^2} + \delta^2 + 2\delta \cdot \frac{2q_i}{a_i} \right)} \geq \frac{q_i^2}{b_i \left(\frac{(2q_i)^2}{(a_i)^2} + 2\delta \cdot \frac{2q_i}{a_i} \right)} \quad (\text{B.20})$$

$$\frac{\partial p_i \cdot q_i}{\partial c_i^q(p_i)} \geq \frac{a_i^2}{b_i (4 + 4\delta \cdot \frac{2a_i}{q_i})} = \frac{a_i^2}{4b_i} \frac{1}{(1 + \delta \frac{a_i}{q_i})} \quad (\text{B.21})$$

Note that increasing c_i^M would result in an equal decrease in $c_{i'}^M > 0$ for product i' . Thus, any increase in c_i^M must result in a profit increase higher than profit lost due to decrease of $c_{i'}^M > 0$. Remember that we define $\pi^{min} = \min_k \{p_k \cdot d_k(p_k) | c_k^M > 0\}$ is the minimum profit lost due to the decrease of $c_{i'}^M$. Therefore:

$$\frac{\partial p_i \cdot q_i}{\partial c_i^q(p_i)} \geq \pi^{min} \quad (\text{B.22})$$

Then,

$$\frac{a_i^2}{b_i (4 + 4\delta \cdot \frac{2a_i}{q_i})} = \frac{a_i^2}{4b_i} \frac{1}{(1 + \delta \frac{a_i}{q_i})} \geq \pi^{min} \quad (\text{B.23})$$

Equivalently,

$$(1 + \delta \frac{a_i}{q_i}) \leq \frac{\frac{a_i^2}{4b_i}}{\pi^{min}} \quad (B.24)$$

Then, by the definition of σ :

$$\delta \leq (\sigma - 1) \frac{q_i}{a_i} \quad (B.25)$$

On the other hand, Let us define $\Delta\pi_i$ as increase in profit provided by product i due to by increasing c_i^M from $c_i^q(p_i)$ to $c_i^q(p_i) + \delta$. Then:

$$\Delta\pi_i = p'_i \cdot q_i - p_i \cdot q_i = q_i \left(\frac{q_i}{c_i^q(p_i)b_i} - \frac{q_i}{(c_i^q(p_i) + \delta)b_i} \right) = \frac{q_i^2}{b_i} \left(\frac{\delta}{(c_i^q(p_i) + \delta)c_i^q(p_i)} \right) \quad (B.26)$$

Let $c_i^q(p_i^*) = \frac{2q_i}{a_i}$ be the total consideration need to sold out a product at optimal price p_i^* . Note that for any $c_i^M < c_i^q(p_i^*)$ it is not optimal to sold out the product. Then:

$$\Delta\pi_i = \delta \left(\frac{q_i^2}{(c_i^q(p_i) + \delta)c_i^q(p_i)} b_i \right) = \delta \left(\frac{\frac{a_i^2 c_i^q(p_i)^{*2}}{4} q_i^2}{(c_i^q(p_i) + \delta)c_i^q(p_i)} b_i \right) = \delta \left(\frac{\frac{a_i^2 c_i^q(p_i)^{*2}}{4} q_i^2}{(c_i^q(p_i^*) + \nu + \delta)(c_i^q(p_i^*) + \nu)} b_i \right) \quad (B.27)$$

$$\Delta\pi_i = \delta \left(\frac{1}{1 + \frac{2c_i^q(p_i^*)\nu + \nu^{*2} + \delta(c_i^q(p_i^*) + \nu)}{c_i^q(p_i)^{*2}}} \cdot \frac{a_i^2}{4b_i} \right) \quad (B.28)$$

$$\Delta\pi_i < \delta \left(\frac{1}{1 + \frac{2\nu + \delta}{c_i^q(p_i^*)}} \cdot \frac{a_i^2}{4b_i} \right) \quad (B.29)$$

Let assume that $j = \operatorname{argmin}_k \{p_k \cdot d_k(p_k) | c_k^M > 0\}$. Then, increasing c_i^M from $c_i^q(p_i)$ to $c_i^q(p_i) + \delta$ would result in a minimum profit lost equal to Let us define $\Delta\pi_j$ as follows:

$$\Delta\pi_j = \delta p_j \cdot d_j(p_j) = \delta \frac{a_j^2}{4b_j} \quad (B.30)$$

Totally, Let $\Delta\pi_{ij}$ denote the total profit change resulted by increasing c_i^M from $c_i^q(p_i)$ to $c_i^q(p_i) + \delta$ (that results in decreasing c_j^M i the same amount):

$$\Delta\pi_{ij} < \delta \left(\frac{1}{1 + \frac{2\nu + \delta}{c_i^q(p_i^*)}} \cdot \frac{a_i^2}{4b_i} - \frac{a_j^2}{4b_j} \right) \quad (B.31)$$

$$\Delta\pi_{ij} < \delta \left(\frac{1}{1 + \frac{2\nu+\delta}{c_i^q(p_i^*)}} \cdot \sigma \cdot \pi^{min} - \pi^{min} \right) \quad (\text{B.32})$$

On the other hand, total profit provided by two products is as follows:

$$\pi_i + \pi_j = c_i^M \frac{a_i^2}{4b_i} + c_j^M \frac{a_j^2}{4b_j} = c_i^M \cdot \sigma \cdot \pi^{min} + c_j^M \cdot \pi^{min} \quad (\text{B.33})$$

Remember that we show that δ is limited in Equation B.25. Also, δ is limited by c_j^M as well. Then:

$$\frac{\Delta\pi_{ij}}{\pi_i + \pi_j} < \frac{(\sigma - 1) \frac{q_i}{a_i}}{(\sigma + 1) c_j^M \cdot \pi^{min}} \cdot \left(\frac{1}{1 + \frac{((\sigma-1) \frac{q_i}{a_i})}{c_i^q(p_i^*)}} \cdot \sigma \cdot \pi^{min} - \pi^{min} \right) = \frac{(\sigma - 1) \frac{q_i}{a_i}}{(\sigma + 1) c_j^M} \cdot \left(\frac{1}{1 + \frac{((\sigma-1) \frac{q_i}{a_i})}{c_i^q(p_i^*)}} \cdot \sigma - 1 \right) \quad (\text{B.34})$$

Note that $c_j^M < c_j^q(p_i)$ and $c_i^q(p_i^*) = \frac{2q_i}{a_i}$. Then:

$$\frac{\Delta\pi_{ij}}{\pi_i + \pi_j} < \frac{(\sigma - 1)}{(\sigma + 1)} \cdot \left(\frac{1}{1 + \frac{(\sigma-1)}{2}} \cdot \sigma - 1 \right) = \frac{(\sigma - 1)}{(\sigma + 1)} \cdot \left(\frac{2\sigma}{1 + \sigma} - 1 \right) = \left(\frac{\sigma - 1}{\sigma + 1} \right)^2 \quad (\text{B.35})$$