DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NUMERICAL SOLUTIONS OF LINEAR AND NONLINEAR EIGENVALUE PROBLEMS USING RESIDUAL METHOD

by
Ayşe BELER

December, 2023
İZMİR

NUMERICAL SOLUTIONS OF LINEAR AND NONLINEAR EIGENVALUE PROBLEMS USING RESIDUAL METHOD

A Thesis Submitted to the
Graduate School of Natural And Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

by
Ayşe BELER

December, 2023
İZMİR

Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled **“NUMERICAL SOLUTIONS OF LINEAR AND NONLINEAR EIGENVALUE PROBLEMS USING RESIDUAL METHOD”** completed by **AYŞE BELER** under supervision of **ASSOC.PROF.DR. MELTEM ADIYAMAN** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Assoc.Prof.Dr. Meltem ADIYAMAN

Supervisor

Prof. Dr. Çetin DİŞİBÜYÜK

Thesis Committee Member

Prof. Dr. Neslihan DEMİREL

Thesis Committee Member

Assoc.Prof.Dr. Sevin GÜMGÜM

Examining Committee Member

Prof. Dr. Ali KONURALP

Examining Committee Member

Prof. Dr. Okan FISTIKOĞLU

Director

Graduate School of Natural and Applied Science

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor Assoc. Prof. Dr. Meltem Adıyaman for her continual presence, invaluable guidance, encouragement, generously providing knowledge, help and never ending patience during the course of this research.

I am also grateful to Dr. Volkan Oger for his patience and technical supports. I would also like to thank to Prof. Dr. Gözde Yazgı Tütüncü who always supports me and gives me happiness with her valuable suggestions. Special thanks to all members of İzmir University of Economics Department of Mathematics and to all members of Dokuz Eylül University Department of Mathematics.

I was supported by a grant BİDEB-2211 from The Scientific and Technological Research Council of Turkey (TÜBİTAK) during my Ph.D. research. So I would like to express my gratitude to TUBİTAK for its financial support.

Finally, I am grateful to my family for their endless patience, understanding and confidence to me throughout my life.

Ayşe BELER

NUMERICAL SOLUTIONS OF LINEAR AND NONLINEAR EIGENVALUE PROBLEMS USING RESIDUAL METHOD

ABSTRACT

The aim of this thesis is to find numerical solutions and approximate eigenvalues of linear and nonlinear eigenvalue problems by applying improved residual method. The residual method is based on the construction of the approximate solution by using the Bézier curves. In this thesis, at first, the residual method, which was developed for initial value problems, is improved to find the unknown coefficients of approximate solution explicitly without the need for any system solution. Most significant advantage of the method is finding approximate solutions of nonlinear problems without any linearization or solving any system of equations. Later, the adaptation of improved residual method is given to find approximate eigenvalues of regular, singular Sturm-Liouville and nonlinear eigenvalue problems, such as Euler eigenvalue problem, Paine problem, Laplace tidal wave equation, Dunsch equation, Boyd equation and Bratu problem. Error analysis for regular Sturm-Liouville eigenvalue problems is demonstrated for a special case. For singular problems, according to the location of singularity, different strategies are presented to find the approximate eigenvalues. Comparisons of the obtained numerical results with the theoretical findings and numerical results obtained by using various methods in literature are shown in graphs and tables. Observed orders are demonstrated for regular Sturm-Liouville and Bratu eigenvalue problems in tables, which show that observations are well confirm with theoretical ones. Comparisons and theoretical observations show that the improved and adapted method is very convenient and successful in solving linear and nonlinear eigenvalue problems and finding high index eigenvalues approximately with high accuracy.

Keywords: Bézier curves, Bernstein polynomials, nonlinear eigenvalue problems, regular Sturm Liouville eigenvalue problems, singular Sturm Liouville eigenvalue problems, Euler eigenvalue problem, Paine problem, Laplace tidal wave equation, Dunsch equation, Boyd equation, Bratu problem.

KALINTI METODU KULLANIMI İLE DOĞRUSAL VE DOĞRUSAL OLMAYAN ÖZDEĞER PROBLEMLERİNİN SAYISAL ÇÖZÜMLERİ

ÖZ

Bu tezin amacı, geliştirilmiş kalıntı yöntemini uygulayarak doğrusal ve doğrusal olmayan özdeğer problemlerinin sayısal çözümlerini ve yaklaşık özdeğerlerini bulmaktır. Kalıntı yöntemi, Bézier eğrileri kullanılarak yaklaşık çözümün oluşturulmasına dayanmaktadır. Bu tezde, öncelikle başlangıç değer problemleri için geliştirilen kalıntı yöntemi, herhangi bir sistem çözümüne ihtiyaç duymadan, yaklaşık çözümün bilinmeyen katsayılarını açık bir şekilde bulmak için geliştirildi. Yöntemin en önemli avantajı, herhangi bir doğrusallaştırmaya veya denklem sistemini çözmeye gerek kalmadan doğrusal olmayan problemlerin yaklaşık çözümlerini bulabilmesidir. Daha sonra, geliştirilmiş kalıntı yönteminin uyarlaması Euler özdeğer problemi, Paine problemi, Laplace gelgit dalgası denklemi, Dunsch denklemi, Boyd denklemi ve Bratu problemi gibi düzenli, tekil Sturm-Liouville ve doğrusal olmayan özdeğer problemlerinin yaklaşık özdeğerlerini bulmak için verilmiştir. Düzenli Sturm-Liouville özdeğer problemleri için hata analizi özel bir durum için gösterilmiştir. Tekil problemler için, yaklaşık özdeğerleri bulmak amacıyla tekilliğin konumuna göre farklı stratejiler sunulmaktadır. Elde edilen sayısal sonuçların teorik bulgular ve literatürdeki çeşitli yöntemler kullanılarak elde edilen sayısal sonuçlarla karşılaştırılması grafik ve tablolarla gösterilmiştir. Düzenli Sturm-Liouville ve Bratu özdeğer problemleri için gözlemlenen hata mertebeleri tablolarda örneklendirilmiştir, bu da çıkarımların teorik olanlarla örtüştüğünü göstermektedir. Karşılaştırmalar ve teorik gözlemler, geliştirilen ve uyarlanan yöntemin, doğrusal ve doğrusal olmayan özdeğer problemlerini çözmede ve yüksek indeksli özdeğerleri yaklaşık olarak yüksek doğrulukla bulmada çok uygun ve başarılı olduğunu göstermektedir.

Anahtar kelimeler: Bézier eğrileri, Bernstein polinomları, doğrusal olmayan özdeğer problemleri, düzgün Sturm Liouville özdeğer problemleri, tekil Sturm Liouville özdeğer problemleri, Euler özdeğer problemi, Paine problemi, Laplace gelgit dalgası denklemi, Dunsch denklemi, Boyd denklemi, Bratu problemi.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

Page

LIST OF SYMBOLS

c_j^i	unknown control points
n	degree of Bézier curves
$\triangle$	difference operator
S_i	equally spaced subintervals
$R_i(x)$	Residual function
$B_j^n(x;[a_{i-1},a_i])$	Bernstein polynomials
N	number of intervals
h	stepsize
$y(x)$	exact solution
$u(x)$	approximate solution
$ord(h)$	order of the method
$M_{k,\beta}(z)$	Whittaker function
λ	eigenvalue
$\wedge_k$	exact eigenvalues
λ_k	approximate eigenvalues obtained by the improved residual method
$\lambda_{k,n}$	approximate eigenvalues obtained by the symmetrical weighted sequential splitting method
$\lambda_{k,n}^{(f)}$	approximate eigenvalues obtained by the finite difference method
λ_k^*	approximate eigenvalues obtained by the trigonometric polynomial
λ_k°	approximate eigenvalues obtained by Numerov method
$\lambda_k^\bullet$	approximate eigenvalues obtained by corrected Numerov method
$\lambda_k^{\sim}$	approximate eigenvalues obtained by the regularized sampling method
$\lambda_{\overline{k}}$	approximate eigenvalues obtained by the half interval method

$\lambda_k^\star$ approximate eigenvalues obtained by Chebyshev spectral collocation method

λ_c critical eigenvalue of Bratu problem

CHAPTER ONE
INTRODUCTION

Eigenvalue problems are one of the fundamental problems in mathematics and mathematical physics. These type of problems arise in many areas such as computational science and engineering, including various fields such as acoustics, control theory, fluid mechanics, quantum mechanics and structural engineering. In such problems, analytical solutions are generally not available, so these lead us to develop numerical methods. Thus, numerous numerical methods have been developed to solve these type of problems approximately. These techniques are generally applied to singular and nonlinear eigenvalue problems, since solving these problems analytically have several difficulties arising from singularity and nonlinearity.

In this thesis, we consider the numerical computation of eigenvalues and eigenfunctions of regular Sturm-Liouville eigenvalue problems, singular Sturm-Liouville eigenvalue problems and nonlinear eigenvalue problems.

Consider the regular Sturm Liouville eigenvalue problems

$$
\begin{aligned}
&-y''(x) + r(x)y(x) = \lambda y(x), \quad 0 \leqslant x_0 < x < x_N, \\
&y(x_0) = 0, \quad y(x_N) = 0,
\end{aligned} \tag{1.1}
$$

where $r(x) \in C^{n-2}[x_0, x_N]$. These eigenvalue problems have an importance in applied mathematics. Many biological, chemical and physical processes are described by using models based on the Sturm-Liouville problems. These problems are named as Sturm-Liouville eigenvalue problems after Swiss mathematician Jacques Sturm (1803-1855) and French mathematician Joseph Liouville (1809-1882), who studied these problems and the properties of their solutions. For more detailed information about the Sturm-Liouville theory, we refer the reader to check Atkinson (1964), Pryce (1993), Zaitsev & Polyanin (1995) and Agarwal & O'Regan (2008).

After that, we handle singular eigenvalue problems which are of the form

$$y''(x) + p(x)y'(x) + (w(x)\lambda - q(x))y(x) = 0$$
$$y(a) = 0, \quad y(b) = 0, \tag{1.2}$$

where $p(x)$ and $w(x)$ are nonzero functions with same sign on the interval (a, b). These types of problems are singular (Bailey et al. (1991)), if (a, b) is an infinite interval, or at least one of p, q, w is not integrable in $[a, b]$.

In literature, so many techniques have been presented by many researchers to find the approximate solutions of singular Sturm-Liouville eigenvalue problems. These methods can be listed as: the Haar wavelet method Bujurke et al. (2008), the shooting-type algorithms Ledoux et al. (2009), the modified Neumann schemes Ledoux & Van Daele (2010), the Chebyshev polynomial expansions method Uğur (2015), the variable step finite difference schemes Amodio & Settanni (2015), A Neumann series of Bessel functions representation Kravchenko & Torba (2018), the Sinc Galerkin method and the variational iteration method (VIM) Al-Khaled & Hazaimeh (2020), the spectral matrix method Magherini (2020), a new modification of finite difference method (FDM) Mukhtarov et al. (2021), the spectral collocation technique Gheorghiu (2021), the variational estimation method Cipu & Barbu (2022), the boundary shape function method Chang et al. (2022), the half-interval method Liu (2022) and spectrum completion technique Kravchenko (2023). In addition to the mathematical methods developed for solving Sturm-Liouville eigenvalue problems numerically, various mathematical codes are available in literature. Among them, we mention SLEIGN Bailey et al. (1978), NAG Pryce (1993), SLEDGE Pruess & Fulton (1993), SLEIGN2 Bailey et al. (2001) and mathlab package Ledoux et al. (2005).

Afterwards, we discuss the one-dimensional "Bratu problem" which is a nonlinear eigenvalue problem and is defined as follows

$$y'' + \lambda e^{y} = 0, \quad 0 < x < 1,$$
$$y(0) = 0 \quad \text{and} \quad y(1) = 0. \tag{1.3}$$

This nonlinear eigenvalue problem has known two-bifurcated solutions for $\lambda < \lambda_c$ and only one solution for $\lambda = \lambda_c$, but no solution for $\lambda > \lambda_c$. The two dimensional Bratu problem

$$\Delta u + \lambda e^{u} = 0, \ \text{ on } \ \Omega : \{(x, y) \in 0 \leq x \leq 1, \quad 0 \leq y \leq 1\},$$
$$u = 0 \quad \text{on} \quad \partial\Omega \tag{1.4}$$

is a nonlinear elliptical partial differential equation and appears in various fields such as the fuel ignition model found in thermal combustion theory Frank-Kamenetski (1995), the Chandrasekhar model of the expansion of the universe Chandrasekhar (1957), chemical reactor theory and nanotechnology. The problem arises via the study of solid fuel ignition model

$$v_t = \Delta v + \lambda(1 - \epsilon v)^{m} e^{v/(1+\epsilon v)}$$
$$v = 0, \quad x \in \partial\Omega \tag{1.5}$$

where λ is a Frank-Kamanetskii parameter, v is a dimensionless temperature and $1/\epsilon$ is the activation energy. Nontrivial solutions of Bratu problem (1.4) arise as steady-state solutions of solid fuel ignition model within the approximation $\epsilon \leq 1$. A brief history and the importance of Bratu problem is given in Jacobsen & Schmitt (2002) by Jacobsen and Schmitt. The problem is also a nonlinear eigenvalue problem, that is, often used as a comparison tool for numerical methods owing to the bifurcation nature of the solution for $\lambda < \lambda_c$. Many techniques have been performed by researchers to find the approximate solutions of this problem such as: the Taylor's decomposition method Adiyaman & Somali (2010), the Mexican Hat wavelet method Masood et al.

(2017), the iterative differential quadrature method Ragb et al. (2017), the quartic B-spline method Roul & Thula (2019), the special case of Hermite interpolation technique Karamollahi et al. (2021), the quintic B-spline collocation method Lodhi et al. (2022) and the collocation method based on Genocchi polynomials Ghomanjani (2022). These approaches have all been adapted independently to solve the Bratu model numerically.

In order to find approximate solutions of the problems listed above, first we improve the residual method, which is given in Adiyaman & Oger (2017) and based on the construction of the approximate solution of an initial value problem using Bézier curves. Properties of Bézier curves may be found in Farin & Hansford (2000). In Adiyaman & Oger (2017), the unknown control points are calculated from a lower triangular system, which is encountered by minimizing the Taylor's series expansion of the residual function at certain points with non-zero diagonals. However, in the current study, it is aimed to calculate the control points without the need for employing any system solution. For this purpose, an explicit formula for unknown control points is obtained. After that, the improved method is adapted to boundary value problems and used to find approximate eigenvalues and eigenfunctions of the problems. Unlike other weighted residual methods, which mostly have nonlinear system of equations to find the unknown constants, the proposed method provides finding unknown constants or unknown control points explicitly in terms of the first or the last two control points. Hence, approximate results of nonlinear problems can be found with no need for any linearization or solving any system of equations in use. This is the most significant advantage and novelty of the proposed method. Another advantage is that the control of the error depends on two parameters. The error can be decreased by increasing the order, decreasing the stepsize or both. Because of these advantages, we recommend the proposed method for nonlinear initial, boundary or eigenvalue problems. The method also allow us to obtain approximate eigenvalues from the roots of a polynomial. However, in this respect, the method has a weakness; the greater the number of eigenvalues to be approximated, the higher the degree of our polynomial; therefore, the higher the degree of Bernstein polynomials.

The outline of this thesis is as follows:

In Chapter 2, improvements of the residual method for second order non-linear initial value problems are exhibited. In Chapter 3, the improved residual method is adapted to regular Sturm-Liouville eigenvalue problems. Furthermore, the demonstration of the convergency of the method for $r(x) = 0$ is mentioned. Also, obtained numerical results are compared with some other methods in literature in this chapter. Afterwards, in Chapter 4, the improved residual method is implemented to several significant singular eigenvalue problems, such as Boyd equation, the Laplace tidal wave equation and Dunsch equation. In Chapter 5, the improved method is applied to find approximate solutions of the Bratu's problem which is a nonlinear eigenvalue problem. In the conclusion chapter, a brief summary of our work and suggestions for future works are presented. Considering the appendix sections; in A.1, definition and properties of Bézier curves are demonstrated. The construction of the method to the initial value problems is presented in A.2. Error analysis of the method for initial value problems is mentioned in A.3. The adaptation of the residual method to the boundary value problems is given in A 4. Error analysis of the method is accentuated in A 5. In A 6, some important lemmas which used to prove some theorems in the thesis are displayed.

CHAPTER TWO
IMPROVEMENT OF RESIDUAL METHOD FOR SECOND ORDER NONLINEAR INITIAL VALUE PROBLEMS

2.1 Residual Method

In this section, we give the improvement of the residual method for second order initial value problems

$$y'' = F(x, y, y') \tag{2.1}$$

with the initial conditions

$$y(a) = \alpha, \quad y'(a) = \beta, \tag{2.2}$$

where $F \in C^{n-2}[a,b] \times C(D_1) \times C(D_2)$, D_1 and D_2 are the closed intervals in $\mathbb{R}$, α and β are finite constants and n is an integer.

The method is firstly introduced in Adiyaman & Oger (2017). In the residual method, the approximate solutions are constructed using Bézier curves

$$u_i(x) = \sum_{j=0}^{n} c_j^i B_j^n\left(x; [a_{i-1}, a_i]\right) \tag{2.3}$$

$x \in S_i$, where $S_i = [a_{i-1}, a_i]$, $a_i = a + ih$, $i = 1, 2, ..., N$, $h = (b-a)/N$, N is a positive integer, n is the degree of Bézier curve and $B_j^n\left(x; [a_{i-1}, a_i]\right)$ are the Bernstein polynomials, such that

$$B_j^n\left(x; [a_{i-1}, a_i]\right) = \binom{n}{j} \frac{1}{h^n} (x - a_{i-1})^j (a_i - x)^{n-j}.$$

First two control points are obtained using initial conditions (2.2). Hence $n-1$ control points has to be determined. Unknown control points c_j^i are obtained by minimizing Taylor series of residual fuction at certain points. This minimization yields the following linear equations for $k = 0, 1, \ldots, n-2$

$$\frac{n(n-1)\ldots(n-k-1)}{h^{k+2}} \Delta^{k+2} c_0^i - F^{(k)}\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0, \tag{2.4}$$

where

$$F^{(k)}(x,y,z) = \frac{d^k}{dx^k} F(x, y(x), z(x))$$

and

$$\Delta^{k+2} c_0^i = \sum_{j=0}^{k+2} \binom{k+2}{j} (-1)^{k+2-j}\, c_j^i. \tag{2.5}$$

Linear system of equations can be written $(n-1) \times (n-1)$ dimensional lower triangular system with non zero diagonals to find the unknown control points. Detailed information of the method is given in Appendix A.2.

2.2 Improvement of Residual Method

In this work, we formulate the control points c_k^i explicitly in terms of c_0^i and c_1^i. To do this, at first, we rewrite the equation (2.4) as follows,

$$\triangle^{k+2} c_0^i = \frac{(n-(k+2))!}{n!} h^{k+2} F^{(k)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big). \tag{2.6}$$

Using equation (2.5), left hand side of the above equation can be written as

$$\triangle^{k+2} c_0^i = \sum_{j=0}^{k+2} \binom{k+2}{j} (-1)^{k+2-j} c_j^i$$

$$= \sum_{j=0}^{k+1} \binom{k+2}{j} (-1)^{k+2-j} c_j^i + \binom{k+2}{k+2} c_{k+2}^i.$$

Thus we have,

$$\triangle^{k+2} c_0^i = \sum_{j=0}^{k+1} \binom{k+2}{j} (-1)^{k+2-j} c_j^i + c_{k+2}^i. \tag{2.7}$$

If we combine (2.6) and (2.7), we get

$$\sum_{j=0}^{k+1} \binom{k+2}{j} (-1)^{k+2-j} c_j^i + c_{k+2}^i = \frac{(n-(k+2))!}{n!} h^{k+2} F^{(k)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big),$$

which gives

$$c^i_{k+2} = \frac{(n-(k+2))!}{n!} h^{k+2} F^{(k)}\big(a_{i-1}, c^i_0, \frac{n}{h}(c^i_1 - c^i_0)\big) - \sum_{j=0}^{k+1} \binom{k+2}{j} (-1)^{k+2-j} c^i_j,$$

or

$$c^i_k = \frac{(n-k)!}{n!} h^k F^{(k-2)}\big(a_{i-1}, c^i_0, \frac{n}{h}(c^i_1 - c^i_0)\big) - \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k-j} c^i_j. \tag{2.8}$$

By using equation (2.8), we can formulate the control points c^i_k explicitly in terms of c^i_0 and c^i_1. To show this, we need to prove some Binomial identities.

Proposition 2.2.1. *For any integer* $k \geqslant 1$,

$$\sum_{j=1}^{k} \binom{k}{j-1} (-1)^{k-j} = 1. \tag{2.9}$$

Proof. Another proof of this identity is given in Simşek (2018). We prove this equation using Binomial theorem. The following equation

$$(1+(-1))^k = 0,$$

yields

$$(1+(-1))^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 1^j$$

$$= \sum_{j=1}^{k+1} \binom{k}{j-1} (-1)^{k+1-j}.$$

If we expand the $(k+1)^{th}$ term, we get

$$\sum_{j=1}^{k} \binom{k}{j-1} (-1)^{k+1-j} + \binom{k}{k} = 0,$$

which implies that

$$\sum_{j=1}^{k} \binom{k}{j-1} (-1)^{k-j} = 1.$$

□

Lemma 2.2.2. *For any integer k and $m = 0, \ldots, k$,*

$$\sum_{j=m}^{k} \binom{k+1}{j} \binom{j}{m} (-1)^{k-j} = \binom{k+1}{m}. \tag{2.10}$$

Proof. For any integer k and $m = 0, \ldots, k$,

$$\binom{k+1}{j} \binom{j}{m} = \frac{(k+1)!}{j!(k+1-j)!} \frac{j!}{(j-m)!m!}.$$

Multiplying the right hand side of the last equation with $\dfrac{(k+1-m)!}{(k+1-m)!}$ gives that

$$\binom{k+1}{j} \binom{j}{m} = \frac{(k+1)!(k+1-m)!}{(k+1-j)!(k+1-m)!(j-m)!m!}$$

$$= \binom{k+1}{m} \frac{(k+1-m)!}{(k+1-j)!(j-m)!}.$$

Thus we have

$$\sum_{j=m}^{k} \binom{k+1}{j} \binom{j}{m} (-1)^{k-j} = \binom{k+1}{m} \sum_{j=m}^{k} \frac{(k+1-m)!}{(k+1-j)!(j-m)!} (-1)^{k-j}.$$

Shifting j from m to 1 yields

$$\sum_{j=m}^{k} \binom{k+1}{j} \binom{j}{m} (-1)^{k-j} = \binom{k+1}{m} \sum_{j=1}^{k+1-m} \frac{(k+1-m)!(-1)^{k+1-m-j}}{(k+2-j-m)!(j-1)!}$$

$$= \binom{k+1}{m} \sum_{j=1}^{k+1-m} \binom{k+1-m}{j-1} (-1)^{k+1-m-j}. \tag{2.11}$$

Applying Proposition (2.2.1), we get

$$\sum_{j=1}^{k+1-m} \binom{k+1-m}{j-1} (-1)^{k+1-m-j} = 1. \tag{2.12}$$

Hence, we get the desired result by substituting equation (2.12) in eqaution (2.11) as follows

$$\sum_{j=m}^{k} \binom{k+1}{j} \binom{j}{m} (-1)^{k-j} = \binom{k+1}{m}.$$

□

Theorem 2.2.3. *Let n be the degree and c_k^i be the control points of the Bézier curves* (2.3)*, that are the approximate solutions of* (2.1) *and* (2.2) *in interval S_i, and $F(x, y, z)$ be the function given in* (2.1)*, then for $k = 0, 1, ..., n$ and $i = 1, 2, \dots, N$,*

$$c_k^i = kc_1^i - (k-1)c_0^i + \sum_{j=2}^{k} \binom{k}{j} \frac{(n-j)!}{n!} h^j F^{(j-2)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big), \tag{2.13}$$

where $F^{(k)}(x, y(x), z(x)) = \dfrac{d^k}{dx^k} F(x, y(x), z(x))$ and h is the stepsize.

Proof. We prove the theorem by induction on k. It is obvious that (2.13) is true for $k = 0$ and $k = 1$. For $k = 2$, from equation (2.8) we have

$$c_2^i = 2c_1^i - c_0^i + \frac{(n-2)!}{n!} h^2 F\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big)$$

$$= 2c_1^i - c_0^i + \sum_{j=2}^{2} \binom{2}{j} \frac{(n-j)!}{n!} h^j F^{(j-2)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big).$$

Thus, the statement holds for $k = 2$. Suppose that the equation holds for k. Then we need to prove that the statement is true for $k + 1$. By using (2.8), we have

$$c_{k+1}^i = \frac{(n-(k+1))!}{n!} h^{k+1} F^{(k-1)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) - \sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} c_j^i. \tag{2.14}$$

Since we assume that (2.13) is true for $j = 0, 1, ..., k$, we obtain

$$c_{k+1}^i = \frac{(n-(k+1))!}{n!} h^{k+1} F^{(k-1)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big)$$
$$- \sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} \Bigg(j c_1^i - (j-1) c_0^i$$
$$+ \sum_{m=2}^{j} \binom{j}{m} \frac{(n-m)!}{n!} h^m F^{(m-2)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \Bigg)$$

and this implies that

$$c_{k+1}^i = \frac{(n-(k+1))!}{n!} h^{k+1} F^{(k-1)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big)$$
$$- \sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} j c_1^i + \sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} (j-1) c_0^i$$
$$- \sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} \Bigg(\sum_{m=2}^{j} \binom{j}{m} \frac{(n-m)!}{n!} h^m F^{(m-2)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \Bigg). \tag{2.15}$$

Now, we investigate each summation in (2.15) one by one. Since

$$\binom{k+1}{j} j = \frac{(k+1)!}{j!(k+1-j)!} j$$

$$= \frac{(k+1)k!}{j(j-1)!(k-(j-1))!} j$$

$$= (k+1)\binom{k}{j-1},$$

first summation in (2.15) becomes

$$\sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} j c_1^i = c_1^i (k+1) \sum_{j=0}^{k} \binom{k}{j-1} (-1)^{k+1-j}.$$

Since the summation is 0 for $j = 0$, we have

$$\sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} j c_1^i = c_1^i (k+1) \sum_{j=1}^{k} \binom{k}{j-1} (-1)^{k+1-j}. \tag{2.16}$$

By using the result of Proposition (2.2.1) in equation (2.16), we have

$$c_1^i (k+1) \sum_{j=1}^{k} \binom{k}{j-1} (-1)^{k+1-j} = -c_1^i (k+1).$$

Hence we obtain

$$\sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} j c_1^i = -c_1^i (k+1). \tag{2.17}$$

We can compute the second summation of equation (2.15) as follows

$$\begin{aligned} \sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} (j-1) c_0^i = \sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} j c_0^i \\ - \sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} c_0^i. \end{aligned} \tag{2.18}$$

Using (2.9) and (2.17), we get

$$\begin{aligned} \sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} (j-1) c_0^i &= -(k+1) c_0^i + c_0^i \\ &= -k c_0^i. \end{aligned} \tag{2.19}$$

Finally, we deal with the last summation

$$\sum_{j=0}^{k} \binom{k+1}{j} (-1)^{k+1-j} \left(\sum_{m=2}^{j} \binom{j}{m} \frac{(n-m)!}{n!} h^m F^{(m-2)} \big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \right)$$

of equation (2.15). Since above summation is 0 for $j = 0$ and $j = 1$, it starts from

$j = 2$. Let's use the notation Ψ to represent the summation

$$\Psi = \sum_{j=2}^{k} \binom{k+1}{j} (-1)^{k+1-j} \Big(\sum_{m=2}^{j} \binom{j}{m} \frac{(n-m)!}{n!} h^m F^{(m-2)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \Big).$$

First we expand the terms of Ψ as

$$\begin{aligned}
\Psi = & \binom{k+1}{2} (-1)^{k-1} \Big(\binom{2}{2} \frac{(n-2)!}{n!} h^2 F^{(0)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \Big) \\
& + \binom{k+1}{3} (-1)^{k-2} \Big(\binom{3}{2} \frac{(n-2)!}{n!} h^2 F^{(0)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \\
& + \binom{3}{3} \frac{(n-3)!}{n!} h^3 F^{(1)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \Big) \\
& + \cdots + \binom{k+1}{k} (-1)^{1} \Big(\binom{k}{2} \frac{(n-2)!}{n!} h^2 F^{(0)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \\
& + \cdots + \binom{k}{k} \frac{(n-k)!}{n!} h^k F^{(k-2)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \Big).
\end{aligned}$$

Rearranging the terms of Ψ gives that

$$\begin{aligned}
\Psi = & \frac{(n-2)!}{n!} h^2 F^{(0)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \Big(\binom{k+1}{2} \binom{2}{2} (-1)^{k-1} \\
& + \binom{k+1}{3} \binom{3}{2} (-1)^{k-2} + \cdots + \binom{k+1}{k} \binom{k}{2} (-1) \Big) \\
& + \frac{(n-3)!}{n!} h^3 F^{(1)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \Big(\binom{k+1}{3} \binom{3}{3} (-1)^{k-2} \\
& + \binom{k+1}{4} \binom{4}{3} (-1)^{k-3} + \cdots + \binom{k+1}{k} \binom{k}{3} (-1) \Big) \\
& + \cdots + \frac{(n-k)!}{n!} h^k F^{(k-2)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \Big(\binom{k+1}{k} \binom{k}{k} (-1) \Big)
\end{aligned}$$

$$\begin{aligned}
= & \frac{(n-2)!}{n!} h^2 F^{(0)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \sum_{j=2}^{k} \binom{k+1}{j} \binom{j}{2} (-1)^{k+1-j} \\
& + \frac{(n-3)!}{n!} h^3 F^{(1)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \sum_{j=3}^{k} \binom{k+1}{j} \binom{j}{3} (-1)^{k+1-j} \\
& + \cdots + \frac{(n-k)!}{n!} h^k F^{(k-2)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big) \sum_{j=k}^{k} \binom{k+1}{j} \binom{j}{k} (-1)^{k+1-j}.
\end{aligned}$$

By Lemma (2.2.2), we have

$$\begin{aligned}\Psi = &-\frac{(n-2)!}{n!}h^2F^{(0)}\big(a_{i-1},c_0^i,\frac{n}{h}(c_1^i-c_0^i)\big)\binom{k+1}{2}\\ &-\frac{(n-3)!}{n!}h^3F^{(1)}\big(a_{i-1},c_0^i,\frac{n}{h}(c_1^i-c_0^i)\big)\binom{k+1}{3}\\ &-\cdots-\frac{(n-k)!}{n!}h^kF^{(k-2)}\binom{k+1}{k}.\end{aligned}$$

Thus, we deduce that

$$\Psi = -\sum_{j=2}^{k}\binom{k+1}{j}\frac{(n-j)!}{n!}h^jF^{(j-2)}\big(a_{i-1},c_0^i,\frac{n}{h}(c_1^i-c_0^i)\big). \tag{2.20}$$

Substituting (2.17), (2.19) and (2.20) into (2.15), we get

$$\begin{aligned}c_{k+1}^i = &\frac{(n-(k+1))!}{n!}h^{k+1}F^{(k-1)}\big(a_{i-1},c_0^i,\frac{n}{h}(c_1^i-c_0^i)\big)+(k+1)c_1^i-kc_0^i\\ &+\sum_{j=2}^{k}\binom{k+1}{j}\frac{(n-j)!}{n!}h^jF^{(j-2)}\big(a_{i-1},c_0^i,\frac{n}{h}(c_1^i-c_0^i)\big).\end{aligned} \tag{2.21}$$

Since

$$\frac{(n-(k+1))!}{n!}h^{k+1}F^{(k-1)}\big(a_{i-1},c_0^i,\frac{n}{h}(c_1^i-c_0^i)\big)$$

is $(k+1)^{th}$ term of

$$\sum_{j=2}^{k+1}\frac{(n-j)!}{n!}h^jF^{(j-2)}\big(a_{i-1},c_0^i,\frac{n}{h}(c_1^i-c_0^i)\big)\binom{k+1}{j},$$

we can add it to the summation. Thus, we get the desired result as follows

$$c_{k+1}^i = (k+1)c_1^i-kc_0^i+\sum_{j=2}^{k+1}\binom{k+1}{j}\frac{(n-j)!}{n!}h^jF^{(j-2)}\big(a_{i-1},c_0^i,\frac{n}{h}(c_1^i-c_0^i)\big).$$

□

Hence by the theorem an explicit formula for control points is obtained by using first two control points. By this way, we can compute all control points explicitly without the need for employing any system solution.

Similarly, we will try to represent the unknown control points by using the last two control points. To do this, we use the information that Bézier curves interpolate both initial and end points. At first, we equate the piecewise residual function to zero at $x = a_i$ as follows

$$R_i^{(k-2)}(a_i) = u_i^{(k)}(a_i) - F^{(k-2)}(a_i, u_i(a_i), u_i'(a_i)) = 0. \tag{2.22}$$

By using equation (A.3), we have

$$\frac{1}{h^k}\frac{n!}{(n-k)!} \triangle^k c_{n-k}^i - F^{(k-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)) = 0.$$

Thus

$$\triangle^k c_{n-k}^i = h^k \frac{(n-k)!}{n!} F^{(k-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)).$$

Since

$$\triangle^k c_{n-k}^i = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} c_{n-k+j}^i, \tag{2.23}$$

we may write

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} c_{n-k+j}^i = h^k \frac{(n-k)!}{n!} F^{(k-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)). \tag{2.24}$$

Expanding the summation for $j = 0$ of equation (2.24), we get

$$(-1)^k \binom{k}{0} c_{n-k}^i + \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} c_{n-k+j}^i = h^k \frac{(n-k)!}{n!} F^{(k-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)),$$

that is

$$(-1)^k c_{n-k}^i = h^k \frac{(n-k)!}{n!} F^{(k-2)} a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)) - \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} c_{n-k+j}^i.$$

Multiplying both sides of above equation with $(-1)^k$ gives that

$$c_{n-k}^i = (-1)^k h^k \frac{(n-k)!}{n!} F^{(k-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)) - \sum_{j=1}^{k} (-1)^{2k-j} \binom{k}{j} c_{n-k+j}^i.$$

Hence we have

$$c^i_{n-k} = (-1)^k h^k \frac{(n-k)!}{n!} F^{(k-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1})) + \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} c^i_{n-k+j} \tag{2.25}$$

or

$$c^i_k = (-1)^{n-k} h^{n-k} \frac{k!}{n!} F^{(n-k-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1})) + \sum_{j=1}^{n-k} (-1)^{j+1} \binom{n-k}{j} c^i_{k+j}. \tag{2.26}$$

To compute the control points c^i_k in terms of last two control points, we need to prove some binomial identities given below.

Lemma 2.2.4. *For any* $k \geq 2$*,*

$$\sum_{j=2}^{k} (-1)^j \binom{k-1}{j-1} = 1. \tag{2.27}$$

Proof. We will demonstrate the statement by using the following equation

$$\binom{k}{j} = \binom{k-1}{j} + \binom{k-1}{j-1},$$

which yields

$$\binom{k-1}{j-1} = \binom{k}{j} - \binom{k-1}{j}.$$

Thus, we have

$$\sum_{j=2}^{k} (-1)^j \binom{k-1}{j-1} = \sum_{j=2}^{k} (-1)^j \left(\binom{k}{j} - \binom{k-1}{j} \right)$$

$$= \sum_{j=2}^{k} (-1)^j \binom{k}{j} - \sum_{j=2}^{k} (-1)^j \binom{k-1}{j}. \tag{2.28}$$

We use binomial theorem to find the summations of the right hand side of (2.28). Since we have

$$0 = (1 + (-1))^k$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^j$$

$$= \binom{k}{0} + \binom{k}{1} + \sum_{j=2}^{k} (-1)^j \binom{k}{j}$$

$$= 0,$$

we obtain

$$\sum_{j=2}^{k} (-1)^j \binom{k}{j} = k - 1. \tag{2.29}$$

Similarly,

$$\sum_{j=2}^{k-1} (-1)^j \binom{k-1}{j} = k - 2. \tag{2.30}$$

We get the desired result by substituting equations (2.29) and (2.30) in equation (2.28) as

$$\sum_{j=2}^{k} (-1)^j \binom{k-1}{j-1} = k - 1 - k + 2 = 1.$$

□

Lemma 2.2.5. *For any integers $k \geq 2$ and $i = 1, 2, ..., k$,*

$$\sum_{j=2}^{k+2-i} (-1)^j \binom{k+1}{j-1} \binom{k+2-j}{i} = \binom{k+1}{i}. \tag{2.31}$$

Proof. We use the following equations to prove the lemma.

$$\binom{n}{k} = \binom{n}{n-k}, \tag{2.32}$$

and

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}. \tag{2.33}$$

By using (2.32), we have

$$\binom{k+1}{j-1} = \binom{k+1}{k+2-j}.$$

If we use the above result in the expression of the left hand side of (2.31), we get

$$\sum_{j=2}^{k+2-i} (-1)^j \binom{k+1}{j-1}\binom{k+2-j}{i} = \sum_{j=2}^{k+2-i} (-1)^j \binom{k+1}{k+2-j}\binom{k+2-j}{i}. \tag{2.34}$$

By using (2.33), we have

$$\binom{k+1}{k+2-j}\binom{k+2-j}{i} = \binom{k+1}{i}\binom{k+1-i}{k+2-j-i}.$$

If we substitute this result into (2.34), we get

$$\sum_{j=2}^{k+2-i} (-1)^j \binom{k+1}{j-1}\binom{k+2-j}{i} = \sum_{j=2}^{k+2-i} (-1)^j \binom{k+1}{i}\binom{k+1-i}{k+2-j-i}. \tag{2.35}$$

Again by using equation (2.32), we have

$$\binom{k+1-i}{k+2-j-i} = \binom{k+1-i}{j-1}$$

and if we substitute this result into (2.35), we obtain

$$\sum_{j=2}^{k+2-i} (-1)^j \binom{k+1}{j-1}\binom{k+2-j}{i} = \binom{k+1}{i}\sum_{j=2}^{k+2-i} (-1)^j \binom{k+1-i}{j-1}.$$

By using Lemma $(2.2.4)$, we have

$$\sum_{j=2}^{k+2-i} (-1)^j \binom{k+1-i}{j-1} = 1.$$

Hence we obtain the desired result as follows

$$\sum_{j=2}^{k+2-i} (-1)^j \binom{k+1}{j-1} \binom{k+2-j}{i} = \binom{k+1}{i}.$$

□

Theorem 2.2.6. *Let n be the degree and c_k^i be the control points of the Bézier curves* (2.3)*, that are the approximate solutions of* (2.1) *and* (2.2) *in interval S_i, and $F(x, y, z)$ be the function given in* (2.1)*, then for $k = 0, 1, ..., n$ and $i = 1, 2, \ldots, N$,*

$$\begin{aligned} c_k^i =& (n-k)c_{n-1}^i - (n-k-1)c_n^i + \sum_{j=2}^{n-k} (-1)^j h^j \binom{n-k}{j} \\ & \times \frac{(n-j)!}{n!} F^{(j-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)), \end{aligned} \tag{2.36}$$

where $F^{(k)}(x, y(x), z(x)) = \dfrac{d^k}{dx^k} F(x, y(x), z(x))$ and h is the stepsize.

Proof. We use induction on k from $k = n$ to $k = 0$ to prove the theorem. It is obvious that (2.36) is true for $k = n$ and $k = n - 1$. For $k = n - 2$, from equation (2.26) we have

$$c_{n-2}^i = h^2 \frac{(n-2)!}{n!} F^{(0)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)) + \sum_{j=1}^{2} (-1)^{j+1} \binom{2}{j} c_{n-2+j}^i$$

$$= h^2 \frac{(n-2)!}{n!} F^{(0)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)) + 2c_{n-1}^i - c_n^i,$$

which yields

$$c_{n-2}^i = 2c_{n-1}^i - c_n^i + \sum_{j=2}^{2} (-1)^j h^j \binom{2}{j} \frac{(n-j)!}{n!} F^{(j-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)).$$

So the statement holds for $k = n - 2$. Suppose that the statement holds for $n - k$, then we try to prove it for $n - (k + 1)$. By using equation (2.26) for $k = n - (k + 1)$, we have

$$c^i_{n-(k+1)} = (-1)^{k+1}h^{k+1}\frac{(n-(k+1))!}{n!}F^{(k-1)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))$$
$$+\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}c^i_{n-(k+1)+j}. \tag{2.37}$$

We use the notation Φ to represent the summation part of the equation (2.37)

$$\Phi = \sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}c^i_{n-(k+1)+j}.$$

Expanding the terms of Φ gives that

$$\Phi = \binom{k+1}{1}c^i_{n-k} - \binom{k+1}{2}c^i_{n-(k-1)} + \ldots + (-1)^{k+1}\binom{k+1}{k}c^i_{n-1}$$
$$+(-1)^{k+2}\binom{k+1}{k+1}c^i_n.$$

By induction hypothesis, we can generate all the unknown control points in terms of c^i_{n-1} and c^i_n for $j = n-k, \ldots, n$. Thus we have

$$\begin{aligned}\Phi =&\binom{k+1}{1}\Bigg(kc^i_{n-1} - (k-1)c^i_n \\ &+\sum_{j=2}^{k}(-1)^j h^j\binom{k}{j}\frac{(n-j)!}{n!}F^{(j-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\Bigg) \\ &-\binom{k+1}{2}\Bigg((k-1)c^i_{n-1} - (k-2)c^i_n \\ &+\sum_{j=2}^{k-1}(-1)^j h^j\binom{k-1}{j}\frac{(n-j)!}{n!}F^{(j-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\Bigg) \\ &+\ldots + (-1)^{k+1}\binom{k+1}{k}c_{n-1} + (-1)^{k+2}\binom{k+1}{k+1}c^i_n.\end{aligned}$$

Rearranging the terms of Φ gives us that

$$
\begin{aligned}
\Phi = c^i_{n-1}&\Bigg(\binom{k+1}{1}k - \binom{k+1}{2}(k-1) + \ldots + (-1)^{k+1}\binom{k+1}{k}\Bigg)\\
&- c^i_n\Bigg(\binom{k+1}{1}(k-1) - \binom{k+1}{2}(k-2) + \ldots\\
&+ (-1)^k\binom{k+1}{k-1} - (-1)^{k+2}\binom{k+1}{k+1}\Bigg)\\
&+ \binom{k+1}{1}\sum_{j=2}^{k}(-1)^j h^j\binom{k}{j}\frac{(n-j)!}{n!}F^{(j-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\\
&- \binom{k+1}{2}\sum_{j=2}^{k-1}(-1)^j h^j\binom{k-1}{j}\frac{(n-j)!}{n!}F^{(j-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\\
&+ \ldots + (-1)^k\binom{k+1}{k-1}\sum_{j=2}^{2}(-1)^j h^j\binom{2}{j}\frac{(n-j)!}{n!}F^{(j-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))
\end{aligned}
$$

$$
\begin{aligned}
&= c^i_{n-1}\sum_{j=1}^{k}(-1)^{j+1}\binom{k+1}{j}(k+1-j) - c^i_n\Bigg(\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}(k-j)\Bigg)\\
&+ \binom{k+1}{1}\Bigg(h^2\binom{k}{2}\frac{(n-2)!}{n!}F^{(0)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\\
&- h^3\binom{k}{3}\frac{(n-3)!}{n!}F^{(1)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\\
&+ \ldots + (-1)^k h^k\binom{k}{k}\frac{(n-k)!}{n!}F^{(k-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\Bigg)\\
&- \binom{k+1}{2}\Bigg(h^2\binom{k-1}{2}\frac{(n-2)!}{n!}F^{(0)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\\
&- h^3\binom{k-1}{3}\frac{(n-3)!}{n!}F^{(1)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\\
&+ \ldots + (-1)^{k-1}h^{k-1}\binom{k-1}{k-1}\frac{(n-(k-1))!}{n!}F^{(k-3)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\Bigg)\\
&+ \ldots + (-1)^k\binom{k+1}{k-1}h^2\frac{(n-2)!}{n!}F^{(0)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))
\end{aligned}
$$

$$
\begin{aligned}
&= c^i_{n-1}\sum_{j=1}^{k}(-1)^{j+1}\binom{k+1}{j}(k+1-j) - c^i_n\Bigg(\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}(k-j)\Bigg)\\
&+ h^2\frac{(n-2)!}{n!}F^{(0)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\Bigg(\binom{k+1}{1}\binom{k}{2}\\
&- \binom{k+1}{2}\binom{k-1}{2} + \ldots + (-1)^k\binom{k+1}{k-1}\binom{2}{2}\Bigg)\\
&- h^3\frac{(n-3)!}{n!}F^{(1)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\Bigg(\binom{k+1}{1}\binom{k}{3}\\
&- \binom{k+1}{2}\binom{k-1}{3} + \ldots + (-1)^{k-1}\binom{k+1}{k-2}\binom{3}{3}\Bigg)\\
&+ \ldots + (-1)^k h^k F^{(k-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\frac{(n-k)!}{n!}\binom{k+1}{1}\binom{k}{k}
\end{aligned}
$$

$$
\begin{aligned}
&= c^i_{n-1}\sum_{j=1}^{k}(-1)^{j+1}\binom{k+1}{j}(k+1-j) - c^i_n\Bigg(\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}(k-j)\Bigg)\\
&+ h^2F^{(0)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\frac{(n-2)!}{n!}\sum_{j=2}^{k}(-1)^j\binom{k+1}{j-1}\binom{k-j+2}{2}\\
&- h^3F^{(1)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\frac{(n-3)!}{n!}\sum_{j=2}^{k-1}(-1)^j\binom{k+1}{j-1}\binom{k-j+2}{3}\\
&+ \ldots + (-1)^k h^k F^{(k-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\frac{(n-k)!}{n!}\sum_{j=2}^{2}(-1)^j\binom{k+1}{j-1}\binom{k-j+2}{k}.
\end{aligned}
$$

We use Lemma (2.2.5) to simplify Φ

$$
\begin{aligned}
\Phi &= c^i_{n-1}\sum_{j=1}^{k}(-1)^{j+1}\binom{k+1}{j}(k+1-j) - c^i_n\Bigg(\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}(k-j)\Bigg)\\
&+ h^2\binom{k+1}{2}\frac{(n-2)!}{n!}F^{(0)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\\
&- h^3\binom{k+1}{3}\frac{(n-3)!}{n!}F^{(1)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1}))\\
&+ \ldots + (-1)^k h^k\binom{k+1}{k}\frac{(n-k)!}{n!}F^{(k-2)}(a_i, c^i_n, \frac{n}{h}(c^i_n - c^i_{n-1})).
\end{aligned}
$$

So we have

$$\Phi = c_{n-1}^i \sum_{j=1}^{k} (-1)^{j+1} \binom{k+1}{j} (k+1-j) - c_n^i \left(\sum_{j=1}^{k+1} (-1)^{j+1} \binom{k+1}{j} (k-j) \right)$$
$$+ \sum_{j=2}^{k} (-1)^j h^j \binom{k+1}{j} \frac{(n-j)!}{n!} F^{(j-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i)).$$

Now we simplify the coefficients of c_n and c_{n-1}. To do this, we use the following equalities

$$\binom{k+1}{j}(k+1-j) = \frac{(k+1)k!}{j!(k+1-j)(k-j)!}(k+1-j)$$

$$= (k+1)\binom{k}{j}$$

and

$$\sum_{j=1}^{k} (-1)^j \binom{k}{j} = -1.$$

Substituting these results into the coefficient of c_{n-1}, we get

$$\sum_{j=1}^{k} (-1)^{j+1} \binom{k+1}{j} (k+1-j) = -(k+1) \sum_{j=1}^{k} (-1)^j \binom{k}{j}$$
$$= k+1. \tag{2.38}$$

Now let us consider the coefficient of c_n

$$\sum_{j=1}^{k+1} (-1)^{j+1} \binom{k+1}{j} (k-j) = \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k+1}{j} k - \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k+1}{j} j$$

$$= -k \sum_{j=1}^{k+1} (-1)^j \binom{k+1}{j} - \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k+1}{j} j.$$

By using Proposition (2.2.1), the last expression becomes

$$\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}(k-j) = k - \sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}j. \tag{2.39}$$

Since

$$\binom{k+1}{j}j = \frac{(k+1)!}{(k+1-j)!j!}j$$

$$= \frac{(k+1)k!}{(k-(j-1))!j(j-1)!}j$$

$$= (k+1)\binom{k}{j-1},$$

we have

$$\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}j = \sum_{j=1}^{k+1}(-1)^{j+1}(k+1)\binom{k}{j-1}$$

$$= (k+1)\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k}{j-1}$$

$$= (k+1)\sum_{j=0}^{k}(-1)^{j+2}\binom{k}{j}$$

$$= (k+1)\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}$$

$$= (k+1)\left(1+\sum_{j=1}^{k}(-1)^{j}\binom{k}{j}\right)$$

$$= 0$$

by Proposition (2.2.1). Thus we get

$$\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}(k-j)=k. \tag{2.40}$$

Substituting (2.38) and (2.40) into Φ we get

$$\begin{aligned}\sum_{j=1}^{k+1}(-1)^{j+1}\binom{k+1}{j}c_{n-(k+1)-j} &= (k+1)c_{n-1}-kc_n^i\\ &+\sum_{j=2}^{k}(-1)^j h^j\binom{k+1}{j}\frac{(n-j)!}{n!}F^{(j-2)}(a_i,c_n^i,\frac{n}{h}(c_n^i-c_{n-1}^i)).\end{aligned}$$

If we substitute this result into the equation (2.37), we get

$$\begin{aligned}c_{n-(k+1)}^i &= (-1)^{k+1}h^{k+1}\frac{(n-(k+1))!}{n!}F^{(k-1)}(a_i,c_n^i,\frac{n}{h}(c_n^i-c_{n-1}^i))+(k+1)c_{n-1}^i\\ &-kc_n^i+\sum_{j=2}^{k}(-1)^j h^j\binom{k+1}{j}\frac{(n-j)!}{n!}F^{(j-2)}(a_i,c_n^i,\frac{n}{h}(c_n^i-c_{n-1}^i))\end{aligned}$$

which gives us the desired result

$$c_{n-(k+1)}^i=(k+1)c_{n-1}^i-kc_n^i+\sum_{j=2}^{k+1}(-1)^j h^j\binom{k+1}{j}\frac{(n-j)!}{n!}F^{(j-2)}(a_i,c_n^i,\frac{n}{h}(c_n^i-c_{n-1}^i)),$$

or

$$\begin{aligned}c_k^i =&(n-k)c_{n-1}^i-(n-k-1)c_n^i+\sum_{j=2}^{n-k}(-1)^j h^j\binom{n-k}{j}\\ &\times\frac{(n-j)!}{n!}F^{(j-2)}(a_i,c_n^i,\frac{n}{h}(c_n^i-c_{n-1}^i)).\end{aligned}$$

Hence, by this theorem we can use Residual method at the end point instead of using the starting point of the interval. □

The bound of the error of the proposed method is given in the following theorem which is proved in Oger (2016).

Theorem 2.2.7. *Let $y(x)$ be the exact solution of the second order nonlinear initial value problem* (2.1)-(2.2) *and $u(x)$ be the corresponding n^{th} degree approximate function* (2.3) *obtained by improved residual method, then*

$$|y(x) - u(x)| \leq Mh^{n-1}, \quad x \in [a, b], \tag{2.41}$$

where

$$M = K(b-a)^2,$$

$$K = \frac{1}{n!} \max_{x \in [a,b]} |y^{(n+1)}(x)|,$$

h is the stepsize and n is the degree of Bézier curve.

CHAPTER THREE
IMPROVED RESIDUAL METHOD FOR REGULAR STURM-LIOUVILLE EIGENVALUE PROBLEMS

3.1 Improved Residual Method for Regular Sturm-Liouville Eigenvalue Problems

The improved residual method, which is described in previous section, is mainly developed for initial value problems. In this section, in order to find approximate eigenvalues and corresponding eigenfunctions of Sturm-Liouville eigenvalue problems, we adapt the method to regular Sturm-Liouville eigenvalue problems of the form

$$-y''(x) + r(x)y(x) = \lambda y(x), \quad 0 \leq x_0 < x \leq x_N, \tag{3.1}$$

with the boundary conditions

$$y(x_0) = y(x_N) = 0, \tag{3.2}$$

where $r(x) \in C^{n-2}[x_0, x_N]$.

Using the shoothing method idea, at first, we define the following initial value problem:

$$-y''(x) + r(x)y(x) = \lambda y(x), \quad 0 \leqslant x_0 < x < x_N \tag{3.3}$$

$$y(x_0) = 0, \qquad y'(x_0) = s, \tag{3.4}$$

where s is a parameter.

From the theory of Sturm-Liouville eigenvalue problems, starting with an initial guess s does not change the result of the problem. So by choosing any initial value s, we apply the improved residual method with high degree polynomial to (3.3) and (3.4) with large stepsize $h < 1$. In order to get approximate solution of the boundary value problem (3.1) and (3.2) from the initial value problem (3.3) and (3.4); solution of (3.3) and (3.4) $u(x; \lambda)$ must satisfy the boundary condition at $x = b$. So we have the following nonlinear equation

$$u(b;\lambda) = 0. \tag{3.5}$$

Using end point property of Bézier curves and (2.13), we get

$$\begin{aligned}
0 &= u(b,\lambda) \\
&= c_n \\
&= nc_1 - (n-1)c_0 + \sum_{j=2}^{n} \binom{n}{j} \frac{(n-j)!}{n!} h^j F^{(j-2)}(a,0,s;\lambda),
\end{aligned}$$

where n is the degree of Bézier curve (2.3) and h is the stepsize. Since $c_0 = 0$ and $c_1 = \frac{h}{n}s$ by $(A.2)$, we have

$$0 = hs - \sum_{j=2}^{n} \frac{1}{j!} h^j F^{(j-s)}(a,0,s;\lambda), \tag{3.6}$$

where

$$F(x,y(x),z(x);\lambda) = -\lambda y(x) + r(x)y(x)$$

and

$$F^{(k)}(x,y(x),z(x);\lambda) = \frac{d^k}{dx^k} F(x,y(x),z(x);\lambda).$$

The roots of equation (3.6) give us the approximate eigenvalues. After finding approximate eigenvalues and choosing initial value s, we apply the improved residual method to obtain the approximate solution of initial value problem (3.3) and (3.4).

3.2 Demonstration of the covergency of the method for $r(x) = 0$

In this section, we demonstrate the convergency of the method for $r(x) = 0$. Consider the following boundary value problem

$$-y''(x) = \lambda y(x), \quad 0 < x < 1,$$

$$y(0) = 0, \quad y(1) = 0. \tag{3.7}$$

First, we write the n^{th} order derivative of $y(x)$ explicitly by the following lemma.

Lemma 3.2.1. *Let $y(x)$ be the solution of* (3.7). *Then the following recuerrence relations hold*

$$y^{(2k)} = (-1)^k \lambda^k y(x) \tag{3.8}$$

and

$$y^{(2k+1)} = (-1)^k \lambda^k y'(x) \tag{3.9}$$

for $k = 0, 1, 2, \ldots$.

Proof. We prove the statements by induction on k. It is obvious that the statements hold for $k = 0$. Suppose that the statements are true for k, then we need to prove them for $k + 1$. Using induction hypothesis for k, we have

$$y^{(2k+2)}(x) = (y^{(2k+1)})'(x)$$

$$= ((-1)^k \lambda^k y'(x))'$$

$$= (-1)^k \lambda^k y''(x)$$

$$= (-1)^{k+1} \lambda^{k+1} y(x).$$

So, (3.8) holds for $k+1$.

$$y^{(2k+3)}(x) = (y^{(2k+2)})'(x)$$

$$= ((-1)^{k+1}\lambda^{k+1}y(x))'$$

$$= (-1)^{k+1}\lambda^{k+1}y'(x).$$

Therefore (3.9) holds for $k+1$. $\square$

Now we can apply improved residual method, which is described in previous section without dividing the interval by taking $h = 1$. Thus, we have $c_0 = 0$, $c_1 = \dfrac{s}{n}$ and $c_n = 0$. Substitution of these values into the equation (2.13) gives

$$c_n = nc_1 - (n-1)c_0 + \sum_{k=2}^{n} \binom{n}{k} \frac{(n-k)!}{n!} h^k F^{(k-2)}(x_0, y_0, s),$$

$$= s + \sum_{k=2}^{n} \frac{1}{k!} F^{(k-2)}(x_0, y_0, s).$$

Since $F^{(2k-2)}(x_0, y_0, s) = (-1)^k \lambda^k y_0$ and $F^{(2k-1)}(x_0, y_0, s) = (-1)^k \lambda^k s$ by Lemma 3.2.1, we have

$$c_n = s + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2k)!} F^{(2k-2)}(a, y_0, s) + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{(2k+1)!} F^{(2k+1-2)}(a, y_0, s),$$
$$= s + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{(2k)!} (-1)^k \lambda^k y_0 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{(2k+1)!} (-1)^k \lambda^k s.$$

Since $y_0 = 0$, we have

$$c_n = s\left(1 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{(2k+1)!} \frac{(-1)^k(\sqrt{\lambda})^{2k+1}}{\sqrt{\lambda}}\right)$$

$$= \frac{s}{\sqrt{\lambda}}\left(\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{(2k+1)!}(-1)^k(\sqrt{\lambda})^{2k+1}\right). \tag{3.10}$$

Taking limits of both sides and using $c_n = 0$, we get

$$0 = \lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{s}{\sqrt{\lambda}}\left(\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{(2k+1)!}(-1)^k(\sqrt{\lambda})^{2k+1}\right)$$

$$= \frac{s}{\sqrt{\lambda}}\left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k(\sqrt{\lambda})^{2k+1}\right).$$

Since

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(-1)^k(\sqrt{\lambda})^{2k+1} = \sin\sqrt{\lambda},$$

we get

$$0 = \frac{s}{\sqrt{\lambda}} \sin\sqrt{\lambda}. \tag{3.11}$$

In order to get nontrivial solution, we must have $s \neq 0$. The roots of equation (3.11) are $\lambda = k^2\pi^2$ for $k = 1, 2, ...$ After finding eigenvalues, now we find the eigenfunctions of the problem (3.7) by using improved residual method. For any $\overline{x} \in [0, 1]$, $h = \overline{x}$ and $c_1 = \dfrac{\overline{x}s}{n}$, approximate solution becomes

$$u(\overline{x}) = c_n$$

$$= s\overline{x} + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{(2k+1)!} \frac{(-1)^k(\sqrt{\lambda})^{2k+1}}{\sqrt{\lambda}} h^{2k+1} s,$$

$$= \frac{s}{\sqrt{\lambda}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^k}{(2k+1)!}(\sqrt{\lambda}\overline{x})^{2k+1}.$$

Limits of both sides give that

$$\lim_{n\to\infty} u(\overline{x}) = \lim_{n\to\infty} c_n$$

$$= \lim_{n\to\infty} \frac{s}{\sqrt{\lambda}} \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^k}{(2k+1)!} (\sqrt{\lambda}\overline{x})^{2k+1}$$

$$= \frac{s}{\sqrt{\lambda}} \sin \sqrt{\lambda}\overline{x} = y(\overline{x}).$$

Hence, we have $\lim_{n\to\infty} u(\overline{x}) = y(\overline{x})$. Therefore, we show that obtained approximate eigenvalues and eigenfunctions using proposed method converge to the exact solutions for $r(x) = 0$.

3.3 Numerical Results and Discussions

In this section, we apply the proposed method to some famous regular Sturm Liouville eigenvalue problems and compare our approximate results with other results from literature. Throughout this section, λ_k represents the k^{th} approximate eigenvalue, which is obtained by using the improved residual method (IRM), of k^{th} exact eigenvalue $\wedge_k$. The observed orders $ord(h)$ are computed using the following formula

$$ord(h) = \frac{ln \frac{y_{4h} - y_{2h}}{y_{2h} - y_h}}{ln 2}, \tag{3.12}$$

where y_{4h}, y_{2h} and y_h are the approximate eigenfunctions at x_k to the corresponding eigenvalue λ_k when the problems are solved with stepsizes $4h, 2h$ and h, respectively. In this study, all numerical calculations and figures are performed by using Mathematica.

Example 3.3.1. Let us consider the following equation

$$-y''(t) + 2y(t) = \lambda y(t),$$

$$y(0) = y(1) = 0. \tag{3.13}$$

The equation (3.13) has exact eigenvalues $\wedge_k$ given by the explicit formula

$$\wedge_k = k^2\pi^2 + 2, \quad k = 1, 2, \ldots. \tag{3.14}$$

In Table 3.1, we compare approximate eigenvalues λ_k obtained by IRM with the approximate eigenvalues obtained by finite difference method (FDM) $\lambda_{k,20}^{(f)}$ for choosen 20 number of intervals and the approximate eigenvalues obtained by symmetrical weighted sequential splitting method $\lambda_{k,2}$ for choosen 2 number of intervals Güzel et al. (2019). It is observed that, the results of the proposed method is better than those of the finite difference method and the symmetrical weighted sequential splitting method. It is also seen from Table 3.1 that, we even compute approximate high eigenvalues accurately.

Table 3.1 Comparison of the errors of approximate eigenvalues obtained by IRM, FDM and the splitting method for Example 3.3.1 for $h = 1$ and $n = 1000$.

k	$\wedge_k$	$\lvert \wedge_k - \lambda_k \rvert$	$\lvert \wedge_k - \lambda_{k,20}^{(f)} \rvert$	$\lvert \wedge_k - \lambda_{k,2} \rvert$
1	11.86960	0.0	2.0277×10^{-2}	9.7745×10^{-2}
3	90.82643	0.0	1.6317	1.2824×10^{-2}
5	248.74011	0.0	12.4255	4.6873×10^{-3}
7	485.61061	2.27374×10^{-13}	46.8030	2.4017×10^{-3}
9	801.43795	0.0	124.5855	1.4554×10^{-3}
11	1196.22213	0.0	269.0746	9.7516×10^{-4}
13	1669.96314	9.09495×10^{-13}	504.7707	6.9855×10^{-4}
15	2222.66099	0.0	854.9756	5.2486×10^{-4}
17	2854.31567	1.81899×10^{-12}	1339.5105	4.0871×10^{-4}
50	24676.01100	0.0	-	-
100	98698.04401	1.455199×10^{-11}	-	-

Example 3.3.2. Consider the Titchmarch equation

$$-y''(x) + x^{2k}y(x) = \lambda y(x),$$

$$y(0) = y(1) = 0, \tag{3.15}$$

where k is a nonnegative integer.

The exact solution of Titchmarch problem for $k = 1$ and $k = 2$ do not exist. Thus, we compare the approximate eigenvalues of the problem with the approximate eigenvalues of different methods from literature. It is seen that, our results are well coincide with other results. For $k = 1$ in Titchmarch problem, we give approximate eigenvalues λ_k obtained by IRM, the approximate eigenvalues λ_k^* from Birkhoff & Varga (1970) obtained by trigonometric polynomial, the approximate eigenvalues $\lambda_{k,20}^{(f)}$ obtained by finite difference method and the approximate eigenvalues $\lambda_{k,l}$ obtained by symmetrical weighted sequential splitting method from Güzel et al. (2019) in Table 3.2. In $\lambda_{k,l}$, l denotes the number of intervals.

Table 3.2 Comparison of the errors of first 10 eigenvalues obtained using IRM with symmetrical weighted sequential splitting method and FDM in Example 3.3.2 for $k = 1$, $h = 1$ and $n = 500$.

k	l	λ_k	λ_k^*	$\lvert\lambda_k - \lambda_k^*\rvert$	$\lvert\lambda_{k,l} - \lambda_k^*\rvert$	$\lvert\lambda_{k,20}^{(f)} - \lambda_k^*\rvert$
1	7	10.15116	10.15116	0.0	5.99769×10^{-3}	2.0291×10^{-2}
2	7	39.79939	39.79939	0.0	5.39722×10^{-3}	3.2365×10^{-1}
3	5	89.15434	89.15434	0.0	3.00800×10^{-3}	1.63169
4	6	158.24396	158.24396	0.0	1.80503×10^{-3}	5.12731
5	2	247.07150	247.07150	0.0	1.82758×10^{-3}	12.42560
6	4	355.63774	355.63774	0.0	2.68230×10^{-3}	25.53406
7	2	483.94295	483.94295	0.0	9.30714×10^{-4}	46.80315
8	5	631.98725	631.98725	0.0	1.39727×10^{-3}	78.86847
9	2	799.77069	799.77069	0.0	5.62593×10^{-5}	124.58579
10	7	987.29328	987.29328	0.0	7.50294×10^{-5}	186.96079
20		3948.17497	-	-	-	-
30		8882.97724	-	-	-	-
40		15791.70034	-	-	-	-
50		24674.34431	-	-	-	-

In Table 3.3, we give computed first eigenvalue of Titchmarch problem by using

IRM, the Taylor's Decomposition Method (TDM) Adiyaman & Somali (2012), the Haar wavelet series method (HWSM) and FDM Bujurke et al. (2008) for $k = 2$ and $h = 0.0625$.

Table 3.3 Comparison of the first eigenvalue and approximate results of Example 3.3.2 obtained using IRM, TDM, HWSM and FDM for $k = 2$, $h = 1/16$ and $n = 50$.

x	*HWSM*	*FDM*	*TDM*	*IRM*
0	0	0	0	0
0.0625	0.27521	0.27756	0.27563	0.27756
0.125	0.54181	0.54434	0.54434	0.54434
0.1875	0.78949	0.78996	0.78995	0.78995
0.25	1.00485	1.00488	1.00487	1.00487
0.3125	1.18153	1.18075	1.18074	1.18074
0.375	1.31286	1.31082	1.31076	1.31076
0.4375	1.39372	1.38996	1.38994	1.38994
0.5	1.42102	1.41527	1.41529	1.41529
0.5625	1.39371	1.38591	1.38598	1.38598
0.625	1.31285	1.30323	1.30334	1.30334
0.6875	1.18154	1.18066	1.17081	1.17081
0.75	1.00480	0.99361	0.99379	0.99379
0.8125	0.77949	0.77917	0.77936	0.77936
0.875	0.53481	0.53577	0.53593	0.53593
0.9375	0.27726	0.27277	0.27288	0.27288
1	0	0	0	0
λ_1	10.3452	9.95067	9.98317	9.98317

By Table 3.4, we present approximate values of eigenfunctions corresponding to the first eigenvalue at $x = 1/2$, where N represents number of subintervals of the proposed method. The reason for choosing $x = 1/2$ is that the maximum error in the given interval occurs at that point. Orders of the improved residual method for different values of N are obtained by using equation (3.12). We see that, the observed orders are well confirm with the theoretical results, since the order of the method is $n - 1$, where n is the degree of approximate Bézier curve.

Table 3.4 The approximate value of eigenfunctions and observed orders of Titchmarch problem 3.3.2 for $k = 2$ at $x = 1/2$ corresponding to the first eigenvalue for different number of subintervals using IRM.

	$n = 2$	$n = 3$	$n = 4$
$N = 8$	1.65214	1.40179	1.41246
$N = 16$	1.53065	1.41128	1.41493
$N = 32$	1.47178	1.41420	1.41524
$N = 64$	1.44319	1.41501	1.41528
$N = 128$	1.42915	1.41529	1.40611
Observed Orders			
$ord(1/8)$	1.04549	1.69820	2.96513
$ord(1/16)$	1.04185	1.85970	2.99114
$ord(1/32)$	1.02562	1.93242	2.99765

Example 3.3.3. Consider the Euler differential equation

$$x^2y'' + 3xy' + \lambda y = 0$$

$$y(1) = y(2) = 0 \tag{3.16}$$

with exact eigenfunctions

$$y_k(x) = x^{-1} \sin\left(\frac{k\pi}{\ln 2} \ln x\right)$$

corresponding to exact eigenvalues $\wedge_k = 1 + \left(\frac{k\pi}{ln2}\right)^2$ for $k = 1, 2, 3, \ldots$.

Figure 3.1 demonstrates the graphs of obtained eigenfunctions for first four eigenvalues, respectively, by the proposed method for Euler differential equation (3.16). In Figure 3.2, graphs of the error functions of the problem are given for the same first four approximate eigenfunctions corresponding to first four approximate eigenvalues.

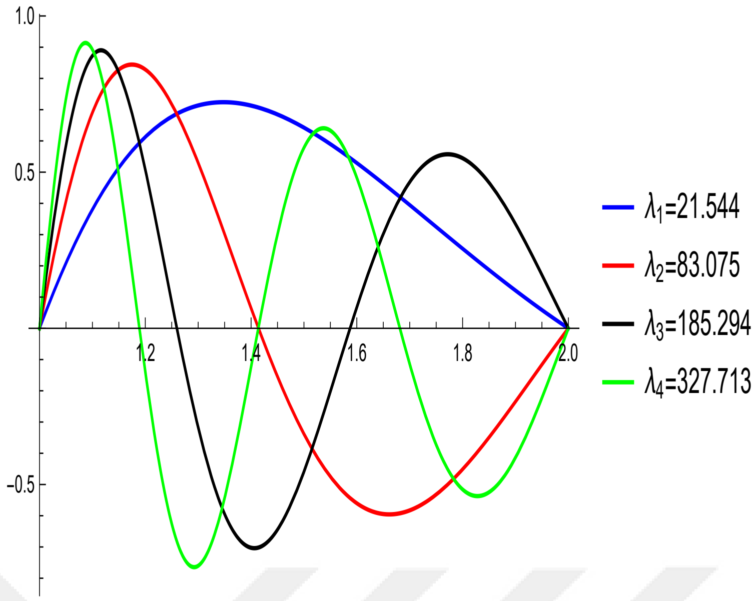

Figure 3.1 Graphs of obtained eigenfunctions of Euler differential equation by IRM.

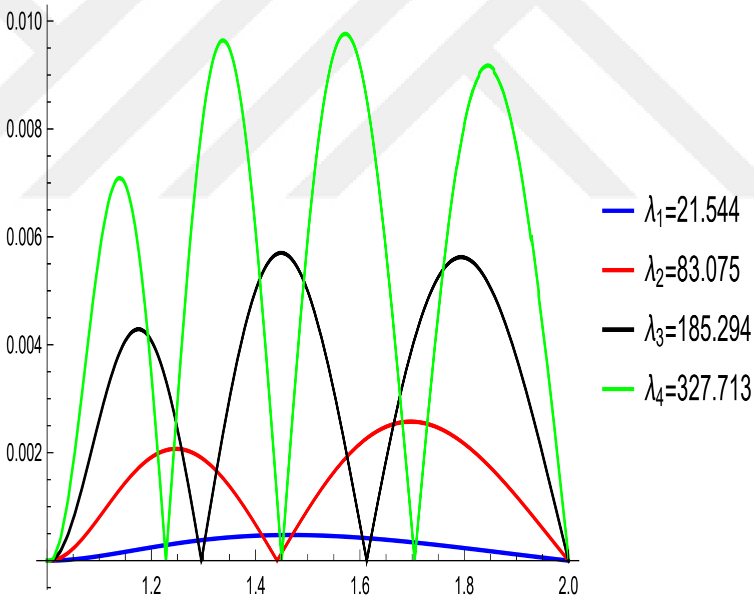

Figure 3.2 Graphs of the error functions for Euler differential equation by IRM.

In Table 3.5, we give approximate eigenvalues λ_k and exact eigenvalues $\wedge_k$ of the given problem. λ_k is found by the improved residual method by dividing the interval into 300 equally spaced intervals and choosing the degree of the Bézier curves as $n = 3$. As it is seen from the table that, our results are well confirm with the exact results.

Table 3.5 Comparison of exact eigenvalues of Euler differential equation with approximate eigenvalues obtained by IRM using $h = 1/300$ and $n = 3$.

k	$\wedge_k$	λ_k *IRM*
1	21.54229	21.54240
2	83.16915	83.16442
3	185.88060	185.85055
4	329.67661	329.57499
5	514.557 21	514.30164

In Table 3.6, approximate eigenvalues are computed by using equation (2.13) and equation (2.36) given in Theorem 2.2.3 and Theorem 2.2.6, respectively. We can say that, no matter which formula we use, we get the approximate results accurately.

Table 3.6 Comparison of eigenvalues computed by IRM using the result of Theorem 2.2.3 and Theorem 2.2.6 for $h = 1/300$ and $n = 3$.

k	λ_k *Theorem* (2.2.3)	λ_k *Theorem* (2.2.6)
1	21.54240	21.54240
2	83.16442	83.16464
3	185.85055	185.85180
4	329.57499	329.57905
5	514.30164	514.31169

Example 3.3.4. Consider the following regular Sturm-Liouville eigenvalue problem

$$-y''(t) + e^t y(t) = \lambda y(t),$$

$$y(0) = y(\pi) = 0. \tag{3.17}$$

In Table 3.7, we give the first 20 exact eigenvalues $\wedge_k$ (see Andrew & Paine (1985)) of (3.17) and computed approximate eigenvalues λ_k obtained by improved residual method. We also give the absolute errors obtained by IRM, Numerov method and corrected Numerov method given in Andrew & Paine (1985). The error in the estimate of the k^{th} eigenvalue of a regular Sturm-Liouville problem obtained by Numerov's method is $O(k^6 h^4)$ and corrected Numerov's method is $O(k^3 h^4)$, where h represents the mesh length. In the mentioned table, approximate eigenvalues of Numerov method and corrected Numerov method are denoted by λ_k° and $\lambda_k^{\bullet}$, respectively. The results given in the table illustrate that the suggested technique has accurate approximate eigenvalues than the other techniques.

Table 3.7 Comparison of the errors of first 20 eigenvalues obtained using Numerov method, corrected Numerov method and improved residual method for Example 3.3.4 for $h = 1$ and $n = 500$.

k	$\wedge_k$	λ_k	$\vert \wedge_k - \lambda_k^{\circ}\vert$	$\vert \wedge_k - \lambda_k^{\bullet}\vert$	$\vert \wedge_k - \lambda_k\vert$
1	4.89667	4.89666	2.82×10^{-6}	2.7×10^{-6}	9.37997×10^{-6}
2	10.04518	10.04519	4.268×10^{-5}	3.25×10^{-5}	1.06746×10^{-7}
3	16.01926	16.01926	2.2720×10^{-4}	1.137×10^{-4}	2.50492×10^{-7}
4	23.26627	23.26627	8.8366×10^{-4}	2.317×10^{-4}	5.99777×10^{-8}
5	32.26370	32.26370	2.88017×10^{-3}	3.879×10^{-4}	4.58045×10^{-8}
6	43.22002	43.22001	8.04318×10^{-3}	5.82×10^{-4}	3.59466×10^{-7}
7	56.18159	56.18159	1.96872×10^{-2}	8.158×10^{-4}	2.28476×10^{-8}
8	71.15299	71.15299	4.32849×10^{-2}	1.0913×10^{-3}	7.53706×10^{-6}
9	88.13211	88.13211	8.72765×10^{-2}	1.4108×10^{-3}	1.91546×10^{-7}
10	107.11668	107.11667	1.64024×10^{-1}	1.7778×10^{-3}	3.86173×10^{-6}
11	128.10502	128.10502	2.90917×10^{-1}	2.1962×10^{-3}	1.27333×10^{-6}
12	151.09604	151.09604	4.91634×10^{-1}	2.67090×10^{-3}	3.74560×10^{-6}
13	176.08900	176.08899	7.97568×10^{-1}	3.20820×10^{-3}	3.19056×10^{-6}
14	203.08337	203.08337	1.24940	3.8153×10^{-3}	1.03863×10^{-6}
15	232.07881	232.07881	1.89885	4.5015×10^{-3}	1.98486×10^{-6}
16	263.07507	263.07506	2.81051	5.2781×10^{-3}	2.03987×10^{-6}
17	296.07196	296.07195	4.06387	6.1589×10^{-3}	3.26256×10^{-6}
18	331.06930	331.06934	5.75541	7.1615×10^{-3}	4.39831×10^{-5}
19	368.06713	368.06712	8.00064	8.3076×10^{-3}	9.76823×10^{-7}
20	407.06520	407.06523	1.09363	9.6248×10^{-3}	4.73266×10^{-6}

Example 3.3.5. Consider the Paine problem with the boundary conditions

$$-y''(t) + (t + 0.1)^{-2} y(t) = \lambda y(t),$$

$$y(0) = y(\pi) = 0. \tag{3.18}$$

This problem is almost singular. In Table 3.8, we give the first 10 exact eigenvalues $\wedge_k$ (see Andrew & Paine (1985)) of (3.18) and approximate eigenvalues λ_k obtained by IRM. We compare absolute errors of suggested technique with Numerov method in (Andrew & Paine (1985)) whose eigenvalues are denoted by $\lambda^\circ_{k,79}$ and$\lambda^\circ_{k,19}$ with mesh points 19 and 79, respectively. The absolute errors for suggested technique are smaller than the other results.

Table 3.8 Comparison of errors of first 10 eigenvalues obtained by IRM and Numerov method for Paine problem for $h = 1/2$ and $n = 500$.

k	$\wedge_k$	λ_k	$\lvert \wedge_k - \lambda^\circ_{k,19} \rvert$	$\lvert \wedge_k - \lambda^\circ_{k,79} \rvert$	$\lvert \wedge_k - \lambda_k \rvert$
1	1.51986	1.51987	4.325×10^{-4}	4×10^{-6}	2.12691×10^{-8}
2	4.94330	4.94331	2.7664×10^{-3}	2.4×10^{-5}	2.32045×10^{-8}
3	10.28466	10.28466	8.6229×10^{-3}	7.3×10^{-5}	3.51706×10^{-7}
4	17.55995	17.55995	1.9436×10^{-2}	1.62×10^{-4}	2.46603×10^{-7}
5	26.78286	26.78286	3.6391×10^{-2}	2.99×10^{-4}	1.71287×10^{-7}
6	37.96442	37.96442	6.0481×10^{-2}	4.86×10^{-4}	1.17988×10^{-7}
7	51.11335	51.11335	9.2608×10^{-2}	7.29×10^{-4}	4.61195×10^{-7}
8	66.23644	66.23645	0.1337	1.029×10^{-3}	3.39523×10^{-6}
9	83.33896	83.33911	0.18481	1.386×10^{-3}	1.56612×10^{-4}
10	102.42499	102.41639	0.24730	1.802×10^{-3}	8.59671×10^{-3}

CHAPTER FOUR
IMPROVED RESIDUAL METHOD FOR SINGULAR EIGENVALUE PROBLEMS

Consider the following second order boundary value problem

$$-(p_1(x)y'(x))' + q_1(x)y(x) = \lambda w_1(x)y(x), \quad -\infty < a \le x \le b < \infty$$

$$\alpha_1 y(a) + \alpha_2 p_1(a)y'(a) = 0, \tag{4.1}$$

$$\beta_1 y(b) + \beta_2 p_1(b)y'(b) = 0$$

with $p_1(x), w_1(x) > 0$ on the interval (a, b), where $\alpha_1, \alpha_2, \beta_1$ and β_2 are constants such that $|\alpha_1| + |\alpha_2| > 0$ and $|\beta_1| + |\beta_2| > 0$. If (a, b) is an infinite interval, or at least one of $1/p_1, q_1, w_1$ is not integrable in any neighborhood of the endpoints a or b, then the problem (4.1) is called singular problem. In this section, in order to find approximate eigenvalues of (4.1), we rewrite the problem in the following form

$$y''(x) + p(x)y'(x) + (w(x)\lambda - q(x))y(x) = 0,$$
$$y(a) = 0, \quad y(b) = 0, \tag{4.2}$$

where $p(x) = \dfrac{p_1'(x)}{p_1(x)}$, $q(x) = \dfrac{q_1(x)}{p_1(x)}$ and $w(x) = \dfrac{w_1(x)}{p_1(x)}$.

4.1 Adaptation of Residual Method of Singular Sturm-Liouville Eigenvalue Problems

Throughout this section, we investigate the problems that have finite intervals. The singularity of a problem can occur in three ways. It can be either at the initial point, at the end point, or at both the initial and end points. The adaptation of the proposed method varies according to the location of the singularity. So we classify the problems in three types.

Type 1: If the initial point is regular and the end point is singular, we define the following initial value problem

$$u''(x) + p(x)u'(x) + (w(x)\lambda - q(x))u(x) = 0, \quad a < x < b$$
$$u(a) = 0 \quad \text{and} \quad u'(a) = s, \tag{4.3}$$

where s is a parameter. In this case, we use the same technique which is described in the previous chapter. The approximate eigenvalues of (4.2) from (4.3) are found by application of improved residual method using (2.13) in Theorem 2.2.3. This application yields the following equation whose solutions are the approximate eigenvalues of the problem (4.3):

$$0 = hs + \sum_{j=2}^{n} \frac{1}{j!} h^j F^{(j-2)}(a, 0, s; \lambda), \tag{4.4}$$

where

$$F(x, y(x), z(x); \lambda) = -p(x)z(x) + (q(x) - w(x)\lambda)y(x)$$

and

$$F^{(k)}(x, y(x), z(x); \lambda) = \frac{d^k}{dx^k} F(x, y(x), z(x); \lambda).$$

Type 2: If the singularity is in the initial point, we define the following problem

$$u''(x) + p(x)u'(x) + (w(x)\lambda - q(x))u(x) = 0, \quad a < x < b$$
$$u(b) = 0 \quad \text{and} \quad u'(b) = s, \tag{4.5}$$

where s is a parameter. The solution of (4.5) is either the solution of (4.2), if the solution of (4.5) $u(x, \lambda)$ satisfies the condition at $x = a$. So we have the following nonlinear equation

$$u(a; \lambda) = 0. \tag{4.6}$$

Before applying improved residual method to (4.5), we choose a value for the parameter s. After using equation (2.36) in Theorem $2.2.6$ in application of the method, we have the following equation

$$
\begin{aligned}
0 &= u(a,\lambda)\\
&= c_0\\
&= nc_{n-1} - (n-1)c_n + \sum_{j=2}^{n} (-1)^j \binom{n}{j} \frac{(n-j)!}{n!} h^j F^{(j-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i); \lambda).
\end{aligned}
\tag{4.7}
$$

Substituting $c_n = 0$, $c_{n-1} = \frac{h}{n}s$ to (4.7), we obtain the following equation

$$
0 = -sh + \sum_{j=2}^{n} (-1)^j h^j \frac{1}{j!} F^{(j-2)}(b, 0, s; \lambda), \tag{4.8}
$$

whose roots are the approximate eigenvalues of (4.5).

Type 3: If the singularities of the problem are in both initial and end points, we divide the interval $[a,b]$ into two subintervals $[a,\sigma]$ and $[\sigma,b]$ by choosing a point $\sigma \in (a,b)$. Then we define the following problems:

$$
\begin{aligned}
&u_1''(x) + p(x)u_1'(x) + (w(x)\lambda - q(x))u_1(x) = 0, \quad a < x < \sigma,\\
&u_1(\sigma) = s_1, \qquad u_1'(\sigma) = s_2,
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
&u_2''(x) + p(x)u_2'(x) + (w(x)\lambda - q(x))u_2(x) = 0, \quad \sigma < x < b,\\
&u_2(\sigma) = s_1, \quad u_2'(\sigma) = s_2,
\end{aligned}
\tag{4.10}
$$

where s_1 and s_2 are parameters. The piecewise function

$$u(x) = \begin{cases} u_1(x,\lambda) & a < x < \sigma, \\ u_2(x,\lambda) & \sigma < x < b, \end{cases}$$

which is obtained using the solutions of (4.9) and (4.10) $u_1(x,\lambda)$ and $u_2(x,\lambda)$, respectively, is the solution of (4.2), if it satisfies the boundary conditions at $x = a$ and $x = b$. So we have the following nonlinear equations

$$u_1(a;\lambda) = 0 \tag{4.11}$$

and

$$u_2(b;\lambda) = 0. \tag{4.12}$$

Before the application of the method, we choose a value for one of the parameters s_1 and s_2, and we denote the unknown one as s. Then we use the improved residual method which is described previously. The usage of the method in (4.9) is the same as described in Type 2 with a parameter s. Since $c_n = s_1$ and $c_{n-1} = s_1 - s_2\frac{h}{n}$, we get

$$\begin{aligned} 0 &= u_1(a,\lambda) \\ &= c_0 \\ &= nc_{n-1} - (n-1)c_n + \sum_{j=2}^{n}(-1)^j\binom{n}{j}\frac{(n-j)!}{n!}h^jF^{(j-2)}(a_i, c_n^i, \frac{n}{h}(c_n^i - c_{n-1}^i);\lambda) \\ &= n(s_1 - s_2\frac{h}{n}) - (n-1)s_1 + \sum_{j=2}^{n}(-1)^jh^j\frac{1}{j!}F^{(j-2)}(\sigma, s_1, s_2;\lambda), \end{aligned}$$

which yields

$$0 = s_1 - s_2h + \sum_{j=2}^{n}(-1)^jh^j\frac{1}{j!}F^{(j-2)}(\sigma, s_1, s_2;\lambda). \tag{4.13}$$

For the approximate eigenvalues of problem (4.10), as in Type 1 we expand the left hand side of the equation (4.12) and using equation (2.13) in Thereom 2.2.3:

$$
\begin{aligned}
0 &= u_2(b,\lambda) \\
&= c_n \\
&= nc_1 - (n-1)c_0 + \sum_{j=2}^{n} \binom{n}{j} \frac{(n-j)!}{n!} h^j F^{(j-2)}\big(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\big); \lambda) \\
&= n(s_1 + s_2\frac{h}{n}) - (n-1)s_1 + \sum_{j=2}^{n} h^j \frac{1}{j!} F^{(j-2)}(\sigma, s_1, s_2; \lambda).
\end{aligned}
$$

The conditions of (4.10) give $c_0 = s_1$ and $c_1 = s_1 + s_2\frac{h}{n}$, so we have

$$
0 = s_1 - s_2 h + \sum_{j=2}^{n} h^j \frac{1}{j!} F^{(j-2)}(\sigma, s_1, s_2; \lambda). \tag{4.14}
$$

The roots of the system of equations (4.13) and (4.14) give us the approximate eigenvalues of (4.2) and the unknown parameter s.

4.2 Numerical Examples

In this section, we deal with the following famous problems as, Boyd equation, Laplace tidal wave equation and Dunsch equation to present the accuracy and the efficiency of the suggested technique. Since the technique for the problems in type 1 is very similar to the technique for the problems which are given in the previous chapter, we give examples for Type 2 and Type 3.

Example 4.2.1. Boyd Equation: The problem is defined as

$$-y''(x) - \frac{1}{x}y(x) = \lambda y(x), \quad x \in (0,1]$$
$$y(0) = 0, \quad y(1) = 0. \tag{4.15}$$

Boyd's problem arise in the context of a problem of classical physics, the eddy motion of the atmosphere about zonally averaged winds and the study of singular potentials in quantum mechanics. The exact eigenvalues of this problem are the square of the zeros of the transcendental equation given by

$$\Phi(\mu) = (2i\mu)^{-1} M_{(2i\mu)^{-1},1/2}(2i\mu) = 0, \tag{4.16}$$

where $M_{k,\beta}(z)$ is a Whittaker function given in Boyd (1981). In equation (4.15), the boundary point 1 is regular, and the initial point 0 is singular, so the problem is in Type 2. Therefore we use equation (4.8) to approximate the eigenvalues of Boyd's problem. In Table 4.1, we give $\wedge_k$ as square of exact solutions of transcendental equation (4.16), λ_k approximate eigenvalues obtained by IRM, $\lambda_{\tilde{k}}$ obtained by the regularized sampling method (RSM) in Chanane (2007) and $\lambda_{\bar{k}}$ obtained by the half interval method (HIM) in Liu (2022). Also in the mentioned table, absolute errors of IRM, RSM and HIM are displayed. One can see that the results of IRM agree with the exact eigenvalues and the accuracy is better than RSM and HIM.

Table 4.1 Exact eigenvalues and approximate eigenvalues obtained by IRM of Boyd equation with the comparison of absolute errors of IRM, RSM and HIM $h = 1$ and $n = 1000$.

k	$\wedge_k$	λ_k	$\lvert\lambda_k - \wedge_k\rvert$	$\lvert\lambda_{\tilde{k}} - \wedge_k\rvert$	$\lvert\lambda_{\bar{k}} - \wedge_k\rvert$
1	7.37399	7.37399	0.0	1.3744×10^{-7}	5.14×10^{-12}
2	36.33601	36.33602	0.0	4.8118×10^{-7}	3.48×10^{-12}
3	85.29258	85.29258	0.0	1.0268×10^{-6}	2.57×10^{-11}
4	154.09862	154.09862	0.0	1.7780×10^{-6}	1.1×10^{-10}
5	242.70556	242.70556	0.0	2.7400×10^{-6}	-
6	351.09117	351.09117	5.68434×10^{-14}	-	-
7	479.24341	479.24341	5.68434×10^{-14}	-	-
10	982.23909	982.23909	0.0	-	-
20	3942.42966	3942.42966	0.0	-	-
30	8876.82700	8876.82700	0.0	-	-
40	15785.26264	15785.26264	0.0	-	-
50	24667.68359	24667.68359	7.27596×10^{-12}	-	-

Example 4.2.2. The Laplace Tidal Wave Equation: The problem is defined as

$$-(\frac{1}{x}y'(x))' + (\frac{m}{x^2} + \frac{m^2}{x})y(x) = \lambda y(x), \quad x \in (0,1),$$
$$y(0) = 0, \quad y(1) = 0, \tag{4.17}$$

where the nonzero parameter $m \in \mathbb{R}$ (Bailey et al. (1991)).

In equation (4.17), endpoint 1 is regular and the initial point 0 is singular, so the problem is in Type 2. Therefore we use equation (4.8) to compute approximate eigenvalues of the problem. There are no representations of solutions of (4.17) in terms of the well known special functions. So, we present our results together with the results of a recent work from the literature. In Table 4.2, the approximate eigenvalues λ_k obtained by IRM, $\lambda_k^\star$ obtained by Chebyshev spectral collocation method from Taher (2019) is given for $m = 1$. Here, we prefer to give results of Chebyshev spectral collocation method, since it computes high-index eigenvalues of singular Sturm-Liouville problems. The given table shows us that, the approximate eigenvalues are well coincide with the results of other technique.

Table 4.2 Approximate eigenvalues of the Laplace tidal wave equation obtained by IRM and Chebyshev spectral collocation method for $m = 1$for $h = 1$ and $n = 500$.

k	λ_k	$\lambda_k^\star$
0	30.39581	30.39581
1	102.44212	102.44212
4	582.80935	582.80935
9	1843.526867	1843.52684
14	4417.85077	4417.85075
19	8102.04342	8102.04327
24	12896.29893	12896.29892
29	18799.70689	18800.70689
39	35698.35801	35698.36794
49	55722.92447	55722.92441

Example 4.2.3. Dunsch Equation: To demonstrate the efficiency and accuracy of the the proposed technique for the problems in Type 3, the Dunsch problem which has exact eigenvalues is chosen to study. The Dunsch equation is

$$y''(x) = \frac{2x}{1-x^2}y'(x) + \frac{1}{1-x^2}\Big(-\lambda + \frac{2\mu^2}{(1+x)} + \frac{2\gamma^2}{(1-x)}\Big)y(x), \tag{4.18}$$

$$y(-1) = 0, \quad y(1) = 0,$$

where the independent parameters $\mu, \gamma \in [0, \infty)$.

Dunsch equation (4.18) has exact eigenvalues given by the explicit formula in Pryce (1993) and Ledoux et al. (2005)

$$\wedge_k = (k + \mu + \gamma + 1)(k + \mu + \gamma), \quad k = 0, 1, 2, \dots. \tag{4.19}$$

In equation (4.18), the end points -1 and 1 are singular, so the problem is in Type 3. Therefore, we use equations (4.13) and (4.14) to approximate the unknown eigenvalues. In Table 4.3, we give computed eigenvalues and absolute errors of the Dunsch problem for $\mu = \gamma = 1$. It is easily seen from the table that, the presented method is in good accuracy, since absolute errors are zero.

Table 4.3 The first fifty eigenvalues and the related absolute errors for the Dunsch Problem, with $\mu = \gamma = 1$ for $h = 1$ and $n = 54$.

k	λ_k	$\lvert\lambda_k - \wedge_k\rvert$	k	λ_k	$\lvert\lambda_k - \wedge_k\rvert$
0	6	0	26	812	0
1	12	0	27	870	0
2	20	0	28	930	0
3	30	0	29	992	0
4	42	0	30	1056	0
5	56	0	31	1122	0
6	72	0	32	1190	0
7	90	0	33	1260	0
8	110	0	34	1332	0
9	132	0	35	1406	0
10	156	0	36	1482	0
11	182	0	37	1560	0
12	210	0	38	1640	0
13	240	0	39	1722	0
15	306	0	40	1806	0
16	342	0	41	1892	0
17	380	0	42	1980	0
18	420	0	43	2070	0
19	462	0	44	2162	0
20	506	0	45	2256	0
21	552	0	46	2352	0
22	600	0	47	2450	0
23	650	0	48	2550	0
24	702	0	49	2652	0
25	756	0	50	2756	0

CHAPTER FIVE
BRATU PROBLEM

Consider the following boundary value problem

$$y'' + \lambda e^y = 0, \quad 0 < x < 1,$$
$$y(0) = 0 \quad \text{and} \quad y(1) = 0, \tag{5.1}$$

which is referred to as the one-dimensional Bratu Problem. This nonlinear eigenvalue problem has known two-bifurcated solution for $\lambda < \lambda_c$ and only one solution for $\lambda = \lambda_c$, but no solution for $\lambda > \lambda_c$. In Buckmire (2003), the exact solution of one dimensional Bratu problem (5.1) is given as

$$y(x) = -2\ln\left[\frac{\cosh((x-\frac{1}{2})\frac{\theta}{2})}{\cosh(\frac{\theta}{4})}\right], \tag{5.2}$$

where θ and λ satisfy

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{5.3}$$

Equation (5.3) has two solutions for $0 < \lambda < \lambda_c$, only one solution for $\lambda = \lambda_c$ which is called critical eigenvalue, no solution for $\lambda > \lambda_c$. The mentioned critical eigenvalue λ_c solves

$$1 = \sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right)\frac{1}{4}. \tag{5.4}$$

From (5.3) and (5.4), one may obtain the followings

$$\theta/4 = \coth(\theta/4) \quad \text{and} \quad \theta_c = 4.798714561030. \tag{5.5}$$

We propose the improved residual method to approximate the critical eigenvalue and eigenfunctions of the one-dimensional Bratu problem (5.1).

5.1 Computation of Eigenfunctions and Critical Eigenvalue by Improved Residual Method

We divide the interval $[0,1]$ into equally spaced subintervals $[0,1/2]$ and $[1/2,1]$ to find approximate initial slopes and critical eigenvalue of Bratu problem. Then, we define the initial value problems in these intervals:

$$
\begin{aligned}
u_1'' + \lambda e^{u_1} = 0, \quad 0 < x < 1/2, \\
u_1(0) = 0, \qquad u_1'(0) = s_1
\end{aligned}
\tag{5.6}
$$

and

$$
\begin{aligned}
u_2'' + \lambda e^{u_2} = 0, \quad 1/2 < x < 1, \\
u_2(1) = 0, \qquad u_2'(1) = s_2.
\end{aligned}
\tag{5.7}
$$

By the continuity of the approximate solution, we have

$$u_1(1/2) = u_2(1/2), \tag{5.8}$$

which implies

$$c_0^2 - c_n^1 = 0, \tag{5.9}$$

where c_n^1 and c_0^2 are the last and first control points of $u_1(x)$ and $u_2(x)$, respectively. Using $c_0^1 = 0$, $c_1^1 = \dfrac{s_1}{2n}$ and Theorem 2.2.3, we have

$$c_n^1 = \frac{s_1}{2} + \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k F^{(k-2)}(0,0,s_1). \tag{5.10}$$

Similarly, using $c_n^2 = 0$, $c_{n-1}^2 = -\dfrac{s_2}{2n}$ and Theorem 2.2.6, we get

$$c_0^2 = -\frac{s_2}{2} + \sum_{k=2}^{n} (-1)^k \frac{1}{k!}(1/2)^k F^{(k-2)}(1,0,s_2). \tag{5.11}$$

Substituting equations (5.10) and (5.11) into equation (5.9), we obtain

$$c_0^2 - c_n^1 = -\frac{s_1+s_2}{2} + \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k \Big((-1)^k F^{(k-2)}(1,0,s_2) - F^{(k-2)}(0,0,s_1) \Big) = 0. \tag{5.12}$$

By using equation (A.36), equation (5.12) becomes

$$\begin{aligned} 0 &= -\frac{s_1+s_2}{2} + (s_1+s_2) \\ &\times \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k \Big((-1)^k \sum_{i=0}^{b_{k-1}} (-1)^{i+1} a_{k-1,i} \lambda^{i+1} \Big[\sum_{j=0}^{k-2i-3} (-1)^j s_2^{k-2i-3-j} s_1^j \Big] \Big), \end{aligned}$$

where $b_j = \lfloor \frac{j-1}{2} \rfloor$ and

$$a_{j,i} = \begin{cases} 1, & i = 0 \\ (i+1)a_{j-1,i} + (j-2i)a_{j-1,i-1} & 1 \le i \le b_{j-1} \\ 0 & else, \end{cases}$$

for $j = 1, \ldots, n-1$. This yields that

$$\begin{aligned} 0 = -(s_1+s_2)\Big(\frac{1}{2} + \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k (-1)^{k-1} \\ \times \Big(\sum_{i=0}^{b_{k-1}} (-1)^{i+1} a_{k-1,i} \lambda^{i+1} \Big[\sum_{j=0}^{k-2i-3} (-1)^j s_2^{k-2i-3-j} s_1^j \Big] \Big) \Big). \end{aligned}$$

Therefore to satisfy equation (5.9) for all n, we must have $s_1 + s_2 = 0$, that is,

$$s_1 = -s_2, \tag{5.13}$$

which coincides with the properties of the exact solution of Bratu problem. In order to satisfy the continuity of first derivative of the approximate solution, we have

$$u_1'(1/2) = u_2'(1/2),$$

which implies that

$$c_n^1 - c_{n-1}^1 = c_1^2 - c_0^2. \tag{5.14}$$

By Theorem 2.2.3, we have

$$c_{n-1}^1 = (n-1)c_1^1 - (n-2)c_0^1 + \sum_{k=2}^{n-1} \binom{n-1}{k} \frac{(n-k)!}{n!} h^k F^{(k-2)}(0,0,s_1)$$

$$= (n-1)\frac{s_1}{2n} + \sum_{k=2}^{n-1} \frac{(n-k)}{nk!}(1/2)^k F^{(k-2)}(0,0,s_1) \tag{5.15}$$

and by Theorem 2.2.6, we have

$$c_1^2 = (n-1)c_{n-1}^2 - (n-2)c_n^2 + \sum_{k=2}^{n-1}(-1)^k \binom{n-1}{k} \frac{(n-k)!}{n!} h^k F^{(k-2)}(1,0,s_2)$$

$$= (n-1)(-\frac{s_2}{2n}) + \sum_{k=2}^{n-1}(-1)^k \frac{(n-k)}{nk!}(1/2)^k F^{(k-2)}(1,0,s_2). \tag{5.16}$$

Substituting c_n^1, c_{n-1}^1, c_1^2 and c_0^2 in equation (5.14) gives

$$\begin{aligned}
0 &= c_n^1 - c_{n-1}^1 - c_1^2 + c_0^2 \\
&= \frac{s_1}{2} + \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k F^{(k-2)}(0,0,s_1) \\
&- (n-1)\frac{s_1}{2n} - \sum_{k=2}^{n-1} \frac{(n-k)}{nk!}(1/2)^k F^{(k-2)}(0,0,s_1) \\
&- (n-1)(-\frac{s_2}{2n}) - \sum_{k=2}^{n-1}(-1)^k \frac{(n-k)}{nk!}(1/2)^k F^{(k-2)}(1,0,s_2) \\
&- \frac{s_2}{2} + \sum_{k=2}^{n}(-1)^k \frac{1}{k!}(1/2)^k F^{(k-2)}(1,0,s_2).
\end{aligned}$$

Thus,

$$\begin{aligned}0 =& \frac{s_1 - s_2}{2} - \frac{n-1}{2n}(s_1 - s_2)\\ &+ \sum_{k=2}^{n} \frac{1}{k!}(1/2)^k \Big(F^{(k-2)}(0,0,s_1) + (-1)^k F^{(k-2)}(1,0,s_2) \Big)\\ &- \sum_{k=2}^{n-1} \frac{(n-k)}{nk!}(1/2)^k \Big(F^{(k-2)}(0,0,s_1) + (-1)^k F^{(k-2)}(1,0,s_2) \Big).\end{aligned}$$

Since second sum is 0 for $k = n$, we obtain the following equation

$$\begin{aligned}&(s_1 - s_2)(\frac{1}{2} - \frac{n-1}{2n}) + \frac{1}{n!}(1/2)^n (F^{(n-2)}(0,0,s_1) + (-1)^n F^{(n-2)}(1,0,s_2))\\ &+ \sum_{k=2}^{n-1} (\frac{1}{k!} - \frac{(n-k)}{nk!})(1/2)^k \Big(F^{(k-2)}(0,0,s_1) + (-1)^k F^{(k-2)}(1,0,s_2) \Big) = 0,\end{aligned}$$

that is,

$$\frac{s_1 - s_2}{2n} + \sum_{k=2}^{n} (\frac{1}{n(k-1)!})(1/2)^k \Big(F^{(k-2)}(0,0,s_1) + (-1)^k F^{(k-2)}(1,0,s_2) \Big) = 0.$$

Shifting k from 1 to $n-1$, we get

$$\frac{1}{n}\Bigg(\frac{s_1 - s_2}{2} + \sum_{k=1}^{n-1} \frac{1}{k!}(1/2)^{k+1} \Big(F^{(k-1)}(0,0,s_1) + (-1)^{k+1} F^{(k-1)}(1,0,s_2) \Big) \Bigg) = 0. \tag{5.17}$$

Since $s_1 = -s_2$, by equation (A.37) we have

$$\begin{aligned}F^{(k-1)}(0,0,s_1) + (-1)^{k+1} F^{(k-1)}(1,0,s_2) &= -((-1)^k F^{(k-1)}(1,0,s_2) - F^{(k-1)}(0,0,s_1))\\ &= 2F^{(k-1)}(0,0,s_1).\end{aligned} \tag{5.18}$$

Substituting (5.18) into (5.17) and using $s_1 = -s_2$, we get

$$\frac{1}{n}\Bigg(s_1 + \sum_{k=1}^{n-1} \frac{1}{k!}(1/2)^k F^{(k-1)}(0,0,s_1) \Bigg) = 0.$$

That is

$$s_1 + \sum_{k=1}^{n-1} \frac{1}{k!}(1/2)^k F^{(k-1)}(0,0,s_1) = 0. \tag{5.19}$$

We name the left-hand side of the equation (5.19) as

$$G(s_1, \lambda) = s_1 + \sum_{k=1}^{n-1} \frac{1}{k!}(1/2)^k F^{(k-1)}(0,0,s_1). \tag{5.20}$$

Drawing the implicit equation $G(s_1, \lambda) = 0$ gives Figure 5.1 which demonstrates that λ has a maximum value. In order to find the mentioned maximum value, the following equation must be satisfied

$$\frac{d\lambda}{ds_1} = -\frac{\partial G/\partial s_1}{\partial G/\partial \lambda} = 0 \quad \text{that is} \quad \frac{\partial G}{\partial s_1} = 0.$$

The nonlinear equations

$$G(s_1, \lambda) = 0 \quad \text{and} \quad \frac{\partial G}{\partial s_1} = 0$$

are solved approximately by Newton's method, so the critical eigenvalue $\lambda_c = 3.51383$ and the corresponding initial value $s1 \simeq y'(0)$ are found approximately. The approximate initial slopes of Bratu problem for the corresponding eigenvalues $\lambda \leq \lambda_c$ are given in Table 5.1. From Figure 5.1 and Table 5.1, it can be seen that there isn't any s_1 for $\lambda > \lambda_c$, there is a unique solution corresponding to the initial value $s_1^1 = s_1$ for $\lambda = \lambda_c$, and there are two solutions corresponding to the initial values s_1^1 and s_1^2 for $\lambda < \lambda_c$ as in the theoretical knowledge of Bratu problem. It is concluded that, the numerical results obtained using improved residual method agree with the exact solution of Bratu problem given in Khuri (2004).

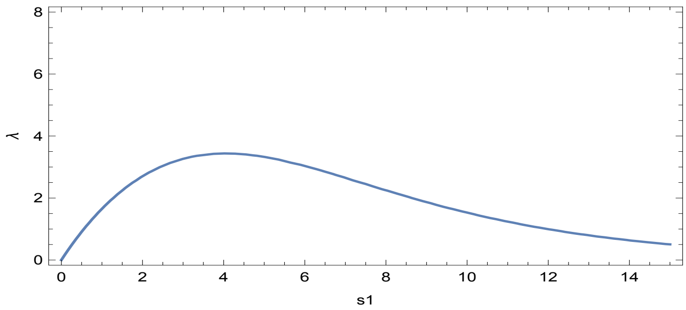

Figure 5.1 Graph of the equation $G(s_1, \lambda) = 0$.

Table 5.1 The approximate initial slopes s_1^1 and s_1^2 corresponding to various $\lambda \leq \lambda_c$ obtained from $G(s_1, \lambda) = 0$ and $\partial G/\partial s_1 = 0$.

λ	s_1^1	s_1^2
0.5	0.26128	13.00803
1	0.54935	10.84669
2	1.24822	8.26876
3	2.31960	6.10338
3.513830719225065	4.	-

5.2 Numerical Results for Bratu Problem

The improved residual method is applied to the one dimensional Bratu Problem. In this section, we give graphs of the errors of the obtained numerical results. We also give the tables of the maximum error moduli and observed orders of the errors. From tables and figures, it can be seen that proposed method has high order accuracy for large step size. It can be also seen that the obtained numerical orders are well confirmed with the theoretical findings for the proposed method. The observed orders are computed using the formula given in equation (3.12).

Table 5.2 Maximum error moduli and observed errors for $\lambda = 1$ and $s_1^1 = 0.549353$.

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$N = 8$	2.96128×10^{-3}	2.6025×10^{-4}	1.48175×10^{-5}	5.06533×10^{-7}
$N = 16$	1.49719×10^{-3}	7.17721×10^{-5}	1.93077×10^{-6}	3.96614×10^{-8}
$N = 32$	7.50268×10^{-4}	1.8823×10^{-5}	2.45018×10^{-7}	2.52492×10^{-9}
Observed Orders				
$ord(1/8)$	0.98396	1.85840	2.94005	3.67485
$ord(1/16)$	0.99678	1.93092	2.97822	3.97343

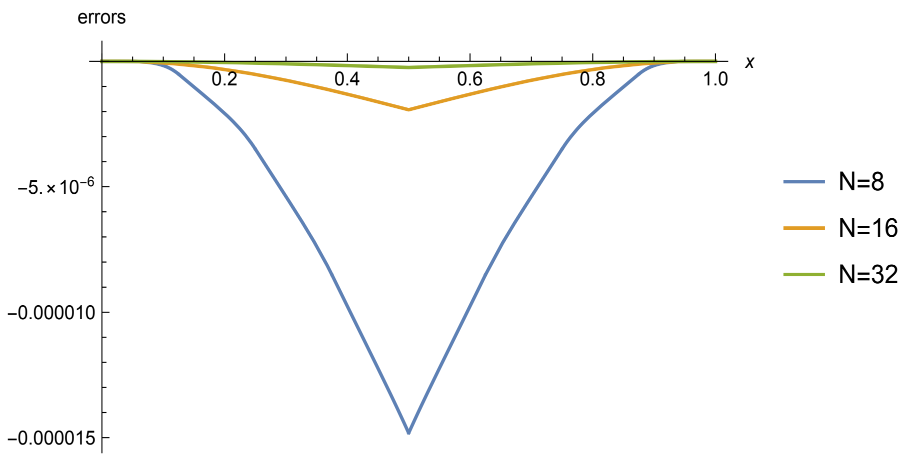

Figure 5.2 Errors for s_1^1 when $\lambda = 1$.

Table 5.3 Maximum error moduli and observed errors for $\lambda = 1$ and $s_1^2 = 10.84669$.

	n=2	**n=3**	**n=4**	**n=5**
$N = 8$	0.43367	0.08392	5.10122×10^{-3}	3.48684×10^{-3}
$N = 16$	0.22631	0.02093	6.19774×10^{-4}	4.87454×10^{-4}
$N = 32$	0.11306	0.00492	5.83269×10^{-5}	6.85824×10^{-5}
Observed Orders				
$ord(1/8)$	0.93827	2.00362	3.04103	2.83858
$ord(1/16)$	1.00120	2.08915	3.40951	2.82936

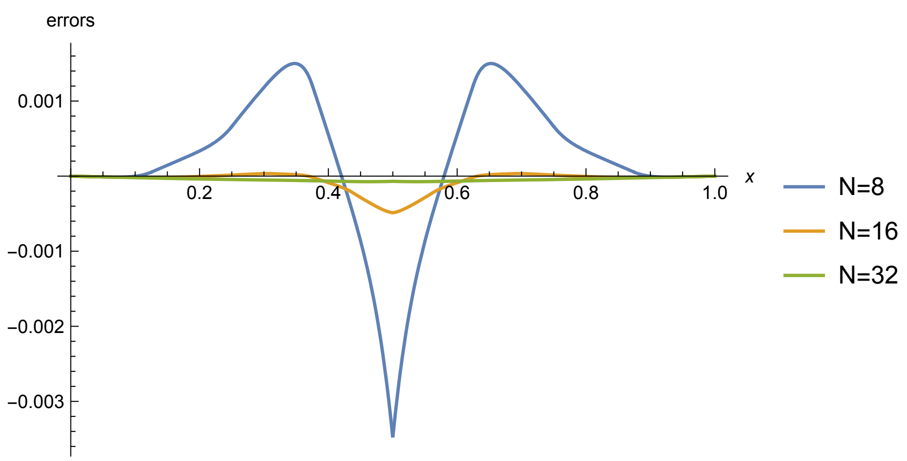

Figure 5.3 Errors for s_1^2 when $\lambda = 1$.

Table 5.4 Maximum error moduli and observed errors for $\lambda = 3$ and $s_1^1 = 2.31960$.

	n=2	**n=3**	**n=4**	**n=5**
$N = 8$	0.04540	9.03418×10^{-4}	6.26397×10^{-4}	4.27077×10^{-5}
$N = 16$	0.02293	3.56031×10^{-4}	8.81698×10^{-5}	2.27439×10^{-6}
$N = 32$	0.01146	1.07455×10^{-4}	1.1604×10^{-5}	1.16448×10^{-7}
Observed Orders				
$ord(1/8)$	0.98564	1.34339	2.82872	4.23094
$ord(1/16)$	1.00002	1.72826	2.92566	4.28772

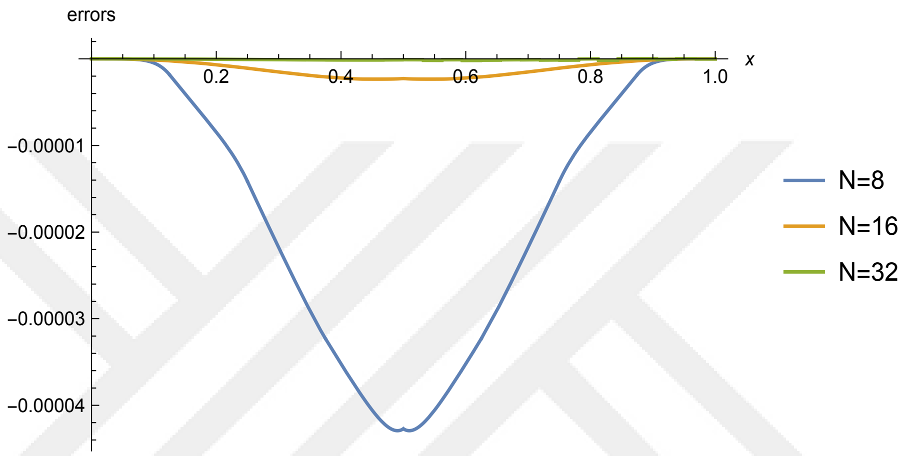

Figure 5.4 Errors for s_1^1 when $\lambda = 3$.

Table 5.5 Maximum error moduli and observed errors for $\lambda = 3$ and $s_1^2 = 6.10338$.

	n=2	**n=3**	**n=4**	**n=5**
$N = 8$	0.21865	0.02046	2.19945×10^{-3}	1.07686×10^{-3}
$N = 16$	0.11106	0.00449	4.82462×10^{-4}	7.73086×10^{-5}
$N = 32$	0.05535	0.00101	7.34435×10^{-5}	4.73904×10^{-6}
Observed Orders				
$ord(1/8)$	0.97732	2.18951	2.18865	3.80006
$ord(1/16)$	1.00473	2.14763	2.71571	4.02796

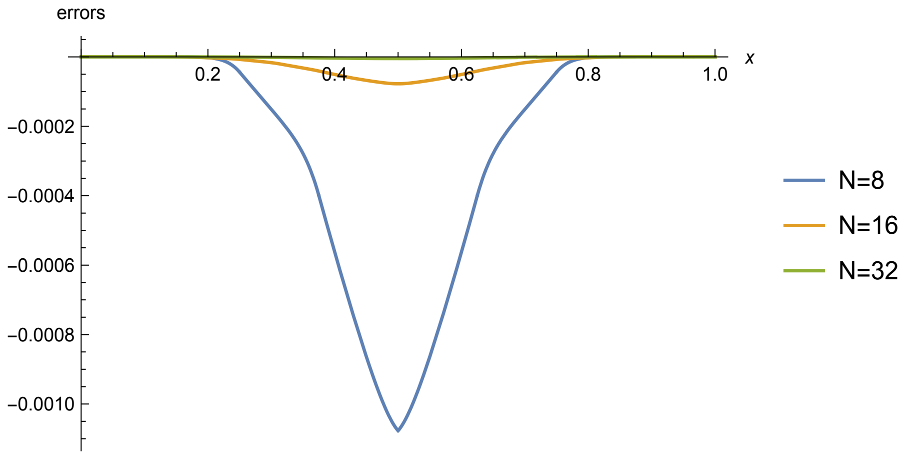

Figure 5.5 Errors for s_1^2 when $\lambda = 3$.

Table 5.6 Maximum error moduli and observed errors for critical eigenvalue $\lambda = 3.513830719$ and $s = 4$.

	n=2	n=3	n=4	n=5
$N = 8$	0.11534	4.95501×10^{-3}	1.69011×10^{-3}	3.20816×10^{-4}
$N = 16$	0.05832	1.12227×10^{-3}	2.68653×10^{-4}	2.02561×10^{-5}
$N = 32$	0.02911	2.66845×10^{-4}	3.71078×10^{-5}	1.23786×10^{-6}
Observed Orders				
$ord(1/8)$	0.98392	2.14247	2.65324	3.98532
$ord(1/16)$	1.00248	2.07235	2.85595	4.03244

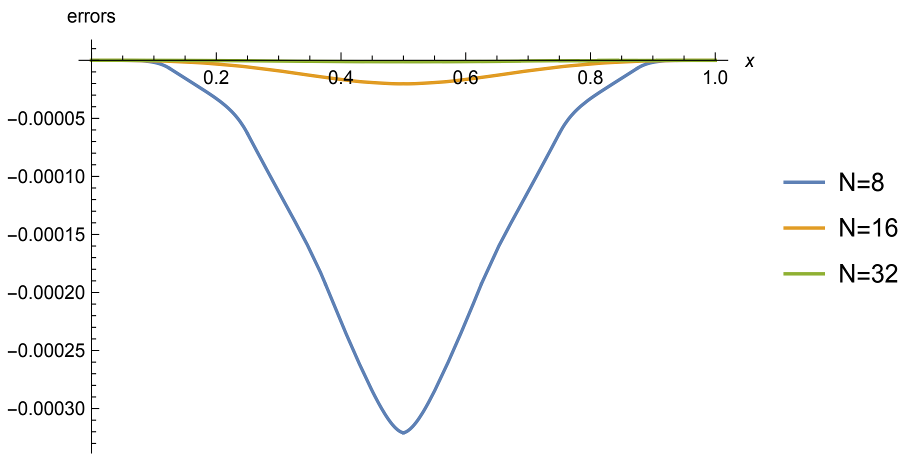

Figure 5.6 Errors for $\lambda_c = 3.513830719$.

CHAPTER SIX
CONCLUSION

In this thesis, at first, the residual method is improved to find the approximate solutions of second order non-linear initial value problems. For this quest, we introduce two new theorems: Theorem 2.2.3 and Theorem 2.2.6. To prove these theorems, we give nice equations by using some classical binomial identities. These theorems enable us to calculate the unknown control points in terms of first or last two control points. By this way, we compute unknown control points explicitly without any linearization or solving any system of equations. Then, we adapt the improved residual method for regular Sturm-Liouville eigenvalue problems to find approximate eigenvalues and approximate eigenfunctions. Furthermore, the convergency of the method for $r(x) = 0$ is demonstrated. It is seen that the numerical results are well confirm with theoretical aspects. We also modified the mentioned method for singular eigenvalue problems. The adaptation of the proposed method varies according to the location of the singularity. Hence, we classify the problems in three types: singularity at the initial point, at the end point, or at both the initial and end points. We deal with some famous problems as, Boyd equation, Laplace tidal wave equation and Dunsch equation to present the accuracy and the efficiency of the suggested technique. At last, we apply the improved method to find approximate solutions of the Bratu's problem which is a nonlinear eigenvalue problem. Comparisons of the numerical results with the theoretical findings show that the improved and adapted method is very effective, so convenient and highly accurate in solving boundary value problems and eigenvalue problems approximately.

For further studies; we suggest that, the proposed method can be improved to find the approximate solutions of n^{th} order initial value problems. Improved residual method can also be modified to find approximate solutions of partial differential equations and chaotic system of equations.

REFERENCES

Adiyaman, M., & Oger, V. (2017). A residual method using Bézier curves for singular nonlinear equations of Lane-Emden type. *Kuwait journal of science*, *44*.

Adiyaman, M., & Somali, S. (2010). Taylor's decomposition on two points for one-dimensional Bratu problem. *Numerical Methods for Partial Differential Equations*, *26*, 412-425.

Adiyaman, M., & Somali, S. (2012). A new approach for linear eigenvalue problems and nonlinear Euler Buckling problem. *Abstract and Applied Analysis*, *2012*, 1085-33750.

Agarwal, R. P., & O'Regan, D. (2008). *An Introduction to Ordinary Differential Equations*. New York: Springer.

Al-Khaled, K., & Hazaimeh, A. (2020). Comparison methods for solving non-linear Sturm–Liouville eigenvalues problems. *Symmetry*, *12*(7 (1179)).

Amodio, P., & Settanni, G. (2015). Variable-step finite difference schemes for the solution of Sturm-Liouville problems. *Communications in Nonlinear Science and Numerical Simulation*, *20*(3), 641-649.

Andrew, A. L., & Paine, J. W. (1985). Correction of Numerov's eigenvalue estimates. *Numerische Mathematik*, *47*, 289-300.

Atkinson, F. V. (1964). *Discrete and Continuous Boundary Value Problems*. New York, London: Academic Press.

Bailey, P., Everitt, W., & Zettl, A. (1991). Computing eigenvalues of singular Sturm-Liouville problems. *Results in Mathematics*, *20*.

Bailey, P. B., Everitt, W., & Zettl, A. (2001). Algorithm 810: The SLEIGN2 Sturm-Liouville code. *ACM Transactions on Mathematical Software*, *27*.

Bailey, P. B., Gordon, M. K., & Shampine, L. F. (1978). Automatic solution of the Sturm-Liouville problem. *ACM Transactions on Mathematical Software*, *4*(3), 193–208.

Birkhoff, G., & Varga, E. S. (1970). *Numerical solution of field problems in continuum physics*. Amer Mathematical Society.

Boyd, J. (1981). Sturm–Liouville eigenproblems with an interior pole. *Journal of Mathematical Physics*, *22*.

Buckmire, R. (2003). On exact and numerical solutions of the one-dimensional planar Bratu problem. *Mathematics eJournal*.

Bujurke, N. M., Salimath, C. S., & Shiralashetti, S. C. (2008). Computation of eigenvalues and solutions of regular Sturm-Liouville problems using Haar wavelets. *Journal of Computational and Applied Mathematics*.

Chanane, B. (2007). Computing the eigenvalues of singular Sturm–Liouville problems using the regularized sampling method. *Applied Mathematics and Computation*, *184*, 972-978.

Chandrasekhar, S. (1957). *An introduction to the study of stellar structure*. New York: Dover Publications Inc.

Chang, J. R., Shen, J. H., & Chen, Y. W. (2022). A boundary shape function method for computing eigenvalues and eigenfunctions of Sturm–Liouville problems. *Mathematics*, *10*, 3689.

Cipu, E. C., & Barbu, C. D. (2022). Variational estimation methods for Sturm–Liouville problems. *Mathematics*, *10(20)*, 3728.

Farin, G., & Hansford, D. (2000). *Essentials of CAGD*. A.K. Peters Ltd.

Frank-Kamenetski, D. A. (1995). *Difussion and heat exchange in chemical kinetics*. Princeton NJ: Princeton University Press.

Gheorghiu, C. I. (2021). Accurate spectral collocation computation of high order eigenvalues for singular Schrödinger equations. *Computation*, *9*(1).

Ghomanjani, F. (2022). A numerical method for solving Bratu's problem. *Palestine Journal of Mathematics*, *2*, 372–377.

Güzel, İ., Adiyaman, M., & Somali, S. (2019). Operator splitting methods for computation of eigenvalues of regular Sturm-Liouville problems.

Jacobsen, J., & Schmitt, K. (2002). The Liouville-Bratu-Gelfand problem for radial operators. *Journal of Differential Equations*, *184*, 283–298.

Karamollahi, N., Loghmani, G. B., & Heydari, M. (2021). A computaional method to find dual solutions of the one-dimensional Bratu problem. *Journal of Computational and Applied Mathematics*, *388*, 113309.

Khuri, S. (2004). A new approach to Bratu's problem. *Applied Mathematics and Computation - AMC*, *147*, 131-136.

Kravchenko, V., & Torba, S. (2018). A Neumann series of Bessel functions representation for solutions of Sturm-Liouville equations. *Calcolo*, *55*.

Kravchenko, V. V. (2023). Spectrum completion and inverse Sturm–Liouville problems. *Mathematical Methods in the Applied Sciences*, *46*(5), 5821-5835.

Ledoux, V., & Van Daele, M. (2010). Solution of Sturm–Liouville problems using modified Neumann schemes. *SIAM Journal of Scientific Computing*, *32*, 563-584.

Ledoux, V., Van Daele, M., & Berghe, G. (2005). Matslise: A matlab package for the numerical solution of Sturm-Liouville and Schrödinger equations. *ACM Transactions on Mathematical Software (TOMS)*, *31*, 532-554.

Ledoux, V., Van Daele, M., & Vanden Berghe, G. (2009). Efficient computation of high index Sturm–Liouville eigenvalues for problems in physics. *Computer Physics Communications*, *180*(2), 241-250.

Liu, C. S. (2022). Accurate eigenvalues for the Sturm-Liouville problems, involving generalized and periodic ones. *Journal of Mathematics Research*, *14*.

Lodhi, R., Aldosary, S. F., Nisar, K. S., & Alsaadi, A. (2022). Numerical solution of non-linear Bratu-type boundary value problems via quintic B-spline collocation method. *AIMS Mathematics*, *7*, 7257–7273.

Magherini, C. (2020). A corrected spectral method for Sturm–Liouville problems with unbounded potential at one endpoint. *Journal of Computational and Applied Mathematics*, *364*, 112349.

Masood, Z., Majeed, K., Samar, R., & Raja, M. (2017). Design of Mexican hat wavelet neural networks for solving Bratu type nonlinear systems. *Neurocomputing*, *221*, 1–14.

Mukhtarov, O. S., Çavuşoğlu, S., & Pandey, P. K. (2021). Development of the finite difference method to solve a new type Sturm-Liouville problems. *Tbilisi Mathematical Journal*, *14*(3), 141–154.

Oger, V. (2016). Numerical solutions of differential equations using residual method. *Phd Thesis*.

Pruess, S., & Fulton, C. T. (1993). Mathematical software for Sturm-Liouville problems. *19*(3).

Pryce, J. D. (1993). *Numerical Solution of Sturm-Liouville Problems*. New York: Oxford University Press.

Ragb, O., Seddek, L. F., & Matbuly, M. S. (2017). Iterative differential quadrature solutions for Bratu problem. *Computers nad Mathematics with Applications*, *74(2)*, 249–257.

Roul, P., & Thula, K. (2019). A fourth-order B-spline collocation method and its error analysis for Bratu-type and Lane–Emden problems. *International Journal of Computer Mathematics*, *96*, 85–104.

Simşek, Y. (2018). Combinatorial sums and binomial identities associated with the beta-type polynomials. *Hacettepe Journal of Mathematics and Statistics*, *47*(5), 1144–1155.

Taher, A. H. S. (2019). Computing high-index eigenvalues of singular Sturm–Liouville problems. *International Journal of Applied and Computational Mathematics*, *5*, 1–13.

Uğur, Y. (2015). Numerical approximations of Sturm–Liouville eigenvalues using Chebyshev polynomial expansions method. *Cogent Mathematics*, *2*(1), 1045223.

Zaitsev, V. F., & Polyanin, A. D. (1995). *Handbook of Exact Solutions for Ordinary Differential Equations*. CRC Press.

APPENDICES

A.1 Bézier curves

We use Bézier curves for the construction of the residual method. Thus, we need to know some properties of Bézier curves. One of the most important mathematical representation of curves and surfaces used in computer graphics and computer-aided design is the Bézier curves. The original development of Bézier curves took place in the automobile industry between 1958 and 1960 by Pierre Bezier and Paul de Casteljau.

Bézier curves are polynomial curves which have a particular mathematical representation. These curves have some nice mathematical properties which facilitate their manipulation and analysis.

A.1.1 The General Bézier Curve

Given $n+1$ control points $c_0, c_1, ..., c_n$ the Bézier curve of degree n is defined as:

$$u(t) = \sum_{i=0}^{n} c_i B_i^n(t; [a, b]), \tag{A.1}$$

where

$$B_i^n(t; [a, b]) = \binom{n}{i} \left(\frac{t-a}{b-a}\right)^i \left(\frac{b-t}{b-a}\right)^{n-i}$$

for $i = 0, 1, ..., n$ are Bernstein polynomials over the interval $[a, b]$. The representasion $(A.1)$ is called Bernstein-Bézier form (or Bézier form for brevity). If $u(t)$ is a vector valued polynomial, $u(t)$ is a parametric Bézier curve and the Bézier coefficeints c_j are the control points. If $u(t)$ is a scalar valued polynomial, we call the function $y = u(t)$ an explicit Bézier curve given by $(t, u(t))$. The control points of an explicit curve are $(j/n, c_j)$, where the x coordinates j/n are called the Grevilla abscissa.

Bézier curves interpolate first and last control points. This property is called the end point interpolation which is

$$u(a) = c_0$$

and

$$u(b) = c_n.$$

Many operators on the Bézier curves can be performed using control points. For example the derivatives of Bézier curves can be written as

$$\frac{d^k}{dt^k}u(t) = \frac{1}{(b-a)^k}\frac{n!}{(n-k)!}\sum_{j=0}^{n-k} \triangle^k c_j B_j^{n-k}(t;[a,b])$$

for $k = 0, 1, ..., n,$ where $\triangle$ is the forward difference operator such that $\triangle c_j = c_{j+1} - c_j$. In particular, derivatives at the end points can be written simply as

$$\left.\frac{d^k}{dt^k}u(t)\right|_{t=a} = \frac{1}{(b-a)^k}\frac{n!}{(n-k)!}\triangle^k c_0 \tag{A.2}$$

$$\left.\frac{d^k}{dt^k}u(t)\right|_{t=b} = \frac{1}{(b-a)^k}\frac{n!}{(n-k)!}\triangle^k c_{n-k}. \tag{A.3}$$

These formulas help us to connect Bézier segments with C^k continuity and to form a Bézier spline. For example, the curve which consists of the following two Bézier segments

$$u_1(t) = \sum_{j=0}^{n} c_j^1 B_j^n(t;[a,b])$$

and

$$u_2(t) = \sum_{j=0}^{n} c_j^2 B_j^n(t;[a,b])$$

satisfies C^1 continuity, if control points satisfy the following equations

$$c_0^2 = c_n^1$$

and

$$c_1^2 - c_0^2 = c_n^1 - c_{n-1}^1.$$

A.2 Residual Method to the Initial Value Problems

In this section, we give the brief description of the residual method for second order non-linear initial value problems, which is constructed in (Adiyaman & Oger (2017) and Oger (2016)). Consider the second order non-linear initial value problems

$$y''(x) = F(x, y(x), y'(x)) \tag{A.4}$$

with initial conditions

$$y(a) = \alpha \quad y'(a) = \beta, \tag{A.5}$$

where $F \in C^{n-1}([a, b] \times D_1 \times D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$, α, β are finite constants and n is the degree of approximate Bézier curve. We use

$$a_i = a + ih, \quad i = 1, 2, \ldots, N,$$

to divide the interval $[a, b]$ into equally spaced sub-intervals $[a_{i-1}, a_i]$, where $h = \dfrac{b-a}{N}$ and N is the number of subintervals. We write the initial value problem (A.4) and (A.5) piecewisely as

$$y_i''(x) = F(x, y_i(x), y_i'(x)), \quad x \in S_i, \tag{A.6}$$

where $S_i = [a_{i-1}, a_i]$ for $i = 1, \ldots, N$ and

$$\begin{gathered} y_1(a_0) = \alpha, \quad y_1'(a_0) = \beta, \\ y_i(a_{i-1}) = y_{i-1}(a_{i-1}), \quad y_i'(a_{i-1}) = y_{i-1}'(a_{i-1}) \end{gathered} \tag{A.7}$$

for $i = 2, \ldots, N$. Our aim is to approximate to the exact solution $y_i(x)$ over the interval S_i using a Bézier curve, so we define the n^{th} degree Bézier curve as

$$u_i(x) = \sum_{j=0}^{n} c_j^i B_j^n(x; S_i), \quad x \in S_i, \tag{A.8}$$

where

$$B_j^n(x;S_i) = \binom{n}{j}\frac{1}{h^n}(x-a_{i-1})^j(a_i-x)^{n-j}$$

is the Bernstein polynomials over the interval S_i and c_j^i are unknown control points. Notice that in (A.8), there are $n+1$ number of unknown control points. Applying the initial conditions to the approximate solution, we obtain

$$u_1(a_0) = \alpha, \quad u_1'(a_0) = \beta. \tag{A.9}$$

Since the given differential equation is second order, we need continuously differentiable approximate solution. For this, $u_i(x)$ must satisfy

$$u_i(a_{i-1}) = u_{i-1}(a_{i-1}), \quad u_i'(a_{i-1}) = u_{i-1}'(a_{i-1}) \tag{A.10}$$

for $i = 2, \ldots, N$. Thus, we can write $u(x)$ piecewise as

$$u(x) = \begin{cases} u_1(x), & \text{if } x \in S_1 \\ u_2(x), & \text{if } x \in S_2 \\ \vdots & \\ u_N(x), & \text{if } x \in S_N, \end{cases} \tag{A.11}$$

which is continuously differentiable. To find the approximate solution $u(x)$, we have to evaluate the unknown control points. First, we calculate the first two control points from the initial conditions in each sub-interval. In order to do this, we use the end point property and the end point derivative property of Bézier curves in $(A.7)$ which gives

$$\begin{aligned} c_0^1 &= \alpha, \\ c_1^1 &= \beta\frac{h}{n} + \alpha \end{aligned} \tag{A.12}$$

for the first interval. In the other sub-intervals, using (A.2) and (A.3) first two control points are obtained as

$$\begin{aligned} c_0^i &= c_n^{i-1}, \\ c_1^i &= 2c_n^{i-1} - c_{n-1}^{i-1}, \end{aligned} \tag{A.13}$$

for $i = 2, \ldots, N$. Thus, C^1 continuity of Bézier curves are satisfied and from on now there are $n - 1$ unknown control points for each sub-interval S_i.

Putting (A.8) into the given differential equation (A.6), we define the piecewise residual function as

$$R(x) = \begin{cases} R_1(x), & \text{if } x \in S_1 \\ R_2(x), & \text{if } x \in S_2 \\ \vdots \\ R_N(x), & \text{if } x \in S_N, \end{cases} \tag{A.14}$$

where

$$R_i(x) = u_i''(x) - F(x, u_i(x), u_i'(x)), \quad x \in S_i. \tag{A.15}$$

The purpose is to find unknown control points c_j^i that minimize the Taylor's series expansion of sufficiently differentiable residual function $R_i(x)$ at $x = a_{i-1}$. For this, the strategy is the same as in Chapter 2. We force the first $n-1$ terms in Taylor's expansion of $R_i(x)$ at $x = a_{i-1}$ be zero to minimize the residual function $R_i(x)$. So, we consider

$$\begin{aligned} R_i(a_{i-1}) &= 0 \\ R_i'(a_{i-1}) &= 0 \\ R_i''(a_{i-1}) &= 0 \\ &\vdots \\ R_i^{(n-2)}(a_{i-1}) &= 0. \end{aligned} \tag{A.16}$$

Then, using equations (A.2), (A.3), (A.12) and (A.13), for $k = 1, \ldots, n-2$, equations

(A.16)

$$R_i^{(k)}(a_{i-1}) = u_i^{(k+2)}(a_{i-1}) - F^{(k)}(a_{i-1}, u_i(a_{i-1}), u_i'(a_{i-1})) = 0$$

become linear equations

$$\frac{n!}{(n-k-2)!\ h^{k+2}}\Delta^{k+2}c_0^i - F^{(k)}\left(a_{i-1}, c_0^i, \frac{n}{h}(c_1^i - c_0^i)\right) = 0,$$

where

$$F^{(k)}(x, y(x), z(x)) = \frac{d^k}{dx^k}F(x, y(x), z(x))$$

and

$$\Delta^{k+2}c_0^i = \sum_{j=0}^{k+2}\binom{k+2}{j}(-1)^{k+2-j}c_j^i.$$

Above linear equations give the following lower triangular system

$$\begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ -3 & 1 & 0 & \dots & 0 \\ 6 & -4 & 1 & 0 & \vdots \\ \vdots & & & \ddots & 0 \\ (-1)^{n-2}\binom{n}{n-2} & (-1)^{n-3}\binom{n}{n-3} & \dots & (-1)\binom{n}{1} & 1 \end{bmatrix} \begin{bmatrix} c_2^i \\ c_3^i \\ c_4^i \\ \vdots \\ c_n^i \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{bmatrix},$$

where

$$g_j = (-1)^{j-1}\ c_0^i + (-1)^{j-2}\ j\ c_1^i - \frac{(n-j)!}{n!}\ h^{j+1}\ F^{(j-2)}\left(a_{i-1}, \alpha, \frac{n}{h}(c_1^i - c_0^i)\right).$$

The unknown control points c_j^i are determined using above lower triangular system in each subintervals for $j = 2, 3, \dots, n$. Thus, we have the approximate solution $u_i(x)$ such that $R_i(x)$ will be minimum in S_i, for $i = 1, \dots, N$.

A.3 Error Analysis of the Residual Method for Initial Value Methods

In this section, using some lemmas and theorems given in Adiyaman & Oger (2017), Adiyaman & Somali (2010) and Oger (2016), we give an upper bound for the error $|y(x) - u(x)|$, where $y(x)$ is the exact solution of the second order non-linear initial value problem (A.4) and $u(x)$ be the corresponding n^{th} degree approximate function (A.11). The proofs of following lemmas and theorem are given in Oger (2016).

Lemma A.3.1. *Residual functions $R_i(x)$ are order of $n-1$ for $i = 1, \dots, N$, where n is the degree of Bézier curve and N is the number of intervals.*

Lemma A.3.2. *Let $\tilde{u}_i(x)$ be the n^{th} degree auxiliary approximate solution of piecewise initial value problem (A.6) with initial conditions (A.7) as shown in Figure A.1. Then, we have*

$$y_i^{(l)}(a_{i-1}) = \tilde{u}_i^{(l)}(a_{i-1}) \tag{A.17}$$

where $y_i(x)$ is the corresponding exact solution.

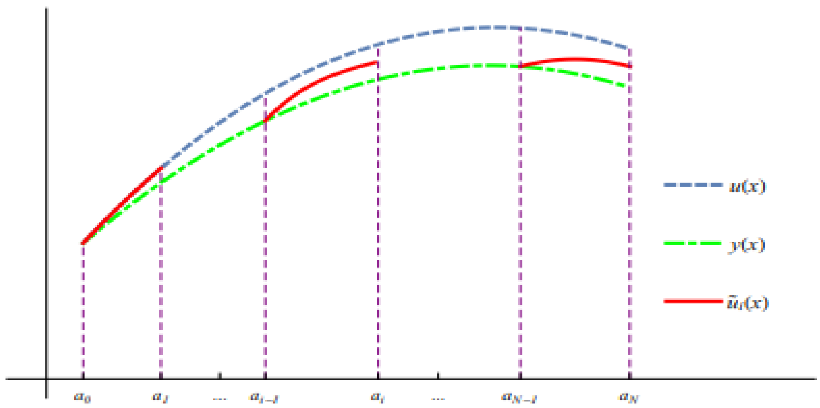

Figure A.1 An illustration of auxiliary approximate solutions $\tilde{u}_i(x)$, approximate solution $u(x)$ and exact solution $y(x)$ of non-linear initial value problem (A.4) on $[a_0, a_N]$.

Lemma A.3.3. *Let $\tilde{u}_i(x)$ be the n^{th} degree auxiliary approximate solution and $y_i(x)$ be the exact solution of piecewise initial value problem (A.6) with initial conditions (A.7), then*

$$|y_i(x) - \tilde{u}_i(x)| \leq K h^{n+1} \tag{A.18}$$

and

$$|y_i'(x) - \tilde{u'}_i(x)| \leq (n+1)Kh^n, \tag{A.19}$$

where $x \in S_i$ *and*

$$K = \frac{1}{(n+1)!} \max_{x\in[a,b]} |y^{(n+1)}(x)|. \tag{A.20}$$

Lemma A.3.4. *Let* $\tilde{u}_i(x)$ *be the* n^{th} *degree auxiliary approximate solution of piecewise initial value problem (A.6) with initial conditions (A.7) and* $u_i(x)$ *be the approximate solution of (A.6) with initial conditions (A.9) or (A.10), then*

$$\begin{aligned} |y_i(x) - \tilde{u}_i(x)| \leq |u_{i-1}(a_{i-1}) - y_{i-1}(a_{i-1})| \left(1 + (\frac{1}{h^2!} + \ldots + \frac{1}{h^n!})C_1\right) \\ + |u_{i-1}'(a_{i-1}) - y_{i-1}'(a_{i-1})| \left(1 + (\frac{1}{h^2!} + \ldots + \frac{1}{h^n!})C_2\right), \end{aligned} \tag{A.21}$$

where $x \in S_i$*,*

$$C_1 = \max_{k=0,1,\ldots,n-2} \left\{ \max_{(x,v,z)\in\Omega} |F_v^{(k)}(x,v,z)| \right\} \tag{A.22}$$

and

$$C_2 = \max_{k=0,1,\ldots,n-2} \left\{ \max_{(x,v,z)\in\Omega} |F_z^{(k)}(x,v,z)| \right\} \tag{A.23}$$

for $\Omega = \{(x,v,z)|x \in [a,b], v \in D_1, z \in D_2\}$*,* D_1 *and* D_2 *are closed intervals in* $\mathbb{R}$*.*

Lemma A.3.5. *Let* $y_i(x)$ *be the exact solution of (A.6) with initial conditions (A.7),* $\tilde{u}_i(x)$ *be the* n^{th} *degree auxiliary approximate solution of piecewise initial value problem (A.6) with initial conditions (A.7) and* $u_i(x)$ *be the approximate solution of (A.6) with initial conditions (A.9) or (A.10), then*

$$\begin{aligned} |y_i'(x) - \tilde{u'}_i(x)| \leq |u_{i-1}(a_{i-1}) - y_{i-1}(a_{i-1})| \left(h + (\frac{h^2}{2!} + \ldots + \frac{h^{n-1}}{(n-1)!})C_1\right) \\ + |u_{i-1}'(a_{i-1}) - y_{i-1}'(a_{i-1})| \left(1 + (\frac{h^2}{2!} + \ldots + \frac{h^{n-1}}{(n-1)!})C_2\right) \\ + (n+1)Kh^n, \end{aligned} \tag{A.24}$$

where $x \in S_i$*,* C_1 *and* C_2 *are given as in (A.22) and (A.23) and* K *is given as in Equation* $(A.20)$.

Theorem A.3.6. *Let $y(x)$ be the exact solution of the second order non-linear initial value problem (A.4) and $u(x)$ be the corresponding n^{th} degree approximate function (A.11). Then, we have the inequality*

$$|y(x) - u(x)| \leq Mh^{(n-1)}, \qquad x \in [a, b], \tag{A.25}$$

where $M = K(b-a)^2(n+1)$ and K is defined as in (A.20).

A.4 Adaptation of the Residual Method to the Boundary Value Problems

In this section, we give an adaptation of the residual method to the boundary value problems, which is given in Oger (2016) in the form

$$y''(x) = F(x, y(x), y'(x)), \tag{A.26}$$

with the boundary conditions

$$y(a) = \alpha, \qquad y(b) = \beta, \tag{A.27}$$

where $F \in C^{n-2}([a, b] \times D_1 \times D_2)$, D_1 and D_2 are closed intervals in $\mathbb{R}$, $a,\ b,\ \alpha,\ \beta$ are finite constants and n be the degree of the Bézier curves. Consider the following initial value problem

$$\begin{aligned} y'' &= F(x, y, y'), \quad x \in [a, b], \\ y(a) &= \alpha, \qquad y'(a) = s, \end{aligned} \tag{A.28}$$

where s is the root of equation

$$y(b; s) - \beta = 0. \tag{A.29}$$

Thus, the solutions of the initial value problem (A.28) satisfying the condition (A.29) are also the solutions of the boundary value problem (A.26) and (A.27). In this subsection, it is aimed to find the approximate value s^* of s and approximate solution $u(x; s^*)$ of $y(x; s)$ using the improved residual method. To achieve this goal, the method is applied to (A.28) with a high degree approximating polynomial and large step-size $h < 1$. Therefore, the unknown coefficients c_k^i of the approximate polynomial $u(x; s)$ are computed in terms of the parameter s. In order to be an approximate solution for the boundary value problem (A.26) and (A.27), the approximate solution $u(x; s)$, which depends on the parameter s, is substituted into (A.29),

$$u(x; s) - \beta = 0. \tag{A.30}$$

Equation (A.30) is solved approximately by Newton's method with the starting value α. Thus, the approximate value s^* of s is found. Finally, the following perturbed initial value problem

$$y''(x) = F(x, y(x), y'(x)), \qquad x \in [a, b]$$
$$y(a) = \alpha \qquad y'(a) = s^*, \tag{A.31}$$

where s^* is the approximate root of (A.30), is obtained. Now, the improved residual method can be applied to (A.31) to find a more accurate approximate solution to (A.26) and (A.27).

A.5 Error Analysis of Residual Method for Boundary Value Problems

In this section, we present the following lemma and theorem, which prove that using accurate approximate value for initial slope does not change the order of convergence, given in Oger (2016).

Lemma A.5.1. *Let the function $F(x, v, z)$ in* (A.26) *satisfy*

$$0 < \frac{\partial F}{\partial v} < L_1$$

for some positive constant L_1. Then

$$|s - y'(a)| \leq Ch^n.$$

where C is a positive constant, y is the exact solution of the boundary value problem (A.26) and (A.27), s is the approximate root of (A.30) *and h is the step size of residual method.*

Lemma A.5.2. *Let the function $F(x, v, z)$ in* (A.26) *satisfy $0 < \frac{\partial F}{\partial v} < L_1$ for some positive constant L_1. Then*

$$|s^* - y'(a)| \leq \widetilde{K}h^n, \tag{A.32}$$

where $\widetilde{K}$ is a positive constant, y is the exact solution of the boundary value problem (A.26) *and* (A.27)*, s^* is the approximate root of* (A.30) *and h is the step size of the residual method.*

Theorem A.5.3. *Let $y(x)$ be the exact solution of the second order non-linear boundary value problem* (A.26) *and* (A.27)*, and $u(x)$ be the n^{th}degree approximate solution of (A.31), then we have the inequality*

$$|y(x) - u(x)| \leq \overline{M} h^{n-1}, \qquad x \in [a, b], \tag{A.33}$$

where

$$\overline{M} = \overline{K}(b-a)^2(n+1), \qquad \overline{K} = \widetilde{K} + K,$$

$$K = \frac{1}{(n+1)!} \max_{x \in [a,b]} |y^{(n+1)}(x)|$$

and $\widetilde{K}$ is a positive constant given in Lemma A.5.2.

A.6 Some Important Lemmas

We need the following lemmas which are given and proved in Adiyaman & Somali (2010) for Bratu Problems .

Lemma A.6.4. *For $j = 1, \ldots, n$, let $F^{(j)}(x, y, z)$ satisfies the recurrence relation*

$$F^{(j)}(x, y, z) = z\frac{\partial F^{(j-1)}(x, y, z)}{\partial y} - \lambda e^{y}\frac{\partial F^{(j-1)}(x, y, z)}{\partial z} \tag{A.34}$$

with $F^{(0)}(x, y(x), z(x)) = -\lambda e^{y}$ and $z(x) = y'(x)$. Then

$$F^{(j-1)}(x, y(x), z(x)) = \sum_{i=0}^{b_j}(-1)^{i+1}a_{j,i}\lambda^{i+1}(e^{y})^{i+1}z^{j-2i-1}, \tag{A.35}$$

where $b_j = \lfloor\frac{j-1}{2}\rfloor$ and

$$a_{j,i} = \begin{cases} 1, & i = 0 \\ (i+1)a_{j-1,i} + (j-2i)a_{j-1,i-1} & 1 \le i \le b_{j-1} \\ 0 & else. \end{cases}$$

Lemma A.6.5. *If $F^{(j)}(x, y, z)$ satisfies* (A.35)*, then for $j = 2, \ldots, n+1$, the following relations hold: (i)*

$$\begin{aligned}(-1)^{k}F^{(k-2)}(1, 0, s_2) - F^{(k-2)}(0, 0, s_1) = (s_1 + s_2)& \\ (-1)^{k}\sum_{i=0}^{b_{k-1}}(-1)^{i+1}a_{k-1,i}\lambda^{i+1}\left(\sum_{j=0}^{k-2i-3}(-1)^{j}s_2^{k-2i-3-j}s_1^{j}\right)&,\end{aligned} \tag{A.36}$$

for any fixed s_1 and s_2, (ii)

$$(-1)^{k}F^{(k-1)}(1, 0, s_1) - F^{(k-1)}(0, 0, s_2) = -2F^{(k-1)}(0, 0, s_2) \tag{A.37}$$

for $s_2 = -s_1$.