REPUBLIC OF TÜRKİYE
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

SUBALGEBRAS OF LATTICE ORDERED ALGEBRAS

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Dedicated to my best friend...

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Ebru KILIÇ

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LIST OF SYMBOLS

$\{x\}$	Band generated by x
$\cong$	Congruence
$\rightarrow$	Converging
$\downarrow$	Decreasing
$\forall$	For all
$\uparrow$	Increasing
$\wedge$ and $\bigwedge$	Infimum
$\mathbb{N}$	Natural numbers
$\mathcal{I}_x$	Order ideal generated by x
$\mathcal{A}^+$	Positive cone of $\mathcal{A}$
$\mathbb{R}$	Real Numbers
C($\mathcal{X}$)	Set of all real continuous functions on $\mathcal{X}$
$C_b(\mathcal{X})$	Set of all bounded real continuous functions on $\mathcal{X}$.
$\vee$ and $\bigvee$	Supremum

LIST OF ABBREVIATIONS

Iff	If and only if
inf	Infimum
$Ker(\varphi)$	Kernel of γ
ℓ-ideal	Order and ring ideal
o-dense	Order dense
o-ideal	Order ideal
$\mathcal{Orth}(\mathcal{A})$	All orthomorphisms on $\mathcal{A}$
RHS	Right hand side
R(γ)	Image (Range) space of γ
r^u-closed	Relatively uniformly closed
r^u-complete	Relatively uniformly complete
r-ideal	Ring ideal
sup	Supremum
u-convergent	Uniform convergent

ABSTRACT

SUBALGEBRAS OF LATTICE ORDERED ALGEBRAS

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Department of Mathematics
Master of Science Thesis

Supervisor: Prof. Dr. Ömer GÖK

This study explores the relationships between vector sublattices and subalgebras of Archimedean unitary f-algebras. The primary focus is on analysing the sufficient and necessary conditions under which a vector sublattice becomes a subalgebra. The 'Stone condition' emerges as a pivotal criterion, proving to be both necessary and sufficient for these relationships, particularly demonstrating that a r^u-closed vector sublattice of an Archimedean unitary f-algebra, possessing a unity becomes a subalgebra. Furthermore, the study delves into the relationship between a lattice homomorphism and an algebra homomorphism, by making observations regarding a Markov operator which maps two Archimedean unitary f-algebras. In addition to these, with a specific focus on their image spaces, our objective is to enhance the comprehension of lattice homomorphisms that map f-algebras. By introducing a product structure into the image space of a lattice homomorphism, it can be considered as an Archimedean f-algebra. Moreover, under specific conditions, the image space can be identified as an f-subalgebra of a semiprime f-algebra. Focusing on these key aspects, this work attempts to reveal valuable insights about lattice homomorphisms between f-algebras.

Keywords: Archimedean property, Riesz space, order unit, Riesz homomorphism, lattice ordered subalgebra, f-algebra.

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ÖZET

LATİS SIRALI CEBİRLERİN ALT CEBİRLERİ

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Matematik Anabilim Dalı
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Bu çalışma, latis alt uzayları ile Arşimet birimsel f-cebirlerinin alt cebirleri arasındaki ilişkileri araştırmaktadır. Birincil odak noktası, bir lattice alt uzayının bir alt cebir haline gelmesi için gerekli ve yeterli koşulu analiz etmektir. 'Stone koşulu' bu ilişki için hem gerekli hem de yeterli olan çok önemli bir kriter olarak ortaya çıkmakta ve özellikle, birim eleman içeren nispeten düzgün kapalı bir lattice alt uzayının, Arşimet birimsel f-cebirinin bir alt cebiri haline geldiğini göstermektedir. Ayrıca çalışma, iki Arşimet birimsel f-cebiri arasındaki Markov operatörüne ilişkin gözlemler yaparak, f-cebirlerinin cebir homomorfizması ile lattice homomorfizmaları arasındaki bağı araştırmaktadır. Daha sonra, özellikle görüntü uzayına odaklanılarak f-cebirleri arasındaki latis homomorfizmalarının anlaşılmasına katkıda bulunulması amaçlanmaktadır. Bir latis homomorfizminin görüntü uzayına bir çarpım yapısı getirilerek, görüntü uzayının bir Arşimet f-cebiri olarak düşünülebileceği ve buna ek olarak, belirli koşullar altında görüntü uzayının, yarı asal bir f-cebirinin bir f-alt cebiri olarak tanımlanabilceği gösterilmektedir. Bu temel yönlere odaklanan söz konusu çalışma, f-cebirleri arasındaki latis homomorfizmaları hakkında değerli bilgiler ortaya çıkarmaya hedeflemektedir.

Anahtar Kelimeler: Arşimet özelliği, Riesz uzayı, sıra birim, Riesz homomorfizması, lattis sıralı alt cebir, f-cebiri

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YILDIZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

1
INTRODUCTION

The term "f-algebra" was introduced by G. Birkhoff and R.S. Pierce in 1956 and it is a concept within the broader field of functional analysis that combines vector lattice structures with algebraic structure. Some notable examples of f-algebras include $C(X)$ (where X is a topological space), spaces of Baire functions and measurable functions. In addition, lattice homomorphisms, which preserve both the lattice and vector space operations, play a crucial role in the study of f-algebras. An important theorem, established by Hager and Robertson [1], asserts that a lattice homomorphism between two Archimedean unitary f-algebras, that preserves the unity, is also an algebra homomorphism. Both f-algebras and lattice homomorphisms still continue to be significant in functional analysis and related areas of mathematics.

This study comprises three main chapters.

In Chapter 2, we delve into the exploration of f-algebras, vector lattices, and related subjects, offering fundamental insights. A comprehensive review of existing literature is provided, highlighting crucial theorems and definitions that form the basis for subsequent chapters.

Moving to Chapter 3, we closely examine the content of the article [2], focusing on the f-algebras and vector lattices and the relationship between them in detail. A key theorem emerges, showing that every r^u-closed vector sublattice of an Archimedean unitary f-algebra becomes a subalgebra.

In Chapter 4, the exploration shifts to the image space of a lattice homomorphism between f-algebras. Specifically, for γ mapping from an Archimedean f-algebra $\mathscr{A}$ to an Archimedean vector lattice $\mathscr{B}$, we demonstrate that the formula $\gamma(w) * \gamma(v) = \gamma(wv)$ defines a multiplication in R(γ), making it an Archimedean f-algebra.

Assuming $\mathscr{A}$ is unitary and $\mathscr{B}$ is semi-prime f-algebras, we examine the necessary and satisfactory conditions for R(γ) to transform into an f-subalgebra of $\mathscr{B}$.

2

PRELIMINARIES

Definition 2.1. Let $\emptyset \neq \mathscr{E}$ be a set where both scalar multiplication and addition are defined. In that case, $\mathscr{E}$ is called a real vector space if the axioms below are satisfied $\forall v, w$ and $u \in \mathscr{E}$ and $\forall a, b \in \mathbb{R}$.

1. $w + u \in \mathscr{E}$ *(closure under addition)*,
2. $w + u = u + w$ *(commutative property)*,
3. $w + (u + v) = (w + u) + v$ *(associative property)*,
4. $\mathscr{E}$ has an *additive identity (zero element)* such that $w + 0 = 0 + w = 0$,
5. Each w has an *additive inverse* in $\mathscr{E}$ such that $w + (-w) = (-w) + w = 0$,
6. $aw \in \mathscr{E}$ *(closure under scalar multiplication)*,
7. $a(w + u) = aw + au$ *(distributive property)*,
8. $(a + b)w = aw + bw$ *(distributive property)*,
9. $a(bw) = (ab)w$ *(associative property)*,
10. $1 \in \mathbb{R}$ is a *scalar identity* such that $1t = t$.

Definition 2.2. The relation denoted as "$\leq$" is termed as an order relation within the vector space $\mathscr{E}$ whenever the following conditions are met.

1. $w \leq v$ and $v \leq u$ imply $w \leq u$ (*transitive*),
2. $w \leq w$ holds for each $w \in \mathscr{E}$ (*reflexive*),
3. $w \leq v$ and $v \leq w$ imply $w = v$ (*antisymetric*).

Definition 2.3. A real vector space $\mathscr{E}$ is said to be an ordered vector space when the following conditions are met.

1. "$\leq$" is an order relation on $\mathscr{E}$,
2. $\mathscr{E}$ is linearly ordered, i.e., if $w \leq v \in \mathscr{E}$ then $w + u \leq v + u$ and $\alpha w \leq \alpha v$ hold for all $r \in \mathscr{E}$ and all real numbers $\alpha \geq 0$.

Definition 2.4. An ordered vector space $\mathscr{E}$ is termed as a vector lattice (or Riesz space) when $\forall w, v \in \mathscr{E}$, $\sup\{w, v\}$ and $\inf\{w, v\}$ are present in E. The notations used in this study are as follows:

$$w \vee u = \sup\{w, u\} \quad \text{and} \quad w \wedge u = \inf\{w, u\}.$$

Notice that, if the infimum exists then the supremum exists as well and vice versa.

Theorem 2.1. *The statements below are mutually equivalent for all w, u in the vector lattice $\mathscr{E}$.*

1. $w \leq u$.
2. $w \wedge u = w$.
3. $w \vee u = u$.

Theorem 2.2. *The following are satisfied for all w, v and u in the vector lattice $\mathscr{E}$.*

1. $w \vee w = w$ *and* $w \wedge w = w$,
2. $w \leq w \vee u$ *and* $u \leq w \vee u$,
3. $w \geq w \wedge u$ *and* $u \geq w \wedge u$,
4. $w \vee u = u \vee w$ *and* $w \wedge u = u \wedge w$,
5. $w \wedge (u \vee v) = (w \wedge u) \vee (w \wedge v)$ *and* $w \vee (u \wedge v) = (w \vee u) \wedge (w \vee v)$.

Definition 2.5. The set $\mathscr{E}^+ = \{w \in \mathscr{E} : w \geq 0\}$ is called the positive cone of the vector lattice $\mathscr{E}$ and elements belonging to $\mathscr{E}^+$ are called the positive elements of $\mathscr{E}$.

Definition 2.6. (Definition 11.6, [3]) Suppose $\mathscr{E}$ is a vector lattice and $w \in \mathscr{E}$;

1. $0 \leq w^+ = w \vee 0$ is called the positive part of w,
2. $0 \leq w^- = w \wedge 0$ is called the negative part of w,

3. $|w| = (-w) \vee (w)$ is called the absolute value (or modulus) of w.

Theorem 2.3. *(Theorem 11.4, [3]) The following are satisfied for any w in the vector lattice $\mathscr{E}$.*

1. *$w = w^+ - w^-$ with $w^+ \wedge w^- = 0$; $|w| = w^+ + w^-$, and so $|w| \in \mathscr{E}^+$*
2. *$(\alpha w)^+ = \alpha w^+$ and $(\alpha w)^+ = \alpha w^-$ for $\alpha \geq 0$.*
   $(\alpha w)^+ = -\alpha w^+$ and $(\alpha w)^- = -\alpha w^-$ for $\alpha \leq 0$.
   $|\alpha w| = |\alpha||w|$ for $\alpha \in \mathbb{R}$.
3. *For $w, u \in \mathscr{E}$. Then $w \leq u$ holds, iff $w^+ \leq u^+$ and $w^- \geq u^-$.*

Theorem 2.4. *(Theorem 11.5, Theorem 11.8, [3]) The following identities are valid $\forall w, v$ and u in the vector lattice $\mathscr{E}$.*

1. *$-(w \vee u) = (-w) \wedge (-u)$.*
2. *$-(w \wedge u) = (-w) \vee (-u)$.*
3. *$w + (u \vee v) = (w + u) \vee (w + v)$*
4. *$w + (u \wedge v) = (w + u) \wedge (w + v)$.*
5. *$w + u = w \vee u + w \wedge u$.*
6. *$\alpha(w \vee u) = (\alpha w) \vee (\alpha u)$ holds for $\alpha \geq 0$.*
7. *$\alpha(w \wedge u) = (\alpha w) \wedge (\alpha u)$ holds for $\alpha \geq 0$.*

Theorem 2.5. *(Theorem 2.4, [3]) The following statements are valid for all w, v and u in the vector lattice $\mathscr{E}$.*

1. *If $w = v - u$ and $v \wedge u = 0$ then $v = w^+$, $u = w^-$.*
2. *$|w - u| = w \vee u - w \wedge u$.*
3. *$|w \vee u - u \vee v| \leq |w - v|$ (Birkhoff's inequality).*
4. *$|w \wedge u - u \wedge v| \leq |w - v|$ (Birkhoff's inequality).*

Theorem 2.6. *(Definition 1.1.25, [4]) [Riesz Decomposition Property] If $\mathscr{E}$ is a vector lattice and $w, v, u \in \mathscr{E}^+$ with $0 \leq w \leq v + u$. Then, there are $t, s \in \mathscr{E}^+$ such that $w = t + s$ and $t \leq v$, $s \leq u$.*

Theorem 2.7. *The inequalities $|v_1 \wedge ... \wedge v_n| \leq |v_1| \wedge ... \wedge |v_n|$ and $|v_1 \vee ... \vee v_n| \leq |v_1| \vee ... \vee |v_n|$ are satisfied for arbitrary $v_1, ..., v_n$ in a vector lattice $\mathscr{E}$.*

Definition 2.7. Let $\emptyset \neq \mathcal{S}$ be a subset of the vector lattice $\mathscr{E}$ and $w, u \in \mathscr{E}$;

1. $\mathcal{S}$ is called bounded above if for each $w \in \mathcal{S}$ there is a $u \in \mathcal{S}$ ensuring that $w \leq u$ holds.
2. $\mathcal{S}$ is called bounded below if for each $w \in \mathcal{S}$ there is a $u \in \mathcal{S}$ ensuring that $u \leq w$ holds.
3. $\mathcal{S}$ is called bounded when $\mathcal{S}$ is bounded both from below and above.
4. $\mathscr{E}$ is called Dedekind complete when each $\mathcal{S}$ that is bounded below has an infimum and each $\mathcal{S}$ that is bounded above has a supremum.
5. $\mathscr{E}$ is called Dedekind σ-complete when each countable (or finite) $\mathcal{S}$ that is bounded below has an infimum and each countable (or finite) $\mathcal{S}$ that is bounded above has a supremum.
6. Assume $w, u \in \mathscr{E}$. The set $[w, u] = \{v \in \mathscr{E} : w \leq v \leq u\}$ is called an order interval in $\mathscr{E}$.
7. $\mathcal{S}$ is called order bounded whenever $\mathcal{S}$ is included in an order interval.
8. w, v are called disjoint if $|w| \wedge |u| = 0$ and it is denoted by $w \perp v$.
9. The set $S^d = \{w \in \mathscr{E} : w \perp u \text{ for all } w \in \mathcal{S}\}$ is called the disjoint complement of $\mathcal{S}$.

Definition 2.8. Let $\{u_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be the sequences in the vector lattice $\mathscr{E}$.

1. If $u_1 \leq u_2 \leq ...$ hold, then $\{u_n\}_{n=1}^{\infty}$ is called decreasing; similarly if $u_1 \geq u_2 \geq ...$ hold, then $\{u_n\}_{n=1}^{\infty}$ is called increasing . In this work, we will use the notations $u_n \downarrow$ and $u_n \uparrow$ respectively.
2. If $u_n \downarrow$ and $u = \inf u_n$, we will use the notation $u_n \downarrow v$; similarly if $u_n \uparrow$ and $u = \sup u_n$, then we will use the notation $u_n \uparrow u$.
3. If $u_n \uparrow u$ then $\alpha u_n \uparrow \alpha u$ for real $\alpha \geq 0$.
4. $u_n \uparrow u$ and $w_n \uparrow w$ imply $u_n + w_n \uparrow u + w$.
5. $u_n \uparrow u$ implies $u_n^+ \uparrow u^+$ and $u_n^- \downarrow u^-$.
6. $u_n \uparrow u$ and $w_n \uparrow w$ imply that $u_n \vee w_n \uparrow u \vee w$ and $u_n \wedge w_n \uparrow u \wedge w$ hold.

Definition 2.9. The vector lattice $\mathscr{E}$ is called Archimedean whenever the decreasing sequence $vn^{-1} \downarrow 0$ $(n = 1, 2, ...)$ for all $v \in \mathscr{E}^+$. That is to say, $\mathscr{E}$ is called Archimedean when for arbitrary $v, w \in \mathscr{E}^+$ such that $nw \leq v$ $(n = 1, 2, ...)$ implies $w = 0$.

Definition 2.10. Assume $\{v_n\}_{n=1}^{\infty}$ is a sequence in the vector lattice $\mathscr{E}$. For a fixed $\alpha \in \mathscr{E}^+$, the sequence $\{v_n\}_{n=1}^{\infty}$ is said converging α-uniformly towards $v \in \mathscr{E}$ whenever, for all $\delta > 0$ there exists a corresponding $N_\delta \in \mathbb{N}$ ensuring that $|v_n - v| \leq \delta\alpha$ holds for all $n \geq N_\delta$. This type of convergence is denoted by $x_n \to x(\alpha)$. In this context, the element α is referred to as the convergence regulator.
Additionally, the sequence $\{v_n\}_{n=1}^{\infty}$ is said converging relatively uniformly to $v \in \mathscr{E}$, if $v_n \to v(\alpha)$ for some $\alpha \in \mathscr{E}$. Here, the notation $v_n \to v(r.u.)$ is employed which does not explicitly identify the regulator.

Theorem 2.8. *If* $w_n \to w(r.u.)$ *and* $v_n \to v(r.u.)$ *in the Archimedean vector lattice* $\mathscr{E}$*, then* $v = w$.

Theorem 2.9. *Let* $w_n \to w(r.u.)$ *and* $u_n \to u(r.u.)$ *in the vector lattice* $\mathscr{E}$*. Under these conditions, the subsequent assertions are valid:*

1. $\alpha w_n + \beta u_n \to \alpha w + \beta u(r.u.)$ *for* $\alpha, \beta \in \mathbb{R}$,
2. $w_n \wedge u_n \to w \wedge u(r.u.)$,
3. $w_n \vee u_n \to w \vee u(r.u.)$.

Definition 2.11. Suppose $\mathscr{E}$ is an Archimedean vector lattice and $\alpha \in \mathscr{E}^+$. Then, the series $\{v_n\}_{n=1}^{\infty}$ is called an α-uniform Cauchy sequence in $\mathscr{E}$ whenever, $\forall \delta > 0$ there exists a corresponding $N_\delta \in \mathbb{N}$ ensuring that $|v_n - v_m| \leq \delta\alpha$ holds for all $m, n \geq N_\delta$.

Definition 2.12. Suppose $\mathscr{E}$ is an Archimedean vector lattice and $\emptyset \neq \mathcal{S} \subseteq \mathscr{E}$.

1. $\mathcal{S}$ is termed as relatively uniformly closed (r^u-closed) if every relative uniform limit is contained in $\mathcal{S}$.
2. $\mathcal{S}$ is called relatively uniformly complete (r^u-complete) if each relatively uniformly Cauchy sequence in $\mathscr{E}$ has a limit.

Definition 2.13. Consider the vector lattice $\mathscr{E}$,

1. The linear subspace $\mathcal{S}$ of $\mathscr{E}$ is called a vector sublattice (or Riesz subspace) in $\mathscr{E}$ if for any pair $w, u \in \mathcal{S}$, $\sup\{w, u\}$ and $\inf\{w, u\} \in \mathcal{S}$.

2. The linear subspace $\mathscr{S}$ of $\mathscr{E}$ is called an order ideal (ideal or o-ideal) if $\mathscr{S}$ is a solid subspace of $\mathscr{E}$, that is to say, for any $w \in \mathscr{S}$ and $u \in \mathscr{E}$ satisfying $|u| \leq |w|$ implies that $u \in \mathscr{S}$.

3. The o-ideal $\mathscr{S}$ of $\mathscr{E}$ is termed as a band whenever $\mathscr{K} \subset \mathscr{S}$ and $w = \sup \mathscr{K}$ implies $w \in \mathscr{K}$.

Example 2.1. *Consider the subset $\mathscr{S} \neq \emptyset$ of a vector lattice $\mathscr{E}$. In this case, $\mathscr{S}^d$ (the disjoint complement of $\mathscr{S}$) is also an o-ideal of $\mathscr{E}$.*

Definition 2.14. Suppose $\mathscr{E}$ is a vector lattice and $\emptyset \neq \mathscr{S} \subseteq \mathscr{E}$.

1. The smallest o-ideal including $\mathscr{S}$ is termed as an o-ideal spanned by $\mathscr{S}$ and it is denoted by $A_{\mathscr{S}}$. If $\mathscr{S}$ is a singleton including v then the o-ideal spanned by v is termed as a principal o-ideal and it is denoted by $\mathscr{J}_v$.

2. The smallest band including $\mathscr{S}$ is said to be a band spanned by $\mathscr{S}$ and it is represented with by $\{S\}$. If $\mathscr{S}$ is a singleton including v then the band spanned by v is said to be a principal band and it is denoted by $\{v\}$.

Definition 2.15. In a vector lattice $\mathscr{E}$, the element $e \geq 0$ is termed as a weak order unit if $\{e\} = \mathscr{E}$, i.e., $\{e\}^d = \{0\}$. In particular e is termed as a weak order unit whenever for any $w \in \mathscr{E}$, $e \wedge w = 0$ implies $w = 0$.

Definition 2.16. In a vector lattice $\mathscr{E}$, the element $e \in \mathscr{E}$ is termed as a strong order unit in $\mathscr{E}$ if $\mathscr{J}_e = \mathscr{E}$. That is, for every $v \in \mathscr{E}$ there is an element $n \in \mathbb{N}$ satisfying $|v| \leq ne$.

Definition 2.17. The vector sublattice $\mathscr{S}$ of a vector lattice $\mathscr{E}$ is called order dense (or o-dense) in $\mathscr{E}$ if for each $v > 0$ in $\mathscr{E}$ there exists $w \in \mathscr{E}$ ensuring that $0 < w \leq v$ holds.

Theorem 2.10. *The o-ideal $\mathscr{S}$ in a vector lattice $\mathscr{E}$ is termed o-dense iff $\mathscr{S}^d = \{0\}$ (i.e.,$\mathscr{S}^{dd} = \mathscr{E}$).*

Theorem 2.11. *For a vector lattice $\mathscr{E}$, the assertions below hold:*

1. *If $\mathscr{J}$ is an o-ideal in $\mathscr{E}$ then $\mathscr{J}$ is also a vector sublattice in $\mathscr{E}$.*

2. *Any vector sublattice of an Archimedean vector lattice is Archimedean.*

3. *$\{0\}$ and $\mathscr{E}$ itself are bands in $\mathscr{E}$.*

4. *If $\mathscr{J}$ is an o-ideal of $\mathscr{E}$ then $v \in \mathscr{J}$ iff $|v| \in \mathscr{J}$.*

Theorem 2.12. *(Theorem 22.3, [5]) For a vector lattice $\mathscr{E}$, the assertions below are equivalent:*

1. *$\mathscr{E}$ is Archimedean.*
2. *$\{\mathcal{S}\} = \{\mathcal{S}\}^{dd}$ holds when $\emptyset \neq \mathcal{S} \subseteq \mathscr{E}$.*
3. *For any band $\mathcal{S}$ in $\mathscr{E}$, $\mathcal{S} = \mathcal{S}^{dd}$ holds.*

Definition 2.18. Let $\mathscr{E}$ be a vector sublattice of the Dedekind complete vector lattice $\mathcal{K}$. Then, $\mathcal{K}$ is termed as a Dedekind completion of $\mathscr{E}$ if for each $w \in \mathcal{K}$ satisfies

$$w = \sup(v : v \in \mathscr{E}, v \leq w) = \inf(u : u \in \mathscr{E}, u \geq w).$$

Definition 2.19. The linear operator $\gamma : \mathscr{E} \to \mathscr{F}$ between the given vector lattices is termed as a lattice homomorphism (or Riesz homomorphism) iff the equalities $\gamma(w \wedge v) = \gamma(w) \wedge \gamma(v)$ or similarly $\gamma(w \vee v) = \gamma(w) \vee \gamma(v)$ are satisfied for all elements $w, v \in \mathscr{E}$.

Theorem 2.13. *(Theorem 2.14, [6]) The statements below are equivalent for the linear operator $\gamma : \mathscr{E} \to \mathscr{F}$ between the given vector lattices $\mathscr{E}$ and $\mathscr{F}$.*

1. *γ is a lattice homomorphism.*
2. *$\gamma(w^+) = \gamma(w)^+$ satisfies $\forall w \in \mathscr{E}$.*
3. *$|\gamma(w)| = \gamma(|w|)$ holds $\forall w \in \mathscr{E}$.*
4. *$\gamma(w \wedge u) = \gamma(w) \wedge \gamma(u)$ satisfies $\forall w, u \in \mathscr{E}$.*
5. *It follows from $w \wedge u = 0$ that $\gamma(w \wedge u) = 0 \; \forall w, u \in \mathscr{E}$.*

Notice that,

$$\gamma(w^+) = \gamma(w \vee 0) = \gamma(w) \vee \gamma(0) = \gamma(w)^+ \geq 0$$

for all $w \in \mathscr{E}^+$, that is to say, the lattice homomorphism γ is also a positive operator.

Definition 2.20. A linear, bijective Riesz homomorphism between vector lattices is called a Riesz isomorphism (or lattice isomorphism). Moreover, asssume $\mathscr{E}$ and $\mathscr{F}$ are vector lattices. They are considered lattice isomorphic if there is a Riesz isomorphism such that $\gamma : \mathscr{E} \to \mathscr{F}$.

Theorem 2.14. *(Theorem 2.15, [6]) Let* $\gamma : \mathscr{E} \to \mathscr{F}$ *be a linear, one to one, onto operator between vector lattices. Then* γ *is termed as a lattice isomorphism iff* γ *and* γ^{-1} *are both positive operators.*

Definition 2.21. The order bounded operator τ from the vector lattice $\mathscr{E}$ into itself is called an orthomorphism if $w \perp v$ implies $\tau(w) \perp v$ holds $\forall w, v \in \mathscr{E}$.
Note that, all orthomorphisms are d-homomorphism, which means $|\tau(w)| = |\tau|w||$ $\forall w \in \mathscr{E}$.

Theorem 2.15. *In the vector lattice* $\mathscr{E}$*, every positive orthomorphism is also a lattice homomorphism.*

Theorem 2.16. *Any orthomorphism* τ *in an Archimedean vector lattice is called order continuous.*

Theorem 2.17. *Let* $Orth(\mathscr{E})$ *be the set of all orthomorphisms on an Archimedean vector lattice* $\mathscr{E}$*. Then* $Orth(\mathscr{E})$ *is also an Archimedean vector lattice, such that*

$$(\tau_1 \vee \tau_2)w = \tau_1 w \vee \tau_2 w \ \ and \ \ (\tau_1 \wedge \tau_2)w = \tau_1 w \wedge \tau_2 w$$

hold for $w \in \mathscr{E}^+$ *and for any* $\tau_1, \tau_2 \in Orth(\mathscr{E})$*. In particular,*

$$|\tau|w = |\tau w|, (\tau^+)w = (\tau w)^+ and (\tau^-)w = (\tau w)^-$$

hold for $w \in \mathscr{E}^+$ *and for any* $\tau \in Orth(\mathscr{E})$*.*

Theorem 2.18. *Suppose* $\mathscr{E}$ *is a* r^u*-complete Riesz space then* $Orth(\mathscr{E})$ *is* r^u*-complete as well.*

Theorem 2.19. *(Theorem 1, [7]) Let* $\mathcal{S}$ *be a subset of the vector lattice* $\mathscr{E}$ *such that* $\mathcal{S}^{dd} = \mathscr{E}$ *and let* τ_1*,* τ_2 *be the orthomorphisms in* $\mathscr{E}$*. If* $\tau_1 w = \tau_2 w$ *for all* $w \in \mathcal{S}$ *then* $\tau_1 = \tau_2$ *on* $\mathscr{E}$*. That is to say, if* e *is the weak order unit of* $\mathscr{E}$ *and* $\tau_1 e = \tau_2 e$ *holds then* $\tau_1 = \tau_2$ *on* $\mathscr{E}$*.*

Definition 2.22. The vector lattice $\mathscr{E}$ is termed as a lattice ordered algebra (or Riesz algebra) when there is an associative multiplication in $\mathscr{E}$ that adheres to standard algebraic axioms such that $wv \in \mathscr{E}^+$ for every $w, v \in \mathscr{E}^+$.

Definition 2.23. The lattice algebra $\mathscr{A}$ is termed as an almost f-algebra if $w \wedge v = 0$ in $\mathscr{A}$ implies $wv = 0$.

Definition 2.24. The vector lattice $\mathscr{A}$ is termed as an f-algebra if $w \wedge v = 0$ in $\mathscr{A}$ implies $wu \wedge v = 0$ and $uw \wedge v = 0$ hold $\forall u \in \mathscr{A}^+$.

Theorem 2.20. *In an Archimedean vector lattice $Orth(\mathscr{E})$, there exists a multiplicative structure ensuring that*

$$(\tau_1\tau_2)w = \tau_1(\tau_2)w$$

holds and $Orth(\mathscr{E})$ is an Archimedean unitary f-algebra (with respect to the multiplication above) with the identity mapping as unity.

Theorem 2.21. *Let τ be an orthomorphism on the Archimedean unitary f-algebra $\mathscr{A}$. Then there is a $\tau_w \in Orth(\mathscr{A})$ for a uniquely determined w in $\mathscr{A}$.*

Theorem 2.22. *In a unitary, r^u-complete f-algebra $\mathscr{A}$ for any $w \in \mathscr{A}$ there exists a uniquely determined $v \in \mathscr{A}$ ensuring that $v^2 = w$ holds. This can also be expressed by $v = \sqrt{w}$.*

3

VECTOR SUBLATTICES AND SUBALGEBRAS IN ARCHIMEDEAN F-ALGEBRAS

Theorem 3.1. *(Theorem 142.1,[5])In any Archimedean f-algebra $\mathscr{A}$ the statements below hold.*

1. $w \perp u$ *implies* $wu = 0$.
2. $ww^+ = (w^+)^2 \geq 0$ *and* $w^2 = (w^+)^2 + (w^-)^2 \geq 0$*, for all* $w \in \mathscr{A}^+$.
3. $(w \vee u)^2 = w^2 \vee u^2$ *and* $(w \wedge u)^2 = w^2 \wedge u^2$ *holds* $\forall w, u \in \mathscr{A}^+$

4. $|wu| = |w||u|$ *hold* $\forall w, u \in \mathscr{A}$.

Proof.

1. Given any w, u in $\mathscr{A}$ satisfying $w \perp u$, then by Definition 2.24 $wu \perp u$ and similarly $wu \perp wu$. Hence $wu = 0$.
2. Considering $w = w^+ - w^-$ we write

$$\begin{aligned} w^2 &= ww \\ &= w^+w^+ - w^+w^- - w^-w^+ + w^-w^- \\ &= (w^+)^2 - w^+w^- - w^-w^+ + (w^-)^2. \end{aligned}$$

   Thus, by Theorem 2.3.2 and Theorem 3.1.1, we obtain $w^2 = (w^+)^2 + (w^-)^2 \geq 0$.

3. For the first one

$$\begin{aligned} (w \wedge u)^2 &= \{w(w \wedge u)\} \wedge \{u(w \wedge u)\} \\ &= w^2 \wedge wu \wedge uw \wedge u^2 \\ &= (w^2 \wedge u^2) \wedge \{(wu) \wedge (uw)\} \end{aligned}$$

By Theorem 3.4.vi, [8] we have, $w^2 \wedge u^2 \leq (wu) \wedge (uw)$. Hence, we obtain

$$(w^2 \wedge u^2) \wedge \{(wu) \wedge (uw)\} = w^2 \wedge u^2.$$

For the second one,

$$\begin{aligned}(w \vee u)^2 &= \{w(w \vee u)\} \vee \{u(w \vee u)\} \\ &= w^2 \vee wu \vee uw \vee u^2 \\ &= (w^2 \vee u^2) \vee \{(wu) \vee (uw)\}\end{aligned}$$

From Theorem 3.4.vi, [8] we have $w^2 \vee u^2 \geq (wu) \vee (uw)$. Hence we obtain,

$$(w^2 \vee u^2) \vee \{(wu) \vee (uw)\} = w^2 \vee u^2.$$

4. Considering $w^+ \perp w^-$ and using Theorem 14.2.iii, [3] we write,

$$(w^+u^+ + w^-u^-) \perp (w^+u^- + w^-u^+).$$

Let $f = w^+u^+ + w^-u^-$ and $g = w^+u^- + w^-u^+$ then,

$$\begin{aligned}f - g &= w^+u^+ + w^-u^- - w^+u^- - w^-u^+ \\ &= w^+(u^+ - u^-) - w^-(u^+ - u^-) \\ &= wu.\end{aligned}$$

Notice that, f, g are disjoint then, Theorem 2.5.1 yields us $f = (wu)^+$ and $g = (wu)^-$. Consequently,

$$\begin{aligned}|wu| &= (wu)^+ + (wu)^- = f + g = (w^+u^+ + w^+u^-) + (w^+u^- + w^-u^+) \\ &= (w^+ + w^-)(u^+ + u^-) \\ &= |w||u|.\end{aligned}$$

■

Theorem 3.2. *(Theorem 3.7, [8])]) For a semi-prime f-algebra $\mathscr{A}$, the assertions below hold.*

1. $w \perp u$ *iff* $wu = 0$.
2. *For all* $w, u \in \mathscr{A}^+$ $w^2 \leq u^2$ *holds iff* $w \leq u$. *Hence,* $w^2 = u^2$ *iff* $w = u$

Proof. 1. In the previous theorem, we have proved, $w \perp u$ implies $wu = 0$.

   Conversely, suppose now that $wu = 0$. Then, we have

$$(|w| \wedge |u|)^2 = (|w| \wedge |u|)(|w| \wedge |u|) \leq |w||u| = |wu| = 0.$$

   As $\mathscr{A}$ is semi-prime, $|w| \wedge |u| = 0$, that is to say, $w \perp u$.

2. Let $0 \leq w \leq u$. Then, we have $w^2 = ww \leq wu \leq uu = u^2$.
   Now, for the converse, assume that $w^2 \leq u^2$, but $0 \leq w \leq u$ does not hold. We have $w \wedge u \leq w$. Subsequently we can find $v > 0 \in \mathscr{A}$ satisfying $v + w \wedge u = w$. In this case, $v^2 + (w \wedge u)^2 \leq w^2$ with $v^2 > 0$. Since $\mathscr{A}$ is semi-prime, $w^2 = w^2 \wedge u^2 = (w \wedge u)^2 \leq w^2$, which means we now have a contradiction.

■

Theorem 3.3. *(Theorem 142.5.iii, [3]) Every Archimedean unitary f-algebra is semi-prime.*

Let $\mathscr{A}$ be an Archimedean unitary f-algebra. Suppose that $w \in \mathscr{A}$ such that $w^2 = 0$. Define an orthomorphism τ_w in $\mathscr{A}$ by $\tau_w v = wv$ for all $v \in \mathscr{A}$. Then, $\tau_w w = w^2 = 0$ and $\tau_w v = wv = 0$ for all $v \in \{w\}^d$. Hence τ_w equals 0 on the set $\mathcal{D} = \{w, \{w\}^d\}$. Since $\mathcal{D}^{dd} = \mathscr{A}$ and $\tau_w = 0$ on $\mathcal{D}$, by Theorem 2.19 $\tau_w = 0$ on $\mathscr{A}$.

Theorem 3.4. *[9] Every Archimedean f-algebra is commutative.*

Proof. Let $\mathscr{A}$ be an Archimedean unitary f-algebra. We will first show that all positive elements are commutative in $\mathscr{A}$. Note that $w \wedge v = 0$ in $\mathscr{A}$ implies $wv = 0$. Furthermore since $\mathscr{A}$ is an f-algebra $w \wedge v = 0$ implies $wv \wedge v = 0$, that is to say $wv \in \{v\}^d$. Moreover, it follows from $v \perp \{v\}^d$ that $wv \perp \{v\}^d$. Hence $wv = 0$. Similarly, of course, $vw = 0$ as well.
Now let $u \in \mathscr{A}^+$. We show that $uv = vu$ for all $v \in \mathscr{A}^+$. In order to achieve that, we define the operators τ_l and τ_r in $\mathscr{A}$ by $\tau_l u = uf$ and $\tau_r u = fu$ for all $f \in \mathscr{A}$. Note that from p. 61, [8], τ_l and τ_r are positive orthomorphisms in $\mathscr{A}$. For all positive element $v \in \{u\}^d$ we have $v \wedge u = 0$. Hence, from the result in the previous paragraph $uv = vu = 0$. That is to say, $\tau_l v = \tau_r v = 0$ for all $0 \leq v \in \{u\}^d$, which means $\tau_l f = \tau_r f$ for all $f \in \{u\}^d$. Additionally, $\tau_l u = \tau_r u = u^2$. Let $\mathcal{D} = \{u, \{u\}^d\}$ be the subset of $\mathscr{A}$. Then, $\mathcal{D}^d = \{0\}$, (i.e, $\mathcal{D}$ is an o-dense subset of $\mathscr{A}$). Moreover, $\tau_l = \tau_r$ on $\mathcal{D}$. Therefore, by Theorem 2.19, we obtain $uv = vu$ holds $\forall u, v \in \mathscr{A}^+$.

For the second part, to prove all elements commute in $\mathscr{A}$, take arbitrary $w, z \in \mathscr{A}$,

$$\begin{aligned} wz &= (w^+ - w^-)(z^+ - z^-) \\ &= w^+z^+ - w^+z^- - w^-z^+ + w^-z^- \\ &= z^+w^+ - z^-w^+ - z^+w^- + z^-w^- \\ &= (z^+ - z^-)(w^+ - w^-) \\ &= zw. \end{aligned}$$

■

In this paper, we focus solely on Archimedean f-algebras and Archimedean vector lattices. Moving forward, whenever we mention f-algebras or vector lattices, we mean they are Archimedean.

Proposition 3.1. *(Proposition 2.1, [2]) Assume $\mathscr{A}$ is an Archimerdean f-algebra possessing the unity e. In that case, the statements below hold true $\forall w \in \mathscr{A}^+$.*

1. $0 \leq w - w \wedge (\alpha e) \leq \alpha^{-1} w^2,\ \alpha \in \mathbb{N}$, *so* $w \wedge (\alpha e) \uparrow w(w^2)$
2. $0 \leq w - w \wedge (\alpha w^2) \leq \alpha^{-1} e,\ \alpha \in \mathbb{N}$, *so* $w \wedge (\alpha w^2) \uparrow w(e)$

Proof. 1. From Theorem 3.1.2 above, $(w - \alpha e)(w - \alpha e)^+ \geq 0$, and so

$$\begin{aligned} 0 &\leq w - w \wedge (\alpha e) \\ &= w + (-w) \vee (-\alpha e) \\ &= 0 \vee (w - \alpha e) \\ &= (w - \alpha e)^+ \\ &\leq \alpha^{-1} w (w - \alpha e)^+ \leq \alpha^{-1} w^2,\ \alpha \in \mathbb{N}. \end{aligned}$$

2. This statement is proven in the same way.

■

Lemma 3.1. *(Lemma 2.1, [10]) Suppose that $\{w_k; k = 1, ...n\}$ is a sequence in the vector latticce $\mathscr{E}$ and let $u \in \mathscr{E}^+$ such that*

1. $|w_k - w_{k+1}| \leq u$,
2. $\inf(w_1, ..., w_n) \leq u$ *and* $\sup(w_1, ..., w_n) \geq -u$.

Then, $\inf(|w_1|, ..., |w_n|) \leq u$.

Lemma 3.2. *(Lemma 3.1, [2]) Let* $0 \leq w \leq v$ *be elements in the Riesz space* $\mathscr{E}$. *Then,*

$$\bigwedge_k \left\{|w - \frac{k}{n}v| : k = 0, 1, ..., n\right\} \leq n^{-1}v$$

holds for all $n \in \mathbb{N}$.

Proof. Writing $u_k = w - \frac{k}{n}v, (k = 0, ..., n)$. It is evident that u_k is a decreasing sequence since $u_1 \geq u_2 \geq ... \geq u_n$. On the other hand,

1. $|u_k - u_{k+1}| = |w - \frac{k}{n}v - w + \frac{k+1}{n}v| = \frac{1}{n}v$,
2. $u_1 \wedge u_2 \wedge ... \wedge u_n = u_n = w - \frac{n}{n}v = w - v \leq 0 \leq \frac{1}{n}v$,
3. $u_1 \vee u_2 \vee ... \vee u_n = u_1 = w - \frac{1}{n}v \geq \frac{-1}{n}v$.

In view of the previous lemma, we conclude that

$$|u_1| \wedge |u_2| \wedge ... \wedge |u_n| \leq \frac{1}{n}v$$

Which means

$$\bigwedge_k \left\{|w - \frac{k}{n}v| : k = 0, 1, ..., n\right\} \leq n^{-1}v \quad \forall\, n \in \mathbb{N}.$$

■

Theorem 3.5. *(Theorem 3.2, [2]) Suppose* $\mathscr{A}$ *is an* f*-algebra and* $w \leq v \in \mathscr{A}^+$. *In this case there exists elements* u_n *in the vector sublattice which is spanned by* w *and* v *such that*

$$0 \leq w^2 - u_n v \leq n^{-2}v^2 \quad \forall\, n \in \mathbb{N}.$$

Proof. Writing, for $n = 1, 2, ...$

$$\begin{aligned} u_n &= \bigvee_k \{2\alpha w - \alpha^2 v : \alpha = \frac{k}{n}, k = 0, ..., n\} \\ &= \bigvee_k \{(2\alpha w - \alpha^2 v)^+ : \alpha = \frac{k}{n}, k = 0, ..., n\} \end{aligned}$$

Hence, $u_n \geq 0$. Since vector spaces are closed under scaler multiplication and addition, u_n is in the vector sublattice which is spanned by w and v. Furthermore,

$$\begin{aligned} w^2 - u_n v &= w^2 - \bigvee_\alpha \{2\alpha w - \alpha^2 v\} v \\ &= w^2 + \bigwedge_\alpha \{\alpha^2 v - 2\alpha w\} v \\ &= w^2 + \bigwedge_\alpha \{\alpha^2 v^2 - 2\alpha w v\} \\ &= \bigwedge_\alpha \{w^2 - 2\alpha w v + \alpha^2 v^2\} \\ &= \bigwedge_\alpha \{(w - \alpha v)^2\}. \end{aligned}$$

Squaring the inequality in Lemma 3.2 we obtain

$$\bigwedge_k \{(w - \alpha v)^2 : \alpha = \frac{k}{n},\ k = 0, 1, ..., n\} \leq n^{-2} v^2.$$

In summary,

$$0 \leq w^2 - u_n v \leq n^{-2} v^2 \quad \forall n \in \mathbb{N}.$$

■

Corollary 3.1. *([8]) Suppose that $\mathscr{A}$ is an f-algebra possessing unity e.*

1. *If $w \leq e \in \mathscr{A}^+$ then, we can obtain a sequence u_n in the vector sublattice $\mathscr{E}$ which is spanned by e and w satisfying $u_n \rightarrow w^2(e)$.*
2. *If $w \geq e$, then we can obtain a sequence u_n in the vector sublattice $\mathscr{E}$ which is spanned by e and w satisfying $u_n w \rightarrow e(w^2)$.*

Theorem 3.6. *(Corollory 3.12.i, [11]) Assume $\mathscr{A}$ is a semi-prime, r^u-complete f-algebra and $w \leq v \in \mathscr{A}^+$. In that case, there exists an element u in $\mathscr{A}^+$ satisfying $w^2 = uv$.*

Proof. The elements u_n

$$u_n = \bigvee_k \{2\alpha w - \alpha^2 v : \alpha = \frac{k}{n}, k = 0, ..., n\}, \ \ \forall n \in \mathbb{N}$$

satisfy $0 \leq w^2 - u_n v \leq n^{-2} v^2$, which means $u_n v \rightarrow w^2(v^2)$. Notice that, u_n is a

v^2-uniform Cauchy sequence. In that case, for all $m, n \geq 0$,

$$\begin{aligned}|u_m v - u_n v| &\leq n^{-2} v^2 \\ |u_m v - u_n v| - n^{-2} v^2 &\leq 0 \\ (|u_m - u_n| - n^{-2} v) v &\leq 0.\end{aligned}$$

We conclude from the last inequality that

$$(|u_m - u_n| - n^{-2} v)^+ v = 0.$$

Being that $\mathscr{A}$ is semi-prime, we can write

$$(|u_m - u_n| - n^{-2} v)^+ \wedge v = 0.$$

Moreover, for all $0 \leq \alpha \leq 1$,

$$\begin{aligned}-1 \leq \alpha - 1 &\leq 0 \\ 0 \leq (\alpha - 1)^2) &\leq 1 \\ 0 \leq 2\alpha - \alpha^2 &\leq 1.\end{aligned}$$

It follows from

$$2\alpha w - \alpha^2 v \leq 2\alpha w - \alpha^2 w = (2\alpha - \alpha^2) w \leq w \leq v$$

that $u_n \leq v \; \forall n \in \mathbb{N}$. Hence $(|u_m - v_n| - n^{-2} v)^+ \leq v$.

$$|u_m - u_n| \leq n^{-2} v$$

holds for all $m \geq n$. Thereby, we can write $u_n \to u(v)$ for some $u \in \mathscr{A}^+$. Likewise $u_n v \to uv(v^2)$. In conclusion, from Theorem 2.8 $w^2 = uv$. ■

The second application focuses on the r^u-complete vector sublattices. It has been proved in (Theorem 3, [12]) that a r^u-complete, unitary vector sublattice $\mathscr{E}$ of a unitary f -algebra $\mathscr{A}$, becomes a subalgebra. This finding is extended in the subsequent theorem, thus, revealing that a 'Stone-condition' on $\mathscr{E}$ is both required and satisfactory for it to be classified as a subalgebra.

Theorem 3.7. *(Stone condition) (Theorem 3.5, [2]) Assume $\mathscr{A}$ is an f-algebra with unity e and $\mathscr{E}$ is a r^u-closed vector sublattice in $\mathscr{A}$. In that case, $\mathscr{E}$ is classified as a subalgebra of $\mathscr{A}$ iff the condition $w \wedge e \in \mathscr{E}$ holds for all $w \in \mathscr{E}^+$. In general, every r^u-closed vector sublattice containing unity is a subalgebra.*

Proof. As a beginning suppose that $w \wedge e \in \mathscr{E}$ for all $w \in \mathscr{E}$ and let $v \in \mathscr{E}^+$ such that $0 \leq v \leq e$. According to Corollary 3.1.1

$$u_n = \bigvee_k \{(2\alpha v - \alpha^2 e)^+ : \alpha = \frac{k}{n}, k = 0, ..., n\}$$

satisfy $u_n \to v^2(e)$. In addition to this, considering $\mathscr{E}$ is a vector space, i.e., it is closed under scaler multiplication and addition,

$$(2\alpha v - \alpha^2 e)^+ = 2\alpha v - \alpha^2(2\alpha^{-1} w \wedge e) \ (\alpha > 0).$$

So, $u_n \in \mathscr{E} \ \forall \ n \in \mathbb{N}$. Thus, by r^u-closedness of $\mathscr{E}$, we get $v^2 \in \mathscr{E}$. Moreover, consider an arbitrary $v \in \mathscr{E}^+$. As stated in Proposition 3.1.1, $v \wedge ne \uparrow v(v^2)$. Therefore, $(v \wedge ne)^2 \uparrow v^2(v^3)$. We have $v \wedge ne \leq e \forall n \in \mathbb{N}$ and by Theorem 3.1.3 we get $(v \wedge ne)^2 \in \mathscr{E} \ \forall n \in \mathbb{N}$. By using r^u-closedness again, we conclude that $v^2 \in \mathscr{E}$. Given the fact that $h^2 = (h^+)^2 + (h^-)^2$, we derive $v^2 \in \mathscr{E}$ whenever $v \in \mathscr{E}$. Furthermore, for all $y, x \in \mathscr{E}$, the equation

$$yx = \frac{1}{2}((y + x)^2 - y^2 - x^2)$$

holds, so that $yx \in \mathscr{E}$ for all $y, x \in \mathscr{E}$.
For the reverse direction, suppose that $\mathscr{E}$ is a subalgebra of $\mathscr{A}$. It will be enough to show that $w \wedge e \in \mathscr{E} \ \forall w \in \mathscr{E}^+$. To do this, define

$$v_n = \bigvee_k \left\{2\alpha(w^2 \wedge w) - \alpha^2 w^2 : \alpha = \frac{k}{n}, \ k = 0, ..., n\right\}.$$

Then, writing

$$2\alpha(w^2 \wedge w) - \alpha^2 w^2 = 2\alpha w[(w \wedge e) - (\alpha^2 w))],$$

we obtain $v_n \in \mathscr{E}$. Furthermore, for $n = 1, 2, ...$

$$\begin{aligned}
w^2 \wedge e - v_n &= w^2 \wedge e - \bigvee_k \left\{2\alpha(w^2 \wedge w) - \alpha^2 w^2 : \alpha = \frac{k}{n}, k = 0, ..., n\right\} \\
&= w^2 \wedge e + \bigwedge_k \left\{\alpha^2 w^2 - 2\alpha(w^2 \wedge w) : \alpha = \frac{k}{n}, k = 0, ..., n\right\} \\
&= \bigwedge_k \left\{w^2 \wedge e - 2\alpha w(w \wedge e) + \alpha^2 w^2 : \alpha = \frac{k}{n}, k = 0, ..., n\right\} \\
&= \bigwedge_k \left\{(w \wedge e - \alpha w)^2 : \alpha = \frac{k}{n} : k = 0, ..., n\right\}.
\end{aligned}$$

By Lemma 3.2, we derive that

$$0 \leq w^2 \wedge e - v_n \leq n^{-2} w^2$$

which means $v_n \uparrow w^2 \wedge e$. Since $\mathscr{E}$ is r^u-closed, then, $w^2 \wedge e \in \mathscr{E}$. So, $(nw^2) \wedge e \in \mathscr{E}$. Further, from Proposition 3.1.2

$$(n(w \wedge e)^2) \wedge (w \wedge e) = (n(w^2 \wedge e) \wedge (e \wedge w) = (nw^2 \wedge e) \wedge w = nw^2 \wedge e \wedge w \uparrow w \wedge e(e).$$

Consequently, It follows, $(nw^2) \wedge e \wedge w \in \mathscr{E}$ that $w \wedge e \in \mathscr{E}$ $\forall n \in \mathbb{N}$.

■

Example 3.1. *Consider the f-algebra $\mathscr{A} = C([0,1])$. Then, $\mathscr{E}$, which contains all piecewise linear functions, is a subspace of $\mathscr{A}$. Despite the fact that $\mathscr{E}$ contains the unity, it is not r^u-closed, so $\mathscr{E}$ cannot be classified as a subalgebra of $\mathscr{A}$.*

Example 3.2. *Suppose $\mathscr{A}$ is a unitary f- algebra which is defined by*

$$\mathscr{A} = \{\kappa \in C(\mathbb{R}^+) : \textit{ there is a polynomial } \rho_\kappa \textit{ satisfying } \kappa(w) = \rho_\kappa(w)\}.$$

Consider the subspace $\mathscr{E} = \{\kappa \in \mathscr{A} : \rho_\kappa(0) = 0\}$. Notice that $\mathscr{E}$ is not r^u-closed. Indeed, let $i(w) = w$ for all $w \in \mathbb{R}^+$. Then, $(n^{-1}i) \vee e \in \mathscr{E}$ and $(n^{-1}i) \vee e \to e(i)$ $\forall n \in \mathbb{N}$. However $e \notin \mathscr{E}$. Consequently the Stone condition is not applicable to $\mathscr{E}$, since $i \in \mathscr{E}$ and $e \wedge i \notin \mathscr{E}$.

Proposition 3.2. *Let $\mathscr{E}$ be a Dedekind σ-complete vector lattice with $w \leq v \in \mathscr{E}^+$. Then, there exists $\tau \in Orth(\mathscr{E})$ such that $\tau v = w$.*

Remark 3.1. Consider the vector lattice $\mathscr{E}$, which is r^u-complete and possessing the strong o-unit e. Then, e is also a strong o-unit of the Dedekind completion $\tilde{\mathscr{E}}$ of $\mathscr{E}$. The foregoing proposition implies the existence of the unique orthomorphism $\tau_{\tilde{w}} \in Orth(\tilde{\mathscr{E}})$ for any $\tilde{w}$ in $\tilde{\mathscr{E}}$, ensuring that $\tau_{\tilde{w}} e = \tilde{w}$ holds, since $\{e\}$ is o-dense in $\tilde{\mathscr{E}}$. Defining

$$\tilde{w}\tilde{v} = \tau_{\tilde{w}} \tilde{v} \tag{3.1}$$

we establish that $\tilde{\mathscr{E}}$ is an f-algebra with unity e. The application of Theorem 3.7 is crucial, as it reveals that $\mathscr{E}$, a r^u-closed vector sublattice of $\tilde{\mathscr{E}}$ containing e becomes a subalgebra of $\tilde{\mathscr{E}}$ under the multiplication 3.1. In summary, introducing a multiplication in $\mathscr{E}$ transforms it into an f-algebra with e as the strong o-unit and the multiplicative identity. This multiplication also extends seamlessly to the vector space complexification $\mathscr{E} + i\mathscr{E}$ of $\mathscr{E}$.

Let $\mathscr{E}$ be a r^u-complete vector lattice. In that case, we can add a modulus into $\mathscr{E}+i\mathscr{E}=\{\vartheta:\vartheta=w+iv,w,v,\in\mathscr{E}\}$ (p. 166, [2]) with respect to the formula

$$|\vartheta| = |w{+}iv| = \sup\{Re(e^{-i\theta}\,\vartheta) : 0 \leq \theta \leq 2\pi\} = \sup\{wcos\theta{+}vsin\theta : 0 \leq \theta \leq 2\pi\}.$$

To demonstrate that the supremum on the RHS exists, let $f(\theta) = |wcos\theta + vsin\theta|$, $e = |w| + |v|$ and define

$$f_n = \sup\{f(2\pi k 2^{-n}) : k = 0, 1, ..., 2^n\}\ \forall n \in \mathbb{N}.$$

Let $m < n$ we have

$$|f_n - f_m| = |\sup\{f(2\pi k 2^{-n})\} - \sup\{f(2\pi k^{-m})\} : k = 0, 1, ..., 2^n|$$

and applying Birkhoff's inequality we get

$$\begin{aligned}
&\leq \sup\left\{|f(2\pi k 2^{-n})\} - \{f(2\pi k^{-m})|\right\} \\
&= \sup\left\{||w\cos(2\pi k 2^{-n}) + v\sin(2\pi k 2^{-n})| - |w\cos(2\pi k 2^{-m}) + v\sin(2\pi k 2^{-m})||\right\}
\end{aligned}$$

for $k = 0, 1, ..., 2^n$. Moreover, applying $||t| - |s|| \leq |t - s|$ we obtain

$$\begin{aligned}
&\leq \sup\{|w\cos(2\pi k 2^{-n}) + v\sin(2\pi k 2^{-n}) - w\cos(2\pi k 2^{-m}) - v\sin(2\pi k 2^{-m})|\} \\
&= \sup\{|w(\cos(2\pi k 2^{-n}) - \cos(2\pi k 2^{-m})) + v(\sin(2\pi k 2^{-n}) - \sin(2\pi k 2^{-m}))|\} \\
&= \sup\{| - 2w\sin(\pi k(2^{-n} + 2^{-m}))\sin(\pi k(2^{-n} - 2^{-m})) \\
&\qquad\qquad\qquad + 2v\cos(\pi k(2^{-n} + 2^{-m}))\sin(\pi k(2^{-n} - 2^{-m}))|\} \\
&\leq \sup\{| - 2w||\pi k(2^{-n} - 2^{-m})| + |2v||\pi k(2^{-n} - 2^{-m})|\} \\
&= [| - 2w\pi(2^{-n} - 2^{-m}) + 2v\pi(2^{-n} - 2^{-m})|]\sup|k| \\
&= [| - 2w + 2v|(\pi(2^{-n} - 2^{-m})]2^n.2 \\
&\leq [(|w| + |v|)(\pi(2^{-n} - 2^{-m}))]2^n.2^2 \\
&\leq (|w| + |v|)\pi\frac{\varepsilon}{\pi 2^2.2^n}2^2.2^n = e\varepsilon.
\end{aligned}$$

Consequently, we have showed that f_n is an increasing e-uniform Cauchy sequence, that is, $f_n \uparrow f(e)$ for some $f \in \mathscr{E}$. Therefore, we have obtained the desired supremum f.

Theorem 3.8 (The complex Riesz decomposition property)**.** *Assume $\mathscr{E} + i\mathscr{E}$ is the complexification of a r^u-complete Riesz space $\mathscr{E}$ furnished with the given modulus. If $w, v \in \mathscr{E}$ and $\vartheta = y + iz \in \mathscr{E} + i\mathscr{E}$ with $|\vartheta| \leq w + v$ then there exist ϑ_1 , $\vartheta_2 \in \mathscr{E} + i\mathscr{E}$ such that $\vartheta = \vartheta_1 + \vartheta_2$ and $|\vartheta_1| \leq w, |\vartheta_2| \leq v$.*

Proof. Let $e = w + v$. The principal o-ideal $\mathscr{J}_e = \{y \in \mathscr{E} : |y| \leq ke, k \in \mathbb{R}^+\}$ spanned in $\mathscr{E}$, is a r^u-complete vector lattice possesing the strong o-unit e. Then, we have a r^u-complete vector lattice furnished with (.) such that (p. 166, [2])

$$\mathscr{J}_e + i\mathscr{J}_e = \{\phi \in \mathscr{E} + i\mathscr{E} : |\phi| \leq ke, k \in \mathbb{R}^+\}$$

By Stone condition $\mathscr{J}_e + i\mathscr{J}_e$ turns into an f-algebra, under the above multiplication, with e as a unity. Notice that we can write $\vartheta_1 = \vartheta w$ and $\vartheta_2 = \vartheta v$ (Theorem 12.10.i, Theorem 12.12, p. 90, [8]. A simple arithmetic reveals that

$$\vartheta_1 + \vartheta_2 = \vartheta w + \vartheta v = \vartheta(w + v) = \vartheta e = \vartheta$$

On the other hand, observe that multiplication by a positive element in an f-algebra is order continuous (Theorem 8.6, [8]). Then,

$$\begin{aligned} |\vartheta_1| = |\vartheta w| &= |(y.w) + i(z.w)| \\ &= \sup\{(y.w)\cos\theta + (z.w)\sin\theta \;\; 0 \leq \theta \leq 2\pi\} \\ &= \sup\{y\cos\theta + z\sin\theta \;\; 0 \leq \theta \leq 2\pi\}.w \\ &= |\vartheta|.w \leq e.w = w. \end{aligned}$$

For the other one,

$$\begin{aligned} |\vartheta_2| = |\vartheta v| &= |(y.v) + i(z.v)| \\ &= \sup\{(y.v)\cos\theta + (z.v)\sin\theta \;\; 0 \leq \theta \leq 2\pi\} \\ &= \sup\{y\cos\theta + z\sin\theta \;\; 0 \leq \theta \leq 2\pi\}.v \\ &= |\vartheta|.v \leq e.v = v. \end{aligned}$$

■

Remark 3.2. Consider two semi-prime f-algebras $\mathscr{A}, \mathscr{B}$ together with the algebra homomorphism $\gamma : \mathscr{A} \to \mathscr{B}$. Notice that, if $w \wedge v = 0$ in $\mathscr{A}$ we have $wv = 0$. It follows from Theorem 3.2 and $\gamma(wv) = \gamma(w).\gamma(v) = 0$ that $\gamma(w) \perp \gamma(v)$. So γ is disjointness preserving. Furthermore, from the equality

$$(\gamma|w|)^2 = \gamma|w|^2 = \gamma w^2 = (\gamma w)^2,$$

we get $|\gamma|w|| = |\gamma w|$ for all $w \in \mathscr{A}$, which means γ is a d-homomorphism (disjointness preserving). Since every positive disjointness preserving homomorphisms are lattice homomorphisms (p. 113, Theorem 2.40, [6]) then the algebra homomorphisms $\gamma : \mathscr{A} \to \mathscr{B}$ said to be a lattice homomorphism iff γ is positive.

Theorem 3.9. *Let $\mathscr{A}$ and $\mathscr{B}$ are semi-prime f-algebras, with the additional assumption that $\mathscr{A}$ is r^u-complete. In that case, every algebra homomorphism $\gamma : \mathscr{A} \rightarrow \mathscr{B}$ is also a lattice homomorphism.*

Proof. Given the foregoing observations, it is sufficient to demonstrate the positivity of γ. Notice that the proof is almost trivial when $\mathscr{A}$ has a unity, via Theorem 2.22. Indeed, for all $w \in \mathscr{A}^+$ there is a uniquely determined $0 \leq v \in \mathscr{A}$ satisfying $v^2 = w$. Thus,

$$\gamma(w) = \gamma(v^2) = \gamma(v)\psi(v) = \gamma(v)^2 \geq 0.$$

However, according to hyphothesis, $\mathscr{A}$ is only semi-prime. So, $\mathscr{A}$ is r^u-complete by Theorem2.18 $Orth(\mathscr{A})$ must be as well. Besides, from Theorem 2.20 $Orth(\mathscr{A})$ is an f-algebra with a unity mapping as a unity. Now let $w \in \mathscr{A}^+$. By Theorem 12.8, [8] $\mathscr{A}$ is an r-ideal in $Orth(\mathscr{A})$, i.e. , $w.w^{\frac{1}{2}} \in \mathscr{A}$, observe that there is $v \in \mathscr{A}^+$ such that $v^2 = w^3$. Hereby,

$$0 \leq \gamma(v)^2 = \psi(v^2) = \psi(w^3) = \psi(w)^3.$$

Moreover, we have $\gamma(w)^+ \perp \psi(w)^-$. Thus,we derive that

$$\gamma(w)^3 = (\gamma(w)^+ - \gamma(w)^-)^3 = (\gamma(w)^+)^3 - (\gamma(w)^-)^3.$$

Then, by Theorem2.5.1. $\gamma(w)^3 = (\gamma(w)^+)^3 - (\gamma(w)^-)^3$ since $(\gamma(w)^+)^3 \perp (\gamma(w)^-)^3$. It follows $\gamma(w)^3 \geq 0$ that, $(\gamma(w)^-)^3 = \gamma(w^3)^- = 0$. Further, by the fact that $\mathscr{B}$ is semi-prime, $(\gamma(w)^-)^3 = (\gamma(w)^-)(\gamma(w)^-)^2 = 0$, i.e., $\gamma(w)^- = 0$ which means $\gamma(w) \geq 0$. ■

Theorem 3.10. *Assume $\mathscr{A}$ is unitary and $\mathscr{B}$ is semi-prime f-algebras. In that case, the algebra homomorphism $\gamma : \mathscr{A} \rightarrow \mathscr{B}$, which is order bounded, is also a lattice homomorphism.*

Proof. As we previously stated, showing that $\gamma : \mathscr{A} \rightarrow \mathscr{B}$ is positive will lead us to the desired result. For this purpose, suppose that unity e and w are the elements in $\mathscr{A}$ such that $0 \leq w \leq e$. Further, let

$$v_n = \frac{1}{2}\bigwedge_k \left\{ \alpha e + \alpha^{-1} w : \alpha = \sqrt{\frac{k}{n}} : k = 1, ..., n \right\},$$

Now, let $w \in \mathscr{A}$ be chosen, satisfying $0 \leq w \leq e$. By Corollary 3.1.1, there is a sequence

$$v_n = \bigvee_k \{2\alpha w - \alpha^2 e : \alpha = \frac{k}{n} : k = 0, ..., n\}$$

such that $0 \leq w^2 - v_n \leq n^{-2}e$. Then,

$$0 \leq \gamma(w^2) - \gamma(v_n) \leq n^{-2}\gamma(e) \; \forall n \in \mathbb{N}.$$

So, $\gamma(v_n) \to \gamma(w^2)(r.u.)$. It follows from γ is a lattice homomorphism that,

$$\gamma(v_n) = \bigvee_k \{2\alpha\gamma(w) - \alpha^2\gamma(e) : \alpha = \frac{k}{n} : k = 0, ..., n\}$$

Using Corollary 3.1.1 for the f-algebra $\{\gamma(e)\}^{dd}$ with unity $\gamma(e)$.

$$0 \leq \gamma(w)^2 - \gamma(v_n) \leq n^{-2}\gamma(e) \;\; \forall n \in \mathbb{N}.$$

which means, $\gamma(v_n) \to \gamma(w)^2(r.u.)$. In view of Theorem 2.8 we derive that, $\gamma(w^2) = \gamma(w)^2 \geq 0$ when $0 \leq w \leq e$.

For the general case, let $w \in \mathscr{A}^+$ be chosen arbitrary. By Proposition 3.1.i. $w \wedge ne \uparrow w(w^2)$,that $w^2 \wedge n^2 e \uparrow w^2(w^3)$. Then, $\gamma(w^2 \wedge n^2 e) \uparrow \gamma(w^2)(r.u.)$. By the above result and Theorem 3.1.3,

$$\gamma(w^2 \wedge n^2 e) = \gamma((w \wedge ne)^2) = (\gamma(w) \wedge \gamma(ne))^2 = \gamma(w)^2 \wedge n^2\gamma(e).$$

Further, we have, $\gamma(w)^2 \wedge n^2\gamma(e) \uparrow \gamma(w)^2$. Therefore, $\gamma(w^2) = \gamma(w)^2 \geq 0$ for all $w \in \mathscr{A}^+$. ■

Corollary 3.2. *(Corollary 5.5, [2]) Asumme that $\mathscr{A}$ and $\mathscr{B}$ are f-algebras, possessing unities e_A and e_B. The positive linear mapping γ from $\mathscr{A}$ to $\mathscr{B}$ satisfying $\gamma(e_A) = e_B$, (which characterizes γ as a Markov operator) is a lattice homomorphism iff γ is an algebra homomorphism.*

Lemma 3.3. *(Lemma 5.6, [2]) Consider the f-algebra $\mathscr{A}$ with unity e and let $w, v \in \mathscr{A}^+$. If for all $\alpha \in \mathbb{R}$*

$$\alpha^2 w + 2\alpha v + e \geq 0$$

holds, then $v^2 \leq w$.

Proof. Let $u_k = |v - \frac{k}{n}|$ $(k = 0, ..., n^2)$. Then,

$$|u_k - u_{k+1}| = |n^{-1}e| \leq n^{-1}(e^2 \vee v^2) \leq= n^{-1}(e \vee v)^2$$

Moreover, since u_k is decreasing $\wedge(u_0, u_1, ...u_{n^2}) = u_{n^2} = v - ne$. Furthermore, the inequalty $-ne \leq -v \wedge ne$ always holds and using Proposition we derive 3.1.i ,

$$v - ne \leq v - v \wedge ne \leq n^{-1}v^2 \leq n^{-1}(v \vee e)^2.$$

Applying Lemma 3.2 and 3.1, we get

$$\bigwedge_k \{|v - \frac{k}{n}| : k = 0, 1, ..., n^2\} \leq n^{-1}(v \vee e)^2 \tag{3.2}$$

Now, using the hypothesis

$$\begin{aligned} \alpha^2 w + 2\alpha v + e &\geq 0 \\ 2\alpha^{-1}v + e\alpha^{-2} &\geq -w \\ v^2 + 2\alpha^{-1}v + e\alpha^{-2} &\geq v^2 - w \\ (v + \alpha^{-1}e)^2 &\geq v^2 - w \end{aligned}$$

By choosing $\alpha = -nk^{-1}$ $(k = 1, 2, ..., n^2)$,

$$v^2 - w \leq (v - nk^{-1}e)^2 (k = 1, 2, ..., n^2).$$

Squaring the inequality 3.2 $\forall n \in \mathbb{N}$ we have

$$\bigwedge_k \{(v - \frac{k}{n})^2 : k = 0, 1, ..., n^2\} \leq n^{-2}(v \vee e)^4$$

So, we derive $v^2 - w \leq n^{-2}(v \vee e)^4$. Consequently, since $\mathscr{A}$ is Archimedean, $v^2 - w \leq 0$. ■

Definition 3.1. (Definition 18.6, [13]) Assume $\mathcal{S}$ is a subset of the linear space $\mathcal{L}$. In that case, $\mathcal{S}$ is said to be a convex set if the conditions $\alpha_1, \alpha_2 \in \mathcal{S}$ and $0 \leq \beta \leq 1$ imply $\beta\alpha_1 + (1 - \beta)\alpha_2 \in \mathcal{S}$.

Definition 3.2. (Lemma 4.4.3, [14]) Assume $\mathcal{L}$ is a linear space and $\mathcal{S}$ be a convex set of $\mathcal{L}$. In this case $\alpha \in \mathcal{S}$ is called the extreme point of $\mathcal{L}$ iff $\alpha_1 + \alpha_2 \in \mathcal{S}$ and $\alpha_1 - \alpha_2 \in \mathcal{S}$ imply $\alpha_2 = 0$.

Theorem 3.11. *(Theorem 5.7, [2]) Suppose $\mathscr{A}$ and $\mathscr{B}$ are f-algebras possessing the unities $e_{\mathscr{A}}$ and $e_{\mathscr{B}}$. Let $\mathscr{M} = \{\gamma : \mathscr{A} \to \mathscr{B} : \gamma$ linear, $\gamma > 0,$ and $\gamma(e_{\mathscr{A}}) =$*

$e_{\mathscr{B}}\}$ *be the convex set of Markov operators. Then,* γ *is a lattice homomorphism with* $\gamma(e_{\mathscr{A}}) = e_{\mathscr{B}}$ *iff* γ *is an extreme point in* $\mathscr{M}$*.*

Proof. Suppose γ is an extreme point in $\mathscr{M}$ and let $0 \leq w \leq e_{\mathscr{A}}$. Then, by the positivity of γ, $0 \leq \gamma(w) \leq \gamma(e_{\mathscr{A}}) = e_{\mathscr{B}}$. Define a linear operator $\kappa : \mathscr{A} \to \mathscr{B}$ such that

$$\kappa w = \gamma(wv) - \gamma w.\gamma v \;\; \forall v \in \mathscr{A}.$$

Then, with some calculation we get

$$(\gamma+\kappa)e_{\mathscr{A}} = \gamma e_{\mathscr{A}}+\kappa e_{\mathscr{A}} = e_{\mathscr{B}}+\gamma(e_{\mathscr{A}}v)-\gamma(e_{\mathscr{A}}).\gamma v = e_{\mathscr{B}}+\gamma v-\gamma v = e_{\mathscr{B}}. \tag{3.3}$$

Furthermore, for $0 \leq w \in \mathscr{A}$

$$(\gamma+\kappa)w = \gamma w+\kappa w = \gamma w+\gamma(wv)-\gamma w.\gamma v = \gamma w(e_{\mathscr{B}}-\gamma v)+\gamma(wv) \geq 0. \tag{3.4}$$

Using equality 3.3 and inequality 3.4, we derive that $\gamma + \kappa \in \mathscr{M}$. Similarly, $\gamma - \kappa \in \mathscr{M}$. Thus, we get $\kappa = 0$. Then $\gamma(wv) = \gamma(w).\gamma(v)$ $\forall w \in \mathscr{A}$ such that $0 \leq w \leq e_{\mathscr{A}}$. Furthermore, Proposition3.1.i yields us $\gamma(wv) = \gamma(w).\gamma(v)$ $\forall w, v \in \mathscr{A}$. Thus, by Corollary 3.2, γ is a lattice homomorphism.
Conversely, assume ψ is a lattice homomorphism with $\psi(e_{\mathscr{A}}) = e_{\mathscr{B}}$ and γ is positive, so $\gamma \in \mathscr{M}$. Moreover, let $\gamma = \frac{1}{2}(\gamma_1 + \gamma_2)$ with $\gamma_1, \gamma_1 \in \mathscr{M}$. For $(n = 1, 2)$ we have

$$0 \leq \gamma_n((\alpha w + e_{\mathscr{A}})^2) = \alpha^2\gamma_n(w^2) + 2\alpha.\gamma_n(w) + e_{\mathscr{A}} \; \forall \alpha \in \mathbb{R}.$$

Applying Lemma 3.3, we get $(\gamma_n w)^2 \leq \gamma_n(w^2)$ $(k = 1, 2)$. Moreover,

$$\gamma(w^2) = \frac{1}{2}(\gamma_1 + \gamma_2)(w^2) = \frac{1}{2}\gamma_1(w^2) + \frac{1}{2}\gamma_2(w^2) \geq \gamma_1(w)^2 + \frac{1}{2}\gamma_2(w)^2.$$

Since γ is an algebra homomorphism then $(\gamma w)^2 = \gamma(w^2)$. So

$$\begin{aligned}(\gamma w)^2 = (\frac{1}{2}(\gamma_1(w) + \frac{1}{2}\gamma_2(w))^2 &= \frac{1}{4}(\gamma_1 w)^2 + \frac{1}{2}(\gamma_1 w)(\gamma_2 w) + \frac{1}{4}(\gamma_2 w)^2 \\ &\geq \frac{1}{2}\gamma_1(w)^2 + \frac{1}{2}\gamma_2(w)^2\end{aligned}$$

Consequently, we derive

$$\frac{1}{4}(\gamma_1 w)^2 - \frac{1}{2}(\gamma_1 w)(\gamma_2 w) + \frac{1}{4}(\gamma_2 w)^2 = \frac{1}{2}(\gamma_1(w) - \gamma_2(w))^2 \leq 0.$$

It follows squares in *f*-algebras are positive, that $(\gamma_1(w) - \gamma_2(w))^2 = 0$. Finally, as $\mathscr{A}$ is semi-prime, $\gamma_1(w) = \gamma_2(w)$ for all $0 \leq w \in \mathscr{A}$. ■

This section deals with the conditions for a r^u-closed subalgebra in a unitary f-algebra to be regarded as an o-ideal. While Theorem 3.7 confirmed that any r^u-closed o-ideal is also a two sided ring ideal (or algebra ideal), we now focus on the reverse scenario – when a r^u-closed algebra ideal can be considered an o-ideal?

Theorem 3.12. *(Theorem 6.1, [2]) Assume that $\mathscr{A}$ is an f-algebra which possessing the unity e. If $\mathscr{J}$ is a r^u-closed, two-sided ring ideal (or algebra ideal) in $\mathscr{A}$, then $\mathscr{J}$ is also an o-ideal in $\mathscr{A}$.*

Proof. First, let $0 \leq w \leq v$ with $v \in \mathscr{J}$ and from Corollary 3.1.ii. we have $u_n \in \mathscr{A}^+$ satisfying $u_n(v + \varepsilon e) \to e(\varepsilon^{-2}\alpha)$ for some $\alpha \geq e$. Now if we write $\alpha = (v + e)^2$ we obtain, $|u_k(v + \varepsilon e) - e| \leq \varepsilon\alpha$ for a proper $k \in \mathbb{N}$. Since $\mathscr{J}$ is an algebra ideal, $wu_kv \in \mathscr{J}$. Furthermore

$$u_k w \leq u_k(v + \varepsilon e) \leq e + \varepsilon\alpha \leq 2\alpha$$

So, using $\alpha \geq e$ and $\alpha \geq v \geq w$ we derive

$$\begin{aligned} |w - wu_kv| = w|e - u_kv| \leq w|e - u_k(v + \varepsilon e)| + \varepsilon u_k w &\leq \varepsilon w\alpha + 2\varepsilon\alpha \\ &\leq \varepsilon\alpha^2 + 2\varepsilon\alpha^2 \qquad (3.5) \\ &= 3\varepsilon\alpha^2 \end{aligned}$$

Now, If we put $\varepsilon = n^{-1}$, we get the sequence $z_n = wu_nv$ in $\mathscr{J}$ such that $z_n \to w(\alpha^2)$. By the hypothesis, $\mathscr{J}$ is r^u-closed, thus, $w \in \mathscr{J}$, i.e., $\mathscr{J}$ is solid.
For the second part, let w be an arbitrary element in $\mathscr{J}$. So $w^2 \in \mathscr{J}$ as well. Observe that $0 \leq |w| \wedge nw^2 \leq nw^2$, $\forall n \in \mathbb{N}$. By the foregoing result we have $|w| \wedge nw^2 \in \mathscr{J}$, $\forall n \in \mathbb{N}$. From Proposition 3.1.ii, we get $|w| \wedge nw^2 \uparrow |w|(e)$. Finally, $\mathscr{J}$ is r^u-closed thus, $|w| \in \mathscr{J}$. That is to say, $\mathscr{J}$ is a subspace of $\mathscr{A}$. ■

Example 3.3. *(Example 6.2, [2]) Consider the non-unitary f-algebra $\mathscr{A} = \{\delta \in C([0, 1]) : \delta \leq n_\delta i$ for some $n_\delta \in \mathbb{N}\}$ and define r^u-closed algebra ideal in $C([0, 1])$, $\mathscr{J} = \{wi : w \in C([0, 1])\}$. Let $f \in \mathscr{A}$ such that*

$$f(w) = \begin{cases} w|\sin w^{-1}| & \textit{if } 0 < w \leq 1, \\ 0 & \textit{if } w = 0, \end{cases}$$

Notice that, $0 \leq |\sin w| \leq 1$ which means, $0 \leq f(w) \leq w = i(w)$. However, f is not continous at 0, that is, $f \notin C([0, 1])$. Therefore, $\mathscr{J}$ is not an o-ideal in $\mathscr{A}$.

Herewith, Example 3.3 has shown that the assumption in Theorem 3.12 that $\mathscr{A}$ is unitary is essential.

Theorem 3.13. *(Theorem 6.3, [2]) Let $\mathcal{S}$ be a r^u-closed subalgebra of $C(\mathcal{X})$ with the additional assumption that $\mathcal{X}$ is a topological space. In that case, $\mathcal{S}$ is a vector sublattice of $C(\mathcal{X})$.*

Proof. Initially, take $\mathcal{X}$=$\mathbb{R}$ and consider any r^u-closed subalgebra $\mathcal{S}$ of $C(\mathcal{X})$ with $i(w) = w$. Notice that the function

$$|i| = |w| = \begin{cases} w & \text{if } 0 \geq w \\ -w & \text{if } w \leq 0, \end{cases}$$

is continuous in $\mathbb{R}$ thus $|i| \in C(\mathbb{R})$. Then, from the well-known Stone–Weierstrass theorem, we could obtain a polynomial without constant term such that

$$|p_n(w) - |w|| \leq 4^{-n}, \ \forall n \in \mathbb{N}$$

and for all $w \in [-n, n]$. Since $i \in \mathcal{S}$, $p_n \in \mathcal{S}$, $\forall n \in \mathbb{N}$ as well. Moreover,

$$2^n |p_n(w) - |w|| \leq 2^{-n} \ \text{ on } \ [-n, n].$$

Then,

$$f(w) = \sum_{n=1}^{\infty} 2^n |p_n(w) - |w||$$

converges pointwise on $\mathbb{R}$. Consequently We obtain that $f \in C(\mathbb{R})$. Furthermore, by the following expression

$$|p_n(w) - |w|| \leq 2^{-n} f$$

we get $p_n \to |i|(f)$, so $|i| \in \mathcal{S}$.
For the general case, consider an arbitrary r^u-closed subalgebra $\mathcal{S}$ of $C(\mathbb{R})$ and let $h \in \mathcal{S}$. Define $\mathcal{L}$ by

$$\mathcal{L} = \{g \in C(\mathbb{R}) : g \circ h \in \mathcal{S}\}$$

Let $f, g \in \mathcal{L}$ which means $f \circ h$ and $g \circ h \in \mathcal{S}$. Then,

$$(f \circ g) \circ h = f \circ (g \circ h) \in \mathcal{S}$$

Thus, $(f \circ g) \in \mathcal{L}$. Thus, $\mathcal{L}$ is a subalgebra of $C(\mathbb{R})$. Furtermore, notice that $(i \circ h) = h \in \mathcal{S}$ then also $i \in \mathcal{L}$. Consequently, by the foregoing paragraph we get $|i| \in \mathcal{L}$. Hence, $||i| \circ h| = |h| \in \mathcal{S}$ and so $\mathcal{S}$ is a vector sublattice of $C(\mathcal{X})$. ■

Theorem 3.14. *(Theorem 6.5, [2]) Assume that $\mathscr{A}$ is an f-algebra possessing the unity e and $\mathcal{S}$ be a r^u-closed subalgebra of $\mathscr{A}$. In that case, $\mathcal{S}$ classified as a sub-*

space of $\mathscr{A}$ *iff* $w \wedge e \in \mathcal{S}$ *for all* $w \in \mathcal{S}^+$.

Proof. Theorem 3.7 has shown that when a r^u-closed vector sublattice $\mathcal{S}$ is also a subalgebra then $w \wedge e \in \mathcal{S}$ for all $w \in \mathcal{S}^+$. Conversely, suppose now that $\mathcal{S}$ is a r^u-closed subalgebra of $\mathscr{A}$ with $w \wedge e \in \mathcal{S}$ for all $w \in \mathcal{S}^+$. In that case, it is sufficient to prove that $|u| \in \mathcal{S}\ \forall u \in \mathcal{S}$.
Take $u \in \mathcal{S}$ so, $u^2 \in \mathcal{S}^+$. Then, by the hypothesis $u^2 \wedge e \in \mathcal{S}$. Moreover the following series

$$\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} [(u^2 \wedge e - e)^n - e^n]$$

converges-e uniformly to $|u| \wedge e$ (pp. 307-313, [15]). The partial sum of the series above are polynomials in $u^2 \wedge e$ and thus, members of $\mathcal{S}$. So, by r^u-closedness of $\mathcal{S}$ we obtain $|u| \wedge e$. Take $u = n^{-1}u$, we get $|u| \wedge |ne| \in \mathcal{S}$ $(n = 1, 2, ..., n)$. By means of Proposition 3.1.ii, we write $|u| \wedge |ne| \uparrow |u|(u^2)$. Therefore, $|u| \in \mathcal{S}$. ■

4

THE IMAGE SPACE OF RIESZ HOMOMORPHISM DEFINED ON F-ALGEBRAS

Consider an Archimedean f-algebra $\mathscr{A}$. Then, by Theorem 2.20 $Orth(\mathscr{A})$ is also an Archimedean f-algebra possessing unity I (identity orthomorphism). Let τ_w denote the multiplication by an arbitrary $w \in \mathscr{A}$. Then, it follows $\mathscr{A}$ is an f-algebra that $\tau_w \in Orth(\mathscr{A})$. Thus, we can define an operator ϕ as follow

$$\begin{aligned} \phi :&\mathscr{A} \to Orth(\mathscr{A}) \\ &w \to \tau_w \end{aligned}$$

Proposition 4.1. *(Proposition 12.1, [8]) For an operator $\phi : \mathscr{A} \to Orth(\mathscr{A})$ the statements below are valid.*

1. *ϕ is Riesz and algebra homomorphism*
2. *ϕ is injective iff $\mathscr{A}$ is semi-prime*
3. *ϕ is bijective iff $\mathscr{A}$ is unitary.*

Proof. 1. Take $w, v \in \mathscr{A}$ with $w \wedge v = 0$. Then, for all $z \geq 0$ we have.

$$(\tau_w \wedge \tau_v)z = (\tau_w z) \wedge (\tau_v z) = wz \wedge vz = (w \wedge v)z = 0z = 0$$

So, $\tau_w \wedge \tau_v = 0$, which means, ϕ is a lattice homomorphism. Further, for arbitrary $f, g \in \mathscr{A}$ we can write

$$\tau_{fg}h = (fg)h = f(gh) = \tau_f(gh) = \tau_f\tau_g h$$

forall $h \in \mathscr{A}$. Thus, $\tau_{fg} = \tau_f\tau_g$, that is, ϕ is an algebra homomorphism.

2. Assume ϕ is semi-prime and $\phi(w) = 0$. Then, $\phi(w) = \tau_w = 0$. Therefore, $\tau_w w = w^2 = 0$. It follows from ϕ is semi-prime that, $w = 0$. Hence ϕ is injective.

   For the opposite direction, suppose ϕ is injective and $w^2 = 0$ in c. Then, $\tau_w w = w^2 = 0$ on $\{w\} \cup \{w\}^d$. Since $\{w\} \cup \{w\}^d$ is an order dense subset of $\mathscr{A}$ then, $\tau_w = 0$ on $\mathscr{A}$. It follows from ϕ is injective that $w = 0$. So ϕ is semi-prime.

3. For the forward direction, assume ϕ is bijective and let I be the identity mapping in $\mathscr{A}$. Then, $\phi^{-1}(I)$ becomes the unity of $\mathscr{A}$.
   For the other one, assume $\mathscr{A}$ is unitary so, $\mathscr{A}$ is also semi-prime. Then, by the above ϕ is injective. Moreover, for all τ in $Orth(\mathscr{A})$ there is a $w \in \mathscr{A}$ such that $\tau_w = \tau$, which means ϕ is surjective.

■

Proposition 4.2. *(Proposition 2.2, [16]) Suppose $\mathscr{A}$ is an f-algebra and $w, u \in \mathscr{A}^+$. Then, $0 \leq wu - wu \wedge nu \leq n^{-1}wu^2$ holds $\forall n \in \mathbb{N}$.*

Proof.

$$\begin{aligned}
0 &= (wu - nu)^+ \wedge (wu - nu)^- \\
&= [(wu - nu) \vee 0] \wedge [(nu - wu) \vee 0] \\
&= [wu + (-nu) \vee (-wu)] \wedge [nu + (-wu) \vee (-nu)] \\
&= [wu - (nu) \wedge (wu)] \wedge [nu - (wu) \wedge (nu)]
\end{aligned}$$

Since $\mathscr{A}$ is an f-algebra, we may write

$$\begin{aligned}
0 &= [wu - (nu) \wedge (wu)] \wedge [wu - (n^{-1}w^2u) \wedge (wu)] \\
&= [-wu + (nu) \wedge (wu)] \wedge [-wu + (n^{-1}w^2u) \wedge (wu)]
\end{aligned}$$

So

$$wu = [nu \wedge wu] \vee [n^{-1}w^2u \wedge wu]$$

Using the equality in Theorem 2.4.5, we get

$$nu \wedge wu + w^2un^{-1} \wedge wu = wu + [nu \wedge wu] \wedge [w^2un^{-1} \wedge wu]$$

Since $[(nu) \wedge (wu)] \wedge [(w^2un^{-1}) \wedge (wu)] \geq 0$, we write the following inequality

$$wu \leq (nu) \wedge (wu) + (w^2un^{-1}) \wedge (wu)$$

Consequently, we obtain

$$wu - nu \wedge wu \leq n^{-1}w^2u \wedge wu \leq n^{-1}w^2u.$$

Finally, $nu \wedge wu \leq wu$ always holds, i.e., $0 \leq wu - nu \wedge wu$. Hence

$$0 \leq wu - nu \wedge wu \leq n^{-1}w^2u.$$

■

Lemma 4.1. *(Lemma 3.1, [16]) If $\gamma : \mathscr{A} \to \mathscr{B}$ is a lattice homomorphism from f-algebra to vector lattice $\mathscr{B}$. Then, $Ker(\gamma)$ is a ring and o-ideal (ℓ-ideal) in $\mathscr{A}$.*

Proof. $Ker(\gamma)$ is a subspace of $\mathscr{A}$ since it is closed under addition and scaler multiplication. In addition to this, we have to show that $Ker(\gamma)$ is solid. Assume that $0 \leq |w| \leq |v|$ and $v \in Ker(\gamma)$. It follows from lattice homomorphisms are positive, that $\gamma(0) \leq \gamma(|w|) \leq \gamma(|v|)$. Then, γ is a lattice homomorphism, so $\gamma(0) \leq |\gamma(w)| \leq |\gamma(|v|)|$ holds. Consequently, we get $0 \leq |\gamma(w)| \leq 0$, which means $w \in Ker(\gamma)$. Therefore, we conclude that $Ker(\gamma)$ is an o-ideal in $\mathscr{B}$.
For the other part, take $w, v \in Ker(\gamma)$.

$$\gamma(w.v) = \gamma(w).\gamma(v) = 0.$$

Therefore, $Ker(\psi)$ is also an r-ideal.iden ■

We are now turning our attention to the range of γ, denoted as $R(\gamma)$. Consider an arbitrary $\gamma(w) \in R(\gamma)$. Since γ is a lattice homomorphism, $|\gamma(w)| = \gamma(|w|)$ holds. It follows from $|w| \in \mathscr{A}$, that $\gamma(|w|) \in R(\gamma)$. Hence $R(\gamma)$ is a vector sublattice of $\mathscr{B}$.
The Riesz operations in $R(\gamma)$ are

$$\gamma(w) \wedge \gamma(v) = \gamma(w \wedge v) \text{ and } \gamma(w) \vee \gamma(v) = \gamma(w \vee v)$$

for all $w, v \in \mathscr{A}$. Moreover, we can introduce the multiplication $*$ in $R(\gamma)$ formulated as

$$\gamma(w) * \gamma(v) = \gamma(wv)$$

for all $w, v \in \mathscr{A}$. This multiplication is well-defined. Indeed, If $w, v, y, z \in \mathscr{A}$ such that $\gamma(w) = \gamma(y)$ and $\gamma(v) = \gamma(z)$. So, $\psi(w - y) = \gamma(w) - \gamma(y) = 0$ and $\gamma(v - z) = \gamma(v) - \gamma(z) = 0$ which means, $(w - y)$ and $(v - z) \in Ker(\gamma)$.

Furthermore, since $Ker(\gamma)$ is ℓ-ideal we deduce that

$$(w-y)v+y(v-z)=wv-yz\in Ker(\gamma)$$

Then, $\gamma(wv-yz)=\gamma(wv)-\gamma(yz)=0$. Hence, $\gamma(wv)=\gamma(yz)$, that is,

$$\gamma(w)*\gamma(v)=\gamma(y)*\gamma(z).$$

Moreover, "$*$" is associative. Take t, s and $r\in\mathscr{A}$. It follows, $\mathscr{A}$ is a Riesz algebra, that $ts\in\mathscr{A}$. Then,

$$\begin{aligned}\gamma(t)*(\gamma(s)*\gamma(r)) &= \gamma(t)*(\gamma(sr))\\ &= \gamma(t(sr))\\ &= \gamma((ts)r)\\ &= \gamma(ts)*\psi(r)\\ &= (\gamma(t)*\gamma(s))*\gamma(r).\end{aligned}$$

Proposition 4.3. *(Proposition 3.2, [16]) Assume, $\mathscr{A}$ is an f-algebra, $\mathscr{B}$ is a vector lattice and $\psi:\mathscr{A}\to\mathscr{B}$ is a lattice homomorphism. In that case, the image space $R(\gamma)$ of γ becomes an f-algebra under the multiplication $*$ defined as follow*

$$\gamma(w)*\gamma(v)=\gamma(wv)$$

$\forall w,v\in\mathscr{A}$. Besides, if $\mathscr{A}$ has unity e then, $R(\gamma)$ has $\gamma(e)$ as unity.

Proof. Let $w,v,z\in R(\gamma)$ with $z\geq 0$ and $w\wedge v=0$ hold. It follows from γ is a lattice homomorphism that, there exist s,t and $r\in\mathscr{A}$ such that $\gamma(s)=w,\gamma(t)=v$ and $\gamma(r)=z$ hold. Since $w\wedge v=0$ then,

$$w\wedge v=\gamma(s)\wedge\gamma(t)=\gamma(s\wedge t)=0.$$

Thus, $s\wedge t\in Ker(\gamma)$. Furthermore, it follows from $Ker(\gamma)$ is a r-ideal that $r(s\wedge t)\in Ker(\gamma)$, which means $\gamma(r(s\wedge t))=0$. So

$$\gamma(s)=\gamma(s)-\gamma(s\wedge t)=\gamma(s-s\wedge t) \tag{4.1}$$

$$\gamma(rt)=\gamma(rt)-\gamma(r(s\wedge t))=\gamma(rt-r(s\wedge t)) \tag{4.2}$$

Additionally, notice that $(s-t)^+ \wedge (s-t)^- = 0$ always holds, that is,

$$\begin{aligned}0 =&((s-t) \vee 0) \wedge ((t-s) \vee 0)\\ =&(s+(-t) \vee (-s)) \wedge (t+(-s) \wedge (-t))\\ =&(s-(s \wedge t)) \wedge (t-(s \wedge t)).\end{aligned}$$

Since $\mathscr{A}$ is an f-algebra, we can write

$$(s-(s \wedge t)) \wedge (rt - r(s \wedge t)) = 0.$$

Thus,

$$\gamma(s-(s \wedge t)) \wedge \gamma(rt - r(s \wedge t)) = 0. \tag{4.3}$$

Together with 4.1, 4.2 and 4.3, we get

$$\begin{aligned}w \wedge (z * v) =&\gamma(s) \wedge (\gamma(r)) * \gamma(t) = \gamma(s) \wedge \gamma(rt)\\ =&\gamma(s - s \wedge t) \wedge \gamma(rt - r(s \wedge t)) = 0.\end{aligned}$$

Therefore, $R(\gamma)$ is an f-algebra. Further, assume that $\mathscr{A}$ possesses the unity e. Then,

$$\gamma(s) * \gamma(e) = \gamma(se) = \gamma(s)$$

$\forall s \in \mathscr{A}$. Thus, $\gamma(e)$ is the unity of $R(\gamma)$. ■

Corollary 4.1. *(Corollary 3.3, [16]) Suppose $\mathscr{E}$ is a vector lattice. The existence of a lattice homomorphism from $Orth(\mathscr{A})$ onto $\mathscr{E}$ implies that $\mathscr{E}$ and $Orth(\mathscr{A})$ are isomorphic as f-algebras.*

Proof. Let γ be the lattice homomorphism from $Orth(\mathscr{E})$ onto $\mathscr{E}$ which means $\mathscr{E} = R(\gamma)$. In the previous proposition we have proved that $R(\gamma)$ is a unitary f-algebra. Hence, from Proposition 4.1.3, we conclude that γ is a bijective lattice homomorphism. ■

Example 4.1. *(Example 3.5, [16])Consider the f-algebra $C_b([1,\infty))$ with unity $i(w) = w$, $\forall w \in [1,\infty)$. Notice that*

$$\mathscr{A} = \{h \in C_b([1,\infty)) : ih \in C_b([1,\infty))\}$$

is a semi-prime f-subalgebra of $C_b([1,\infty))$. Then,

$$\begin{aligned}\tau_h :&\mathscr{A} \to \mathscr{A}\\ &g \to hg\end{aligned}$$

is an orthomorphism of $\mathscr{A}$ for an arbitrary $h \in \mathscr{A}$. So, we can define a mapping as follow

$$\begin{aligned} \tau : C_b([1,\infty)) \to & Orth(\mathscr{A}) \\ h \to & \tau_h \end{aligned}$$

$\forall h \in C_b([1,\infty))$. *Observe that $\tau(hf) = \tau(h).\tau(f)$ holds for $h, f \in C_b([1,\infty))$ so τ is an algebra homomorphism. Moreover for $h, f \in C_b([1,\infty))$ such that $h \wedge f = 0$ hold. So we have $\tau(h \wedge f) = 0$. Therefore, τ is also a lattice homomorphism. To show the surjectivity of τ, let $\pi \in Orth(\mathscr{A})$ and set $h = i\pi(i^{-1})$ for all $w \in [1,\infty)$. Then, $h \in C_b[1,\infty)$ and $\tau_h(i^{-1}) = \pi(i^{-1})$. It follows from i^{-1} is a weak order unit in $\mathscr{A}$ that $\tau_h = \pi$. Hence, $C_b([1,\infty)) \cong Orth(\mathscr{A})$.*
Let us define the operator $\psi : Orth(\mathscr{A}) \to \mathscr{A}$ such that $\psi(h(w)) = h(2w)/w$ holds $\forall w \in [1,\infty)$ and all $h \in C_b([1,\infty))$. Since $|\psi h(w)| = \psi|h(w)|$ holds then, ψ is a lattice homomorphism. However, define

$$h(w) = \begin{cases} wg(w/2)/2 & \text{if } w \in [2,\infty) \\ g(1) & \text{if } w \in [1,2). \end{cases}$$

Notice that, $h(2) = h(1)$. Hence, ψ is not injective.

Example 4.2. *(Example 3.5, [16]) Consider the multiplication structure on $C(\mathbb{R})$ formulated by*

$$f \bullet h = wf(w)h(w)$$

$\forall f, h \in C(\mathbb{R})$ and $w \in \mathbb{R}$. Notice that, $C(\mathbb{R})$ is a semi-prime f-algebra under the multiplication $\bullet$. Indeed, If $f \bullet f = wf(w)f(w) = wf^2(w) = 0$ for an arbitrary $f \in C(\mathbb{R})$ and $w \in C(\mathbb{R})$ then, $f^2(w) = 0$ which means $f(w) = 0$.
Furthermore, take $f, h \in C(\mathbb{R})$ with $f \wedge h = f(w) \wedge g(w) = 0$. Then, we obtain $g \bullet f \wedge h = wg(w)f(w) \wedge h(w) = 0\ \forall w \in \mathbb{R}$
Moreover, let γ be the lattice homomorphism from $C(\mathbb{R})$ to semi-prime f-algebra $\mathscr{B}$ defined by

$$\gamma(f)(w) = f(0).$$

Notice that, $\gamma(f) * \gamma(h) = \gamma(f \bullet h) = 0$ holds $\forall f, h \in C(\mathbb{R})$, which means, $R(\gamma)$ is not semi-prime.

Example 4.3. *(Example 3.6, [16]) Consider the unitary f-algebra of real sequences with the standard operations and order $\mathbb{R}^{\mathbb{N}}$ and $\gamma : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined as $\gamma(\{a_n\}_{n\geq 0}) = \{a_1 n\}_{n\geq 0}$. Then, $|\gamma(\{a_n\}_{n\geq 0})| = |\{a_1 n\}_{n\geq 0}| = \gamma(|\{a_n\}_{n\geq 0}|)$ hence γ is a lattice homomorphism. Observe that, the image space $R(\gamma) = \{\{an\}_{n\geq 0} :$*

$a \geq 0\}$ *with the multiplication* $*$ *formulated as*

$$\{\alpha n\}_{n\geq 0} * \{\beta n\}_{n\geq 0} = \{\alpha\beta n\}_{n\geq 0}$$

is not an f*-algebra. Indeed, given* $\{\alpha n\}_{n\geq 0}$ *and* $\{\beta n\}_{n\geq 0} \in R(\gamma)$*. Notice that* $\{\alpha n\}_{n\geq 0} \wedge \{\beta n\}_{n\geq 0} = 0$ *does not imply* $\{\alpha n\}_{n\geq 0} * \{\beta n\}_{n\geq 0} = \{\alpha\beta n\}_{n\geq 0} = 0$ *which means* $R(\gamma)$ *is not almost* f*-algebra. Thus,* $R(\gamma)$ *is not an* f*-algebra.*

Proposition 4.4. *(Proposition 3.7, [16]) Let* $\mathscr{A}$ *be a semi-prime* f*-algebra and* $\mathscr{E}$ *be a Riesz subspace of* $\mathscr{A}$*. Suppose that* $\mathscr{E}$ *is equipped with the multiplicative structure* $*$*, ensuring that* $\mathscr{E}$ *becomes an* f*-algebra with unity* e*. In that case, the assertions below are equivalent.*

1. $\mathscr{E}$ *is an* f*-subalgebra of* $\mathscr{A}$ *equipped with* $*$*, that is,* $*$ *coincides with the multiplication in* $\mathscr{A}$*.*
2. e *is idempotent in* $\mathscr{A}$*.*

Proof. For the first part, assume 1. holds then, $e * e = e$. For the other one, assume e is idempotent. It follows from $\mathscr{A}$ is a r-ideal in $Orth(\mathscr{A})$ that $\mathscr{A}$ has a unity i. In addition to this, for an arbitrary $w \in \mathscr{E}^+$ such that $0 \leq w \leq e$. We obtain

$$0 \leq |we - w| = |e - i|w \leq |e - i|e = |e^2 - e| = 0$$

which means $we = w$. Moreover using Lemma 3.2 we get the following inequality

$$\bigwedge_k \{|w - kn^{-1}e| : k = 0, 1, ..., n\} \leq n^{-1}e, \quad n \in \mathbb{N}.$$

Squaring the both sides, we get

$$\begin{aligned} w * w - \bigvee_k \left\{\frac{2k}{n}w - \frac{k^2}{n^2}e : k = 0, ..., n\right\} &= \bigwedge_k \left\{w * w - \frac{2k}{n}w + \frac{k^2}{n^2}e : k = 0, ..., n\right\} \\ &= \bigwedge_k \left\{\left(w - \frac{k}{n}e\right) * \left(w - \frac{k}{n}e\right) : k = 0, ..., n\right\} \\ &\leq \frac{1}{n^2}e, \end{aligned}$$

for $n \in \mathbb{N}$. In other respects, we can write

$$\begin{aligned} w^2 - \bigvee_k \left\{ \frac{2k}{n} w - \frac{k^2}{n^2} e : k = 0, ..., n \right\} &= \bigwedge_k \left\{ w^2 - \frac{2k}{n} w + \frac{k^2}{n^2} e : k = 0, ..., n \right\} \\ &= \bigwedge_k \left\{ \left(w - \frac{k}{n} \right)^2 : k = 0, ..., n \right\} \leq \frac{1}{n^2} e \end{aligned}$$

for $n \in \mathbb{N}$. From the foregoing inequalites, we get $w * w = w^2$ holds $\forall\, w \in \mathscr{E}^+$ whenever $0 \leq w \leq e$. Subsequently, in view of Proposition 3.1

$$\begin{aligned} &0 \leq w - w \wedge ne \leq w * w \frac{1}{n}, \;\; n \in \mathbb{N} \\ &0 \leq w - w \wedge ne \leq w^2 \frac{1}{n}, \;\; n \in \mathbb{N} \end{aligned}$$

we obtain $w * w = w^2, \forall w \in \mathscr{E}^+$. Furthermore, for an arbitrary $w \in \mathscr{E}$ we can write $w = w^+ - w^-$ and $w^+ * w^- = w^+ w^- = 0$. In conclusion, $w * w = w^2$ holds $\forall w \in \mathscr{E}$. Moreover, observe that, for arbitrary $s, t \in \mathscr{E}$ we have

$$\begin{aligned} s * t = &\left(\frac{s+t}{2} \right) * \left(\frac{s+t}{2} \right) - \left(\frac{s-t}{2} \right) * \left(\frac{s-t}{2} \right) \\ = &\left(\frac{s+t}{2} \right)^2 - \left(\frac{s-t}{2} \right)^2 \\ = &st. \end{aligned}$$

This means, * coincides with the multiplication in $\mathscr{A}$. ■

Theorem 4.1. *(Theorem 3.8, [16]) Let $\gamma : \mathscr{A} \to \mathscr{B}$ be a lattice homomorphism from f-algebra $\mathscr{A}$ to semi-prime f-algebra $\mathscr{B}$. The range space $R(\gamma)$ is a Riesz subspace and f- algebra with unity $\gamma(e)$ under the multiplication * formulated by*

$$\gamma(w) * \gamma(v) = \gamma(wv) \; \forall w, v \in \mathscr{A}.$$

Then, the assertions below are identical.

1. *R(γ) is an f-subalgebra of $\mathscr{B}$.*
2. *γ is an into algebra homomorphism.*
3. *$\gamma(e)^2 = \gamma(e)$, that is to say, $\gamma(e)$ is idempotent under the multiplication *.*

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5

CONCLUSION

This study, conducted with f-algebras, can be explored for its potential applicability to almost f-algebras or d-algebras in the future.

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1. Ö. Gök, E. Kılıç,"Subalgeras of lattice ordered algebras", *10th International Congress on Fundamental and Applied Sciences*, 2023.