

**ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL**

**CONSTRUCTION OF REVERSIBLE COMPLEMENT DNA CODES  
OVER A FAMILY OF CHAIN RINGS**



**M.Sc. THESIS**

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**Mathematical Engineering Program**

**DECEMBER 2023**



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**İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ**

**BİR ZİNCİR HALKA AİLESİ ÜZERİNDE  
TERSİNİR TÜMLER DNA KODLARI İNŞAASI**

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*To my family,*



## FOREWORD

I wish to express my gratitude to my thesis advisor Ergün YARANERİ, for his patient and caring mentorship and wisdom. Being a member of Istanbul Technical University Mathematical Engineering department is and always will be an honor and key to my success. I wish to thank all of my professors and coworkers for their support. Moreover, I wish to extend my appreciation to my loving family for always having my back through my journey. Lastly, I wish to thank Scientific and Technological Research Council of Turkey (TÜBİTAK) for 2210 Scholarship Program.

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## ABBREVIATIONS

**DNA** : Deoxyribonucleic Acid  
**GC-Content** : Guanin Cytosin Content  
**NP-Complete**: Nondeterministic Polynomial-Time Complete  
**WCC** : Watson Crick Complement  
**PIR** : Principal Ideal Ring





## SYMBOLS

$\mathbb{F}_q$	: The field of order $q$ where $q$ is a power of a prime.
$S$	: A chain ring of the form $\mathbb{F}_4[u]/\langle u^k \rangle$
$R$	: A chain ring of the form $\mathbb{F}_4[v]/\langle v^k \rangle$
$S_{D_4}$	: The DNA alphabet $\{A, C, G, T\}$
$A_n$	: The polynomial quotient ring $A[x]/\langle x^n - 1 \rangle$
$R_n$	: The polynomial quotient ring $R[x]/\langle x^n - 1 \rangle$
$p^*(x)$	: The reciprocal of the polynomial $p(x)$
$\deg f$	: The degree of the polynomial $f(x)$





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# CONSTRUCTION OF REVERSIBLE COMPLEMENT DNA CODES OVER A FAMILY OF CHAIN RINGS

## SUMMARY

DNA codes, which consist of codewords over the alphabet  $\{A, C, G, T\}$ , are widely used in bioscience, computer engineering, data storage and DNA computing. Depending on the field of application, it is common to study DNA codes with certain combinatorial properties (reversibility, Watson-Crick complement, minimum Hamming distance, constant GC-Content, Hamming distance between reverse complements, etc.). Defining DNA codes via commutative rings is a common approach since the algebraic structure of these rings allow one to obtain codes with the foresaid features.

In this study, the aim is to obtain reversible complement DNA codes, which are images of codes on a particular chain ring family  $S = \mathbb{F}_4[u]/\langle u^k \rangle$ . The elements of these rings  $S$  are mapped bijectively onto elements of another chain rings  $R = \mathbb{F}_4[v]/\langle v^{tk} \rangle$  whose structures are more convenient to derive DNA codes.

Some nice maps are constructed from the elements of the above rings to DNA codewords, allowing one to check whether a DNA code satisfies some combinatorial properties. These maps provide codes with a large spectrum of length, dimension and distance. Moreover, using a family of rings instead of one specific ring provides extreme versatility and novelty to the field.

Reversibility conditions are established for codes over arbitrary commutative rings. Subsequently, necessary and sufficient conditions are proposed for reversibility of DNA codes yielded from cyclic codes of odd length.

In addition to expressing the theoretical method of constructing these DNA codes, a variety of sample codes are presented. The samples include repetition and punctured codes, over a diverse collection of rings in the family. These codes are also analysed and thereafter compared with each other and previously constructed codes in the literature with respect to their size, transmission rate, Hamming distance and relative minimum distance.



## BİR ZİNCİR HALKA AİLESİ ÜZERİNDE TERSİNİR TÜMLER DNA KODLARI İNŞAASI

### ÖZET

$\{A, C, G, T\}$  alfabeti kullanılarak oluşturulan DNA kodları biyobilim, bilgisayar mühendisliği, veri depolama, DNA programlama, hata düzeltme teorisi gibi birçok alanda yaygın olarak kullanılmaktadır. Kullanım alanına bağlı olarak bu kodların belirli kombinatoriyal özelliklere sahip olması beklenir. Bu özellikler arasında tersinirlik, Watson-Crick tümlerlik, minimum Hamming uzaklığı, sabit GC içeriği, ters tümler kod sözcükleri arası Hamming uzaklığı sayılabilir. DNA kodları değişmeli halkalar üzerinde tanımlamak sıkça kullanılan bir yöntemdir ve bu halkaların cebirsel yapıları sayesinde belirtilen özelliklere sahip kodlar elde edilebilmektedir.

Bu tez çalışmasında, genelleştirilmiş bir zincir halka ailesi üzerinde tanımlanmış tersinirlik ve tümlerlik özelliklerini sağlayan DNA kodları elde edilmiş ve anlatılmıştır. Bu halka ailesi  $S = \mathbb{F}_4[u]/\langle u^k \rangle$ 'dir.  $S$  halkalarının elemanları öncelikle başka bir zincir halka ailesi olan  $R = \mathbb{F}_4[v]/\langle v^k \rangle$ 'nin elemanlarına eşlenmiştir.  $R$  halkası, cebirsel yapısı sebebiyle DNA kodları inşası için daha elverişlidir.

Zincir halkalarının elemanlarından DNA kod sözcüklerine tanımlanan birebir ve örten fonksiyonlar inşaa edilerek oluşan kodların kombinatoriyal özellikleri kontrol edilmiştir. Bu fonksiyonlar ile geniş bir yelpazede uzunluk, boyut ve uzaklık değerlerine sahip kodlar elde edilmiştir. Ayrıca, belirli bir halka yerine bir halka ailesinin kullanılması, çalışmaya çok yönlülük ve özgünlük sağlamıştır.

Çalışmada, keyfi değişmeli halkalar üzerinde kodlar için tersinirlik koşulları oluşturulmuştur. Ardından, tek uzunluklu devirli kodlardan elde edilen DNA kodlarının tersinirliği için gerek ve yeter şartlar oluşturulmuştur.

DNA molekülleri hücrenin genetik depolama birimleridir. Çift zincirlidirler. Her zincir 4 farklı tip nükleotit içerir: Adenin (A), Guanin (G), Sitozin (C) ve Timin (T). DNA'nın iki zinciri, nükleotit çiftleri arasındaki hidrojen bağları sayesinde birbirlerine bağlıdır. Bu eşleşme, Watson-Crick Tümlerlik özelliğine göre gerçekleşir. Watson-Crick Tümlerlik, Adenin tümlerini Timin, Guanininin tümlerini Sitozin olarak tanımlar.  $\bar{A} = T$ ,  $\bar{T} = A$ ,  $\bar{C} = G$  and  $\bar{G} = C$  ile gösterilir.

DNA'nın özelliklerinin hesaplamalı bilimlerde kullanılışı, Adleman tarafından yapılan bir çalışmaya dayanır. Bu çalışmada, NP-tam bir Hamilton yol problemi Watson-Crick Tümlerlik özelliği kullanılarak cevaplanmıştır. O günden bu yana, zorlu hesaplamaların ve programlamaların çözümünde DNA moleküllerinin kombinatoriyal yönleri kullanılmaya devam etmiştir.

DNA kodları,  $S_{D_4} = \{A, C, G, T\}$  alfabeti kullanılarak yazılan kod sözcükleri kümeleridir. Yukarıda bahsedilen kombinatoriyal özellikleri sağlayan, belirli uzunluk

ve Hamming uzaklığındaki en fazla sözcük sayısına sahip DNA kodları oluşturmak amaçlanır. Böylece hata düzeltme kapasitesi ve iletim hızı yüksek kodlara ulaşılır. Kodlama teorisinde genel olarak hem algoritmik, hem de cebirsel metodlar sıkça kullanılmaktadır. DNA kodlarının inşasında kullanılan halkalar ve cisimler arasında ise  $\mathbb{F}_2 + u\mathbb{F}_2$ ,  $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ ,  $\mathbb{F}_2[u]/(u^{2k} - 1)$ ,  $\mathbb{Z}_4[v]/(v^2 - v)$ ,  $\mathbb{Z}_4[u]/\langle u^2 - 1 \rangle$ ,  $\mathbb{Z}_4 + v\mathbb{Z}_4$ ,  $\mathbb{F}_2[u]/(u^4 - 1)$ ,  $\mathbb{F}_4[u]/\langle u^3 \rangle$ , grup matris halkaları ve Galois halkaları sayılabilir.

Bu tez çalışmasında seçilen halka ailesi olan  $S = \mathbb{F}_{4^t}[u]/\langle u^k \rangle$ 'den  $tk$  uzunluğundaki DNA kod sözcüklerine fonksiyonlar aşağıdaki şemada özetlenmiştir:

$$\begin{array}{ccccc} S & \xrightarrow{\gamma} & R & \xrightarrow{\eta} & \mathbb{F}_4^{tk} & \xrightarrow{\tilde{\sigma}} & S_{D_{4^{tk}}} \\ & & & & \uparrow & & \\ & & & & \mathbb{F}_4 & \xrightarrow{\sigma} & S_{D_4} \end{array}$$

Bileşke fonksiyon  $\theta = \tilde{\sigma} \circ \eta$ 'nin  $n$  boyutlu uzaya genişletilmesiyle oluşturulan  $\Theta : R^n \rightarrow S_{D_{4^{tk}}}^n$  sayesinde ise  $tkn$  uzunluğunda kod sözcükleri oluşturulmuştur. Bu fonksiyon şeması, nihai kodların her  $t, k$  için tümlerlik ve tersinirlik özelliklerine sahip olması sağlanacak şekilde kurgulanmıştır.

Her  $R = \mathbb{F}_4[v]/\langle v^{tk} \rangle$  halkasında

$$\kappa := (\omega + 1) \sum_{i=\lfloor \frac{tk}{2} \rfloor}^{tk-1} v^i$$

elemanı tanımlanmıştır. Bu eleman üzerinden halkadaki keyfi bir  $a \in R$  ya da  $\mathbf{a} \in R^n$  elemanın DNA kod sözcüğü görüntüsünün tümeleri için  $a + \bar{a} = \kappa$  ve  $\Theta(\bar{\mathbf{a}}) = \Theta(\mathbf{a} + \kappa \mathbf{1}_n)$  özelliği sağlatılmıştır.  $R$  halkası üzerinde  $n$  uzunlukta bir  $C$  lineer kodunun tümler kod olması için gerek ve yeter şart  $\kappa \mathbf{1}_n$  elemanının  $C$ 'de olması olarak belirlenmiştir. Bu koşul,  $\Theta(C)$  DNA kodunun tümlerlik şartını belirlemek için de kullanılmıştır.

$Z$  herhangi bir alfabe olmak üzere  $T : Z^n \rightarrow Z^n$  fonksiyonu herhangi bir  $\mathbf{a} = (a_1, a_2, \dots, a_n) \in Z^n$

için  $T(\mathbf{a}) = \mathbf{a}^r := (a_n, \dots, a_2, a_1)$  olarak tanımlansın.  $C$  ise  $Z$  üzerinde  $n$  uzunluğunda bir kod olsun.  $T(C) = C$  ise  $C$  koduna tersinirdir denir.

Benzer şekilde, her  $R = \mathbb{F}_4[v]/\langle v^{tk} \rangle$  halkasında

$$\rho := 1 + \sum_{i=\lfloor \frac{tk}{2} \rfloor}^{tk-1} v^i$$

elemanı tanımlanmıştır. Bu eleman üzerinden halkadaki keyfi bir  $a \in R$  ya da  $\mathbf{a} \in R^n$  elemanın DNA kod sözcüğü görüntüsünün tersinin  $\theta(a)^r = \theta(\rho a)$  ve  $\Theta(\mathbf{a})^r = \Theta(\rho \mathbf{a}^r)$  ile hesaplanabilmesi sağlanmıştır.

Keyfi bir  $A$  halkası üzerinde kodların tersinirliği üzerine genel şartlar sunulmuştur. Bunu takiben, keyfi  $A$  halkası üzerinde devirli kodların tersinirliği ele alınmış, bu

devirli kodlar  $A[x]/\langle x^n - 1 \rangle$ 'in idealleri olarak tanımlanmıştır. Bir polinomun tersi tanımlanıp, üreteç polinomların kendine ters olması ile üretilen DNA kodlarının tersinirliği arası ilişkiler ortaya konmuştur.

$R$  halkası, karakteristiği 2, tek maksimal ideali  $\langle v \rangle$ , bu idealin nilpotentliği  $tk$  olan sonlu zincir halkasıdır.  $R/\langle v \rangle$  bölüm halkası  $\mathbb{F}_4$ 'e izomorftur ve  $\mathbb{F}_4$ ,  $R$ 'in birimsel alt halkasıdır.  $n$  tek sayı,  $s = tk$  olmak üzere,  $R[x]/\langle x^n - 1 \rangle$ 'in herhangi bir ideali  $C = \langle g_0, vg_1, \dots, v^{s-1}g_{s-1} \rangle$  olacak şekilde  $\mathbb{F}_4$  üzerinde  $g_{s-1}|g_{s-2}| \cdots |g_1|g_0|x^n - 1$  koşulunu sağlayan  $g_0, g_1, \dots, g_{s-1}$  monik polinomları vardır.  $C$  devirli kodunun tersinir olması için gerek ve yeter şartın, her  $g_i$  polinomunun kendine ters polinom olması olduğu kanıtlanmıştır.  $\Theta(C)$  DNA kodunun tersinir olması için gerek ve yeter şartın her  $g_i$  polinomunun kendine ters polinom olması ve  $\kappa(1 + x + \dots + x^n)$ 'nin  $C$ 'nin elemanı olması olduğu gösterilmiştir.

DNA kodların inşasının teorik metodlarının yanısıra, çeşitli örnek kodlar sunulmuştur. Bu örneklerin içerisinde, tekrarlı ve delikli kodlar dahil olmak üzere, muhtelif halka aileleri üzerinde tanımlı kodlar bulunmaktadır. Bu kodlar, uzunluk, aktarım hızı, Hamming uzunluğu ve göreceli minimum uzaklık parametreleri bakımından hem kendi içlerinde hem de literatürdeki örneklerle karşılaştırılmıştır.

Örneklerin de desteklediği gibi, bu tez çalışması, cebirsel yapıların özellikleri kullanılarak, isteğe göre ayarlanmış uzunlukta, yüksek kod hızına ve göreceli minimum uzaklığa, böylece daha yüksek iletim hızına ve hata düzeltme kapasitesine sahip DNA kodları oluşturmanın mümkün olduğunu göstermektedir. İleriki araştırma alanları arasında, lineer kodlardan elde edilen DNA kodlarının tersinirliği, sabit GC içeriğine sahip DNA kodlarının cebirsel türetimi, ve keyfi halkalar üzerinde çift uzunlukta DNA kodları sayılabilir.



# 1. INTRODUCTION

## 1.1 Purpose of Thesis

The main purpose of this thesis is to construct DNA codes over a family of chain rings and establish conditions for reversibility and complementary properties. By choosing a family of rings with variable parameters instead of one specific ring, it is aimed to obtain codes with high transmission rates and error-correcting capabilities.

## 1.2 Literature Review

DNA molecules are the genetic data storage units of cells, made up of two strands. Each strand is a sequence of nucleotides of four types: Adenine (A), Guanine (G), Cytosine (C) and Thymine (T). The two strands of a DNA molecule are connected to each other by hydrogen bonds between nucleotide pairs, respecting the Watson-Crick Complement (WCC) Property, in which Adenine pairs with Thymine and Cytosine pairs with Guanine. Using properties of DNA in computing, as well as bio-science and chemical array analysis, dates back to a study of Adleman [1], in which WCC Property of DNA molecules is used to answer an NP-Complete Hamiltonian path problem. Since then, computationally challenging problems are continued to be dealt by employing the combinatorial aspects of DNA molecules such as WCC Rule, Hamming distance constraint, reversibility, fixed GC-content constraint. DNA codes are sets of codewords over the alphabet  $\{A, C, G, T\}$  which respect these combinatorial conditions, in line with the scope of application. In recent decades, DNA codes have come into prominence in DNA computing [2]–[5], data storage [6] and error correction theories [7,8].

Construction of DNA codes with the greatest possible number of codewords of pre-determined length and constraints has become one of the primary objectives in coding theory, addressed by both algorithmic and algebraic methods [9,10].

Algebraic construction of codes over fields and rings is an established approach [11,12]. Similarly, construction of DNA codes are studied over the rings  $\mathbb{F}_2 + u\mathbb{F}_2$  in [13,14],  $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$  in [15],  $\mathbb{F}_2[u]/\langle u^{2k} - 1 \rangle$  in [16],  $\mathbb{Z}_4[v]/\langle v^2 - v \rangle$  in [17],  $\mathbb{Z}_4[u]/\langle u^2 - 1 \rangle$  in [18],  $\mathbb{Z}_4 + v\mathbb{Z}_4$  in [19],  $\mathbb{F}_2[u]/\langle u^4 - 1 \rangle$  in [20] and  $\mathbb{F}_4[u]/\langle u^3 \rangle$  in [21,22]. Further, DNA codes are studied over group matrix rings in [23] and Galois rings in [24].

### 1.3 Hypothesis

In this thesis, the focus is DNA images of codes over chain rings  $S = \mathbb{F}_{4^t}[u]/\langle u^k \rangle$ . Beginning with the mapping of the elements of  $S$  to  $R = \mathbb{F}_4[v]/\langle v^k \rangle$  and using  $R$  for the formulation of DNA codes for the rest of the work, each element of  $R$  is related to a DNA strand of length  $4^k$  using the Gray map  $\eta$  and one-to-one correspondence  $\sigma$  to the DNA alphabet  $S_{D_4}$ . Contrary to the previous works, due to the variability of  $t$  and  $k$ , DNA codes of desired length and size can be obtained. After defining the convenient maps, the condition for being a complement code for the DNA image of  $R^n$  is given. As well as some generalized conditions for reversibility of the DNA image of an arbitrary ring  $A$ , particular results regarding the reversibility of the DNA image of cyclic codes over  $R$  for odd length  $n$  is obtained. Lastly, with examples of DNA codes for numerous choices of  $t$ ,  $k$ ,  $n$  and examining these codes for their size and Hamming distance, codes with high transmission rates and error correcting capabilities are demonstrated.

## 2. PRELIMINARIES

### 2.1 Brief Information on Algebraic Structure of Codes

Coding theory originated from the need of electronic information to be transmitted and stored without errors. This information used to be viewed as an array of ones and zeros, then extended to an alphabet. The connections between coding theory and algebra, geometry and combinatorics are founded quickly and using finite fields and rings as the alphabet gained popularity. Here, some fundamental definitions and basic theory [11,25] are introduced as a guide for the comprehension of the work to be presented.

**Definition 1.** Let  $A$  be any finite set. A code  $C$  over the alphabet  $A$  of length  $n$  is a subset of  $A^n$ . Here,  $n$  is the length and  $M = |C|$  is the size of the code.

**Definition 2.** A code  $C$  over  $\mathbb{F}_q$  of length  $n$  is a subset of  $\mathbb{F}_q^n$ . Here,  $\mathbb{F}_q$  is a field of  $q$  elements where  $q$  is a prime power. If  $C$  has size  $M$  and distance  $d$ , then we say that  $C$  is a  $(n, M, d)$  code.

**Definition 3.** Let  $\mathbf{v}, \mathbf{w} \in A^n$  where  $A$  is any set. Then,

$$d_H(\mathbf{v}, \mathbf{w}) = |\{i | v_i \neq w_i\}|$$

is the Hamming distance between codewords  $\mathbf{v}$  and  $\mathbf{w}$ . The distance of a code  $C$  is defined as

$$d(C) = \min\{d_H(\mathbf{v}, \mathbf{w}) | \mathbf{v}, \mathbf{w} \in C, \mathbf{v} \neq \mathbf{w}\}.$$

**Definition 4.** For an  $(n, M, d)$  code over  $\mathbb{F}_q$ , the relative minimum distance is defined as  $\delta = \frac{d}{n}$  and the information transmission rate (i.e. code rate) is defined as  $R = \frac{\log_q M}{n}$ .

**Definition 5.** Let  $R$  be a nonempty set with two operations  $(+, \cdot)$ , called 'addition' and 'multiplication' respectively, satisfying the following axioms. For all  $a, b, c \in R$ ,

- $R$  is closed under  $+$  and  $\cdot$ . i.e.  $a + b \in R$  and  $a \cdot b \in R$ .

- Commutativity.  $a + b = b + a$  and  $a \cdot b = b \cdot a$ .
- Associativity.  $(a + b) + c = a + (b + c)$  and  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- Distributive law.  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- Additive identity element.  $a + 0 = a$ .
- Multiplicative identity element.  $a \cdot 1 = a$  and  $a \cdot 0 = 0$ .
- For any  $a$ , there exists an additive inverse element  $(-a) \in R$  such that  $a + (-a) = 0$ .

Then,  $R$  is called a (commutative) ring.

Throughout, a ring means a commutative ring with 1.

**Definition 6.** Let  $R$  be a ring. A linear code  $C$  over the alphabet  $R$  of length  $n$  is a  $R$ -submodule of  $R^n$ .

When  $R$  is a field, the linear code  $C$  over  $R$  of length  $n$  with Hamming distance  $d$ , is a vector space over  $R$ , and so it has a basis. The dimension of the linear code  $C$  is defined to be the dimension of this vector space. If the dimension is  $k$ , then the code is said to be an  $[n, k, d]$ -linear code.

An  $[n, k, d]$ -linear code can detect up to  $d - 1$  errors and correct up to  $\lfloor (d - 1)/2 \rfloor$  errors.

For "better" codes,  $n$  is desired to be small for fast transmission,  $k$  and  $d$  to be large for variety of messages and higher error detecting, correcting capabilities respectively.

The Main Problem in Coding Theory is to maximize  $A_q(n, d)$ , which stands for the maximal number of codewords one can get in an  $[n, k, d]$ -code over  $\mathbb{F}_q$  while  $k$  is ranging.

**Definition 7.** Let  $C$  be a code over an alphabet  $R$  of length  $n$ . The cyclic shift of the codeword  $(x_1, x_2, \dots, x_{n-1}, x_n)$  is defined to be  $(x_n, x_1, x_2, \dots, x_{n-1})$ . The code  $C$  is called cyclic if it is invariant under the cyclic shift.

**Definition 8.** A principal ideal ring (PIR) is a ring  $R$ , all of whose ideals are generated by a single element. i.e. Each ideal can be represented as  $I = \langle a \rangle$  for some element  $a \in R$ .

**Definition 9.** A chain ring is a PIR whose ideals are linearly ordered by set theoretic containment. It is known that a ring is a chain ring if and only if it is Artinian, local, and PIR. There exists an element  $a \in R$  that generates the unique maximal ideal and creates the chain:  $\{0\} \subseteq \langle a^{e-1} \rangle \subseteq \langle a^{e-2} \rangle \subseteq \dots \subseteq \langle a \rangle \subseteq R$ .

Here,  $e$  is the index of nilpotency of the maximal ideal  $\langle a \rangle$ .

**Definition 10.** In this thesis, any code over the alphabet  $\{A, C, G, T\}$  will be referred as a DNA code.

## 2.2 Setup of Preliminary Structures and Maps

Throughout the thesis,  $\mathbb{F}_q$  denotes a field of order  $q$  where  $q$  is a power of a prime. The local rings  $S = \mathbb{F}_{4^t}[u]/\langle u^k \rangle$  and  $R = \mathbb{F}_4[v]/\langle v^{tk} \rangle$  of order  $4^{tk}$  which are both vector spaces over the field  $\mathbb{F}_4 = \{0, 1, \omega, \omega + 1\}$  where  $\omega$  is a root of  $x^2 + x + 1 = 0$  are considered. To study DNA codes over  $R$  and  $S$ , construction of bijections from  $R$  and  $S$  to the set  $S_{D_4^{tk}} = \{A, C, G, T\}^{tk}$  are needed, satisfying some certain properties where  $S_{D_4} = \{A, C, G, T\}$  is the DNA alphabet. The constructions of the bijections  $\tilde{\sigma} \circ \eta \circ \gamma : S \rightarrow S_{D_4^{tk}}$  and  $\tilde{\sigma} \circ \eta : R \rightarrow S_{D_4^{tk}}$ , satisfying properties in Lemma 4, are not immediate, see [26]. The following bijections  $\gamma, \eta, \sigma, \tilde{\sigma}$  are used where  $\gamma, \eta$  are  $\mathbb{F}_4$ -linear maps.

$$\begin{array}{ccccccc} S & \xrightarrow{\gamma} & R & \xrightarrow{\eta} & \mathbb{F}_4^{tk} & \xrightarrow{\tilde{\sigma}} & S_{D_4^{tk}} \\ & & & & & \uparrow & \\ & & & & \mathbb{F}_4 & \xrightarrow{\sigma} & S_{D_4} \end{array}$$

To define the linear bijections  $\gamma, \eta$ , firstly the following respective bases

$$\{u^i \alpha^j \mid i = 0, \dots, k-1, j = 0, \dots, t-1\}, \{v^i \mid i = 0, \dots, tk-1\}, \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{tk}\}$$

of  $S, R, \mathbb{F}_4^{tk}$  over  $\mathbb{F}_4$  are fixed. Here,  $\alpha$  is a primitive root of the simple field extension  $\mathbb{F}_4 \subseteq \mathbb{F}_{4^t}$  so that  $\mathbb{F}_{4^t} = \mathbb{F}_4[\alpha]$  and  $\mathbf{e}_i$  is the standard basis vector of  $\mathbb{F}_4^{tk}$  whose  $j$  th component is the Kronecker symbol  $\delta_{ij}$ . The map from elements of  $S$  to  $R$  is defined

by  $\gamma(u^i \alpha^j) = v^{i+j}$  for all  $i, j$ . The definition of the Gray map  $\eta$  depends on the parity of  $tk$ . If  $tk = 2s$  is even,

$$\eta(v^i) = \begin{cases} \mathbf{e}_{i+1} + \sum_{j=s+1}^{2s-i-1} \mathbf{e}_j, & \text{if } i \leq s-1 \\ \mathbf{e}_{i-s+1} + \mathbf{e}_{3s-i}, & \text{if } i \geq s \end{cases}$$

for all  $i$ . If  $tk = 2s - 1$  is odd,

$$\eta(v^i) = \begin{cases} \mathbf{e}_{i+1} + \sum_{j=s+1}^{2s-i-2} \mathbf{e}_j, & \text{if } i \leq s-1 \\ \mathbf{e}_{i-s+1} + \mathbf{e}_{3s-i-1}, & \text{if } i \geq s \end{cases}$$

for all  $i$ . In the definition of  $\eta$ , the convention that if the lower index of the summation symbol is greater than its upper index then the summation is zero is used. The definition of the linear maps  $\gamma$  and  $\eta$  are completed by extending linearly. The bijection  $\tilde{\sigma}$  is the componentwise extension of  $\sigma$  which is defined by letting  $\sigma(0) = A$ ,  $\sigma(1) = C$ ,  $\sigma(\omega) = G$  and  $\sigma(\omega + 1) = T$ . So  $\tilde{\sigma}(x_1, \dots, x_{tk}) = (\sigma(x_1), \dots, \sigma(x_{tk}))$  for all  $(x_1, \dots, x_{tk}) \in \mathbb{F}_4^{tk}$ .

*Example 1.* Let  $\mathbf{a} = \sum_{i=0}^{tk-1} a_i v^i \in R$ . If  $tk = 10$  then  $\eta(\mathbf{a}) = (a_0 + a_5, a_1 + a_6, a_2 + a_7, a_3 + a_8, a_4 + a_9, a_0 + a_1 + a_2 + a_3 + a_9, a_0 + a_1 + a_2 + a_8, a_0 + a_1 + a_7, a_0 + a_6, a_5)$ . If  $tk = 11$  then  $\eta(\mathbf{a}) = (a_0 + a_6, a_1 + a_7, a_2 + a_8, a_3 + a_9, a_4 + a_{10}, a_5, a_0 + a_1 + a_2 + a_3 + a_{10}, a_0 + a_1 + a_2 + a_9, a_0 + a_1 + a_8, a_0 + a_7, a_6)$ .

To simplify the notation, put  $\theta = \tilde{\sigma} \circ \eta$  and let  $\Theta : R^n \rightarrow S_{D_{4^{tk}}}^n$  be the componentwise extension of  $\theta$  so that  $\Theta(\mathbf{r}) = (\theta(r^{(0)}), \theta(r^{(1)}), \dots, \theta(r^{(n-1)}))$  for all  $\mathbf{r} = (r^{(0)}, r^{(1)}, \dots, r^{(n-1)}) \in R^n$ . The map  $\Theta$  is used to obtain codes  $\Theta(C)$  of length  $tkn$  over the DNA alphabet by using codes  $C$  of length  $n$  over  $R$ . The so obtained codes  $\Theta(C)$  are called DNA codes over  $R$ . It is tried to relate the properties of codes  $C$  and  $\Theta(C)$ , namely WCC property and reversibility, as mentioned in the previous section. The complements of the DNA letters are defined as  $\bar{A} = T$ ,  $\bar{T} = A$ ,  $\bar{C} = G$  and  $\bar{G} = C$ . One may define the respective complements  $\bar{s}$ ,  $\bar{r}$ ,  $\bar{x}$  of elements of  $s \in S$ ,  $r \in R$ ,  $x \in \mathbb{F}_4$  in such a way that they commute with the maps in the above diagram. That is,  $\sigma(\bar{x}) = \overline{\sigma(x)}$ ,  $\theta(\bar{r}) = (\bar{r}_1, \bar{r}_2, \dots, \bar{r}_{tk})$  where  $r_i$  are the DNA letters such that  $(r_1, r_2, \dots, r_{tk}) = \theta(r)$  and  $\gamma(\bar{s}) = \overline{\gamma(s)}$ . Furthermore, the complement  $(\overline{x_1, x_2, \dots, x_m})$  of any tuple  $(x_1, x_2, \dots, x_m)$  of elements of  $R, S, \mathbb{F}_4$  or of DNA letters are defined

**Table 2.1 :** DNA Strands corresponding to the elements of  $R = \mathbb{F}_4[v]/\langle v^3 \rangle$ .

R	$\eta(R)$	$\theta(R)$	R	$\eta(R)$	$\theta(R)$
0	(0,0,0)	AAA	$\omega v^2$	( $\omega, 0, \omega$ )	GAG
1	(1,0,0)	CAA	$\omega v^2 + 1$	( $\omega + 1, 0, \omega$ )	TAG
$\omega$	( $\omega, 0, 0$ )	GAA	$\omega v^2 + \omega$	(0,0, $\omega$ )	AAG
$\omega + 1$	( $\omega + 1, 0, 0$ )	TAA	$\omega v^2 + \omega + 1$	(1,0, $\omega$ )	CAG
$v$	(0,1,0)	ACA	$\omega v^2 + v$	( $\omega, 1, \omega$ )	GCG
$v + 1$	(1,1,0)	CCA	$\omega v^2 + v + 1$	( $\omega + 1, 1, \omega$ )	TCG
$v + \omega$	( $\omega, 1, 0$ )	GCA	$\omega v^2 + v + \omega$	(0,1, $\omega$ )	ACG
$v + \omega + 1$	( $\omega + 1, 1, 0$ )	TCA	$\omega v^2 + v + \omega + 1$	(1,1, $\omega + 1$ )	CCT
$\omega v$	(0, $\omega, 0$ )	AGA	$\omega v^2 + \omega v$	( $\omega, \omega, \omega$ )	GGG
$\omega v + 1$	(1, $\omega, 0$ )	CGA	$\omega v^2 + \omega v + 1$	( $\omega + 1, \omega, \omega$ )	TGG
$\omega v + \omega$	( $\omega, \omega, 0$ )	GGA	$\omega v^2 + \omega v + \omega$	(0, $\omega, \omega$ )	AGG
$\omega v + \omega + 1$	( $\omega + 1, \omega, 0$ )	TGA	$\omega v^2 + \omega v + \omega + 1$	(1, $\omega, \omega$ )	CGG
$(\omega + 1)v$	(0, $\omega + 1, 0$ )	ATA	$\omega v^2 + (\omega + 1)v$	( $\omega, \omega + 1, \omega$ )	GTG
$(\omega + 1)v + 1$	(1, $\omega + 1, 0$ )	CTA	$\omega v^2 + (\omega + 1)v + 1$	( $\omega + 1, \omega + 1, \omega$ )	TTG
$(\omega + 1)v + \omega$	( $\omega, \omega + 1, 0$ )	GTA	$\omega v^2 + (\omega + 1)v + \omega$	(0, $\omega + 1, \omega$ )	ATG
$(\omega + 1)v + \omega + 1$	( $\omega + 1, \omega + 1, 0$ )	TTA	$\omega v^2 + (\omega + 1)v + \omega + 1$	(1, $\omega + 1, \omega$ )	CTG
$v^2$	(1,0,1)	CAC	( $\omega + 1$ ) $v^2$	( $\omega + 1, 0, \omega + 1$ )	TAT
$v^2 + 1$	(0,0,1)	AAC	( $\omega + 1$ ) $v^2 + 1$	( $\omega, 0, \omega + 1$ )	GAT
$v^2 + \omega$	( $\omega + 1, 0, 1$ )	TAC	( $\omega + 1$ ) $v^2 + \omega$	(1,0, $\omega + 1$ )	CAT
$v^2 + \omega + 1$	( $\omega, 0, 1$ )	GAC	( $\omega + 1$ ) $v^2 + \omega + 1$	(0,0, $\omega + 1$ )	AAT
$v^2 + v$	(1,1,1)	CCC	( $\omega + 1$ ) $v^2 + v$	( $\omega + 1, 1, \omega + 1$ )	TCT
$v^2 + v + 1$	(0,1,1)	ACC	( $\omega + 1$ ) $v^2 + v + 1$	( $\omega, 1, \omega + 1$ )	GCT
$v^2 + v + \omega$	( $\omega + 1, 1, 1$ )	TCC	( $\omega + 1$ ) $v^2 + v + \omega$	(1,1, $\omega + 1$ )	CCT
$v^2 + v + \omega + 1$	( $\omega, 1, 1$ )	GCC	( $\omega + 1$ ) $v^2 + v + \omega + 1$	(0,1, $\omega + 1$ )	ACT
$v^2 + \omega v$	(1, $\omega, 1$ )	CGC	( $\omega + 1$ ) $v^2 + \omega v$	( $\omega + 1, \omega, \omega + 1$ )	TGT
$v^2 + \omega v + 1$	(0, $\omega, 1$ )	AGC	( $\omega + 1$ ) $v^2 + \omega v + 1$	( $\omega, \omega, \omega + 1$ )	GGT
$v^2 + \omega v + \omega$	( $\omega + 1, \omega, 1$ )	TGC	( $\omega + 1$ ) $v^2 + \omega v + \omega$	(1, $\omega, \omega + 1$ )	CGT
$v^2 + \omega v + \omega + 1$	( $\omega, \omega, 1$ )	GGC	( $\omega + 1$ ) $v^2 + \omega v + \omega + 1$	(0, $\omega, \omega + 1$ )	AGT
$v^2 + (\omega + 1)v$	(1, $\omega + 1, 1$ )	CTC	( $\omega + 1$ ) $v^2 + (\omega + 1)v$	( $\omega + 1, \omega + 1, \omega + 1$ )	TTT
$v^2 + (\omega + 1)v + 1$	(0, $\omega + 1, 1$ )	ATC	( $\omega + 1$ ) $v^2 + (\omega + 1)v + 1$	( $\omega, \omega + 1, \omega + 1$ )	GGT
$v^2 + (\omega + 1)v + \omega$	( $\omega + 1, \omega + 1, 1$ )	TTC	( $\omega + 1$ ) $v^2 + (\omega + 1)v + \omega$	(1, $\omega + 1, \omega + 1$ )	CTT
$v^2 + (\omega + 1)v + \omega + 1$	( $\omega, \omega + 1, 1$ )	GTC	( $\omega + 1$ ) $v^2 + (\omega + 1)v + \omega + 1$	(0, $\omega + 1, \omega + 1$ )	ATT

componentwise so that  $\overline{(x_1, x_2, \dots, x_m)} = (\overline{x_1}, \overline{x_2}, \dots, \overline{x_m})$ . For instance, the complements in  $\mathbb{F}_4$  are given by  $\overline{0} = w + 1$  and  $\overline{1} = w$ .

The three letter long DNA Strands, i.e. codons, that are images of elements of  $R$  under  $\theta$  for  $(n, k, t) = (1, 3, 1)$  are listed in Table 2.1.

*Example 2.* Let  $t = k = 2$ . For  $u\alpha \in S$  and  $v \in R$  one easily sees that  $\overline{u\alpha} = (w + 1)u + wu\alpha$  and  $\overline{v} = v + (w + 1)v^2 + (w + 1)v^2$ .

Whenever it is convenient, the results are presented in a more general setting, such as over arbitrary commutative rings. For the rest, let  $A$  be any commutative unital ring such that  $1 \neq 0$ . The  $A$ -module isomorphism  $\pi : A^n \rightarrow A[x]/\langle x^n - 1 \rangle$ , given for any  $\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \in A^n$  by  $\pi(\mathbf{a}) = p_{\mathbf{a}}(x) + \langle x^n - 1 \rangle$  where  $p_{\mathbf{a}}(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$ , allows to identify codewords  $\mathbf{a}$  of length  $n$  with polynomials  $p_{\mathbf{a}}(x)$  of degree  $\leq n - 1$ . A subset  $C$  of  $A^n$  is called a cyclic code over  $A$  of length  $n$  if  $\pi(C)$  is an ideal of the quotient ring  $A[x]/\langle x^n - 1 \rangle$ . By abuse of notation, it is said that any ideal of  $A[x]/\langle x^n - 1 \rangle$  is a cyclic code over  $A$ , and the elements  $g(x) + \langle x^n - 1 \rangle$  of the quotient  $A[x]/\langle x^n - 1 \rangle$  are still denoted by  $g(x)$ . For simplicity,  $A_n = A[x]/\langle x^n - 1 \rangle$  and  $R_n = R[x]/\langle x^n - 1 \rangle$  for any positive integer  $n$ .



### 3. COMPLEMENT CODES

Let  $C$  be a code over  $R$  and  $D$  be a code over  $S_{D_4}$ . It is said that  $C$  is a complement code if  $\bar{a} \in C$  for all  $a \in C$ , and  $D$  is a complement DNA code if  $\bar{a} \in D$  for all  $a \in D$ . By its definition, its obvious that  $\overline{\Theta(a)} = \Theta(\bar{a})$  for all  $a \in R^n$ . Therefore,

*Remark 1.* For any code  $C$  over  $R$ ,  $C$  is a complement code if and only if  $\Theta(C)$  is a complement DNA code.

The above remark remains true for codes over  $S$  and their images under the componentwise expansion of  $\theta \circ \gamma$ . As the sum of any two distinct elements of  $\mathbb{F}_4$  is equal to the sum of the other two elements, one sees that  $a + \bar{a}$  is the same for all  $a \in R$ . Indeed,

**Lemma 1.** *Let*

$$\kappa := (\omega + 1) \sum_{i=\lfloor \frac{tk}{2} \rfloor}^{tk-1} v^i$$

and  $\mathbf{1}_n$  be the all-ones vector in  $R^n$ . Then,  $a + \bar{a} = \kappa$  and  $\overline{\Theta(a)} = \Theta(a + \kappa \mathbf{1}_n)$  for all  $a \in R$  and  $a \in R^n$ .

*Proof.* Let  $a \in R$ . Then  $\eta(a) = (b_1, \dots, b_{tk})$  and  $\eta(\bar{a}) = (c_1, \dots, c_{tk})$  for some  $b_i, c_i \in \mathbb{F}_4$ . As  $\eta(\bar{a}) = \overline{\eta(a)}$ , one sees that  $c_i = \bar{b}_i$  for each  $i$ . For any  $i$ , the possibilities for the set  $\{b_i, c_i\}$  are  $\{0, w + 1\}$  and  $\{1, w\}$ . In any case one sees that  $b_i + c_i = w + 1$ . Therefore,  $\eta(a + \bar{a}) = (w + 1) \mathbf{1}_{tk}$  where  $\mathbf{1}_{tk}$  is the all-ones vector in  $\mathbb{F}_4^{tk}$ . As  $\eta^{-1}(\mathbf{1}_{tk}) = \sum_{i=\lfloor \frac{tk}{2} \rfloor}^{tk-1} v^i$ , the proof is finished.  $\square$

As  $\gamma$  is a linear bijection commuting with complements, one sees that  $b + \bar{b} = \gamma^{-1}(\kappa)$  for all  $b \in S$  and so a result similar to the previous lemma holds for  $S$ .

*Example 3.* Let  $n = 1$ ,  $R = \mathbb{F}_4[v]/\langle v^7 \rangle$  and  $a = (\omega + 1) + \omega v^2 + \omega v^3 + (\omega + 1)v^4 + v^5 + (\omega + 1)v^6 \in R^1$ . Then, one sees that

$$\begin{aligned} \kappa &= (\omega + 1)(v^3 + v^4 + v^5 + v^6), & \eta(a) &= (0, 1, \omega + 1, \omega, \omega, \omega, \omega + 1) \\ \theta(a) &= ACTGGGT, & a + \kappa &= \omega + 1 + \omega v^2 + v^3 + \omega v^5 + \omega v^6 \\ \eta(a + \kappa) &= (\omega + 1, \omega, 0, 1, 1, 1, 0), & \theta(a + \kappa) &= TGACCCA = \overline{\theta(a)} \end{aligned}$$

**Proposition 2.** For any linear code  $C$  over  $R$  of length  $n$ ,  $C$  is a complement code if and only if  $\kappa \mathbf{1}_n \in C$ .

*Proof.* Follows from Lemma 1 stating that  $\mathbf{a} + \bar{\mathbf{a}} = \kappa \mathbf{1}_n$ . □

**Proposition 3.** Let  $C$  be an ideal of  $R_n$ . Then,  $C$  is a complement code if and only if  $\kappa(1 + x + \cdots + x^{n-1}) \in C$ .

*Proof.* As the  $\mathbf{1}_n$  vector corresponds to  $(1 + x + \cdots + x^{n-1})$ , the result follows from Proposition 2. □



#### 4. REVERSIBLE (CYCLIC) CODES

Let  $Z$  be any alphabet (i.e., DNA alphabet or a commutative ring). A code  $C$  over  $Z$  of length  $n$  is called reversible if  $T(C) = C$  where  $T$  is the map  $T : Z^n \rightarrow Z^n$  given for any  $\mathbf{a} = (a_1, a_2, \dots, a_n) \in Z^n$  by  $T(\mathbf{a}) = \mathbf{a}^r := (a_n, \dots, a_2, a_1)$ . Reversible codes over the DNA alphabet are called reversible DNA codes.

The difficulty of characterizing linear codes over  $A$  with respect to their minimal generating sets depends on the structure of the ring  $A$ . Leaving the elaborations to another work, the following are only mentioned without justification.

*Remark 2.* Let  $C$  be a reversible linear code over  $A$  such that  $A$  cannot be written as the direct sum of two nonzero reversible linear subcodes.

1. If 2 is invertible in  $A$ , then either  $\mathbf{c}^r = \mathbf{c}$  for all  $\mathbf{c} \in C$  or  $\mathbf{c}^r = -\mathbf{c}$  for all  $\mathbf{c} \in C$ .
2. If  $A$  is a finite local ring of characteristic 2 and the maximal ideal  $\mathfrak{m}$  of  $A$  is principal, then either  $C$  is a free  $R$ -module of rank 2 with a basis  $\{\mathbf{c}_1, \mathbf{c}_2\}$  such that  $\mathbf{c}_1^r = \mathbf{c}_2$  or else  $\mathbf{c}^r \equiv \mathbf{c} \pmod{\mathfrak{m}C}$  for all  $\mathbf{c} \in C$ .

Note that if an alphabet  $Z$  has a module structure over a ring (i.e., a vector space over a field), then  $T : Z^n \rightarrow Z^n$  is a module isomorphism. Reversing elements of  $R$  and  $S_{D_{4^k}}$  have the following commuting property.

**Lemma 4.** *Let*

$$\rho := 1 + \sum_{i=\lceil \frac{tk}{2} \rceil}^{tk-1} v^i.$$

*Then,  $\theta(a)^r = \theta(\rho a)$  and  $\Theta(\mathbf{a})^r = \Theta(\rho \mathbf{a}^r)$  for all  $a \in R$  and  $\mathbf{a} \in R^n$ .*

*Proof.* Justification is done only for  $\theta$ , from which the result for  $\Theta$  follows easily. For any  $\mathbf{x} = (x_1, \dots, x_{tk}) \in \mathbb{F}_4^{tk}$  note that  $\tilde{\sigma}(\mathbf{x})^r = (\sigma(x_1), \dots, \sigma(x_{tk}))^r = (\sigma(x_{tk}), \dots, \sigma(x_1)) = \tilde{\sigma}(\mathbf{x}^r)$ . As  $\theta = \tilde{\sigma} \circ \eta$ , it is enough to justify that  $\eta(a)^r = \eta(\rho a)$  for all  $a \in R$ . As  $T$  and  $\eta$  are  $\mathbb{F}_4$ -linear, it is only needed to justify that  $\eta(v^i)^r = \eta(\rho v^i)$  for all  $i$ . This can

be done easily by using the definition of  $\eta$  and the fact that the basis elements  $\mathbf{e}_i$  of  $\mathbb{F}_4^{tk}$  satisfy  $\mathbf{e}_i^r = \mathbf{e}_{tk-i+1}$ .  $\square$

The following remark follows from Lemma 4.

*Remark 3.* Let  $C$  be a linear code over  $R$ . Then,  $\Theta(C)$  is a reversible DNA code if and only if  $C$  is reversible.

Now, reversibility of cyclic codes is considered. It begins with the classification of reversible cyclic codes  $C$  in terms of the ideal generators of  $\pi(C)$ . To do so, first some elementary technicalities are needed. The reciprocal  $p^*(x)$  of a nonzero polynomial  $p(x) \in A[x]$  is defined to be the polynomial  $p^*(x) := x^{\deg p(x)} p(1/x)$ . A polynomial  $p(x)$  is called self-reciprocal if  $p^*(x) = ap(x)$  for some unit  $a \in A$ .

**Lemma 5.** Let  $a(x), b(x), f(x), g(x), h(x)$  be nonzero polynomials over  $A$ . Then

1. If  $\deg ab = \deg a + \deg b$  then  $(a(x)b(x))^* = a^*(x)b^*(x)$ .
2. If  $\deg(a + b) = \deg a$ , then  $(a(x) + b(x))^* = a^*(x) + x^{\deg a - \deg b} b^*(x)$ .
3. If  $f(0) \neq 0$ , then  $\deg f^*(x) = \deg f(x)$  and  $(f^*(x))^* = f(x)$ .
4. If  $\deg f(x) \leq n - 1$  and  $\mathbf{a}_f \in A^n$  is the codeword such that  $\pi(\mathbf{a}_f) = f(x)$ , then  $\pi(\mathbf{a}_f^r) = x^{n-1-m} f^*(x)$  where  $m = \deg f(x)$ .
5. If the leading coefficient of  $h(x)$  is not a zero divisor and  $\deg f > \deg g$ , then  $f(x) \equiv g(x) \pmod{h(x)}$  implies that  $f^*(x) \equiv x^{\deg f - \deg g} g^*(x) \pmod{h^*(x)}$ .
6. Let  $C$  be an ideal of  $A_n$ . For any polynomial  $f(x) \in A[x]$  and any positive integer  $m$ ,  $f(x) \in C$  if and only if  $x^m f(x) \in C$ .

*Proof.* The first two parts can be found in [14,24], however here some conditions are added to guarantee that these properties are satisfied in an arbitrary polynomial ring. For instance,  $a = x^2 + x + 1$  and  $b = -x^2 - 1$  satisfies the condition  $\deg a \geq \deg b$  given in the cited papers but  $(a + b)^* \neq a^* + b^*$ . (3) is easy to justify.

(4) Let  $f(x) = a_0 + \cdots + a_m x^m$ . The codeword corresponding to  $f$  under the bijection  $\pi$  is  $\mathbf{a}_f = (a_0, a_1, \dots, a_m, 0, \dots, 0) \in C$ . Its reverse codeword  $\mathbf{a}_f^r = (0, \dots, 0, a_m, \dots, a_1, a_0)$  corresponds to the polynomial  $x^k f^*(x)$  under  $\pi$  where  $k = n - 1 - m$ .

(5) There is a polynomial  $q \in A[x]$  such that  $f - g = hq$ . Using parts (1) and (2) one sees that  $f^* - x^{\deg f - \deg g} g^*(x) = h^* q^*$ .

(6) Assume  $x^m f \in C$ . There is a positive integer  $m'$  such that  $n$  divides  $m' + m$ . As  $C$  is an ideal,  $x^{m'+m} f \in C$ . Since  $x^n \equiv 1 \pmod{x^n - 1}$ , it follows that  $f \in C$ .

□

**Proposition 6.** *Let  $C$  be an ideal of  $A_n$ . Then,  $C$  is a reversible code if and only if  $f^*(x) \in C$  for all nonzero polynomials  $f(x)$  over  $A$  such that  $f(x) \in C$ .*

*Proof.* It follows from part (4) of Lemma 5 that any code  $D$  is reversible if and only if  $x^{n-1-m} g^*(x) \in D$  for all nonzero polynomial  $g(x)$  over  $A$  of degree  $m \leq n - 1$  such that  $g(x) \in D$ . As  $C$  is cyclic, using part (6) of Lemma 5 one sees that  $C$  is a reversible code if and only if  $g^*(x) \in C$  for all nonzero polynomials  $g(x)$  over  $A$  of degree  $\leq n - 1$  such that  $g(x) \in C$ .

Let  $f$  be a nonzero polynomial over  $A$  of degree  $\geq n$  such that  $f \in C$ . If  $f$  is a multiple of  $x^n - 1$  then so is  $f^*$ , implying that  $f^* \in C$  regardless of whether  $C$  is reversible. It is assumed that  $f$  is not a multiple of  $x^n - 1$ . As  $x^n - 1$  is monic, the division algorithm may be used to divide  $f$  by  $x^n - 1$  to obtain a nonzero remainder  $r$  with degree  $\leq n - 1$ . Note that  $r \in C$ . In part (5) of Lemma 5 by letting  $h = x^n - 1$  and  $g = r$  one sees that  $f^* \in C$  if and only if  $x^m r^* \in C$  for some positive integer  $m$ , equivalently  $r^* \in C$  by part (6) of Lemma 6. The result follows from the first paragraph by letting  $g = r$ . □

Proposition 6 may be refined as follows.

**Corollary 7.** *Let  $\mathcal{A}$  be a set of nonzero polynomials in  $A[x]$  and  $C = \langle \mathcal{A} \rangle$  be the ideal of  $A_n$  generated by  $\mathcal{A}$  (i.e., by  $\{a + \langle x^n - 1 \rangle \mid a \in \mathcal{A}\}$ ). Then,  $g^* \in C$  for all  $g \in \mathcal{A}$  if and only if  $C$  is a reversible code.*

*Proof.* Assume that  $g^* \in C$  for all  $g \in \mathcal{A}$ . Let  $f(x)$  be a nonzero polynomial over  $A$  such that  $f(x) \in C$ . Then

$$f(x) \equiv \sum_{i=1}^m c_i a_i \pmod{x^n - 1}$$

for some polynomials  $a_i \in \mathcal{A}$  and  $c_i \in A[x]$ . As  $c_i \equiv c_i + (x^n - 1)^{n_i} \pmod{x^n - 1}$  where  $n_i$  is any non-negative integer, replacing  $c_i$  with  $c_i + (x^n - 1)^{n_i}$  if necessary one may assume that  $d_1 > d_2 > \dots > d_m$  and  $\deg f < d_1$  where  $d_i = \deg c_i a_i$  and assume that each  $c_i$  is monic (so that one may apply parts (1) and (2) of Lemma 5 to  $c_i a_i$  and  $\sum c_i a_i$ ). Part (5) of Lemma 5 with  $h = x^n - 1$  implies that

$$x^s f^*(x) \equiv \left( \sum_{i=1}^m c_i a_i \right)^* \pmod{x^n - 1}$$

where  $s = d_1 - \deg f$ . Using parts (1) and (2) of Lemma 5 successively one sees that

$$x^s f^*(x) \equiv \sum_{i=1}^m x^{d_1 - d_i} c_i^* a_i^* \pmod{x^n - 1}.$$

As  $a_i^* \in C$  by the assumption,  $x^s f^* \in C$  and so  $f^* \in C$  by part (6) of Lemma 6. Hence  $C$  is reversible by Proposition 6.  $\square$

The previous result allows to produce many reversible cyclic codes. For instance, for any nonzero  $f(x) \in A[x]$ , the ideal  $\langle f(x), f^*(x) \rangle$  of  $A_n$  is a reversible cyclic code over  $A$  of length  $n$ . However, the determination of its parameters may not be easy.

**Corollary 8.** *Let  $f(x) \in A[x]$  be monic such that  $f(x) | x^n - 1$  in  $A[x]$ . Then, the cyclic code  $C = \langle f(x) \rangle$  over  $A$  of length  $n$  is reversible if and only if  $f(x)$  is self-reciprocal.*

*Proof.* Suppose that  $C$  is reversible. Then  $f^*(x) \in C$  by Proposition 6 and so  $f^*(x) \equiv g(x)f(x) \pmod{x^n - 1}$  for some  $g(x) \in A[x]$ . As  $f(x) | x^n - 1$ ,  $f^*(x) = h(x)f(x)$  in  $A[x]$  for some  $h(x) \in A[x]$ . As the leading coefficient and the constant term of  $f(x)$  are unit in  $A$ , considering the degrees one sees that  $h(x)$  is constant and unit in  $A$ . So  $f(x)$  is self-reciprocal.

Converse direction follows from Corollary 7.  $\square$

As the following example shows, the condition  $f(x) | x^n - 1$  in the previous result can not be dropped.

*Example 4.* Let  $A = \mathbb{F}_4$ ,  $n = 3$ ,  $f = x^2 + x$  and  $C = \langle x^2 + x \rangle$ . Then  $C = \langle x + 1 \rangle$  too, since  $x^2(x^2 + x) \equiv x + 1 \pmod{x^3 - 1}$  and  $x(x + 1) = x^2 + x$ . Although  $f$  is not self-reciprocal, Corollary 8 implies that  $C$  is reversible.

**Corollary 9.** *Let  $I$  and  $J$  be ideals of  $A_n$  and  $a \in A$ . Then,*

1. *If  $I$  and  $J$  are reversible codes, then so is  $IJ$ .*
2. *The ideal  $\langle a \rangle$  of  $A_n$  is reversible.*
3. *The ideal  $\langle af(x) \rangle$  of  $A_n$  is reversible for any self-reciprocal monic polynomial  $f(x) \in A[x]$  such that  $f(x) \mid x^n - 1$ .*

*Proof.* (1) Follows from Proposition 6 and Corollary 7 because  $IJ$  is generated by the set  $\mathcal{A} = \{fg \mid f \in I, g \in J\}$ , and as in the proof of Corollary 7 one may assume that  $f$  and  $g$  are monic, and so  $(fg)^* = f^*g^* \in IJ$ .

(2) Follows from Proposition 6.

(3) Follows from the previous parts and Corollary 8. □

**Lemma 10.** *Let  $a$  be a nilpotent element of  $A$  with nilpotency  $s$  and  $g_0, g_1, \dots, g_{s-1}$  be monic polynomials over  $A$  such that  $g_{s-1} \mid g_{s-2} \mid \dots \mid g_1 \mid g_0 \mid x^n - 1$ . Assume for any  $j$  with  $0 \leq j \leq s - 1$  and for any unit  $z \in A$  that the annihilator of  $a^{s-1-j}$  in  $A$  is  $\langle a^{j+1} \rangle$ , and  $a^{s-1-j}g_j^*(x) = a^{s-1-j}zg_j(x)$  implies  $g_j$  is self-reciprocal. Then the ideal  $\langle g_0, ag_1, \dots, a^{s-1}g_{s-1} \rangle$  of  $A_n$  is a reversible if and only if each  $g_i$  is self-reciprocal.*

*Proof.* Assume that the given ideal  $C$  is reversible. It follows from Proposition 6 that  $g_i^* \in C$  for all  $i$ . There are polynomials  $h_i$  in  $A[x]$  such that

$$g_i^* \equiv h_0g_0 + h_1ag_1 + \dots + h_{s-1}a^{s-1}g_{s-1} \pmod{x^n - 1}.$$

Multiplying through  $a^{s-1-i}$  one gets

$$a^{s-1-i}g_i^* \equiv a^{s-1-i}(h_0g_0 + h_1ag_1 + \dots + h_i a^i g_i) \pmod{x^n - 1}.$$

As  $g_i$  divides each of  $g_0, g_1, \dots, g_{i-1}$ ,

$$a^{s-1-i}g_i^* \equiv a^{s-1-i}hg_i \pmod{x^n - 1}$$

for some polynomial  $h$  in  $A[x]$ . Using the fact that  $g_i$  divides  $x^n - 1$  one concludes that

$$a^{s-1-i} g_i^* = a^{s-1-i} q g_i$$

for some polynomial  $q$  in  $A[x]$ . Erasing the coefficients of  $q$  annihilated by  $a^{s-1-i}$  one may assume that the leading coefficient of  $q$  is not annihilated by  $a^{s-1-i}$ . As  $g_i$  is a divisor of  $x^n - 1$ , the leading coefficient  $c$  of  $g_i^*$  is a unit. Comparing the degrees and the coefficients of  $x^{d_i}$  where  $d_i = \deg g_i$ , one sees that  $q$  is constant and  $c - q \in \langle a^{i+1} \rangle$ , implying that  $q$  is a unit in  $A$ . Hence  $g_i$  is self-reciprocal.  $\square$

The structure of ideals of  $A_n$  when  $A$  is a finite chain ring is studied in [27]–[29].

*Remark 4.* [27] Let  $A$  be a finite chain ring with the maximal ideal  $\langle a \rangle$  and let  $s$  be the nilpotency of  $a$ . Assume that the characteristic of the field  $A/\langle a \rangle$  does not divide  $n$ . Reduction of the coefficients modulo  $\langle a \rangle$  induces a surjective ring homomorphism  $\mu : A[x] \rightarrow (A/\langle a \rangle)[x]$ . For any nonzero ideal  $C$  of  $A_n$  there are monic polynomials  $g_0, g_1, \dots, g_{s-1}$  over  $A$  such that  $g_{s-1} | g_{s-2} | \dots | g_1 | g_0 | x^n - 1$  and  $C = \langle g_0, a g_1, \dots, a^{s-1} g_{s-1} \rangle$ . Furthermore, if one writes  $x^n - 1 = h_1 h_2 \dots h_r$  over the field  $A/\langle a \rangle$  for some monic irreducible polynomials  $h_i$  which are necessarily coprime (as the characteristic of  $A/\langle a \rangle$  does not divide  $n$ ), then by the Hensel's Lemma there are pairwise coprime monic irreducible polynomials  $H_i$  over  $A$  satisfying  $\mu(H_i) = h_i$  such that  $x^n - 1 = H_1 H_2 \dots H_r$  in  $A[x]$ . The polynomials  $g_j$  appeared in the generators of  $C$  are products of certain polynomials among  $H_1, H_2, \dots, H_r$ .

The next result follows from the definition of the ring  $R$  and Remark 4.

*Remark 5.* Assume the notations of Remark 4. If  $A$  has a unital subring isomorphic to  $A/\langle a \rangle$ , then by regarding the coefficients of  $h_i$  as elements of  $A$  one may see that  $h_i \in A[x]$  and so one may assume  $H_i = h_i$ . The ring  $R$  is a finite chain ring of characteristic 2 with the unique maximal ideal  $\langle v \rangle$  and the nilpotency of  $v$  is  $tk$ . The quotient field  $R/\langle v \rangle$  is isomorphic to  $\mathbb{F}_4$  and  $\mathbb{F}_4$  is a unital subring of  $R$ . If  $n$  is odd, for any ideal  $C$  of  $R_n$  there are monic polynomials  $g_0, g_1, \dots, g_{s-1}$  over  $\mathbb{F}_4$  such that  $g_{s-1} | g_{s-2} | \dots | g_1 | g_0 | x^n - 1$  and  $C = \langle g_0, v g_1, \dots, v^{s-1} g_{s-1} \rangle$  where  $s = tk$ .

**Theorem 11.** *Let  $C = \langle g_0, vg_1, \dots, v^{s-1}g_{s-1} \rangle$  be an ideal of  $R_n$  where  $n$  is odd,  $s = tk$  and the generators  $v^i g_i$  are given as in Remark 5. Then,  $C$  is reversible if and only if each  $g_i$  is self-reciprocal.*

*Proof.* It is enough to justify that the conditions in Lemma 10 are satisfied for  $A = R$  and  $a = v$ . As the ideals of  $R$  are  $0 \subset \langle v \rangle \subset \dots \subset \langle v^{s-1} \rangle$  and  $s$  is the nilpotency of  $v$ , it is clear that the annihilator of  $v^{s-1-j}$  is  $\langle v^{j+1} \rangle$ . For the second condition let  $v^{s-1-j}g_j^*(x) = v^{s-1-j}zg_j(x)$  for some unit  $z$  in  $R$ . Then  $z = \sum c_i v^i$  for some  $c_i \in \mathbb{F}_4$  with  $c_0 \neq 0$  and  $g_j = \sum_{i=0}^m \lambda_i x^i$  for some  $\lambda_i \in \mathbb{F}_4$ . Comparing the coefficients of  $x^i$ , one sees that  $v^{s-1-j}\lambda_{m-i} = v^{s-1-j}z\lambda_i$ . As  $1, v, \dots, v^{s-1}$  are linearly independent over  $\mathbb{F}_4$ , it follows that  $\lambda_{m-i} = c_0\lambda_i$ . Hence,  $g_j^* = c_0g_j$  and  $g_j$  is self-reciprocal.  $\square$

The following is immediate from Proposition 3 and Theorem 11.

**Corollary 12.** *Let  $C = \langle g_0, vg_1, \dots, v^{s-1}g_{s-1} \rangle$  be an ideal of  $R_n$  where  $n$  is odd,  $s = tk$  and the generators  $v^i g_i$  are given as in Remark 5. Then,  $\Theta(C)$  is reversible complement DNA code if and only if each  $g_i$  is reversible and  $\kappa(1 + x + \dots + x^{n-1}) \in C$ .*



## 5. EXAMPLES OF DNA CODES

*Example 5.* Let  $n = 5$  and  $S = \mathbb{F}_{16}[u]/\langle u^2 \rangle$ . Then,

$$R = \mathbb{F}_4[v]/\langle v^4 \rangle, \quad \Theta : R^5 \rightarrow S_{D_{256}}^5 = \{A, C, G, T\}^{20}, \quad \kappa = (\omega + 1)(v^2 + v^3) \in R.$$

Let  $C = \langle \kappa(1 + x + x^2 + x^3 + x^4) \rangle \subset R^5$ . The reversible complement DNA code  $\Theta(C)$  of length  $l = 20$  is given in Table 5.1. It has size  $M = 16$  and Hamming distance  $d = 10$ .

**Table 5.1 :** DNA Code of length 20 obtained from  $C = \langle \kappa(1 + x + x^2 + x^3 + x^4) \rangle$ .

AAAAAAAAAAAAAAAAAAAA
TTTTTTTTTTTTTTTTTT
GGGGGGGGGGGGGGGGGG
CCCCCCCCCCCCCCCCCC
ATTAATTAATTAATTAATTA
TAATTAATTAATTAATTAAT
GCCGGCCGGCCGGCCGGCCG
CGGCCGGCCGGCCGGCCGGC
ACCAACCAACCAACCAACCA
TGGTTGGTTGGTTGGTTGGT
GTTGGTTGGTTGGTTGGTTG
CAACCAACCAACCAACCAAC
AGGAAGGAAGGAAGGAAGGA
TCCTTCCTTCCTTCCTTCCT
GAAGGAAGGAAGGAAGGAAG
CTTCCTTCCTTCCTTCCTTC

*Example 6.* Let  $n = 9$  and  $S = \mathbb{F}_4[u]/\langle u^3 \rangle$ . Then,

$$R = \mathbb{F}_4[v]/\langle v^3 \rangle, \quad \Theta : R^9 \rightarrow S_{D_{64}}^9 = \{A, C, G, T\}^{27}, \quad \kappa = (\omega + 1)(v + v^2) \in R.$$

Let  $C = \langle \kappa(1 + x^3 + x^6) \rangle \subset R^9$ . The DNA code  $\Theta(C)$  of length  $l = 27$  is given in Table 5.2. It has size  $M = 106$  and Hamming distance  $d = 3$ . For this DNA code, code (transmission) rate  $\mathcal{R} = \frac{\log_q M}{l} \approx 0.125$ , and relative minimum distance  $\delta = \frac{d-1}{l} \approx 0.074$ .



**Table 5.3 :** Part of the DNA Code of length 27 obtained from  $C = \langle \kappa(1 + x + x^2) \rangle$ .

AAAAAAAAAAAAAAAAAAAAAAAA	AAAAACAAAAAAAAACAAAAACAAA
AAAAATAAAAAAAAAATAAAAAATAAA	AAAACAAAAAAAAACAAAAAACAAAA
AAAACGAAAAAAAAACGAAAAAAAAACGAAA	AAAACATAAAAAAACTAAAAAACTAAA
AAAAGCAAAAAAAAAAGCAAAAAAAAAAGCAAAA	AAAAGGAAAAAAAAAGGAAAAAAAAAGGAAA
AAAATAAAAAAAAAATAAAAAATAAAAA	AAAATCAAAAAAAATCAAAAAATCAAAA
AAAATTAATAAAAAATTAATAAAAAATTAATA	AAACAATAAAAAACAAAAAAACAAAAA
AAACAGAAAAAAAAACGAAAAAAAAACGAAA	AAACATAAAAAACATAAAAAACATAAAA
AAACCCAAAAAAAAACCCAAAAAACCCAAA	AAACCGAAAAAAAAACCGAAAAAACCGAAA
AAACGAAAAAAAAACGAAAAAAAAACGAAAA	AAACGCAAAAAACGCAAAAAACGCAAAA
AAACGTAAAAAAAAACGTAAAAAACGTAAA	AAACTAAAAAAAACTAAAAAACTAAAA
AAACTGAAAAAAAAACTGAAAAAACTGAAA	AAACTTAAAAAACTTAAAAAACTTAAA
AAAGCAAAAAAAAAAGCAAAAAAGCAAAA	AAAGGAAAAAAAAAGGAAAAAGGAAAA
AAAGCAAAAAAGCAAAAAAGCAAAA	AAAGGTAATAAAAAAGTAAAAAGGTAAA
AAAGCTAAAAAGCTAAAAAGCTAAA	AAAGTGAAAAAGTGAAAAAGTGAAA
AAAGGGAAAAAGGGAAAAAGGGAAA	AAATCAAAAAATCAAAAAATCAAAA
AAAGTCAAAAAAGTCAAAAAAGTCAAAA	AAATCAAAAAATCAAAAAATCAAAA
AAATAAAAAATAAAAAATAAAAA	AAATCTAAAAATCTAAAAATCTAAA
AAATATAAAAAATATAAAAAATATAAAA	AAATGAAAAATGAAAAATGAAA
AAATCGAAAAATCGAAAAATCGAAA	AAATTAATAAAATTAATAAAATTAATA
AAATGCAAAAAATGCAAAAAATGCAAAA	AAAAAGAAAAAGAAAAAGAAAA
AAATTAATAAAATTAATAAAATTAATA	AAAAAGAAAAAGAAAAAGAAAA
AAAAAGAAAAAGAAAAAGAAAA	AAAAATGAAAAATGAAAAATGAAA
AAAAAGAAAAAGAAAAAGAAAA	AAACCAAAAAACCAAAAAACCAAAA
AAAAATGAAAAATGAAAAATGAAA	AAACCTAAAAAACCATAAAAAACCTAAA
AAACCAAAAAACCAAAAAACCAAAA	AAACTCAAAAAACTCAAAAAACTCAAAA
AAACGGAAAAACGGAAAAACGGAAA	AAAGATAAAAAAGATAAAAAAGATAAAA
AAAGAAAAAGAAAAAGAAAA	AAAGGCAAAAAAGGCAAAAAAGGCAAAA
AAAGCGAAAAAGCGAAAAAGCGAAA	AAAGTTAAAAAGTTAAAAAGTTAAA
AAAGTAAAAAGTAAAAAGTAAA	AAATCCAATAATCCAATAATCCAATA
AAATAGAAAAATAGAAAAATAGAAA	AAATGTAATAATGTAATAATGTAATA
AAATGAAAAATGAAAAATGAAA	

*Example 7.* Let  $n = 3$  and  $S = \mathbb{F}_{64}[u]/\langle u^3 \rangle$ . Then,

$$R = \mathbb{F}_4[v]/\langle v^9 \rangle, \quad \Theta : R^3 \rightarrow S_{D_{49}}^3 = \{A, C, G, T\}^{27},$$

$$\kappa = (\omega + 1)(v^4 + v^5 + v^6 + v^7 + v^8) \in R.$$

Let  $C = \langle \kappa(1 + x + x^2) \rangle \subset R^3$ . Part of the DNA code  $\Theta(C)$  of length  $l = 27$  is given in Table 5.3. It has size  $M = 1024$  and Hamming distance  $d = 3$ . Here, code rate  $\mathcal{R} \approx 0.185$ , and relative minimum distance  $\delta \approx 0.074$ . Comparing with the code in Example 6, for a fixed length, a code with a 48% higher transmission rate is built.

*Example 8.* Let  $n = 11$  and  $S = \mathbb{F}_{64}[u]/\langle u^2 \rangle$ . Then,

$$R = \mathbb{F}_4[v]/\langle v^6 \rangle, \quad \Theta : R^{11} \rightarrow S_{D_{46}}^{11} = \{A, C, G, T\}^{66},$$

$$\kappa = (\omega + 1)(v^3 + v^4 + v^5) \in R.$$

Let  $C = \langle \kappa(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10}) \rangle \subset R^{11}$ . The DNA code  $\Theta(C)$  of length  $l = 66$  is listed in Table 5.4. This code has size  $M = 64$  and Hamming distance  $d = 22$ . Here, code rate  $\mathcal{R} \approx 0.045$ , and relative minimum distance  $\delta \approx 0.318$ .

*Example 9.* Let  $n = 11$  and  $S = \mathbb{F}_{256}[u]/\langle u^2 \rangle$ . Then,

$$R = \mathbb{F}_4[v]/\langle v^8 \rangle, \quad \Theta : R^{11} \rightarrow S_{D_{48}}^{11} = \{A, C, G, T\}^{88},$$

$$\kappa = (\omega + 1)(v^4 + v^5 + v^6 + v^7) \in R.$$

Let  $C = \langle \kappa(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10}) \rangle \subset R^{11}$ . 80 codewords of the punctured DNA code  $\Theta_P(C)$  of length  $l = 66$ , obtained by omitting every second

**Table 5.4 : The DNA Code  $\Theta(C)$  of length 66.**

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AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA
AACC AAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAACCAA
AAGG AAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGG
AATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA
ACAACAACAACAACAACAACAACAACAACAACAACAACAACAACAACAACAACAACAACAACAACA
ACCCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCCCA
ACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCA
ACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCA
AGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGA
AGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGA
AGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGA
AGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGA
ATAATAATAATAATAATAATAATAATAATAATAATAATAATAATAATAATAATAATAATAATAATA
ATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTA
ATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTA
ATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTA
CAAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAACCAAAC
CACCACCACCACCACCACCACCACCACCACCACCACCACCACCACCACCACCACCACCACCACCACC
CAGGACAGGACAGGACAGGACAGGACAGGACAGGACAGGACAGGACAGGACAGGACAGGACAGGAC
CATTACCATTACCATTACCATTACCATTACCATTACCATTACCATTACCATTACCATTACCATTAC
CCAACCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCCCAACCC
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCGGCCGGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCC
CCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCC
CGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGCCGAAGC
CGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCC
CGGGCCGGGGCCGGGGCCGGGGCCGGGGCCGGGGCCGGGGCCGGGGCCGGGGCCGGGGCCGGGGCC
CGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGC
CTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATCCTAATC
CTCCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTC
CTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCT
CTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTTC
GAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAGGAAAAG
GACCAGGACCAGGACCAGGACCAGGACCAGGACCAGGACCAGGACCAGGACCAGGACCAGGACCAGG
GAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAG
GATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAG
GCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACGGCAACG
GCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGGCCCGG
CGGGCCGGCGGGCCGGCGGGCCGGCGGGCCGGCGGGCCGGCGGGCCGGCGGGCCGGCGGGCCGGCG
GCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCG
GGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGGGGAAGG
GGCCGGGCCGGGCCGGGCCGGGCCGGGCCGGGCCGGGCCGGGCCGGGCCGGGCCGGGCCGGGCCGG
GGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGG
GGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGG
GTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATGGTAATG
GTCCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTGGTCTG
GTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGT
GTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGG
TAAATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA AATTA
TACCATTACCATTACCATTACCATTACCATTACCATTACCATTACCATTACCATTACCATTACCATT
TAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATTAGGATT
TATTATTATTATTATTATTATTATTATTATTATTATTATTATTATTATTATTATTATTATTATTATT
TCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCAACTTCA
TCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTTCCCTT
TCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCTTCGGCT
TCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCT
TGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTTGAAGTT
TGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTTGCCGTT
TGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTTGGGGTT
TGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTT
TAAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTTAATTT
TCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTTCCCTTT
TGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTTGGTTTT
TTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTT

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and fourth letter of  $\Theta(C)$ , are listed in Table 5.5. This code has size  $M = 256$  and Hamming distance  $d = 11$ . The DNA code has code rate  $\mathcal{R} \approx 0.061$ , and relative minimum distance  $\delta \approx 0.151$ . This puncturing method allows us to get codes of desired length with larger number of codewords. As expected, the code has 35% higher code rate than the code in Example 8.

*Example 10.* Let  $n = 15$  and  $S = \mathbb{F}_4[u]/\langle u^5 \rangle$ . Then,

$$\begin{aligned} R &= \mathbb{F}_4[v]/\langle v^5 \rangle, & \Theta : R^{15} &\rightarrow S_{D_{1024}}^{15} = \{A, C, G, T\}^{75}, \\ \kappa &= (\omega + 1)(v^2 + v^3 + v^4) \in R. \end{aligned}$$

Let  $C = \langle \kappa(1 + x + \dots + x^{14}) \rangle \subset R^{15}$ . The punctured DNA code  $\Theta_P(C)$  of length 45, obtained by omitting every first two letters of five in  $\Theta(C)$ , is given in Table 5.6. This code has size  $M = 64$  and Hamming distance  $d = 15$ . The code has the same length and distance with the code presented in Example 2 of [21]. However, here it was possible to get a code with larger size, hence 50% greater code rate, thanks to the variability of  $t$  and  $k$  values in the rings.



**Table 5.6 : DNA Code  $\Theta_P(C)$  of length 45.**

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AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA
ACAACAACAACAACAACAACAACAACAACAACAACA
AGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGA
ATAATAATAATAATAATAATAATAATAATAATAATA
AACAAACAACAACAACAACAACAACAACAACAACA
ACCACCACCACCACCACCACCACCACCACCACCACC
AGCAGCAGCAGCAGCAGCAGCAGCAGCAGCAGCAGC
ATCATCATCATCATCATCATCATCATCATCATCATC
AAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAG
ACGACGACGACGACGACGACGACGACGACGACGACG
AGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGG
ATGATGATGATGATGATGATGATGATGATGATGATG
AATAATAATAATAATAATAATAATAATAATAATAAT
ACTACTACTACTACTACTACTACTACTACTACTACT
AGTAGTAGTAGTAGTAGTAGTAGTAGTAGTAGTAGT
ATTATTATTATTATTATTATTATTATTATTATTATT
CAACAACAACAACAACAACAACAACAACAACAACA
CCACCACCACCACCACCACCACCACCACCACCACCA
CGACGACGACGACGACGACGACGACGACGACGACGA
CTACTACTACTACTACTACTACTACTACTACTACTA
CACCACCACCACCACCACCACCACCACCACCACCAC
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CGCCGCGCCGCGCCGCGCCGCGCCGCGCCGCGCCG
CTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCT
CAGCAGCAGCAGCAGCAGCAGCAGCAGCAGCAGCAG
CCGCGCGCCGCGCCGCGCCGCGCCGCGCCGCGCCG
CGGCGGCGGCGGCGGCGGCGGCGGCGGCGGCGGCG
CTGCTGCTGCTGCTGCTGCTGCTGCTGCTGCTGCTG
CATCATCATCATCATCATCATCATCATCATCATCAT
CCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCT
CGTCGTCGTCGTCGTCGTCGTCGTCGTCGTCGTCG
CTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTT
GAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGAAGA
GCAGCAGCAGCAGCAGCAGCAGCAGCAGCAGCAGCA
GGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGA
GTAGTAGTAGTAGTAGTAGTAGTAGTAGTAGTAGTA
GACGACGACGACGACGACGACGACGACGACGACGAC
GCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCC
GGCGGCGGCGGCGGCGGCGGCGGCGGCGGCGGCGG
GTCGTGCTGCTGCTGCTGCTGCTGCTGCTGCTGCTG
GAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAGGAG
GCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCCGCC
GGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGG
GTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGT
GATGATGATGATGATGATGATGATGATGATGATGAT
GCTGCTGCTGCTGCTGCTGCTGCTGCTGCTGCTGCT
GGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGT
GTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTT
TAATAATAATAATAATAATAATAATAATAATAATA
TCATCATCATCATCATCATCATCATCATCATCATCA
TGATGATGATGATGATGATGATGATGATGATGATGA
TTATTATTATTATTATTATTATTATTATTATTATTA
TACTACTACTACTACTACTACTACTACTACTACTACT
TCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCCTCC
TGCTGCTGCTGCTGCTGCTGCTGCTGCTGCTGCTG
TTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTT
TAGTAGTAGTAGTAGTAGTAGTAGTAGTAGTAGTAG
TCGTCGTCGTCGTCGTCGTCGTCGTCGTCGTCGTCG
TGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGGTGG
TTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTG
TATTATTATTATTATTATTATTATTATTATTATTAT
TCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCTTCT
TGTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTTGTT
TTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTT

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## 6. CONCLUSIONS

In this thesis, construction of reversible complement DNA codes over a family of chain rings is presented, and further, reversibility conditions for cyclic DNA codes over arbitrary commutative rings are included. Both by making use of the architecture of algebraic structures and keeping the relevant variables as parameters, the work establishes the possibility of obtaining DNA codes of customized length, with better code rates and relative minimum distances, resulting in higher transmission rates and error-correcting capabilities.

Areas of further research might include reversibility of DNA codes derived from linear codes, construction of fixed GC content DNA codes with algebraic approach and DNA codes of even length over arbitrary rings. Moreover, primer construction for DNA coding, secondary structure free DNA codes and codes that avoid polymer runs are possible directions for later studies.



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