DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

NEAT AND CONEAT SUBGROUPS

by
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İZMİR

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We have read the thesis entitled **"NEAT AND CONEAT SUBGROUPS"** completed by **SALAHATTİN ÖZDEMİR** under supervision of **ASSIST. PROF. DR. ENGİN MERMUT** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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NEAT AND CONEAT SUBGROUPS

ABSTRACT

We survey the properties of neat subgroups of abelian groups. Coneat subgroups are always neat subgroups. Conversely, if a torsion group A with all but finitely many primary components zero is a neat subgroup of a group B, then it is coneat in B. We cannot generalize this to any torsion group A. We give another proof for the structure of neat-injective abelian groups using the structure of reduced algebraically compact groups. This proof is generalized to give the structure of c-injective modules over Dedekind domains. A submodule V of a module M is a coneat submodule of M if and only if V is a Rad-supplement of a submodule U of M in M. We investigate some properties of Rad-supplemented modules over any ring R. A module M is Rad-supplemented if and only if $M/P(M)$ is Rad-supplemented, where $P(M)$ denotes the sum of all radical submodules of M. A reduced Rad-supplemented module is weakly supplemented. A reduced module is totally Rad-supplemented if and only if it is totally supplemented. A reduced Rad-supplemented module is coatomic. For a left reduced ring R, R is left perfect if and only if every R-module is Rad-supplemented. For a commutative noetherian ring R, if an R-module M is reduced and Rad-supplemented, then it is supplemented. We also investigate some properties of Rad-supplemented modules over discrete valuation rings and Dedekind domains. For a discrete valuation ring R which is not a field, an R-module M is Rad-supplemented if and only if M is the direct sum of its divisible part, a finitely generated free R-module and a bounded R-module. A module M over a Dedekind domain R is Rad-supplemented if and only if the quotient module of M by its divisible part is supplemented. For a Dedekind domain R, a reduced R-module is Rad-supplemented if and only if it is supplemented.

Keywords: Rad-supplement, coneat, neat, supplement, complement , c-injective, coatomic module, proper class, closed submodule, high subgroup.

DÜZENLİ VE KODÜZENLİ ALT GRUPLAR

ÖZ

Abel grupların düzenli alt gruplarının özellikleri incelendi. Kodüzenli alt gruplar daima düzenlidir. Tersine eğer bir A grubu sonlu sayıdaki asal parçaları sıfırdan farklı bir burulmalı grup ise ve bir B grubunun düzenli bir alt grubu ise A grubu B'de kodüzenlidir. Bunu herhangi bir burulmalı grup A için genelleyemeyiz. İndirgenmiş cebirsel kompakt grupların yapısını kullanarak, düzenli-injektif abel grupların yapısı farklı bir yolla ispatlandı. Bu ispat Dedekind tamlık bölgesi üzerinde c-injektif modüllerin yapısını vermek için genelleştirildi. M modülünün bir V alt modülü M'de kodüzenlidir ancak ve ancak V alt modülü M'nin bir U alt modülünün M'de bir Rad-tümleyeni ise. Herhangi bir halka üzerinde, Rad-tümlenen modüllerin bazı özellikleri hakkında araştırma yapıldı. Bir M modülü Rad-tümlenendir ancak ve ancak $M/P(M)$ Rad-tümlenen ise, burada $P(M)$, M'nin radikal olan alt modüllerinin toplamını ifade etmektedir. İndirgenmiş Rad-tümlenen bir modül zayıf tümlenendir. İndirgenmiş bir modül tamamen zayıf Rad-tümlenendir ancak ve ancak tamamen tümlenen ise. İndirgenmiş Rad-tümlenen bir modül koatomiktir. Sol indirgenmiş bir R halkası icin, R sol mükemmel bir halkadır ancak ve ancak bütün R-modüller Rad-tümlenen ise. Değişmeli bir noether R halkası icin, bir R-modül indirgenmiş Rad-tümlenen ise tümlenendir. Ayrıca bir ayrık değerli tamlık bölgesi ve bir Dedekind tamlık bölgesi üzerinde, Rad-tümlenen modüllerin bazı özellikleri incelendi. Bir ayrık değerli tamlık bölgesi üzerinde, bir M modülü Rad-tümlenendir ancak ve ancak M, bölünebilir kısmının, sonlu türetilmiş serbest bir R-modülün ve sınırlı bir R-modülün direkt toplamı ise. Bir Dedekind tamlık bölgesi üzerinde, bir M modülü Rad-tümlenendir ancak ve ancak M'in bölünebilir kısmına bölüm modülü tümlenen ise. Bir Dedekind tamlık bölgesi üzerinde, indirgenmiş bir modül Rad-tümlenendir ancak ve ancak tümlenen ise.

Anahtar Sözcükler: Rad-tümleyen, kodüzenli, düzenli, tümleyen, tamamlayan, c-injektif, koatomik modül, öz sınıf, kapalı alt modül, yüksek alt grup.

CONTENTS

CHAPTER ONE
INTRODUCTION

In this introductory chapter, we will give the motivating ideas for our thesis problems and summarize what we have done. To explain these problems and results we will summarize what proper classes are in Section 1.3. See Section 1.1 for the definition of *complement* and *supplement*. What is assumed as preliminaries notions is sketched in Section 1.2; see also Chapter 2. See Section 1.4 for the definition of *neat* subgroups and *neat* submodules. For the definition of *coneat* submodules see Section 1.5. In section 1.6, we will summarize the main results of this thesis.

The proper class $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ [$\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$] consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms such that $\mathrm{Im}(f)$ is a complement [resp. supplement] in B. $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ [$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$] consists of all short exact sequences of R-modules and R-module homomorphisms with respect to which every simple module is projective [resp. every module with zero radical is injective].

In Chapter 4, we survey the properties of neat subgroups and we seek for the neat subgroups which are coneat in Section 4.6. We deal with Rad-supplemented modules for an associative ring R with unity, using the known results for supplemented modules, in Chapter 6. In Section 6.1, we deal with Rad-supplemented modules for an arbitrary ring R. The next step is to give a characterization of Rad-supplemented modules for the case R is a Dedekind domain in Section 6.4. We deal with c-injective modules in Chapter 5. Another proof for the structure of neat-injective abelian groups is given using the structure of reduced algebraically compact groups (see Section 5.2). This is generalized to give the structure of c-injective modules over Dedekind domains in Mermut et al. (2007).

1.1 Complements and Supplements

We try to understand a module through its submodules, or better to say through its relation with its submodules. More precisely, let R denote an associative ring with unity, B be an R-module and let K be a submodule of B. It would be best if K is a *direct summand* of B, that is if there exists another submodule A of B such that $B = K \oplus A$; that means,

$$B = K + A \quad \text{and} \quad K \cap A = 0.$$

When K is not a direct summand, we hope at least to retain one of these conditions. These give rise to two concepts: *complement* and *supplement*.

If A is a submodule of B such that $B = K + A$ (that is the above first condition for direct sum holds) and A is *minimal* with respect to this property (that is there is *no* submodule $\tilde{A}$ of B such that $\tilde{A} \subsetneqq A$ but still $B = K + \tilde{A}$), then A is called a *supplement of* K *in* B and K is said to *have a supplement in* B. Equivalently, $K + A = B$ and $K \cap A$ is *small* (=*superfluous*) in A (which is denoted by $K \cap A \ll A$, meaning that for *no* proper submodule X of A, $K \cap A + X = A$). The module K need *not* have a supplement in B. If a module B is such that every submodule of it has a supplement in it, then it is called a *supplemented module* . For the definitions and related properties see Wisbauer (1991, §41) and Clark et al. (2006, §20).

If A is a submodule of B such that $K \cap A = 0$ (that is the above second condition for direct sum holds) and A is *maximal* with respect to this property (that is there is *no* submodule $\tilde{A}$ of B such that $\tilde{A} \supsetneqq A$ but still $K \cap \tilde{A} = 0$), then A is called a *complement of* K (or K-high) *in* B and K is said to *have a complement in* B. When $K = B^1$, the first Ulm subgroup of B (i.e., $\bigcap_{n=1}^{\infty} nB$, that is, the subgroup of elements of infinite height in B), A will be called *high* in B. By Zorn's Lemma, it is seen that K always has a complement in B (unlike the case for supplements). In fact, by Zorn's Lemma, we know that if we have a submodule A' of B such that $A' \cap K = 0$, then there exists a complement A of K in B such that $A \supseteq A'$. See the monograph Dung et al. (1994) for a survey of results in the related concepts.

A submodule A of a module B is said to be a *complement in* B if A is a complement of some submodule of B; shortly, we also say that A is a *complement submodule of* B in this case and denote this by $A \leq_c B$. It is said that A is *closed in* B if A has *no* proper essential extension in B, that is, there exists *no* submodule $\tilde{A}$ of B such that $A \subsetneqq \tilde{A}$ and A is *essential* in $\tilde{A}$ (which is denoted by $A \trianglelefteq \tilde{A}$ and meaning that for every *nonzero* submodule X of $\tilde{A}$, we have $A \cap X \neq 0$). We also say in this case that A is a *closed submodule* and it is known that closed submodules and complement submodules in a module coincide (see Dung et al. (1994, §1)). So the "c" in the notation $A \leq_c B$ can be interpreted as complement or closed. Dually, a submodule A of a module B is said to be a *supplement in* B if A is a supplement of some submodule of B; shortly, we also say that A is a *supplement submodule of* B.

1.2 Preliminaries, Terminology and Notation

Throughout this thesis, by a *ring* we mean an associative ring with unity; R will denote such a general ring, unless otherwise stated. So, *if nothing is said about* R *in the statement of a theorem, proposition, etc., then that means* R *is just an arbitrary ring.* We consider unital left R-modules; an R-module M will mean *left* R-module M and is denoted by ${}_RM$. A *right* R-module M is denoted by M_R. R-$\mathcal{M}od$ denotes the category of all *left* R-modules. $\mathcal{M}od$-R denotes the category of *right* R-modules. $\mathbb{Z}$ denotes the ring of integers. *Group* will mean *abelian group* ($\mathbb{Z}$-module) only. *Integral domain*, or shortly *domain*, will mean a nonzero ring without zero divisors, *not* necessarily commutative. But following the general convention a *principal ideal domain* (shortly *PID*) will mean a *commutative* domain in which every ideal is principal, i.e., generated by one element. A *unique factorization domain* (shortly *UFD*) is a commutative domain R in which every nonzero element $r \in R$ which is not a unit can be written as a finite product of irreducibles p_i of R (not necessarily distinct): $r = p_1p_2 \cdots p_n$ and this decomposition is *unique up to associates.* Also a Dedekind domain is commutative.

All definitions not given here can be found in Wisbauer (1991), Dung et al. (1994), Anderson & Fuller (1992), Clark et al. (2006) and Fuchs (1970).

*The **notation** we use have been given on pages (vii- viii) just before this chapter.*

*The **index** have been given on pages 165-168 at the end of this thesis.*

We do not delve into the details of definitions of every term in modules, rings and homological algebra. Essentially, we accept fundamentals of module theory, categories, pullback and pushout, the Hom and tensor ($\otimes$) functors, projective modules, injective modules, flat modules, homology functor, the functor $\mathrm{Ext}_R = \mathrm{Ext}_R^1$ are known.

For more details in homological algebra see the books Alizade & Pancar (1999), Rotman (1979), Cartan & Eilenberg (1956) and Maclane (1963). We will explain most of the terms and summarize the necessary concepts. See Chapter 2 for properties for essential submodules, small submodules, radical, socle, localization, artinian rings, noetherian rings, discrete valuation rings, Dedekind domains, perfect rings and some facts in modules.

1.3 Proper Classes of R-modules for a ring R

Let $\mathcal{P}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.3.1}$$

belongs to $\mathcal{P}$, then f is said to be a $\mathcal{P}$-*monomorphism* and g is said to be a $\mathcal{P}$-*epimorphism* (both are said to be $\mathcal{P}$-*proper* and the short exact sequence is said to be a $\mathcal{P}$-*proper* short exact sequence.). The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions (see Buschbaum (1959), Maclane (1963, Ch. 12, §4), Stenström (1967a, §2) and Sklyarenko (1978, Introduction)):

(i) If a short exact sequence $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.

(ii) $\mathcal{P}$ contains all splitting short exact sequences.

(iii) The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined. The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.

(iv) If g and f are monomorphisms, and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism. If g and f are epimorphisms, and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

An important example for proper classes in abelian groups is $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$: The proper class of all short exact sequences (1.3.1) of abelian groups and abelian group homomorphisms such that $\operatorname{Im}(f)$ is a pure subgroup of B, where a subgroup A of a group B is *pure* in B if $A \cap nB = nA$ for all integers n (see Fuchs (1970, §26-30) for the important notion of purity in abelian groups). The short exact sequences in $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ are called *pure-exact sequences* of abelian groups.

The smallest proper class of R-modules consists of only *splitting* short exact sequences of R-modules which we denote by $\mathcal{S}plit_{R\text{-}\mathcal{M}od}$. The largest proper class of R-modules consists of *all* short exact sequences of R-modules which we denote by $\mathcal{A}bs_{R\text{-}\mathcal{M}od}$ (*absolute purity*).

For a proper class $\mathcal{P}$ of R-modules, call a submodule A of a module B a $\mathcal{P}$-submodule of B, if the inclusion monomorphism $i_A : A \to B$, $i_A(a) = a$, $a \in A$, is a $\mathcal{P}$-monomorphism.

1.3.1 Projectives and Injectives with respect to a Proper Class

Take a short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms.

An R-module M is said to be *projective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the epimorphism* g if any of the following equivalent conditions holds:

(i) every diagram

$$\begin{tikzcd} \mathbb{E}: & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\ & & & & M \arrow[ul, dashed, "\tilde{\gamma}"] \arrow[u, "\gamma"'] & \end{tikzcd}$$

where the first row is $\mathbb{E}$ and $\gamma : M \longrightarrow C$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\gamma} : M \longrightarrow B$; that is, for every homomorphism $\gamma : M \longrightarrow C$, there exits a homomorphism $\tilde{\gamma} : M \longrightarrow B$ such that $g \circ \tilde{\gamma} = \gamma$.

(ii) The sequence

$$\mathrm{Hom}(M, \mathbb{E}) : \quad 0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \longrightarrow 0$$

is exact.

Dually, an R-module M is said to be *injective with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the monomorphism* f if any of the following equivalent conditions holds:

(i) every diagram

$$\begin{tikzcd} \mathbb{E}: & 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[dl, dashed, "\tilde{\alpha}"] & C \arrow[r] & 0 \\ & & M & & & \end{tikzcd}$$

where the first row is $\mathbb{E}$ and $\alpha : A \longrightarrow M$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\alpha} : B \longrightarrow M$; that is, for every homomorphism $\alpha : A \longrightarrow M$, there exists a homomorphism $\tilde{\alpha} : B \longrightarrow M$ such that $\tilde{\alpha} \circ f = \alpha$.

(ii) The sequence

$$\mathrm{Hom}(\mathbb{E}, M) : \quad 0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{g^*} \mathrm{Hom}(B, M) \xrightarrow{f^*} \mathrm{Hom}(A, M) \longrightarrow 0 \text{ is exact.}$$

An R-module M is said to be *$\mathcal{P}$-projective* [*$\mathcal{P}$-injective*] if it is projective [injective] with respect to all short exact sequences in $\mathcal{P}$. Denote all $\mathcal{P}$-projective [$\mathcal{P}$-injective] modules by $\pi(\mathcal{P})$ [$\iota(\mathcal{P})$].

1.3.2 Projectively generated Proper Classes

For a given class $\mathcal{M}$ of modules, denote by $\pi^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\mathrm{Hom}(M,\mathbb{E})$ is exact for all $M \in \mathcal{M}$, that is,

$$\pi^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} |\, \mathrm{Hom}(M,\mathbb{E}) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\pi^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective. It is called the proper class *projectively generated* by $\mathcal{M}$.

1.3.3 Injectively generated Proper Classes

For a given class $\mathcal{M}$ of modules, denote by $\iota^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $\mathrm{Hom}(\mathbb{E},M)$ is exact for all $M \in \mathcal{M}$, that is,

$$\iota^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} |\, \mathrm{Hom}(\mathbb{E},M) \text{ is exact for all } M \in \mathcal{M}\}.$$

$\iota^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-injective. It is called the proper class *injectively generated* by $\mathcal{M}$.

An injective module is called *elementary* if it coincides with the injective envelope of some *cyclic* submodule. Such modules form a set and every injective module can be embedded in a direct product of elementary injective modules (Sklyarenko, 1978, Lemma 3.1). A subclass $\mathcal{M}$ of a class $\overline{\mathcal{M}}$ of modules is called an *injective basis* for $\overline{\mathcal{M}}$ if every module in $\overline{\mathcal{M}}$ is a direct summand of a direct product of modules in $\mathcal{M}$ and of certain elementary injective modules. If $\mathcal{M}$ is a set, then for the proper class $\iota^{-1}(\mathcal{M})$,

$\mathcal{M}$ is an injective basis for the class of all $\iota^{-1}(\mathcal{M})$-injective modules (Sklyarenko, 1978, Proposition 3.3).

1.3.4 Flatly generated Proper Classes

When the ring R is *not* commutative, we must be careful with the sides for the tensor product analogues of projectives and injectives with respect to a proper class. Remember that by an R-module, we mean a *left* R-module.

Take a short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms. We say that a *right* R-module M is *flat with respect to the short exact sequence* $\mathbb{E}$, or *with respect to the monomorphism* g if

$$M \otimes \mathbb{E}: \quad 0 \longrightarrow M \otimes A \xrightarrow{1_m \otimes f} M \otimes B \xrightarrow{1_m \otimes g} M \otimes C \longrightarrow 0$$

is exact.

A *right* R-module M is said to be $\mathcal{P}$-*flat* if M is flat with respect to every short exact sequence $\mathbb{E} \in \mathcal{P}$, that is, $M \otimes \mathbb{E}$ is exact for every $\mathbb{E}$ in $\mathcal{P}$.

For a given class $\mathcal{M}$ of *right* R-modules, denote by $\tau^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $M \otimes \mathbb{E}$ is exact for all $M \in \mathcal{M}$:

$$\tau^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} | M \otimes \mathbb{E} \text{ is exact for all } M \in \mathcal{M}\}.$$

$\tau^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ of (left) R-modules for which each $M \in \mathcal{M}$ is $\mathcal{P}$-flat. It is called the proper class *flatly generated* by the class $\mathcal{M}$ of *right* R-modules.

When the ring R is commutative, there is no need to mention the sides of the modules since a right R-module may also be considered as a left R-module and vice versa.

1.4 Neat Subgroups and Neat Submodules

The classes $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ defined in the introduction to this chapter really form proper classes as has been shown more generally by Generalov (1978, Theorem 1), Generalov (1983, Theorem 1), Stenström (1967b, Proposition 4 and Remark after Proposition 6). In Stenström (1967b), following the terminology in abelian groups, the term 'high' is used instead of complements and low' for supplements. Generalov (1978, 1983) use the terminology 'high' and 'cohigh' for complements and supplements, and give more general definitions for proper classes of complements and supplements related to another given proper class (motivated by the considerations as pure-high extensions and neat-high extensions in Harrison et al. (1963)); 'weak purity' in Generalov (1978) is what we denote by $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$. See also Erdoğan (2004, Theorem 2.7.15 and Theorem 3.1.2) for the proofs of $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ being proper classes.

A subgroup A of a group B is said to be *neat* in B if $A \cap pB = pA$ for all prime numbers p (see Fuchs (1970, §31); the notion of neat subgroup has been introduced in Honda (1956, pp. 43-44)). This is a weaker condition than being a pure subgroup. What is important for us is that neat subgroups of an abelian group coincide with complements in that group (see Theorem 4.2.4). Denote by $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$, the proper class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian groups and abelian group homomorphisms where $\mathrm{Im}(f)$ is a neat subgroup of B; call such short exact sequences *neat-exact sequences* of abelian groups (like the terminology for pure-exact sequences). The following result is one of the motivations for us to deal with complements and its dual supplements: The proper class $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ is projectively generated, flatly generated and injectively generated by simple groups $\mathbb{Z}/p\mathbb{Z}$, p a prime number:

$$\begin{aligned}
\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} &= \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} = \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}) \\
&= \tau^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\}) = \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\})
\end{aligned}$$

(see Theorem 4.4.8).

The second equality $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od} = \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} | p \text{ prime}\})$ was the motivation to define for each ring R, as said in the introduction to this chapter,

$$\begin{aligned} \mathcal{N}eat_{R\text{-}\mathcal{M}od} &\overset{def.}{=} \pi^{-1}(\{\text{all simple } R\text{-modules}\}) \\ &= \pi^{-1}(\{R/P | P \text{ maximal left ideal of } R\}), \end{aligned}$$

following Stenström (1967a, 9.6) (and Stenström (1967b, §3)).

For a submodule A of an R-module B, say that A is a *neat submodule* of B, or say that A is *neat in* B, if A is a $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule. We always have $\mathcal{C}ompl_{R\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{R\text{-}\mathcal{M}od}$ for every ring R (by Stenström (1967b, Proposition 5)).

If R is a commutative noetherian ring in which every *nonzero* prime ideal is maximal, then

$$\mathcal{C}ompl_{R\text{-}\mathcal{M}od} = \mathcal{N}eat_{R\text{-}\mathcal{M}od},$$

by Stenström (1967b, Corollary to Proposition 8).

Generalov (1978, Theorem 5) gives a characterization of this equality in terms of the ring R:

$$\mathcal{C}ompl_{R\text{-}\mathcal{M}od} = \mathcal{N}eat_{R\text{-}\mathcal{M}od} \qquad \textit{if and only if} \qquad R \text{ is a left C-ring.}$$

The notion of C-ring has been introduced by Renault (1964): A ring R is said to be a *left* C-ring if for every (left) R-module B and for every essential proper submodule A of B, $\mathrm{Soc}(B/A) \neq 0$, that is B/A has a simple submodule. Similarly *right* C-rings are defined. For example, a commutative noetherian ring in which every *nonzero* prime ideal is maximal is a C-ring. So, of course, in particular a Dedekind domain and therefore a PID is also a C-ring (see for example Mermut (2004, Proposition 5.1.6)).

1.5 Coneat Submodules

1.5.1 *The Proper Class* $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$

We have,

$$\begin{aligned}\mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \pi^{-1}(\{\text{all semisimple } R\text{-modules}\}) \\ &= \pi^{-1}(\{M \mid \operatorname{Soc} M = M,\ M \text{ an } R\text{-module}\}).\end{aligned}$$

Dualizing this, we obtain the proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ as said in the introduction to this chapter by

$$\begin{aligned}\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \iota^{-1}(\{\text{all } R\text{-modules with zero radical}\}) \\ &= \iota^{-1}(\{M \mid \operatorname{Rad} M = 0,\ M \text{ an } R\text{-module}\}).\end{aligned}$$

The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ has been introduced in Mermut (2004, §3.4).

If A is a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule of an R-module B, the we say that A is a *coneat submodule* of B, or that the submodule A of the module B is *coneat in* B.

For any ring R (see Mermut (2004, Proposition 3.4.1)),

$$\mathcal{S}uppl_{R\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} \subseteq \iota^{-1}(\{\text{ all (semi-)simple } R\text{-modules}\}).$$

Without using the terminology relative homological algebra, we give the definitions of neat and coneat submodules as follows:

A monomorphism $f : K \to L$ is called *neat* if any simple module S is *projective* relative to the projection $L \to L/\operatorname{Im} f$, that is, the Hom sequence

$$\operatorname{Hom}(S,\, L) \to \operatorname{Hom}(S,\, L/\operatorname{Im} f) \to 0$$

is exact.

Dually, a monomorphism $f : K \to L$ is called *coneat* if any module M with $\operatorname{Rad} M = 0$ is injective with respect to it, that is, the Hom sequence

$$\operatorname{Hom}(L,\, M) \to \operatorname{Hom}(K,\, M) \to 0$$

is exact (see Clark et al. (2006, §10)).

Remark 1.5.1. The notion *neat defined in this thesis* and the notion *neat introduced in* Bowe (1972) does not coincide. Similarly, the notion *coneat homomorphism*, dualization of the neat homomorphism introduced in Bowe (1972), and the notion *coneat defined in this thesis* does not coincide. Zöschinger (1978) introduced , which does not coincide with our coneat monomorphism defined in this thesis.

1.5.2 Rad-supplement Submodules

The notion τ-complement and dually τ-supplement has been introduced in Clark et al. (2006, 10.11):

Let τ be a radical for $\sigma[M]$ with associated classes $\mathbb{T}_\tau$ and $\mathbb{F}_\tau$. Then for a submodule $K \leq L$ where $L \in \sigma[M]$, the following statements are equivalent:

(i) Every $N \in \mathbb{F}_\tau$ is injective with respect to the inclusion $K \hookrightarrow L$;

(ii) there exists a submodule $U \leq L$ such that

$$U + K = L \text{ and } U \cap K = \tau(K);$$

(iii) there exists a submodule $U \leq L$ such that

$$U + K = L \text{ and } U \cap K \leq \tau(K).$$

If these conditions are satisfied, then K is called a *τ-supplement* in L.

For more details for the radical τ for $\sigma[M]$, see Clark et al. (2006, Ch. 2). Now we consider the case $\tau = \mathrm{Rad}$ for R-$\mathcal{M}od$. The condition (iii) is like being a coneat, so let us define being a *Rad-supplement* of a submodule in the module: for submodules U, V of a module M, V is said to be a Rad-supplement of U in M or U is said to have a Rad-supplement V in M if

$$U + V = M \text{ and } U \cap V \leq \mathrm{Rad}\, V.$$

So, a submodule V of a module M is said to be a *Rad-supplement* in M if V is a Rad-supplement of a submodule U of M in M.

Being a coneat submodule is like being a supplement: For a submodule V of a module M, V is coneat in M if and only if there exists a submodule $U \leq M$ such that

$$U + V = M \quad \text{and} \quad U \cap V \leq \operatorname{Rad} V$$

(Proposition 3.3.1).

So, a submodule V of a module M is a coneat submodule of M if and only if V is a Rad-supplement of a submodule $U \leq M$ in M.

Dual to the relationship between the neat and complement submodules, there is a relationship between *coneat* submodules and *supplement* submodules; every supplement submodule is coneat (see Mermut (2004, Proposition 3.4.1)).

1.6 Main Results of This Thesis

1.6.1 Neat Subgroups which are Coneat

Coneat subgroups are always neat (see Mermut (2004, Proposition 3.4.1)). We deal with neat subgroups which are coneat. We search for the answer of the question that in which case neat subgroups and coneat subgroups coincide. A torsion group A of a group B such that all but finitely many primary components of A are zero is neat if and only if it is coneat. We cannot generalize this result to any torsion group A. The torsion subgroup $\bigoplus_p \mathbb{Z}/p\mathbb{Z}$ of the group $\prod_p \mathbb{Z}/p\mathbb{Z}$ is neat in $\prod_p \mathbb{Z}/p\mathbb{Z}$, but it is not coneat (Example 4.6.7).

1.6.2 c-injective Modules over a Dedekind Domain

Let X and M be R-modules. The module X is called *M-c-injective* if, for every *closed* submodule A of M, every homomorphism $f : A \longrightarrow X$ can be lifted to M, i.e., there exists a homomorphism $\tilde{f} : M \longrightarrow X$ such that $\tilde{f}|_A = f$:

$$\begin{tikzcd} A \leq_c M \arrow[d, "f"'] \arrow[d, dashed, from=1-1, "\tilde{f}"] \\ X \end{tikzcd}$$

A module M is called *self-c-injective* if M is M-c-injective. For a discussion of c-injectivity and related problems see Santa-Clara & Smith (2000), Smith (2000a) and Santa-Clara & Smith (2004).

We say that a module X is ***c-injective*** if it is M-c-injective for every module M.

Since neat-injective abelian groups are nothing but c-injective abelian groups (since $\mathcal{C}ompl_{\mathbb{Z}\text{-}\mathcal{M}od} = \mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$), we have given the details for the structure of neat-injective abelian groups in Harrison et al. (1963, Lemma 4) and we have given another proof of this result suggested by Bill Wickless (see Section 5.2). So following this proof c-injective modules over Dedekind domains are similarly described in Mermut et al. (2007) (see Section 5.3).

1.6.3 Rad-Supplemented Modules over a Dedekind Domain

A module M is called *Rad-supplemented* if every submodule of M has a Rad-supplement.

The notion *Rad-supplemented* modules was introduced in Wang & Ding (2006) as *generalized supplemented modules* , or shortly, GS-modules. Firstly, we investigate some properties of Rad-supplemented modules over any ring R in Section 6.1. Every supplement submodule is a Rad-supplement, and so every Rad-supplemented module is supplemented (Proposition 6.1.4). If a module M is reduced and Rad-supplemented, then it is weakly supplemented (Proposition 6.2.15). We give a very useful lemma that if a module M is reduced and Rad-supplemented, then it is coatomic (Lemma 6.1.19). For a left reduced ring R, R is left perfect if and only if every left R-module is Rad-supplemented (Theorem 6.1.27). For a commutative noetherian ring R, if a module M is reduced and Rad-supplemented, then it is supplemented (Lemma 6.2.22). For all details see Chapter 6.

For a discrete valuation ring R with quotient field $K \neq R$, an R-module M is Rad-supplemented if and only if M is isomorphic to $M_1 \oplus M_2 \oplus M_3 \oplus M_4$, where M_1, M_2, M_3, M_4 are R-modules such that $M_1 \cong R^a$, $M_2 \cong K^{(I_1)}$, $M_3 \cong (K/R)^{(I_2)}$, and $M_4 p^b = 0$ for some integers $a, b \geq 0$ and arbitrary index sets I_1, I_2 (Theorem 6.3.4). Over a Dedekind domain R, a module M is Rad-supplemented if and only if the quotient of M by its divisible part is supplemented (Theorem 6.4.2). We give a proof of this using the structure of supplemented modules over a Dedekind domain.

CHAPTER TWO
PRELIMINARIES

In this chapter we shall give definitions and properties of some terms in rings and modules which will be used frequently. In the first two sections, we give some elementary properties in abelian groups. For more details in abelian groups see the books Fuchs (1970), Kaplansky (1969) and Griffith (1970). Some useful properties for essential submodules and small submodules are given in section 2.4. In Section 2.5, we survey the properties of radical and socle. In Sections 2.13, we give the definitions of cosmall inclusions and coclosed submodules; for more details see Clark et al. (2006). In Section 2.12, we give some properties for coatomic and reduced modules; for more details see the papers Zöschinger (1974a), Zöschinger (1980). Some facts for supplemented modules are given in 2.11; for more details see the books Wisbauer (1991) and Clark et al. (2006). In Section 2.7, we shall give some properties of artinian rings and noetherian rings. In the other sections of this chapter, we have given some properties for localization, discrete valuation rings, Dedekind domains. For more details for modules and rings see for example the books Anderson & Fuller (1992), Lam (2001, 1999), Facchini (1998), Atiyah & Macdonald (1969), Kasch (1982) and Dummit & Foote (2004).

2.1 Inverse Limit of abelian groups

Definition 2.1.1. A partially ordered set I is called a *directed set* if for every $i, j \in I$ there exists always a $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 2.1.2. Assume A_i $(i \in I)$ is a system of groups, indexed by a directed set I, and for each pair $i, j \in I$ with $i \leq j$, there is given a homomorphism

$$\pi_i^j : A_j \to A_i \quad (i \leq j)$$

such that

(i) π_i^j is the identity map of A_i, for each $i \in I$,

(ii) for all $i \leq j \leq k$ in I, we have $\pi_i^j \pi_j^k = \pi_i^k$.

Then the system

$$A = \{A_i \, (i \in I) \mid \pi_i^j\}$$

is called an *inverse system.*

Definition 2.1.3. The *inverse limit* of the inverse system defined above

$$A^* = \lim_{\leftarrow (i \in I)} A_i$$

is defined to consists of all vectors $a = (\cdots, a_i, \cdots)$ in the direct product $A = \prod_{i \in I} A_i$, for which $\pi_i^j a_j = a_i \quad (i \leq j)$ holds.

2.2 Topological Group

For more details for topological groups, see for example Călugăreanu et al. (2003).

Definition 2.2.1. Given a set A, $P(A) = \{X \mid X \subseteq A\}$, we call $\tau \subseteq P(A)$ a *topology* on A if $\varnothing, A \in \tau$ and τ is closed under unions and finite intersections.

We say that (A, τ) is a *topological space.*

Definition 2.2.2. Let us consider a group $(G, +)$ and the maps $g_1 : G \times G \to G$, $g_1(x, y) = x + y$, and $g_2 : G \to G$, $g_2(x) = -x$. A topological space G is called a *topological group* if g_1 is continuous in both variables and g_2 is continuous.

Definition 2.2.3. Given a topological group G a topology on G is a *linear topology* if it has a base of neighborhoods of 0 whose elements are subgroups.

Definition 2.2.4. The $\mathbb{Z}$-*adic topology* on an abelian group G is the one induced by setting $\{nG \mid n > 0\}$ as a neighborhood base of $0 \in G$.

An abelian group G is Hausdorff in the $\mathbb{Z}$-adic topology if and only if its first Ulm subgroup $G^1 = \bigcap_{n>0} nG = 0$.

Definition 2.2.5. The *p-adic topology* on an abelian group G is the one induced by setting $\{p^k G \mid k \geq 0\}$ as a neighborhood base of $0 \in G$ for a prime number p.

Definition 2.2.6. A group A is called *complete* in a given topology if it is Hausdorff, and every Cauchy net in A has a limit in A.

2.3 Basic Notions in Modules

Definition 2.3.1. An R-module M is said to be *π-projective* if for every two submodules U, V of M with $U + V = M$, there exists $f \in \mathrm{End}\ (M)$ such that $\mathrm{Im}\, f \leq U$ and $\mathrm{Im}(1 - f) \leq V$.

Definition 2.3.2. An R-module N is said to be an *M-generated* module if there exists an R-module epimorphism $f : \bigoplus_{\lambda \in \Lambda} M \longrightarrow N$ for some index set Λ.

Definition 2.3.3. An R-module N is said to be a *finitely M-generated* module if there exists an R-module epimorphism $f : \bigoplus_{i=1}^{n} M_i \longrightarrow N$ with finitely many R-modules $M_1, M_2, \ldots, M_n$.

Definition 2.3.4. Let M be an R-module. A module N is called *subgenerated* by M, or *M-subgenerated*, if N is isomorphic to a submodule of an M-generated module.

The book Wisbauer (1991) gives the concepts in module theory relative to the category σ[M] for a module M. This category σ[M] is the full subcategory of R-$\mathcal{M}od$ consisting of all M-subgenerated modules. This category reflects the properties of the module M. For example, $\sigma[R] = R$-$\mathcal{M}od$, where the ring R is considered as a left R-module.

Proposition 2.3.5. *(Modular Law)(see for example Kasch (1982, 2.3.15)) Let A, B and C be submodules of a module M such that $B \leq C$. Then*

$$(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B.$$

Definition 2.3.6. An R-module M is *bounded* if $rM = 0$ for some nonzero $r \in R$.

Definition 2.3.7. Suppose that M is a free R-module with a finite basis of n elements, we call M a free module of *rank* n.

Definition 2.3.8. Let R be a commutative domain and M be an R-module. An element $x \in M$ is called *torsion* if $rx = 0$ for some nonzero $r \in R$. The *torsion submodule* of M is the set of torsion elements, and is denoted by $T(M)$. M is a *torsion* module if $T(M) = M$, and M is *torsion-free* if $T(M) = 0$, that is, $rx = 0$ implies that either $r = 0$ or $x = 0$.

2.4 Essential and Small Submodules

Definition 2.4.1. A submodule A of an R-module M is called *small (=superfluous)* in M and is denoted by $A \ll M$ if for every submodule U of M,

$$A + U = M \quad \text{implies} \quad U = M.$$

Definition 2.4.2. Let A and B be R-modules. An epimorphism $\alpha : A \to B$ is called *small* if $\operatorname{Ker} \alpha \ll A$; such an α is called a *small cover*.

Definition 2.4.3. An R-module M is called *simple* if $M \neq 0$ and for every submodule $A \leq M$, $A = 0$ or $A = M$, that is, $M \neq 0$, and 0 and M are the only submodules of M; equivalently if $M \neq 0$ and for every $0 \neq m \in M$, $Rm = M$.

Definition 2.4.4. An R-module M is called *semisimple* if every submodule of M is a direct summand of M (see for example Kasch (1982, Ch.8)).

Since an elementary group (i.e., every element of the group has a square-free order) is a semisimple $\mathbb{Z}$-module we have:

Proposition 2.4.5. *(see for example Călugăreanu et al. (2003, S 1.61)) Any subgroup of an elementary group is a direct summand.*

Theorem 2.4.6. *(Kasch (1982, 8.1.3)) For an R-module M, the following conditions are equivalent:*

(i) *Every submodule of M is a sum of simple submodules,*

(ii) *M is sum of simple submodules,*

(iii) *M is a direct sum of simple submodules,*

(iv) *Every submodule of M is a direct summand of M, that is, M is semisimple.*

Corollary 2.4.7. *(Kasch, 1982, 8.1.5)*

(i) *Every submodule of a semisimple module is semisimple.*

(ii) *Every epimorphic image of a semisimple module is semisimple.*

(iii) *Every sum of a semisimple modules is semisimple.*

Definition 2.4.8. An R-module M is said to be a *homogenous* (*isotypic*) *semisimple* R-module if M is a semisimple R-module whose simple submodules are all isomorphic, that is, $M = \bigoplus_{\lambda\in\Lambda} S_\lambda$ for some index set Λ and simple submodules S_λ of M such that for some maximal left ideal P of R, $S_\lambda \cong R/P$ for every $\lambda \in \Lambda$.

The following properties for small submodules will be frequently used.

Theorem 2.4.9. *(Kasch, 1982, Ch.5, §1) Let M be an R-module and $A, B \leq M$ be submodules. Then:*

(i) *Let $A \leq B \leq M$. If $A \ll B$, then $A \ll M$ and if $B \ll M$, then $A \ll M$.*

(ii) *$A \ll M$ if and only if $A + U \neq M$ for every proper submodule U of M.*

(iii) *If $M \neq 0$ and $A \ll M$, then $A \neq M$.*

(iv) $0 \ll M$.

(v) *If M is semisimple, then 0 is the only small submodule of M.*

(vi) *If $\operatorname{Jac} R = 0$, then $\operatorname{Rad} F = 0$ for every free R-module F, i.e., only the trivial submodule 0 is a small submodule.*

(vii) *Let $A \leq B \leq M \leq N$ for an R-module N. If $B \ll M$, then $A \ll N$.*

(viii) *Let $A_i \leq M$ be submodules for $i = 1, 2, \ldots, n$. If $A_i \ll M$ for each i, then* $\sum_{i=1}^{n} A_i \ll M$.

(ix) *If $A \ll M$ and $\varphi : M \to N$ is a homomorphism, then $\varphi(A) \ll \varphi(M)$ and so $\varphi(A) \ll N$.*

(x) *If $\alpha : A \to B$ and $\beta : B \to C$ are small epimorphisms, then $\beta\alpha : A \to C$ is also a small epimorphism.*

(xi) *For $a \in M$, Ra is not small in M if and only if there is a maximal submodule $C \leq M$ with $a \notin C$.*

(xii) *(Clark et al. (2006, 2.2-(7))) If $A \ll M$, then M is finitely generated if and only if M/A is finitely generated.*

(xiii) *(see Anderson & Fuller (1992, §5)) $N \ll M$ if and only if $K \ll M$ and $N/K \ll M/K$.*

(xiv) *$H + K \ll M$ if and only if $H \ll M$ and $K \ll M$.*

(xv) *Let $K' \leq N' \leq M$ and let $M = N \oplus N'$. Then;*
$K \oplus K' \ll N \oplus N'$ if and only if $K \ll N$ and $K' \ll N'$.

Definition 2.4.10. A submodule A of an R-module M is called *essential* (or *large* or *big*) in M and is denoted by $A \trianglelefteq M$ if for every submodule U of M,

$$A \cap U = 0 \quad \text{implies} \quad U = 0,$$

equivalently $A \cap U \neq 0$ for every nonzero submodule $U \leq M$.

Definition 2.4.11. Let A and B be R-modules. A monomorphism $\alpha : A \to B$ is called *essential* if $\operatorname{Im} \alpha \trianglelefteq B$.

The following properties for essential submodules will be frequently used.

Theorem 2.4.12. *(see for example Anderson & Fuller (1992, §)) Let M be an R-module with submodules $K \leq N \leq M$ and $H \leq M$. Then:*

(i) $M \trianglelefteq M$.

(ii) $K \trianglelefteq M$ *if and only if* $K \trianglelefteq N$ *and* $N \trianglelefteq M$.

(iii) $H \cap K \trianglelefteq M$ *if and only if* $H \trianglelefteq M$ *and* $K \trianglelefteq M$.

(iv) *If* M *is semisimple, then* M *is the only essential submodule of* M.

(v) *Let* $K' \leq N' \leq M$ *and let* $M = N \oplus N'$. *Then;*
$K \oplus K' \trianglelefteq N \oplus N'$ *if and only if* $K \trianglelefteq N$ *and* $K' \trianglelefteq N'$.

(vi) *If* H *is a complement of* N *in* M, *then;*

(a) $H \oplus N \trianglelefteq M$,

(b) $(H \oplus N)/H \trianglelefteq M/H$.

2.5 Radical and Socle

Definition 2.5.1. The *radical* of an R-module M, denoted by $\operatorname{Rad} M$, is defined as the intersection of all maximal submodules of M, or equivalently, as the sum of all small submodules of M as usual we take $\operatorname{Rad} M = M$ when M has no maximal submodules.

Definition 2.5.2. The *socle* of an R-module M, denoted by $\operatorname{Soc} M$, is defined as the intersection of all essential submodules of M, or equivalently, as the sum of all its simple submodules.

Note that $\operatorname{Soc} A = A \cap \operatorname{Soc} M$ for every submodule $A \leq M$. Moreover, M is semisimple precisely when $M = \operatorname{Soc} M$.

Proposition 2.5.3. *(Fuchs, 1970, Exercise 16.10) A subgroup* E *is essential in a group* G *exactly if* $\operatorname{Soc}(G) \leq E$ *and* G/E *is torsion.*

Proof. See Călugăreanu et al. (2003, Exercise M 1.6). $\square$

Proposition 2.5.4. *For an abelian group* A,

(i) $\operatorname{Soc} A$ *consists of all* $a \in A$ *such that* $o(a)$ *is a square-free integer.*

(ii) $\operatorname{Soc} A = A$ *if and only if* A *is an* elementary *group in the sense that every element has a square-free order.*

(iii) *For a prime number* p*, if* A *is a* p*-group, then* $\operatorname{Soc} A = A[p]$.

We collect together several elementary properties of the radical in the following theorem.

Theorem 2.5.5. *(Kasch, 1982, Ch.9, §1-2) Let* M *and* N *be* R*-modules and* $C \leq M$ *be a submodule. Then we have:*

(i) *For* $m \in M$*,* $Rm \ll M$ *if and only if* $m \in \operatorname{Rad} M$*.*

(ii) *If* $\varphi : M \to N$ *a homomorphism, then* $\varphi(\operatorname{Rad} M) \leq \operatorname{Rad} N$*. In particular, if* $\operatorname{Ker} \varphi \ll M$*, then* $\varphi(\operatorname{Rad} M) = \operatorname{Rad} N$*.*

(iii) $\operatorname{Rad}(M/\operatorname{Rad} M) = 0$*, and if* $\operatorname{Rad}(M/C) = 0$*, then* $\operatorname{Rad} M \leq C$*.*

(iv) *If* $C \leq M$*, then* $\operatorname{Rad} C \leq \operatorname{Rad} M$*.*

(v) *For a family of* R*-modules* $\{M_i \mid i \in I\}$*, if* $M = \bigoplus_{i \in I} M_i$ *then* $\operatorname{Rad} M = \bigoplus_{i \in I} \operatorname{Rad} M_i$ *and* $M/\operatorname{Rad} M \cong \bigoplus_{i \in I} (M_i/\operatorname{Rad} M_i)$*.*

(vi) *If* M *is semisimple, then* $\operatorname{Rad} M = 0$*.*

(vii) *If* M *is finitely generated, then* $\operatorname{Rad} M \ll M$*.*

(viii) $\operatorname{Rad}({}_R R) \ll_R R$*.*

(ix) *If* M *is finitely generated and* $C \leq \operatorname{Rad}({}_R R)$ *then* $CM \ll M$ *(Nakayama's Lemma).*

(x) *If* M *is finitely generated and* $M \neq 0$*, then* $\operatorname{Rad} M \neq M$*.*

(xi) $\operatorname{Rad}({}_R R)$ *is a two-sided ideal of* R*.*

(xii) $(C + \operatorname{Rad} M)/C \leq \operatorname{Rad}(M/C)$*.*

Lemma 2.5.6. *(Dlab, 1960, Theorem 2) Let $G = \prod_{i\in I} G_i$ be a direct product decomposition of a group G. Then*

$$\operatorname{Rad} G = \prod_{i\in I} \operatorname{Rad}(G_i).$$

Definition 2.5.7. The *Frattini* subgroup of an abelian group A is the intersection of all the maximal subgroups of the group A. When A is considered as a $\mathbb{Z}$-module, the Frattini subgroup of abelian group A is nothing but $\operatorname{Rad} A$.

Proposition 2.5.8. *(Fuchs, 1970, Exercise 3.4)*

(i) *The intersection of all maximal subgroups of A of the same prime index p is pA.*

(ii) *The Frattini subgroup of A is the intersection of all pA with p running over all primes p, i.e.,* $\operatorname{Rad} A = \bigcap_{p\ prime} pA$.

Proof. (i) Let $\mid A/K \mid = p$ for each maximal subgroup K of A. Then $p(A/K) = K$ for every $K \underset{max.}{\leq} A$, so $pA + K = K$, i.e., $pA \leq K$. Hence $\bigcap_{K \underset{max.}{\leq} A} K \geq pA$. Conversely,

$$\bigcap_{K \underset{max.}{\leq} A} K = \bigcap_{K \underset{max.}{\leq} A} p(A/K) \leq pA.$$

(ii) Lemma 2.5.11 is the general case of this.

□

Generalizing this to modules:

Definition 2.5.9. A ring R is said to be a *left quasi-duo ring* if each maximal left ideal is two-sided ideal.

Proposition 2.5.10. *(Hungerford, 1980, Theorem 2.4) Every vector space V over a division ring D has a basis and is therefore a free D-module. More generally every linearly independent subset of V is contained in a basis of V.*

Lemma 2.5.11. *(Generalov, 1983, Lemma 3) Let R be a quasi-duo ring. Then for each module M,*

$$\operatorname{Rad} M = \bigcap_{P \underset{max.}{\leq} R} PM$$

where the intersection is over all maximal left ideals of R.

Proof. Let K denote the maximal submodule of M. Then we know that

$$\operatorname{Rad} M = \bigcap_{\substack{K \leq M \\ max.}} K.$$

Let P be any maximal left ideal of R. Since R is a left quasi-duo ring, P is a two-sided ideal. So R/P is a ring. Moreover R/P is a division ring since the only left ideals of R/P are 0 and R/P as P is a maximal left ideal of R. So the quotient module M/PM can be considered as an R/P-module, since $P(M/PM) = 0$. Therefore M/PM is a vector space over the division ring R/P, and so M/PM is a free R/P-module. As R/P-modules, we have an internal direct sum

$$M/PM = \bigoplus_{\lambda \in \Lambda} M_\lambda/PM,$$

where for each $\lambda \in \Lambda$, $M_\lambda/PM \cong R/P$ as R/P-modules, for some indexing set Λ. These isomorphism equalities hold also as R-modules. So for each $\lambda_0 \in \Lambda$,

$$(M/PM)/[\bigoplus_{\lambda \in \Lambda \setminus \{\lambda_0\}} M_\lambda/PM] \cong M_{\lambda_0}/PM \cong R/P$$

is a simple R-module. Let $N_{\lambda_0} = \sum_{\lambda \in \Lambda \setminus \{\lambda_0\}} M_\lambda$ for each $\lambda_0 \in \Lambda$. Thus

$$N_{\lambda_0}/PM = \bigoplus_{\lambda \in \Lambda \setminus \{\lambda_0\}} \big(M_\lambda/PM\big)$$

is a maximal submodule of M/PM and

$$(\bigcap_{\lambda_0 \in \Lambda} N_{\lambda_0})/PM = \bigcap_{\lambda_0 \in \Lambda} \big(N_{\lambda_0}/PM\big) = \bigcap_{\lambda_0 \in \Lambda} \left(\bigoplus_{\lambda \in \Lambda \setminus \{\lambda_0\}} \big(M_\lambda/PM\big) \right) = 0.$$

Then $\bigcap_{\lambda_0 \in \Lambda} N_{\lambda_0} = PM$. Since N_{λ_0} is maximal in M for every $\lambda_0 \in \Lambda$, we obtain that

$$\operatorname{Rad} M = \bigcap_{\substack{K \leq M \\ max.}} K \leq \bigcap_{\lambda_0 \in \Lambda} N_{\lambda_0} \leq \bigcap PM.$$

Conversely, since M/K is simple for each maximal submodule $K \leq M$, we have $M/K \cong R/P$ where P is a maximal left ideal of R. Then $P(M/K) = 0$ implies $PM \leq K$ for each maximal left ideal $P \leq R$. Hence

$$\bigcap PM \leq \bigcap_{\substack{K \leq M \\ max.}} K = \operatorname{Rad} M.$$

□

Definition 2.5.12. A ring R is a *left max ring* if $\operatorname{Rad} M \ll M$ for all (left) R-modules M, equivalently every R-module M has a maximal submodule.

Proposition 2.5.13. *For a p-group G for a prime number p, $\operatorname{Rad} G = pG$.*

Proof. Since G_p is q-divisible for every prime $q \neq p$, i.e., $qM_p = M_p$ we obtain

$$\operatorname{Rad} G_p = \bigcap_{q \text{ prime}} qG_p = pG_p.$$

□

Definition 2.5.14. The *Jacobson radical* of R is the intersection of all maximal ideals of R and is denoted by $\operatorname{Jac} R$.

The Jacobson radical is analogous to the Frattini subgroup of a group (i.e., the intersection of all the maximal subgroups of the group).

Proposition 2.5.15. *(Kasch, 1982, 9.3.1) For $A \leq {}_RR$,*

$$A \ll {}_RR \quad \textit{if and only if} \quad A \leq \operatorname{Rad}({}_RR).$$

Proposition 2.5.16. *(Clark et al., 2006, 2.8, (8)-(9))*

(i) *If $M/\operatorname{Rad} M$ is semisimple and $\operatorname{Rad} M \ll M$, then every proper submodule of M is contained in a maximal submodule of M, that is, M is coatomic (see Section 2.12).*

(ii) $\operatorname{Soc}(\operatorname{Rad} M) \ll M$.

For completeness we shall also prove the following well-known result.

Proposition 2.5.17. *Let M be an R-module and K be submodule of M. If K is finitely generated and $K \leq \operatorname{Rad} M$, then $K \ll M$.*

Proof. Since $K \leq \operatorname{Rad} M = \sum_{S \ll M} S$ and K is finitely generated, we obtain that $K \leq \sum_{i=1}^{n} S_i$ where $S_i \ll M$ for each $i = 1, 2, \ldots, n$. Hence $K \ll M$ since $\sum_{i=1}^{n} S_i \ll M$ (by Theorem 2.4.9). □

For completeness we shall also prove the following well-known result.

Proposition 2.5.18. *Let M be R-module and K be a submodule of M. If $K \leq \operatorname{Rad} M$, then $\operatorname{Rad}(M/K) = (\operatorname{Rad} M)/K$.*

Proof. $\operatorname{Rad}(L/K) = \bigcap_{T/K \underset{max.}{\leq} L/K} T/K = \Big(\bigcap_{K \leq T \underset{max.}{\leq} L} T\Big)/K = (\operatorname{Rad} L)/K$ because since $K \leq \operatorname{Rad} L = \bigcap_{N \underset{max.}{\leq} L} N$, every maximal submodule of L contains K. Hence

$$\bigcap_{K \leq T \underset{max.}{\leq} L} T = \bigcap_{N \underset{max.}{\leq} L} N = \operatorname{Rad} L.$$

□

Lemma 2.5.19. *Let $f : M \longrightarrow N$ be a homomorphism of R-modules M and N. Let U be a nonzero submodule of M. If $\operatorname{Rad} U = U$, then $\operatorname{Rad}(f(U)) = f(U)$.*

Proof. Define a homomorphism $g : U \longrightarrow f(U)$ such that $g(a) = f(a)$ for all $a \in U$. Clearly, g is an epimorphism. Then $U/\operatorname{Ker} g \cong f(U)$. Since U is a radical module it has no maximal submodule. Thus $U/\operatorname{Ker} g$ has no maximal submodule, and so $f(U)$ has no maximal submodule. Hence $\operatorname{Rad}(f(U)) = f(U)$. □

Proposition 2.5.20. *(Wisbauer, 1991, 21.6-(3)) If every proper submodule of M is contained in a maximal submodule of M, then $\operatorname{Rad} M \ll M$.*

Proof. Suppose $(\operatorname{Rad} M) + X = M$ for some $X \leq M$ and suppose for the contrary that $X \neq M$. Then by hypothesis, X is contained in a maximal submodule N of M. Since $\operatorname{Rad} M \leq N$ already, we obtain $M = X + \operatorname{Rad} M \leq N$. This contradicts the fact that N is a proper submodule of M as it is maximal in M. So we must have $X = M$. Thus $\operatorname{Rad} M \ll M$. □

2.6 Localization

The idea of *localization at a prime* in a ring is an extremely powerful and pervasive tool in algebra for isolating the behavior of the ideals in a ring.

Definition 2.6.1. Let R be a ring and S be a multiplicative subset (that is, $1 \in S$, and $st \in S$ for all $s, t \in S$) of R. Introduce the following relation $\sim$ on $R \times S$:

$$(r, s) \sim (r', t) \text{ if and only if there exists } u \in S \text{ such that } u(rt - r's) = 0;$$

it will be proved shortly that $\sim$ is an equivalence relation. Write r/s for the class of (r, s). Then the *ring of fractions* of R with respect to S is

$$S^{-1}R = (R \times S)/\sim$$

with ring operations defined by the usual arithmetic operations on fractions:

$$\frac{r}{s} \underline{+} \frac{r'}{t} = \frac{(rt \underline{+} r's)}{st} \text{ and } \frac{r}{s}.\frac{r'}{t} = \frac{rr'}{st}.$$

Proposition 2.6.2. *(see for example Reid (1995, §6.1, Proposition))*

(i) $\sim$ *is an equivalence relation.*

(ii) *The ring operations are well defined, and $S^{-1}R$ is a ring.*

(iii) $\varphi : R \to S^{-1}R$ *given by* $r \mapsto r/1$ *is a ring homomorphism.*

Localization is a particular case of ring of fractions.

Definition 2.6.3. (see for example Dummit & Foote (2004), Atiyah & Macdonald (1969), Reid (1995)) Let P be a *prime ideal* in any commutative ring R (i.e., for all $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$) and let $S = R \setminus P$. Since P is a prime ideal, S is a multiplicative set . Passing to the ring $S^{-1}R$ in this case is called *localization* of R (or *localizing* R) at P and the ring $S^{-1}R$ is denoted by R_P.

When S is the set of all nonzero elements of a commutative domain R, $S^{-1}R$ is called *field of fractions* or *quotient field* of R.

Every element of $R \setminus P$ becomes a unit in R_P. For example, if $R = \mathbb{Z}$ and $P = (p)$ is a prime ideal where p is a prime number, then

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } p \nmid b \right\} \leq \mathbb{Q}$$

and every integer b not divisible by p is a unit.

Definition 2.6.4. A *local* ring is a commutative ring with identity which has a unique maximal ideal.

For example, if p is prime and $n \geq 1$, then $\mathbb{Z}/p^n\mathbb{Z}$ is a local ring with unique maximal ideal $p\mathbb{Z}/p^n\mathbb{Z}$.

If I an ideal in R, then the ideal $S^{-1}I$ in R_P is denoted by I_P.

Proposition 2.6.5. *(see for example Reid (1995, §6.2, Proposition)) Let R be a ring and S be a multiplicative subset of R. If P is a prime ideal of R and $P \cap S = \varnothing$, then $S^{-1}P$ is a prime ideal of $S^{-1}R$.*

Proposition 2.6.6. *(see for example Hungerford (1980, ch.III, Theorem 4.11)) Let P be a prime ideal in a commutative ring R with identity. Then:*

(i) *There is a one-to-one correspondence between the set of prime ideals of R which are contained in P and the set of prime ideals of R_P;*

(ii) *the ideal P_P in R_P is the unique maximal ideal of R_P.*

Theorem 2.6.7. *(see for example Hungerford (1980, Ch.III, Theorem 4.13)) If R is a commutative ring with identity, then the following conditions are equivalent:*

(i) *R is local ring;*

(ii) *all nonunits of R are contained in some ideal $M \neq R$;*

(iii) *the nonunits of R form an ideal.*

Definition 2.6.8. An R-module M is called *semilocal* if $M/\operatorname{Rad} M$ is semisimple. Thus a ring R is *semilocal* if it is semilocal as a left R-module, that is, if $R/\operatorname{Jac} R$ is a semisimple ring.

Proposition 2.6.9. *(see Lam (2001, 20.2)) For any ring R, consider the following two conditions:*

(i) *R is semilocal,*

(ii) *R has finitely many maximal left ideals.*

We have, in general, (ii)⇒ (i). The converse holds if $R/\operatorname{Jac} R$ is commutative.

We obtain the following Corollary from Proposition 2.6.9.

Corollary 2.6.10. *A commutative ring R is semilocal if and only if R has finitely many maximal ideals.*

2.7 Noetherian Rings and Artinian Rings

Definition 2.7.1. A commutative ring R is said to be *noetherian* or to satisfy the *ascending chain condition on ideals* (shortly *ACC on ideals*) if there is no infinite increasing chain of ideals in R, i.e., whenever $I_1 \leq I_2 \leq I_3 \leq \cdots$ is an increasing chain of ideals of R, then there is a positive integer m such that $I_k = I_m$ for all $k \geq m$.

Every quotient R/I of a commutative noetherian ring R by an ideal I is again a noetherian ring. Any homomorphic image of a commutative noetherian ring is noetherian.

Proposition 2.7.2. *(see for example Dummit & Foote (2004, Ch.15, Theorem 2)) The following are equivalent for a commutative ring R:*

(i) *R is a noetherian ring.*

(ii) *Every nonempty set of ideals of R contains a maximal element under inclusion.*

(iii) *Every ideal of R is finitely generated.*

Definition 2.7.3. For any commutative ring R, the *Krull dimension* (or simply the *dimension*) of R is the maximum possible length n of a chain $P_0 \leq P_1 \leq P_2 \leq \cdots \leq P_n$ of distinct prime ideals in R. The dimension of R is said to be infinite if R has arbitrarily long chains of distinct prime ideals.

A ring with finite dimension must satisfy both the ascending and descending chain conditions on prime ideals (although not necessarily on all ideals). A field has dimension 0 and a principal ideal domain that is not a field has dimension 1.

Definition 2.7.4. A commutative ring R is said to be *artinian* or to satisfy the *descending chain condition on ideals* (shortly *DCC on ideals*) if there is no infinite decreasing chain of ideals in R, i.e., whenever $I_1 \geq I_2 \geq I_3 \geq \cdots$ is decreasing chain of ideals of R, then there is a positive integer m such that $I_k = I_m$ for all $k \geq m$. Similarly, an R-module M is said to be artinian if it satisfies DCC on submodules.

It is immediate from the Lattice Isomorphism Theorem that every quotient R/I of a commutative artinian ring R by an ideal I is again an artinian ring.

Proposition 2.7.5. *(see for example Dummit & Foote (2004, Ch.16, Proposition 2)) The following are equivalent:*

(i) *R is an artinian ring.*

(ii) *Every nonempty set of ideals of R contains a minimal element under inclusion.*

Definition 2.7.6. Let R be a commutative ring. An element $x \in R$ is *nilpotent* if $x^n = 0$ for some integer n.

The set of all nilpotent elements of R is an ideal of R, the *nilradical* of R and is denoted by $\operatorname{nilrad} R$.

The next result gives the main structure theorem for artinian rings.

Theorem 2.7.7. *(see for example Dummit & Foote (2004, Ch.16, Theorem 3)) Let R be an artinian ring. Then we have:*

(i) *There are only finitely many maximal ideals in R.*

(ii) *The quotient $R/\operatorname{Jac} R$ is a direct product of a finite number of fields. More precisely, if $M_1, \ldots, M_n$ are the finitely many maximal ideals in R, then*

$$R/\operatorname{Jac} R \cong K_1 \times K_2 \times \cdots \times K_n,$$

where K_i is the field R/M_i, for $1 \leq i \leq n$.

(iii) *Every prime ideal of R is maximal, i.e., R has Krull dimension 0. The Jacobson radical of R equals the nilradical of R and is a nilpotent ideal: $(\operatorname{Jac} R)^m = 0$ for some $m \geq 1$.*

(iv) *The ring R is isomorphic to the direct product of a finite number of artinian local rings.*

(v) *Every commutative artinian ring is noetherian.*

Corollary 2.7.8. *(see for example Dummit & Foote (2004, Corollary 4, Ch.16)) A commutative ring R is artinian if and only if it is noetherian and has Krull dimension 0.*

2.8 Discrete Valuation Rings

In the previous section we showed that the commutative artinian rings are the noetherian rings having Krull dimension 0. We now consider the noetherian rings of dimension 1.

Definition 2.8.1.

(1) A *discrete valuation* on a field K is a function $v : K^* \to \mathbb{Z}$, where $K^* = K \setminus \{0\}$ satisfying

(i) v is surjective,

(ii) $v(xy) = v(x) + v(y) \quad$ for all $x, y \in K^\times$,

(iii) $v(x + y) \geq \min\{v(x), v(y)\} \quad$ for all $x, y \in K^\times$ with $x + y \neq 0$.

The subring $\{x \in K \mid \quad v(x) \geq 0\} \cup \{0\}$ is called the *valuation ring* of v.

(2) A commutative domain R is called a *discrete valuation ring* (DVR) if R is the valuation ring of a discrete valuation v on the field of fractions of R.

The valuation v is often extended to all of K by defining $v(0) = +\infty$, in which case (ii) and (iii) hold for all $a, b \in K$ with $n + (+\infty) = (+\infty) + n = +\infty$ and $n < \infty$ for all $n \in \mathbb{Z}$.

Example 2.8.2. The localization $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid \quad a, b \in \mathbb{Z},\ p \nmid b\} \leq \mathbb{Q}$ of the ring $\mathbb{Z}$ of integers at any nonzero prime ideal $(p) = p\mathbb{Z}$ is a DVR with respect to the discrete valuation v_p on $\mathbb{Q}$ (which is the field of fractions of $\mathbb{Z}$, and also of $\mathbb{Z}_{(p)}$) defined as follows:
Every element $a/b \in \mathbb{Q}^{\times}$ can be written uniquely in the form $p^n(a_1/b_1)$ where $n \in \mathbb{Z}$, $a_1/b_1 \in \mathbb{Q}^{\times}$ and both a_1 and b_1 are integers relatively prime to p. Define

$$v_p\big(\frac{a}{b}\big) = v_p\big(p^n \frac{a_1}{b_1}\big) = n.$$

Here, n may be negative or zero as well as positive. We call v_p the *p-adic valuation* on $\mathbb{Q}$. The corresponding valuation ring is the set of rational numbers a/b where b is not divisible by p, which is $\mathbb{Z}_{(p)}$ (see for example Reid (1995, §1.14), where $p = 5$).

Definition 2.8.3. Let A be a subring of the ring R and let $x \in R$. We say that x is *integral over* A if x is a root of a monic polynomial with coefficients in A.

Definition 2.8.4. If A is a subring of the ring R, the *integral closure* of A in R is the set A_c of elements of R that are integral over A.

We say that A is *integrally closed* in R if $A_c = A$.

Theorem 2.8.5. *(see for example Dummit & Foote (2004, Theorem 7, Ch.16)) The following properties of a commutative ring R are equivalent:*

(i) *R is a discrete valuation ring,*

(ii) *R is a PID with unique maximal ideal $P \neq 0$,*

(iii) *R is a UFD with a unique (up to associates) irreducible element t,*

(iv) *R is noetherian commutative domain that is also a local ring whose Krull dimension is 1, i.e., R has a unique nonzero prime ideal M.*

Corollary 2.8.6. *(see for example Dummit & Foote (2004, Ch.16, Corollary)) If R is any noetherian, integrally closed, commutative domain and P is a minimal nonzero prime ideal of R, then the localization R_P of R at P is a discrete valuation ring.*

Proposition 2.8.7. *(see for example Dummit & Foote (2004, Ch.16, Proposition 11)) Suppose the commutative domain R is a local ring that is not a field. Then R is a discrete valuation ring if and only if every nonzero fractional ideal of R is invertible.*

The next lemma exhibits a useful link between the topological and algebraic aspects of R-modules.

Lemma 2.8.8. *(Kaplansky, 1969, Lemma 19) Let R be a DVR and M be an R-module with no elements of infinite height, S a submodule of M. Then S is dense if and only if M/S is divisible.*

Definition 2.8.9. Let R be a DVR with quotient field $K \neq R$ and maximal ideal Rp and let M be an R-module. M is called a *basic submodule* of M if it satisfies three conditions:

(i) M/S is divisible (equivalently S is dense in M),

(ii) S is pure in M (i.e., $Sp^n = S \cap Mp^n$ for all n),

(iii) S is a direct sum of cyclic modules.

Definition 2.8.10. By a *p-basic submodule* B of a module A, we mean a submodule of A satisfying the following three conditions:

(i) B is a direct sum cyclic p-modules and infinite cyclic modules;

(ii) B is p-pure in A, i.e., $p^k B = B \cap p^k A$ for a prime p and $k \in \mathbb{Z}^+$;

(iii) A/B is divisible (i.e., B is dense in A).

2.9 Dedekind Domains

Definition 2.9.1. A *Dedekind Domain* is a noetherian, integrally closed, commutative domain of Krull dimension 1.

Definition 2.9.2. For every commutative domain R with fraction field K, a *fractional ideal* of R is an R-submodule A of K such that $rA \leq R$ for some nonzero $r \in R$ (equivalently, a submodule of the form $r^{-1}I$ for some nonzero $r \in R$ and ideal I of R). We will call r a *denominator* of A.

It follows from definition that the product of fractional ideals with denominators r and s respectively is fractional ideal with denominator rs.

Definition 2.9.3. The fractional ideal A is said to be *invertible* if there exists a fractional ideal B with $AB = R$, in which case B is called the *inverse* of A and denoted A^{-1}.

The following theorem gives a number of important equivalent characterizations of Dedekind Domains.

Theorem 2.9.4. *(see for example Dummit & Foote (2004, Ch.16, Theorem 15)) Suppose R is a commutative domain with fraction field $K \neq R$. The following are equivalent conditions for R to be a Dedekind Domain:*

(i) *The ring R is noetherian, integrally closed, and every nonzero prime ideal is maximal.*

(ii) *The ring R is noetherian and for each nonzero prime P of R the localization R_P is a Discrete Valuation Ring.*

(iii) *Every nonzero fractional ideal of R in K is invertible.*

(iv) *Every nonzero fractional ideal of R in K is a projective R-module.*

(v) *Every nonzero proper ideal I of R can be written as a finite product of prime ideals: $I = P_1P_2 \cdots P_n$ (not necessarily distinct).*

When the condition in (v) holds, the set of primes $\{P_1, \ldots, P_n\}$ is uniquely determined and so every nonzero proper ideal I of R can be written uniquely (up to order) as a product of powers of distinct prime ideals.

Proposition 2.9.5. *(see for example Dummit & Foote (2004, Ch.16, Corollary 19)) Suppose I is an ideal in the Dedekind Domain R. Then*

(i) *there is an ideal J of R relatively prime to I (i.e., $I + J = R$) such that the product $IJ = (a)$ is a principal ideal of R,*

(ii) *if I is nonzero, then every ideal in the quotient R/I is principal; equivalently, if I_1 is an ideal of R containing I, then $I_1 = I + Rb$ for some $b \in R$, and*

(iii) *every ideal in R can be generated by two elements; in fact if I is nonzero and $0 \neq a \in I$, then $I = Ra + Rb$ for some $b \in I$.*

Corollary 2.9.6. *(see for example Dummit & Foote (2004, Corollary 20, Ch.16)) If R is a Dedekind Domain, then R is a PID if and only if R is a UFD*

Theorem 2.9.7. *(see for example Dummit & Foote (2004, Ch.22, Theorem 22)) Suppose M is a finitely generated module over the Dedekind Domain R. Let $n \geq 0$ denote the rank of M and let $T(M)$ be the torsion submodule of M. Then*

$$M \cong \underbrace{R \oplus R \oplus \cdots \oplus R \oplus I}_{n\ factors} \oplus T(M)$$

for some ideal I of R, and

$$T(M) \cong R/P_1^{e_1} \times R/P_2^{e_2} \times \cdots \times R/P_s^{e_s}$$

for some $s \geq 0$ and powers $P_i^{e_i}$, $e_i \geq 1$ for $i = 1, 2, \ldots, s$, of (not necessarily distinct) prime ideals. The ideals $P_i^{e_i}$ for $i = 1, \ldots, s$ are unique and the ideal I is unique up to multiplication by a principal ideal.

Corollary 2.9.8. *(see for example Dummit & Foote (2004, Ch.16, Corollary 23)) A finitely generated module over a Dedekind Domain is projective if and only if it is torsion-free.*

Definition 2.9.9. Let R be a Dedekind domain and let K be its field of fractions. Take the torsion R-module K/R and split it into its primary parts; the summand for the prime ideal P is called the *module of type* P^∞. For example, the Prüfer group $\mathbb{Z}_{p^\infty}$ is $\mathbb{Z}$-module of $(p)^\infty$, for a prime number p.

Theorem 2.9.10. *(Kaplansky, 1952, Theorem 5) Let R be a Dedekind domain, M an R-module, and S a pure submodule of bounded order (i.e., $rS = 0$ for some nonzero $r \in R$). Then S is a direct summand of M.*

Theorem 2.9.11. *(Kaplansky, 1952, Theorem 7) Let R be a Dedekind domain with quotient field K. Then any divisible R-module is the direct sum of a vector space over K and modules of type P^{∞} for various prime ideals P.*

Like in abelian groups, we have the following theorem:

Theorem 2.9.12. *(Kaplansky, 1952, Theorem 8) Any module M over a Dedekind domain possesses a unique largest divisible submodule D; $M = D \oplus E$ where E has no divisible submodules.*

Theorem 2.9.13. *(by Cohn (2002, Proposition 10.6.9)) Any torsion $\mathbb{R}$-module M over a Dedekind domain R is a direct sum of its primary parts in a unique way:*

$$M = \bigoplus_{0 \neq P \underset{max.}{\leq} R} M_P,$$

where for each nonzero prime ideal P of R (so P is a maximal ideal of R),

$$M_P = \{x \in M | P^n x = 0 \textit{ for some } n \in \mathbb{Z}^+\}$$

*is the P-*primary part *of the R-module M.*

Theorem 2.9.14. *(by Alizade et al. (2001, Lemma 4.4)) Let R be a Dedekind domain which is not a field and let M be an R-module. Then the following conditions are equivalent:*

(i) *M is injective,*

(ii) *M is divisble (i.e., $M = rM$ for every nonzero $r \in R$),*

(iii) *$M = PM$ for every maximal ideal of R,*

(iv) *M does not contain a maximal submodule (i.e., $\operatorname{Rad} M = M$).*

Proof. $(i) \Leftrightarrow (ii)$ follows from Sharpe & Vámos (1972, Propositions 2.6 and 2.10).

$(ii) \Rightarrow (iii)$: $PM \leq M$ clearly for every maximal ideal P of R. Now since $P \neq 0$, there is a nonzero element X in P. Since M is divisible, we have $M = xM$. This means $M \leq PM$. Hence $M = PM$.

$(iii) \Rightarrow (ii)$: If $0 \neq r \in R$ is a unit, then $rM = M$ is immediate. Indeed, for $0 \neq m \in M$, $m = r(r^{-1}m) \in rM$. Now if $0 \neq r \in R$ is a nonunit element, then Rr is nonzero proper ideal of R. So $Rr = P_1P_2 \cdots P_n$ for prime ideals $P_1, P_2, \ldots, P_n$ of R (see Theorem 2.9.4). Since R is a Dedekind domain, every prime ideal of R is maximal. Thus $rM = (Rr)M = P_1P_2 \cdots P_n = P_1P_2 \cdots P_{n-1}M$ since $M = P_nM$ by hypothesis. Going on this way we obtain that $rM = p_1M = M$. Then for every nonzero $r \in R$, we have $rM = M$, that is, M is divisible.

$(iii) \Rightarrow (iv)$: Suppose for the contrary that M contains a maximal submodule K. By hypothesis $PM \leq K$ must be hold. Hence $K = PM = M$ contradicting the maximality of K.

$(iv) \Rightarrow (iii)$: Since M does not contain a maximal submodule, we have $\operatorname{Rad} M = M$. Then $\bigcap_{P \underset{max.}{\leq} R} PM = \operatorname{Rad} M = M$ implies that $M \leq PM$ for every maximal ideal P of R (see Lemma 2.5.11). Hence $M = PM$ for every maximal ideal P of R since $PM \leq M$ already.

$\square$

2.10 Perfect Rings

Definition 2.10.1. A subset A of a ring R is called *left T-nilpotent* if for every sequence of elements $\{a_1, a_2, \ldots, \} \leq A$, there exists an integer $n \geq 1$ such that $a_1a_2 \cdots a_n = 0$.

Definition 2.10.2. A pair (P, π) is a *projective cover* of a left R-module M if P is a projective left R-module and $\pi : P \longrightarrow M$ is a small epimorphism (i.e. $\operatorname{Ker} \pi \ll P$).

Definition 2.10.3. Let R be a ring. Then R is called a *left perfect* ring if every (left)

R-module has a projective cover.

Definition 2.10.4. A module M is called *semiperfect* if every factor module of M has a projective cover.

Proposition 2.10.5. *(see for example Anderson & Fuller (1992, Theorem 28.4)) Let R be a ring with radical $J = \mathrm{Jac}(R)$. Then the following statements are equivalent:*

(i) *R is left perfect;*

(ii) *R/J is semisiple and J is left T-nilpotent;*

(iii) *R/J is semisimple and every nonzero left R-module contains a maximal submodule.*

Proposition 2.10.6. *Let R be a ring. Then:*

(i) *If R is left noetherian, then it is left reduced.*

(ii) *If R is left perfect, then (every) left R-module is reduced.*

Proof. (i) Since R is left noetherian, all left ideals of R are finitely generated. Then $\mathrm{Rad}\, I \ll I$ for every left ideal I of R. This implies that $\mathrm{Rad}\, I \neq I$ for every nonzero left ideal I of R because if $\mathrm{Rad}\, I = I$ for a left ideal I of R, then $\mathrm{Rad}\, I + 0 = I$ implies $I = 0$ since $\mathrm{Rad}\, I \ll I$. Thus R is left reduced.

(ii) Since R is left perfect, every nonzero (left) R-module contains a maximal submodule (by Proposition 2.10.5). Then $\mathrm{Rad}\, U \ll U$ for every R-module U. So for every R-module M and for every nonzero submodule U of M, $\mathrm{Rad}\, U \neq U$. This means every R-module is reduced.

□

Proposition 2.10.7. *A ring R is left artinian if and only if it is left noetherian and left perfect.*

Proof. ($\Rightarrow$): If R is left artinian it is left noetherian by Cohn (2002, Corollary 5.3.10). Hence by Anderson & Fuller (1992, Corollary 28.8), R is left perfect.

($\Leftarrow$): If R a is left noetherian and left perfect ring, then R is left artinian by for example Büyükaşık (2005, Lemma 2.3.5). $\square$

2.11 Supplemented Modules

Definition 2.11.1. An R-module M is called *lifting* if every submodule N of M lies over a direct summand , that is, N contains a direct summand X of M such that $N/X \ll M/X$.

Definition 2.11.2. An R-module M which is both π-projective and supplemented is called a *quasi-discrete* module.

Proposition 2.11.3. *(see Clark et al. (2006, 26.7)) For an R-module M, the following are equivalent:*

(i) *M is quasi-discrete,*

(ii) *M is lifting and π-projective,*

(iii) *M is lifting and for each direct summands U, $V \leq M$ with $M = U + V$, $U \cap V$ is a direct summand of M.*

Theorem 2.11.4. *(by Wisbauer (1991, 43.9)) For a ring R the following assertions are equivalent:*

(i) *R is a left perfect ring;*

(ii) *every (left) R-module (or only $R^{(\mathbb{N})}$) is semiperfect;*

(iii) *every (left) R-module is (amply) supplemented;*

(iv) *The (left) R-module $R/(\operatorname{Jac} R)$ is semisimple and $\operatorname{Rad} R^{(\mathbb{N})} \ll R^{(\mathbb{N})}$.*

Definition 2.11.5. Let M be an R-module. A submodule of N of M is said to have *ample supplements* in M if for every submodule $L \leq M$ with $N + L = M$, there is a supplement L' of N in M with $L' \leq L$.

Definition 2.11.6. An R-module M is called *amply supplemented* if all submodules of it have ample supplements in M.

Definition 2.11.7. An R-module M is said to be *totally supplemented* if every submodule of M is supplemented.

2.12 Coatomic and Reduced Modules

Definition 2.12.1. An R-module M is called a *radical module* if

$$\operatorname{Rad} M = M.$$

Definition 2.12.2. (Zöschinger, 1974a, Definition, pp. 47) An R-module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule, equivalently if it has no nonzero radical factor module, that is, for every submodule $U \leq M$, if $\operatorname{Rad}(M/U) = M/U$, then $M/U = 0$.

Definition 2.12.3. (Zöschinger, 1974a, Definition, pp. 47) An R-module M is called *reduced* if it has no nonzero radical submodule, that is, for every submodule $U \leq M$, if $\operatorname{Rad} U = U$, then $U = 0$.

It is easily seen that:

Proposition 2.12.4. (i) *Every submodule of a reduced module is reduced.*

(ii) *Every factor module of coatomic module is coatomic.*

Proposition 2.12.5. *For every coatomic module M,* $\operatorname{Rad} M \ll M$.

Proof. Follows immediately from Proposition 2.5.20, since every submodule of M is contained in a maximal submodule of M. $\square$

Definition 2.12.6. A ring R is called *left reduced* if $\operatorname{Rad} I = I$ implies $I = 0$ for every left ideal I of R, that is, if it is a reduced module as a left R-module.

Definition 2.12.7. Let M be an R-module. By $P(M)$ we denote *the sum of all radical submodules* of M, that is,

$$P(M) = \sum\{U \leq M \mid \operatorname{Rad} U = U\}.$$

Clearly, M is reduced if and only if $P(M) = 0$.

Lemma 2.12.8. *A direct sum of R-modules is reduced if and only if each component is reduced.*

Proof. Let M be a module such that $M = \bigoplus_{i\in I} M_i$ for a family $\{M_i\}_{i\in I}$ of submodules of M. For each $j \in I$, let $p_i : M \to M_j$ be the projection onto M_j: $p(\sum_i m_i) = m_j$, where $\sum_i m_i \in \bigoplus_{i\in I} M_i = M$ ($m_i \in M_i$ for all $i \in I$).

($\Leftarrow$): Assume M_i is reduced for all $i \in I$. Suppose for the contrary that M is not reduced. Then there exists a nonzero submodule U of M such that $\operatorname{Rad} U = U$. Since $U \neq 0$, we have $p_j(U) \neq 0$ for some $j \in I$. But by Lemma 2.5.19, $\operatorname{Rad}(p_j(U)) = p_j(U)$ since $\operatorname{Rad} U = U$. Then $p_j(U) = 0$ since M_j is reduced and $p_j(U) \leq M_j$. This contradiction shows that M must be reduced.

($\Rightarrow$): Assume M_j is not reduced for some $j \in I$. Then there exists a nonzero submodule U of M_j such that $\operatorname{Rad} U = U$. But U is also a nonzero radical submodule of $M = \bigoplus_{i\in I} M_i$ which shows that M is not reduced. □

Lemma 2.12.9. *Let M be an R-module. Then $P(M)$ is a radical submodule of M.*

Proof. Clearly, $\operatorname{Rad}(P(M)) \leq P(M)$. Conversely, let $x \in P(M)$. Then $x = n_1 + n_2 + \cdots + n_k$, where $n_i \in N_i \leq M$ and $\operatorname{Rad} N_i = N_i$ for each $i = 1, 2, \ldots, k$. Therefore $Rn_i \leq N_i = \operatorname{Rad} N_i$ and so $Rn_i \ll N_i$ since Rn_i is finitely generated (see Proposition 2.5.17). Then $Rn_i \ll N_1 + N_2 + \cdots + N_k$ for each $i = 1, 2, \ldots, k$. Then by Theorem 2.4.9, we obtain

$$\sum_{i=1}^{k} Rn_i \ll \sum_{i=1}^{k} N_i.$$

Thus

$$Rx \leq \sum_{i=1}^{k} Rn_i \ll \sum_{i=1}^{k} N_i \leq P(M).$$

Then $Rx \ll P(M)$ and so $Rx \leq \operatorname{Rad}(P(M))$ which implies $x \in \operatorname{Rad}(P(M))$. Thus $P(M) \leq \operatorname{Rad}(P(M))$ and so $\operatorname{Rad}(P(M)) = P(M)$. $\square$

Lemma 2.12.10. *Let M be an R-module and V be a submodule of M. If $P(M) \leq V$, then $P(M) \leq \operatorname{Rad} V$.*

Proof. $P(M) \leq V$ implies $\operatorname{Rad}(P(M)) \leq \operatorname{Rad} V$. Hence $P(M) = \operatorname{Rad}(P(M)) \leq \operatorname{Rad} V$ by Lemma 2.12.9, that is, $P(M) \leq \operatorname{Rad} V$. $\square$

Proposition 2.12.11. *Let M be an R-module. Then $M/P(M)$ is reduced.*

Proof. Suppose $U/P(M)$ is a radical submodule of $M/P(M)$, i.e., $\operatorname{Rad}(U/P(M)) = U/P(M)$, where U is a submodule of M such that $P(M) \leq U$. Then $P(M) \leq \operatorname{Rad} U$ by Lemma 2.12.10. So we obtain that

$$\operatorname{Rad}(U/P(M)) = (\operatorname{Rad} U)/P(M)$$

by Proposition 2.5.18. Thus $U/P(M) = \operatorname{Rad}(U/P(M)) = (\operatorname{Rad} U)/P(M)$ which implies that $\operatorname{Rad} U = U$. So, U is a radical submodule of M and thus by definition of $P(M)$, we have $U \leq P(M)$. Hence $U/P(M) = 0$, since $P(M) \leq U$ already. $\square$

Proposition 2.12.12. *(Zöschinger, 1980, Lemma 1.1) Let R be a commutative noetherian ring and M be an R-module. If M is coatomic, then every submodule of M is coatomic.*

2.13 Cosmall Inclusions and Coclosed Submodules

Definition 2.13.1. Given submodules $K \leq N \leq M$, the inclusion $K \leq N$ is called *cosmall* in M if $N/K \ll M/K$; we denote this by $K \xhookrightarrow[M]{cs} N$, equivalently if for each submodule $X \leq M$, $N + X = M$ implies $K + X = M$ (see (Clark et al., 2006, 3.2-(1))).

Definition 2.13.2. A submodule $L \leq M$ is called *coclosed* in M, we write $L \xhookrightarrow{cc} M$, if L has no proper submodule K for which $K \xhookrightarrow[M]{cs} L$, that is, for every submodule $K \leq L$, $K \xhookrightarrow[M]{cs} L$ implies $K = L$. Thus we have $L \xhookrightarrow{cc} M$ if and only if, for every proper submodule K of L, there is a submodule N of M such that $L + N = M$ but $K + N \neq M$ (see Clark et al. (2006, 3.6)).

Definition 2.13.3. Let $L \leq M$ be a submodule. A submodule $K \leq L$ is called a *coclosure* of L in M if $K \xhookrightarrow[M]{cs} L$ and $K \xhookrightarrow{cc} M$.

Proposition 2.13.4. *(Clark et al., 2006, 3.7-(3)) Let M be a module and K, L be submodules of M such that $K \leq L$. If $L \xhookrightarrow{cc} M$, then $K \ll M$ implies $K \ll L$; hence* $\operatorname{Rad} L = L \cap \operatorname{Rad} M$.

CHAPTER THREE
NEAT AND CONEAT SUBMODULES

In Section 3.1, we give equivalent definitions for being a neat submodule. In Section 3.2, we investigate the relation between the basic submodules of a module M over DVR and the supplements of the radical of M (see Zöschinger (1976)). In the last section, we give some known properties for coneat submodules (see Mermut (2004)).

3.1 Neat Submodules

For the definition of neat submodules see Section 1.4.

Proposition 3.1.1. *(Mermut, 2004, Corollary 3.2.5) Let M be an R-module and $N \leq M$ be a submodule. Then the following are equivalent:*

(i) *N is neat in M;*

(ii) *For every maximal left ideal P of R, for every $m \in M$, if $Pm \leq N$, then there exists $n \in N$ such that $P(m-n) = 0$.*

Lemma 3.1.2. *(Fuchs & Salce, 2001, Lemma 1.8.4) Suppose*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r, "g"] & C \arrow[r] & 0 \\
0 \arrow[r] & A_1 \arrow[u, "\alpha"] \arrow[r, "f_1"'] & B_1 \arrow[u] \arrow[r] & C_1 \arrow[u, "\beta"'] \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of modules and module homomorphisms with exact rows. Then, β can be lifted to a homomorphism $C_1 \longrightarrow B$ if and only if α can be extended to a map $B_1 \longrightarrow A$, that is, there exists $\tilde{\beta} : C_1 \longrightarrow B$ such that $g \circ \tilde{\beta} = \beta$ if and only if there exists $\tilde{\alpha} : B_1 \longrightarrow A$ such that $\tilde{\alpha} \circ f_1 = \alpha$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\
0 \arrow[r] & A_1 \arrow[u, "\alpha"] \arrow[r, "f_1"'] & B_1 \arrow[u] \arrow[ul, dotted, "\tilde{\alpha}"] \arrow[r, "g_1"'] & C_1 \arrow[u, "\beta"'] \arrow[ul, dotted, "\tilde{\beta}"] \arrow[r] & 0
\end{tikzcd}$$

Proof. ($\Rightarrow$): If $\widetilde{\beta} : C_1 \to B$ satisfies $g\widetilde{\beta} = \beta$, then $\psi = \gamma - \widetilde{\beta}g_1$ satisfies $g\psi = g\gamma - g\widetilde{\beta}g_1 = \beta g_1 - \beta g_1 = 0$ by commutativity of the diagram. Then we have $\operatorname{Im}\psi \leq \operatorname{Ker} g = \operatorname{Im} f \cong A$. Thus there exists a map $\widetilde{\alpha} : B_1 \to A$) with $f\widetilde{\alpha} = \psi$. Indeed, since $\psi(b_1) \in \operatorname{Im} f$ for some $b_1 \in B$, we can define $\psi(b_1) = f(\widetilde{\alpha}(b_1))$. Therefore it satisfies $f\widetilde{\alpha}f_1 = \psi f_1 = \gamma f_1 - \widetilde{\beta}g_1 f_1 = f\alpha$. Then since $g_1 f_1 = 0$, $f(\widetilde{\alpha}f_1 - \alpha) = 0$ and so $\widetilde{\alpha}f_1 = \alpha$, since f is monic.

($\Leftarrow$): Conversely, if $\widetilde{\alpha} : B_1 \to A$ satisfies $\widetilde{\alpha}f_1 = \alpha$, then $(\gamma - f\widetilde{\alpha})f_1 = \gamma f_1 - f\widetilde{\alpha}f_1 = f\alpha - f\alpha = 0$. Then $\operatorname{Ker} g_1 = \operatorname{Im} f_1 \leq \operatorname{Ker}(\gamma - f\widetilde{\alpha})$. Thus there exists a map $\widetilde{\beta} : C_1 \to B$ such that $\widetilde{\beta}g_1 = \gamma - f\widetilde{\alpha}$, that is, the diagram

$$\begin{tikzcd} B_1 \arrow[r, "g_1"] \arrow[d, "\gamma - f\widetilde{\alpha}"'] & C_1 \arrow[dl, dotted, "\widetilde{\beta}"] \\ B & \end{tikzcd}$$

is commutative [indeed, *the map* $g_1 : B_1 \to C_1$ *exists*: since g_1 is epic, $C_1 = \operatorname{Im} g_1 \cong B_1/\operatorname{Ker} g_1$, and $\widetilde{\beta} : C_1 \to B$ is *well-defined*: if $g_1(b_1) = g_1(b_1')$, then $b_1 - b_1' \in \operatorname{Ker} g_1 \leq \operatorname{Ker}(\gamma - f\widetilde{\alpha})$, so $(\gamma - f\widetilde{\alpha})(b_1 - b_1') = 0$, hence $(\gamma - f\widetilde{\alpha})(b_1) = (\gamma - f\widetilde{\alpha})(b_1')$ implies that $\widetilde{\beta}(g_1(b_1)) = \widetilde{\beta}(g_1(b_1'))$]. Finally, we obtain from $g\widetilde{\beta}g_1 = g\gamma - gf\widetilde{\alpha} = \beta g_1$ [since $g\gamma = \beta g_1$ and $gf = 0$ by exactness of commutative diagram] that $g\widetilde{\beta} = \beta$, since g_1 is epic. □

Lemma 3.1.3. *(Sklyarenko, 1978, Lemma 6.1) For every right ideal I of R the condition $IA = A \cap IB$ is equivalent to the map*

$$R/I \otimes A \xrightarrow{1_{R/I} \otimes i_A} R/I \otimes B$$

being a monomorphism.

Proof. IA coincides with the image of $h : I \otimes A \to R \otimes A = A$.

Clearly, $A \cong R \otimes A$ by defining a homomorphism $\phi : A \to R \otimes A$ such that $\phi(a) = 1 \otimes a$. Now define $f : I \times A \to A$ such that $f((i, a)) = ia$, then f is bilinear. So for every $i \in I$ and $a \in A$, there exists a homomorphism $h : I \otimes A \to A$ such that,

$$h(i \otimes a) = f((i, a)) = ia.$$

Thus $\operatorname{Im} h = \{h(i \otimes a) = ia \mid i \otimes a \in I \otimes A\} = IA$. Hence we have the following diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & IA \arrow[r, "i_{IA}"] \arrow[d, "f"] & A \arrow[r, "\beta"] \arrow[d, "i_A"] & R/I \otimes A \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & IB \arrow[r, "i_{IB}"] & B \arrow[r, "g"] & R/I \otimes B \arrow[r] & 0 \end{tikzcd}$$

We know that $IA \leq A \cap IB$ is always true. Now take any $x \in \operatorname{Ker} \alpha$, then $\alpha(x) = 0$. Since β is an epimorphism, there exists a $y \in A$ such that $\beta(y) = x$. Then by commutative diagram $\alpha(\beta(y)) = g(i_A(y)) = 0$, thus $i_A(y) = y \in \operatorname{Ker} g = \operatorname{Im} i_{IB} = IB$, that is, $y \in A \cap IB$. Now we want to show that $y \in IA$ (i.e., $A \cap IB \leq IA$) if and only if $x = 0$ (i.e., α is monic).

Suppose $y \in IA$. Then $\beta(i_{IA}(y)) = 0$, $x = \beta(y) = 0$. Conversely, let $x = 0$. Then $x = \beta(y) = 0$ implies $y \in \operatorname{Ker} \beta = \operatorname{Im} i_{IA} = IA$.

Hence $x = 0$ (α is monic) if and only if $IA = A \cap IB$ (A is neat in B). $\square$

Theorem 3.1.4. *(by Warfield (1969, Proposition 2)) For an element r in a ring R, the following are equivalent for a short exact sequence*

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms where A is a submodule of B and i_A is the inclusion map:

(i) $\operatorname{Hom}(R/Rr, B) \xrightarrow{g_*} \operatorname{Hom}(R/Rr, C)$ *is epic (i.e., R/Rr is projective relative to the short exact sequence $\mathbb{E}$)*

(ii) $R/rR \otimes A \xrightarrow{1_{R/rR} \otimes i_A} R/rR \otimes B$ *is monic (i.e., R/rR is flat relative to the short exact sequence $\mathbb{E}$)*

(iii) $A \cap rB = rA$.

Proof. $(ii) \Leftrightarrow (iii)$: This equivalence holds for every right ideal I, by Lemma 3.1.3.

$(i) \Rightarrow (iii)$: Clearly, $rA \leq A \cap rB$. We shall show that $A \cap rB \leq rA$:
Suppose $a_0 = rb_0 \in A \cap rB$ for some $a_0 \in A$, $b_0 \in B$ and for each $r \in R$. Then we

can define $f : R/rR \to B/A$ by $f(1 + rR) = b_0 + A$ and $f(s + rR) = sb_0 + A$. f is well-defined: If $s + rR = s' + rR$, then $s - s' \in rR$. Let $s - s' = rt$ for some $t \in R$. So $trb_0 = (s - s')b_0 \in A$ since $rb_0 \in A$. Thus $sb_0 + A = s'b_0 + A$.

Now define $h : R \to B$ by $h(s) = sb_0$ for every $s \in R$. Then clearly, $gh = f\beta$ where β: $R \to R/rR$ and $g : B \to B/A \cong C$ are canonical epimorphisms [indeed, for $s \in R$, $gh(s) = g(h(s)) = h(s) + A = sb_0 + A = f(s + rR) = f\beta(s)$]. Therefore, $gh(rR) = f\beta(rR) = f(\beta(rR)) = f(0 + rR) = A = 0_B/A$. Then $h(rR) \leq \operatorname{Ker} g = \operatorname{Im} i_A = A$, so $\widetilde{h} : rR \to A$ exists. Thus $\widetilde{h}(r) = h(r) = rb_0$ and clearly, $i_A\widetilde{h} = h\alpha$, where $\alpha : rR \to R$ is inclusion map. Since R/rR is projective, there exists $\widetilde{f} : R/rR \to B$ such that $g\widetilde{f} = f$. Hence there exists a $\widetilde{g} : R \to A$ such that $\widetilde{g}\alpha = \widetilde{h}$, by Lemma 3.1.2. Let $\widetilde{g}(1) \in A$. Then $rb_0 = \widetilde{h}(r) = (\widetilde{g}\alpha)(r) = \widetilde{g}(\alpha(r)) = \widetilde{g}(r) = r\widetilde{g}(1) \in rA$. This shows that $A \cap rB \leq rA$.

Let us show the work done above in a diagram:

The diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_A"] & B \arrow[r, "g"] & C \cong B/A \arrow[r] & 0 \\
0 \arrow[r] & rR \arrow[r, "\alpha"'] \arrow[u, "\widetilde{h}"] \arrow[ur, phantom, "\circlearrowleft"] & R \arrow[r, "\beta"'] \arrow[u, "h"'] \arrow[ul, dotted, "\widetilde{g}"'] \arrow[ur, phantom, "\circlearrowleft"] & R/rR \arrow[r] \arrow[u, "f"'] \arrow[ul, dotted, "\widetilde{f}"'] & 0
\end{tikzcd}$$

is commutative with exact rows.

$(iii) \Rightarrow (i)$: Conversely, suppose that $rA = A \cap rB$. Then define $\widetilde{g} : R \to A$ by $\widetilde{g}(s) = sa_0$. Then $\widetilde{g}\alpha = \widetilde{h}$ [indeed, $\widetilde{g}\alpha(r) = \widetilde{g}(\alpha(r)) = \widetilde{g}(r) = ra_0 = rb_0$ for some $b_0 \in B$, since $ra_0 \in rA = A \cap rB$, and so $\widetilde{g}\alpha(r) = rb_0 = \widetilde{h}(r)$]. Thus by Lemma 3.1.2, there exists a $\widetilde{f} : R/rR \to B$ such that $g\widetilde{f} = f$. Hence R/rR is projective with respect to E. $\square$

3.2 Neat Submodules and Supplements over a DVR

Theorem 3.2.1. *(Zöschinger, 1976, Introduction) Let R be a DVR with quotient field $K \neq R$ and maximal ideal Rp and let M be an R-module. A submodule V of M is a supplement of radical if and only if it satisfies the following three conditions:*

(i) *V is dense in M,*

(ii) *V is neat in M (i.e., $Vp = V \cap Mp$ for all prime p),*

(iii) *V is coatomic (i.e., it is a direct sum of a finitely generated and a bounded module).*

If such a V exists, M is called *radical-supplemented.*

Note that radical-supplemented modules and Rad-supplemented modules does not coincide (see Section 6.1 for Rad-supplemented modules).

Lemma 3.2.2. *(Zöschinger, 1974c, Lemma 3.2) Let R be a DVR with quotient field $K \neq R$ and maximal ideal Rp. For an R-module M, we have the following statements:*

(i) *The class of radical-supplemented R-modules is closed under factor modules, pure submodules and group extensions.*

(ii) *If M is radical-supplemented and M/U is reduced, then U is also radical-supplemented.*

(iii) *Every submodule of M is radical-supplemented if and only if $T(M)$ is supplemented and $M/T(M)$ has finite rank.*

Theorem 3.2.3. *(Zöschinger, 1976, Proposition 1) Let R be a DVR with quotient field $K \neq R$ and maximal ideal Rp. For an R-module M, the following are equivalent:*

(i) $\operatorname{Rad} M$ *has ample supplements in M,*

(ii) *Every neat submodule of M is radical-supplemented,*

(iii) *If $M/\operatorname{Rad} M$ is not finitely generated, then every submodule of M is radical-supplemen-ted.*

Proof. $(i) \Rightarrow (ii)$: Let U be a neat submodule of M, i.e., $\operatorname{Rad} U = U \cap \operatorname{Rad} M$ (being equal $Up = U \cap Mp$, since R has a unique maximal Rp). Construct $M/U = D(M/U) \oplus M_1/U$, hence M_1 is also neat in M [because: M_1/U is neat in M/U, since every direct summand is neat, and hence $M_1 \leq_{neat} M$ by Lemma 4.1.21] and $M_1 + \operatorname{Rad} M = M$ [indeed, for $m+U \in M/U$, $m+U = (x+U)+(m_1+U)$, where $x+U \in D(M/U), m_1 \in M_1$, then $m - x - m_1 \in U$. So there is $u \in U$ such that $m = x + m_1 + u$. Thus since $x+U = (x'+U)p$ for some $x' \in M$, we have $x - x'p \in U \leq M_1$, then $x = x'p+x-x'p \in Mp+M_1$. This shows that $m = x+m_1+u \in Mp+M_1$ ($\operatorname{Rad} M = Mp$)]. By assumption there is a supplement V of $\operatorname{Rad} M$ in M with $V \leq M_1$, that is, $V + \operatorname{Rad} M = M$ and $V \cap \operatorname{Rad} M \ll V$. We shall show that V is a supplement of $\operatorname{Rad} M_1$ in M_1. Indeed $M_1 \cap (V + \operatorname{Rad} M) = M_1$, then by modular law, $V + (M_1 \cap \operatorname{Rad} M) = M_1$, i.e., $V + (M_1 \cap Mp) = M_1$, then $V + M_1 p = M_1$ since $M_1 \leq_{neat} M$, i.e., $V + \operatorname{Rad} M_1 = M_1$. And for $X \leq V$, let $X + (V \cap \operatorname{Rad} M_1) = V$. Then since $M_1 \leq_{neat} M$, $X + [V \cap (M_1 \cap \operatorname{Rad} M)] = X + (V \cap \operatorname{Rad} M) = V$. Hence by minimality of $V \cap \operatorname{Rad} M$, $X = V$, that is, $V \cap \operatorname{Rad} M_1 \ll V$. This implies M_1 is radical-supplemented. Then also U is by Lemma 3.2.2, since M_1/U is reduced.

$(ii) \Rightarrow (iii)$: As a radical-supplemented, M is of the form $M = D \oplus B \oplus F$ where D is divisible torsion, B is bounded, F is torsion-free and $F/\operatorname{Rad} F$ is finitely generated (see Zöschinger (1974c, Satz 3.1)). Assume now $M/\operatorname{Rad} M$ is not finitely generated, so $B = B_0 \oplus (\bigoplus_{i=1}^{\infty} B_i)$ where B_i cyclic and not all zero for $i \geq 1$. We must show by Lemma 3.2.2-(c) that F and D have finite rank [because, $T(M) = D \oplus B$ is supplemented if D has finite rank and $M/T(M) \cong F$ must have finite rank]. Assume that F has no finite rank, i.e., there is an $X \leq F$ with $X \cong R^{(\mathbb{N})}$, so choose $Y \leq \operatorname{Rad} X$ with $X/Y \cong \bigoplus_{i=1}^{\infty} B_i$. Obviously, $F \times X/Y$ is a direct summand in M up to isomorphism, since $B \cong B_0 \oplus X/Y$ and we have the following We shall show that the monomorphism $f : X \to F \times X/Y$, by $x \mapsto (x, x+Y)$ is neat. Let $(x, x+Y) = p(a, b+Y)$ for some $a \in F$, $b \in X$. Then $x = pa$ and $x - pb \in Y \leq \operatorname{Rad} X = pX$. So, $x - pb = px'$ for some $x' \in X$. Then $x = p(b + x') \in pX$, say $x = pt$ where $t = b + x' \in X$. Thus $(x, x+Y) = (pt, pt+Y) = p(t, t+Y)$, where $(t, t+Y) \in \operatorname{Im} f$. Hence $X \cong \operatorname{Im} f$ is neat $F \times X/Y$. Therefore, X is a neat submodule of M up to isomorphism, since $X \leq_{neat} F \times X/Y \leq_{neat} M$. But since X is not radical-supplemented (since X is torsion-free and $X/\operatorname{Rad} X$ is not finitely generated, see Zöschinger (1974c, Satz 3.1)), we have a contradiction. Now assume there is a decomposition $D = D_0 \oplus (\bigoplus_{i=1}^{\infty} D_i)$ with $D_i \cong K/R$ for all $i \geq 1$, so choose $Y_i \leq X_i \leq D_i$ with $Y_i \cong R/<p^i>$ and $X_i/Y_i \cong B_i$ for all $i \geq 1$. Define $Y = \oplus Y_i$ and $X = \oplus X_i$, then $Y \leq \operatorname{Rad} X$ [since, for each i, $R/<p^i> \leq X_i$, and so $p(R/<p^{i+1}>) \leq pX_{i+1}$] and $X/Y \cong \bigoplus_{i=1}^{\infty} X_i/Y_i \cong \bigoplus_{i=1}^{\infty} B_i$. So, as above X is a neat submodule of M up to isomorphism, although not radical supplemented (see Zöschinger (1974c, Satz 3.1)). Contradiction.

$(iii) \Rightarrow (i)$: Let $X + \operatorname{Rad} M = M$. We shall show that $\operatorname{Rad} M$ has ample supplements in M. In case, $M/\operatorname{Rad} M$ is finitely generated there is immediately a finitely generated $X_1 \leq X$ with $X_1 + \operatorname{Rad} M = M$, because:

$$M/\operatorname{Rad} M = (X+\operatorname{Rad} M)/\operatorname{Rad} M = \{\sum_{i=1}^{n} R((x_i+y_i)+\operatorname{Rad} M) \mid \quad x_i \in X, y_i \in \operatorname{Rad} M\}$$

Then for $m \in M$,

$$m + \operatorname{Rad} M = \sum_{i=1}^{n} r_i(x_i + \operatorname{Rad} M) = (\sum_{i=1}^{n} r_i x_i) + \operatorname{Rad} M$$

and so $m - \sum_{i=1}^{n} r_i x_i \in \operatorname{Rad} M$. Thus $m \in (\sum_{i=1}^{n} Rx_i) + \operatorname{Rad} M$ implies that $M = X_1 + \operatorname{Rad} M$ by saying $X_1 = \sum_{i=1}^{n} Rx_i$.

In the other case, choose a basic submodule X_1 of X which is radical supplemented by assumption, and of course $X_1 + \operatorname{Rad} M = M$ holds, because: by definition of basic submodule, X_1 is pure in X and X/X_1 is divisible. Then $p(X/X_1) = X/X_1$, and so $(pX + X_1) = X$. Thus $pX + X_1 + pM = X + pM = M$, i.e., $X_1 + \operatorname{Rad} M = M$ since $pX \leq pM$. In both cases X_1 is supplemented and a supplement of $X_1 \cap \operatorname{Rad} M$ in X_1 (exists as X_1 is supplemented) is also a supplement of $\operatorname{Rad} M$ in M which is contained in X, because: let V be a supplement of $X_1 \cap \operatorname{Rad} M$ in X_1, that is, $V + (X_1 \cap \operatorname{Rad} M) = X_1$ and $V \cap (X_1 \cap \operatorname{Rad} M) = V \cap \operatorname{Rad} M \ll V$. Then, $V + \operatorname{Rad} M = V + \operatorname{Rad} M + (X_1 \cap \operatorname{Rad} M) = X_1 + \operatorname{Rad} M = M$. Hence V is a supplement of $\operatorname{Rad} M$ in M such that $V \leq X_1$. $\square$

An example for a radical supplemented module in which the radical does not have ample supplements is as follows:

Example 3.2.4. (Zöschinger, 1976, pp. 3) Let R be a DVR with quotient field $K \neq R$ and maximal ideal Rp. An R-module M such that $M = [K/R \times R/Rp]^{\mathbb{N}}$ is radical supplemented but it does not have ample supplements. Indeed, M can be written as

$$M = (K/R \times K/R \times \cdots) \oplus (R/Rp \times R/Rp \times \cdots).$$

Then $\operatorname{Rad} M = Mp = (K/R \times K/R \times \cdots) \oplus (0 \times 0 \times \cdots)$. Thus $\operatorname{Rad} M + V = M$ where $V = (0 \times 0 \times \cdots) \oplus (R/Rp \times R/Rp \times \cdots)$, and

$$V \cap \operatorname{Rad} M = (0 \times 0 \times \cdots) \ll V.$$

Hence V is a supplement of $\operatorname{Rad} M$, i.e., M is radical supplemented. But $\operatorname{Rad} M$ does not have ample supplements.

Proposition 3.2.5. *(Zöschinger, 1976, Remark) Let R be a DVR with quotient field $K \neq R$ and maximal ideal Rp and let M be an R-module. An arbitrary submodule U of M satisfies the following:*

If V is a supplement of U in M, where $V = v_0R \oplus V'$ with $v_0 \neq 0$ and $u_0 \in U$, then $W = (v_0 + u_0)R + V'$ is also a supplement of U in M. Therefore this allows us to describe at once the contrary situation of Theorem 3.2.3 :

If $\operatorname{Rad} M$ has only one supplement in M, then M is either coatomic or divisible.

Theorem 3.2.6. *(Zöschinger, 1976, Proposition 2) Let R be a DVR with quotient field $K \neq R$ and maximal ideal Rp. For an R-module M, the following are equivalent:*

(i) *M has ample basic submodules,*

(ii) *Every dense neat submodule of M is pure in M,*

(iii) *In case M is not coatomic, $T(M)$ is divisible.*

Proof. $(iii) \Rightarrow (ii)$: If M is coatomic, then M is the only dense in M. Indeed, let S be a dense submodule in M, i.e., M/S is divisible. Then $\operatorname{Rad}(M/S) = M/S$ and so $M = S$ since M is coatomic. If M is not coatomic, then by assumption $T(M)$ is divisible. So, for every neat submodule U of M, $Up^n = U \cap Mp^n$; because: by induction, for $n = 1$, clearly $Up = U \cap Mp$ since U is neat in M. Suppose $Up^k = U \cap Mp^k$ holds for $k < n$. Then we want to show that $Up^n = U \cap Mp^n$. Clearly, $Up^n \leq U \cap Mp^n$. Now let $x = mp^n \in U \cap Mp^n$. Then $x = (mp^{n-1})p$, and so $x = up$ for some $u \in U$ since U is neat in M. This implies $mp^n = up$, and, then $(u - mp^{n-1})p = 0$. So, $(u - mp^{n-1}) \in T(M)$. Since $T(M)$ is divisible, $u - mp^{n-1} = p^{n-1}y$ for some $y \in T(M)$. Then $u = (m + y)p^{n-1} \in U \cap Mp^{n-1}$. Thus by induction hypothesis, $u = \widetilde{u}p^{n-1}$ for some $\widetilde{u} \in U$. Hence $x = up = \widetilde{u}p^{n-1}p = \widetilde{u}p^n \in Up^n$.

$(ii) \Rightarrow (i)$: Let $X + \operatorname{Rad} M = M$. The set $\Gamma = \{U \leq X \mid \quad U \text{ is neat in } M\}$ has a maximal element X_1 by Zorn's lemma, because: if $N_1 \leq N_2 \leq \ldots$ is a chain of neat submodules N_i in Γ, then $\bigcup N_i$ is also neat, and so if $N_i \leq X$, then $\bigcup N_i \leq X$. This implies $X_1 \in \Gamma$ is maximal element. Since X/X_1 does not contain any nonzero neat submodule of M/X_1 (suppose A/X_1 is neat in M/X_1 and $A/X_1 \leq X/X_1$, then $A \leq X$ implies that A is not neat, otherwise since $X_1 \leq A$, the neatness of A now contradicts the maximality of X_1. So, $pA \neq A \cap pM$, and so $p(A/X_1) = (pA + X_1)/X_1 \neq (A/X_1) \cap p(X/X_1)$. Hence A/X_1 is not neat. A contradiction), we have $X/X_1 \leq \operatorname{Rad}(M/X_1)$. Hence $X_1 + \operatorname{Rad} M = M$, because:
Since $X/X_1 \leq \operatorname{Rad}(M/X_1) = p(M/X_1)$, $X \leq pM + X_1$ and so $M = pM + X \leq pM + X_1$ implies $M = X_1 + \operatorname{Rad} M$. Now choose a basic submodule X_2 of X_1. We shall show that X_2 is dense and neat in M:
Since $X_2 \leq_{neat} X_1 \leq_{neat} M$, we have $X_2 \leq_{neat} M$. Now since X_1/X_2 is divisible, $pX_1 + X_2 = X_1$. Then $pX_1 + X_2 + pM = X_1 + pM = M$, and so $pM + X_2 = M$. Hence $p(M/X_2) = M/X_2$ implies M/X_2 is divisible, i.e., X_2 is dense in M. So, by assumption, X_2 is pure in M, i.e., X_2 is a basic submodule of M.

$(i) \Rightarrow (ii)$: Let U be dense and neat in M. By assumption M has a basic submodule, say S, with $S \leq U$. Since M/S is divisible (as S is basic in M) and $T(M/S)$ is pure in M/S, $T(M/S)$ is also divisible. We also have U/S is neat in M/S (as U is neat in M). Thus by using the idea in the proof of $(iii) \Rightarrow (ii)$, we have $(U/S)p^n = U/S \cap (M/S)p^n$, i.e., U/S is pure in M/S. Hence S is pure in M (as S is basic in M) implies that U is pure in M.

$(ii) \Rightarrow (iii)$: *Claim 1-* $N = K/R \times R/ < p^t >$ where $t \geq 1$ does not satisfy the condition (ii). Indeed, choose an intermediate module $0 \neq Y \leq X \leq K/R$ with $X/Y \cong R/ < p^t >$. Then the diagonal $f : X \to K/R \times X/Y$ is a neat monomorphism with dense image. Really,

$$(K/R \times X/Y)/\operatorname{Im} f = (K/R \times X/Y)/(X \times X/Y) \cong (K/R)/X$$

is divisible. This means $\operatorname{Im} f$ is dense. But its image is not isomorphic to $R/ < p^t >$, so

it is not pure. *Claim 2*- If $N = N_0 \oplus N'$ satisfies the condition (ii) where $N_0 \cong R/ < p^t >$ and N' is torsion-free, then N' is finitely generated. Indeed, suppose for the contrary that N' is not coatomic, so there is a basic submodule $V' \neq N'$ [since N'/V' is divisible, $\mathrm{Rad}(N'/V') = N'/V' \neq 0$ for some V' since N' is not coatomic]. Further, choose $0 \neq a \in N'$ with $aR \cap V' = 0$ and $N_0 = v_0R$, $u_0 = ap$. Hence $(u_0 + v_0)R \cong R$, by $r \mapsto (u_0+v_0)r$, and $(u_0+v_0)R \cap V' = 0$; because clearly, the map is an epimorphism, and if $(u_0+v_0)r = (u_0+v_0)r'$, then $v_0(r-r') = u_0(r'-r) \in N_0 \cap N'$, since $u_0R = apR \leq N'$. Thus $r = r'$ implies the map is an isomorphism. And now let $v' = (u_0 + v_0)r$, then $v' - u_0r = v_0r \in N' \cap N_0 = 0$. Thus $v' = u_0r = apr \in V' \cap aR = 0$ implies $v' = 0$. Thus $W = (u_0 + v_0)R \oplus V'$ is dense and neat in N:
Indeed, let $w = (r, v') = p(n_0, n')$ for some $(n_0, n') \in N$, $v' \in V'$, $r \in R$ and prime p. Then $w = (pv_0r', pn')$, and so $v' = pn' = pv''$ for some $v'' \in V'$, since V' is neat in N' (as it is pure). Thus $w = p(v_0r', v'')$ where $(v_0r', v'') \in W$. This shows that W is neat in N . By assumption W is pure in N, so it is a basic submodule [since W is dense and direct sum of cyclic modules as V' is direct sum of cyclic modules]. Hence W is isomorphic to a basic submodule $V = v_0R \oplus V'$ [(i) since $N_0 = v_0R$, $N'/V' \cong (N/N_0)/(V/N_0) \cong N/V$ is divisible (as V' is dense in N'), and so V is dense in N. (ii) Let $v = (v_0r, v') = p^k(v_0r', n')$, then $v' = p^kn'$ implies $v' = p^kv''$ since V' is pure in N'. Hence $v = p^k(v_0r', v'')$ where $(v_0r', v'') \in V$, that is, V is pure in N. (iii) V is direct sum of cyclic modules as V' is direct sum of cyclic modules] which is not torsion-free [really, since $v_0R \cong R/ < p^t >$, $V \cong R/ < p^t > \oplus V'$ and so there is $v = (x, 0) \in V$ such that $p^tv = 0$]. This is not possible, since $W \cong R \oplus V'$ is torsion-free as $V' \leq N'$ is torsion-free. This gives a contradiction. Hence N' is finitely generated. *Claim 3*- $N = \bigoplus_{i=0}^{\infty}$ with $N_i = R/ < p^{t_i} >$ and $0 < t_0 < t_1 \cdots$ does not satisfy the condition (ii). choose two basic submodules V', V'' in $N' = \bigoplus_{i=1}^{\infty}$ with $V' \cap V'' = 0$ as in Mitchell & Mitchell (1967, Lemma 1). Let $a \in V''$ with $ap^{t_0+1} \neq 0$, $N_0 = v_0R$ and $u_0 = ap$. Then it follows from $(v_0 + u_0)p^{t_0} \neq 0$ [since $u_0p^{t_0} = app^{t_0} = ap^{t_0+1} \neq 0$] that $(v_0+u_0)R \cap V' = 0$ [if $v' = (v_0+u_0)r \in (v_0+u_0)R \cap V'$, then $v_0r = v' - u_0r \in N_0 \cap N' = 0$. Thus $v' = u_0r = apr \in V' \cap V'' = 0$ since $a \in V''$,

and so $v' = 0$], hence $W = (v_0 + u_0)R \oplus V'$ is dense and neat in N (see the proof of Claim 2). But since $W \ncong V = v_0R \oplus V'$ (as in the proof of Claim 2; W is torsion-free but V is not torsion-free) W is not pure in N (since W is dense but not basic submodule of N). Now consider (ii) for M, and assume that $T(M)$ is not divisible. We shall show that M is coatomic. Since $T(M)$ is not divisible, $M = M_0 \oplus M_1$ with $M_0 \cong R/ < p^t >$. Since $f : M \to M_0 \times (M_1/T(M_1))$ is an isomorphism with pure kernel. Indeed, clearly $\text{Ker}\, f = \{m_0+m_1 \in M \mid \quad (m_0, m_1+T(M_1)) = 0\} = \{m_0+m_1 \mid m_0 = 0,\ m_1 \in T(M_1)\} = T(M_1)$ and $T(M_1)$ is pure in M_1. Hence M_1 is pure in M (as it is a direct summand) implies $T(M_1)$ is pure in M. So we have the condition (ii)) for this module. So, $M_1/T(M_1)$ is finitely generated by Claim 2. It remains to show that $T(M)$ is bounded. By Claim 1, it is reduced and a basic submodule S of $T(M)$ again satisfies (ii). So it is bounded by Claim 3, hence $T(M)$ is bounded, too. □

Theorem 3.2.7. *(Zöschinger, 1976, Proposition 3) Let R be a DVR with quotient field $K \neq R$ and maximal ideal Rp. For a radical supplemented R-module M the following are equivalent:*

(i) *Every supplement of radical is a basic submodule,*

(ii) *The supplements of radical are isomorphic to each other,*

(iii) *In case M is not coatomic, $T(M)$ is divisible.*

3.3 Coneat Submodules

For the definition of coneat submodules, see Section 1.5.

The following proposition gives a characterization of coneat submodules.

Proposition 3.3.1. *(Mermut, 2004, Proposition 3.4.2) For a submodule $V \leq M$, the following are equivalent:*

(i) *$V \leq M$ is a coneat submodule;*

(ii) *There exists a submodule* $U \leq M$ *such that*

$$U + V = M \quad \text{and} \quad U \cap V = \operatorname{Rad} V;$$

(iii) *There exists a submodule* $U \leq M$ *such that*

$$U + V = M \quad \text{and} \quad U \cap V \leq \operatorname{Rad} V.$$

Proposition 3.3.2. *(Mermut, 2004, Proposition 3.4.1) For any ring* R*, if a submodule* V *of an* R*-module* M *is supplement in* M*, then it is coneat (= Rad-supplement) in* M*.*

Proof. Let $V \leq M$ be a supplement in M. Then there exists a submodule $U \leq M$ such that $U + V = M$ and $U \cap V \ll V$. Thus clearly $U \cap V \leq \operatorname{Rad} V$, since $\operatorname{Rad} V$ is the sum of all small submodules of V. Hence V is a Rad-supplement of U in M. □

Proposition 3.3.3. *(Mermut, 2004, Proposition 3.4.5) Let* $V \leq M$ *be a submodule and suppose* $\operatorname{Rad} V \ll V$ *or* $\operatorname{Rad} V = 0$*. Then* V *is coneat if and only if it is a supplement in* M*.*

Proof. ($\Leftarrow$) : Follows immediately from Proposition 3.3.2.

($\Rightarrow$) : Suppose V is coneat in M. Then there exists a submodule $U \leq M$ such that $U + V = M$ and $U \cap V \leq \operatorname{Rad} V$. Now if $\operatorname{Rad} V \ll V$, then $U \cap V \ll V$ and so V is a supplement in M. On the other hand, if $\operatorname{Rad} V = 0$, then clearly, $\operatorname{Rad} V \ll V$. This completes the proof. □

Proposition 3.3.4. *(Mermut, 2004, Proposition 3.8.3) For a left max ring* R*, a submodule* $V \leq M$ *is a supplement in* M *if and only if it is a coneat in* M*.*

Proof. Follows from Proposition 3.3.3 □

By characterization of left perfect rings as in for example Anderson & Fuller (1992, Theorem 28.4), a ring R is *left perfect* if and only if R is a *semilocal left max* ring. So: it follows from Proposition 3.3.4 that for a left perfect ring R, V is a supplement in M if and only if V is a coneat in M.

Theorem 3.3.5. *(Generalov, 1983, Corollaries 1 and 6) For a Dedekind domain R, if V is a supplement in an R-module M, then it is a complement. Moreover, since a Dedekind domain is C-ring, V is a complement in M if and only if it is neat in M (see Section 1.4).*

For finitely generated torsion modules over a Dedekind domain, we have the following results from Mermut (2004):

Theorem 3.3.6. *(Mermut, 2004, Theorems 5.3.1, 5.3.2 and 5.4.9) Let R be a Dedekind domain. Take an R-module M and a submodule $N \leq M$. If M or N is finitely generated and torsion, then:*

(i) *N is a complement in M if and only if N is a supplement in M,*

(ii) *N is neat in M if and only if N is coneat in M.*

Theorem 3.3.7. *(Mermut, 2004, Theorem 5.4.6) Let R be a Dedekind domain which is not a field.*

(i) *If $\operatorname{Jac} R = 0$, then we have:*

  (a) There exists a module M_0 and a submodule N_0 of M_0 such that N_0 is coneat in M_0 but it is not supplement in M_0.

  (b) There exists a module M_0 and a submodule N_0 of M_0 such that N_0 is neat in M_0 but it is not coneat.

(ii) *If $\operatorname{Jac} R \neq 0$, then we have:*

  (a) There exists a module M_0 and a submodule N_0 of M_0 such that N_0 is coneat in M_0 but it is not supplement in M_0.

  (b) For every R-module M and for every submodule $N \leq M$, N is coneat in M if and only if it is neat in M.

CHAPTER FOUR
NEAT AND CONEAT SUBGROUPS

In this chapter, we survey the properties of neat subgroups and we seek for neat subgroups which are coneat in the last section. In particular, we give some elementary properties for neat subgroups in Section 4.1. We also give some results by adapting the properties for pure subgroups in this section. In Section 4.3, a characterization of intersection of neat subgroups is given (Rangaswamy (1965)). In the other sections, we give some properties for N-high subgroups (= complement = neat) for an abelian group N, for neat-exact sequences of abelian groups and for Frattini-high subgroups.

4.1 Neat Subgroups

For the definition of neat subgroups, see Section 1.4.

Definition 4.1.1. For a prime number p, a subgroup H of a group G is said to be *p-neat* in G if

$$pH = H \cap pG.$$

The concept as well as some of the properties stated here are due to Honda (1956).

Theorem 4.1.2. *(Fuchs, 1958, §28, properties (a), (c), (f) and (b), pp. 92) The following elementary properties of neat subgroups are of some interest:*

(i) *Every direct summand, and more generally, every pure subgroup is neat. But the converse fails to be true.*

(ii) *The property of being a neat subgroup is transitive.*

(iii) *The union of an ascending chain of neat subgroups is again neat.*

(iv) *Neat subgroups of torsion-free groups are pure.*

Proof. (i) Let $A = B \oplus C$. We shall show that B is a neat subgroup in A. Clearly, $pB \leq B$ and since $B = B + 0 \leq A$, $pB \leq pA$. Then $pB \leq B \cap pA$.

Conversely, let $b = pa \in B \cap pA$ for some $a \in A$, $b \in B$. Then $b = pa = p(b' + c') = pb' + pc'$, where $a = b' + c'$ for some $b' \in B$ and $c' \in C$. Then $b - pb' = pc' \in B \cap C = 0$, so $pc' = 0$ already. Thus $b = pb' \in pB$. Hence $B \cap pA \leq pB$, and so B is a neat subgroup of A.

Now we shall give an example of a neat subgroup which is not pure.
Let $G =< a > \oplus < b >$ such that $\mathrm{o}(a) = p^3$ and $\mathrm{o}(b) = p$. Suppose $N =< pa+b >$. Then N is neat in G but not pure. To show that N is neat in G, we shall show that every element $k(pa + b)$ of N is divisible by p in G implies is divisible by p in N. Let $k(pa + b) = p(ta + sb)$ where $ta + sb \in G$. Then $kpa + kb = pta + psb = pta$, since $\mathrm{o}(b) = p$. Then $kb = (pt - kp)a \in< a > \cap < b >= 0$ implies that $kb = 0$. Since $\mathrm{o}(b) = p$, p must divide k, i.e., $k = pk'$ for some $k' \in \mathbb{Z}$. Thus $k(pa + b) = pk'(pa + b)$. It means that $k(pa + b)$ is divisible by p in N. On the other hand N is a cyclic group which has order p^2, i.e., $p^2N = 0$. But $N \cap p^2G$ has a nonzero element p^2a (i.e., $p^2a = p(pa + 0) \in N \cap p^2G$). Hence $0 = p^2N \neq N \cap p^2G \neq 0$ for an integer p^2, that is, N is not pure in G.

(ii) Let N and H be subgroups of a group G. Suppose $N \leq_{neat} H$ and $H \leq_{neat} G$. Then for every prime p we have $pN = N \cap pH$ and $pH = H \cap pG$. Thus

$$pN = N \cap (H \cap pG) = (N \cap H) \cap pG = N \cap pG.$$

Hence N is neat in G.

(iii) Let $N_1 \leq N_2 \leq N_3 \ldots$ be an ascending chain of neat subgroups of G. Set $N = \bigcup_{i=1}^{\infty} N_i$. If $n = pg \in N$ for some $g \in G$, then there exist some N_k such that $n = pg \in N_k$. Since N_k is neat in G, $n = pn_k$ for some $n_k \in N_k$. Clearly $n_k \in \bigcup_{i=1}^{\infty} = N$, and so n is divisible by p in N, that is, N is neat in G.

(iv) Suppose B is a neat subgroup of a torsion-free group A. Let $b = na$ for some $a \in A$ and for every integer n. We shall show that $b = nb'$ for some $b' \in B$.
Let $n = (p_1 \ldots p_m)$ where p_i are primes but not necessarily distinct. Since B is a

neat subgroup of A, there exists $b_1 \in B_1$ such that $b = p_1 b_1$. Then $b = na = p_1 b_1$ implies $p_1 b_1 - (p_1 \ldots p_m)a = 0$, so $p_1(b_1 - p_2 \ldots p_m a) = 0$. Since A is torsion-free, $b_1 - p_2 \ldots p_m a = 0$, that is, $b_1 = p_2 \ldots p_m a$. By applying the same method above, we get $b_2 = p_3 \ldots p_m a$ for some $b_2 \in B$, and so $b = na = p_1 b_1 = p_1 p_2 b_2$. Going on this way we obtain that $b = p_1 p_2 \ldots p_m b_m = n b_m$ for some $b_m \in B$.

$\square$

Theorem 4.1.3. *(Fuchs, 1958, §28, (g), pp. 92) If N is neat subgroup of G and either N itself is an elementary p-group or the factor group G/N is elementary, then N is a direct summand of G.*

Proof. Suppose N is an elementary p-group which is neat in G. Then $N \cap pG = pN = 0$, since each element in N has order p. By Zorn's Lemma there exists a subgroup M of G maximal with respect to the properties $pG \leq M$ and $M \cap N = 0$ (i.e., M is N-high subgroup of G). Indeed, consider the partially ordered set

$$\Gamma = \{K \leq G \mid pG \leq K, K \cap N = 0\}$$

with " $\leq$ " , $\Gamma \neq 0$ since $pG \in \Gamma$. If we take a chain Λ from Γ, then it has an upper bound $\bigcup K_i$ in Γ ($K_i \in \Lambda$). Thus by Zorn's lemma, there is a maximal element in Γ, say M. Now let us prove that $G = M \oplus N$. To apply Lemma 4.2.3, let $qa = b + c$ with $a \in G$, $b \in N$, $c \in M$ and q a prime. We shall show that $b = qb'$ for some $b' \in N$.
If $q \neq p$, then since N is a p-group, N is q-divisible (Since $(q,p) = 1, qu + pv = 1$ for some integers u, v. Then $qua + pva = a$, $a \in N$, and so $a = q(ua)$ since $pva = 0$).
If $q = p$, then $pa = b + c$ or equivalently $b = pa - c \in N \cap M = 0$ (since $pa \in pG \leq M$). So b=0 already. Thus b is divisible by p in N. Hence by Lemma 4.2.3, $G = N \oplus M$, i.e., N is direct summand of G.

Now suppose G/N is elementary, i.e., $G/N = \bigoplus_{i \in I} < a_i + N >$, where $\mathrm{o}(a_i + N) = p_i$ is prime for each $i \in I$ for an arbitrary index set I. Let $< a_i + N >= G_i/N$. By Fuchs (1970, Lemma 9.4) it suffices to prove that N is a direct summand of G_i for every $i \in I$. Since N is neat in G, if $n = p_i a_i \in N$ for $a_i \in G_i$, then $n = p_i n'$ for some $n' \in N$.

Then $p_i a_i = p_i n'$ or equivalently $p_i(a_i - n') = 0$. Then for the element $a = a_i - n'$, we have $p_i a = 0$ and $a - a_i \in N$. Thus o(a) =o$(a_i + N) = p_i$, and so $< a > \cong < a_i + N >$. Moreover, for the homomorphism $f :< a_i + N > \longrightarrow G_i$ defined by $f(k(a_i + N)) = ka$, we have $\sigma \circ f = 1_{<a_i+N>}$, where $\sigma : G_i \longrightarrow G_i/N$ is a natural epimorphism. Therefore the short exact sequence

$$0 \longrightarrow N \longrightarrow G_i \xrightarrow{\sigma} G_i/N \longrightarrow 0$$

splits, and so N is a direct summand of G_i for every $i \in I$. □

Example 4.1.4. For a group $B = \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$, the subgroup $A = \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$ is neat in B, but A is not a direct summand of G. In particular, A is the torsion part of B, but it is not a direct summand of B.

Firstly, we shall show that A is *neat* in B. Let

$$(a_1, a_2, \ldots, a_n, 0, 0, \ldots) = p(b_1, b_2, \ldots, b_n, b_{n+1}, \ldots).$$

So there exists m such that $b_{m+j} = 0$ $(j \geq 0)$, i.e.,

$$(b_1, b_2, \ldots, b_n, \ldots) = (b_1, b_2, \ldots, b_m, 0, 0, \ldots) \in A.$$

Hence A is neat in B.

Now let us show that A is not a direct summand of B. If it were a direct summand, then $B = A \oplus C$ for some $C \leq B$. Since B is reduced, so is its direct summand $C \cong B/A$. But this contradicting B/A is divisible.

It remains only to show that B/A is divisible. For this it suffices to show that $p(B/A) = B/A$ for all primes p. Let $b + A \in B/A$ and p an arbitrary prime. Then $b = (b_1, b_2, \ldots, b_p, b_{p+1}, \ldots) = (b_1, b_2, \ldots, 0, b_{p+1}, \ldots) + (0, 0, \ldots, 0, b_p, 0, \ldots) = c + d$. Then $b + A = c + d + A = c + A$ since $d \in A$. Since $\mathbb{Z}/q\mathbb{Z}$ is p-divisible for all primes $q \neq p$, we have $b_q = pb'_q$ for some $b'_q \in \mathbb{Z}/q\mathbb{Z}$. So c is p-divisible, that is, $c = pc'$ for some $c' \in B$. Hence $b + A = c + A = p(c' + A)$ implies B/A is divisible.

Remark 4.1.5. (Imam, 2000, Remark 1.2.2) An elementary neat subgroup need not be a direct summand (by the above example).

Proposition 4.1.6. *(Fuchs, 1958, §28, (d), pp. 92) Let M be a subgroup of a group G. If M is a complement in G, then M is neat in G.*

Proof. Suppose $0 \neq m = pg$ for some $m \in M$, $g \in G$, but $g \notin M$. Then $(M+ < g >) \gneqq M$. Since M is maximal with the property $M \cap H = 0$, $(M+ < g >) \cap H \neq 0$. So there exists $0 \neq h = m' + kg \in (M+ < g >) \cap H$ for some $m' \in M$ and $k \in \mathbb{Z}$, where k is not divisible by p (really, if $p \mid k$, then $k = pk'$ for some $k' \in \mathbb{Z}$, and so $h = m' + pk'g = m' + k'pg = m' + k'm \in H \cap M = 0$ contradicting $h \neq 0$). Thus $\gcd(p, k) = 1$, so there exist $u, v \in \mathbb{Z}$ such that $pu + kv = 1$. Then $upg + vkg = g$ implies that $um + v(h - m') = um + vh - vm' = g$, since $kg = h - m'$ and $pg = m$. Therefore $pum + pvh - pvm' = pg$, so $pvh = pg - pum + pvm' \in M \cap H = 0$ since $pg = m \in M$. Hence $m = pg = p(um + vm')$, i.e., m is divisible by p in M $(um + vm' \in M)$, that is, M is neat in G. □

Definition 4.1.7. We call a subgroup A of a group G an *absolute direct summand* of G, if for every complement B of A in G, $A \oplus B = G$.

Proposition 4.1.8. *(Fuchs, 1958, (e), pp. 92) A subgroup A of a group G is an absolute direct summand of G if and only if for every neat subgroup N of G with $A \cap N = 0$, the sum $A \oplus N$ is also neat in G.*

Proof. (⇒): If A is an absolute direct summand of G and N is a neat subgroup with $A \cap N = 0$, then $G = A \oplus B$, where B is a subgroup of G maximal with respect to the property $A \cap B = 0$ (clearly $N \leq B$). We shall show that $A + N$ is neat in G.
Suppose $a' + c = pg$ for some $g \in G$, $a' \in A$, $c \in N$ and an arbitrary prime p. Since $g = a + b$ for some $a \in B, b \in B$, we get $pg = pa + pb$. So $pg = a' + c = pa + pb$ or equivalently $a' - pa = pb - c \in A \cap B = 0$ since $c \in N \leq B$. Thus we obtain $pa = a'$ and $pb = c$. Since N is neat in G, $pb = c \in N$ implies that $c = pc'$ for some $c' \in N$. Hence $pg = a + c = pa + pc' = p(a + c')$, where $a + c' \in A + N$ which implies $A + N$ is neat in G.

(⇐): Conversely, let A be a subgroup of G which satisfies the stated conditions and let B be any subgroup maximal with respect to the property $B \cap A = 0$ (i.e., B is A-high)(existence of B by Zorn's lemma). Then by Proposition 4.1.6, B is neat in G, and so by hypothesis $A + B$ is again neat in G. Let $pg \in A + B$ for some $g \in G$, i.e., $pg = a + b$ ($a \in A$, $b \in B$). Since $A + B$ is neat in G, we have $pg = a + b = p(a' + b')$ for some $a' \in A$ and $b' \in B$. Thus $a - pa' = pb' - b \in A \cap B = 0$. Hence $a = pa'$ implies that $G = A \oplus B$, by Lemma 4.2.3. $\square$

Theorem 4.1.9. *(Fuchs, 1958, §28, (h), pp. 92) If E is a minimal divisible group containing a group G, then N is a neat subgroup of G if and only if $N = G \cap D$ where D is a divisible subgroup of E.*

Proof. ($\Leftarrow$): Let us suppose $N = G \cap D$ where D is a divisible subgroup of E and E is a minimal divisible group containing G. We shall show that $N = G \cap D$ is neat in G. Suppose $pg \in N = G \cap D \leq D$ for some $g \in G$ and an arbitrary prime p. Since D is divisible, there exist some $x \in D$ such that $pg = px$ or equivalently $p(g - x) = 0$. In case $x \neq g$, $g - x \in E[p] = G[p]$, and so $x \in G$. Since $x \in D$ also, we obtain $x \in G \cap D = N$. Hence N is neat in G.

($\Rightarrow$): Conversely, let N be neat in G and D be a minimal divisible subgroup of E, containing N. Clearly $N \leq G \cap D$. Now let $a \in G \cap D$ and suppose $a \notin N$ Then we can choose some least $0 \neq n \in \mathbb{Z}$ such that $na \in N$. Choose $n > 0$ as small as possible and write $n = pn_1$. Then $b = n_1 a \notin N$. This implies $pb = pn_1 a = na \in N$, and so $pb = pc$ for some $c \in N \leq D$ since N is neat in G. So $p(b - c) = 0$ shows that $b - c \in D[p] = N[p]$, since $b - c \in D$ as $b = n_1 a \in D$ and $c \in N \leq D$. Thus $b - c \in N$, and so $b \in N$ since $c \in N$. Contradicting $b \notin N$. Hence $a \in N$, that is, $G \cap D \leq N$. $\square$

Corollary 4.1.10. *(Fuchs, 1958, §28, (i), pp. 92) Every subgroup H of a group G can be embedded in a minimal neat subgroup N of G.*

Proof. Take a minimal divisible subgroup D of E (E as defined in Theorem 4.1.9) which contains H. Then by Theorem 4.1.9, $N = G \cap D$ will be a neat subgroup containing H and will be minimal. $\square$

We next list some useful facts concerning neat subgroups. These facts follow by adapting the related properties for pure subgroups in Fuchs (1970, pp. 114)

Theorem 4.1.11. *(Bilhan, 1995, Ch. 1) :*

(i) *Torsion part of a group is a pure subgroup so also a neat subgroup.*

(ii) *Let A be a group, G be a subgroup of A. If A/G is torsion-free, then G is pure in A so also neat in A.*

(iii) *For every prime number p, a p-neat, p-subgroup of a group is neat.*

(iv) *Let A and B be groups and $f : A \longrightarrow B$ be an isomorphism. Let C be a subgroup of A. If C is neat in A, then $f(C)$ is neat in B.*

(v) *Let A, B, C be groups satisfying $A \leq B \leq C$. If A is neat in C, then A is neat in B.*

(vi) *Let B and C be groups and A be a subgroup of B. If $f(A)$ is neat in $f(B)$ for some monomorphism $f : B \longrightarrow C$, then A is neat in B.*

(vii) *In torsion-free groups, intersections of neat subgroups is neat.*

In general, the intersection of two neat subgroups need not be neat. We give an example from Rangaswamy (1965, Introduction):

Example 4.1.12. Let G be a group such that $G =< a > \oplus < b >$ for some $a, b \in G$, where $\mathrm{o}(a) = p^2$ and $\mathrm{o}(b) = p$. Take $S_1 =< a >$ and $S_2 =< a + b >$. Then S_1 and S_2 are neat subgroups of G, but $S_1 \cap S_2 =< pa >$ is not neat in G. Indeed, let $ka = p(ta + sb) \in S_1$ for some $ta + sb \in G$ and integers k, t, s. Then $ka = p(ta)$ since $psb = 0$ as $\mathrm{o}(b) = p$. Thus S_1 is neat in G since $ta \in S_1$.
Now suppose $k(a+b) = p(ta+sb) \in S_2$, then $ka+kb = pta$ since $psb = 0$. So $(pt-k)a = kb \in< a > \cap < b >= 0$, so $kb = 0$. Since $\mathrm{o}(b) = p$, p must divide k, i.e., $k = pk'$ for some integer k'. Thus $k(a + b) = pk'(a + b)$ which means that $k(a + b) \in S_2$ is divisible by p. Hence S_2 is also neat in G. Finally, $p(S_1 \cap S_2) = p < pa >= 0$ since $\mathrm{o}(a) = p^2$. On the other hand, $(S_1 \cap S_2) \cap pG =< pa > \cap pG \neq 0$ since $0 \neq pa \in< pa > \cap pG$. Hence $p(S_1 \cap S_2) \neq (S_1 \cap S_2) \cap pG$, i.e, $S_1 \cap S_2$ is not neat in G.

Theorem 4.1.13. *A subgroup B of a group A is neat in A if and only if every coset of A modulo B of prime order contains an element of the same order.*

Proof. Following the proof for pure subgroups in Fuchs (1970, Theorem 28.1), we argue as follows.

($\Rightarrow$): Suppose $B \leq_{neat} A$. Take a coset $a^* =< a + B >\in A/B$ of prime order p, for some $a \in A$. Then for every $g \in a^*$, $g = t(a + B) = ta + b$, for some $b \in B$ and $t \in \mathbb{Z}$. Now we have $pg = pta + pb \in B$, since $\mathrm{o}(< a + B >) = p$ implies $pta + B = B$, i.e., $pta \in B$. Since B is neat in A, $pg = pb'$ for some $b' \in B$. Thus $p(g - b') = 0$, and so $\mathrm{o}(g - b') \leq p$. Since $g - b' = ta + b - b' \in< a + B >= a^*$, $\mathrm{o}(g - b') = p$ must be hold.

($\Leftarrow$): Conversely, if the stated condition holds, and if $pg = b \in B$, for some $g \in A$, then choose $a = g + b \in< g + B >$ of order of this coset. Then $pa = 0$, since $\mathrm{o}(< g + B >) = p$. Thus $p(g - a) = pg - pa = b$, where $g - a = -b \in B$. Hence B is neat in A. $\square$

The results in the following theorem are inferred from some exercises on pure subgroups from Fuchs (1970):

Theorem 4.1.14. *(Fuchs, 1970, Exercises 26.1, 26.2, 26.6, 26.8, 28.3)*

(i) *If $G \cap H$ and $G + H$ are neat subgroups of A, then so are G and H.*

(ii) *Let G be a neat subgroup of a group A. Then:*

   (a) $\operatorname{Rad} G = G \cap \operatorname{Rad} A$.

   (b) $(G + \operatorname{Rad} A)/\operatorname{Rad} A$ *is neat in* $A/\operatorname{Rad} A$.

   (c) $G \leq \operatorname{Rad} A$ *implies* $pG = G$ *for all prime numbers* p.

(iii) *The only essential neat subgroup of a group G is G itself.*

(iv) *If $A = B + C$ and $B \cap C$ is neat in B, then $A[p] = B[p] + C[p]$ for every prime number p.*

(v) *If G is a neat subgroup of each member of a chain $\cdots \leq B_i \leq \cdots$, then G is neat in $\bigcup B_i$.*

Proof. (i) Let $pa \in G$ for some $a \in A$ and an arbitrary prime p. Then $pa = pa + 0 \in G + H$. Since $G + H$ is neat in A, there exists $g + h \in G + H$ such that $pa = p(g+h) = pg+ph$ or equivalently $pa-pg = ph \in G \cap H$. Since $G \cap H$ is also neat in A, there exists $x \in G \cap H \leq G$ such that $ph = px$. Thus $pa - pg = px$, and so $pa = p(g + x)$, where $g + x \in G$. Hence G is neat in A. Similarly, H is neat in A.

(ii) (a) Since G is neat in A,

$$\operatorname{Rad} G = \bigcap_{p \in P} pG = \bigcap_{p \in P} (G \cap pA) = G \cap (\bigcap_{p \in P} pA) = G \cap \operatorname{Rad} A.$$

(b) For $x \in (G + \operatorname{Rad} A)/\operatorname{Rad} A$, let $x = p(a + \operatorname{Rad} A)$ for some $a \in A$. We shall show that $x \in p((G+\operatorname{Rad} A)/\operatorname{Rad} A)$. Here $x = g+a_1+\operatorname{Rad} A = g+\operatorname{Rad} A$, $g \in G$, $a_1 \in \operatorname{Rad} A$. So $x = g + \operatorname{Rad} A = pa + \operatorname{Rad} A$ or equivalently $pa-g \in \operatorname{Rad} A$. Say $pa-g = pa'$ for every $p \in P$ and $a' \in A$, so $g = p(a-a')$. Since $a - a' \in A$ and since $G \leq_{neat} A$, there exists an element $g' \in G$ such that $g = p(a-a') = pg'$. Thus $x = g+\operatorname{Rad} A = pg'+\operatorname{Rad} A = p(g'+\operatorname{Rad} A) \in p((G + \operatorname{Rad} A)/\operatorname{Rad} A)$ completes the proof.

(c) Clearly, $pG \leq G$. Conversely, let $g \in G$. Then by assumption $g \in \operatorname{Rad} A$, and so $g = pa$ for every $p \in P$ and $a \in A$. Since G is neat in A, there exists $g' \in G$ such that $g = pa = pg'$. Hence $g \in pG$. This means $G \leq pG$.

(iii) Following the proof for pure subgroups in Călugăreanu et al. (2003, S 3.12), we argue as follows. Let H be a neat subgroup of G. Suppose H is essential in G, then we have $\operatorname{Soc} H = \operatorname{Soc} G$ by Proposition 2.5.3 (because if $x \in \operatorname{Soc} G \leq H$, then $x \in H$ has a square-free order, this means $x \in \operatorname{Soc} H$). Using Theorem 4.1.13, it results that if $g + H \in G/H$ with $\mathrm{o}(g + H) = p$ for some prime p then there exists $h \in H$ such that $\mathrm{o}(g + h) = p$, so $g + h \in \operatorname{Soc} G = \operatorname{Soc} H$, thus $g \in H$. Hence we deduce that $G = H$.

(iv) ($\Rightarrow$): Take any element $a = b + c$ of $A[p]$, for some $b \in B$, $c \in C$. Then $pa = pb + pc = 0$, so $-pc = pb \in B \cap C$. Since $B \cap C \leq_{neat} B$, we have $-pc = pb = px$ for some $x \in B \cap C$. Then $p(b-x) = 0$, and so $b - x \in B$ implies that $b - x \in B[p]$. Moreover, $pa = pb + pc = px + pc = 0$ since $pb = px$, and so $p(x + c) = 0$ implies $x + c \in C[p]$, since $x + c \in C$. Hence $a = b + c = (b - x) + (x + c) \in B[p] + C[p]$.

($\Leftarrow$): Clearly, $B[p] + C[p] \leq B + C = A$. Then for $b + c \in B[p] + C[p]$ there exists an $a \in A$ such that $b + c = a$. Then $pa = p(b + c) = pb + pc = 0$. Thus $b + c = a \in A[p]$.

(v) Let $pb \in G$ for some $b \in \bigcup B_i$ and an arbitrary prime p. Then $b \in B_j$ for some j, since $\cdots \leq B_i \leq \cdots$ is a chain. Now since $G \leq_{neat} B_j$, $pb = pg$ for some $g \in G$ which implies the neatness of G in $\bigcup B_i$.

□

Proposition 4.1.15. *If H and K are neat subgroups of a group G and H is essential in K, then $H = K$.*

Proof. Since H is neat in G, we have by Theorem 4.1.11 H is neat in K. Thus the result follows immediately from the Theorem 4.1.14. □

Lemma 4.1.16. *Let K be a group and H be a neat subgroup of a group G. Then we have:*

(i) *$H + D$ is neat in G, where D is the largest divisible subgroup of G.*

(ii) *$K \oplus H$ is neat in $K \oplus G$.*

Proof. (i) We have $pH = H \cap pG$ and $pD = D$. We shall show that $p(H + D) = (H + D) \cap pG$.

Solution 1. By modularity,

$$(H + D) \cap pG = (H + pD) \cap pG = pD + (H \cap pG) = pD + pH = p(H + D).$$

Solution 2. Let $h + d = pg \in (H + D) \cap pG$ for some $h \in H$, $d \in D$. Since D is divisible there exists $d' \in D$ such that $d = nd'$ for every positive integer n, and so does for a prime p, ie.e., $d = pd'$. Thus $pg = h + d = h + pd'$ and so $h = p(g - d')$. Since H is neat in G there is $h' \in H$ such that $h = p(g - d') = ph'$. Therefore $pg = h + d = ph' + pd' = p(h' + d') \in p(H + D)$ as desired. Hence $H + D$ is neat in G since the reverse inclusion always holds.

(ii) Since $pK \leq K \oplus H$, using the modularity of the subgroup lattice of a group, we have

$$p(K \oplus G) \cap (K \oplus H) = (pK \oplus pG) \cap (K \oplus H) = pK \oplus (pG \cap (K \oplus H)).$$

We shall show that these equalities equal to $p(K \oplus H)$. Let

$$pg = k + h \in pG \cap (K \oplus H)$$

for some $k \in K$, $h \in H$ and $g \in G$. Then $k = pg - h \in G \cap K = \{0\}$, and so $pg = h$. Thus there exists $h' \in H$ such that $pg = h = ph' \in pH$ since H is neat in G. Hence it follows that

$$p(K \oplus G) \cap (K \oplus H) = pK \oplus (pG \cap (K \oplus H)) \leq pK \oplus pH = p(K \oplus H).$$

Since the reverse inclusion always holds, the proof is completed.

□

Theorem 4.1.17. *In a group G, every subgroup is neat if and only if G is an elementary group (i.e.,* $\operatorname{Soc} G = G$*).*

Proof. ($\Rightarrow$): If every subgroup of the group G is neat, then G has only one essential subgroup that is G itself (see Theorem 4.1.14). Since the socle of a group is the intersection of all essential subgroups, we deduce that $G = \operatorname{Soc} G$, that is, G is elementary.

($\Leftarrow$): Conversely, let G be an elementary group. Then every subgroup of G is a direct summand, by Proposition 2.4.5, and so it is clearly neat in G. □

Theorem 4.1.18. *Let H be a subgroup of a group G. Then the following conditions are equivalent:*

(i) *H is neat in G;*

(ii) $(G/H)[p] = (G[p] + H)/H$ *for all prime* p*;*

(iii) $p^{-1}H = H + p^{-1}\{0\}$ *for all prime* p*.*

Proof. Following the proof for pure subgroups in Călugăreanu et al. (2003, S 3.2), we argue as follows.

$(i) \Rightarrow (ii)$: Clearly, $(G/H)[p] \supseteq (G[p] + H)/H$. Conversely, if $g + H \in (G/H)[p]$, then $p(g + H) = pg + H = H$, and so $pg \in H$. But $pg \in H \cap pG = pH$ since H is neat in G. So, there exists an element $h \in H$ such that $pg = ph$ or equivalently $p(g-h) = 0$. This means that $g - h \in G[p]$, and so $g + H = (g - h) + H \in (G[p] + H)/H$.

$(ii) \Rightarrow (iii)$: The inclusion $p^{-1}H \supseteq H + p^{-1}\{0\}$ is immediate, because if $h + x \in H + p^{-1}\{0\}$, then $p(h + x) = ph + px = ph \in H$ since $px = 0$, and so $h + x \in p^{-1}H$. Conversely, let us take $g \in p^{-1}H$ i.e., $pg \in H$, that is, $p(g + H) = H$. Then $g + H \in (G/H)[p] = (G[p] + H)/H$ (by hypothesis) which implies the existence of an element $g' \in G[p]$ such that $g + H = g' + H$. Hence $g = g' + h$ for some $h \in H$. Since $pg' = 0$ (i.e., $g' \in p^{-1}\{0\}$) it follows that $g \in H + p^{-1}\{0\}$.

$(iii) \Rightarrow (i)$: If $h \in H \cap pG$, then $h = pg$ for some $g \in G$ and an any prime p. It follows that $g \in p^{-1}H = H + p^{-1}\{0\}$, and so there exist $h' \in H$ and $x \in p^{-1}\{0\}$ such that $g = h' + x$. Hence $h = pg = p(h' + x) = ph' \in pH$ since $px = 0$. Since the reverse inclusion always holds, $H \cap pG = pH$, that is, H is neat in G. □

Proposition 4.1.19. *(see for example Călugăreanu et al. (2003, M 3.9)) If* M *is an essential subgroup of a group* G *such that* $G = A \oplus F$ *for some subgroups* A *and* F *of* G*, then* $A \cap M$ *is neat in* M*.*

Proof. We have to verify that $A \cap pM = (A \cap M) \cap pM \leq p(A \cap M)$ (the reverse inclusion always holds). Take an element $a = pg \in A \cap pM$ $(a \in A,\ g \in M)$ and $g = a_1 + f$ $(a_1 \in A,\ f \in F)$. Then $a = pg = p(a_1 + f) = pa_1 + pf$, or, $a - pa_1 = pf \in A \cap F = 0$, and so $pf = 0$, i.e., $a = pa_1$. Hence it suffices to prove $a_1 \in M$.

Case (i): $f = 0$;, then $g = a_1 \in M$.

Case (ii): $f \neq 0$;, then $M \cap < f > \neq 0$ since M is essential, and so there is an $m \in \mathbb{Z}^+$

such that $0 \neq mf \in M$. But $pf = 0$, so we may suppose $0 < m < p$ and hence $\gcd(m, p) = 1$. Then $um + vp = 1$ for some integers u, v. Finally, $f = 1.f = (um + vp)f = umf \in M$ since $pf = 0$, whence $a_1 = g - f \in M$. $\square$

The following theorem is a modification of the exercise for pure subgroups in Fuchs (1970, 27.12).

Theorem 4.1.20. *Let A be a neat subgroup of a group G such that $G = H \oplus K$ for some subgroups H and K of G. If $A \cap K$ is essential in both A and K, then $G = H \oplus A$.*

Proof. Following the proof for pure subgroups in Călugăreanu et al. (2003, M 3.1), we argue as follows. First observe that $H \cap A = 0$. Indeed, if $x \in H \cap A$, then $(A \cap K) \cap < x > \leq H \cap K = 0$. Since $A \cap K$ is essential in A, we obtain that $< x >= 0$, and so $x = 0$ already. Thus the sum $H + A$ is direct.

Observe (see Theorem 4.1.14) that it suffices to prove that: (i) $H + A$ is essential in G; and (ii) $H + A$ is neat in G.

(i): Notice that $\operatorname{Soc} G = \operatorname{Soc}(H \oplus K) = \operatorname{Soc} H \oplus \operatorname{Soc} K \leq H + A$ (because $A \cap K$ is essential in K implies that $\operatorname{Soc} K \leq A \cap K \leq A$, see Proposition 2.5.3).

Now we show that $G/(H + A)$ is a torsion group:

Since $A \cap K$ is essential in K it follows from Proposition 2.5.3 that $K/A \cap K \cong (K + A)/A$ is a torsion group. Hence for every $g = h + k \in G$ with $h \in H$, $k \in K$ there exists a positive integer n such that $nk \in A$ (since $n(k + A) = A$ as $(K + A)/A$ is torsion), and it follows $ng = nh + nk \in H + A$ or equivalently $n(g + (H + A)) = H + A$ for every $g + (H + A) \in G/(H + A)$. Hence we obtain from Proposition 2.5.3 that $H + A$ is essential in G.

(ii): Since A is a neat subgroup of G it suffices (see Lemma 4.1.21) to prove that $(H + A)/A$ is neat in G/A. But this follows using the neatness of H (as a direct summand) in G (see Lemma 4.1.21-(ii)). $\square$

Lemma 4.1.21. *(by Bilhan (1995, Lemma 1)) Let B, C be subgroups of A such that $C \leq B \leq A$. Then we have:*

(i) *If B is neat in A, then B/C is neat in A/C.*

(ii) *If* C *is neat in* A *and* B/C *is neat* A/C*, then* B *is neat in* A*.*

Proof. Following the proof for pure subgroups in Fuchs (1970, Lemma 26.1), we argue as follows.

(i) follows from the equalities

$$p(B/C) = (pB+C)/C = [(B\cap pA)+C]/C = [B\cap(pA+C)]/C = (B/C)\cap p(A/C).$$

(ii) Let $b = pa \in B$ for some $a \in A$ and an arbitrary prime p. Now $b + C = pa + C = p(a+C)$ and since $B/C \leq_{neat} A/C$, there exists $b' \in B$ such that $b+C = p(b'+C)$. From $pb' - b \in C$, we get $pb' - b = c$ ($c \in C$). So $pb' - pa = c$, i.e., $p(b' - a) = c$. Since C is neat in A, $c = pc'$ for some $c' \in C$. Finally, $pb' - b = c = pc'$ implies that $p(b' - c') = b$, where $b' - c' \in B$. Hence B is neat in A.

□

Proposition 4.1.22. *(Bilhan, 1995, Theorem 2) If* B *is a subgroup of a group* A *such that* $pB = 0$*, then the following statements are equivalent:*

(i) B *is a neat [p-neat] subgroup of* A*;*

(ii) $B \cap pA = 0$*;*

(iii) B *is a direct summand of* A*.*

Proof. Following the proof for pure subgroups in Fuchs (1970, Proposition 27.1), we argue as follows.

(i)⇒(ii): If B is neat in A, then $B \cap pA = pB = 0$ by hypothesis.

(ii)⇒(iii): Let C be a B-high subgroup of A (existence of C by Zorn's Lemma). Since C is maximal with the property $C \cap B = 0$, we have $pA \leq C$. If, for $a \in A$, $pa = b + c$ ($b \in B, c \in C$), then by Lemma 4.2.3 it suffices to show that $b = pb'$ for some $b' \in B$. Then $pa - c = b \in C \cap B = 0$, since $pa \in C$, and so $b = 0$ already. Since $pb' \in pB = 0$ for some $b' \in B$, we obtain that $0 = b = pb'$. Hence $A = B \oplus C$.

(iii)⇒(i) is trivial. □

Corollary 4.1.23. *(Bilhan, 1995, Corollary 3) For a prime p, every pA-high subgroup of a group A is direct summand of A.*

Proof. Let B be a pA-high subgroup of A. Then B is neat in A by Proposition 4.1.6. Thus $pB = B \cap pA = 0$ since B is pA-high. Hence the result follows immediately from Proposition 4.1.22. □

Theorem 4.1.24. *(Bilhan, 1995, Theorem 4) The following are equivalent conditions for a subgroup B of a group A:*

(i) *B is neat in A;*

(ii) *B/pB is a direct summand of A/pB for every prime p;*

(iii) *If $C \leq B$ and $pB \leq C$, then B/C is a direct summand of A/C.*

Proof. (i)⇒ (ii): Assume (i), then $B \cap pA = pB \leq B$; thus by Lemma 4.1.21, B/pB is also neat in A/pB. Since B/pB is a cyclic group of order p, by Proposition 4.1.22, it is a direct summand of A/pB.

(ii)⇒(iii): Suppose B/pB is a direct summand of A/pB, i.e., $A/pB = (B/pB) \oplus (K/pB)$, where $pB \leq K$. Also Suppose $C \leq B$ such that $pB \leq C$. Then there exists a projection map $f : A/pB \to B/pB$ defined by $f(a+pB) = b+pB$, where $a \in A$, $b \in B$. Let us define $F : A/C \to B/C$ as $F(a + C) = (g \circ f)(a + pB)$, where $pB \leq C \leq B$ (by hypothesis) and $g : B/pB \to B/C$ such that $g(b + pB) = b + C$. Clearly, g is well-defined since $pB \leq C$.

Let us show that F is also well-defined: Take $a + C = a' + C$ for some $a, a' \in A$, i.e., $a - a' \in C$. Thus we have,

$$F(a + C) = (g \circ f)(a + pB) = g(f(a + pB)) = g(b + pB) = b + C$$

and

$$F(a' + C) = g(f(a' + pB)) = g(b' + pB) = b' + C.$$

Now we shall show that $b + C = b' + C$, i.e., $b - b' \in C$ when $a - a' \in C$.
Since $A/pB = (B/pB) \oplus (K/pB)$, for $a + pB \in A/pB$, we have $a = b + k + pb_1$, and for $a' + pB \in A/pB$, we have $a' = b' + k' + pb_2$ for some $b_1, b_2 \in B$ and $k, k' \in K$. Then

$a - a' = (b - b') + (k - k') + p(b_1 - b_2)$. We know that $a - a' \in C$, $p(b_1 - b_2) \in pB \leq C$ and $k - k' \in C$ (indeed, $k - k' = (a - a') + (b' - b) + p(b_2 - b_1) \in B$ since $a - a' \in C \leq B$, and so $k - k' \in B \cap K = pB \leq C$ since $(B/pB) \cap (K/pB) = 0)$. Thus $b - b' \in C$ which implies $b + C = b' + C$. Hence F is well-defined. Since F is a projection map, B/C is a direct summand of A/C.

(iii)⇒(i): Let $b = pa \in B$, for some $a \in A$. Since $pB \leq B$ and $pB \leq pB$, B/pB is a direct summand of A/pB by hypothesis. Then there exists a projection map $f : A/pB \to B/pB$ defined by $f(a + pB) = b + pB$, where $a \in A$ and $b \in B$. Then $b + pB = f(b + pB) = f(pa + pB) = pf(a + pB) = p(b' + pB) = pb' + pB$ for some $b' \in B$. So $b - pb' \in pB$, and, then $b - pb' = pb''$ for some $b'' \in B$. Thus $b = p(b' + b'')$, where $b' + b'' \in B$. Hence B is neat in A. □

4.2 N-high Subgroups

High subgroups of abelian groups were first introduced by Irwin (1961). Since, then lots of research on various properties and representation of high subgroups have been conducted. See Irwin & Walker (1961), Irwin et al. (1962), Megibben (1964), Irwin & Benabdallah (1968) Benabdallah et al. (1974), Benabdallah & Irwin (1975), etc.

For the definition of high subgroups, see Section 1.1.

Definition 4.2.1. (i) A minimal divisible group E containing A is called the *divisible (injective) envelope* of A.

(ii) A neat subgroup N minimally containing S is called a *neat envelope* of S.

Lemma 4.2.2. *(Fuchs, 1970, Lemma 9.8) If N is a subgroup of G and K is a N-high subgroup of G, then $a \in G$, $pa \in K$ [p a prime] implies $a \in N \oplus K \leqq G$.*

Proof. If $a \in K$, then clearly $a \in N \oplus K$. Because $a = 0 + k \in N + K$ for some $k \in K$ and $N \cap K = 0$, since K is N-high subgroup of G. If $a \notin K$, then $< a > +K \gneqq K$ and since K is N-high in G, $(< a > +K) \cap N \neq 0$. Thus there exists an element $0 \neq b \in N$, i.e., $b = ta+k$ for some $k \in K$ and integer t. Here $(t,p) = 1$ since $pa \in K$ and $N \cap K = 0$ [otherwise, if $(p,t) \neq 1$, then $(p,t) = p$ since p is prime, so $t = pl$ for some integer l, thus $0 \neq b = ta+k = lpa+k \in N \cap K = 0$ gives a contradiction]. Thus $ut+vp = 1$ for some integers u, v, and so $a = u(ta)+v(pa) = u(b-k)+v(pa) = ub+(-uk+vpa) \in N \oplus K$ since $b \in N$ and $(-uk+vpa) \in K$. $\square$

Lemma 4.2.3. *(Fuchs, 1970, Lemma 9.9) Let G, N, K be as in Lemma 4.2.2. Then $G = N \oplus K$ if and only if $pa = b+k$ $(a \in G, b \in N, k \in K)$ implies $pb' = b$ for some $b' \in N$.*

Proof. Suppose $G = N \oplus K$ and $pa = b+k$ $(a \in G, b \in N, k \in K)$. Then there exist $b' \in N$ and $k' \in K$ such that $a = b'+k'$. Thus $pb'+pk' = p(b'+k') = pa = b+k$, and so $pb'-b = k-pk' \in N \cap K = 0$ implies $b = pb'$.

Conversely, if $pa = b+k$ $(a \in G, b \in N, k \in K)$ implies $pb' = b$ for some $b' \in N$, then we have:

(i) The quotient group $G/(N \oplus K)$ contains no elements of prime order, and therefore it is torsion-free:

Suppose that there is an element $a+(N \oplus K)$ with prime order p, i.e., $p(a+N \oplus K) = N \oplus K$. Then $pa \in N \oplus K$, and so $pa = b+k$ for some $b \in N, k \in K$. Thus by hypothesis, there exists $b' \in N$ such that $pb' = b$, and so $pa = pb'+k$ or equivalently $p(a-b') = k \in K$. Then by Lemma 4.2.2, $a-b' \in N \oplus K$. This implies $a \in N \oplus K$ since $b' \in N \oplus K$ already. So $a+(N \oplus K) = N \oplus K$. This shows that $G/(N \oplus K)$ has no nontrivial element of prime order, i.e., it is torsion-free.

(ii) The quotient group $G/(N \oplus K)$ is torsion:

Take any nonzero element $x+(N \oplus K) \in G/(N \oplus K)$, i.e., $x \notin N \oplus K$, then $(< x > +K) \gneqq K$ since $x \notin K$, and so $(< x > +K) \cap N \neq 0$ since K is N-high in G. Therefore there is $0 \neq b'' \in N$ such that $b'' = k''+lx$ for some $k'' \in K$ and integer l. Clearly $l \neq 0$ since $N \cap K = 0$. Thus $lx = b''-k'' \in N \oplus K$, and so $lx+(N \oplus K) = N \oplus K$ or equivalently $l(x+(N \oplus K)) = N \oplus K$. This means $G/(N \oplus K)$ is torsion.

Hence $G/(N \oplus K)$ is both torsion and torsion-free group. We conclude that $G = (N \oplus K)$.

$\square$

The following result immediate since it holds for modules over commutative noetherian ring in which every nonzero prime ideal is maximal (see Section 1.4), and for the direct proof of this important fact see also Imam (2007).

Theorem 4.2.4. *A is a neat subgroup of B if and only if A is a K-high subgroup of B (or a complement of K), for some $K \leq B$.*

Proof. ($\Rightarrow$): Let A be a neat subgroup of B and prove that A is a K-high subgroup of B for some $K \leq B$. Applying Zorn's lemma to the set

$$\Gamma = \{T \leq B \mid T \cap A = 0\}$$

we find an A-high subgroup K of B. Now taking the set

$$\Gamma' = \{S \leq B \mid S \cap K = 0, A \leq S\}$$

again by Zorn's lemma we obtain a K-high subgroup M of B with $A \leq M$. We shall show that $A = M$.

Suppose for the contrary that $M \neq A$. Then there exists $m \in M \setminus A$. So, if $< m > \cap A = 0$, then $(K+ < m >) \cap A = 0$. Indeed, if $a = k + tm \in (K+ < m >) \cap A$ for some $k \in K, t \in \mathbb{Z}$, then $k = a - tm \in K \cap M = 0$ (since $a \in A \leq M$), and so $a = tm \in < m > \cap A = 0$ implies $a = 0$. But this contradicting the fact that K is A-high in B since $(< m > +K) \gneqq K$ (as $m \in M$ implies $m \notin K$). Now we have $< m > \cap A \neq 0$, so there exists $0 \neq a = sm \in < m > \cap A$, where $s = p_1 p_2 \ldots p_n$ for prime numbers p_i (not necessarily distinct). We know $m \notin A$, but $a = (p_1 p_2 \ldots p_n)(m) \in A$ implies that $x \notin A$, but $px \in A$ for some $x \in B$ and prime p. Then $px \in A \cap pB = pA$ since $A \leq_{neat} B$, that is, $px = pa'$ ($a' \in A$) or equivalently $p(x - a') = 0$. Put $y = x - a' \in M \setminus A$, then $\mathrm{o}(y) = p$, and so $< y > \cap A = 0$. Indeed, if $0 \neq ty \in A$, then $(t, p) = 1$, i.e., $tu + pv = 1$ for some integers u, v. So $y = uty \in A$ since $vpy = 0$ as

$o(y) = p$, contradicting $y \notin A$. Thus $(K+ < y >) \cap A = 0$ gives a contradiction, since K is A-high in B and $(< y > \cap K \gneqq K)$ (as $y \in M$ implies $y \notin K$). Hence m must be an element of A, that is, $M = A$.

($\Leftarrow$): Conversely, we assume that A is a K-high subgroup of B for some $K \leq B$ and prove that A is neat in B, i.e., $pA = A \cap pB$ for every prime p. Now $pA \leq A \cap pB$ is always true. Let $a = pb \in A \cap pB$ for some $a \in A, b \in B$. By Lemma 4.2.2 $b \in A \oplus K$ since $pb \in A$. Thus $b = a' + k$ for some $a' \in A, k \in K$. Hence $a = p(a' + k) = pa' + pk$, and so $pk = a - pa' \in A \cap K = 0$ or equivalently $a = pa' \in pA$. So $A \cap pB \leq pA$, completes the proof.
For another direct proof of this part, without using Lemma 4.2.2, see Proposition 4.1.6. □

In fact, we have the following Theorem from Harrison et al. (1963).

Theorem 4.2.5. *(Harrison et al., 1963, after Definition 3, pp. 327) if A is a subgroup of a group G, the following statements are equivalent:*

(i) *A is a neat subgroup of G,*

(ii) *A is maximal disjoint from some subgroup K of G (i.e., A is K-high in G),*

(iii) *If K is a subgroup of G maximal disjoint from A, , then A is maximal disjoint from K,*

(iv) *$A \cap pG = pA$.*

Theorem 4.2.6. *(Irwin, 1961, Lemma 1) Let G be a primary group with H an N-high subgroup of G, D is divisible envelope of G, A any divisible envelope of H in D (this means that $A \leq D$), and B any divisible envelope of N in D. Then:*

(i) *$D = A \oplus B$*

(ii) *$A \cap G = H$, and H and $B \cap G$ are neat in G.*

(iii) *$D[p] = (H \oplus N)[p] = H[p] \oplus N[p] = G[p]$.*

(iv) *All N-high subgroups H of G may be obtained as $E \cap G$, E is a complementary direct summand of a divisible envelope F of N in D.*

Proposition 4.2.7. *(Akıncı, 1996) Let G be a torsion group, $G = \bigoplus_p G_p$ where G_p's are p-groups belonging to different primes p and N be a subgroup of G. Then if H is an N-high subgroup of G, H_p is an N_p-high subgroup of G_p.*

Theorem 4.2.8. *(Irwin & Walker, 1961, Theorem 12) Let H and K be any two high subgroups of a group G. Then:*

(i) *G/H is divisible*

(ii) *G/H is a divisible envelope of $(G^1 \oplus H)/H \cong G^1$*

(iii) *$G/H \cong G/K$.*

The following theorem gives a characterization in terms divisible envelope of N-high subgroups of a torsion group G.

Theorem 4.2.9. *(Irwin & Walker, 1961, Theorem 3) Let N be any subgroup of a torsion group G, and E be a divisible envelope of G with D a divisible envelope of N in E. Then the set of N-high subgroups of G is the set of intersections of G with complementary summands of D in E.*

Proof. Let $H = A \cap G$, where $A \oplus D = E$. We shall show that H is N-high in G. Now by Theorem 4.2.6, and Fuchs (1958, pp. 67), we have for each prime p,

$$G[p] = E[p] = A[p] \oplus D[p] = (A \cap G)[p] \oplus N[p] = H[p] \oplus N[p].$$

Thus by Theorem 4.1.9, H is neat in G and finally H is N-high in G by Theorem 4.2.4. For the converse, suppose H is N-high in G, so that H is maximal subgroup with respect to $H \cap N = 0$. Now $H \cap D = 0$, because: notice that $(H \cap D) \cap N = H \cap N = 0$, and by Fuchs (1958, Lemma 20.3) (Kulikov's lemma), $H \cap D = 0$. Since D is an absolute direct summand, there exists a group A containing H with $A \oplus D = E$. But $H \leq A \cap G$, and since $(A \cap G) \cap N = 0$, by the maximality of H with respect to $H \cap N = 0$, we have $H = A \cap G$. $\square$

Theorem 4.2.10. *(Irwin & Walker, 1961, Theorem 7) Let H and K be high subgroups of a torsion group G. Then if H is a direct sum of cyclic groups, so is K. Moreover $H \cong K$.*

Now we give some results of *purity of N-high subgroups of torsion groups*:

Theorem 4.2.11. *(Irwin, 1961, Lemma 2) Let N be a subgroup of a primary group G, H an N-high subgroup of G, and let H contain a basic subgroup B of G. Then H is pure in G.*

Lemma 4.2.12. *(Irwin, 1961, Lemma 12) Let G be a torsion group and let H be a high subgroup of G. Then H contains a basic subgroup of G.*

Theorem 4.2.13. *(Irwin, 1961, Theorem 3) If H is a high subgroup of a torsion group G, then H is pure in G.*

Theorem 4.2.14. *(Irwin & Walker, 1961, Lemma 5-6) Let N be a subgroup of a torsion group G. Then*

(i) *If H and K both N-high subgroups of G then $((H \oplus N)/N)[p] = ((K \oplus N)/N)[p]$ for each relevant prime p.*

(ii) *If H is an N-high subgroup of G with $N \leq G^1$, then $((H \oplus N)/N)$ is pure in G/N.*

Proposition 4.2.15. *(Irwin & Walker, 1961, Theorem 5) Let N be a subgroup of a torsion group G with $N \leq G^1$, and let H be an N-high subgroup of G. Then H is pure in G.*

Theorem 4.2.16. *(Irwin & Walker, 1961, Lemma 7) Let H and K be any two N-high subgroups of a primary group G with N a subgroups of G^1. Then for all positive integers n, we have:*

(i) *$p^n H$ is N-high in $p^n G$,*

(ii) *$p^n H$ is pure in $p^n G$,*

(iii) *$p^n((H \oplus N)/N)[p] = (p^n((K \oplus N)/N))[p]$,*

(iv) *H, K, and G have the same nth Ulm invariants (see Kaplansky (1969)).*

Theorem 4.2.17. *(Irwin & Walker, 1961, Theorems 1-2) Let $G = \sum G_\alpha$ be an arbitrary direct sum of torsion groups, where H_α and N_α are subgroups of G_α for each α. Then:*

(i) *If H_α is N_α-high in G_α for each α and $N = \sum N_\alpha$, then $H = \sum H_\alpha$ is N-high in G.*

(ii) *If H_α is a high subgroup of G_α for each α, then $H = \sum H_\alpha$ is high in G.*

Theorem 4.2.18. *(Irwin & Walker, 1961, Theorems 13) Let T be the torsion subgroup of an abelian group G, H be a high subgroup of G, and T_H be the torsion subgroup of H. Then T_H is high in T.*

The Theorem 4.2.13 is generalized by Irwin & Walker (1961) as follows:

Theorem 4.2.19. *(Irwin & Walker, 1961, Theorem 14) Let H be a high subgroup of an abelian group G. Then H is pure in G.*

Proof. Let T be a torsion subgroup of G, and T_H be the torsion subgroup of H. Now by Theorem 4.2.18, T/T_H is divisible, so that $G/T_H = T/T_H \oplus R/T_H$, where R is chosen such that R/T_H contains H/T_H. Since T_H is pure in G by Irwin & Walker (1961, Corollary to Theorem 13), R is pure in G (as R/T_H is pure, since a direct summand). Thus since H is neat in G (as a high subgroup), H/T_H is neat in G/T_H, and so H/T_H is neat in R/T_H (see Theorem 4.1.11). But R/T_H is torsion-free. This implies that H/T_H is pure in R/T_H by Theorem 4.1.2. Hence we have that H is pure in R, so that H is pure in G (since R is pure in G), and the proof is complete. $\square$

The concept of $\sum$-groups was introduced by Irwin & Walker (1961).

Definition 4.2.20. A $\sum$-*group* is any group G all of whose high subgroups are direct sums of cyclic groups. This means that in a torsion $\sum$-group every high subgroup is basic.

The next theorem shows that torsion $\sum$-groups are closed under direct sums.

Theorem 4.2.21. *(Irwin & Walker, 1961, Theorem 8) For torsion groups, a direct sum of $\sum$-groups is a $\sum$-group.*

The next theorem gives a characterization of torsion $\sum$-groups of a torsion group. $\sum$-groups in terms of their basic subgroups.

Theorem 4.2.22. *(Irwin & Walker, 1961, Theorem 9) A torsion group G is a $\sum$-group if and only if G contains a maximal basic subgroup.*

The next Theorem is a result concerning the $\sum$-groups of a torsion group.

Theorem 4.2.23. *(Irwin & Walker, 1961, Theorem 10) Every torsion group G contains a $\sum$-subgroup R pure in G such that $R^1 = G^1$.*

Proposition 4.2.24. *(Irwin & Walker, 1961, Theorem 11) Every subgroup L of a torsion $\sum$-group G with $L^1 = L \cap G^1$ (and so every pure subgroup of such G) is a $\sum$-group.*

Definition 4.2.25. Let $T(G)$ denote the torsion subgroup of a group G. Then G *splits* if $T(G)$ is a direct summand of G.

Proposition 4.2.26. *(Irwin et al., 1962, Theorem 1) Let H be a high subgroup of a group G, and suppose $H = H_t \oplus L$, where H_t is the torsion subgroup of H. Then $G = M \oplus L$, where $M/T(G)$ is the divisible part of $G/T(G)$.*

From the Theorem 4.2.26, we obtain the following necessary and sufficient condition that a reduced group split.

Theorem 4.2.27. *(Irwin et al., 1962, Theorem 2) Let G be reduced. Then G splits if and only if $G/T(G)$ is reduced and some high subgroup of G splits, where $T(G)$ is the torsion part of G.*

Theorem 4.2.28. *(Irwin et al., 1962, Corollary, pp. 191) If one high subgroup of a group G is a direct sum of cyclic groups then G is a $\sum$-group, and any two high subgroups of G are isomorphic.*

Theorem 4.2.29. *(Megibben, 1964, Theorem 3) If G is a $\sum$-group, then every high subgroup of G is an endomorphic image of G. More generally, if the high subgroup H of G splits and the torsion subgroup of H is a direct sum of cyclic groups, then H is an endomorphic image of G.*

The following theorem gives a characterization of N-high subgroups of a group G, with $N \leq G^1$.

Theorem 4.2.30. *(Irwin & Benabdallah, 1968, Theorem 2.4) Let G be a group, N a subgroup of G^1 and K a pure subgroup of G containing N. Then for every N-high subgroup H of G,*

$$G = H + K.$$

Theorem 4.2.31. *(Irwin & Benabdallah, 1968, Theorem 2.5) Let G be a group, N a subgroup of G^1 and H a subgroup of G disjoint from N. Then H is N-high in G if and only if H is pure, G/H divisible and $G = H + K$ for every pure subgroup K of G containing N.*

Now we obtain a criterion for pure subgroups of a group G to be summands of G.

Theorem 4.2.32. *(Irwin & Benabdallah, 1968, Theorem 3.1) Let G be a group, K a pure subgroup of G containing a subgroup N of G^1. Then K is a direct summand of G if and only if there exists an N-high subgroup H of G such that $H \cap K = M$ is a direct summand of H.*

Definition 4.2.33. A p-group G is called *pure complete* if every subsocle $S \leq G[p]$ supports a pure subgroup of G.

Theorem 4.2.34. *(Benabdallah et al., 1974, Theorem 6) If G is a pure complete p-group, and $N \leq G$, then all N-high subgroups of G are direct sums of cyclics if and only if all pure N-high subgroups are direct sums of cyclics.*

Corollary 4.2.35. *(Benabdallah et al., 1974, Corollary 15) Let G be a p-group, and N a subgroup of G. Then there exists an N-high subgroup of G which is pure in G.*

Theorem 4.2.36. *(Benabdallah et al., 1974, Lemma 8) Let K be a subgroup of a p-group G and $S \leq K[p]$. Then K/S is $(G[p]/S)$-high in G/S if and only if $K[p] = S$ and K is neat in G.*

Proof. $(\Rightarrow)$: Let K/S be $(G[p]/S)$-high in G/S. Let $pg \in K$ and $g \notin K$. Then

$$(< g + S > +K/S) \cap (G[p]/S) \neq 0 \text{ since } K/S \text{ is } (G[p]/S)\text{-high.}$$

So there exists $g + k + S \in G[p]/S$ for some $k \in K$ and $g + k \in G[p]$. Then $pg + pk = 0$, $pg = -pk$. Thus K is neat in G. Now, let $x \in K[p] \leq G[p]$. Then $x + S \in K/S \cap G[p]/S = 0$, and so $x \in S$ which implies that $K[p] = S$.

$(\Leftarrow)$: Conversely, suppose K is neat in G, then $K/S \cap G[p]/S = 0$, because: let $k + S = g + S$ for some $k \in K$, $g \in G[p]$. Then $k - g \in S = K[p]$ and so $p(k - g) = 0$. Hence $pk = 0$, since $pg = 0$. This implies $k \in K[p] = S$, so $k + S = 0$.
And $(G/S)[p] = (G[p]/S) \oplus (K/S)[p]$, because: for $g + S \in (G/S)[p]$, we have $p(g + S) = S$, and so$pg \in S \leq K$. Since K is neat in G, $pg = pk'$ for some $k' \in K$. So, $p(g - k') = 0$ or equivalently $g - k' \in G[p]$. Hence $g + S = (g - k' + S) + (k' + S) \in G[p]/S + K/S$. Now K is neat in G implies K/S is neat G/S. Hence K/S is maximal subgroup by $(K/S)[p]$ and is $(G[p]/S)$-high in G/S. ∎

Theorem 4.2.37. *(Benabdallah et al., 1974, Theorem 10) All neat subgroups supported by a subsocle S of a p-group G are dense if and only if S is dense in $G[p]$.*

The following proposition is a modification of the exercise on pure subgroups in Fuchs (1970, Exercise 28.2)

Proposition 4.2.38. *(Benabdallah et al., 1974, Lemma 12) If H is a neat subgroup of a p-group G, then*

$$(G/H)[p] \cong G[p]/H[p].$$

Proof. Following the proof for pure subgroups in Călugăreanu et al. (2003, Exercise S 3.14), we argue as follows. Consider the group homomorphism

$$f \quad : \quad G[p] \longrightarrow (G/H)[p], \quad f(a) = a + H$$

for every $a \in G$ (the restriction of the canonical projection). Obviously f is well-defined (since $pa = 0$ implies p(a+H)=H) and, moreover, H being neat in G, it is surjective by using Theorem 4.1.13. Finally,

$$\operatorname{Ker} f = \{a \in G[p] \mid \ a + H = H, \text{ or equivalently, } a \in H\} = H \cap G[p] = H[p],$$

so our claim follows by the First Isomorphism Theorem. □

4.3 Intersection of Neat Subgroups

Now we give a few characterizations of Neat Subgroups of abelian groups from Rangaswamy (1965) :

Definition 4.3.1. (i) A subgroup S of a group G which can be represented as the intersection of neat subgroups of G is called π-*subgroup*.

(ii) Let p be a prime. A subgroup S is p-*absorbing* if $px \in S$ implies $x \in S$.

(iii) If G is a group, a subgroup which is maximally disjoint with $T(G)$ (torsion subgroup of G) is denoted by G_∞.

Lemma 4.3.2. *(Rangaswamy, 1965, (1.6)) If S is essential in G (i.e., $S \cap A \neq 0$ for every nonzero subgroup A of G), then the only neat subgroup containing S is G itself.*

Proposition 4.3.3. *(Rangaswamy, 1965, Theorem 2) Let G be a p-group. If $S \neq G$, then S is a π-subgroup if and only if there exists a nonzero subgroup H of G disjoint with S.*

Proposition 4.3.4. *(Rangaswamy, 1965, Corollary) Let G be a torsion-free group. Then a subgroup S of G is a π-subgroup if and only if S is neat.*

Theorem 4.3.5. *(Rangaswamy, 1965, Theorem 5) Let G be an arbitrary divisible group. Then the following are equivalent:*

(i) *S is the intersection of divisible subgroups of G (S is π-subgroup).*

(ii) *If $px \in S$, $x \notin S$, then there exists $z \notin S$ with $pz = 0$.*

(iii) *S is p-absorbing whenever $S[p] = G[p]$.*

(iv) *For a prime p, S_p (p-primary component of S_t) and S/S_p are π-subgroups of G and G/S_p respectively.*

(v) *For each relevant prime p, either (i) $S_p = G_p$ and $p^{-1}(S_\infty) \leq S$ for every S_∞ or (ii) there exists a nonzero p-subgroup disjoint with S.*

4.4 Neat Exact Sequences

Definition 4.4.1. A group G is said to be a *neat-projective* if for every neat-exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ and homomorphism $\phi : G \to C$ there exist $\psi : G \to B$ such that $\beta \circ \psi = \phi$.

$$\begin{tikzcd} & & & G \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\ 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

Lemma 4.4.2. *(Imam, 2000, Lemma 1.4.2) A group P is neat-projective if and only if $Next(P, G) = 0$ for each group G.*

Theorem 4.4.3. *(Bilhan, 1995, Theorem 8) P is neat-projective if and only if $P = P' \oplus F$, where P' is a direct sum of cyclic groups of prime order and F is free.*

Definition 4.4.4. A group E is said to be a *neat-injective* if for every neat-exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ and homomorphism $\xi : A \to E$ there exist $\eta : B \to E$ such that $\eta \circ \alpha = \xi$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\xi"'] & B \arrow[r, "\beta"] \arrow[dl, dashed, "\eta"] & C \arrow[r] & 0 \\ & E & & & \end{tikzcd}$$

Lemma 4.4.5. *(Imam, 2000, Lemma 1.4.6) A group E is neat-injective if and only if* $\operatorname{Next}(G, E) = 0$ *for each group G.*

Proposition 4.4.6. *(Imam, 2000, Proposition1.4.8) Let p be a prime. An elementary p-group is neat-injective and neat-projective.*

Another important characterization for being a neat subgroup is the following:

Lemma 4.4.7. *(Bilhan, 1995, Theorem 7) Let A be a subgroup of a group B. Then A is neat in B if and only if $\mathbb{Z}/p\mathbb{Z}$ is injective with respect to the inclusion homomorphism $A \hookrightarrow B$.*

Proof. ($\Rightarrow$): Suppose A is neat in B, i.e., $A \hookrightarrow B$ is a neat homomorphism. Let $\phi : A \to \mathbb{Z}/p\mathbb{Z}$ be any homomorphism. Since $\mathbb{Z}/p\mathbb{Z}$ has no proper subgroup (as a simple subgroup) $\phi(A) = 0$ or $\phi(A) = \mathbb{Z}/p\mathbb{Z}$. If $\phi(A) = 0$, then ϕ is a zero homomorphism and, then it can be extended to $\psi : B \to \mathbb{Z}/p\mathbb{Z}$, where $\psi(x) = 0$ for every $x \in B$. So we shall consider the case when $\phi(A) = \mathbb{Z}/p\mathbb{Z}$, i.e., ϕ is an epimorphism. Since $A/\operatorname{Ker}\phi \cong \mathbb{Z}/p\mathbb{Z}$ (By Fundamental isomorphism theorem), we obtain $p(A/\operatorname{Ker}\phi) = 0$, that is, $pA \leq \operatorname{Ker}\phi$, and so we have a homomorphism $g : A/pA \to A/\operatorname{Ker}\phi$ such that $g(a+pA) = a+\operatorname{Ker}\phi$. Clearly g is well-defined. Since A is neat in B, A/pA is a direct summand of B/pA by Theorem 4.1.24. So there exists a projection $\sigma : B/pA \to A/pA$. Thus we have the following diagram

$$\begin{tikzcd}
A \arrow[r, hook, "\alpha"] \arrow[dr, "\pi_1"] \arrow[dd, "\phi"'] & B \arrow[r, "\pi_2"] & B/pA \arrow[dl, dashed, "\sigma"] \\
& A/pA \arrow[dl, "g"] & \\
\mathbb{Z}/p\mathbb{Z} \cong A/\operatorname{Ker}\phi & &
\end{tikzcd}$$

where π_1, π_2 are natural epimorphisms. Therefore for $a \in A$, $(\sigma \circ \pi_2 \circ \alpha)(a) = (\sigma \circ \pi_2)(a) = \sigma(\pi_2(a)) = \sigma(a + pA) = a + pA = \pi_1(a)$, since σ is projection and α is inclusion. So we have $\sigma \circ \pi_2 \circ \alpha = \pi_1$. Moreover, $(g \circ \pi_1)(a) = g(a + pA) = \phi(a)$, and so $g \circ \pi_1 = \phi$. Finally, let us define $\psi : B \to \mathbb{Z}/p\mathbb{Z}$ such that $\psi = g \circ \sigma \circ \pi_2$. It is clearly well-defined since g, σ, π_2 are well-defined. Hence for every $a \in A$, we have

$$(\psi \circ \alpha)(a) = (g \circ \sigma \circ \pi_2 \circ \alpha)(a) = (g \circ \pi_1)(a) = \phi(a),$$

that is, $\mathbb{Z}/p\mathbb{Z}$ is injective with respect to the neat monomorphism $A \hookrightarrow B$.

($\Leftarrow$): Conversely, suppose that every $\mathbb{Z}/p\mathbb{Z}$ has injective property with respect to the short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

Now assume that A is not neat in B, that is, $\mathbb{E}$ is not neat-exact. Then $A \cap pB \neq pA$, so there exists an $a \in (A \cap pB) \setminus pA$. Let us define the set

$$\Gamma = \{D \leq B \mid pA \leq D \leq A, \, a \notin D\}.$$

Clearly, every chain $\{D_i\}$ has an upper bound, namely $\bigcup D_i$. Therefore by Zorn's lemma Γ has a maximal element, say M.

Consider A/M. Since $pA \leq M$, we have $p(A/M) = M$, and so $A/M \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p since A/M is simple (as M is maximal in A). Then A/M is injective with respect to $\mathbb{E}$ (by hypothesis), so there exists $\psi : B \to A/M$ such that the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\pi"] & B \arrow[r, "\beta"] \arrow[dl, "\psi"] & C \arrow[r] & 0, \\ & A/M & & & \end{tikzcd}$$

that is, $\psi \circ \alpha = \pi$, where π is natural projection. Since α is inclusion map $\alpha(a) = a$, and $\psi(a) \in \psi(pB)$, since $a \in A \cap pB$ for some $a \in A$. We know that $\psi(pB)$ is a subgroup of $p(A/M)$, then $\psi(a) \in p(A/M) = M$. Thus $\psi(a) = \psi(\alpha(a)) = \pi(a) \notin M$ since $a \notin M$. Contradicting $\psi(a) \in\in M$. Hence A is neat in B. $\square$

Theorem 4.4.8. *Let*

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence for abelian groups A and B. Then the following are equivalent:

(i) *A is neat subgroup of B, i.e., $pA = A \cap pB$ for every prime p;*

(ii) *$\mathbb{Z}/p\mathbb{Z}$ has the projective property with respect to $\mathbb{E}$, for every prime p;*

(iii) *$\mathbb{Z}/p\mathbb{Z}$ has the injective property with respect to $\mathbb{E}$, for every prime p;*

(iv) *$\mathbb{Z}/p\mathbb{Z}$ is flat with respect to $\mathbb{E}$, for every prime p.*

Proof. (i)⇔(ii)⇔(iv): Follows immediately from Proposition 3.1.4.

(i)⇔(iii): This equivalence follows from Lemma 4.4.7. □

By using the idea of short pure-exact sequences for example in Griffith (1970, Lemma 26), we get the following Theorem for short neat-exact sequences:

Theorem 4.4.9. (i) *A pushout of a short neat-exact sequence is neat-exact.*

(ii) *A pull back of a short neat-exact sequence is neat-exact.*

(iii) *The Baer sum of two short neat-exact sequences is neat-exact.*

Proof. (i) Suppose that a short exact sequence $\mathbb{E} : 0 \longrightarrow A \xrightarrow{i} B \xrightarrow{v} C \longrightarrow 0$ is neat and that the diagram

$$\begin{tikzcd}
\mathbb{E} : 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "\psi"] & C \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{E}' : 0 \arrow[r] & E \arrow[r, "\pi"] & M \arrow[r, "\rho"] & C \arrow[r] & 0,
\end{tikzcd}$$

represents a *pushout* of $\mathbb{E}$. Without loss of generality, assume A is a subgroup of B and i is the inclusion homomorphism. Recall that
$M = (E \oplus B)/N$ where $N = \{(f(a), -i(a)) : a \in A\}$, and $\pi : e \mapsto (e, 0) + N$, $\psi : b \mapsto (0, b) + N$ and $\rho : (e, b) + N \mapsto v(b)$, for some $e \in E$, $b \in B$. We want to show that $\mathbb{E}'$ is neat-exact, i.e., $\pi(E)$ is neat in M.
Suppose that $\pi(e_1) = pm$ for some $e_1 \in E$, $m \in M$. Since $m = (e, b) + N$ for some $e \in E$, $b \in B$, we have $(e_1, 0) + N = \pi(e_1) = pm = p[(e, b) + N] = p(e, b) + N$. Then $p(e, b) - (e_1, 0) \in N$. Thus by definition of N, there exists $a \in A$ such that $p(e, b) - (e_1, 0) = (f(a), -i(a))$, and so $(pe, pb) = (e_1 + f(a), -i(a))$. Then

$a = i(a) = -pb = p(-b)$. Since $A \leq_{neat} B$ $a = pa_1$ for some $a_1 \in A$. We also have $pe = e_1 + f(a)$, then $e_1 = pe - f(a) = pe - f(pa_1) = p(e - f(a_1))$. Since $e - f(a_1) \in E$, we can write $e/f(a_1) = pe'$ for some $e' \in E$. Thus $\pi(e_1) = (e_1, 0) + N = (pe', 0) + N = p((e', 0) + N) = p\pi(e')$, and this shows that $\pi(E)$ is neat in M.

(ii) For a short neat-exact sequence $\mathbb{E} : 0 \longrightarrow A \xrightarrow{i} B \xrightarrow{v} C \longrightarrow 0$, suppose that the commutative diagram

$$\begin{tikzcd}
\mathbb{E}' : 0 \arrow[r] & A \arrow[r, "\pi"] \arrow[d, equal] & M \arrow[r, "\rho"] \arrow[d, "\psi"] & E \arrow[r] \arrow[d, "f"] & 0 \\
\mathbb{E} : 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "v"] & C \arrow[r] & 0,
\end{tikzcd}$$

represents a *pullback* of $\mathbb{E}$. Without loss of generality, assume A is a subgroup of B and i is the inclusion homomorphism. Recall that
$M = \{(b, e) \in B \oplus E : v(b) = f(e)\}$, and $\pi : a \mapsto (a, 0)$, $\rho : (b, e) \mapsto e$, and $\psi : (b, e) \mapsto b$ for some $a \in A$, $b \in B$, $e \in E$. We want to show that $\mathbb{E}'$ is neat-exact, i.e., $\pi(A)$ is neat in M.
Suppose that $\pi(a) = pm$ for some $a \in A$, $m \in M$. Since $m = (b, e)$ for some $b \in B$, $e \in E$, we have $\pi(a) = (a, 0) = p(b, e) = (pb, pe)$. Then $a = pb$, and so $a = pa_1$ for some $a_1 \in A$ since $A \leq_{neat} B$ (as $i(A) = A$). Hence $\pi(a) = \pi(pa_1) = p\pi(a_1)$ shows that $\pi(A)$ is neat in M as required.

(iii) Suppose that

$$\mathbb{E}_1 : 0 \longrightarrow B \xrightarrow{i} X \xrightarrow{j} A \longrightarrow 0 \text{ and } \mathbb{E}_2 : 0 \longrightarrow B \xrightarrow{\rho} Y \xrightarrow{\pi} A \longrightarrow 0$$

are neat-exact sequences and that

$$\mathbb{E} : 0 \longrightarrow B \xrightarrow{\tau} W \xrightarrow{\sigma} A \longrightarrow 0$$

represents the *Baer sum* of these sequences $\mathbb{E}_1$ and $\mathbb{E}_2$, where
$W = M/N$, $M = \{(x, y) \in X \oplus Y : j(x) = \pi(y)\}$, $N = \{(-i(b), \rho(b)) : b \in B\}$, and $\tau : b \mapsto (i(b), 0) + N$ and $\sigma : (x, y) + N \mapsto j(x)$.
Without loss of generality, assume B is a subgroup of X and Y, and i, ρ are the

inclusion homomorphisms. We want to show that $\mathbb{E}$ is neat-exact, i.e., $\tau(B)$ is neat in W.

Suppose that $\tau(b) = pw$ for some $w \in W = M/N$. Let $w = (x, y) + N$ for some $x \in X$ and $y \in Y$. Then $\tau(b) = (i(b), 0) + N = p(x, y) + N = (px, py) + N$ for some $b \in B$. So $(i(b), 0) - (px, py) \in N$. Then there exists $(-i(b_1), \rho(b_1)) \in N$ $(b_1 \in B)$ such that $(i(b), 0) - (px, py) = (-i(b_1), \rho(b_1)) = (-b_1, b_1)$ as i, ρ are inclusions. Thus we have, $b_1 = -py = p(-y)$ and so $b_1 = pb'$ for some $b' \in B$, since $B \leq_{neat} Y$. And we also have, $i(b) - px = -b_1$, that is, $b + b_1 = px$. Then $b + b_1 = pb''$ for some $b'' \in B$, since $B \leq_{neat} X$. So, $b = pb'' - b_1 = pb'' - pb' = p(b'' - b')$. Hence $\tau(b) = (i(b), 0) + N = (b, 0) + N = (p(b'' - b'), 0) + N = p(b'' - b', 0) + N = p[(i(b'' - b'), 0) + N] = p\tau(b'' - b')$, i.e., $\tau(b) = p(\tau(b'' - b'))$. This shows that $\tau(B)$ is neat in W.

$\square$

Theorem 4.4.10. *(Imam (2000, Proposition1.3.2)-without proof) For a short exact sequence*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

the following conditions are equivalent:

(i) $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is neat-exact;*

(ii) $0 \longrightarrow pA \xrightarrow{\alpha_1} pB \xrightarrow{\beta_1} pC \longrightarrow 0$ *is exact for every prime p;*

(iii) $0 \longrightarrow A[p] \xrightarrow{\alpha_2} B[p] \xrightarrow{\beta_2} C[p] \longrightarrow 0$ *is exact for every prime p;*

(iv) $0 \longrightarrow A/pA \xrightarrow{\alpha_3} B/pB \xrightarrow{\beta_3} C/pC \longrightarrow 0$ *is exact for every prime p;*

(v) $0 \longrightarrow A/A[p] \xrightarrow{\alpha_4} B/B[p] \xrightarrow{\beta_4} C/C[p] \longrightarrow 0$ *is exact for prime p.*

Proof. (i)$\Leftrightarrow$(ii): By exactness of (i), we have α is monic and β is epic. Since $\operatorname{Ker}\alpha_1 = \operatorname{Ker}\alpha \cap pA = 0$ (as $\operatorname{Ker}\alpha = 0$), α_1 is monic. Indeed,

$$\operatorname{Ker}\alpha_1 = \{x \in pA \mid \alpha_1(x) = \alpha|_{pA}(x) = 0\} = \{x \in pA \mid x \in \operatorname{Ker}\alpha\} = \operatorname{Ker}\alpha \cap pA.$$

And $\operatorname{Im}\beta_1 = \operatorname{Im}\beta \cap pC = pC$ since β is epic, and so β_1 is epic. Finally,

$$\operatorname{Ker}\beta_1 = \operatorname{Ker}\beta \cap pB = \operatorname{Im}\alpha \cap pB$$

which is equal to $p(\operatorname{Im}\alpha) = \alpha(pA) = \operatorname{Im}\alpha|_{pA} = \operatorname{Im}\alpha_1$ (i.e., (ii) is exact) if and only if $\operatorname{Im}\alpha$ is neat B, i.e., (i) is neat-exact.

(i)$\Leftrightarrow$(iii): α_2 is always monic, while β_2 is epic for every p exactly if every element of $C[p]$ is an image of an element of $B[p]$. This is, by Theorem 4.1.13, equivalent to neatness of (i). Finally,

$$\operatorname{Ker}\beta_2 = \operatorname{Ker}\beta \cap B[p] = \operatorname{Im}\alpha \cap B[p] = \alpha(A[p]) = \operatorname{Im}\alpha_2,$$

that is, (i) is neat-exact if and only if (iii) is exact.

(i)$\Leftrightarrow$(iv) and (i)$\Leftrightarrow$(v) follows from 3x3-lemma. Indeed, we have the following commutative diagram with all three columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & pA \arrow[r, "\alpha_1"] \arrow[d, "i_1"] & pB \arrow[r, "\beta_1"] \arrow[d, "i_2"] & pC \arrow[r] \arrow[d, "i_3"] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\sigma_1"] & B \arrow[r, "\beta"] \arrow[d, "\sigma_2"] & C \arrow[r] \arrow[d, "\sigma_3"] & 0 \\
0 \arrow[r] & A/pA \arrow[r, "\alpha_3"] \arrow[d] & B/pB \arrow[r, "\beta_3"] \arrow[d] & C/pC \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd},$$

where i_k is inclusion map and σ_k is canonical epimorphism for $k = 1, 2, 3$. Clearly, all columns of the diagram are exact. Assume (i). We have proved that (i)$\Leftrightarrow$(ii). This means the first two rows of the diagram are exact. Thus by the 3×3-lemma (see for example Fuchs (1970, Lemma 2.4)), the remaining row is also exact, that is, (iv) is exact. Conversely, suppose (iv) is exact, that is, α_3 is a monomorphism, β_3 is an epimorphism and $\operatorname{Ker}\beta_3 = \operatorname{Im}\alpha_3$. We shall show that (i) holds, i.e., $\alpha(pA) = \alpha(A) \cap pB$. $\alpha(pA) \leq \alpha(A) \cap pB$ always holds. Let $\alpha(a) = pb \in \alpha(A) \cap pB$ for some $a \in A$ and an arbitrary prime p. Then $(\sigma_2 \circ \alpha)(a) = (\alpha_3 \circ \sigma_1)(a)$ by commutativity of the diagram. Thus we obtain,

$$(\alpha_3 \circ \sigma_1)(a) = (\sigma_2 \circ \alpha)(a) = \sigma_2(\alpha(a)) = \sigma_2(pb) = p(b + pB) = pB = 0_{B/pB},$$

that is, $\sigma_1(a) \in \operatorname{Ker} \alpha_3 = 0_{A/pA} = pA$ since α_3 is a monomorphism. Then $a + pA = \sigma_1(a) = pA$ implies $a \in pA$, i.e., $\alpha(a) \in \alpha(pA)$. Hence $\alpha(A) \cap pB \leq \alpha(pA)$ as required. Similarly, (i)⇔(iv) follows from 3×3-lemma. □

We know (see Fuchs (1970, §53)) that the extensions corresponding to pure-exact sequences form a subgroup of Ext which coincides with the first Ulm subgroup of Ext. This leads a functor Pext.

Proposition 4.4.11. *(Fuchs, 1970, Theorem 53.3) Let the exact sequence*

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

represent an element of $\operatorname{Ext}(C, A)$. *Then* E *belongs to* $n \operatorname{Ext}(C, A)$ *if and only if* $\alpha(nA) = \alpha(A) \cap nB$ *(i.e.,* $\operatorname{Im} \alpha \leq_{neat} B$*). It belongs to the first Ulm subgroup of* $\operatorname{Ext}(C, A)$ *if and only if* $\mathbb{E}$ *is a pure-exact sequence. This asserts:*

$$\operatorname{Pext}(C, A) = \operatorname{Ext}(C, A)^1 = \bigcap_n n \operatorname{Ext}(C, A).$$

Like the functor Pext, the extensions corresponding to neat-exact sequences form a subgroup of Ext which coincides with the Frattini subgroup of Ext. This leads a new functor Next.

Theorem 4.4.12. *(Bilhan, 1995, Theorem 10) A short exact sequence*

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is neat if and only if $\mathbb{E}$ *is an element of* $p \operatorname{Ext}(C, A)$ *for every prime* p. *This asserts:*

$$\operatorname{Next}(C, A) = \operatorname{Rad}(\operatorname{Ext}(C, A)) = \bigcap_p p \operatorname{Ext}(C, A)$$

(see Fuchs (1970, Exercise 53.4) and Nunke (1959, Theorem 5.1)).

Proposition 4.4.13. *(Imam, 2000, Corollary 1.4.13) If* $\{\mathbb{E}_i : i \in I\}$ *is a family of short neat-exact sequence*

$$\mathbb{E}_i : 0 \longrightarrow A_i \xrightarrow{\alpha} B_i \xrightarrow{\beta} C_i \longrightarrow 0,$$

then their direct sum

$$\bigoplus_{i\in I} \mathbb{E}_i : 0 \longrightarrow \bigoplus_{i\in I} A_i \xrightarrow{\alpha} \bigoplus_{i\in I} B_i \xrightarrow{\beta} \bigoplus_{i\in I} C_i \longrightarrow 0$$

is also neat-exact.

4.5 Frattini-high Subgroups

A Frattini-high subgroup, a notion parallel to high subgroups, was introduced by Ahmad (2006).

Definition 4.5.1. A subgroup H of a group G is *Frattini-high* subgroup if H is maximal with respect to the property $H \cap \operatorname{Rad} G = 0$, that is, H is a complement of $\operatorname{Rad} G$.

The following theorem shows that a Frattini-high subgroup of a torsion group when combined with pure subgroups generates the group.

Theorem 4.5.2. *(Ahmad, 2006, Theorem 2) Let* G *be a torsion group and* K *be a pure subgroup of* G *containing* $\operatorname{Rad} G$*. Then for every* Φ*-high subgroup* H *of* G*, we have* $G = H + K$*.*

4.6 Coneat Subgroups

For the definition of coneat subgroups, see Section 1.5.

Theorem 4.6.1. *(Mermut, 2004, Theorem 4.3.1) Let* A *be a finite subgroup of an abelian group* B*. Then* A *is a complement in* B *if and only it is a supplement in* B*.*

We cannot generalized this theorem to include finitely generated abelian groups as the following example in Mermut (2004, Example 4.3.2) shows.

Example 4.6.2. Consider the following short exact sequence of abelian groups,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z}/p^2\mathbb{Z} \longrightarrow 0$$

where $f(k) = k \cdot (-1 + p\mathbb{Z}, p)$, $k \in \mathbb{Z}$, and $g(a + p\mathbb{Z}, b) = (pa + b) + p^2\mathbb{Z}$, $a, b \in \mathbb{Z}$. Then $\operatorname{Im} f$ is a neat (complement) subgroup of $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z}$, but not a coneat subgroup of $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z}$ and so not a supplement in $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z}$.

Proposition 4.6.3. *(Mermut, 2004, Corollary 4.3.3) Let B be a group. If a subgroup A of B is coneat in B, then it is neat in B. But the converse fails to be true always (see Example 4.6.2).*

Proposition 4.6.4. *(Mermut, 2004, Proposition 4.6.1) Let A be a subgroup of a group B. If A is a supplement in B, then it is coneat in B and so neat in B (equal being a complement).*

Theorem 4.6.5. *(Mermut, 2004, Theorem 4.6.6) A finite subgroup A of a group B is coneat in B if and only if it is neat in B.*

For a *torsion* group B, neat subgroups and coneat subgroups coincide:

Theorem 4.6.6. *(Mermut, 2004, Theorem 4.6.8) Let B be a torsion group, and A any subgroup of B. Then A is neat in B if and only if A is coneat in B.*

In general, a torsion and neat subgroup of a group need not be coneat as the following example shows:

Example 4.6.7. Let $A = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ be a torsion subgroup of a $B = \prod_p \mathbb{Z}/p\mathbb{Z}$. Then A is neat in B (see Example 4.1.4), but it is not coneat in B. We shall show that A is not *coneat* in B:

If A were coneat in B, then since $\operatorname{Rad} A = \bigoplus_p \operatorname{Rad}(\mathbb{Z}/p\mathbb{Z}) = \bigoplus_p p(\mathbb{Z}/p\mathbb{Z}) = \bigoplus_p 0 =$

0, we would have that A is a direct summand of B. Indeed, if the inclusion monomorphism $A \hookrightarrow B$ is coneat, then A is injective with respect to it (since $\operatorname{Rad} A = 0$). This means the short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

splits, that is, A is a direct summand of B. Contradicting A is not a direct summand of B (see Example 4.1.4).

In particular, we can generalize the Theorem 4.6.5 as follows:

Theorem 4.6.8. *Let A be a torsion subgroup of a group B such that all but finitely many primary components of A are zero. Then A is neat in B if and only if it is coneat in B.*

Proof. ($\Leftarrow$) always holds for every group B by Proposition 4.6.3. Conversely, suppose A is neat in B. To show that A is coneat in B, we must show that for every module M with $\operatorname{Rad} M = 0$, any homomorphism $f : A \longrightarrow M$ can be extended to B. Since A is torsion, so is $f(A)$. So, without loss of generality, we may suppose that M is also a torsion group. Let T denote the torsion part of B. Decompose A, T and M into their p-primary components: $A = \bigoplus_{k=1}^{n} A_{p_k}$, where $p_1, \ldots, p_n$ are distinct primes, and $T = \bigoplus_p T_p$ and $M = \bigoplus_p M_p$, where the index p runs through all prime numbers (see for example Fuchs (1970, Theorem 8.4) for p-primary components of a torsion group). For each prime p_k, let $f_{p_k} : A_{p_k} \longrightarrow M_{p_k}$ be the restriction of f to A_{p_k}, with range restricted to M_{p_k} also (note that $f(A_{p_k}) \leq M_{p_k}$). Since

$$0 = \operatorname{Rad} M = \bigoplus_p \operatorname{Rad} M_p = \bigoplus_p pM_p$$

(as M_p is a p-group), we have $pM_p = 0$ for each prime p. So, each M_p is a neat-injective abelian group by the structure of neat-injective abelian groups (see Theorem 5.2.1 in §5.2). Now let $\widetilde{M} = \bigoplus_{k=1}^{n} M_{p_k}$. Then $\widetilde{M}$ is neat-injective since it is a finite direct sum of neat-injective groups. Since

$$\operatorname{Im} f = f(A) = \sum_{i=1}^{n} f(A_{p_k}) \leq \sum_{i=1}^{n} M_{p_k} = \widetilde{M},$$

we can define $f_1 = \oplus f_{p_k} : A \longrightarrow \widetilde{M}$ by $f_1(x) = f(x)$ for each $x \in A$. Since A is neat in B, it is also neat in T (by Theorem 4.1.11). So there exists a homomorphism $g : T \longrightarrow \widetilde{M}$ extending $f_1 : A \longrightarrow \widetilde{M}$ (since $\widetilde{M}$ is neat-injective). Moreover, since T is neat in B as a torsion part of B, there exists a homomorphism $h : B \longrightarrow \widetilde{M}$ extending $g : T \longrightarrow \widetilde{M}$ (since $\widetilde{M}$ is neat-injective). Thus we have the following diagram:

$$
\begin{tikzcd}
A = \bigoplus_{k=1}^{n} A_{p_k} \arrow[r, hook, "i_1"] \arrow[dr, "f_1"'] \arrow[dd, "f"'] & T \arrow[r, hook, "i_2"] \arrow[d, dashed, "g"] & B \arrow[dl, dashed, "h"] \\
 & \widetilde{M} = \bigoplus_{k=1}^{n} M_{p_k} \arrow[dl, "i_3"] & \\
M = \bigoplus_{p} M_p & &
\end{tikzcd}
$$

where i_1, i_2, i_3 are inclusion homomorphisms. Therefore we have from the commutative diagrams that $h \circ i_2 = g$, $g \circ i_1 = f_1$ and $i_3 \circ f_1 = f$. Finally, let us define $\widetilde{f} : B \longrightarrow M$ such that $\widetilde{f} = i_3 \circ h$. Ir is clearly well-defined since h is well-defined. Hence $\widetilde{f} \circ (i_2 \circ i_1) = i_3 \circ (h \circ i_2) \circ i_1 = i_3 \circ g \circ i_1 = i_3 \circ f_1 = f$, that is, $f : A \longrightarrow M$ is extended to B by $\widetilde{f}$. This means A is coneat in B. $\square$

CHAPTER FIVE

c-INJECTIVE MODULES AND ENVELOPES

In Section 5.1, we give some results for c-injective modules which will be used in other sections. In Section 5.2, we have given the details for the structure of neat-injective abelian groups in Harrison et al. (1963) and we have given another proof of this result suggested by Bill Wickless. c-injective modules over Dedekind domains are similarly described in Section 5.3. In Section 5.4, we give the result for the structure of neat-injective envelopes of an abelian group in Onishi (1984) and for the structure of neat-injective envelopes of an abelian group A in terms of the basic subgroups$B_p(A)$ of p-component $T_p(A)$ of A and $A/T(A)$ in Alizade et al. (2004). In the last section, a structure of c-injective envelopes of modules over a Dedekind domain is given (see Imam (2007)).

5.1 c-Injective Modules

We will approach to a *relative injectivity* problem for modules over Dedekind domains using *relative homological algebra*.

For definition of c-injective module, see Section 1.6.

Santa-Clara & Smith (2004, Theorem 6) shows that for a Dedekind domain R, every direct product of simple R-modules is self-c-injective. Santa-Clara & Smith (2004, after Theorem 6) has also noted that: for a Dedekind domain R, if M is a direct product of homogeneous semisimple R-modules, then M is self-c-injective and any simple R-module is M-c-injective.

We shall describe c-injective modules by the general theorems for *injectively generated proper classes* since for a Dedekind domain R, the *proper class* $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ defined below is injectively generated by homogenous semisimple R-modules and c-injective modules are the same concept with $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-injective modules.The relative homological algebra approach goes as follows:

For terminology and notation in *proper classes*, we shall follow Sklyarenko (1978). When R is a Dedekind domain R, using mainly the results in Nunke (1959, Lemmas 4.4 and 5.2, and Theorem 5.1), Mermut (2004) that the proper class $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ is both projectively generated, injectively generated and flatly generated:

Theorem 5.1.1. *(see Mermut (2004, Theorem 5.2.2))* $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ *equals the following for a Dedekind domain R:*

(i) $\mathcal{N}eat_{R\text{-}\mathcal{M}od} \overset{def.}{=} \pi^{-1}(\{R/P | P \text{ maximal ideal of } R\})$,

(ii) $\iota^{-1}(\{M | M \in R\text{-}\mathcal{M}od \text{ and } PM = 0 \text{ for some maximal ideal } P \text{ of } R\})$,
$= \iota^{-1}(\{M | M \text{ is a homogenous semisimple } R\text{-module}\})$,

(iii) $\iota^{-1}(\{R/P | P \text{ maximal ideal of } R\})$,

(iv) $\tau^{-1}(\{R/P | P \text{ maximal ideal of } R\})$

(v) *The proper class of all short exact sequences of R-modules and R-module homomorphisms of the form* $\mathbb{E}: 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *such that*

$$A' \cap PB = PA', \qquad \textit{where } A' = \operatorname{Im} f,$$

for every maximal ideal P of R (or $A \cap PB = PA$ when A is identified with its image and f is taken as the inclusion homomorphism).

A c-injective module X is nothing but a $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-injective, that is, X is injective with respect to every short exact sequence $\mathbb{E}$ in $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$, so:

Corollary 5.1.2. *(Mermut, 2004, Theorem 5.2.4) For a Dedekind domain R, every* c-*injective R-module is a* direct summand *of a direct product of homogeneous semisimple R-modules and of injective envelopes of cyclic R-modules, that is, every* c-*injective R-module is a* direct summand *of a module E such that*

$$E = D \oplus \prod_{P \underset{max.}{\leq} R} G_P$$

where D is a divisible module and the product is over all maximal ideals of R such that for every maximal ideal P of R, G_P is a homogeneous semisimple R-module having simple components isomorphic to R/P, i.e.,

$$PG_P = 0 \quad \textit{for all maximal ideals } P \leq R.$$

5.2 Structure of Neat-Injective Abelian Groups

Tho following theorem gives the structure of neat-injective abelian groups.

Theorem 5.2.1. *(Harrison et al., 1963, Lemma 4) A group E is neat-injective if and only if $E = D \oplus \prod_p T_p$, where D is divisible, $pT_p = 0$, and p ranges over all primes.*

For the proof of Theorem 5.2.1, we shall firstly give the following lemmas:

There are enough neat-injectives, that is, given any group, we can embed it into a neat-injective group as a neat subgroup; in fact, we do have the following particular embedding:

Lemma 5.2.2. *(by Harrison et al. (1963, proof of Lemma 5)) For an abelian group X, if we embed X into a divisible group D_1, then the homomorphism*

$$\begin{aligned} f : X &\longrightarrow D_1 \oplus \prod_{p \text{ prime}} (X/pX) \\ f(x) &= (x, (x + pX)_{p \text{ prime}}), \quad x \in X, \end{aligned}$$

is a $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$-monomorphism, that is, $\operatorname{Im} f$ is a neat subgroup of $D_1 \oplus \prod_p (X/pX)$ and $D_1 \oplus \prod_p (X/pX)$ is neat-injective.

Lemma 5.2.3. *Let the group E be of the form*

$$E = D \oplus \prod_{p \text{ prime}} T_p,$$

where D is a divisible *group, and in the product, p ranges over all prime numbers and for every prime number p, T_p is a direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$, i.e.,*

$$p\,T_p = 0 \quad \textit{for all prime numbers } p.$$

Then:

(i) $\prod_p T_p$ *is reduced and the divisible part of* E *is* $D \oplus 0$*:*

$$E = D \oplus \prod_p T_p = \underbrace{(D \oplus 0)}_{\textit{divisible part of } E} \oplus \underbrace{(0 \oplus \prod_p T_p)}_{\textit{reduced}}$$

(ii) $E = D \oplus \prod_p T_p$ *has* no *nonzero torsion-free reduced direct summands.*

Proof. (i) The group $\prod_p T_p$ is reduced: Suppose for the contrary that it is *not* reduced. Then it has a nonzero divisible subgroup D_1. Suppose $(g_p)_p$ is a nonzero element of D_1. Then it must be divisible in D_1, and so in $\prod_p T_p$, by every positive integer. Since $(g_p)_p$ is a nonzero element, $g_q \neq 0$ for some prime number q. Then in particular this element $(g_p)_p$ must be divisible by q in $\prod_p T_p$. But, then $(g_p)_p = q(h_p)_p$ for some $(h_p)_p \in \prod_p T_p$. This gives a contradiction when we look at the qth component: $0 \neq g_q = qh_q = 0$ since $pT_p = 0$ for all prime numbers p.

Thus in $E = D \oplus \prod_p T_p$, we have that $D \oplus 0$ is the divisible part of E.

(ii) Suppose A is a torsion-free reduced direct summand of $E = D \oplus \prod_p T_p$. So $A \oplus B = E$ for some subgroup $B \leq E$. Let $\pi_A : E \longrightarrow A$ and $\pi_B : E \longrightarrow B$ be projections corresponding to the direct sum $E = A \oplus B$. Since $D \oplus 0$ is a divisible subgroup of E, we have that $\pi_A(D \oplus 0)$ is also divisible. But $\pi_A(D \oplus 0) \leq A$ and A is reduced by hypothesis. So we must have $\pi_A(D \oplus 0) = 0$. Then

$$D \oplus 0 \leq \pi_A(D \oplus 0) + \pi_B(D \oplus 0) = 0 + \pi_B(D \oplus 0) = \pi_B(D \oplus 0) \leq B.$$

So,

$$B = B \cap E = B \cap [(D \oplus 0) \oplus (0 \oplus \prod_p T_p)] = (D \oplus 0) \oplus [B \cap (0 \oplus \prod_p T_p)],$$

and, then we obtain

$$(D \oplus 0) \oplus (0 \oplus \prod_p T_p) = E = A \oplus B = A \oplus (D \oplus 0) \oplus [B \cap (0 \oplus \prod_p T_p)].$$

Taking quotient groups by $D \oplus 0$ of both sides gives

$$\prod_p T_p \cong A \oplus [B \cap (0 \oplus \prod_p T_p)].$$

Hence we can assume that A is a direct summand of $\prod_p T_p$.

Let $M = \prod_p T_p$. Let's show that $\prod_p T_p$ has no nonzero torsion-free reduced direct summands. Suppose A is a torsion-free reduced direct summand of $M = \prod_p T_p$. So $A \oplus B = M$ for some subgroup $B \leq M$. Since A is torsion-free, we have that the torsion part $T(M)$ of M is contained in B:

$$T(M) = T(A \oplus B) = T(A) \oplus T(B) = 0 \oplus T(B) = T(B) \leq B.$$

Then:

$$M/T(M) = (A \oplus B)/T(M) = [(A \oplus T(M))/T(M)] \oplus [B/T(M)].$$

Since $M = \prod_p T_p$ where for every prime number p, $pT_p = 0$, we have that

$$T(M) = \bigoplus_p T_p,$$

because if $(a_p)_p \in T(M) = T(\prod_p T_p)$, then for some $n \in \mathbb{Z}^+$, $na_p = 0$ for all prime numbers p. Since $pa_p = 0$ for all prime numbers p already (as $pT_p = 0$), this implies that for all prime numbers p that does not divide n (thus for all but finitely many prime numbers p), we must have $a_p = 0$. Hence $(a_p)_p \in \bigoplus_p T_p$.

We shall now show that $M/T(M)$ is divisible. For any prime number q,

$$qM = \prod_p qT_p = \prod_p \widetilde{T_p},$$

where for all prime numbers $p \neq q$, $\widetilde{T_p} = qT_p = T_p$ since T_p is a p-group so it is q-divisible and $\widetilde{T_q} = qT_q = 0$. Then

$$qM + T(M) = \prod_p \widetilde{T_p} + \bigoplus_p T_p = \prod_p T_p = M$$

which implies that

$$q(M/T(M)) = (qM + T(M))/T(M) = M/T(M),$$

that is, $M/T(M)$ is q-divisible. This holds for all prime numbers q. Thus $M/T(M)$ is divisible. Then the direct summand $(A \oplus T(M))/T(M)$ of $M/T(M)$

is also divisible. But $(A \oplus T(M))/T(M) \cong A$ is reduced. Thus we must have $A \cong (A \oplus T(M))/T(M) = 0$, i.e., $A = 0$ as required.

□

The notion of cotorsion groups is fundamental in the study of Ext. It was discovered by Harrison et al. (1963) and found independently by Nunke (1959) and Fuchs (1960).

Definition 5.2.4. A group is called *cotorsion* if $\operatorname{Ext}(J, G) = 0$ for every torsion-free group J, equivalently if every extension of G by a torsion-free group splits.

Proof of Theorem 5.2.1.

A group E of the form $E = D \oplus \prod_p T_p$, where D is divisible, $pT_p = 0$, and p ranges over all primes, is clearly a neat-injective abelian group (= c-injective abelian group) by Corollary 5.1.2. So assume conversely that E is a neat injective abelian group. Then by Corollary 5.1.2, E is a *direct summand* of a group E' such that

$$E' = D' \oplus \prod_{p \text{ prime}} G_p$$

where D' is a *divisible* group, and in the product, p ranges over all prime numbers and for every prime number p, G_p is a direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$, i.e.,

$$p\,G_p = 0 \quad \text{for all prime numbers } p.$$

By Lemma 5.2.3, the group $\prod_p G_p$ is reduced and in $E' = D' \oplus \prod_p G_p$, we have that $D' \oplus 0$ is the divisible part of E':

$$E' = D' \oplus \prod_p G_p = \underbrace{(D' \oplus 0)}_{\text{divisible part of } E'} \oplus \underbrace{(0 \oplus \prod_p G_p)}_{\text{reduced}}$$

Let D be the *divisible part* of E. Then

$$E = D \oplus C$$

for some subgroup C of E such that C is *reduced*.

Let C_t be the *torsion part* of C and for each prime p, let T_p be the *p-primary component* of the torsion part C_t:

$$C_t = \bigoplus_{p \text{ prime}} T_p, \qquad T_p \text{ is } p\text{-primary for each prime number } p.$$

We shall follow the below steps:

Step 1. For all prime numbers p, $pT_p = 0$.

Step 2. C is *cotorsion*, i.e., $\mathrm{Ext}(J, C) = 0$ for every torsion-free abelian group J (or equivalently $\mathrm{Ext}(\mathbb{Q}, C) = 0$).

Step 3. C has *no* nonzero torsion-free direct summands.

Step 4. $C \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, C)$ since C is cotorsion and reduced.

Step 5. $C \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, C_t)$ since C is *adjusted* cotorsion.

Step 6. $C \cong \prod_p \mathrm{Ext}((\mathbb{Q}/\mathbb{Z})_p, C_t) \cong \prod_p \mathrm{Ext}(\mathbb{Z}_{p^\infty}, T_p) \cong \prod_p T_p$, where for each prime number p, $(\mathbb{Q}/\mathbb{Z})_p = \mathbb{Z}_{p^\infty}$ is the p-primary part of $\mathbb{Q}/\mathbb{Z}$.

Proof of Step 1: *For all prime numbers p, $pT_p = 0$.*

Since $E = D \oplus C$ is a direct summand of $E' = D' \oplus \prod_p G_p$, we have $E' = E \oplus H$ for some subgroup H of E'. Let D'' be the divisible part of H. Then $H = D'' \oplus C''$ for some subgroup C'' of H. Let $C' = \prod_p G_p$. We have seen that C' is reduced, $E' = (D' \oplus 0) \oplus (0 \oplus C)'$. Thus:

$$\begin{aligned}
\underbrace{(D' \oplus 0)}_{\text{divisible part of } E'} \oplus \underbrace{(0 \oplus \prod_p G_p)}_{\text{reduced}} &= E' = E \oplus H = (D \oplus C) \oplus (D'' \oplus C'') \\
&= \underbrace{(D \oplus D'')}_{\text{divisible}} \oplus \underbrace{(C \oplus C'')}_{\text{reduced}}
\end{aligned}$$

since the sum of divisible groups is divisible and the direct sum of reduced groups is reduced. By uniqueness of the divisible part of an abelian group, we must have

$$D' \oplus 0 = D \oplus D'',$$

and so $(C \oplus C'') \cap (D' \oplus 0) = (C \oplus C'') \cap (D \oplus D'') = 0$. Thus we obtain

$$C \cap (D' \oplus 0) = 0.$$

Suppose for the contrary that for some prime number q, $qT_q \neq 0$. Since T_q is the q-primary part of the torsion part C_t of C, the order of every element of T_q is a power of q. Since $qTq \neq 0$, there exists an element $x \in T_q$ and an integer $n \geq 2$ such that

$$q^n x = 0 \quad \text{and} \quad q^{n-1}x \neq 0$$

Since $x \in T_q \leq \bigoplus_p T_p = C_t \leq C \leq D \oplus C = E \leq E \oplus H = E' = D' \oplus \prod_p G_p$, we have

$$x = (d', (g_p)_p) \qquad \text{for some } d' \in D' \text{ and } (g_p)_p \in \prod_p G_p,$$

where for each prime number p, $g_p \in G_p$ and so $pg_p = 0$ since $pG_p = 0$. Then $0 = q^n x = (q^n d', (q^n g_p)_p)$ implies that

$$q^n d' = 0 \quad \text{and} \quad q^n g_p = 0 \text{ for all prime numbers } p.$$

For each prime number $p \neq q$, since $pg_p = 0$ already and we have $q^n g_p = 0$, we obtain $g_p = 0$ by relative primality of p and q^n:

$$g_p = 0 \quad \text{for all prime numbers } p \neq q.$$

Since $0 \neq q^{n-1}x = (q^{n-1}d', (q^{n-1}g_p)_p)$, we obtain

$$q^{n-1}d' \neq 0 \quad \text{or} \quad q^{n-1}g_q \neq 0$$

because for all prime numbers $p \neq q$, we have $g_p = 0$ and so $q^{n-1}g_p = 0$ already. But $qg_q = 0$ since $qG_q = 0$. So we must have $q^{n-1}g_q = 0$ since $n \geq 2$. Thus the only possibility is that

$$q^{n-1}d' \neq 0.$$

Then we obtain the contradiction

$$0 \neq \underbrace{q^{n-1}x}_{\in T_q \text{ since } x \in T_q} = \underbrace{(q^{n-1}d', (0)_p)}_{\in D' \oplus 0} \in T_q \cap (D' \oplus 0) \leq C \cap (D' \oplus 0) = 0$$

by the initial observation about the divisible part of E'.

The direct proof of Step 1 in Harrison et al. (1963, Lemma 4), without using Corollary 5.1.2, goes as follows:

Suppose for the contrary that T_p has an element of order p^2. Then it has a cyclic summand $\mathbb{Z}/p^n\mathbb{Z}$ of order p^n, $n > 1$ (i.e., a cyclic summand of a basic subgroup of T_p).

Indeed, since T_p is torsion it contains a basic subgroup, say B_p (i.e., B_p is pure in T_p, T_p/B_p divisible and B_p is direct sum of cyclics of p-power order). Now suppose for the contrary that B_p is a direct sum of cyclic groups of order p. Then since B_p is pure in T_p, we have

$$(pT_p) \cap B_p = pB_p = 0.$$

Since T_p/B_p is divisible, we obtain

$$T_p/B_p = p(T_p/B_p) = (pT_p + B_p)/B_p,$$

that is, $T_p = (pT_p) + B_p$. Thus $T_p = (pT_p) \oplus B_p$, and so $T_p/B_p \cong pT_p$ is divisible which contradicts T_p is reduced. Hence B_p contains a cyclic summand p^n, $n > 1$.

Now let $A =< a >$ and $B =< b >$ be two cyclic groups such that $\mathrm{o}(a) = p^{n+1}$ and $\mathrm{o}(b) = p$. Since $< pa + b >$ is a cyclic group of order n (indeed, suppose there is an integer $k < n$ such that $p^k(pa + b) = 0$, then $p^{k+1}a = 0$ and so $n + 1$ must divide $k + 1$ since $\mathrm{o}(a) = n + 1$, but this is impossible),

$$\mathbb{Z}/p^n\mathbb{Z} \cong < pa + b > \leq A \oplus B.$$

Let us show that $< pa + b >$ is neat subgroup of $A \oplus B$, but it is not pure:
Suppose $k(pa+b) = p(ta+sb)$ for some $k(pa+b) \in < pa+b >$, $(ta+sb) \in A \oplus B$, integers k, t, s and any prime p. Then $kpa + kb = pta + psb = pta$ since $psb = 0$ as

$o(b) = p$. So $kb = pta - kpa = (pt - kp)a \in < a > \cap < b > = A \cap B = 0$. Thus $p \mid k$ since $o(b) = p$, that is, $k = pk'$ for some $k' \in \mathbb{Z}$. Therefore $k(pa+b) = pk'(pa+b) = p[k'(pa + b)]$, where $k'(pa + b) \in < pa + b >$. Hence $< pa + b >$ is neat in $A \oplus B$. On the other hand, since $0 \neq p^n a = p^{n-1}(pa + 0) \in (< pa + b >) \cap p^n(A \oplus B)$ we have

$$0 = p^n(< pa + b >) \neq (< pa + b >) \cap p^n(A \oplus B) \neq 0.$$

This implies that $< pa + b >$ is not pure.

Since $\mathbb{Z}/p^n\mathbb{Z}$ is pure in B_p (as a direct summand) and B_p is pure in T_p (as a basic subgroup), $\mathbb{Z}/p^n\mathbb{Z}$ is pure in T_p. So $\mathbb{Z}/p^n\mathbb{Z}$ is pure in C since T_p is pure in C (as adirect summand). Then there exists a subgroup $S \leq C$ such that $C = \mathbb{Z}/p^n\mathbb{Z} \oplus S$, and so $C \leq A \oplus B \oplus S$ since $\mathbb{Z}/p^n\mathbb{Z} \leq A \oplus B$. Thus clearly we have a neat-exact sequence which does not split

$$0 \longrightarrow E \xrightarrow{f} D \oplus (A \oplus B) \oplus S \longrightarrow X \longrightarrow 0,$$

where $E = D \oplus \mathbb{Z}/p^n\mathbb{Z} \oplus S$ and X any group. Indeed, to show that $\operatorname{Im} f$ is neat in $D \oplus (A \oplus B) \oplus S$, it suffices to show $\operatorname{Im} g$ is neat in $A \oplus B$, where $g : \mathbb{Z}/p^n\mathbb{Z} \longrightarrow A \oplus B$. Since $\operatorname{Im} g \cong \mathbb{Z}/p^n\mathbb{Z} \cong < pa + b >$, we have $\operatorname{Im} g$ is neat in $A \oplus B$. If the sequence splits, then $\mathbb{Z}/p^n\mathbb{Z}$ will be a direct summand of $A \oplus B$. But this is impossible. On the other hand, since E is neat-injective group the sequence must split, a contradiction. Hence every element of T_p has order p., that is, $pT_p = 0$, for all prime p.

Proof of Step 2. *C is* cotorsion*, i.e.,* $\operatorname{Ext}(J, C) = 0$ *for every torsion-free abelian group J (or equivalently* $\operatorname{Ext}(\mathbb{Q}, C) = 0$*).*

The group C is also neat-injective since C is a direct summand of the neat-injective abelian group E.

Let

$$0 \longrightarrow C \xrightarrow{f} B \xrightarrow{g} J \longrightarrow 0$$

be a short exact sequence of abelian groups such that J is a *torsion-free* abelian group. Since $B/\operatorname{Im} f \cong J$ is torsion-free, $\operatorname{Im} f$ is pure in B (by Fuchs (1970, §26, (d), p. 114)) and so neat in B. Thus this short exact sequence is in $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$. Since C is neat-injective, this short exact sequence must split. This shows that $\mathrm{Ext}(J, C) = 0$ for all torsion-free abelian groups J which means that C is cotorsion.

Proof of Step 3. *C has* no *nonzero torsion-free direct summands.*

Suppose for the contrary that $C = X \oplus Y$ for some submodules X and Y of C such that $X \neq 0$, X is *torsion-free*. By Lemma 5.2.2, if we embed X into a divisible group D_1, then for the homomorphism

$$\begin{aligned} f : X &\longrightarrow D_1 \oplus \prod_p (X/pX) \\ f(x) &= (x, (x + pX)_p), \quad x \in X, \end{aligned}$$

we have that $\operatorname{Im} f$ is a neat subgroup of $\prod_p (X/pX)$ and $D_1 \oplus \prod_p (X/pX)$ is neat-injective. Since X is a direct summand of the neat-injective group C, we obtain that $\operatorname{Im} f \cong X$ is also neat-injective. Thus $\operatorname{Im} f$ must be a direct summand of $D_1 \oplus \prod_p (X/pX)$. But by Lemma 5.2.3, the group $D_1 \oplus \prod_p (X/pX)$ has *no* nonzero torsion-free reduced direct summands and this contradicts with $\operatorname{Im} f \cong X$ being a nonzero torsion-free reduced module.

Proof of Step 4. *$C \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, C)$ since C is cotorsion and reduced.*

Since C is cotorsion by Step 2 and C is reduced, we have a natural isomorphism

$$C \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, C)$$

by Fuchs (1970, §54, (H), p. 234).

Proof of Step 5. *$C \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, C_t)$ since C is* adjusted *cotorsion.*

By Fuchs (1970, Theorem 55.5), since C is a reduced cotorsion group (by Step 2), we have a direct decomposition

$$C = A \oplus C$$

for some subgroups A and C of C such that A is torsion-free and $C \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, C_t)$. By Step 3, C has *no* nonzero torsion-free direct summands. Thus we must have

$A = 0$ and so

$$C = C \cong \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, C_t).$$

A cotorsion group that is reduced and has *no* nonzero torsion-free direct summands is called *adjusted*; see Fuchs (1970, §55). We have seen so far that C is adjusted cotorsion.

Proof of Step 6. *$C \cong \prod_p \operatorname{Ext}((\mathbb{Q}/\mathbb{Z})_p, C_t) \cong \prod_p \operatorname{Ext}(\mathbb{Z}_{p^\infty}, T_p) \cong \prod_p T_p$, where for each prime number p, $(\mathbb{Q}/\mathbb{Z})_p = \mathbb{Z}_{p^\infty}$ is the p-primary part of $\mathbb{Q}/\mathbb{Z}$.*

By Step 5, $C \cong \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, C_t)$. Since $\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}_{p^\infty}$, we have by the properties of Ext (see Fuchs (1970, Theorem 52.2)),

$$C \cong \operatorname{Ext}\left(\bigoplus_p \mathbb{Z}_{p^\infty}, C_t\right) \cong \prod_p \operatorname{Ext}(\mathbb{Z}_{p^\infty}, C_t)$$

Since $C_t = \bigoplus_p T_p$, we have for each prime number p,

$$C_t = T_p \oplus \left(\bigoplus_{q \text{ prime},\ q \neq p} T_q\right).$$

Then by the properties of Ext (see Fuchs (1970, Theorem 52.2)), for each prime number p,

$$\operatorname{Ext}(\mathbb{Z}_{p^\infty}, C_t) = \operatorname{Ext}(\mathbb{Z}_{p^\infty}, T_p) \oplus \operatorname{Ext}\left(\mathbb{Z}_{p^\infty}, \bigoplus_{q \neq p} T_q\right).$$

Since $\mathbb{Z}_{p^\infty}$ is a p-group and $\bigoplus_{q \neq p} T_q$ is p-divisible, we obtain by Fuchs (1970, §52, (K), p. 223) that

$$\operatorname{Ext}\left(\mathbb{Z}_{p^\infty}, \bigoplus_{q \neq p} T_q\right) = 0.$$

Thus

$$\operatorname{Ext}(\mathbb{Z}_{p^\infty}, C_t) = \operatorname{Ext}(\mathbb{Z}_{p^\infty}, T_p).$$

So it only remains to show that $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, T_p) \cong T_p$.

By Fuchs (1970, §52, (F), p. 223), we have for each abelian group C and positive integer m, $\operatorname{Ext}(C, \mathbb{Z}/m\mathbb{Z}) \cong \operatorname{Ext}(C[m], \mathbb{Z}/m\mathbb{Z})$, where $C[m] = \{c \in C \mid mc = 0\}$.

For the proof of that it suffices to have $m(\mathbb{Z}/m\mathbb{Z}) = 0$. For $C = \mathbb{Z}_{p^\infty}$ and $m = p$, we follow Fuchs (1970, §52, proof of (F), p. 223). From the exactness of

$$0 \longrightarrow C[p] \xrightarrow{f} C \xrightarrow{g} pC \longrightarrow 0,$$

where the first map f is inclusion and the second map g is multiplication by p (i.e., $g(x) = px$ for all $x \in C$), we obtain the induced exact sequence

$$\operatorname{Ext}(pC, T_p) \xrightarrow{g^*} \operatorname{Ext}(C, T_p) \xrightarrow{f^*} \operatorname{Ext}(C[p], T_p) \longrightarrow 0.$$

We shall show that $\operatorname{Im} g^* = 0$ which implies that f^* is an isomorphism:

$$\operatorname{Ext}(C, T_p) \cong \operatorname{Ext}(C[p], T_p).$$

Let $g_1 : pC \longrightarrow C$ be inclusion homomorphism and $g_2 : C \longrightarrow C$ be multiplication by p (i.e., $g_2(x) = px$ for all $x \in C$). Then $g_2 = g_1 \circ g$. Applying the functor $\operatorname{Ext}(-, T_p)$, we obtain homomorphisms

$$\begin{aligned} g_1^* &: \operatorname{Ext}(C, T_p) \longrightarrow \operatorname{Ext}(pC, T_p), \\ g_2^* &: \operatorname{Ext}(C, T_p) \longrightarrow \operatorname{Ext}(C, T_p), \end{aligned}$$

such that $g_2^* = g^* \circ g_1^*$. So $\operatorname{Im} g^* \leq \operatorname{Im} g_2^*$. By Fuchs (1970, Lemma 52.1), since $g_2 : C \longrightarrow C$ is multiplication by p, the homomorphism $g_2^* : \operatorname{Ext}(C, T_p) \longrightarrow \operatorname{Ext}(C, T_p)$ is also multiplication by p. Hence $\operatorname{Im} g_2^* = p \operatorname{Ext}(C, T_p)$. By Fuchs (1970, §52, (E), p. 223), since $pT_p = 0$, we obtain $p \operatorname{Ext}(C, T_p) = 0$. Thus $\operatorname{Im} g_2^* = 0$ and so $\operatorname{Im} g^* = 0$.

For $C = \mathbb{Z}_{p^\infty}$, since $C[p] = \mathbb{Z}_{p^\infty}[p] \cong \mathbb{Z}/p\mathbb{Z}$, we obtain

$$\operatorname{Ext}(\mathbb{Z}_{p^\infty}, T_p) \cong \operatorname{Ext}(\mathbb{Z}/p\mathbb{Z}, T_p),$$

and, then by Fuchs (1970, §52, (D), p. 222),

$$\operatorname{Ext}(\mathbb{Z}/p\mathbb{Z}, T_p) \cong T_p/pT_p \cong T_p$$

since $pT_p = 0$ by Step 1. Thus $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, T_p) \cong T_p$ and this completes the proof of Step 6.

Another argument to prove $\mathrm{Ext}(\mathbb{Z}_{q^\infty}, T_q) \cong T_q$ for every prime number q is as follows. Since T_q is a homogeneous semisimple $\mathbb{Z}$-module, T_q is neat-injective. In the proof of Step 2, we have indeed shown that neat-injective abelian groups are cotorsion. Thus T_q is cotorsion. Since $T_q \leq C$ and C is reduced, T_q is also reduced. Then as in the proof of Step 4, since T_q is reduced and cotorsion, we have a natural isomorphism

$$T_q \cong \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T_q)$$

by Fuchs (1970, §54, (H), p. 234). Since $\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}_{p^\infty}$, we have by the properties of Ext (see Fuchs (1970, Theorem 52.2)),

$$T_q \cong \mathrm{Ext}\left(\bigoplus_p \mathbb{Z}_{p^\infty}, T_q\right) \cong \prod_p \mathrm{Ext}(\mathbb{Z}_{p^\infty}, T_q)$$

For every prime number $p \neq q$, $\mathbb{Z}_{p^\infty}$ is a p-group and T_q is p-divsible, so by Fuchs (1970, §52, (K), p. 223) that

$$\mathrm{Ext}(\mathbb{Z}_{p^\infty}, T_q) = 0 \qquad \text{for all prime numbers } p \neq q.$$

Thus $\prod_p \mathrm{Ext}(\mathbb{Z}_{p^\infty}, T_q) \cong \mathrm{Ext}(\mathbb{Z}_{q^\infty}, T_q)$ as required.

Another proof of Theorem 5.2.1 suggested by Bill Wickless.

My supervisor E. Mermut has given a talk on c-injective modules over Dedekind domains in Antalya Algebra Days 2006 (Mermut, May 17-21, 2006) and he has asked to Bill Wickless (Wickless, William J.) about the proof for the structure of the neat-injective abelian groups given in Harrison et al. (1963, Lemma 4). Bill Wickless has clarified the details for the case of abelian groups with ideas and suggestions of how it can be generalized in the case of modules over Dedekind domains. Bill Wickless pointed out the structure theorem for reduced algebraically compact groups to obtain the following result: a direct summand of a direct product of homogeneous semisimple abelian groups turns out to be again of this form, that is, it is also a direct product of homogeneous semisimple abelian groups. See Fuchs (1970, Ch. VII) or Wickless (2004, Sections 8.5 and 8.6) for algebraically compact groups.

Theorem 5.2.5. *Structure of reduced algebraically compact groups.*
(by Fuchs (1970, Proposition 40.1 and its proof, §40) or Wickless (2004, Theorem 8.6.1 and its proof, §8.6))
Let A be a reduced algebraically compact group. Then there exist groups A_p for each prime number p such that

$$A \cong \prod_{p\ prime} A_p$$

and for every prime p,

(i) *A_p is a p-*local *group, that is, A_p is q-divisible for all prime numbers $q \neq p$: For all positive integers k,*

$$q^k A_p = A_p \quad \text{for all prime numbers } q \neq p \quad \text{and} \quad \bigcap_{k=1}^{\infty} p^k A_p = 0.$$

(ii) *A_p is complete in its p-adic topology (which equals the $\mathbb{Z}$-adic topology on A_p since A_p is p-local) since A_p is also a reduced algebraically compact group,*

(iii) *A_p is uniquely determined up to isomorphism by A, it is called the* p-adic component *of the reduced algebraically compact group A; more precisely*

$$\begin{aligned} A_p \quad &\cong \quad (\text{ the unique maximal } p\text{-local subgroup of } A\ ) \\ &= \bigcap \{q^k A \mid k \in \mathbb{Z}^+ \text{ and } q \text{ is a prime number such that } q \neq p\}. \\ &\cong \widetilde{A_p} = (\text{ the unique maximal } p\text{-local subgroup of } \prod_{q\ prime} A_q\ ) \end{aligned}$$

where $\widetilde{A_p}$ consists of all elements of $\prod_q A_q$ with all coordinates zero except the pth coordinate (so just a copy of the pth coordinate A_p):

$$\widetilde{A_p} = \{(a_q)_q \in \prod_q A_q \mid a_q = 0 \text{ for all prime numbers } q \neq p\}.$$

We shall follow the beginning of the above proof of Theorem 5.2.1 till the first part of proof of Step 1. In fact it suffices to show the reduced case as the below argument shows.

Let E be a neat-injective abelian group. Then by Corollary 5.1.2, E is a *direct summand* of a group E' such that

$$E' = D' \oplus \prod_{p \text{ prime}} G_p$$

where D' is a *divisible* group, and in the product, p ranges over all prime numbers and for every prime number p, G_p is a direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$, i.e.,

$$p\,G_p = 0 \qquad \text{for all prime numbers } p.$$

By Lemma 5.2.3, the group $\prod_p G_p$ is reduced and in $E' = D' \oplus \prod_p G_p$, we have that $D' \oplus 0$ is the divisible part of E':

$$E' = D' \oplus \prod_p G_p = \underbrace{(D' \oplus 0)}_{\text{divisible part of } E'} \oplus \underbrace{(0 \oplus \prod_p G_p)}_{\text{reduced}}$$

Let D be the *divisible part* of E. Then

$$E = D \oplus C$$

for some subgroup C of E such that C is *reduced*.

Since $E = D \oplus C$ is a direct summand of $E' = D' \oplus \prod_p G_p$, we have $E' = E \oplus H$ for some subgroup H of E'. Let D'' be the divisible part of H. Then $H = D'' \oplus C''$ for some subgroup C'' of H. Let $C' = \prod_p G_p$. We have seen that C' is reduced, $E' = (D' \oplus 0) \oplus (0 \oplus C)'$. Thus:

$$\begin{aligned} \underbrace{(D' \oplus 0)}_{\text{divisible part of } E'} \oplus \underbrace{(0 \oplus \prod_p G_p)}_{\text{reduced}} &= E' = E \oplus H = (D \oplus C) \oplus (D'' \oplus C'') \\ &= \underbrace{(D \oplus D'')}_{\text{divisible}} \oplus \underbrace{(C \oplus C'')}_{\text{reduced}} \end{aligned}$$

since the sum of divisible groups is divisible and the direct sum of reduced groups is reduced. By uniqueness of the divisible part of an abelian group, we must have

$$D' \oplus 0 = D \oplus D''.$$

Thus taking quotient group by $D' \oplus 0 = D \oplus D''$ in of both sides of

$$(D' \oplus 0) \oplus (0 \oplus \prod_p G_p) = (D \oplus D'') \oplus (C \oplus C''),$$

we obtain

$$\prod_p G_p \cong C \oplus C''$$

Thus C is isomorphic to a direct summand of $\prod_p G_p$. So it suffices to show that a direct summand of $\prod_p G_p$ is isomorphic to a group of the form $\prod_p B_p$ where for each prime p, B_p is a group such that $pB_p = 0$.

For every prime number p, since $pG_p = 0$, the group G_p is bounded and so algebraically compact by Fuchs (1970, Theorem 27.5 and §38). Then the direct product $\prod_p G_p$ is also algebraically compact by Fuchs (1970, Corollary 38.3). Let B be a direct summand of the reduced algebraically compact group $\prod_p G_p$. Then B is also algebraically compact and reduced (see Fuchs (1970, §38)). By the structure of reduced algebraically compact groups (Theorem 5.2.5), we know that $B \cong \prod_p B_p$ where for each prime p, B_p is the unique maximal p-local subgroup of B. Then necessarily B_p is contained in the unique maximal p-local subgroup of $\prod_q G_q$ which is isomorphic to G_p. Thus B_p is isomorphic to a subgroup of G_p which is a homogeneous semisimple $\mathbb{Z}$-module. Thus $pB_p = 0$ also since $pG_p = 0$. This ends the proof suggested by Bill Wickless.

5.3 c-injective Modules over a Dedekind Domain

To prove the similar result for the structure of c-injective modules over Dedekind domains, the proof suggested by Bill Wickless in the case of abelian groups is followed in Mermut et al. (2007).

Definition 5.3.1. An R-module N is called *finitely presented* if

(i) N is finitely generated and

(ii) in every exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0,$$

with L is finitely generated R-module, K is also finitely generated R-module.

Definition 5.3.2. A short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of left R-modules is *pure* if the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(E, A) \longrightarrow \mathrm{Hom}(E, B) \longrightarrow \mathrm{Hom}(E, C) \longrightarrow 0$$

is exact for every finitely presented left R-module E.

Definition 5.3.3. A submodule A of an R-module B is a *pure submodule* of B if the canonical exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0 \quad \text{is pure.}$$

Theorem 5.3.4. *(see for example Facchini (1998, Theorem 1.27))Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of (left) R-modules. Then the following conditions are equivalent:

(i) *The sequence* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is pure.*

(ii) *For every finitely presented right R-module F_R, the induced sequence of abelian groups*

$$0 \longrightarrow F \otimes A \longrightarrow F \otimes B \longrightarrow F \otimes C \longrightarrow 0$$

is exact.

(iii) *For every right R-module F_R, the induced sequence of abelian groups*

$$0 \longrightarrow F \otimes A \longrightarrow F \otimes B \longrightarrow F \otimes C \longrightarrow 0$$

is exact.

(iv) *Every system of m linear equations*

$$\sum_{i=1}^{n} x_i r_{ij} = a'_j, \quad j = 1, 2, \ldots, m,$$

with $r_{ij} \in R$ and $a'_j \in f(A)$ ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$), which has a solution in B^n, also has a solution in $(f(A))^n$.

Definition 5.3.5. A submodule A of an R-module B is called *relatively divisible* , or briefly, an *RD-submodule*, if

$$rA = A \cap rB \quad \text{for each } r \in R.$$

Definition 5.3.6. A commutative domain R is said to be a *Prüfer domain* if all its localizations at maximal ideals are valuation domains; thus Prüfer domains are those domains which are *locally* valuation domains.

Proposition 5.3.7. *(see Fuchs & Salce (2001, Ch.IV, §3, pp. 131)) A commutative domain R is h*-local *if and only if the following two conditions are satisfied:*

(i) *R is of finite character (i.e., R/I is semilocal for every ideal $I \neq 0$);*

(ii) *every nonzero prime ideal is contained in only one maximal ideal (i.e., R/I is even local if $I \neq 0$ is a prime ideal).*

See for example Fuchs & Salce (2001, §IV.3) for h-local domains and Fuchs & Salce (2001, Ch. III) for Prüfer domains.

Note that, a Dedekind domain is an h-local Prüfer domain.

Proposition 5.3.8. *(Fuchs & Salce, 2001, Ch.I, Theorem 8.11) Over Prüfer domains (and so over Dedekind domains), relative divisibility and purity are equivalent.*

In the case of abelian groups, a *direct summand* of a direct product of homogeneous semisimple abelian groups and an injective abelian group turns out to be again of this form, that is, it is also a direct product of homogeneous semisimple abelian groups and an injective abelian group. This is seen by Theorem 5.2.1 since neat-injective abelian groups are nothing but c-injective abelian groups:

Like in abelian groups, we have structure theorems for pure-injective modules over Dedekind domains. More generally:

Theorem 5.3.9. *(by Warfield (1969) and Fuchs & Salce (2001, Ch. XIII, Proposition 4.5)) A reduced pure-injective module M over an* h*-local Prüfer domain R decomposes in a unique way, up to isomorphism, into the product of pure-injective R_P-modules, with P ranging over the set of maximal ideals of R:*

$$M \cong \prod_P A_P$$

where the product runs through all maximal ideals P of R and where A_P is an R_P-module such that

$$A_P = \mathrm{Hom}_R(R_P, M)$$

for every maximal ideal P of R.

Since a Dedekind domain is an h-local Prüfer domain, the above result also holds for modules over Dedekind domains.

Lemma 5.3.10. *(Mermut et al., 2007) For a Dedekind domain R which is not a field, let E be an R-module of the form*

$$E = D \oplus \prod_{P \underset{max.}{\leq} R} G_P$$

where D is a divisible module and the product is over all maximal ideals of R such that for every maximal ideal P of R, G_P is a homogeneous semisimple R-module having simple components isomorphic to R/P, i.e.,

$$PG_P = 0 \quad \text{for all maximal ideals } P \leq R.$$

Then:

(i) *$\prod_P G_P$ is reduced and the divisible part of E is $D \oplus 0$:*

$$E = D \oplus \prod_P G_P = \underbrace{(D \oplus 0)}_{\textit{divisible part of } E} \oplus \underbrace{(0 \oplus \prod_P G_P)}_{\textit{reduced}}$$

(ii) *Let $M = \prod_P G_P$. Then $M = \prod_P G_P$ is a reduced pure-injective R-module and for every maximal ideal P of R, G_P is a reduced pure-injective R_P-module such that*

$$G_P \cong \mathrm{Hom}_R(R_P, M).$$

(iii) *Let E' be a direct summand of E. Then $E' = D' \oplus C'$ for an injective submodule D' of E' and a submodule C' of E' such that C' is isomorphic to a direct summand of $\prod_P G_P$.*

Proof. (i) Suppose for the contrary that $\prod_P G_P$ is *not* reduced. Then it has a nonzero divisible submodule D_1. So there exists a nonzero element $(g_P)_P$ of D_1. Then it must be divisible in D_1, and so in $\prod_P G_P$, by every nonzero $r \in R$. Since $(g_P)_P$ is a nonzero element, $g_Q \neq 0$ for some maximal ideal Q of R. Then in particular this element $(g_P)_P$ must be divisible by every nonzero $q \in Q$ in $\prod_P G_P$. But then $(g_P)_P = q(h_P)_P$ for some $(h_P)_P \in \prod_P G_P$. This gives a contradiction when we look at the Qth component: $0 \neq g_Q = qh_Q = 0$ since $PG_P = 0$ for every maximal ideal P of R.

Thus in $E = D \oplus \prod_P G_P$, we have that $D \oplus 0$ is the divisible part of E.

(ii) For every maximal ideal P of R, the module G_P is bounded since $PG_P = 0$. So each G_P is pure-injective (by Theorem 2.9.10). Since $PG_P = 0$, the R-module G_P can be considered as an R_P-module also. Since G_P is a pure-injective R-module, it is also a pure-injective R_P-module. Thus the reduced pure-injective module $M = \prod_P G_P$ have been expressed as a product of pure-injective R_P-modules with P ranging over the set of maximal ideals of R. Then by the uniqueness part of Theorem 5.3.9, we have $G_P \cong \mathrm{Hom}_R(R_P, M)$.

(iii) Let D' be the *divisible part* of E'. Then

$$E' = D' \oplus C'$$

for some submodule C' of E such that C' is *reduced*.

Since $E' = D' \oplus C'$ is a direct summand of $E = D \oplus \prod_P G_P$, we have $E = E' \oplus H$ for some submodule H of E. Let D'' be the divisible part of H. Then $H = D'' \oplus C''$ for some reduced submodule C'' of H. Let $C = \prod_P G_P$. We have seen that C is reduced and $E = (D \oplus 0) \oplus (0 \oplus C)$. Thus:

$$\begin{aligned}\underbrace{(D \oplus 0)}_{\text{divisible part of } E} \oplus \underbrace{(0 \oplus \prod_P G_P)}_{\text{reduced}} &= E = E' \oplus H = (D' \oplus C') \oplus (D'' \oplus C'') \\ &= \underbrace{(D' \oplus D'')}_{\text{divisible}} \oplus \underbrace{(C' \oplus C'')}_{\text{reduced}}\end{aligned}$$

since the sum of divisible modules is divisible and the direct sum of reduced modules is reduced. By uniqueness of the divisible part of a module over a Dedekind domain, we must have

$$D \oplus 0 = D' \oplus D''.$$

Thus taking quotient group by $D \oplus 0 = D' \oplus D''$ in of both sides of

$$(D \oplus 0) \oplus (0 \oplus \prod_P G_P) = (D' \oplus D'') \oplus (C' \oplus C''),$$

we obtain

$$\prod_P G_P \cong C' \oplus C''$$

Thus C' is isomorphic to a direct summand of $\prod_P G_P$.

$\square$

Theorem 5.3.11. *(Mermut et al., 2007) For a Dedekind domain R, every* c*-injective R-module is isomorphic to a direct product of homogeneous semisimple R-modules and an injective module; more precisely, every* c*-injective R-module is isomorphic to a module E of the form*

$$E = D \oplus \prod_{\substack{P \le R \\ max.}} G_P$$

where D is a divisible module and the product is over all maximal ideals of R such that for every maximal ideal P of R, G_P is a homogeneous semisimple R-module having simple components isomorphic to R/P, i.e.,

$$PG_P = 0 \quad \textit{for all maximal ideals } P \leq R.$$

Proof. Let E be a c-injective R-module. Then by Corollary 5.1.2, E is a *direct summand* of an R-module E' of the form

$$E' = D' \oplus \prod_{\substack{P \leq R \\ max.}} G_P$$

Then by Lemma 5.3.10, $E = D' \oplus C$ for some injective submodule D' of E and a submodule C of E such that C is isomorphic to a direct summand of $\prod_P G_P$. Thus it suffices to show that a direct summand of $\prod_P G_P$ is isomorphic to a module of the form $\prod_P A_P$ such that for every maximal ideal P of R, the R-module A_P satisfies $PA_P = 0$.

Let B be a direct summand of $\prod_P G_P$. By Lemma 5.3.10, $\prod_P G_P$ is a reduced pure-injective R-module. Then its direct summand B is also pure-injective (by for example Fuchs & Salce (2001, §XIII.2, (B), p. 430)) and reduced. By the structure of reduced pure-injective R-modules (Theorem 5.3.9), we know that $B \cong \prod_P A_P$ where for each maximal ideal P of R, A_P is a pure-injective R_P-module such that

$$A_P \cong \mathrm{Hom}_R(R_P, B).$$

Since B is a direct summand of M, $M = B \oplus B'$ for some submodule $B' \leq M$. Thus:

$$\begin{aligned} G_P &\cong \mathrm{Hom}_R(R_P, M) = \mathrm{Hom}_R(R_P, B \oplus B') \\ &\cong \mathrm{Hom}_R(R_P, B) \oplus \mathrm{Hom}_R(R_P, B') \\ &\cong A_P \oplus \mathrm{Hom}_R(R_P, B') \end{aligned}$$

Thus A_P is isomorphic to a direct summand of G_P. Since $PG_P = 0$, we also have $PA_P = 0$. $\square$

5.4 Neat-Injective Envelopes of Abelian Groups

It is well known that every abelian group A can be embedded in a minimal injective (i.e., divisible) group which is called *injective envelope* (or *injective hull*) of A (see Fuchs (1970, Ch.IV, §24)). Similar results were proved for pure-injective envelopes (see Fuchs (1970, Ch.VII, §41)). Then neat-injective envelopes were proved by Onishi (1984).

Definition 5.4.1. Let S be the set of all square-free integers. Let G be a group. The subgroups nG $(n \in S)$ form a base of neighborhood about $0 \in G$. We call this linear topology the *S-adic topology* on G. Obviously, S-adic topology is coarser than $\mathbb{Z}$-adic topology.

Theorem 5.4.2. *(see Imam (2000, Corollary 1.4.11)) A group G can be embedded as a neat subgroup in a reduced neat-injective group $\overline{G}$ if and only if its Frattini subgroup* $\operatorname{Rad} G = 0$.

The following lemma clarifies the relationship among neat-injective groups, pure-injective groups and cotorsion groups.

Lemma 5.4.3. *(by Onishi (1984, Lemma 2.3)) If A is a group such that the Frattini subgroup of its reduced part vanishes, then the following properties of A are equivalent:*

(i) *A is neat-injective,*

(ii) *A is pure-injective,*

(iii) *A is cotorsion.*

Proof. $(i) \Rightarrow (ii)$: Since neat-injectivity is stronger than pure-injectivity, A is pure-injective.

$(ii) \Rightarrow (iii)$: If A is pure-injective, then it is algebraically compact by Fuchs (1970, Theorem 38.1). Thus it follows immediately from Fuchs (1970, Proposition 54.1) that A is cotorsion.

$(iii) \Rightarrow (ii)$: If A is cotorsion, then it is an epimorphic image of an algebraically compact group (by Fuchs (1970, Proposition 54.1)). Thus A is an algebraically compact group since an epimorphic image of a pure subgroup is again pure. Hence A is pure-injective.

$(ii) \Rightarrow (i)$: A has a decomposition $A = D \oplus C$ where C is a reduced subgroup and D is the maximal divisible subgroup of A. Since C is reduced pure-injective, it follows from Fuchs (1970, Proposition 40.1) that $C = \prod_p C_p$ where C_p is complete in its p-adic topology. Then each C_p is clearly a direct sum of cyclic groups of order p . Indeed,

$$0 = \operatorname{Rad} C = \bigcap_{q \text{ prime}} qC = \bigcap_{q \text{ prime}} q(\prod_p C_p) = \prod_q qC_q$$

since C_q is p-divisible for every prime $q \neq p$. Thus for every prime q, $qC_q = 0$, and so C_q is a direct summand of cyclic groups of order q. Therefore A is neat-injective by Onishi (1980, Theorem 4.3) (or see Theorem 5.2.1). $\square$

Since the S-adic topology is coarser than the $\mathbb{Z}$-adic topology, if a group A is complete (for details of completeness, see for example Fuchs (1970, §13)) in the S-adic topology, then A is complete in the $\mathbb{Z}$-adic topology by Fuchs (1970, Remark, pp. 163). Conversely, we have:

Proposition 5.4.4. *(Onishi, 1984, Lemma 2.4) Let a group A is complete in the $\mathbb{Z}$-adic topology. If the Frattini subgroup* $\operatorname{Rad} A = 0$*, then A is complete in the S-adic topology.*

Proposition 5.4.5. *(Onishi, 1984, Proposition 2.5) A group is complete in the S-adic topology if and only it is reduced neat-injective group.*

Proposition 5.4.6. *(Onishi, 1984, Proposition 2.6) The inverse limit of reduced neat-injective groups is reduced neat-injective.*

Let N be a group. Define $m \leq n$ for $m, n \in S$ to mean that m is a divisor of n. Then clearly S is a directed set. Moreover, for $m \leq n$, let $\pi_m^n : N/nN \to N/mN$ be the homomorphism that maps $a + nN$ upon $a + mN$ for $a \in N$. Then

$$\{N/nN\,(n \in S) \mid \pi_m^n\,(m \leq n)\}$$

is clearly an inverse system. We shall denote the inverse limit of this system by $\widetilde{N}$. Namely,

$$\widetilde{N} = \lim_{\leftarrow (n \in S)} N/nN.$$

Definition 5.4.7. For any group N, $\widetilde{N} = \lim\limits_{\leftarrow (n \in S)} N/nN$ is called a *S-adic completion* of N.

Lemma 5.4.8. *(Onishi, 1984, Proposition 2.7) For any group N, $\widetilde{N}$ is complete in the S-adic topology.*

Moreover, the mapping μ defined by $\mu(a) = (a + nN)$ $(n \in S, a \in N)$ is a homomorphism from N into $\widetilde{N}$ satisfying the following three properties:

(i) $\operatorname{Ker} \mu = \operatorname{Rad} N$,

(ii) $\mu(N)$ *is neat in* $\widetilde{N}$,

(iii) $\widetilde{N}/\mu(N)$ *is divisible.*

Now we give a useful criterion from Onishi (1984) under which a neat-injective group A containing N as a neat subgroup is minimal.

Lemma 5.4.9. *(Onishi, 1984, Proposition 3.1) Let N be a group. The neat-injective group A containing N as a neat subgroup is minimal if and only if the following two conditions hold:*

(i) *$D(A)$, where $D(A)$ is the maximal divisible subgroup of a group A, is the divisible envelope of the Frattini subgroup* $\operatorname{Rad} N$ *of N;*

(ii) *A/N is divisible.*

By Lemma 5.4.9 we obtain the following Corollary.

Corollary 5.4.10. *(Onishi, 1984, Corollary 3.2) Any two minimal neat-injective groups, containing a given group N as a neat subgroup are isomorphic over N (i.e., there is an isomorphism $f : A \to B$ fixes the elements of N, where A, B are groups containing N).*

It is well known (see Fuchs (1970, Theorem 41.9)) that a minimal pure-injective group containing a given group G as a pure subgroup (i.e., the *pure-injective envelope or pure-injective hull* of G) is isomorphic to the direct sum of the divisible envelope of the Ulm subgroup G^1 of G and the $\mathbb{Z}$-adic completion of G.

The following theorem which gives the characterization of the neat-injective envelope of a given group N can be derived parallel to the Theorem 41.9 in Fuchs (1970).

Theorem 5.4.11. *(Onishi, 1984, Theorem 3.3) Let N be a group. The direct sum of the S-adic completion of N and the divisible envelope of the Frattini subgroup of N (i.e.,* $\operatorname{Rad} N$*) is a minimal neat-injective group containing N as a neat subgroup. Conversely, a minimal neat-injective group containing N as a neat subgroup is isomorphic to the direct sum of the S-adic completion of N and the divisible envelope of* $\operatorname{Rad} N$*.*

Proof. Since $\widetilde{N}$ is complete in its S-adic topology (by Lemma 5.4.8), it follows from Proposition 5.4.5 that $\widetilde{N}$ is reduced neat-injective. Let D be the divisible envelope of $\operatorname{Rad} N$. Since D is also neat-injective, $\widetilde{N} \oplus D$ is neat-injective by Onishi (1980, Proposition 4.2). Finally, we have to prove that
(i) N can be embedded as a neat subgroup into $\widetilde{N} \oplus D$ and (ii) $(\widetilde{N} \oplus D)/N$ is divisible.

Consider the mapping $\mu : N \to \widetilde{N}$ defined by $\mu(a) = (a + nN)$ for some $n \in S$ and $a \in N$. Thus (i) and (ii) follow from Lemma 5.4.8. Hence by Lemma 5.4.9, it follows that $\widetilde{N} \oplus D$ is a minimal neat-injective group containing N as a neat subgroup.

The converse follows immediately from Corollary 5.4.10. $\square$

Alizade et al. (2004) gives a structure of *neat-injective envelopes* of an abelian group A in terms of the basic subgroups$B_p(A)$ of p-component $T_p(A)$ of A and $A/T(A)$.

Definition 5.4.12. Let χ be any class of abelian groups. $I \in \chi$ is called an χ-*envelope* for an abelian group A if there is a homomorphism $\phi : A \to I$ such that the following hold:

(i) For any homomorphism $f : A \to X$ with $X \in \chi$, there is a homomorphism $g : I \to X$ such that $f = g \circ \phi$;

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"'] & I \arrow[dl, dashed, "g"] \\ X & \end{tikzcd}$$

(ii) If an endomorphism $h : I \to I$ is such that $\phi = h \circ \phi$, that is, the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\phi"'] & I \arrow[dl, dashed, "g"] \\ I & \end{tikzcd}$$

is commute, then h is an automorphism.

The following definitions and results in Alizade et al. (2004) can easily be derived parallel to the results given in Fuchs (1970, pp. 170-173) for pure-injective hull.

Let G be a neat subgroup of A, and $K(G, A)$ denote the set of all subgroup $H \leq G$, such that (i) $G \cap H = 0$ and (ii) $(G + H)/H$ is neat in A/H.

$K(G, A)$ is not empty since at least $0 \in K(G, A)$. Indeed, for $H = 0$, (i) $G \cap 0 = 0$ and (ii) $(G + 0)/0 = G$ is neat in $A = A/0$ hold.

Definition 5.4.13. A group A is called *neat-essential extension* of its subgroup G if G is neat in A, and if $K(G, A)$ consists of 0 only.

Definition 5.4.14. A group A is called *maximal neat-essential extension* of G if A' with $A \leq A'$ is never a neat-essential extension of G.

A maximal neat-essential extension of G is a minimal neat-injective group containing G as a neat subgroup.

Definition 5.4.15. A group A is called a *neat-injective envelope* of G if A is minimal neat-injective group containing G as a neat subgroup.

Definition 5.4.16. A group G is called N_t^d*-group* if all its neat extensions by a torsion divisible group split. In other words G is a N_t^d-group if $\mathrm{Next}(\mathbb{Q}/\mathbb{Z}, G) = 0$.

Proposition 5.4.17. *(Imam, 2000, Theorem 3.1.4) Let G be a reduced N_t^d-group. Then the sequence*

$$0 \longrightarrow G \longrightarrow \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \longrightarrow \mathrm{Ext}(\mathbb{Q}, G) \longrightarrow 0$$

is neat-exact with $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ *neat-injective. Moreover,* $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ *is a minimal reduced neat-injective group containing G as a neat subgroup.*

Proposition 5.4.18. *(Imam, 2000, Theorem 3.1.6) If G is an arbitrary N_t^d-group such that $G = D \oplus R$, where D is divisible and R is reduced, then*

$$0 \longrightarrow G \longrightarrow \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, R) \oplus D \longrightarrow \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, R) \oplus D/G \longrightarrow 0$$

is a neat-exact sequence with the middle group neat-injective. Moreover, $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, R) \oplus D$ *is a minimal neat-injective group containing G as a neat subgroup.*

The equivalence of general definition of χ-envelopes and the above definition of neat-injective envelopes is shown in the following proposition.

Proposition 5.4.19. *(Alizade et al., 2004, Proposition 1) Let χ be the set of all neat-injective groups and N be any group. A group A containing N is a neat-injective envelope for N if and only if A is an χ-envelope for N.*

The neat-injective envelope of cyclic groups of prime power order is given in the following Proposition.

Proposition 5.4.20. *(Alizade et al., 2004, Lemma 5) A neat-injective envelope of* $\mathbb{Z}(p^n)$ $(n \gneqq 1)$ *is isomorphic to* $\mathbb{Z}(p) \oplus \mathbb{Z}(p^\infty)$.

Let T_p be any p-group. T_p contains a p-basic subgroup B_p (by Fuchs (1970, Theorem 32.3)). Since B_p is a direct sum of groups isomorphic $\mathbb{Z}(p^n)$ we know the structure of the neat-injective envelope of B_p. On the other hand, T_p/B_p is divisible, so it is a neat-injective envelope for itself.

The following Proposition describes the neat-injective envelope of T_p in terms of those of B_p and T_p/B_p, where B_p is the basic subgroup of T_p and T_p a p-group.

Proposition 5.4.21. *(Alizade et al., 2004, Proposition 3) Let* T_p *be a* p*-group. Let* M *be a neat-injective envelope of the basic subgroup* B_p *of* T_p *and* $K = T_p/B_p$*. Then* $M \oplus K$ *is a neat-injective envelope of* T_p*.*

The following Proposition describes the neat-injective envelope of a torsion group T.

Proposition 5.4.22. *(Alizade et al., 2004, Proposition 4) Let* T *be a torsion group and for each prime* p*,* $M_p \oplus D_p$ *be neat-injective envelope for the* p*-component* T_p *of* T *with* $pM_p = 0$ *and divisible* D_p*. Then* $M = (\prod_p M_p) \oplus (\bigoplus_p D_p)$ *is a neat-injective envelope for* T*.*

The following Proposition describes the neat-injective envelope of a torsion-free group.

Proposition 5.4.23. *(Alizade et al., 2004, Theorem 1) Let* S *be a torsion-free group,* D *be an injective envelope of the Frattini subgroup* $\operatorname{Rad} S$ *of* S *and for every prime* p*,* α_p *be the rank of the* p*-basic subgroup* B_p *of* S*. Then the neat-injective envelope of* S *is isomorphic to* $E = D \oplus (\prod_p \mathbb{Z}(p)^{I_p})$*, with* $|I_p| = \alpha_p$*.*

Finally, knowing the neat-injective envelopes of torsion and torsion-free groups, we are now able to describe neat-injective envelopes of arbitrary group.

Theorem 5.4.24. *(Alizade et al., 2004, Theorem 2) Let A be any group, T be its torsion part, $S = A/T$, I be a neat-injective envelope for T, $M = \prod_p S/pS$, and D be an injective envelope for* $\operatorname{Rad} A/\operatorname{Rad} T$*. Then a neat-injective envelope for A is isomorphic to $J = I \oplus D \oplus M$.*

5.5 c-Injective Envelopes of Modules over a Dedekind Domain

It was proved that given any group can be embedded into a neat-injective group as a neat subgroup by Harrison et al. (1963, Lemma 5). The structure of neat-injective envelopes were proved by Onishi (1984), see Theorem 5.4.11. It is well known that every abelian group has a neat-injective envelope. Moreover, Imam (2007) generalize the notion of neat-injective envelope for a module over any ring R. He proved that in the case of R is a Dedekind domain, every R-module has a c-injective envelope.

Definition 5.5.1. A c-monomorphism $\alpha : L \longrightarrow M$ is called *c-essential* if every $\beta : M \longrightarrow N$, such that $\beta \circ \alpha$ is a c-monomorphism, is a monomorphism.

Definition 5.5.2. A c-essential monomorphism $\alpha : L \longrightarrow M$ is a *maximal c-essential monomorphism* if every monomorphism $\beta : M \longrightarrow N$ with $\beta \circ \alpha$ is c-essential, is an isomorphism.

Definition 5.5.3. A c-essential monomorphism $\alpha : L \longrightarrow M$ with M being c-injective is called a *c-injective envelope*.

Proposition 5.5.4. *(Imam, 2007, Proposition 2.2) If an R-module M is c-injective, then it is a maximal c-essential extension of itself.*

Proposition 5.5.5. *(see Mermut (2004, Theorem 5.2.2)) Let R be a Dedekind domain. Then $\alpha : L \longrightarrow M$ is a c-monomorphism if and only if $a \otimes 1_S : L \otimes S \longrightarrow M \otimes S$ is a monomorphism for every simple R-module S.*

Proposition 5.5.6. *(Imam, 2007, Theorem 2.3) Let R be a Dedekind domain. For every R-module M there is a maximal c-essential extension $\alpha : M \longrightarrow E$.*

Theorem 5.5.7. *(Imam, 2007, Theorem 2.5) Let R be a Dedekind domain. If $\alpha : M \longrightarrow E$ is a maximal c-essential extension, then E is a c-injective module.*

Corollary 5.5.8. *(Imam, 2007, Corollary 2.6) For a Dedekind domain R, every R-module has a c-injective envelope which is unique up to isomorphism.*

CHAPTER SIX
RAD-SUPPLEMENTED MODULES

In this chapter, we give some properties and results on Rad-supplemented modules. In first two sections, we deal with Rad-supplemented modules and weakly Rad-supplemented modules over any ring R. We also collect together the results in Wang & Ding (2006) and Türkmen & Pancar (2007) in these sections. Some of the given properties follow by adapting the related properties for supplemented modules (see Zöschinger (1974a), Zöschinger (1974b), Zöschinger (1974c), Zöschinger (1976), Wisbauer (1991) and Clark et al. (2006)). In Section 6.3, we give a characterization of Rad-supplemented modules over a discrete valuation (DVR). In the last section, we give a characterization of Rad-supplemented modules over Dedekind domains using the structure of supplemented modules over Dedekind domains in Zöschinger (1974a).

6.1 Rad-Supplemented Modules

Definition 6.1.1. Let M be an R-module. A submodule $U \leq M$ is said to have *ample Rad-supplements* in M if for every submodule $X \leq M$ with $U + X = M$, there is a Rad-supplement V of U in M such that $V \leq X$.

Definition 6.1.2. An R-module M is called *amply Rad-supplemented* if every submodule of M has ample Rad-supplements.

Such an R-module M is also called a *generalized amply supplemented* module, shortly GAS-module in Wang & Ding (2006).

The following definition can easily be derived parallel to the definition given in Smith (2000b) for totally supplemented modules.

Definition 6.1.3. An R-module M is said to be *totally Rad-supplemented* if every submodule of M is Rad-supplemented.

Proposition 6.1.4. *If a module M is supplemented, then it is Rad-supplemented.*

Proof. Follows immediately from Proposition 3.3.2. □

But the converse implication of this Proposition fails to be true. Now we shall give an example of a Rad-supplemented module which is not supplemented.

Example 6.1.5. The $\mathbb{Z}$-module $\mathbb{Q}$ is Rad-supplemented but not supplemented.

Proof. (i) We know that there is no maximal submodule of $\mathbb{Q}$ (see for example Kasch (1982, 2.3.7 Proposition)). Thus we obtain $\operatorname{Rad}\mathbb{Q} = \mathbb{Q}$ and this implies that $\mathbb{Q}$ is Rad-supplemented (see Corollary 6.1.22).

(ii) By Clark et al. (2006, 20.12.), it is proved that the $\mathbb{Z}$-module $\mathbb{Q}$ is not supplemented by showing that for a prime number p,

$$\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid p \text{ does not divide } b\}$$

does not have a supplement in $\mathbb{Q}$. □

Theorem 6.1.6. *Let M be an R-module and $V \leq M$ be coneat in M. Then for every submodule $K \leq M$ and $L \leq V$, we have:*

(i) *if $K \leq \operatorname{Rad} M$, then $K \cap V \leq \operatorname{Rad} V$.*

(ii) *if $L \leq \operatorname{Rad} M$, then $L \leq \operatorname{Rad} V$.*

(iii) *(Türkmen & Pancar (2007)) if $K \ll M$, then $K \cap V \leq \operatorname{Rad} V$.*

(iv) *if $L \ll M$ and $L$is finitely generated, then $L \ll V$.*

Proof. (i) Assume that V is a Rad-supplement of a submodule $U \leq M$ in M: Now suppose for the contrary that $K \cap V \nleq \operatorname{Rad} V$. Then there exists a maximal submodule $T \leq V$ such that $K \cap V \nleq T$. So there exists an $m \in (K \cap V) \setminus T$. Since T is a maximal submodule of V, $T + Rm = V$. Thus $M = U + V = U + T + Rm$. Since $Rm \leq K \leq \operatorname{Rad} M$, we obtain $Rm \ll M$ as Rm is a finitely generated submodule of $\operatorname{Rad} M$, and so $M = (U + T) + Rm$ implies that $U + T = M$. Then by modular law $V = V \cap M = V \cap (U + T) = (V \cap U) + T$ and since

$V \cap U \leq \operatorname{Rad} V \leq T$, we obtain $V = T$. This contradicts with T being maximal submodule of V.

(ii) Since $L \leq \operatorname{Rad} M$ we obtain by (i), $L = L \cap V \leq \operatorname{Rad} V$.

(iii) Since $K \ll M$ implies $K \leq \operatorname{Rad} M$, the result follows immediately from (i).

(iv) Since $L \ll M$ implies $L \leq \operatorname{Rad} M$, we obtain by (iii) that $L \leq \operatorname{Rad} V$. Thus $L \ll V$ since L is finitely generated by hypothesis (see Proposition 2.5.17).

□

For a submodule V of an R-module M, it is known that the property

$$\operatorname{Rad} V = V \cap \operatorname{Rad} M$$

holds if V is coclosed in M (see Clark et al. (2006, 3.7-(3))) and so if V is a supplement in M (Clark et al. (2006, 20.4-(7) and 20.2)). The following theorem shows that this property also holds when V is a Rad-supplement in M.

Theorem 6.1.7. *Let M be an R-module. If a submodule $V \leq M$ is coneat in M, then*

$$\operatorname{Rad} V = V \cap \operatorname{Rad} M.$$

Proof. For a submodule $V \leq M$, $\operatorname{Rad} V \leq V \cap \operatorname{Rad} M$ always holds. Conversely, let $x \in V \cap \operatorname{Rad} M$. Then $Rx \leq V$ and $Rx \leq \operatorname{Rad} M$. Then by Theorem 6.1.6, we obtain $Rx \leq \operatorname{Rad} V$. So $x \in \operatorname{Rad} V$ as required. □

The formula $\operatorname{Rad} V = V \cap \operatorname{Rad} M$ holds for all submodules $V \leq M$ only in the case $\operatorname{Rad} M = 0$:

Proposition 6.1.8. *Let M be an R-module. Then the following are equivalent:*

(i) *For every submodule $V \leq M$, $\operatorname{Rad} V = V \cap \operatorname{Rad} M$,*

(ii) $\operatorname{Rad} M = 0$.

Proof. $(i) \Rightarrow (ii)$: Let $x \in \operatorname{Rad} M$ and let $V = Rx$. Then $V \leq \operatorname{Rad} M$ and by hypothesis, $\operatorname{Rad} V = V \cap \operatorname{Rad} M = V$. But $V = Rx$ is cyclic, so it is finitely generated. Then $V \leq \operatorname{Rad} V$ implies $V \ll V$. Therefore, $V + 0 = V$ implies $0 = V$ since $V \ll V$. Hence $Rx = V = 0$, so $x = 0$. Thus $\operatorname{Rad} M = 0$.

$(ii) \Rightarrow (i)$: Let $V \leq M$ be a submodule, $\operatorname{Rad} V \leq V \cap \operatorname{Rad} M$ always holds. If $\operatorname{Rad} M = 0$, then $\operatorname{Rad} V \leq V \cap \operatorname{Rad} M = V \cap 0 = 0$. So $\operatorname{Rad} V = 0 = V \cap \operatorname{Rad} M$ since $\operatorname{Rad} M = 0$ by hypothesis. □

Corollary 6.1.9. *For an R-module M, if $\operatorname{Rad} M \neq 0$, then M has a submodule V_0 which is not a Rad-supplement in M (i.e., not coneat in M) for which $\operatorname{Rad} V_0 \neq V_0 \cap \operatorname{Rad} M$, thus there exists $x \in V_0$ such that for the cyclic submodule Rx of V_0, $Rx \ll M$ but $Rx \not\ll V_0$.*

Corollary 6.1.10. *Let M be an R-module. Every submodule of M is a coneat in M if and only if M is semisimple.*

Proof. $(\Leftarrow)$: It is obvious. Since every submodule of semisimple module is a direct summand, and so a Rad-supplement in M.

$(\Rightarrow)$: Since every submodule of M is Rad-supplement, for each $V \leq M$, we have $\operatorname{Rad} V = V \cap \operatorname{Rad} M$. So $\operatorname{Rad} M = 0$ by Proposition 6.1.8, and so $\operatorname{Rad} V = 0$ already. Thus V is injective with respect to monomorphism $V \hookrightarrow M$ (as V is coneat). Hence V is a direct summand of M, that is, M is semismple. □

Lemma 6.1.11. *Let V be a submodule of an R-module M and let T be a maximal submodule of V. Then the following are equivalent:*

(i) *There exists a maximal submodule L of M such that $L \cap V = T$.*

(ii) *V/T is not small in M/T, that is, the inclusion $T \leq V$ is not cosmall in M.*

Proof. $(i) \Rightarrow (ii)$: $T = L \cap V \leq L$ implies $T \leq L$. Since T is a maximal submodule of V there exists $x \in V \setminus T$. Then $x \notin L$ because if $x \in L$, then we would have $x \in L \cap V = T$, contradicting $x \notin T$. Thus $L + Rx = M$ since L is a maximal submodule of M and

$x \notin L$. Since $Rx \leq V$ as $x \in V$, we obtain $L+V=M$. Then since $T \leq V$ and $T \leq L$,

$$M/T = (L+V)/T = (L/T) + (V/T)$$

and, then $L/T \neq M/T$ because $L \neq M$ as L is maximal submodule of M. This shows that V/T is not small in M/T.

$(ii) \Rightarrow (i)$: Since V/T is not small in M/T, $(V/T)+(L/T) = M/T$ for some submodule $L \leq M$ such that $T \leq L \neq M$. Since V/T is simple as T is a maximal submodule of V, $(V/T) \cap (L/T)$ is either 0 or V/T. If $(V/T) \cap (L/T) = V/T$, then $V/T \leq L/T$ and $M/T = (V/T)+(L/T) = L/T$, contradicting $L \neq M$. Thus

$$(V \cap L)/T = (V/T) \cap (L/T) = 0,$$

and so we have $V \cap L = T$. It remains to show that L is maximal in M. Since $(V/T) \cap (L/T) = 0$, we obtain $M/T = (V/T) \oplus (L/T)$. Hence

$$M/L \cong (M/T)/(L/T) \cong V/T$$

is simple which implies that L is maximal in M. $\square$

Proposition 6.1.12. *Let M be an R-module and $V \leq M$ be coneat in M. If T is a maximal submodule of V, then there exists a maximal submodule L of M such that $L \cap V = T$.*

Proof. Suppose T is a maximal submodule of M. Then V/T is simple. Now since V is a Rad-supplement in M there exists a submodule $U \leq M$ such that $U+V=M$ and $U \cap V \leq \operatorname{Rad} V$. Let $L = U+T$. Then since $T \leq V$, $L+V = U+T+V = U+V = M$. Since T is maximal in V and $\operatorname{Rad} V$ is the intersection of all maximal submodules of V, we have $U \cap V \leq \operatorname{Rad} V \leq T$. Thus

$$L \cap V = (U+T) \cap V = (U \cap V) + T = T.$$

First proof: So,

$$(L/T) \cap (V/T) = (L \cap V)/T = T/T = 0$$

and

$$M/T = (L+V)/T = (L/T) + (V/T).$$

Thus $M/T = (L/T) \oplus (V/T)$. Since $M/L \cong (M/T)/(L/T) \cong V/T$ is simple (as T is maximal in V), we obtain that L is maximal in M.

Second proof: We have $L \neq M$ because if $L = M$, then $V = M \cap V = L \cap V = T$ contradicting T is maximal in V. Thus

$$M/T = (L + V)/T = (L/T) + (V/T)$$

where $L/T \neq M/T$. This shows that V/T is not small in M/T. Hence use Lemma 6.1.11 $\square$

.

Proposition 6.1.13. *Let S be a simple submodule of an R-module M such that S is essential in M. If M/S is Rad-supplemented, then M is Rad-supplemented.*

Proof. By hypothesis, S is contained in every nonzero submodule of M because $S \cap N \neq 0$ (as $S \trianglelefteq M$) for every nonzero submodule $N \leq M$, and since S is simple $S = S \cap N \leq N$. Of course the zero submodule and the whole module M have supplements in M. Now let $U \leq M$ be a submodule such that $U \neq 0$ and $U \neq M$. Then $S \leq U$ and since M/S is Rad-supplemented U/S has a Rad-supplement in M/S, that is, there exists a submodule V of M such that $S \leq V$, $(U + V)/S = (U/S) + (V/S) = M/S$ and $(U/S) \cap (V/S) \leq \operatorname{Rad}(V/S)$. We shall show that V is a Rad-supplement of U in M. Clearly, $U + V = M$ and $(U \cap V)/S = (U/S) \cap (V/S) \leq \operatorname{Rad}(V/S) = (\operatorname{Rad} V)/S$ since $S \leq \operatorname{Rad} V$. Thus $U \cap V \leq \operatorname{Rad} V$. Hence M is Rad-supplemented. $\square$

Theorem 6.1.14. . *Let M be an R-module and K, L be submodules of M such that $K \leq L \leq M$. Then we have:*

(i) *If $K \leq \operatorname{Rad} L$ and L/K is coneat in M/K, then L is coneat in M.*

(ii) *If K a coneat in M, then K is coneat in L.*

(iii) *If L is coneat in M, then L/K is coneat in M/K.*

(iv) *For submodules $K \leq L \leq M$, if K is coneat in L and L is coneat in M, then K is coneat M.*

Proof. (i) Say V/K is a Rad-supplement of L/K in M/K, where V is a submodule of M such that $K \leq V$. Then $(L/K) + (V/K) = M/K$ and $(L/K) \cap (V/K) \leq \operatorname{Rad}(L/K)$. Therefore we have $L + V = M$ clearly and since $K \leq \operatorname{Rad} L$,

$$(L \cap V)/K = (L/K) \cap (V/K) \leq \operatorname{Rad}(L/K) = (\operatorname{Rad} L)/K$$

(by Proposition 2.5.18). Thus $L \cap V \leq \operatorname{Rad} L$, and so L is a Rad-supplement of V in M.

(ii) Since K is a Rad-supplement in M, there exists a submodule $U \leq M$ such that $K + U = M$ and $K \cap U \leq \operatorname{Rad} K$. Then $L = L \cap M = L \cap (K + U) = K + (L \cap U)$ and $K \cap (L \cap U) = K \cap U \leq \operatorname{Rad} K$. Hence K is a Rad-supplement of $L \cap U$ in L.

(iii) Since L is a Rad-supplement in M, there exists a submodule $U \leq M$ such that $L + U = M$ and $L \cap U \leq \operatorname{Rad} L$. Then

$$(L/K) + [(U + K)/K] = (L + U + K)/K = (L + U)/K = M/K$$

and

$$(L/K) \cap [(U + K)/K] = [(L \cap U) + K]/K \leq [(\operatorname{Rad} L) + K]/K \leq \operatorname{Rad}(L/K)$$

(by Theorem 2.5.5) Hence L/K is a Rad-supplement of $(U + K)/K$ in M/K.

(iv) If $f : K \hookrightarrow L$ and $g : L \hookrightarrow M$ are coneat monomorphisms, then $h = g \circ f : K \hookrightarrow M$ is also a coneat homomorphism by the axioms of being a proper class (as $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ is form a proper class). Hence K is coneat in M.

Actually, (ii) also follows from being $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ form a proper class. □

Corollary 6.1.15. *Let M be an R-module.*

(i) *If M is Rad-supplemented, then every factor module of M is Rad-supplemented.*

(ii) *If M is Rad-supplemented, then every direct summand of M is Rad-supplemented.*

Proposition 6.1.16. *Let M be an R-module and N, K be submodules of M. Suppose $M = N + K$. If K is Rad-supplemented, then K contains a Rad-supplement of N in M.*

Proof. Since K is Rad-supplemented, the submodule $N \cap K$ of K has a Rad-supplement in K, that is, there exists a submodule $L \leq K$ such that $K = (N \cap K) + L$ and $(N \cap K) \cap L \leq \operatorname{Rad} L$. Then $M = N + K = N + (N \cap K) + L = N + L$ and $N \cap L = (N \cap K) \cap L \leq \operatorname{Rad} L$. Hence L is a Rad-supplement of N in M. $\square$

Lemma 6.1.17. *An R-module M has ample Rad-supplements in every module containing M if and only if every submodule U of M has a Rad-supplement in every module containing U.*

Proof. Following the proof for supplements in Zöschinger (1974c, Lemma 1.2), we argue as follows. ($\Leftarrow$): Let N be a module containing M: $M \leq N$. Suppose that for a submodule $X \leq N$, $X + M = N$. By hypothesis the submodule $U = X \cap M$ of M has a Rad-supplement V in the module X containing U, that is, $(X \cap M) + V = X$ and $(X \cap M) \cap V \leq \operatorname{Rad} V$. Then $N = X + M = [(X \cap M) + V] + M = V + M$ and $V \cap M = (V \cap X) \cap M = (X \cap M) \cap V \leq \operatorname{Rad} V$. Hence V is a Rad-supplement of M in N.

($\Rightarrow$): Let U be a submodule of M and N be a module containing U: $U \leq N$. Thus we can draw the pushout for the inclusion homomorphisms $i_1 : U \hookrightarrow N$, $U \hookrightarrow M$:

$$
\begin{tikzcd}
M \arrow[r, dashed, "\alpha"] & F \\
U \arrow[u, hook, "i_2"] \arrow[r, hook, "i_1"'] & N \arrow[u, dashed, "\beta"']
\end{tikzcd}
$$

where $\alpha : M \to F$, $\beta : N \to F$ are such that $\beta i_1 = \alpha i_2$. Since i_1 and i_2 are monomorphisms, α and β are also monomorphisms by the properties of pushout (see for example Fuchs (1970, §10)). Let $M' = \operatorname{Im} \alpha$ and $N' = \operatorname{Im} \beta$. Then $M' = \operatorname{Im} \alpha \cong M$, $N' = \operatorname{Im} \beta \cong N$ and $F = M' + N'$ again by the properties of pushout. So by hypothesis, $M' \cong M$ has a Rad-supplement V in the module F containing M such that $V \leq N'$: $M' + V = F$ and $M' \cap V \leq \operatorname{Rad} V$. Therefore V is a Rad-supplement of $M' \cap N'$ in N' because $N' = N' \cap F = N' \cap (M' + V) = (N' \cap M') + V$ and $V \cap (N' \cap M') = V \cap M' \leq \operatorname{Rad} V$.

We shall show that $\beta^{-1}(V)$ is a Rad-supplement of U in N. Since $\beta : N \to F$ is a monomorphism with $N' = \operatorname{Im}\beta$, we have that $\widetilde{\beta} : N \to N'$, $\widetilde{\beta}(x) = \beta(x)$, $x \in N$, is an isomorphism. By this isomorphism, since V is a Rad-supplement of $M' \cap N'$ in N', we obtain that $\widetilde{\beta}^{-1}(V)$ is a Rad-supplement of $\widetilde{\beta}^{-1}(M' \cap N')$ in $\widetilde{\beta}^{-1}(N')$. Clearly, $\widetilde{\beta}^{-1}(V) = \beta^{-1}(V)$, $\widetilde{\beta}^{-1}(N') = \beta^{-1}(N') = N$ and so it remains to show that $\widetilde{\beta}^{-1}(M' \cap N') = U$.

We have $\widetilde{\beta}^{-1}(M' \cap N') = \beta^{-1}(M' \cap N') = \beta^{-1}(M') \cap \beta^{-1}(N') = \beta^{-1}(M') \cap N$. Now let $x \in U$. Then by the pushout diagram $\alpha(x) = \alpha(i_1(x)) = (\alpha \circ i_1)(x) = (\beta \circ i_2)(x) = \beta(i_2(x)) = \beta(x)$. So $\beta(x) = \alpha(x) \in \operatorname{Im}\alpha = M'$, hence $x \in \beta^{-1}(M')$. Since $U \leq N$, $x \in N$ is clear. Thus $x \in \beta^{-1}(M') \cap N$. Conversely, let $x \in \beta^{-1}(M') \cap N$. Then $x \in N$, $\beta(x) \in M' = \operatorname{Im}\alpha = \alpha(U)$ and so $\beta(x) = \alpha(y)$ for some $y \in U$. By the pushout diagram, since $y \in U$, $\alpha(y) = \beta(y)$. So $\beta(x) = \alpha(y) = \beta(y)$. Then $\beta(x) = \beta(y)$ implies that $x = y$ since β is a monomorphism. Thus $x = y \in U$ and so $x \in U$ as desired. $\square$

Lemma 6.1.18. *If an R-module M has a Rad-supplement in every module containing M then every direct summand U of M has a Rad-supplement in every module containing U.*

Proof. Following the proof for supplements in Zöschinger (1974c, Lemma 1.3-(a)), we argue as follows. Suppose $M = A \oplus U$ for some submodule $A \leq M$ and N is a module containing U: $U \leq N$. By hypothesis $M = A \oplus U$ has a Rad-supplement in the module $A \oplus N$ containing M, that is, there exists a submodule V of $A \oplus N$ such that

$$(A \oplus U) + V = A \oplus N \quad \text{and} \quad (A \oplus U) \cap V \leq \operatorname{Rad} V.$$

Let $g : A \oplus N \to N$ be the projection onto N. $g(a, n) = n$, for all $(a, n) \in A \oplus N$. Then $U + g(V) = g(A \oplus U) + g(V) = g((A \oplus U) + V) = g(A \oplus N) = N$ and $U \cap g(V) = g(A \oplus U) \cap g(V) = g(M) \cap g(V) = g(M \cap V)$. Indeed, $g(M \cap V) \leq g(M) \cap g(V)$ always holds. Conversely, let $y \in g(M) \cap g(V)$. Then $y = g(m) = g(v)$ for some $m \in M$ and $v \in V$. Thus $g(m - v) = 0$ implies $m - v \in \operatorname{Ker} g \leq M$ since $\operatorname{Ker} g = A \oplus 0 \leq A \oplus U = M$. So $m - v \in M$. Then $v \in M$ since $m \in M$ already. Hence $y = g(v) \in g(M \cap V)$ since $v \in M \cap V$. Thus $U \cap g(V) = g(M \cap V) = g((A \oplus U) \cap V) \leq g(\operatorname{Rad} V) \leq \operatorname{Rad} g(V)$. So, $g(V)$ is a Rad-supplement of U in N. $\square$

Lemma 6.1.19. *If a module is reduced and Rad-supplemented, then it is coatomic.*

Proof. If $M = 0$, then we are done. Suppose $M \neq 0$. Since M is reduced, $\operatorname{Rad} M \neq M$. Let $U \leq M$, $U \neq M$. Since M is Rad-supplemented, U has a Rad-supplement V in M, i.e., $U + V = M$ and $U \cap V \leq \operatorname{Rad} V$. We shall consider two cases: If $U \leq \operatorname{Rad} M$, then clearly U is contained in a maximal submodule of M since $\operatorname{Rad} M \neq M$. Now suppose $U \nleq \operatorname{Rad} M$. Since $U \neq M$, we cannot have $V = 0$. Thus $\operatorname{Rad} V \neq V$ since M is reduced. So V has a maximal submodule, say T. Then

$$M/(U+T) = (U+V)/(U+T) = [(U+T)+V]/(U+T) \cong V/[V\cap(U+T)] = V/[(U\cap V)+T].$$

Since $U \cap V \leq \operatorname{Rad} V \leq T$ as T is a maximal submodule of V, we obtain that $(U \cap V) + T = T$ and so $M/(U + T) \cong V/T$. Thus $M/(U + T)$ is simple, since V/T is simple. This implies that $U + T$ is a maximal submodule of M and it clearly contains U. So in any case U is contained in a maximal submodule of M. Hence M is coatomic. □

Lemma 6.1.20. *Let V be a submodule of a module M. If $V \leq M$ is coatomic and coneat in M, then V is a supplement in M.*

Proof. Since V is a Rad-supplement in M, there exists a submodule $U \leq M$ such that $U + V = M$ and $U \cap V \leq \operatorname{Rad} V$. Since V is coatomic, $\operatorname{Rad} V \ll V$ by Proposition 2.12.5. Thus $U \cap V \ll V$. Hence V is a supplement of U in M. □

Proposition 6.1.21. *Every submodule of a module M contained in* $\operatorname{Rad} M$ *has a Rad-supplement in M.*

Proof. Let $U \leq \operatorname{Rad} M$. Then clearly, $U + M = M$ and $U \cap M = U \leq \operatorname{Rad} M$. Hence M is a Rad-supplement of U in M. □

Corollary 6.1.22. *Every radical module is Rad-supplemented.*

Proposition 6.1.23. *For a module M, $P(M)$ is Rad-supplemented.*

Proof. Since $P(M)$ is a radical module by Lemma 2.12.9, the results follows from Corollary 6.1.22. □

Theorem 6.1.24. *Let M be an R-module. Then M is Rad-supplemented if and only if $M/P(M)$ is Rad-supplemented.*

Proof. ($\Rightarrow$): It follows from Corollary 6.1.15.

($\Leftarrow$): Let $U \leq M$. Either $U \leq P(M)$ or $U \nleq P(M)$. Assume $U \leq P(M)$. Then M is a Rad-supplement of U in M: $U + M = M$ and $U = U \cap M \leq P(M)$ by assumption and so by Lemma 2.12.10, $P(M) = \operatorname{Rad}(P(M)) \leq \operatorname{Rad} M$ which implies that $U \cap M \leq P(M) \leq \operatorname{Rad} M$. Now assume $U \nleq P(M)$. Then by hypothesis, $(U + P(M))/P(M)$ has a Rad-supplement $V/P(M)$ in $M/P(M)$, where V is a submodule of M such that $P(M) \leq V$. Then

$[(U+P(M))/P(M)]+[V/P(M)] = M/P(M)$ implies $U+V = (U+P(M))+V = M$,

and

$$[(U + P(M))/P(M)] \cap [V/P(M)] \leq \operatorname{Rad}(V/P(M))$$

implies

$[(U\cap V)+P(M)]/P(M) = [(U+P(M))\cap V]/P(M) \leq \operatorname{Rad}(V/P(M)) = (\operatorname{Rad} V)/P(M)$

since $P(M) \leq \operatorname{Rad} V$ by Lemma 2.12.10. Thus

$$U \cap V \leq (U \cap V) + P(M) \leq \operatorname{Rad} V.$$

Hence V is a Rad-supplement of U in M. Thus M is Rad-supplemented. $\square$

Proposition 6.1.25. *Let M be a reduced R-module. Then M is totally Rad-supplemented if and only if it is totally supplemented.*

Proof. ($\Rightarrow$): Let $U \leq M$ be a submodule. We shall show that U is supplemented: Since M is reduced and totally Rad-supplemented, every submodule of M is reduced (by Proposition 2.12.4) and Rad-supplemented. So every submodule of M is coatomic by Lemma 6.1.19. Let K be a submodule of U. Since U is Rad-supplemented, there exists a submodule $L \leq U$ such that $K + L = U$ and $K \cap L \leq \operatorname{Rad} L$. Since every submodule of M is coatomic, the submodule L is also coatomic and so $\operatorname{Rad} L \ll L$ (by Proposition 2.12.5). Thus $K \cap L \ll L$ (by Theorem 2.4.9). Hence L is a supplement of K in U. This means that U is supplemented. Thus M is totally supplemented.

($\Leftarrow$): If every submodule of M is supplemented, then every submodule of M is Rad-supplemented by Proposition 6.1.4. Hence M is totally Rad-supplemented. $\square$

Lemma 6.1.26. *(Wang & Ding, 2006, Proposition 2.6) Let M be an R-module. If M is a Rad-supplemented module, then we have:*

(i) *Every finitely M-generated module is Rad-supplemented.*

(ii) *The module $M/(\operatorname{Rad} M)$ is semisimple.*

Theorem 6.1.27. *Let R be a left reduced ring, i.e., R is reduced as a left R-module. Then the following are equivalent:*

(i) *The ring R is left perfect,*

(ii) *The (left) R-module $R^{(\mathbb{N})}$ is Rad-supplemented,*

(iii) *every (left) R-module is Rad-supplemented.*

Proof. $(i) \Rightarrow (iii)$: Since R is left perfect, by Theorem 2.11.4 every (left) R-module is supplemented, and so Rad-supplemented.

$(iii) \Rightarrow (ii)$: Clear.

$(ii) \Rightarrow (i)$: Suppose $R^{(\mathbb{N})}$ is Rad-supplemented. Since R is reduced, $R^{(\mathbb{N})}$ is also reduced by Lemma 2.12.8. Thus $R^{(\mathbb{N})}$ is coatomic by Lemma 6.1.19. So $\operatorname{Rad}(R^{(\mathbb{N})}) \ll R^{(\mathbb{N})}$ (by Proposition 2.12.5). Since R is Rad-supplemented, $R/\operatorname{Jac}(R)$ is semisimple by Lemma 6.1.26. Hence R is left perfect by Theorem 2.11.4. $\square$

Corollary 6.1.28. *Let R be a left noetherian ring. Then R is left artinian if and only if every left R-module is Rad-supplemented.*

Proof. $(\Rightarrow)$: Since R is left artinian it is left perfect by Proposition 2.10.7. Since R is left noetherian, it is left reduced by Proposition 2.10.6. Thus every left R-module is Rad-supplemented by Theorem 6.1.27.

$(\Leftarrow)$: Suppose every left R-module is Rad-supplemented. By Proposition 2.10.6, the ring R is left reduced since it is left noetherian. R is left perfect by Theorem 6.1.27. Hence R is left artinian by Proposition 2.10.7. $\square$

We collect together the results in Wang & Ding (2006) on Rad-supplemented modules in the following theorems.

Theorem 6.1.29. *(Wang & Ding, 2006, Propositions 2.1, 2.3, 2.4, 2.5 and Remark 2.20) Let M be an R-module. If M is a Rad-supplemented module, then we have:*

(i) *If $L \leq M$ is a submodule with the property $L \cap \operatorname{Rad} M = 0$, then L is semisimple.*

(ii) *$M = N \oplus L$ for submodules N, $L \leq M$ such that N is semisimple and $\operatorname{Rad} L \trianglelefteq L$.*

(iii) *Let U be a submodule of a module K. If $M + U$ has a Rad-supplement in K, then U has a Rad-supplement in K.*

(iv) *If a module T is Rad-supplemented, then $M + T$ is Rad-supplemented.*

(v) *If $\operatorname{Rad} M$ is noetherian (or M satisfies ACC on small submodules), then M is supplemented.*

Theorem 6.1.30. *(Wang & Ding, 2006, Propositions 2.2, 2.7, 2.13, Lemma 2.10, Corollary 2.14 and Theorems 2.15, 2.16) Let M be an R-module. Then:*

(i) *If M is amply Rad-supplemented, then every direct summand of M is also amply Rad-supplemented.*

(ii) *Let $M = M_1 + M_2$ for submodules $M_1, M_2 \leq M$. If M_1 and M_2 have ample Rad-supplements in M, then $M_1 \cap M_2$ also has ample Rad-supplements in M.*

(iii) *If M is totally Rad-supplemented, then it is amply Rad-supplemented.*

(iv) *If U, V are submodules of M such that U is maximal in M and V is a Rad-supplement of U in M, then $U \cap V = \operatorname{Rad} V$ is the unique maximal submodule of V.*

(v) *For any ring R, every R-module is amply Rad-supplemented if and only if every R-module is Rad-supplemented.*

(vi) *If M is π-projective and Rad-supplemented, then it is amply Rad-supplemented.*

(vii) *M is artinian if and only if M is amply Rad-supplemented and satisfies DCC on Rad-supplement submodules and on small submodules.*

Theorem 6.1.31. *(Wang & Ding, 2006, Theorem 2.8) Let M be an R-module and $U \leq M$ be a submodule. Then the following statements are equivalent:*

(i) *There is a decomposition $M = X \oplus X'$ for submodules $X, X' \leq M$ such that $X \leq U$ and $X' \cap U \leq \operatorname{Rad} X'$,*

(ii) *There is an idempotent $e \in End\ (M)$ with $e(M) \leq U$ and $(1-e)U \leq \operatorname{Rad}((1-e)M)$,*

(iii) *There is a direct summand X of M with $X \leq U$ and $U/X \leq \operatorname{Rad}(M/X)$,*

(iv) *U has a Rad-supplement V in M such that $V \cap U$ is a direct summand of U.*

An R-module M said to have property (P^*) if for each submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \leq \operatorname{Rad}(N/K)$ (see Al-Khazzi & Smith (1991)).

Theorem 6.1.32. *(Wang & Ding, 2006, Theorem 2.19) Let M be an R-module with ACC on small submodules. Then:*

(i) *M is amply Rad-supplemented and every Rad-supplement is a direct summand of M if and only if M is lifting module.*

(ii) *M satisfies (P^*) if and only if M is lifting module.*

(iii) *If M is a π-projective Rad-supplemented module, then it is a quasi-discrete module.*

We collect together the results in Türkmen & Pancar (2007) on Rad-supplemented modules in the following theorems.

Theorem 6.1.33. *(Türkmen & Pancar, 2007) Let M be an R-module and $U, V \leq M$ be submodules. Then we have:*

(i) *V is a Rad-supplement of U in M if and only if $U + V = M$ and $Rm \ll V$ for every $m \in U \cap V$.*

(ii) *If V is a Rad-supplement of U in M, then:*

(a) if $L \leq U$ and $L + V = M$, then V is also a Rad-supplement of L in M.

(b) if $K \ll V$ for a submodule $K \leq M$, then V is also a Rad-supplement of $U + K$ in M.

(c) if K is a maximal submodule of V, then $U + K$ is a maximal submodule of M.

(iii) *Let $p : M \to M/\operatorname{Rad} M$ be the canonical epimorphism. If V is a Rad-supplement of U in M and $U \leq \operatorname{Rad} M$, then*

$$M/\operatorname{Rad} M = p(U) \oplus p(V).$$

(iv) *If U has a nonzero Rad-supplement in M and $\operatorname{Rad} M \ll M$, then U is contained in a maximal submodule of M.*

Definition 6.1.34. (Türkmen & Pancar, 2007) An R-module M is called *f-Rad-supplemented* if every finitely generated submodule of M has a Rad-supplement in M.

Theorem 6.1.35. *(Türkmen & Pancar, 2007) Let M be an f-Rad-supplemented.*

(i) *If L is a finitely generated submodule of M, then M/L is also f-Rad-supplemented.*

(ii) *If $\operatorname{Rad} M$ is finitely generated, then every finitely generated submodule of $M/\operatorname{Rad} M$ is a direct summand of $M/\operatorname{Rad} M$.*

6.2 Weakly Rad-Supplemented Modules

Definition 6.2.1. Let M be an R-module and let L be a submodule of M. A submodule $N \leq M$ is said to be a *weak Rad-supplement* of L (or L is said to be a weak Rad-supplement of N) in M or L is said to have a weak Rad-supplement N in M if

$$N + L = M \quad \text{and } N \cap L \leq \operatorname{Rad} M.$$

Definition 6.2.2. (Wang & Ding, 2006, Definition 3.1) An R-module M is called *weakly Rad-supplemented* if every submodule of M has a weak Rad-supplement.

Such an R-module is also called a *weakly generalized supplemented* or briefly a WGS-module in Wang & Ding (2006).

Theorem 6.2.3. *(Wang & Ding, 2006, Propositions 3.2, 3.7 and Lemma 3.6) Let M be a weakly Rad-supplemented module. Then we have:*

(i) *Every supplement submodule of M is weakly Rad-supplemented.*

(ii) *For a module N, if $f : N \to M$ is a small cover of M, then N is also weakly Rad-supplemented.*

(iii) *Every factor module of M is weakly Rad-supplemented.*

(iv) *If M is a submodule of a module L such that a submodule K of L is also weakly Rad-supplemented, then $M + K$ is weakly Rad-supplemented.*

(v) *Suppose M is a submodule of a module N. If $M + U$ has a weak Rad-supplement in N, then U has a weak Rad-supplement in N for every submodule $U \leq N$.*

Theorem 6.2.4. *(Wang & Ding, 2006, Theorem 3.9) Let R be a ring and M be an R-module. The following statements are equivalent:*

(i) *R is semilocal,*

(ii) *If $\operatorname{Rad} M \ll M$, then M is weakly Rad-supplemented,*

(iii) *If M is finitely generated, then it is weakly Rad-supplemented,*

(iv) *If M is cyclic, then it is weakly Rad-supplemented.*

We collect together the results in Türkmen & Pancar (2007) on weakly Rad-supplemented modules in the following theorem.

Definition 6.2.5. (Türkmen & Pancar, 2007) Let M be an R-module. A submodule $L \leq M$ is called *Rad-coclosed* in M if L has no proper submodule K such that $L/K \leq \operatorname{Rad}(M/K)$.

Theorem 6.2.6. *(Türkmen & Pancar, 2007) Let U and V be submodules of a module M. Then:*

(i) V *is a weak Rad-supplement of* U *in* M *if and only if* $U+V=M$ *and* $Rm \ll M$ *for every* $m \in U \cap V$.

(ii) *If* $U \leq V$, V *is Rad-coclosed in* M *and* $U \leq \operatorname{Rad} M$, *then* $U \leq \operatorname{Rad} V$.

(iii) *If* V *is Rad-coclosed in* M, *then* $\operatorname{Rad} V = V \cap \operatorname{Rad} M$.

(iv) *Suppose* V *is a weak Rad-supplement of* U *in* M. *If* V *is Rad-coclosed in* M *or a direct summand in* M, *then* V *is a Rad-supplement of* U *in* M.

(v) *suppose* $U \leq V$.*Then* $V \leq \operatorname{Rad} M$ *if and only if* $U \leq \operatorname{Rad} M$ *and* $V/U \leq \operatorname{Rad}(M/U)$.

(vi) *If* $U \leq \operatorname{Rad} M$ *and* M/U *is weakly Rad-supplemented, then* M *is weakly Rad-supplemented.*

(vii) *If* M *is weakly Rad-supplemented and* V *is Rad-coclosed in* M, *then* V *is also weakly Rad-supplemented.*

Proposition 6.2.7. *(Türkmen & Pancar, 2007) Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of modules.

(i) *If* L *and* N *are weakly Rad-supplemented and* L *has a weak Rad-supplement in* M, *then* M *is weakly Rad-supplemented.*

(ii) *If* M *is weakly Rad-supplemented and* L *is Rad-coclosed in* M, *then* L *and* N *are weakly Rad-supplemented.*

Theorem 6.2.8. *(by Clark et al. (2006, 17.2)) For an R-module* M, *the following statements are equivalent:*

(i) M *is semilocal;*

(ii) M *is weakly Rad-supplemented;*

(iii) *There is a decomposition* $M = M_1 \oplus M_2$ *such that* M_1 *is semisimple and* M_2 *is semilocal with* $\operatorname{Rad} M \trianglelefteq M_2$.

Proof. $(i) \Rightarrow (ii)$: Let $L \leq M$ be a submodule. Since M is semilocal, $M/\operatorname{Rad} M$ is semisimple. So $(L + \operatorname{Rad} M)/\operatorname{Rad} M$ is a direct summand of $M/\operatorname{Rad} M$. Then there exists a submodule $K \leq M$ with $\operatorname{Rad} M \leq K$ such that

$$[(L + \operatorname{Rad} M)/\operatorname{Rad} M] \oplus (K/\operatorname{Rad} M) = M/\operatorname{Rad} M.$$

Then we obtain $L + K = L + K + \operatorname{Rad} M = M$ since $\operatorname{Rad} M \leq K$. Since

$$[(L + \operatorname{Rad} M) \cap K]/\operatorname{Rad} M = 0,$$

$$(L \cap K) + \operatorname{Rad} M = (L + \operatorname{Rad} M) \cap K = \operatorname{Rad} M.$$

Thus $L \cap K \leq \operatorname{Rad} M$ and so K is a weak Rad-supplement of L in M. Hence M is weakly Rad-supplemented.

$(ii) \Rightarrow (i)$: Let $L/\operatorname{Rad} M$ be a submodule of $M/\operatorname{Rad} M$ with $\operatorname{Rad} M \leq L$. Since M is weakly Rad-supplemented there is a submodule $K \leq M$ such that $L + K = M$ and $L \cap K \leq \operatorname{Rad} M$. Thus

$$M/\operatorname{Rad} M = (L + K)/\operatorname{Rad} M = (L/\operatorname{Rad} M) + (K + \operatorname{Rad} M)/\operatorname{Rad} M$$

and

$$(L/\operatorname{Rad} M) \cap [(K + \operatorname{Rad} M)/\operatorname{Rad} M] = [L \cap (K + \operatorname{Rad} M)]/\operatorname{Rad} M = 0$$

because $L \cap (K + \operatorname{Rad} M) = (L \cap K) + \operatorname{Rad} M = \operatorname{Rad} M$, since $\operatorname{Rad} M \leq L$ and $L \cap K \leq \operatorname{Rad} M$. So

$$M/\operatorname{Rad} M = (L/\operatorname{Rad} M) \oplus [(K + \operatorname{Rad} M)/\operatorname{Rad} M].$$

Hence every submodule of $M/\operatorname{Rad} M$ is a direct summand. So $M/\operatorname{Rad} M$ is semisimple which implies that M is semilocal.

$(i) \Rightarrow (iii)$: Let M_1 be a complement of $\operatorname{Rad} M$ in M. Then $M_1 \oplus \operatorname{Rad} M$ is essential in M by Theorem 2.4.12. Since $M/\operatorname{Rad} M$ is semisimple, $M_1 \cong (M_1 \oplus \operatorname{Rad} M)/\operatorname{Rad} M$ is a direct summand in $M/\operatorname{Rad} M$, hence semisimple (see Corollary 2.4.7), and there is a semisimple submodule $M_2/\operatorname{Rad} M$ where $M_2 \leq M$ with $\operatorname{Rad} M \leq M_2$ such that

$$[(M_1 \oplus \operatorname{Rad} M)/\operatorname{Rad} M] \oplus (M_2/\operatorname{Rad} M) = M/\operatorname{Rad} M.$$

So $(M_1 \oplus \operatorname{Rad} M) + M_2 = M$ and $(M_1 \oplus \operatorname{Rad} M) \cap M_2 = \operatorname{Rad} M$. Since $\operatorname{Rad} M \leq M_2$, we have $M = M_1 + \operatorname{Rad} M + M_2 = M_1 + M_2$ and $\operatorname{Rad} M = (M_1 \oplus \operatorname{Rad} M) \cap M_2 = (M_1 \cap M_2) \oplus \operatorname{Rad} M$ which implies that $M_1 \cap M_2 = 0$. Thus $M = M_1 \oplus M_2$. Since M_1 is a complement in M, $\operatorname{Rad} M$ is essential in M_2. Indeed, suppose for the contrary that $\operatorname{Rad} M \ntrianglelefteq M_2$. Then there exists a nonzero submodule $L \leq M_2$ such that $L \cap \operatorname{Rad} M = 0$. Thus we obtain $(M_1 \oplus L) \cap \operatorname{Rad} M = 0$ because if $x = m_1 + l \in (M_1 \oplus L) \cap \operatorname{Rad} M$ for some $x \in \operatorname{Rad} M \leq M_2$, $m_1 \in M_1$ and $l \in L \leq M_2$, then $x - l = m_1 \in M_1 \cap M_2 = 0$ and so $x = l \in L \cap \operatorname{Rad} M = 0$, i.e., $x = 0$. But this contradicts the fact that M_1 is a complement of $\operatorname{Rad} M$.

$(iii) \Rightarrow (i)$: We shall show that $M/\operatorname{Rad} M$ is semisimple.

$$M/\operatorname{Rad} M = (M_1 \oplus M_2)/\operatorname{Rad} M = [(M_1 \oplus \operatorname{Rad} M)/\operatorname{Rad} M] \oplus (M_2/\operatorname{Rad} M)$$

since $\operatorname{Rad} M \leq M_2$. Then $M/\operatorname{Rad} M \cong M_1 \oplus (M_2/\operatorname{Rad} M)$. By assumption M_1 is semisimple, so it suffices to show that $M_2/\operatorname{Rad} M$ is semisimple (by Corollary 2.4.7). But we have,

$$M_2/\operatorname{Rad} M \cong (M_2/\operatorname{Rad} M_2)/(\operatorname{Rad} M/\operatorname{Rad} M_2)$$

and $M_2/\operatorname{Rad} M_2$ is semisimple since M_2 is semilocal by hypothesis. Thus

$$M_2/\operatorname{Rad} M \cong (M_2/\operatorname{Rad} M_2)/(\operatorname{Rad} M/\operatorname{Rad} M_2)$$

is also semisimple since epimorphic image of a semisimple module is again semisimple (by Corollary 2.4.7). □

Proposition 6.2.9. *Every Rad-supplemented module is weakly Rad-supplemented.*

Proof. Let M be a Rad-supplemented module and U be a submodule of M. We shall show that U has a weak Rad-supplement in M. Since M is Rad-supplemented there exists a submodule $V \leq M$ such that $U + V = M$ and $U \cap V \leq \operatorname{Rad} V$. Thus clearly, $U \cap V \leq \operatorname{Rad} M$ since $\operatorname{Rad} V \leq \operatorname{Rad} M$ already. Hence V is a weak Rad-supplement of U in M. □

Proposition 6.2.10. *Let V be a submodule of a module M. Then the following are equivalent:*

(i) *V is a weak Rad-supplement in M and* $\operatorname{Rad} V = V \cap \operatorname{Rad} M$*,*

(ii) *V is a Rad-supplement in M.*

Proof. $(i) \Rightarrow (ii)$: By hypothesis, V is a weak Rad-supplement of a submodule $U \leq M$. Then $U+V=M$ and $U \cap V \leq \operatorname{Rad} M$. Thus $U \cap V = V \cap (U \cap V) \leq V \cap \operatorname{Rad} M = \operatorname{Rad} V$ by hypothesis. Hence V is a Rad-supplement of U in M.

$(ii) \Rightarrow (i)$: Suppose V is a Rad-supplement of U in M, that is, $U+V=M$ and $U \cap V \leq \operatorname{Rad} V$. Since $\operatorname{Rad} V \leq \operatorname{Rad} M$, V is clearly a weak Rad-supplement of U in M. Since V is a Rad-supplement in M, it follows immediately from Theorem 6.1.7 that $\operatorname{Rad} V = V \cap \operatorname{Rad} M$. □

Proposition 6.2.11. *Let M be a weakly Rad-supplemented module. If a submodule $V \leq M$ satisfies the property* $\operatorname{Rad} V = V \cap \operatorname{Rad} M$*, then V is weakly Rad-supplemented.*

Proof. Let $T \leq V$. Since M is weakly rad-supplemented, there exists a submodule N of M such that N is a weak Rad-supplement of T in M. Then $T+N=M$ and $T \cap N \leq \operatorname{Rad} M$. Thus by modular law $V = V \cap M = V \cap (T+N) = T + (V \cap N)$ since $T \leq V$, and we have $T \cap (N \cap V) \leq T \cap N \leq \operatorname{Rad} M$. Since $T \cap (N \cap V) \leq V$ clearly, we obtain $T \cap (N \cap V) \leq V \cap \operatorname{Rad} M = \operatorname{Rad} V$ by hypothesis. Hence $N \cap V$ is a weak Rad-supplement of T in V. This shows that V is weakly Rad-supplemented. □

Proposition 6.2.12. *Let M be a module and N, L be submodules of M such that $L \leq N$ and $L \leq \operatorname{Rad} M$. Assume N/L is weakly Rad-supplemented. If $K \leq M$ is a submodule such that $N+K$ has a weak Rad-supplement in M, then K also has a weak Rad-supplement in M.*

Proof. Following the proof for weakly supplemented modules in Clark et al. (2006, Lemma 17.11), we argue as follows. Let $X \leq M$ be a weak Rad-supplement of $N+K$. Then $(N+K)+X=M$ and $(N+K) \cap X \leq \operatorname{Rad} M$. We have two cases. Firstly, suppose $N \cap (K+X) \leq L$. Then by hypothesis $N \cap (K+X) \leq \operatorname{Rad} M$. Thus

$$(N+X) \cap K \leq N \cap (K+X) + X \cap (N+K) \leq \operatorname{Rad} M,$$

and since $(N+X)+K=(N+K)+X=M$, we obtain that $N+X$ is a weak Rad-supplement of K in M. On the other hand, suppose $N \cap (K+X) \nsubseteq L$ and let Y/L be a weak Rad-supplement of $[L+N \cap (K+X)]/L$ in N/L for some submodule $Y \leq N$ such that $L \leq Y$. Then clearly we have

$$N = Y + L + [N \cap (K+X)] = Y + [N \cap (K+X)],$$

and

$$Y \cap [L + N \cap (K+X)]/L \leq \operatorname{Rad}(N/L) \leq \operatorname{Rad}(M/L) = (\operatorname{Rad} M)/L$$

since $L \leq \operatorname{Rad} M$ (see Proposition 2.5.18). Therefore

$$\operatorname{Rad} M \geq Y \cap [L + N \cap (K+X)] = L + [Y \cap (N \cap (K+X))] = L + [Y \cap (K+X)].$$

So $Y \cap (K+X) \operatorname{Rad} M$. Thus

$$M = N + (K+X) = [Y + (N \cap (K+X))] + (K+X) = (Y+X) + K$$

and

$$(Y+X) \cap K \leq Y \cap (K+X) + X \cap (Y+K) \leq Y \cap (K+X) + X \cap (K+N)$$

since $Y \leq N$. Hence $(Y+X) \cap K \leq \operatorname{Rad} M$ since $Y \cap (K+X) \leq \operatorname{Rad} M$ and $X \cap (K+N) \leq \operatorname{Rad} M$. This shows that $Y+X$ is a weak Rad-supplement of K in M. □

Proposition 6.2.13. *Let M be a module and K, N be submodules of M such that $K \leq N$. If $K \xhookrightarrow[M]{cs} N$ and N has a weak Rad-supplement in M, then $N = K + S$ for some submodule $S \leq N$ such that $S \leq \operatorname{Rad} M$.*

Proof. Following the proof for weak supplements in Clark et al. (2006, Lemma 17.9-(5)), we argue as follows. Suppose L is a weak Rad-supplement of N in M. Then $N+L=M$ and $N \cap L \leq \operatorname{Rad} M$. Since $K \xhookrightarrow[M]{cs} N$, we obtain $K+L=M$ (see Definition 2.13.1). By modular law,

$$N = N \cap M = N \cap (K+L) = K + (N \cap L).$$

Hence for $S = N \cap L$, we have $N = K+S$ and $S \leq \operatorname{Rad} M$. □

Theorem 6.2.14. *Let M be a weakly Rad-supplemented module. Then:*

(i) *If $M = A + B$ for some submodules A and B of M, then A has a weak Rad-supplement C in M such that $C \leq B$.*

(ii) *If every submodule of M has a coclosure in M, then M is amply Rad-supplemented.*

Proof. Following the proof for weakly supplemented modules in Clark et al. (2006, Lemmas 17.9-(6) and 20.25), we argue as follows.

(i) Suppose $M = A+B$. As M is weakly Rad-supplemented, there exists a weak Rad-supplement K of $A \cap B$ in M, that is, $M = K + (A \cap B)$ and $K \cap (A \cap B) \leq \operatorname{Rad} M$. Now $B = B \cap M = B \cap (K + (A \cap B)) = (B \cap K) + (A \cap B)$. Then $M = A + B = A + (B \cap K) + (A \cap B) = A + (B \cap K)$ and $A \cap (B \cap K) = K \cap (A \cap B) \leq \operatorname{Rad} M$. Hence $C = B \cap K \leq B$ is a weak Rad-supplement of A in M.

(ii) Let K, $L \leq M$ with $M = K + L$. Since M is weakly Rad-supplemented, there exists a weak Rad-supplement T of K in M such that $T \leq L$ by (i). Then $T + K = M$ and $T \cap K \leq \operatorname{Rad} M$. By hypothesis, there exists a coclosure of T in M, say N. Since $N \xhookrightarrow[M]{cs} T$, $T + K = M$ implies $N + K = M$. We also have $N \cap K \leq T \cap K \leq \operatorname{Rad} M$. Since $N \xhookrightarrow{cc} M$, $\operatorname{Rad} N = N \cap \operatorname{Rad} M$ (see Proposition 2.13.4). Thus we obtain $N \cap K \leq N \cap \operatorname{Rad} M = \operatorname{Rad} N$. Hence N is a Rad-supplement of K in M such that $N \leq L$.

□

Proposition 6.2.15. *If a module M is reduced and Rad-supplemented, then it is weakly supplemented.*

Proof. Let $U \leq M$. Since M is Rad-supplemented, there exists a Rad-supplement $V \leq M$ of U in M. Then $U + V = M$ and $U \cap V \leq \operatorname{Rad} V$. Since M is coatomic by Lemma 6.1.19, we have $\operatorname{Rad} M \ll M$ by Proposition 2.12.5. So $U \cap V \leq \operatorname{Rad} V \leq \operatorname{Rad} M \ll M$ implies that $U \cap V \ll \operatorname{Rad} M$ (see Theorem 2.5.5). Thus M is weakly supplemented. □

Proposition 6.2.16. *Let M be an R-module with small radical (i.e.,* $\operatorname{Rad} M \ll M$*).*

(i) *If M is Rad-supplemented, then it is weakly supplemented.*

(ii) *M is weakly supplemented if and only if it is weakly Rad-supplemented.*

Proof. (i) Let $U \leq M$ be a submodule. We shall show that U has a weak supplement in M. Since M is Rad-supplemented there exists a submodule $V \leq M$ such that $U + V = M$ and $U \cap V \leq \operatorname{Rad} V$. This implies $U \cap V \leq \operatorname{Rad} M$. Thus $U \cap V \ll M$ since $\operatorname{Rad} M \ll M$ by hypothesis. Hence V is also a weak supplement of U in M.

(ii) ($\Rightarrow$): Let $U \leq M$. Since M is weakly supplemented there exists a submodule $V \leq M$ such that $U + V = M$ and $U \cap V \ll M$. Then it follows immediately that $U \cap V \leq \operatorname{Rad} M$. Thus V is a weak Rad-supplement of U in M. Hence M is weakly Rad-supplemented.

($\Leftarrow$): Let $U \leq M$. Since M is weakly Rad-supplemented there exists a submodule $V \leq M$ such that $U + V = M$ and $U \cap V \leq \operatorname{Rad} M$. Since $\operatorname{Rad} M \ll M$, we obtain that $U \cap V \ll M$. Hence V is a weak supplement of U in M.

$\square$

The following definitions can easily be derived parallel to the definitions given in Alizade et al. (2001) and Alizade & Büyükaşık (2003) for supplemented modules.

Definition 6.2.17. A submodule N of an R-module M is called *cofinite* if M/N is finitely generated (see Alizade et al. (2001)).

An R-module M is called *cofinitely Rad-supplemented* if every cofinite submodule of M has a Rad-supplement in M.

Definition 6.2.18. An R-module M is called *cofinitely weakly Rad-supplemented* if every cofinite submodule of M has a weak Rad-supplement in M.

The class of cofinitely weakly supplemented modules is closed under homomorphic images and small covers (see Clark et al. (2006, 18.5-(1))). Similar result holds for cofinitely weakly Rad-supplemented modules as the following lemma shows:

Lemma 6.2.19. *Let M be a cofinitely weakly Rad-supplemented module. Then:*

(i) *M/N is cofinitely weakly Rad-supplemented for every submodule $N \leq M$.*

(ii) *Let S be a module and $f : S \to M$ be a small cover of M, then S is cofinitely weakly Rad-supplemented.*

Proof. (i) Let T/N be a cofinite submodule of M/N, where T is a submodule of M such that $N \leq T$. Then $M/T \cong (M/N)/(T/N)$ is finitely generated, and so T is a cofinite submodule of M. Since M is cofinitely weakly Rad-supplemented, there exists a weak Rad-supplement L of T in M. Then $T+L = M$ and $T \cap L \leq \operatorname{Rad} M$. Thus

$$M/N = (T+L)/N = (T/N) + [(L+N)/N]$$

and

$$(T/N) \cap [(L+N)/N] = [T \cap (L+N)]/N = [(T \cap L) + N]/N \leq (N + \operatorname{Rad} M)/N,$$

and so we obtain $(T/N) \cap [(L+N)/N] \leq \operatorname{Rad}(M/N)$ since by Theorem 2.5.5 $(N + \operatorname{Rad} M)/N \leq \operatorname{Rad}(M/N)$. This shows that $(L+N)/N$ is a weak Rad-supplement of T/N in M/N. Thus M/N is cofinitely weakly Rad-supplemented.

(ii) Let $f : S \to M$ be a small cover with $K = \operatorname{Ker} f$ ($K \ll S$). Then $S/K \cong \operatorname{Im} f = M$. Now let $L \leq S$ be a cofinite submodule. Then S/L is finitely generated. We shall show that L has a weak Rad-supplement in S. Now consider the epimorphism $g : S/L \to S/(L+K)$, $g(x+L) = x + (L+K)$ for all $x \in S$. Since S/L is finitely generated, its epimorphic image $S/(L+K)$ is also finitely generated. Thus $L+K$ is a cofinite submodule of S and so $(L+K)/K$ is a cofinite submodule of S/K because $(S/K)/(L+K)/K \cong S/(L+K)$ is finitely generated. Since $S/K \cong M$ is cofinitely weakly Rad-supplemented, $(L+K)/K$ has a weak Rad-supplement T/K in S/K, where T is a submodule of N such that $K \leq T$. Then

$$[(L+K)/K] + (T/K) = S/K$$

and

$$[(L+K)/K] \cap (T/K) \leq \operatorname{Rad}(S/K) = (\operatorname{Rad} S)/K$$

since $K \ll S$ (see Proposition 2.5.18). Therefore we obtain $L + K + T = S$ and so $L + T = S$ since $K \ll S$. Finally, since $(L \cap T) + K = (L + K) \cap T \leq \operatorname{Rad} S$, we obtain that $L \cap T \leq \operatorname{Rad} S$. Hence T is a weak Rad-supplement of L in S as desired.

□

Proposition 6.2.20. *If an R-module M is cofinitely weakly Rad-supplemented, then every cofinite submodule of $M/\operatorname{Rad} M$ is a direct summand.*

Proof. Following the proof for supplemented modules in Clark et al. (2006, 18.5-(2)), we argue as follows. By Lemma 6.2.19, $M/\operatorname{Rad} M$ is cofinitely weakly Rad-supplemented . Let $T/\operatorname{Rad} M$ be a cofinite submodule of $M/\operatorname{Rad} M$, where T is a submodule of M such that $\operatorname{Rad} M \leq T$. Then T is a cofinite submodule of M, since $M/T \cong (M/\operatorname{Rad} M)/(T/\operatorname{Rad} M)$ is finitely generated. So by hypothesis T has a weak Rad-supplement L in M, that is, $T + L = M$ and $T \cap L \leq \operatorname{Rad} M$. Since

$$(T/\operatorname{Rad} M) \cap [(L + \operatorname{Rad} M)/\operatorname{Rad} M] = [\operatorname{Rad} M + (T \cap L)]/\operatorname{Rad} M = 0$$

as $T \cap L \leq \operatorname{Rad} M$, we obtain that

$$M/\operatorname{Rad} M = (T + L)/\operatorname{Rad} M = (T/\operatorname{Rad} M) \oplus [(L + \operatorname{Rad} M)/\operatorname{Rad} M].$$

Hence $T/\operatorname{Rad} M$ is a direct summand of $M/\operatorname{Rad} M$. □

Proposition 6.2.21. *Let M be an R-module and $N, L \leq M$ be submodules such that N is cofinite, L is cofinitely Rad-supplemented and $N + L$ has a Rad-supplement in M. Then N has a Rad-supplement in M.*

Proof. Following the proof for supplemented modules in Clark et al. (2006, 20.18-(1)), we argue as follows. Let K be a Rad-supplement of $N+L$ in M, that is, $M = K+N+L$ and $K \cap (N + L) \leq \operatorname{Rad} K$. Then $L \cap (N + K)$ is a cofinite submodule of L. Indeed, consider the epimorphism

$$f : M/N \to M/(N + K), \quad f(x + N) = x + (N + K) \quad (x \in M).$$

Since M/N is finitely generated as $N \leq M$ is cofinite its epimorphic image $M/(N+K)$ is also finitely generated. So the result follows from

$$M/(N+K) = [L + (N+K)]/(N+K) \cong L/[L \cap (N+K)].$$

By hypothesis there exists a Rad-supplement H of $L \cap (N+K)$ in L. Then $L = H + [L \cap (N+K)]$ and $H \cap (N+K) = H \cap [L \cap (N+K)] \leq \operatorname{Rad} H$. Finally, we shall show that $H+K$ is a Rad-supplement of N in M. Clearly,

$$M = K + N + L = (N+K) + [H + L \cap (N+K)] = N + (H+K)$$

and

$$N \cap (H+K) \leq [H \cap (N+K)] + [K \cap (N+H)] \leq [H \cap (N+K)] + [K \cap (N+L)].$$

Hence $N \cap (H+K) \leq \operatorname{Rad} H + \operatorname{Rad} K \leq \operatorname{Rad}(H+K)$ which completes the proof. □

Lemma 6.2.22. *Let R be a commutative noetherian ring and M be an R-module. If M is reduced and Rad-supplemented, then M is supplemented.*

Proof. Let $U \leq M$. Since M is Rad-supplemented, there exists a Rad-supplement V of U in M, i.e., $U + V = M$ and $U \cap V \leq \operatorname{Rad} V$. We shall show that V is a supplement of U in M. For this, it suffices to show that $\operatorname{Rad} V \ll V$. By Lemma 6.1.19, M is coatomic. Since R is a commutative noetherian ring V is also coatomic by Proposition 2.12.12. Then Proposition 2.12.5, $\operatorname{Rad} V \ll V$. □

6.3 Rad-Supplemented Modules over a Discrete Valuation Ring

The next lemma exhibits a useful link between the coatomic modules and supplemented modules over a discrete valuation ring (DVR). It also gives a characterization of coatomic modules over a DVR.

Lemma 6.3.1. *(Zöschinger, 1974a, Lemma2.1) Let R be a DVR with quotient field $K \neq R$, and p be the unique prime element. Then the following are equivalent:*

(i) *M has a small radical (i.e., $\operatorname{Rad} M \ll M$);*

(ii) *M is coatomic;*

(iii) *M is a direct sum of a finitely generated free and a bounded submodule;*

(iv) *M is reduced and supplemented.*

The following theorem gives a characterization of supplemented modules over a DVR.

Theorem 6.3.2. *(Zöschinger, 1974a, Theorem 2.4) Let R be a DVR with quotient field $K \neq R$, and p be the unique prime element. Then an R-module M is supplemented if and only if $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$, where $M_1 \cong R^{n_1}$, $M_2 \cong K^{n_2}$, $M_3 \cong (K/R)^{n_3}$, and $M_4 p^{n_4} = 0$ for some nonnegative integers n_1, n_2, n_3, n_4.*

Over a DVR, supplemented modules and Rad-supplemented modules does not coincide as the following example shows:

Example 6.3.3. Let R be a DVR which is not a field, let K be its field of quotients. The R-module $M = K^{(\mathbb{N})}$ is Rad-supplemented by Corollary 6.1.22 since $\operatorname{Rad} M = M$. Indeed, $\operatorname{Rad} M = \operatorname{Rad} K^{(\mathbb{N})} = \bigoplus_{i\in\mathbb{N}} \operatorname{Rad} K = \bigoplus_{i\in\mathbb{N}} K$ since $\operatorname{Rad} K = K$ as K is divisible (see Theorem 2.9.14). But it is not supplemented by Theorem 6.3.2.

Now we give the main result of this section, using the structure of coatomic modules, in the following theorem.

Theorem 6.3.4. *Let R be a DVR with quotient field $K \neq R$, and p be the unique prime element. Then M is Rad-supplemented if and only if $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$, where $M_1 \cong R^a$, $M_2 \cong K^{(I_1)}$, $M_3 \cong (K/R)^{(I_2)}$, and $M_4 p^b = 0$ for some integers $a, b \geq 0$ and arbitrary index sets I_1, I_2.*

Proof. ($\Rightarrow$): Suppose M is Rad-supplemented. M can be described as follows, $M = M_1' \oplus M_2'$ for submodules M_1' and M_2' of M such that M_1' is the divisible part of M and M_2' is reduced (see Theorem 2.9.12). Since every direct summand of a Rad-supplemented module is again Rad-supplemented by Corollary 6.1.15, M_2' is Rad-supplemented. Thus by Lemma 6.1.19, M_2' is coatomic. Then by Lemma 6.3.1, $M_2' = M_1 \oplus M_4$ for some submodules M_1 and M_4 of M such that $M_1 \cong R^a$ for some integer $a \geq 0$ and M_4 is bounded, say $p^b M = 0$ for some integer $b \geq 0$. Since M_1' is divisible, $M_1' = M_2 \oplus M_3$, where $M_2 \cong K^{(I_1)}$ and $M_3 \cong (K/R)^{(I_2)}$ for some index sets I_1, I_2 (see Theorem 2.9.11). Hence $M = M_1 \oplus M_2 \oplus M_3 \oplus M_4$ is required form.

($\Leftarrow$): Let $M_1' = M_2 \oplus M_3$ and $M_2' = M_1 \oplus M_4$. Since M_1' is a divisible module $\operatorname{Rad} M_1' = M_1'$ by Theorem 2.9.14, i.e., M_1' is a radical module. So by Corollary 6.1.22, M_1' is Rad-supplemented. By Theorem 6.3.2, M_2' is supplemented and so it is Rad-supplemented, too (by Proposition 6.1.4). □

Corollary 6.3.5. *Let R be a DVR and M be an R-module. If M is a reduced module, then M is supplemented if and only if M is Rad-supplemented.*

Corollary 6.3.6. *Let R be a DVR. Let M be an R-module and $D(M)$ be the divisible part of M: $M = D(M) \oplus M'$ for some submodule M' such that M' is reduced. Then the following are equivalent:*

(i) *M is Rad-supplemented,*

(ii) *M' is supplemented,*

(iii) *$M/D(M)$ is supplemented.*

Proof. *(i)* $\Rightarrow$ *(ii)*: Since M is Rad-supplemented M' is also Rad-supplemented by Corollary 6.1.15. Thus M' is supplemented by Corollary 6.3.5.

(ii) $\Rightarrow$ *(iii)*: Since M' is supplemented, $M/D(M) \cong M'$ is supplemented.

(iii) $\Rightarrow$ *(i)*: Since $M' \cong M/D(M)$ is supplemented, M' is Rad-supplemented (by Proposition 6.1.4). Since $\operatorname{Rad} D(M) = D(M)$ (by Theorem 2.9.14), $D(M)$ is also Rad-supplemented by Corollary 6.1.22. Hence $M = D(M) \oplus M'$ is Rad-supplemented (by Theorem 6.1.29). □

6.4 Rad-Supplemented Modules over a Dedekind Domain

The next theorem gives a characterization of supplemented modules over Dedekind domains.

Theorem 6.4.1. *(Zöschinger, 1974a, Theorem 3.1) Let R be a non-local Dedekind domain and M be an R-module. Then M is supplemented if and only if it is torsion and every primary component is supplemented.*

Using previous theorem we give a characterization of Rad-supplemented modules over Dedekind domains.

Theorem 6.4.2. *Let R be a Dedekind domain. Then the following are equivalent:*

(i) *M is Rad-supplemented,*

(ii) *$M = D(M) \oplus M'$, where $D(M)$ is the divisible part of M and M' is a reduced submodule of M such that M' is supplemented,*

(iii) *$M/D(M)$ is supplemented, where $D(M)$ is the divisible part of M.*

Proof. $(i) \Rightarrow (ii)$: The R-module M can be written as follows: $M = D(M) \oplus M'$ for submodules $D(M)$ and M' of M such that $D(M)$ is the divisible part of M and M' is a reduced submodule of M. Since every direct summand of a Rad-supplemented module is again Rad-supplemented (by Corollary 6.1.15), M' is also Rad-supplemented. Since R is a Dedekind domain (and so it is a commutative noetherian ring) , we obtain by Lemma 6.2.22 that M' is supplemented.

$(ii) \Rightarrow (i)$: Suppose $M = D(M) \oplus M'$, where $D(M)$ is the divisible part of M and M' is supplemented. Since R is Dedekind domain and $D(M)$ is a divisible module, $D(M)$ is a radical module, i.e., $\operatorname{Rad} D(M) = D(M)$. So $D(M)$ is Rad-supplemented by Corollary 6.1.22. Now since M' is supplemented, it is clearly Rad-supplemented. Hence $M = D(M) \oplus M'$ is Rad-supplemented (by Theorem 6.1.29).

$(ii) \Leftrightarrow (iii)$: This is easily seen since $M' \cong M/D(M)$.

$\square$

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