ON SOME COMPUTATIONAL PROBLEMS IN ALGEBRAIC GEOMETRY

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ABSTRACT

On Some Computtaional Problems in Algebraic Geometry

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A method for computing a basis of syzygy module for a given generating set of a homogeneous ideal of polynomial ring is introduced. This method, unlike the common way to compute syzygy basis, do not involve a computation of Gröbner basis. The only linear algebraic techniques are used in this method. The new developed method of computing syzygy module and a property of leading forms of polynomials of a generating set of an polynomial ideal suggests an alternative algorithm for computing H-basis of the ideal.

Keywords:H-Bases, Gröbner Bases, Syzygies, Polynomial Ideals, Hilbert Functions

ÖZET

Cebirsel Geometride Bazı Hesaplamalı Problemler Üzerine

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Polinom halkasında homojen bir idealin üreteç kümesinin syzygy (çekirdek) modülünün tabanının hesabı için bir metod geliştirildi. Bu metod, bilinen klasik yöntemin aksine, Gröbner taban hesabı içermemektedir. Bu yeni metodta sadece doğrusal cebir teknikleri kullanılmıştır. Ayrıca bu yeni geliştirilen metod ve polinonom halkasındaki bir idealin üreteçlerinin öncü formlarının bir özelliği yardımıyla bir polinom idealinin H-tabanını hesaplamak için alternatif bir algoritma önerilmiştir.

Anahtar Kelimeler:Homojen Tabanlar, Gröbner Tabanlar, Syzygies, Polinom idealler, Hilbert Fonksiyonu

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CHAPTER 1

BASIC CONCEPTS AND DEFINITIONS

In this chapter we give a number of definitions and theorems used in the subsequent parts of the thesis. More detail can be found in [1],[2] and[3].

1.1 Gröbner Basis for Polynomial Ideals

Let $k[x_1, \ldots, x_n]$ be the ring of the polynomials over a field k.

Definition 1.1 *A* **monomial** *in $x_1, \ldots, x_n$ is a product of the form*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

The total degree of this monomial is the sum $\alpha_1 + \cdots + \alpha_n$.

For brevity, let $x = (x_1, \ldots, x_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the monomial can be written as x^n. We also let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ denote the total degree of the monomial x^α.

Definition 1.2 *A* **polynomial** *f is a finite linear combination of monomials with coefficients in a field k. A polynomial, f, will be written in the form*

$$f = \sum_{\alpha} \mathbf{a}_\alpha \mathbf{x}^\alpha,$$

where the sum is over a finite number of $n-$tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\mathbf{a}_\alpha \in k - \{0\}$. The total degree of f, denoted $\deg(f)$*, is the maximum $|\alpha| = \alpha_1 + \cdots + \alpha_n$ in the expression of f.*

Given a collection of polynomials, $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, we can consider all polynomials which can be written as a linear combination of these polynomials.

Definition 1.3 *Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$. We let $\langle f_1, \ldots, f_s \rangle$ denote the collection*

$$\langle f_1, \ldots, f_s \rangle = \{a_1 f_1 + \cdots + a_s f_s : a_i \in k[x_1, \ldots, x_n], 1 \leq i \leq s\}.$$

It is easy to show that $\langle f_1, \ldots, f_s \rangle$ is an ideal of $k[x_1, \ldots, x_n]$ and is called ideal generated by $f_1, \ldots, f_s$.

The well-known Hilbert Basis Theorem states that every ideal of $k[x_1, \ldots, x_n]$ is finitely generated.

Definition 1.4 *We will call the set $k^n = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in k\}$ the affine n-dimensional space over k.*

Definition 1.5 *(i) Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal in $k[x_1, \ldots, x_n]$. The set of zero locus of polynomials of I,*

$$V(I) = \{(a_1, \ldots, a_n) : f_i(a_1, \ldots, a_n) = 0, 1 \leq i \leq s\} \subset k^n$$

is called variety of ideal I.

(ii) Let $V \subset k^n$ be a variety. The collection of polynomials

$$I(V) = \{f \in k[x_1, \ldots, x_n] : f(a_1, \ldots, a_n) = 0, \forall (a_1, \ldots, a_n) \in V\}$$

is called ideal of the variety V.

We always have $V(I(V)) = V$, but $I(V(I)) = \sqrt{I}$ if k is algebraically closed by Strong Nullstellensatz.

Definition 1.6 *A monomial ordering on $k[x_1, \ldots, x_n]$ is any relation $>$ on $\mathbb{Z}^n_{\geq 0}$ or equivalently, any relation on the set of the monomials x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$ satisfying:*

(i) $>$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$.

(ii) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.

(iii) $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$. This means that every nonempty subset of $\mathbb{Z}^n_{\geq 0}$ has a smallest element under $>$.

Definition 1.7 *A monomial ordering $>$ in $k[x_1, x_2, \ldots, x_n]$ is called a graded monomial ordering if $x^\alpha > x^\beta$ whenever $|\alpha| > |\beta|$.*

Now, we will define the most used monomial orderings. Notice that the last two orderings are graded orderings.

Definition 1.8 *Let x^α and x^β be monomials in $k[x_1, \ldots, x_n]$*

(i) Lexicographic Order: We say $x^\alpha >_{\text{lex}} x^\beta$ if in the difference $\alpha - \beta \in \mathbb{Z}^n$, the left-most nonzero entry is positive.

(ii) Graded Lexicographic Order: We say $x^\alpha >_{\text{grlex}} x^\beta$ if $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$, or $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and $x^\alpha >_{\text{lex}} x^\beta$.

(iii) Graded Reverse Lexicographic Order: We say $x^\alpha >_{\text{gevlex}} x^\beta$ if $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$, or in the difference $\alpha - \beta \in \mathbb{Z}^n$, the right-most nonzero entry is negative.

For example; $x^3y^2z^4 >_{\text{lex}} x^3y^2z$, since $(3,2,4) - (3,2,1) = (0,0,3)$ the left-most nonzero entry is positive, and $x^2y^2z^2 >_{\text{grlex}} xy^4z$ since $2+2+2 = 1+1+4$ but $(2,2,2) - (1,4,1) = (1,-2,1)$, the left-most nonzero entry is positive. However, $x^2y^2z^2 <_{\text{grevlex}} xy^4z$ since $2+2+2 = 1+1+4$ but $(1,4,1) - (2,2,2) = (-1,2,-1)$, the right-most nonzero entry is negative.

Definition 1.9 *Let $f = \sum_\alpha a_\alpha x^\alpha$ be a nonzero polynomial in $k[x_1, x_2, \ldots, x_n]$ and let $>$ be a monomial order.*

(i) The multidegree of f is

$$\text{multideg}(f) = \max(\alpha \in \mathbf{Z}^n_{\geq 0} : a_\alpha \neq 0)$$

the maximum is taken with respect to $>$.

(ii) The leading monomial coefficient of f is

$$\text{LC}(f) = a_{\text{multideg}(f)} \in k.$$

(iii) The leading monomial of f is

$$\mathrm{LM}(f) = x^{\mathrm{multideg}(f)}.$$

(iv) The leading term of f is

$$\mathrm{LT}(f) = \mathrm{LC}(f)\mathrm{LM}(f).$$

To illustrate, let $f = 4xyz + 4z^2 - 5x^3 + 7x^2z^2$. If our order is lexicographic order, $\mathrm{multideg}(f) = (3,0,0), \mathrm{LC}(f) = -5, \mathrm{LM}(f) = x^3$ and $\mathrm{LT}(f) = -5x^3$. If we choice graded lexicographic order, $\mathrm{multideg}(f) = (2,0,2), \mathrm{LC}(f) = 7, \mathrm{LM}(f) = x^2z^2$ and $\mathrm{LT}(f) = 7x^2z^2$.

The following is the well-known division algorithm on $k[x_1, \ldots, x_n]$.

Theorem 1.10 *Fix a monomial order $>$, and let $F = (f_1, \ldots, f_s)$ be an ordered s-tuple of polynomials in $k[x_1, \ldots, x_n]$. Then every $f \in k[x_1, \ldots, x_n]$ can be written as*

$$f = a_1f_1 + \ldots + a_sf_s + r,$$

where $a_i, r \in k[x_1, x_2, \ldots, x_n]$, and either $r = 0$ or r is a $k-$linear combination of monomials, none of which is divisible by any $\mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_s)$. r is called remainder of f on division by F. Furthermore, if $a_if_i \neq 0$, then

$$\mathrm{multideg}(f) \geq \mathrm{multideg}(a_if_i)$$

Since we now have a division algorithm in $k[x_1, \ldots, x_n]$ that seems to have many of the same features as the one-variable version, it is natural to ask if deciding whether a given $f \in k[x_1, \ldots, x_n]$ is a member of a given ideal $I = \langle f_1, \ldots, f_s \rangle$ can be done by computing the remainder on division. One direction is easy. Namely, if the remainder $r = 0$ on dividing by $(f_1, \ldots, f_s)$, then $f = a_1f_1 + \cdots + a_sf_s$. By definition then, $f \in \langle f_1, \ldots, f_s \rangle$. On the other hand, the following example shows that we are not guaranteed to get remainder $r = 0$ for every $f \in \langle f_1, \ldots, f_s \rangle$.

$$p = x^2 + \frac{1}{2}y^2z - z - 1 = (-\frac{1}{2}z + 1)(x^2 + z^2 - 1) + \frac{1}{2}z(x^2 + y^2 + (z-1)^2 - 4)$$

shows $p \in \langle f_1 = x^2 + z^2 - 1, f_2 = x^2 + y^2 + (z-1)^2 - 4\rangle$. However the remainder on division of f by (f_1, f_2) is not zero. For instance, using lexicographic order,we get

$$p = 1f_1 + 0f_2 + \frac{1}{2}y^2z - z^2 - z$$

or

$$p = 0f_1 + 1f_2 + \frac{1}{2}y^2z - y^2 - z^2 + z + 2$$

The remainder is not zero because it contains terms that cannot be removed by division by this particular generators for I. The leading terms of $f_1 = x^2 + z^2 - 1$ and $f_2 = x^2 + y^2 + (z-1)^2 - 4$ the any term of the remainder. In order for division to produce zero remainders for all elements of I, we need to be able to remove all leading terms of elements of I using leading terms of the divisors. That is the motivation for the following definition.

Definition 1.11 *Fix a monomial ordering. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a Gröbner Basis if for every nonzero $f \in I$,* $\mathrm{LT}(f)$ *is divisible by* $\mathrm{LT}(g_i)$ *for some i.*

A Gröbner basis for I is indeed a generating set for I. Existence of Gröbner basis for an arbitrary ideal is an easy consequence of Hilbert Basis theorem. Moreover, remainders computed by division with respect to a Gröbner basis are much better behaved than those computed with respect to arbitrary sets of divisors.

Theorem 1.12 *If $G = \{g_1, \ldots, g_s\}$ is a Gröbner basis for I, then for any $f \in I$, the remainder on division f by G is zero.*

More useful for many purposes than the existence proof for Gröbner bases is an algorithm, due to Buchberger, that takes an arbitrary generating set $\{f_1, \ldots, f_s\}$ for I

and produce a Gröbner basis G for I from it. This algorithm works by forming new elements of I using expressions guaranteed to cancel leading terms and uncover other possible leading terms.

Definition 1.13 *Let $f, g \in k[x_1, x_2, \ldots, x_n]$ be nonzero polynomials.*

(i)If multideg$(f) = \alpha$ *and* multideg$(g) = \beta$*, then let $\gamma = (\gamma_1, \ldots, \gamma_n)$ where $\gamma_i = \max(\alpha_i, \beta_i)$ for each i. x^γ is called the least common multiple of* LM(f) *and* LM(g)*, written as $x^\gamma =$* LCM(LM(f), LM(g)).

(ii)The S-polynomial of f and g is combination

$$S(f,g) = \frac{x^\gamma}{LT(f)}f - \frac{x^{|\gamma}}{LT(g)}g.$$

For example; let $f = x^3y^2 - x^2y^3 + x$ and $g = 3x^4y + y^2$ in $[x, y]$ with the grlex order. Then $\gamma = (4, 2)$ and

$$\begin{aligned} S(f,g) &= \frac{x^4y^2}{x^3y^2}f - \frac{x^4y^2}{3x^4y}g \\ &= xf - (1/3)yg \\ &= -x^3y^3 + x^2 - (1/3)y^3. \end{aligned}$$

The next theorem gives a criterion for when a basis of an ideal is a Gröbner basis.

Theorem 1.14 *Let I be a polynomial ideal. Then a basis $G = \{g_1, \ldots, g_t\}$ for I Gröbner basis for I if and only if for all pairs $i \neq j$, the remainder on division of $S(g_i, g_j)$ by G (listed in some order) is zero.*

Using this criterion above, Buchberger's Algorithm computes a Gröbner basis for an ideal I from a given basis of that ideal.

Algorithm 1.15 *Let $I = \langle f_1, \ldots, f_s \rangle$ be a polynomial ideal. Then a Gröbner basis for I can be constructed in a finite steps by the following algorithm:*

INPUT: $F = \langle f_1, \ldots, f_s \rangle \subset k[x_1, \ldots, x_n]$ with $f_i \neq 0 (1 \leq i \leq s)$

OUTPUT: $G = \{g_1, \ldots, g_t\}$, a Gröbner basis for $\langle f_1, \ldots, f_s \rangle$

INITIALIZATION: $G := F, \mathcal{G} := \{(f_i, f_j) | f_i \neq f_j \in G)$

Fix a monomial order.

WHILE $\mathcal{G} \neq \emptyset$ *DO*

Choose any $(f, g) \in \mathcal{G}$

$\mathcal{G} := \mathcal{G} - \{(f, g)\}$

$h := \overline{S(f,g)}^{G}$

IF $h \neq 0$ *THEN*

$\mathcal{G} := \mathcal{G} \cup \{(u, h) | \text{forall} u \in \mathcal{G}\}$

$G := G \cup \{h\}$,

where $\overline{S(f,g)}^{G}$ *is remainder on division of* $S(f, g)$ *by* G.

Return G

For example; Consider $k[x, y]$ with grlex order, and let $I = \langle f_1, f_2 \rangle = \langle x^3 - 2xy, x^2y - 2y^2 + x \rangle$. Recall that $\{f_1, f_2\}$ is not a Gröbner basis for I since $LT(S(f_1, f_2)) = -x^2 \notin \langle LT(f_1), LT(f_2) \rangle$. To produce a Gröbner basis, one natural idea is to try first to extend the original generating set to a Gröbner basis by adding more polynomials in I. We have $S(f_1, f_2) = -x^2 \in I$, and its remainder on division by $F = (f_1, f_2)$ is $-x^2$ which is nonzero. Hence, we should include that remainder in our generating set, as a new generator $f_3 = -x^2$. If we set $F(f_1, f_2, f_3)$, we can use Theorem 1.14 to test if this new set is a Gröbner basis for I. We compute

$S(f_1, f_2) = f_3$, so

$\overline{S(f_1, f_2)}^{F} = 0$,

$S(f_1, f_3) = (x^3 - 2xy) - (-x)(-x^2) = -2xy$. but

$\overline{S(f_1, f_3)}^{F} = -2xy \neq 0$.

Hence, we must add $f_4 = -2xy$ to our generating set. If we let $F = (f_1, f_2, f_3, f_4)$. then

$\overline{S(f_1, f_2)}^{F} = \overline{S(f_1, f_3)}^{F} = 0$,

$S(f_1, f_4) = y(x^3 - 2xy) - (-1/2)x^2(-2xy) = -2xy^2 = yf_4$, so

$\overline{S(f_1, f_4)}^{F} = 0$.

$S(f_2, f_3) = (x^2y - 2y^2 + x) - (-y)(-x^2) = 2y^2 + x$, but

$$\overline{S(f_2, f_3)}^F = -2y^2 + x \neq 0.$$

Thus, we must also add $f_5 = -2y^2+x$ to our generating set. Setting $F = (f_1, f_2, f_3, f_4, f_5)$, one can compute that

$$\overline{S(f_i, f_j)}^F = 0 for all 1 \leq i < j \leq 5.$$

It follows that a Gröbner basis for I is given by

$$\{f_1, f_2, f_3, f_4, f_5\} = \{x^3 - 2xy, x^2y - 2y^2 + x, -x^2, -2xy, -2y^2 + x\}.$$

1.2 Syzygy Modules

Definition 1.16 *A module over a ring R (or R−module) is a set M together with a binary operation, usually written as addition of R on M, and an operation of R on M, called scalar multiplication, satisfying properties.*

a. M is an Abelian group under addition.

b. For all $a \in R$ and all $f, g \in M, a(f + g) = af + ag$.

c. For all $a, b \in R$ and all $f \in M, (a + b)f = af + bf$.

d. For all $a, b \in R$ and all $f \in M, (ab)f = a(bf)$.

e. If 1 is multiplicative identity in R, $1f = f$ for all $f \in M$.

Definition 1.17 *Let M be a module over a ring R. M is said to be a free module if M has a module basis (that is, a generating set which is R−linearly independent).*

For instance, the $R-$module $M = R^m$ is a free module. The standard vectors $\mathbf{e_1} = (1, 0, 0, \ldots, 0)^T, \mathbf{e_2} = (0, 1, 0, \ldots, 0)^T, \cdots, \mathbf{e_m} = (0, 0, 0, \ldots, 1)^T$ form one basis for M as $R-$module.

One obtains other examples of $R-$modules by considering submodules of R^m, that is, subsets of R^m which are closed under addition and scalar multiplication by elements of R and which are, therefore, modules in their own right.

We might, for example, choose a finite set of vectors $\mathbf{f_1}, \ldots, \mathbf{f_s}$ and consider the set of all vectors which are can be written as an $R-$linear combination of these vectors:

$$\{a_1\mathbf{f_1} + \cdots + a_s\mathbf{f_s} \in R^m, a_1, \ldots, a_s \in R\}.$$

We denote this set $\langle \mathbf{f_1}, \ldots, \mathbf{f_s} \rangle$, and we can easily show that it is a submodule of R^m. This kind of submodules are called finitely generated submodules.

In this thesis, we consider finitely generated submodules of free module R^m where $R = k[x_1, \ldots, x_n]$ is the polynomial ring over some field k. In particular, we will consider what we call syzygy submodules of R^m.

Definition 1.18 *Let $R = k[x_1, \ldots, x_n]$, and let $(f_1, \ldots, f_t)$ be an ordered $t-$tuple of elements $f_i \in R$. The set of all $(a_1, \ldots, a_t)^T \in R^t$ such that $a_1 f_1 + a_2 f_2 + \cdots + a_t f_t = 0$ is an $R-$submodule of R^t, called the (first) syzygy module of $(f_1, \ldots, f_t)$, and denoted* $\mathrm{syz}(f_1, \ldots, f_t)$.

The syzygy modules are finitely generated $R-$modules, and a set of generators for them can be computed. If the starting point is a Gröbner basis, then finding a set of generators for the syzygy module is accomplished using Buchberger's algorithm. While computing a Gröbner basis $G = \{g_1, \ldots, g_s\}$ for an ideal $I \subset R$ with respect to some fixed monomial ordering, a slight modification of the algorithm would actually compute a set of generators for the syzygy module $\mathrm{syz}(g_1, \ldots, g_s)$ as well as the g_i's themselves.

Let $S(g_i, g_j)$ be the $S-$polynomial of g_i and g_j.

$$S(g_i, g_j) = \frac{x^{\gamma_{ij}}}{LT(g_i)} g_i - \frac{x^{\gamma_{ij}}}{LT(g_j)} g_j,$$

where $x^{\gamma_{ij}}$ is the least common multiple of $\mathrm{LM}(g_i)$ and $\mathrm{LM}(g_j)$. Since G is a Gröbner basis, the remainder of $S(g_i, g_j)$ on division by G is zero, and division algorithm gives an expression

$$S(g_i, g_j) = \sum_{k=1}^{s} a_{ijk} g_k,$$

where $a_{ijk} \in R$, and $\mathrm{LT}(a_{ijk}) \leq \mathrm{LT}(S(g_i, g_j)$ for all i, j, k.

Let $\mathbf{a_{ij}} \in \mathbf{R^s}$ denote the column vector

$$\mathbf{a_{ij}} = a_{ij1}\mathbf{e_1} + a_{ij2}\mathbf{e_2} + \cdots + a_{ijs}\mathbf{e_s} = (a_{ij1}, a_{ij2}, \ldots, a_{ijs})$$

and define $\mathbf{s_{ij}} \in R$ by setting

$$\mathbf{s_{ij}} = \frac{x^{\gamma_{ij}}}{LT(g_i)}\mathbf{e_i} - \frac{x^{\gamma_{ij}}}{LT(g_j)}\mathbf{e_j} + \mathbf{a_{ij}} \qquad (*)$$

in R^s. Then the next theorem shows how to find a basis for the syzygy module.

Theorem 1.19 *Let $G = \{g_1, \ldots, g_s\}$ be a Gröbner basis of an ideal I in R with respect to some fixed monomial order, and let $M = \mathrm{Syz}(g_1, \ldots, g_s)$. The collection $\{\mathbf{s_{ij}}, 1 \leq i, j \leq s\}$ from $(*)$ generates M as an $R-$module.*

The next step is to compute $\mathrm{syz}(f_1, \ldots, f_t)$ for a collection $\{f_1, \ldots, f_t\}$ of nonzero polynomial is R which may not from a Gröbner basis. First compute a Gröbner basis $\{g_1, \ldots, g_s\}$ for $\langle f_1, \ldots, f_t\rangle$. Set $F = (f_1, \ldots, f_t)$ and $G = (g_1, \ldots, g_s)$. Since the elements of F and G generator the same ideal, there are a $t \times s$ matrix A and an $s \times t$ matrix B, both with entries in R, such that $G = FA$ and $F = GB$. The matrix B can be compute by applying the division algorithm with respect to G. The matrix A can be obtained by keeping track of reductions of S-polynomials on the division process during Buchberger's Algorithm.

A generating set $\{\mathbf{s_1}, \ldots, \mathbf{s_p}\}$ for $\mathrm{syz}(G)$ (the $\mathbf{s_i}$'s are column vectors in R^s) can be computed using Theorem 1.19. Let $\mathbf{r_1}, \ldots, \mathbf{r_t}$ be the column of the matrix $I_t - AB$.

Theorem 1.20 *With the notation above,*

$$\{A\mathbf{s_1}, A\mathbf{s_2}, \ldots, A\mathbf{s_p}, \mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_t}\}$$

is a generating set for $\mathrm{Syz}(f_1, \ldots, f_t)$.

for example, let $M = \langle f_1, f_2\rangle$ in $R = k[x, y]$ where

$$f_1 = xy + x, \qquad f_2 = y^2 + 1.$$

Using the lex monomial order, the reduced Gröbner basis for M consist of

$$g_1 = x, \qquad g_2 = y^2 + 1.$$

then it is easy to check that

$$f_1 = (y+1)g_1$$

$$g_1 = -(1/2)(y-1)f_1 + (1/2)xf_2,$$

so that

$$G = (g_1, g_2) = (f_1, f_2)\begin{pmatrix} -(y-1)/2 & 0 \\ x/2 & 1 \end{pmatrix} = FA$$

and

$$F = (f_1, f_2) = (g_1, g_2)\begin{pmatrix} y+1 & 0 \\ 0 & 1 \end{pmatrix} = GB$$

Since

$$S(g_1, g_2) = y^2 g_1 - x g_2 = -x = -g_1,$$

by Theorem 1.19 we have that

$$s_{12} = \begin{pmatrix} y^2+1 \\ -x \end{pmatrix}$$

generates $Syz(g_1, g_2)$. Multiplying By A we get

$$As_{12} = \begin{pmatrix} -(y-1)/2 & 0 \\ x/2 & 1 \end{pmatrix}\begin{pmatrix} y^2+1 \\ -x \end{pmatrix}$$

$$= \begin{pmatrix} -(y^3 - y^2 + y - 1)/2 \\ (xy^2 - x)/2 \end{pmatrix}.$$

The columns of $I_2 - AB$ are

$$S_1 = \begin{pmatrix} (y^2+1)/2 \\ -(xy+x)/2 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So by the theorem 1.20

$$Syz(f_1, f_2) = \langle As_{12}, S_1 \rangle.$$

1.3 Hilbert Function and Hilbert Polynomial

The Hilbert function of a homogeneous ideal $I \subset k[x_1, \ldots, x_n]$ of polynomial ring associates to an integer n the dimension of $k[x_1, \ldots, x_n]/I$. For sufficiently large n, the values of this function are given by a polynomial, the Hilbert polynomial. To show this, we use Hilbert-Poincare series, a formal power series in t with coefficients being the values of the Hilbert function. This power series turns out to be rational function.

Definition 1.21 *Let $I \subset k[x_1, \ldots, x_n]$ be a homogeneous ideal. The Hilbert function $H_I : \mathbb{Z} \to \mathbb{Z}$ of I is defined by*

$$H_I(d) := \dim_k(k[x_1, \ldots, x_n]/I),$$

and the Hilbert-Poincare series HP_I of I is defined by

$$HP_I(t) := \sum_{m \in \mathbb{Z}} H_I(k) t^m \in \mathbb{Z}[[t]][t^{-1}]$$

Hilbert-Poincare series of any homogeneous ideal I can be written in the form

$$HP_I(t) := \frac{Q(t)}{(1-t)^r} \quad where \quad Q(t) \in \mathbb{Z}[t].$$

When we cancel all common factors in the numerator, we obtain

$$HP_I(t) = \frac{G(t)}{(1-t)^s}, \quad 0 \leq s \leq n, \quad G(t) = \sum_{m=0}^{r} g_m t^m.$$

such that $g_d \neq 0$ and $G(1) \neq 0$, that is, s is the pole order of $HP_I(t)$. In fact, the degree of $G(t)$ which is equal to r is the number when Hilbert function becomes a polynomial. The Hilbert polynomial of I is

$$\sum_{m=0}^{r} g_m \binom{s-1+d-m}{s-1}$$

which is agree with the Hilbert function of I for $d \geq r$.

Since the Hilbert polynomial and the Hilbert-Poincare series determine each other, it suffices to study and compute Hilbert-Poincare series. The following algorithm computes the Hilbert-Poincare series of a monomial ideal, more precisely, it computes the polynomial $Q(t)$.

Algorithm 1.22 *Input:* $I = \langle m_1, \ldots, m_p \rangle \subset k[x_1, \ldots, x_n]$, m_i *is a monomial.*

Output: A polynomial $Q(t)$ *such that* $Q(t)/(1-t)^n$ *is HP series of* I.

Choose $S = \{x^{\alpha_1}, \ldots, x^{\alpha_s}\}$ *a minimal set of generators of* I.

If $S = \{0\}$ *then return 1.*

If $S\{1\}$ *then return 0.*

If all elements of S *have degree 1 then return* $(1-t)^s$.

Choose $1 \leq i \leq s$ *such that* $\deg(x^{\alpha_i}) > 1$

Choose $1 \leq j \leq n$ *such that* $x_j \mid x^{\alpha_i}$.

Return MonHilbertPoincare $\langle I, x_j \rangle$ + t *MonHilbertPoincare* $(I : x_j)$.

Example 1.23 *Since we have* $\binom{n+d-1}{n-1}$ *monomial of degree* d *in* $k[x_1, \ldots, x_n]$,

$$HP_{k[x_1,\ldots,x_n]}(t) = \sum_{d=0}^{\infty} \binom{n+d-1}{n-1} t^d = \frac{1}{(1-t)^n}$$

Let $I = \langle xz, yz \rangle \subset k[x, y, z]$. *Using algorithm above*

$$\begin{aligned} HP_I(t) = HP_{\langle I,z \rangle}(t) + t.HP_{\langle I:z \rangle}(t) &= HP_{k[x,y]}(t) + t.HP_{k[z]}(t) \\ &= \frac{1}{(1-t)^2} + t\frac{1}{1-t} \end{aligned}$$

$$= \frac{1+t-t^2}{(1-t)^2}.$$

Since

$$\begin{aligned}\frac{1+t-t^2}{(1-t)^2} &= \sum_{d=0}^{\infty}\binom{d+1}{1}t^d + \sum_{d=0}^{\infty}\binom{d+1}{1}t^{d+1} - \sum_{d=0}^{\infty}\binom{d+1}{1}t^{d+2} \\ &= \sum_{d=0}^{\infty}\left(\binom{d+1}{1} + \binom{d}{1} - \binom{d-1}{1}\right)t^d\end{aligned}$$

the Hilbert polynomial of I is

$$H_I(d) = \binom{d+1}{1} + \binom{d}{1} - \binom{d-1}{1} = d+2$$

which is agree with the Hilbert function for $d \geq 1$.

CHAPTER 2

COMPUTATION OF SYZYGY BASIS OF HOMOGENEOUS IDEALS USING LINEAR ALGEBRA

2.1 Introduction

In algebraic geometry over a field k, we study the geometry of varieties through properties of polynomial ring $R = k[x_1, \ldots, x_n]$ and its ideals. It turns out that to study ideals effectively we also need to study more general modules over R. If an ideal $I \subset R$ is given by a set of generators $\{f_1, f_2, \ldots, f_s\}$, the most important submodule of free module R^s is the syzygy module of $(f_1, \ldots, f_s)$. The syzygy module plays important role in the computation of Hilbert polynomials, free resolutions and geometric aspects of variety of ideals such as dimension and degree of varieties. Hence the computation of a basis for syzygy module is crucial. The only way, other than a few exceptional cases, is to use Gröbner basis techniques. In this chapter, we will try to give an alternative method to compute a bases for syzygy modules of homogeneous ideals. Unlike the known method, we will use only linear algebraic techniques.

2.2 Vector Spaces Concerning Homogeneous Ideals

We consider $\mathbf{R}$ the ring of polynomials in $x_1, x_2, \ldots, x_n$ with coefficients from an infinite field k, i.e $\mathbf{R} = k[x_1, x_2, \ldots, x_n]$ and the subsets $\mathbf{R_d}$ of all homogeneous polynomials of degree of d. A direct sum

$$\mathbf{R} = \bigoplus_{d \in \mathbb{N}} \mathbf{R_d}$$

is called the degree grading. Because for all $\alpha, \beta \in \mathbb{N}$

$$f \in \mathbf{R}_\alpha, g \in \mathbf{R}_\beta \Rightarrow f.g \in \mathbf{R}_{\alpha+\beta}$$

Since the decomposition of above is a direct sum, each polynomial $f \neq 0$ has a unique representation.

$$f = \sum_{i=1}^{s} f_{\gamma_i} \qquad\qquad 0 \neq f_{\gamma_i}$$

If we consider $\mathbf{R_d}$ as a vector space over k with monomials of degree d being a basis , then there is a well-known formula

$$dim(\mathbf{R_d}) = \binom{d+n-1}{n-1}.$$

Let $I \subset k[x_1, x_2, \ldots, x_n]$ be a homogeneous ideal. We define the vector space $V_d(I) = \mathbf{R_d} \cap I$. If $\{g_1, \ldots, g_s\}$ is a basis of I involving only homogeneous polynomials, then $V_d(I)$ is generated by all monomial multiples $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} g_i$ with $\alpha_1 + \alpha_2 + \ldots + \alpha_n + deg(g_i) = d$.

Let us define $W_d(I)$ as orthogonal component of $V_d(I)$. Hence $\mathbf{R_d} = V_d(I) \bigoplus W_d(I)$.

Definition 2.1 *For given polynomials $g_1, \ldots, g_s \in \mathbf{R}$, let*

$$Syz(g_1, \ldots, g_s) := \left\{ (h_1, \ldots, h_s) \in \mathbf{R^s} \middle| \sum_{i=1}^{s} h_i g_i = 0 \right\}$$

be the the module of syzygies with respect to $g_1, \ldots, g_s$.

As explained in section 1.2 , finding a generating set for $syz(g_1, \ldots, g_s)$ involves a Gröbner basis computation. In this chapter of the thesis, our aim is to find a generating set for $syz(g_1, \ldots, g_s)$ using techniques of linear algebra. The following observation will be the crucial for this computation.

Lemma 2.2 *For given polynomials $g_1, \ldots, g_s \in \mathbf{R}$, let*

$$Syz(g_1, \ldots, g_s) := \bigoplus_{d \in \mathbb{N}} S_d(g_1, \ldots, g_s)$$

where

$$S_d(g_1, \ldots, g_s) := \left\{ (h_1, \ldots, h_s) \in Syz(g_1, \ldots, g_s) : deg(h_i g_i) = d \text{ or } h_i = 0 \right\} \subseteq Syz(g_1, \ldots, g_s)$$

Then , $dim(S_d(g_1,\ldots,g_s)) = \sum_{i=1}^{s} dim(\mathbf{R_{d-def(g_i)}}) - dimV_d(g_1,\ldots,g_s)$.

Proof. Every syzygy in $S_d(g_1,\ldots,g_s)$ is a linearly dependent relation among the degree d polynomials $x_1^{\alpha_1}\ldots x_n^{\alpha_n}g_i$. Since $dim(V_d(g_1,\ldots,g_s))$ is number of linearly independent polynomials of this kind, the assertion follows □

Let us define

$$\binom{d+n-1}{n-1} x \sum_{i=1}^{s} \binom{d+n-deg(g_i)-1}{n-1}$$

matrix M where each row corresponds to a monomial of degree d and each column corresponds the polynomial of kind $x_1^{\alpha_1}\ldots x_n^{\alpha_n}g_i$ where $\alpha_1+\ldots+\alpha_n+deg(g_i) = d$ for $i = 1,\ldots,s$. In this case, the basis of nullspace of M is a generating set for $S_d(g_1,\ldots,g_s)$. The null space of M can be easily seen from row-reduced echelon of M.

Hence, if we compute the row reduced echelon form of matrices in the above form, then we can obtain a generating set for $S_d(g_1,\ldots,g_s)$ for each d. However we need to know the value of d for which we obtain a generating set for $Syz(g_1\ldots,g_s)$. In other words , we must have an ending criterium of the process. This criterium will come from Hilbert Function of the ideal $I =< g_1,\ldots,g_s >$. To use this criterium, however, we need to know information about the vector space $W_d(I)$. This information can be read off from the row reduced echelon form of the transpose of M. We also have to calculate, thus, the row reduced echelon form of the transpose of M.

On the other hand, if we do row reduction without row interchange operation, a basis of $W_d(I)$ can be obtained from row-reduced form of the matrix M.

Suppose that the rows of M corresponds the monomials of degree d in lexicographic order with $x_1 > x_2 > x_3 >,\ldots,> x_n$.

Lemma 2.3 *Let M be the matrix defined as above. If M is row-reduced without using elementary row operation of interchanging rows, then for each monomial corresponding non-zero rows of M there exists a polynomial in $I =< g_1,\ldots,g_s >$ which has this monomial as leading monomial. Hence the monomials corresponding zero rows of M form a basis for $W_d(I)$.*

Proof.

Suppose that when applying row-reduction process to the matrix M, we also made the following computations in the polynomials corresponding columns of M:

In the first row of M , we find the most-left non-zero entry, if there is one, and divide the first row by its value. The polynomial, f_1 corresponding the column of this non-zero entry has leading monomial x_1^d. Dividing this polynomial by the value of non-zero entry, makes it monic. For convenience, if the first row contains only zero entries define $f_1 := 0$

In the second row , we again find the most-left non-zero entry, say $m_{2j} \neq 0$. This means we have a polynomial $f_2 \in I_d$ involving the monomial $x_1^{d-1}x - 2$. If $m_{1j} = 0$, then the leading monomial of f_2 is $x_1^{d-1}x_2$. Otherwise , define $f_2 := f_2 - m_{1j}f_1$ and thus f_2 has leading monomial of $x_1^{d-1}x_2$. In both cases, dividing polynomials by leading coefficient, makes the monic. As before, if the second row contains only zero entries, define $f_2 := 0$

In general, if the i^{th} row has the most-left non-zero entry m_{ij} then take the monomial corresponding the j^{th} column of M, call if f_i.Define ne $f_i := f_i - \sum_{k=1}^{i-1} m_{kj}f_k$ and divide f_i by its leading coefficient.

Hence the non-zero f_i's would have be a basis of $V_d(I)$ each of which have leading monomial corresponding non-zero rows of M if we made the above computations. Therefore, the monomials corresponds zero-rows form a basis for $W_d(I)$

□

Now, we are ready to give an algorithm to compute syzygy basis of $g_1, \ldots, g_s$. Before that, let us recall definition of Hilbert function of an ideal and observe some facts regarding dimension of $S_d(g_1 \ldots, g_s)$.

Definition 2.4 *Let $I =< g_1, \ldots, g_s > \subseteq R$ be an ideal. The mapping*

$$H_I : \mathbb{N} \to \mathbb{N}, \;\; H_I(d) = dim(W_d(I))$$

is called the Hilbert function of I.

For every ideal $I \subset R$ there exists a constant t_0 such that $H_I(t)$ becomes a polynomial in t for $t \geq t_0$. This is the so called Hilbert polynomial, see 1.3.

As explained in section 1.3, we can easily compute Hilbert polynomial of monomial ideals.

we have the following equalities about the dimensions of vector space which are concerned.

$$dim(V_d(I)) = \binom{d+n-1}{n-1} - H_I(d)$$

$$dim(S_d(g_1, \ldots, g_s)) = H_I(d) - \binom{d+n-1}{n-1} + \sum_{i=1}^{s} \binom{d+n-1-deg(g_i)}{n-1}$$

Now, we are ready to give an algorithm for finding a generating set for $syz(g_1, \ldots, g_s)$. Before that, recall $H_I(d) = H_{LT(I)}(d)$ for every $d \in \mathbb{N}$

2.3 An Algorithm to Compute Syzygy Basis of Homogenous Polynomials

Algorithm 2.5 *The following algorithm computes a syzygy basis of a set of homogeneous polynomials.*

Input: *an s-tuple of homogeneous polynomials* $(g_1, \ldots, g_s)$.

Output: *a generating set for* $syz(g_1, \ldots, g_s)$.

$d_0 := min\{ deg(g_1), \ldots, deg(g_s) \}$, $Syz := \{\}$;

$LM(I) := \{\}$, $DelCol := \{\}$; $lhs := 0$; $rhs = 1$;

WHILE $lhs < rhs$

FORM *the matrix M of dimension* $\binom{d_0+n-1}{n-1} \times \sum_{i=1}^{s} \binom{d_0+n-deg(g_i)-1}{n-1}$;

DELETE *the columns of M corresponding to DelCol.*

READ *the* $S_{d_0}(g_1, \ldots, g_s)$ *from the null space of M;*

$Syz := Syz \cup S_{d_0}$;

SELECT *a polynomial of the form* $x^\alpha g_i \notin DelCol$ *which*

is part of $\mathbf{s} \in Syz$ *for each* $\mathbf{s} \in Syz$.

$DelCol := DelCol \cup x^{\alpha} g_i$.

READ *leadmon, lead monomials from non-zero rows of M;*

$LM(I) := LM(I) \cup leadmon$

MINIMIZE $LM(I)$.

m:= the number of repeated elements in DelCol

$lhs := \sum_{\mathbf{s} \in Syz} \binom{d+n-1-\deg(\mathbf{s})}{n-1} - \binom{d+n-1-m}{n-1}$;

$rhs := H_{LM(I)}(d) - \binom{d+n-1}{n-1} + \sum_{i=1}^{s} \binom{d+n-1-deg(g_i)}{n-1}$ $for\ d \geq d_0$

$d_0 := d_0 + 1$

END WHILE

RETURN *Syz*.

Proof. Regarding dimension of syzygy basis of degree d, we have proved that

$$dim(S_d(g_1, \ldots, g_s)) = \sum_{i=1}^{s} dim(\mathbf{R_{d-deg(g_i)}}) - dim(V_d(g_1, \ldots, g_s)).$$

Using Hilbert polynomial, on the other hand, we have

$$dim(V_d(I)) = \binom{d+n-1}{n-1} - H_{LM(I)}(d).$$

Combining this two equalities gives us,

$$dim(S_d(g_1, \ldots, g_s)) = H_{LM(I)}(d) - \binom{d+n-1}{n-1} + \sum_{i=1}^{s} \binom{d+n-1-deg(g_i)}{n-1}.$$

This is the number we called rhs in the algorithm.

For the other side, Every syzygy $(h_1, \ldots, h_s) \in S_{d_0}(g_1, \ldots, g_s)$ can be extended to $\binom{d+n-1-d_0}{n-1}$ different syzygies in S_d by multiplying by the monomials $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = d - d_0$. Hence

$$dim(S_d(g_1, \ldots, g_s)) = \sum_{\mathbf{s} \in Syz} \binom{d+n-1-\deg(\mathbf{s})}{n-1}$$

We have to check whether there is repeated element in Delcol in each d_0. If m is the number of repeated elements, then

$$\sum_{\mathbf{s}\in Syz} \binom{d+n-1-\deg(\mathbf{s})}{n-1} - \binom{d+n-1-m}{n-1}$$

This is the number we have called *lhs* in the algorithm.

Notice that, in the *rhs*, $\sum_{i=1}^{s} \binom{d+n-1-deg(g_i)}{n-1} - \binom{d+n-1}{n-1}$ is a fixed number. Therefore the only part which can be change in the *rhs* is the Hilbert Polynomial, $H_{LM(d)}(I)$. Since $H_{LM(d)}(I)$ is decreasing when a new monomial is added to LT(I), *rhs* is a decreasing function of d in each step of the algorithm.

On the other hand, when a new syzygy is added to bases, *lhs* is increasing at that step of the algorithm. Hence whenever *lhs* = *rhs*, we shall obtain the syzygy basis and the algorithm can stop. □

Now, we are going to give an example to apply the algorithm given above.

Example 2.6 *Consider the ideal* $I =< g_1 = xy + z^2, g_2 = x^2 - yz, g_3 = xy^3z + z^5 >$.

For $d_0 = 2$

$$\mathbf{M}= \begin{array}{c} \\ x^2 \\ xy \\ xz \\ y^2 \\ yz \\ z^2 \end{array} \begin{array}{c} \begin{array}{cc} g_1 & g_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array}$$

M is already reduced form, there is no syzygy. $LT(I) := \{xy, x^2\}$

For $d_0 = 3$

$$M=\begin{array}{c} \\ x^3 \\ x^2y \\ x^2z \\ xy^2 \\ xyz \\ xz^2 \\ y^3 \\ y^2z \\ yz^2 \\ z^3 \end{array}\begin{array}{c} \begin{array}{cccccc} xg_1 & yg_1 & zg_1 & xg_2 & yg_2 & zg_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Reduced form of **M** *is,*

$$\begin{array}{c} \\ x^3 \\ x^2y \\ x^2z \\ xy^2 \\ xyz \\ xz^2 \\ y^3 \\ y^2z \\ yz^2 \\ z^3 \end{array}\begin{array}{c} \begin{array}{cccccc} xg_1 & yg_1 & zg_1 & xg_2 & yg_2 & zg_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

There is no syzygy again. $LT(I) := \{xy, x^2, xz^2\}$

The new element xz^2 *of the* $LT(I)$ *in fact can be seen from the one step earlier matrix of reduction process. Because the matrix just before reduced form in the process is,*

$$\begin{array}{c} \\ x^3 \\ x^2y \\ x^2z \\ xy^2 \\ xyz \\ xz^2 \\ y^3 \\ y^2z \\ yz^2 \\ z^3 \end{array}\begin{array}{c} \begin{array}{cccccc} xg_1 & yg_1 & zg_1 & xg_2 & yg_2 & zg_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \square & 0 & 0 & 0 & \square & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \square & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

To make zero entry in the position $(\mathbf{2},\mathbf{5})$*, we multiply the sixth row by -1 add the second row. Since the leading element in the second row is in the position* $(\mathbf{2},\mathbf{1})$*, this operation corresponds to* $xg_1 - yg_2 = x(xy + z^2) - y(x^2 - yz) = xz^2 - y^2z$*. Hence* $LT(xz^2 - y^2z) = xz^2 \in LT(I)$*.*

Let us continue the process.

For $d_0 = 4$

$$M = \begin{array}{c} \\ x^4 \\ x^3y \\ x^3z \\ x^2y^2 \\ x^2yz \\ x^2z^2 \\ xy^3 \\ xy^2z \\ xyz^2 \\ xz^3 \\ y^4 \\ y^3z \\ y^2z^2 \\ yz^3 \\ z^4 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} x^2g1 & xyg1 & xzg1 & y^2g1 & yzg1 & z^2g1 & x^2g2 & xyg2 & xzg2 & y^2g2 & yzg2 & z^2g2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Reduced form of $\mathbf{M}$ *is,*

	x^2g_1	xyg_1	xzg_1	y^2g_1	yzg_1	z^2g_1	x^2g_2	xyg_2	xzg_2	y^2g_2	yzg_2	z^2g_2
x^4	0	0	0	0	0	0	1	0	0	0	0	0
x^3y	1	0	0	0	0	0	0	0	0	0	0	1
x^3z	0	0	0	0	0	0	0	0	1	0	0	0
x^2y^2	0	1	0	0	0	0	0	0	0	0	0	0
x^2yz	0	0	1	0	0	0	0	0	0	0	0	0
x^2z^2	0	0	0	0	0	0	0	1	0	0	0	-1
xy^3	0	0	0	1	0	0	0	0	0	0	0	0
xy^2z	0	0	0	0	1	0	0	0	0	0	0	-1
xyz^2	0	0	0	0	0	1	0	0	0	0	0	0
xz^3	0	0	0	0	0	0	0	0	0	0	1	0
y^4	0	0	0	0	0	0	0	0	0	0	0	0
y^3z	0	0	0	0	0	0	0	0	0	1	0	0
y^2z^2	0	0	0	0	0	0	0	0	0	0	0	0
yz^3	0	0	0	0	0	0	0	0	0	0	0	0
z^4	0	0	0	0	0	0	0	0	0	0	0	0

The last column do not have a leading entry. Hence we can see that $z^2g_2 = x^2g_1 - xyg_2 - yzg_1$. *Hence* $\left(x^2 - yz, -xy - z^2, 0\right)$ *is a syzygy.*

Then $LT(I) := \{xy, x^2, xz^2, y^3z\}$

It can be read from the reduced form that $\dim W_4(I) = 4$ *and* $\left\{y^4, y^2z^2, yz^3, z^4\right\}$ *is a basis for* $W_4(I)$. *Now, in order to check that whether the equality condition of the algorithm is satisfied or not, we need to compute Hilbert Function on* $LT(I)$.

Using the technique of section 1.3 the Hilbert polynomial is the fixed number 4. *That means* $\dim(W_4(I)) = 4$

Since we have a syzygy of degree 4, *this syzygy produces*

$\binom{3+(d-4)-1}{3-1} = \frac{1}{2}d^2 - \frac{5}{2}d + 3$ *(This is lhs since m=0)*

syzygies for degree $d \geq 4$.

Now, $n = 3$, $deg(g_1) = 2$, $deg(g_2) = 2$, $deg(g_3) = 5$, *right side gives us*

$4 - \binom{d+3-1}{3-1} + 2\binom{d+3-2-1}{3-1} + \binom{d+3-5-1}{3-1} = d^2 - 6d + 9$ *for* $d \geq 5$ *(this is rhs)*

Since $\frac{1}{2}d^2 - \frac{5}{2}d + 3 = d^2 - 6d + 9$ *is not true for all* $d \geq 5$, *we shall continue to process.*

Recalling the equation $z^2g_2 = x^2g_1 - xyg_2 - yzg_1$, *multiplying this equation by* x, y *and* z *gives us more equality in degree* 5. *Because of this, we will not use the columns corresponding* xz^2g_2, yz^2g_2, z^3g_2 *in the next step.For* $d_0 = 5$,

$$M=\begin{array}{c} \\ x^5 \\ x^4y \\ x^4z \\ x^3y^2 \\ x^3yz \\ x^3z^2 \\ x^2y^3 \\ x^2y^2z \\ x^2yz^2 \\ x^2z^3 \\ xy^4 \\ xy^3z \\ xy^2z^2 \\ xyz^3 \\ xz^4 \\ y^5 \\ y^4z \\ y^3z^2 \\ y^2z^3 \\ yz^4 \\ z^5 \end{array}\begin{array}{c} \begin{array}{ccccccccccccccccccc} x^3g1 & x^2yg1 & x^2zg1 & xy^2g1 & xyzg1 & xz^2g1 & y^3g1 & y^2zg1 & yz^2g1 & z^3g1 & x^3g2 & x^2yg2 & x^2zg2 & xy^2g2 & xyzg2 & y^3g2 & y^2zg2 & g3 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&-1&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&-1&0&0&0&0&1&0\\ 0&1&0&0&0&1&0&0&0&0&0&0&-1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&1&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1 \end{pmatrix} \end{array}$$

Reduced form of the matrix M for $d_0 = 5$ is the following;

$$\begin{array}{c} \\ x^5 \\ x^4y \\ x^4z \\ x^3y^2 \\ x^3yz \\ x^3z^2 \\ x^2y^3 \\ x^2y^2z \\ x^2yz^2 \\ x^2z^3 \\ xy^4 \\ xy^3z \\ xy^2z^2 \\ xyz^3 \\ xz^4 \\ y^5 \\ y^4z \\ y^3z^2 \\ y^2z^3 \\ yz^4 \\ z^5 \end{array}\begin{array}{c} \begin{array}{ccccccccccccccccccc} x^3g1 & x^2yg1 & x^2zg1 & xy^2g1 & xyzg1 & xz^2g1 & y^3g1 & y^2zg1 & yz^2g1 & z^3g1 & x^3g2 & x^2yg2 & x^2zg2 & xy^2g2 & xyzg2 & y^3g2 & y^2zg2 & g3 \end{array} \\ \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}$$

There is no column without leading entry. So there is no new syzygy. On the other

hand, we have new element y^2z^3 *for* $LT(I)$*. If we compute Hilbert function of*

$LT(I) = \left\{xy, x^2, xz^2, y^3z, y^2z^3\right\}$*, the result is again a constant number* 3.

Since we have only one syzygy of degree 4*, the number of syzygies of degree* $d \geq 5$ *is*

$\binom{3+d-4}{2} = \binom{d-1}{2} = \frac{1}{2}d^2 - \frac{3}{2}d + 1$ (lhs)

On the other hand, since Hilbert polynomial is 3,

$3 - \binom{d+3-1}{3-1} + 2\binom{d+3-2-1}{3-1} + \binom{d+3-5-1}{3-1} = d^2 - 6d + 8$ *for* $d \geq 5$ (rhs)

Since $lhs \neq rhs$*, we will continue to process. Now, we are going to compute the reduced form of the matrix* M *for* $d_0 = 6$*.*

For $d_0 = 6$ *,*

M=

	x^4g1	x^3yg1	x^2y^2g1	xy^3g1	y^4g1	x^3zg1	x^2yzg1	xy^2zg1	y^3zg1	x^2z^2g1	xyz^2g1	y^2z^2g1	xz^3g1	yz^3g1	z^4g1	x^4g2	x^3yg2	x^2y^2g2	xy^3g2	y^4g2	x^3zg2	x^2yzg2	xy^2zg2	y^3zg2	$xg3$	$yg3$	$zg3$
x^6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
x^5y	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
x^5z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
x^4y^2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
x^4yz	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0	0
x^4z^2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x^3y^3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
x^3y^2z	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	0	0	0	0
x^3yz^2	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
x^3z^3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x^2y^4	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
x^2y^3z	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	1	1	0	0
$x^2y^2z^2$	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0
x^2yz^3	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x^2z^4	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
xy^5	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
xy^4z	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	0
xy^3z^2	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	1
xy^2z^3	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
xyz^4	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
xz^5	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
y^6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
y^5z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
y^4z^2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0
y^3z^3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
y^2z^4	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
yz^5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0
z^6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1

The reduced form of the matrix **M** *is*

	x^4g_1	x^3yg_1	$x^2y^2g_1$	xy^3g_1	y^4g_1	x^3zg_1	x^2yzg_1	xy^2zg_1	y^3zg_1	$x^2z^2g_1$	xyz^2g_1	$y^2z^2g_1$	xz^3g_1	yz^3g_1	z^4g_1	x^4g_2	x^3yg_2	$x^2y^2g_2$	xy^3g_2	y^4g_2	x^3zg_2	x^2yzg_2	xy^2zg_2	y^3zg_2	xg_3	yg_3	zg_3
x^5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
x^5y	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x^5z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
x^4y^2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x^4yz	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x^4z^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
x^3y^3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
x^3y^2z	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
x^3yz^2	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x^3z^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
x^2y^4	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x^2y^3z	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1
$x^2y^2z^2$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1
x^2yz^3	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1
x^2z^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
xy^5	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
xy^4z	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
xy^3z^2	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
xy^2z^3	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
xyz^4	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
xz^5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1
y^6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
y^5z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
y^4z^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
y^3z^3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	−1
y^2z^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
yz^5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	−1
z^6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

In the reduced form of the matrix M the last column do not have a leading entry. Hence we can see that,

$zg_3 = x^2y^2g_1 + x^2yzg_1 - xy^2zg_1 - xyz^2g_1 - xz^3g_1 + yz^3g_1 + z^4g_1 - xy^2zg_2 - xy^3g_2 + xg_3 - yg_3.$

Hence $\left(\mathbf{x^2y^2 + x^2yz - xy^2z - xyz^2 - xz^3 + yz^3 + z^4, -xy^2z - xy^3, x - y - z}\right)$

is a syzygy.

We have a new element for LT(I) which is yz^5. *So* $LT(I) = \{xy, x^2, xz^2, y^3z, y^2z^3, yz^5\}$

It can be read from the reduced for of M that $dim W_6(I) = 2$ *and* $\left\{y^6, z^6\right\}$ *is a basis for* $W_6(I)$. *Now in order to check that whether the equality condition of the algorithm is satisfied or not, we again need to compute Hilbert function of* $LT(I)$.

Referencing the technique of 1.3 , the Hilbert polynomial is the fixed number 2. *That means* $dim(W_d(I)) = 2,\ d \geq 6$

Since we have a syzygy of degree 4 *and degree* 6 *these syzygies produces* $\binom{3+(d-4)-1}{2} = \frac{1}{2}d^2 - \frac{5}{2}d + 3$ *and* $\binom{3+(d-6)-1}{2} = \frac{1}{2}d^2 - \frac{9}{2}d + 10$ *syzygies for degree* $d \geq 6$ *respectively.*

Since $n = 3, deg(g_1) = 2, deg(g_2) = 2, deg(g_3) = 5$, *the right side give us,*

$2 - \binom{d+3-1}{3-1} + \binom{d+3-2-1}{3-1} + \binom{d+3-2-1}{3-1} + \binom{d+3-5-1}{3-1} = d^2 - 6d + 7$

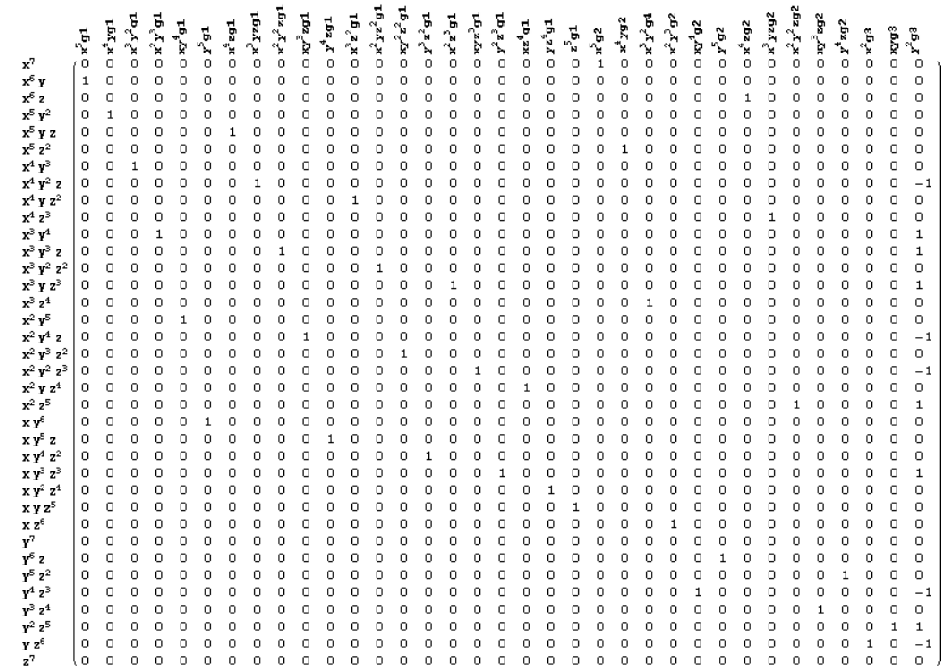
Since lhs $\neq$ *rhs we shall continue the process.*

Recalling the equations $zg_3 = x^2y^2g_1 - xy^2zg_1 - xyz^2g_1 - xz^3g_1 + yz^3g_1 - xy^2zg_2 - xy^3g_2 + xg_3 - yg_3$ *and* $z^2g_2 = x^2g_1 - xyg_2 - yzg_1$, *multiplying these equations by*

monomials of degree 1 and of degree 3 gives us more equality in degree 7. Because of this, we are not going to include the columns corresponding to xzg_3, yzg_3, z^2g_3 *and* $x^3z^2g_2, x^2yz^2g_2, x^2z^3g_2, xy^2z^2g_2, xyz^3g_2, xz^4g_2, y^3z^2g_2, y^3z^2g_2, y^2z^3g_2, yz^4g_2, z^5g_2$.

For $d_0 = 7$

the reduced form of the matrix is,



In the reduced for of the matrix M the last column do not have a leading entry.

Hence we can see that,

$y^2g_3 = -x^3yzg_1 + x^2y^3g_1 + x^2y^2zg_1 + x^2z^3g_1 - xy^3zg_1 - xyz^3g_1 + y^2z^3g_1 + x^2y^2zg_2 - xy^4g_2 - x^2g_3 + xyg_3$

So , from this equation , we have a new syzygy:

$\left(\mathbf{-x^3yz + x^2y^3 + x^2y^2z + x^2z^3 - xy^3z - xyz^3 + y^2z^3, x^2y^2z - xy^4, -x^2 + xy - y^2}\right)$.

$LT(I) = \left\{xy, x^2, xz^2, y^3z, y^2z^3, yz^5\right\}$

Again $dim(W_d(I)) := 2$. *This gives us*

$$rhs = 2 - \binom{d+3-1}{3-1} + \binom{d+3-2-1}{3-1} + \binom{d+3-2-1}{3-1} + \binom{d+3-5-1}{3-1} = d^2 - 6d + 7$$

On the other hand,

$$lhs = \binom{3+(d-4)-1}{3-1} + \binom{3+(d-6)-1}{3-1} + \binom{3+(d-7)-1}{3-1} = \frac{3}{2}d^2 - \frac{25}{2}d + 28$$

We can easily see that lhs > rhs for $d \geq 8$. *Hence dimension of syzygy for* $d = 8$ *is not correct. This means that we have at least one syzygy which is a linear combination of others. Since we find* $y^2g_3 = -x^3yzg_1 + x^2y^3g_1 + x^2y^2zg_1 + x^2z^3g_1 - xy^3zg_1 - xyz^3g_1 + y^2z^3g_1 + x^2y^2zg_2 - xy^4g_2 - x^2g_3 + xyg_3$ *for the last syzygy, we have to delete columns corresponding to* xy^2g_3, y^3g_3, y^2zg_3 *in* $d_0 = 8$. *However for the second syzygy, we find* $zg_3 = x^2y^2g_1 - xy^2zg_1 - xyz^2g_1 - xz^3g_1 + yz^3g_1 - xy^2zg_2 - xy^3g_2 + xg_3 - yg_3$. *Therefore, we have to delete* $x^2zg_3, xyzg_3, xz^2g_3, y^2zg_3, yz^2g_3, z^3g_3$. *So we delete* y^2zg_3 *twice and we have to subtract* $\binom{3+(d-8)-1}{3-1}$ *from lhs. Hence,*
$lhs = \binom{3+(d-4)-1}{3-1} + \binom{3+(d-6)-1}{3-1} + \binom{3+(d-7)-1}{3-1} - \binom{3+(d-8)-1}{3-1} = d^2 - 6d + 7.$

So lhs = rhs. This terminates the algorithm.

The syzygy bases returned is $\Big\{\{x^2 - yz, -xy - z^2, 0\}, \{x^2y^2 + x^2yz - xy^2z - xyz^2 - xz^3 + yz^3 + z^4, -xy^2z - xy^3, x - y - z\}, \{-x^3yz + x^2y^3 + x^2y^2z + x^2z^3 - xy^3z - xyz^3 + y^2z^3, x^2y^2z - xy^4, -x^2 + xy - y^2\}\Big\}$

CHAPTER 3

H-BASIS (HOMOGENEOUS BASIS) ALGORITHM

3.1 Introduction

The H-bases were first introduced by Macaulay [4]. His original motivation was transformation of systems of polynomial equations into simpler ones. the power of this concept was not really understood presumably because of the lack of facilities for symbolic computations. When Computer Algebra Systems come up, Gröbner Bases were used instead of H-bases. these bases, originally invented by Buchberger [5] for computing multiplication tables for factor rings, are now also applied for simplifying some problems in Numerical Analysis. However,all approaches related to Gröbner Bases are fundamentally tied on term orders which leads to asymmetry among the variables to be considered. On the other hand, the concept of H-bases is based solely on homogeneous terms of a polynomial. Because of this, H-bases seem to be better suited for numerical problems.Comparing of two bases in application of numerical analysis can be found in [6] , [7].

It is known that a Gröbner bases with respect to a degree compatible ordering is also a H-basis. However, the Gröbner basis with respect to a degree compatible ordering is al so a H-basis. However, the Gröbner bases usually contains more elements than necessary for a h-bases. The elimination of these unnecessary elements was studied in [8].Maculay gave an example of a H-basis construction in [9] His method consist in a succession of homogenization and dehomogenization of ideals and uses heavily that he knows basis for the module of syzygies. This method clarified by Möller and Sauer in [6]. In there , they showed that the reduction of polynomials can be done only using linear algebra. They do not need to division algorithm to do this reduction. Unfortunately, efficiency of their algorithm depends on the complexity of computing bases for

module of syzygies. However, the only known method for the computing the syzygies involves Gröbner bases techniques described in Chapter 1. Because of this, they claimed that; it is more efficient to compute an H-bases rather by the Buchberger's algorithm using a degree compatible ordering than by the procedure they suggested which use Gröbner bases techniques for the calculation of module of syzygies. However, in chapter 2 we found an alternative method to compute the module of syzygies using only linear algebra.

Combining Möller and Sauer's procedure of computing H-bases with our method of finding a syzygy bases, we are able to give an algorithm which obtains an H-bases from a given basis o an ideal using only linear algebra.

3.2 Background

In this section we will define some terms and state some theorems.

Definition 3.1 *We say that a polynomial $f \in k[x_1, x_2, \ldots, x_n]$ is* **homogeneous of total degree** *d if every term of f has total degree d*

Let $f \in k[x_1, x_2, \ldots, x_n]$. Then $f = \sum_k f_k$, where f_k is the sum of all terms of f with total degree k. Noting that f_k is homogeneous, we can say that any $f \in k[x_1, x_2, \ldots, x_n]$ can be written uniquely as a sum of homogeneous polynomials.

In the next definition we will define the homogenization of a polynomial $g \in k[x_1, x_2, \ldots, x_n]$.

Definition 3.2 *Let $g \in k[x_1, x_2, \ldots, x_n]$ be a polynomial of total degree d and $g = \sum_{i=0}^{d} g_i$, be the decomposition of g as the sum of its homogeneous components, where g_i has total degree i, then*

$$g^h(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{d} g_i(x_1, \ldots, x_n) x_0^{d-i}$$

is called the homogenization of g with respect to x_0

The next proposition reveals some facts about the process of homogenization.

Proposition 3.3 *Let $g(x_1, x_2, \ldots, g_n) \in k[x_1, x_2, \ldots, x_n]$ be a polynomial of total degree d*

(i) The homogenization of g, g^h, is a homogeneous polynomial of total degree d.

(ii) The homogenization of g can be computed using the formula

$$g^h = x_0^d g(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}).$$

(iii) Dehomogenizing g^h yields g. That is, $g^h(1, x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$

(iv) Let $F(x_0, x_1, \ldots, x_n)$ be a homogeneous polynomial and let x_0^e be the highest power of x_0 dividing F. If $f = F(1, x_1, \ldots, x_n)$ is the dehomogenization of F, then $F = x_0^e f^h$.

(v)$f^h g^h = (fg)^h$

Definition 3.4 *An ideal $I \in k[x_1, x_2, \ldots, x_n]$ is said to be homogeneous if for every $f \in I$, the homogeneous components f_i's of f are in I also.*

In the following proposition it is stated another equivalent condition for an ideal to be homogenous.

Proposition 3.5 *Let $I \subset k[x_1, x_2, \ldots, x_n]$ be an ideal. Then the following statements are equivalent:*

(i) I is an homogeneous ideal of $k[x_1, x_2, \ldots, x_n]$

(ii) $I = \langle f_1, f_2, \ldots, f_s \rangle$ for some homogeneous polynomials $f_1, f_2, \ldots, f_s$ in I.

The following lemma is about a property of homogeneous ideals that can be used to find a criterion for a homogeneous basis.

Lemma 3.6 *Let $f_1, f_2, \ldots, f_s$ be homogeneous polynomials of total degree $d_1 \leq d_2 \leq \ldots \leq d_s$ and let $I = \langle f_1, f_2, \ldots, f_s \rangle$ If g is another polynomial of degree d in d, then g must be an element of $I_d = \langle f_i : deg(f_i) \leq d) \rangle \subset I$. Moreover, there exist $q_1, q_2, \ldots, g_t$ with $g = \sum_{i=1}^{t} q_i f_i$ such that each q_i is homogeneous of total degree $d - d_i$.*

Proof. Assume there exists t such that each $\deg(f_i) > d$ for $i > t$. Let

$$g = p_1 f_1 + p_2 f_2 + \ldots p_t f_t + \ldots p_s f_s.$$

Since $\deg(g) = d$ and $\deg(p_i f_i) > d$ for $i \leq t$, $p_t f_t + \ldots + p_s f_s = 0$. Since $f_i \neq 0$, $p_i = 0$ for $i > t$. Therefore

$$g = p_1 f_1 + p_2 f_2 + \ldots p_t f_t$$

If we write p_i as a sum of two polynomials $p_i = q_i + r_i$ where q_i is a homogeneous polynomial of total degree $d - d_i$ and r_i is the other part of p_i, then the multiplication $r_i f_i$ produces monomials which have total degree different from d. Since g is a homogeneous polynomial of total degree d, $r_i f_i = 0$ for all i. Hence

$$g = q_1 f_1 + q_2 f_2 + \ldots + q_t f_t.$$

□

3.3 Homogeneous Basis

Definition 3.7 *Let I be an ideal in $k[x_0, x_1, \ldots, x_n]$. The homogenization of I is defined to be the ideal,*

$$I^h = \langle f^h : f \in I \rangle \subset k[x_0, x_1, \ldots, x_n]$$

where f^h is the homogenization of f with respect to x_0.

Proposition 3.8 *For any ideal $I \subset k[x_1, x_2, \ldots, x_n]$, the homogenization I^h is a homogeneous ideal in $k[x_0, x_1, \ldots, x_n]$.*

Proof.

By the Hilbert Basis Theorem, there exist polynomials $g_1, g_2, \ldots, g_n \in I^h$ such that $I^h = \langle g_1, g_2, \ldots, g_n \rangle$. By the definition of I^h,

$$g_i = \sum a_{ij} f_{ij}^h,$$

where $f_{ij} \in I$. Since f_{ij}^h's are homogeneous and $I^h =< f_{ij}^h >$, I^h is a homogeneous ideal by the proposition 3.5. □

Definition 3.7 defines I^h ,**homogenization of I**, where $I \in k[x_1, x_2, \ldots, x_n]$ but does not give us a generating set for I^h.Let $I = \langle f_1, f_2, \ldots, f_s \rangle \subset k[x_1, x_2, \ldots, x_n]$. It is always true for I^h, **homogenization of I**, that $\langle f_1 h, \ldots, f_s^h \rangle$ is a subset of I^h. But ,as we are going to see is the next example, I^h can be strictly larger than $\langle f_1 h, \ldots, f_s^h \rangle$.

Example 3.9 *Let $I = \langle f_1, f_2 \rangle = \langle y - x^2, z - x^3 \rangle$, the ideal of the twisted cubic in $\mathbb{R}^3$. The homogenization of f_1 and f_2 gives $J = \langle yt - x^2, zt^{2-x^3} \rangle$ in $\mathbb{R}[t, x, y, z]$. To prove $J \neq I^h$, consider the polynomial*

$$f_3 = f_2 - x f_1 = z - x^3 - x(y - x^2) = z - xy \in I.$$

Then $f_3^h = tz - xy$ is a homogeneous polynomial of degree 2 in I^h. Since the generators of J are also homogeneous of degree 2 and 3 respectively, if $f_3^h \in I^h$, then f_3^h should be constant multiple of f_1^h by lemma 3.6 . Since this is obviously false $f_3^h \notin J$ and thus $J \neq I^h$

In the following definition we will define what a **homogeneous basis** is.

Definition 3.10 *Let $I = \langle f_1, f_2, \ldots, f_t \rangle$ be an ideal in $k[x_1, x_2, \ldots, x_n]$. If the homogenization of I, $I^h = \langle f_1^h, \ldots, f_t^h \rangle$, then $f_1, f_2, \ldots, f_t$ is said to be a* **homogeneous basis** *for I.*

Note that there always exists a homogeneous basis for an ideal I by the proposition 3.8.

Example 3.9 shows that not every basis is a homogeneous basis. Therefore we need a criterion to test a basis whether it is homogeneous or not. For this, let us return back to Example 3.9. Multiplying f_2 by t, $t(z - xy)^h = t(z - x^3)^h - x(y - x^2)^h$. Note that the

power of t multiplied is exactly difference between total degrees of two sides.The next lemma generalizes this observation.

Lemma 3.11 *Let $I = \langle f_1, f_2, \ldots, f_m$ and $f = \sum_{i=1}^{m} q_i f_j$. Set*

$$d = max\ \{d_i = deg(q_i f_i) : 1 \leq i \leq m\}$$

and $d' = deg f$. Then $x_0^{d-d'} f^h \in \langle f_1^h, f_2^h, \ldots, f_m^h \rangle$.

The next theorem gives us a necessary and sufficient condition for a basis to be homogeneous.

Theorem 3.12 *Given $I = \langle f_1, f_2, \ldots, f_s \rangle \in k[x_1, x_2, \ldots, x_n]$ $I^h = \langle f_1^h, f_2^h, \ldots, f_s^h \rangle$ if and only if for every $f \in I$ there exist $a_1, a_2, \ldots, a_s \in k[x_1, k_2, \ldots, x_n]$ such that $f = a_1 f_1 + a_2 f_2 + \ldots + a_s f_s$ and $deg(f) = max\{deg(a_i f_i), i = 1, \ldots, s\}$*

Proof. Assume $\{f_1, f_2, \ldots, f_s\}$ is a homogeneous bases. Then $f \in \langle f_1 1, f_2, \ldots, f_s \rangle$ implies $f^h \in \langle f_1^h, f_2^h, \ldots, f_s^h \rangle$. If $deg(f) = d$ then $deg(f^h) = d$. By Lemma 3.6, there exist $A_1, A_2, \ldots, A_s$ such that $f^h = \sum_{i=1}^{s} A_i f_i^h$ and $max\{\, deg(A_i f_i^h)\} = d$. By part (iii) of Proposition3.3 , $f = \sum_{i=1}^{s} A_i(1, x_1, \ldots, x_n) f_i$

Hence,

$$d = deg(f) \leq max\{deg(A_i(1, x_1, \ldots, x_n) f_i)\} \leq max\{deg(A_i f_i^h)\} = deg(f^h) = d.$$

Conversely, if $f = a_1 f_1 + \ldots + a_s f_s$ such that $deg(f) = max\{deg(a_i f_i)\}$ for some a_i's, then $f^h \in \langle f_1^h, \ldots, f_s^h \rangle$ by Lemma 3.11. □

Definition 3.13 *If $\sum_i^d f_i$ is the decomposition of F as the sum of its homogeneous components , Then $LT(F) = f_d$ is called the leading form of f. The homogeneous ideal of leading forms are also defined by $LF(I) = \left\langle LF(f) : f \in I \right\rangle$.*

Lemma 3.14 *Let $I =< f_1, \ldots, f_s >$ be an ideal of $k[x_1, \ldots, x_n]$.The followings are equivalent.*

i) *Every* $f \in I$ *has a representation* $f = \sum_i h_i f_i$ *where* $deg f_i = max\{deg(h_i f_i) : i = 1, \ldots, s\}$.

ii) $LF(I) = \Big\langle LF(f_1, \ldots, LF(f_s)) \Big\rangle$.

Now, in the next theorem, we will give another equivalent condition for a basis of an ideal to be homogeneous.

Theorem 3.15 *Let* $I = \Big\langle f_1, \ldots, f_t \Big\rangle \subset k[x_1, \ldots, x_n]$, *and let the columns of the* **t x l** *matrix* $S = (s_{ij})$ *be a generating set of* $syz\Big(LF(f_1), \ldots, LF(f_t)\Big)$. *Further assume that each* $\sum s_{ij} LF(f_j)$ *is a homogeneous polynomial.*

Then $F = \{f_1, \ldots, f_s\}$*is a homogeneous bases if and only if each*

$$q_i = \sum_{j=1}^{t} s_{ji} f_j = \sum_{j=1}^{t} s_{ji}(f_j - LF(f_j)) = \sum_{j=1}^{t} a_{ji} f_j$$

for some $a_{ji} \in k[x_1, \ldots, x_n]$ *such that* $deg(q_i) = max\{deg(a_{ij} f_i) : j = 1, \ldots, t\}$ *for* $i = 1, \ldots, l$.

3.4 Computation of H-Bases

The Theorem 3.15 suggest the following procedure for computing H-bases.

Given a set of polynomials $F = \{f_1, \ldots, f_m\}$ generating and ideal I, first the elements of a bases of

$$Syz(LF(f_1), \ldots, LF(f_m)) ...(*)$$

have to be computed. Then for each bases element $(g_1, \ldots, g_m)$ the polynomial

$$q = g_1 f_1 + g_2 f_2 + \ldots + g_m f_m$$

has to reduced modulo F as far as possible. If the reduction procedure terminates with a nonzero polynomial f, the set F is enlarged by f. For the new set F a bases for the module of syzygies is computed etc. This procedure differs from Buchberger's

algorithm only by the module of syzygies. Instead of basis of (*) one consider for Gröbner bases computation the bases of

$$Syz(LT(f_1), \ldots, LT(f_m)).$$

As Buchemberg's algorithm terminates correctly with a Gröbner bases, our proposed algorithm terminates with an H-bases. before giving precise statement of the algorithm, we will define reduction process.

Definition 3.16 *For a given $f_1 f_2, \ldots, f_m \in k[x_1, \ldots, x_n]$ we say that f reduces to $\overline{f}$ module $F = \{f_1, \ldots, f_m\}$ if $LF(f) = a_1 LF(f_1) + a_2 Lf(f_2) + \ldots + a_m LF(f_m)$ and*

$$\overline{f} = f - (a_1 f_1 + a_2 f_2 + \ldots + a_m f_m)$$

In this case we write

$$f \underset{F}{\rightarrow} \overline{f}$$

By $\overset{*}{\underset{F}{\rightarrow}}$ we denote the transitive closure of the binary relation $\underset{F}{\rightarrow}$. We also say f reduces modulo F to g if $f \overset{*}{\underset{F}{\rightarrow}} g$

Algorithm 3.17 *Algorithm to compute h-basis*

INPUT : *A bases $F = \{f_1, \ldots, f_m\}$ of an ideal I.*

OUTPUT : *a H- bases $H = \{f_1, \ldots, f_s\}$ of the same ideal I.*

$H := F \;\; ; S := m$

Construct *a bases S for $Syz(LF(H))$*

While $S \neq \{\}$

Select $(g_1, \ldots, g_s) \in S$, $S := S - \{(g_1, \ldots, g_s)\}$

Construct $f = g_1 f_1 + \ldots + g_s f_s$

$P := f \overset{*}{\underset{F}{\rightarrow}}$

If $P \neq 0$, $H := H \cup \{P\}$

Construct *a bases S for Syz(LF(H))*

End While

Return H.

Proof.

The proof that this algorithm terminates, a consequences of the ascending chain condition, and produces a H-bases, which is follows from the theorem 3.15, is exactly the same as for Gröbner bases with respect to a term order.

□

Now, we will give an example for the computation of an H-bases basis of an ideal to illustrate the algorithm given above.

Example 3.18 *Let* $I =< f_1 = x^2 - y, f_2 = xy - z, f_3 = y^2z - x$ *and let*

$$g_1 = LF(f_1) = x^2,\ g_2 = LF(f_2)xy,\ g_3 = LF(f_3) = y^2z, J =< g_1, g_2, g_3 > .$$

Syzygy basis of g_1, g_2, g_3 *is,*

$$\left\langle \begin{pmatrix} y \\ -x \\ o \end{pmatrix}, \begin{pmatrix} 0 \\ yz \\ -x \end{pmatrix} \right\rangle$$

The first syzygy gives

$h = yf_1 - xf_2 = xz - y^2$ $LF(h) = h = xz - y^2$

J_2 *is generated by* $< x^2, xy >$

since $LF(h) \notin J_2, f_4 = h.$

$I =< f_1, f_2, f_3, f_4 >,\ g_i = LF(f_i)\ ,\ J =< g_1, g_2, g_3, g_4 >.$

Syzygy basis of (g_1, g_2, g_3, g_4) *is*

$$\left\langle \begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -z \\ y \\ 0 \\ x \end{pmatrix} \begin{pmatrix} 0 \\ yz \\ -x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -z^2 \\ y \\ yz \end{pmatrix} \right\rangle$$

We have already looked at the first syzygy.

The second syzygy gives,

$h = -zf_1 + yf_2 + xf_4 = 0.$

The third syzygy gives,

$h = yzf_2 - xf_3 = -yz^2 + x^2$, $LF(h) = -yz^2$

J_3 *is generated by* $< x^3, x^2y, x^2z, zy^2, xyz, y^2z, x(xz - y^2), y(xz - y^2), z(xz - y^2) >$.

Since $LF(H) \notin J_3$, $f_5 = h = -yz^2 + x^2$.

$I =< f_1, f_2, f_3, f_4, f_5 >$, $g_i = LF(f_i)$ $J =< g_1, g_2, g_3, g_4, g_5 >$.

Syzygy basis of $(g_1, g_2, g_3, g_4, g_5)$ *is* ,

$$\left\langle \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -z \\ y \\ 0 \\ x \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ yz \\ -x \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z \\ 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ -z^2 \\ 0 \\ 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ y \\ yz \\ x \end{pmatrix} \right\rangle$$

WE have already looked at the first three syzygies.

The fourth syzygy gives,

$h = zf_3 + yf_5 = x^2y - xz \quad LF(h) = x^2y$

$LF(h) = x^2y = yg_1$

$h_1 = h - yg_1 = (x^2y - xz) - y(x^2 - y) = -xz + y^2.$

$LF(h_1) = h_1 = -xz + y^2 = -g_4$

$h_2 = h_1 - f_4 = (-xz + y^2) - (xz - y^2) = 0.$

The fifth syzygy gives

$h = z^2f_2 + xf_5 = x^3 - z^2 \quad LF(h) = x^3 - z^3$

$LF(h) \notin J_3$ *So,* $f_6 = h = x^3 - z^3$.

$I =< f1, f_2, \ldots, f_6 > \quad J =< g_1, g_2, \ldots, g_6 >$

Syzygy basis of $(g_1, g_2, g_3, g_4, g_5, g_6)$ *is,*

$$\left\langle \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -z \\ y \\ 0 \\ x \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ yz \\ -x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z \\ 0 \\ y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -z^2 \\ 0 \\ 0 \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ y \\ yz \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \\ 0 \\ 0 \\ z \\ -y \end{pmatrix}, \begin{pmatrix} x^2 \\ 0 \\ -z \\ z^2 \\ 0 \\ -x \end{pmatrix} \right\rangle$$

we have already looked at the first five syzygies.

The sixth syzygy

$h = yf_3 + yzf_4 + xf_5 = x^3 - xy$. $LF(h) = x^3$.

$LF(h) = x^3 = xg_1$

$h_1 = h - xf_1 = (x^3 - xy) - x(x^2 - y) = 0$.

The seventh syzygy,

$h = xyf_1 + zf_5 - yf_6 = x^2z - xy^2 \quad LF(h) = x^2z - zy^2$

$LF(h) = x^2z - xy^2 = zg_1 - yg_2$

$h_1 = h - zf_1 + yf_2 = x^2z - zy^2 - z(x^2 - y) + y(xy - z) = 0$

The eighth syzygy

$h = x^2f_1 - zf_3 - z^2f_4 - xf_6 = -x^2y + xz \quad LF(h) = -x^2y$

$LF(h) = -yg_1$

$h_1 = h + yf_1 = -x^2y + xz + y(x^2 - y) = xz - y^2$

$LF(h_1) = xz - y^2 = g_3$

$h_2 = h_1 - f_3 = xz - y^2 - (xz - y^2) = 0$

We have found H-basis.

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