BERNSTEIN CENTER FOR GL(n,$\mathbf{Q}_p$)

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ABSTRACT

BERNSTEIN CENTER FOR GL(n,$\mathbf{Q}_p$)

In this thesis we will present J. N. Bernstein's description of the centre of the category of smooth modules over l-groups for the case of $GL(n, \mathbf{Q}_p)$. Firstly we decompose this category into two parts: quasi-cuspidal and non-quasi-cuspidal components. Then we analyse the center of each component separately. In the first part, we follow the line of reasoning in the article "Le Centre de Bernstein" for the general case of l-groups. In the second part, we will analyse the centre of non-quasi-cuspidal components for the special case of $GL(n, \mathbf{Q}_p)$.

ÖZET

GL(n,$\mathbf{Q}_p$) İÇİN BERNSTEIN MERKEZİ

Bu tezde J. N. Bernstein'ın tasvir ettigi l-gruplarının cebirsel modülleri kategorisinin merkezini $GL(n, \mathbf{Q}_p)$ özel durumu için sunacağız. Önce bu kategoriyi iki kısma ayıracağız: "quasi-cuspidal" ve "quasi-cuspidal" olmayan kısımlar. Daha sonra, her kısmın merkezini ayrı ayrı inceleyeceğiz. İlk bölümde "quasi-cuspidal" kısmın merkezini oluştururken, genel olarak l-gruplar için Bernstein'ın "Le Centre de Bernstein" makalesinde izledigi sırayı takip edeceğiz. İkinci bölümdeyse, "quasi-cuspidal" olmayan kısımların merkezini $GL(n, \mathbf{Q}_p)$ özel durumu için inceleyeceğiz.

TABLE OF CONTENTS

LIST OF SYMBOLS/ABBREVIATIONS

$Alg\ G$	The category of smooth representations of G
$(Alg\ G)_f$	The category of finite representations of G
$(Alg\ G)_{nqc}$	The category of non-quasi-cuspidal representations of G.
$(Alg\ G)_{qc}$	The category of quasi-cuspidal representations of G.
$\mathcal{B}(A)$	The power set of A
X^c	The complement of the set X in its container
$C^\infty(X)$	Locally constant complex valued functions on X
$\mathbf{D}_n$	The algebraic subgroup of diagonal matrices in $\mathbf{GL}(n)$
e_K	The idempotent $p_K^*(\mu_K)$ of $\mathcal{H}(G)$
$G^\wedge$	The set of smooth irreducible representations of G
$\mathbf{GL}(n)$	The general linear subgroup of $n \times n$ comlex matrices with the corresponding algebraic structure
$\mathcal{H}(G)$ or $\mathcal{H}$	The Hecke algebra of G
$\mathcal{H}(G)^\wedge$	The completion of the Hecke algebra
i_P^G	The parabolic induction from the parabolic subgroup P
$Ind(G, H, \rho)$	The induced representation by ρ
$ind(G, H, \rho)$	The finitely induced representation by ρ
$int_n(g)$	The conjugate of g by n
$\mathbf{Irr}(Alg\ G)$	The isomorphism class of irreducible representations in $(Alg\ G)$
$JH(\pi)$	The Jordan-Hölder content of π
$ker(f)$, $coker(f)$	Kernel and cokernel of f
N_i	The congruence subgroup of $GL(n, \mathbf{Q}_p)$ generated by p^i
$\mathbf{N}_n$	The index set $\{1, 2, \ldots, n\}$
$N_G(L)$	The normalizer of L in G
$\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{Q_p}, \mathbf{C}$	Respectively the set of natural numbers/integers/rational numbers/p-adic numbers/ complex numbers
$\mathcal{O}(X)$	The algebra of regular functions on the variety X
pr_i	The projection to the ith component

r_P^G	The Jacquet (restriction) functor
$S_c^*(X)$	The space of compactly supported distributions
$stab\ \xi$	The stabilizer of ξ in G
$supp(T)$	The support of the distribution T
$S(X)$	The space of Schwartz functions on X
$S^*(X)$	The space of distributions on X
V^N	The subspace of N invariant vectors in V
V_α	The smooth part of V
V^*	The space of linear functionals on V
$W(G_\alpha, D)$	The set of n's in $N_G(G_\alpha)$ for which D is stable under conjugation by n
$Z(G)$,$Z(\mathcal{H})$,$Z(\mathcal{A})$	Respectively, the center of the group G, the center of the ring $\mathcal{H}$, the center of the category $\mathcal{A}$
Δ_G	Modulus of the Haar measure
δ_g	Dirac distribution of g
$\delta_{i,j}$	The Kronecker delta
μ_G	The Haar measure on G
(π, G, V) or (V, π)	A representation π of G on V
π^{g_0}	The conjugate of π by g_0
χ_X	The characteristic function of the set X
$\mathcal{A} \simeq \mathcal{B}$	The categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent
$T_1 * T_2$	The convolution of the distributions T_1 and T_2
$\lvert z \rvert$	The absolute value of the complex number z

1. INTRODUCTION

In the representation theory of p-adic groups, we have the notion of a Hecke algebra associated to our group which plays the role of the group ring of a finite group. This algebra has the property that a smooth representation of the group is also a nondegenerate module on this algebra and vice versa. The main problems of the representation theory of such a group can be described in two steps: the classification of all irreducible representations and to express a given representation in terms of irreducible ones. The general strategy to attack the second problem is to find a sufficiently big family of operators acting on the representation space on which the group (algebra) acts and to find the joint eigenspaces of these commuting family of operators. Therefore, it is natural to study the center of the category of the nondegenerate modules over the Hecke algebra. J.N. Bernstein proposed in [1] to study the center by decomposing the whole category of smooth representation of G into the product of smaller categories whose centers can be viewed as algebras of regular functions on appropriate algebraic varieties. This way Bernstein was able to use the tools of algebraic geometry to attack the problems in the representation theory of p-adic groups. Our aim in this thesis is to introduce the decomposition of this category into cuspidal and noncuspidal components and study the center of the cuspidal components in detail. In the second part we will also sketch the center of the non-cuspidal components for the sake of completeness.

In the preliminaries, we will give some basic notions from [2] on representations of l-groups that we will essentially use in the thesis. The reader may find the details in the first chapter in the same place. Moreover, we will also directly use further results from [2], by giving reference to them. This article is very fundamental for the understanding of the subject. We will also give some useful definitions from Category Theory. Furthermore, we should note that in appropriate places we will freely use the language of N. Bourbaki. For example, at Proposition 3 in the first part we used the terminology of [7] and [8] without any further preperation.

In the first part we will use the first part of [1]; we will follow his line of reasoning as in the article. We will give the description of the center for l-groups in general. However we will only take the essential parts for the case of $GL(n, \mathbf{Q}_p)$ and we will try to further explain these. The construction of the center will still remain rather straightforward. This is because in the preparation of the thesis, at one point I needed to make a choice between going further to complete the description of the center for the whole category, and to give more details for the quasi-cuspidal part. I chose the former, for it would be more motivating for me to learn new things than to dig for the old ones. Therefore, besides an in some degree detailed analysis of the cuspidal component, part I also includes a preperation for the second part.

In the second part we will study the center of the non-quasi-cuspidal components for $GL(n, \mathbf{Q}_p)$. Again, we will benefit from the second part of Bernstein's article where he deals with the case of reductive p-adic groups in general. We will confine ourselves to the context of $GL(n, \mathbf{Q}_p)$; it is a concrete case and the arguments will be easier. Chapter 4 and 5 of [2] will provide us with the fundamentals of this construction. We will thus begin by introducing some of these results and then conclude the thesis by describing the center following Bernstein [1].

The original article "Le centre de Bernstein" is in french. To provide english-speaking readers with a chance of comparison we will put a translation of the first part of this article in the appendix. A remark on Azumaya Algebra in the original text is omitted; in fact we will not need this concept in any part of the thesis. The translation text will also include extra explanatory remarks besides the original text. There will also be some typographical corrections. The reader may see the french original in J.N. Bernstein's personal web site. I hope the translation do not harass much the article's originality; I ask for your pardon in advance for any mistake that might have occured.

Finally, I claim no originality in the main text in case of a controversy; many times I just rewrite Bernstein's original arguments with few additions. My principal goals in writing this thesis were to expand my knowledge of mathematics in the area, to understand Bernstein's article and to learn how to write a thesis. Due to my limits I may not be able to fully achieve these objectives; however I made a remarkable amount of elementary and specific reading related to the subject some of which are listed in the references. Lastly, I checked the text several times for mistakes. Yet in order to be cautious I put the sentences, which are susceptible to me and I had no opportunity to check over, between the symbols (ZZZ).

2. PRELIMINARIES

We call a topological space an l-space if it is Hausdorff, locally compact, and such that each point has a fundamental system of open compact neigborhoods. We call a topological group G an l-group if as a topological space it is an l-space. An l-group is such that there is a fundamental system of neighborhoods around the identity consisting of open compact subgroups.

Let X be an l-space. We call locally constant complex valued functions on X with compact support *Schwartz functions* on X. We denote the space of these functions by $S(X)$. We call linear functionals on $S(X)$ *distributions* on X. We denote the space of distributions on X by $S^*(X)$. If $f \in S(X)$ and $T \in S^*(X)$, then the value of T at f is denoted by $\langle T, f\rangle$ or $\int_G f(x)dT(x)$ or $\int f dT$. If $x_0 \in X$, the Dirac distribution δ_{x_0} is defined by $\langle \delta_{x_0}, f\rangle = f(x_0)$. We denote the set of locally constant functions on X by the symbol $C^\infty(X)$.

Let Y be an open and Z be a closed subset of X. Let us define the functors $i_Y : S(Y) \to S(X)$ and $p_Z : S(X) \to S(Z)$: $i_Y(f)$ is the continuation of f by zero outside Y and $p_z(f)$ is the restriction of f to Z. The following sequences are exact:

$$0 \longrightarrow S(Y) \xrightarrow{i_Y} S(X) \xrightarrow{p_{X\setminus Y}} S(X \setminus Y) \longrightarrow 0$$

$$0 \longrightarrow S^*(X \setminus Y) \xrightarrow{p^*_{X\setminus Y}} S^*(X) \xrightarrow{i^*_Y} S^*(Y) \longrightarrow 0.$$

For T in $S^*(X)$, $supp(T) := \{x \in X \mid \text{ for any neighborhood } N \text{ of } x,\ i^*_N(T) \neq 0 \}$. Let $S^*_c(X) :$: the set of compactly supported distributions.

If G is an l-group then there exists, up to a factor, only one left invariant distribution $\mu_G \in S^*(G)$. We call such a distribution μ_G a *left invariant Haar measure* on G. A right invariant Haar measure ν_G is defined similarly.

Let G be an l-group. By the convolution of two distributions T_1 and T_2 in $S_c^*(G)$ we mean the distribution $T_1 * T_2 \in S^*(G)$ defined by

$$\int_G f(g)d(T_1 * T_2)(g) = \int_G \left\{ \int_G f(g_1.g_2)dT_1(g_1) \right\} dT_2(g_2) \quad (f \in S(G)).$$

The convolution operation $*$ determines on $S_c^*(G)$ the structure of an associative algebra. The subalgebra of locally constant distributions in $S_c^*(G)$ is called the "Hecke Algebra of G" . We denote it by the symbol $\mathcal{H}(G)$ or simply by $\mathcal{H}$ if there is no confusion. We should note that $S(G)$ is isomorphic to $\mathcal{H}(G)$ via the isomorphism $f \mapsto f\mu_G$.

Let G be an l-group and V be a vector space. A representation π of G on V is called a $G-$ *module*. We denote it by the triple (π, G, V) or shortly (V, π). A representation (π, G, V) is called *algebraic* (*smooth*) if for any $\xi \in V$ the stabilizer $stab\ \xi = \{g \in G \mid \pi(g)\xi = \xi\}$ is open in G. We set $V_\alpha = \{\xi \in V \mid stab\ \xi \text{ is open}\}$. It is a submodule of V. We call it *the algebraic part of* π. A representation (π, G, E) is called *admissible* if it is algebraic and if for any open subgroup $N \subset G$ the subspace E^N consisting of N invariant vectors in E is finite dimensional. We denote *Alg* G the category whose objects are the algebraic modules of G and whose morphisms are intertwining operators.

Let (π_i, G_i, V_i) be representations of l-groups G_i $(i = 1, \ldots, r)$. By the tensor product of π_i's we mean the space $V := \bigotimes_i V_i$ together with the action of $G := \prod_i G_i$ on it, denoted as $\pi := \bigotimes_i \pi_i$, defined as follows: for $g := (g_i) \in G$, $\bigotimes_i \xi_i \in V$;

$$\pi(g)(\bigotimes_i \xi_i) = \bigotimes_i \pi(g_i)\xi_i.$$

Let $(\pi, G, E) \in Alg\ G$. Let E^* be the space of all linear functionals on E. We define the representation $\pi^* = (\pi^*, G, E^*)$ conjugate to π by $\langle \pi^*(g)\xi^*, \xi\rangle = \langle \xi^*, \pi(g^{-1})\xi\rangle$, where $\xi^* \in E^*, \xi \in E$ and $\langle \xi^*, \xi\rangle$ is the value of ξ^* at ξ. The algebraic part of (π^*, G, E^*) is called the *contragradient representation* to π, and is denoted by $(\tilde{\pi}, G, \tilde{E})$. G acts on $S^*(G)$ by right (δ) and left (γ) translations.

For a left invariant Haar measure μ_G of G, the function $\Delta_G : G \longrightarrow \mathbf{C}$ is the homomorphism that assigns to g the unique number $\Delta_G(g)$ such that

$$\delta(g)\mu_G = \Delta_G(g)\mu_G.$$

This homomorphism is called the modulus of G.

Let H be a closed subgroup of an l-group G, and let $(\rho, H, E) \in Alg\ H$. We denote by $L(G, \rho)$ the space of functions $f : G \longrightarrow E$ satisfying the following conditions:

(i) $f(hg) = \rho(h)f(g)$ for all $h \in H$, $g \in G$.
(ii) There exists an open compact subgroup N of G such that $f(g.g_0) = f(g)$ for all $g \in G$ $g_0 \in N$.

The representation $(\pi, G, L(G, \rho))$ acting according to the formula

$$(\pi(g_0)f)(g) = f(g.g_0)$$

is said to be *induced by* ρ. We denote it by $\pi = Ind(G, H, \rho)$. If moreover we replace $L(G, \rho)$ by $S(G, \rho)$ the subspace of the functions f that are finite modulo H(i.e. there is a compact set K such that $supp\ f \subset H.K$) then $(\pi, G, S(G, \rho))$ is said to be *finitely induced by* ρ. We denote it by $\pi = ind(G, H, \rho)$. For a representation (V, π) of G and $g_0 \in G$, $\pi^{g_0}(g) := \pi(g_0^{-1} g g_0)$ $(\forall g \in G)$. It is also a representation and it is called "the conjugate of π by g_0". The set of isomorphism classes of all irreducible subquotients of (V, π) is called the Jordan-Hölder content of V and it is denoted by $JH((V, \pi))$.

For a group G, the central extension of G by a group A is a short exact sequence of groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow G \longrightarrow 0$$

such that the image of A is contained in the centre of B.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Their product category $\mathcal{A}\times\mathcal{B}$ is the category consisting of the objects $\langle A, B\rangle$ where A in $\mathcal{A}$ and B in $\mathcal{B}$ and of the arrows $\langle f, g\rangle$ where f an arrow in $\mathcal{A}$ and g in $\mathcal{B}$.

A category that satisfies the following properties is called *abelian*:

(i) For any two objects A and B arrows from A to B have an abelian group structure under a law called addition on which composition is bilinear.
(ii) It has a null object.
(iii) It has binary products.
(iv) Every arrow has kernel and cokernel.
(v) Every monic arrow is kernel, and every epic is cokernel.

We use the categories of modules over rings. They are abelian since module homomorphisms between two modules have an abelian group structure under addition; 0 module is the null object; products of modules are defined, and kernels and quotients of morphisms are also modules.

Given two abelian categories $\mathcal{A}$ and $\mathcal{B}$ and two functors F and G from $\mathcal{A}$ to $\mathcal{B}$, a natural transformation from F to G is a family of arrows $(z_A)_{A\ in\ \mathcal{A}}$ in $\mathcal{B}$ such that for any two objects A and B in $\mathcal{A}$ and any arrow $f : A \longrightarrow B$ the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle z_A} & & \downarrow{\scriptstyle z_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array} .$$

The set of natural transformations of a functor towards itself together with componentwise addition and composition is a ring. When we talk of the ring of natural transformations of a functor F towards itself we also call it the ring of endomorphisms of F and we denote it by $End(F)$. The centre of an abelian category $\mathcal{A}$, denoted by $Z(\mathcal{A})$, is the ring of endomorphisms of the identity functor. A natural transformation whose components are invertible is called a natural isomorphism.

We say that two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there are functors $F : \mathcal{A} \longrightarrow \mathcal{B}$, $G : \mathcal{B} \longrightarrow \mathcal{A}$ and natural isomorphisms from the identity functors of respective categories to the functors $F \circ G$ and $G \circ F$. We denote the equivalence by the symbol $\mathcal{A} \simeq \mathcal{B}$. For a directed set I, "$lim\ ind_I$" or shortly "$lim_{\rightarrow}$" will denote direct limit over I. And for an ordered set I, "$limproj_I$" or "$lim_{\leftarrow}$" will denote the projective limit or the inverse limit [8].

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. For additive functors, a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be glued from a finite sequence of functors $F_i : \mathcal{A} \longrightarrow \mathcal{B}$ $(i = 1, \ldots n)$ if for each object A in $\mathcal{A}$ there is a filtration $0 = D_0 \subset D_1 \subset \ldots \subset D_n = F(A)$ in $\mathcal{B}$ such that the set of quotients $\{D_i/D_{i-1}\}$ is isomorphic after a permutation to the set $\{F_i(A)\}$ and such that for each morphism $\phi : A \longrightarrow B$ in $\mathcal{A}$ $F_i(\phi)$ carries $F_{i-1}(A)$ to $F_{i-1}(B)$. More generally, for an abelian category $\mathcal{A}$ and $C_1 \ldots C_n$ in $\mathcal{A}$, we say that an object D is glued from $C_1 \ldots C_n$ if there is a filtration $0 = D_0 \subset D_1 \subset \ldots \subset D_n = D$, which is isomorphic after a permutation to the set $\{C_1 \ldots C_n\}$. ([3] : 1.10 & 1.11).

3. PART I : THE CENTRE OF QUASI-CUSPIDAL COMPONENTS

3.1. Hecke Algebra

Let G be a σ-compact unimodular l-group, μ_G be the Haar measure on G and $\mathcal{H}(G)$ be the Hecke algebra of G with coefficients in $\mathbf{C}$. First of all we begin by showing that the Hecke algebra of G that we introduced in the preliminaries is an "idempotented algebra":

Proposition 3.1.1. *For each compact subgroup K of G; let us define the distribution*

$$e_K := p_K^*(\mu_K)$$

where μ_K is the normalized Haar measure on K. It is an idempotent of $\mathcal{H}(G)$. Moreover, for each finite family of elements x_i in $\mathcal{H}(G)$ there is such an idempotent that fixes all x_i such that $x_i * e_K = e_K * x_i$.

Proof. Let $f \in S(G)$. Then

$$\begin{aligned}
\langle e_K * e_K, f\rangle &= \int_G \Big\{ \int_G f(g_1.g_2) de_K(g_2)\Big\} de_K(g_1) \\
&= \int_G \Big\{ \int_G f(g_1.g_2) dp_K^*(\mu_K)(g_2)\Big\} dp_K^*(\mu_K)(g_1) \quad (definition) \\
&= \int_G \Big\{ \int_K (p_{g_1K} f)(g_1.g_2) d\mu_K(g_2)\Big\} dp_K^*(\mu_K)(g_1) \\
&= \int_K \Big\{ \int_K (p_K f)(g_1.g_2) d\mu_K(g_2)\Big\} d\mu_K(g_1) \\
&= \int_K \Big\{ \int_K (p_K f)(g_2) d\mu_K(g_2)\Big\} d\mu_K(g_1) \quad (\mu_K\ is\ left-invariant) \\
&= \int_K (p_K f)(g_2) d\mu_K(g_2) \quad (\mu_K \text{ is normalized}) \\
&= \langle e_K, f\rangle .
\end{aligned}$$

This proves that e_K is an idempotent of $\mathcal{H}(G)$ for any compact subgroup K of G.

Secondly, let $(x_i : i = 1, \cdots, n)$ be a finite family of elements of $\mathcal{H}(G)$. As they are locally constant, there is a compact subgroup K of G such that $\gamma(k).x_i = \delta(k).x_i = x_i$ $(k \in K;\ i = 1, \cdots, n)$; where '$\gamma$', and '$\delta$' denotes the left and right actions of G on $\mathcal{H}(G)$. Then, for $f \in S_k(G),\ \ i = 1, \cdots, n$;

$$\begin{aligned}
\langle x_i * e_K, f\rangle &= \int_G [\int_G f(g_1.g_2)dx_i(g_1)]de_K(g_2) \\
&= \int_G [\int_G f(g_1)d(\delta(g_2).x_i)(g_1)]de_K(g_2) \\
&= \int_K [\int_G f(g_1)d(\delta|_K(g_2).x_i)(g_1)]d\mu_K(g_2) \\
&= \int_K [\int_G f(g_1)d(x_i)(g_1)]d\mu_K(g_2) \qquad (x_i \textit{ is locally constant}) \\
&= [\int_K 1\mu_K(g_2)].[\int_G f(g_1)d(x_i)(g_1)] \\
&= \int_G f(g_1)d(x_i)(g_1) = \langle x_i, f\rangle
\end{aligned}$$

Similarly, using Fubini's Theorem one can prove that

$$\langle e_K * x_i, f\rangle = \langle x_i, f\rangle.$$

□

Lemma 3.1.2. $(V, \pi) \in Alg\ G \Leftrightarrow V$ *is a non-degenerate* $\mathcal{H}(G)$*-module; i.e.* $\mathcal{H}(G).V = V$.

Proof. Suppose (V, π) is a smooth representation of G for an action π of G on V. Let us define the action of $\mathcal{H}(G)$ on V: for $T \in \mathcal{H}(G),\ v \in V;\ T.v := \int_G \pi(g).vdT(g)$. Then, V is an $\mathcal{H}(G)$-module with this action. Let K be an open compact subgroup of G such that $K \subset stab(v)$. Then, $e_K.v = v$; i.e. $\mathcal{H}(G).V = V$. Conversely, if V is a non-degenerate $\mathcal{H}(G)$-module , then by Proposition 3.1.1, for some idempotent e_K of $\mathcal{H}(G)$, $e_K.v = v$. Let us define the action π of G on V by $\pi(g).v := (\delta_g * e_K).v$. Then, as $\delta_g * e_K = e_K$ $(\forall g \in K)$ ([2]: 1.26), we have $\pi(g).v = (\delta_g * e_K).v = (e_K.v) = v$. Hence, (V, π) is an algebraic representation. □

Proposition 3.1.3. *For each idempotent e of* $\mathcal{H}(G)$, *let*

$$\mathcal{H}(G)(1-e) := \{h - he \mid h \in \mathcal{H}(G)\}.$$

It is an ideal, for each idempotent e in $\mathcal{H}(G)$;

$$\mathcal{H}(G) = \mathcal{H}(G)e \oplus \mathcal{H}(G)(1-e).$$

Proof. It is clear that $\mathcal{H}(G)(1-e)$ is an ideal and $\mathcal{H}(G) = \mathcal{H}(G)e \ + \ \mathcal{H}(G)(1-e)$. Moreover, if $h_1.e \ + \ h_2(1-e) = 0$ then $h_2 = (h_2 - h_1).e \in \mathcal{H}(G)e$. Hence $h_2(1-e) = 0$ and $h_1.e = 0$. The conclusion now follows. □

Let T be the topology on $\mathcal{H}(G)$ having $\mathcal{H}(G)(1-e)$'s as fundamental system of neighborhoods around 0.

Proposition 3.1.4. *Let* I *be the set of* e_K*'s of* $\mathcal{H}(G)$ *where* K*'s are compact open subgroups of* G. *It is a directed set with respect to the relation* $e_{K_1} << e_{K_2}$ *iff* $e_{K_1} = e_{K_2} * e_{K_1} * e_{K_2}$. *Let* $\mathcal{H}^\wedge(G)$ *be the completion of the uniform space* $(\mathcal{H}(G), T)$. *Then,*

$$\mathcal{H}^\wedge(G) = limproj_I \ \mathcal{H}(G)e$$

for the morphisms of transition $\mathcal{H}(G)f \longrightarrow \mathcal{H}(G)e : x \mapsto x.e \ (e, f \in I)$.

Proof. I is an ordered set. If $e_{K_1}, e_{K_2} \in I$ then $e_{K_1} << e_{K_3}$ and $e_{K_2} << e_{K_3}$ for $K_3 \subset K_1 \cap K_2$. Thus, I is a directed set.

For each neighborhood V of 0, let us define the set V_d of pairs $(x, y) \in \mathcal{H} \times \mathcal{H}$ such that $x - y \in V$. Let $\mathcal{O}_d := \{V_d \in \mathcal{B}(\mathcal{H} \times \mathcal{H}) \mid V \textit{ is in the} \text{ neighborhood } \textit{filter of } 0\}$. $\mathcal{O}_d$ is a *fundamental system of entourages* ([7]). Thus, $\mathcal{H}$ is endowed with a uniform structure, say $\mathcal{U}$, that induces T.

As subspaces of $\mathcal{H}$, $\mathcal{H}e$ are also uniform spaces([7]). The transitions $\mathcal{H}f \longrightarrow \mathcal{H}e$ are continuous. Because, for an open set $V = \mathcal{H}e \cap \mathcal{H}(1-e')$ $(e' << e)$ in $\mathcal{H}e$ its inverse image $\mathcal{H}f \cap \mathcal{H}(1-e')$ is also an open set in $\mathcal{H}f$.

$(\mathcal{H}e, \mathcal{H}f \longrightarrow \mathcal{H}e)$ is an inverse system of sets. $\mathcal{H}f \longrightarrow \mathcal{H}e$ is uniformly continuous $(\forall\ e \ll f\ in\ I)$: Let V' be an entourage of $\mathcal{H}e$. Then there is an open set V in $\mathcal{H}$ such that $V_d \cap (\mathcal{H}e \times \mathcal{H}e) \subset V'$. Then $V_d \cap (\mathcal{H}f \times \mathcal{H}f)$ is an entourage of $\mathcal{H}f$ such that $([(\mathcal{H}f \longrightarrow \mathcal{H}e) \times (\mathcal{H}f \longrightarrow \mathcal{H}e)](V_d \cap (\mathcal{H}f \times \mathcal{H}f))) \subset V'$.

Hence, ($\mathcal{H}e$, $\mathcal{H}f \longrightarrow \mathcal{H}e$) is an inverse system of uniform spaces, and $limproj\mathcal{H}e$ being the inverse limit of ($\mathcal{H}e$, $\mathcal{H}f \longrightarrow \mathcal{H}e$) (notation: $limproj\mathcal{H}e = lim_{\leftarrow}\mathcal{H}e$) is a uniform space. ([7])

We can embed $\mathcal{H}$ into $limproj\mathcal{H}e$ by the mapping $i : h \mapsto (h.e)_{e\in I}$. $i(\mathcal{H})$ is dense in $limproj\ \mathcal{H}e$. We will show that the pair $(i, limproj\ \mathcal{H}e)$ is the solution of *the universal mapping problem* for $\mathcal{H}$ with respect to the $\Sigma - sets$ complete Hausdorff uniform spaces, the $\sigma - morphisms$ the uniformly continuous mappings, and the $\alpha - morphisms$ the uniformly continuous mappings from $\mathcal{H}$ to complete Hausdorff uniform spaces:

(i) Firstly, we show that $limproj\ \mathcal{H}e$ is a complete uniform space. Let $\mathcal{F}_0$ be a Cauchy filter on $i(\mathcal{H})$. For each e in I there is an x_e in $\mathcal{H}e$ such that $i(x_e + \mathcal{H}(1-e)) \in \mathcal{F}_0$. Then the element $(x_e)_{e\in I}$ is a limit of $\mathcal{F}_0$ in $limproj\ \mathcal{H}e$. Hence, $limproj\ \mathcal{H}e$ is complete $([7]:$ Ch. 2 §3 no.4).

(ii) Secondly, let f be an $\alpha - morphism$ into a complete Hausdorff uniform space Y. We will show that there exists a unique $\sigma - morphism$ $f^\wedge$ such that $f = f^\wedge \circ i$. On an element of $i(\mathcal{H})$, $f^\wedge$ is defined as the value of f on the preimage of this element under i. In general, for $h^\wedge \in limproj\ \mathcal{H}e$, the family of sets $\{h^\wedge(e) \in \mathcal{H} \mid f \ll e \ in\ I\}$, where f runs over idempotents, is a Cauchy filter base on $\mathcal{H}$. Then its image under f is a Cauchy filter in Y. As Y is Hausdorff and complete, this filter has a unique limit point in Y, so that we define $f^\wedge(h^\wedge)$ as this value. In this way we obtain a uniformly continuous map, and this extension is unique as $i(\mathcal{H})$ is dense in $limproj\ \mathcal{H}e$.

Therefore, $\mathcal{H}^\wedge = limproj\ \mathcal{H}e$. □

We can extend the action of $\mathcal{H}$ to its completion :

Proposition 3.1.5. *$(\forall h^\wedge \in \mathcal{H}^\wedge)$ $(\forall v \in V)$ if $e \in I$ and $e.v = v$, then*

$$h^\wedge.v := h^\wedge(e).v.$$

Proof. For all $v \in V$, there is an idempotent e such that $ev = v$. Let $v' \in V$. Then, for all (h, v) such that $h.v = v'$, there is an idempotent e such that $(h + \mathcal{H}(1-e)) \times \{v\}$ is an open neighborhood of (h, v) whose image is v'. Thus the action is continuous if we take V as discrete topological space.

The action of $\mathcal{H}^\wedge$ on V is the continuous extension of the action of $\mathcal{H}$ as the domain of the latter is dense in that of the former. As a result of that we must define $h^\wedge.v = h^\wedge(e).v$. □

Now we will ask to describe the center of the category of algebraic representations of G by means of the completion of the Hecke Algebra. The following two lemmas will give an answer to this question.

Lemma 3.1.6. *Let ω be the forgetful functor*

$$(non-degenerate\ \mathcal{H}-modules) \longrightarrow (abelian\ groups).$$

By Proposition 4 we consider $h^\wedge$ as a group endomorphism on $\omega(V)$ for all $V \in (non-degenerate\ \mathcal{H}-modules)$. Then, we have

$$\mathcal{H}^\wedge \xrightarrow{\sim} End\ (\omega).$$

Proof. Let us recall that the elements (z_V) of $End\ (\omega)$ make the following diagram commute for each $f : V_1 \longrightarrow V_2$ in the category $(non-degenerate\ \mathcal{H}-modules)$:

$$\begin{array}{ccc} \omega(V_1) & \xrightarrow{z_{V_1}} & \omega(V_1) \\ {\scriptstyle \omega(f)} \downarrow & & \downarrow {\scriptstyle \omega(f)} \\ \omega(V_2) & \xrightarrow[z_{V_2}]{} & \omega(V_2). \end{array}$$

It is easy to see that $\mathcal{H}^\wedge$ embeds in $End\ (forgetful\ functor\ \omega)$. We will show that the map is onto.

Suppose V and W are $\mathcal{H}$-module. Then, $V \simeq \mathcal{H}_S^{(T_1)}/V_0$ and $W \simeq \mathcal{H}_S^{(T_2)}/W_0$ for some indices T_1 and T_2 and submodules V_0 and W_0. If a natural transformation coincides with an element $h^\wedge$ of $\mathcal{H}^\wedge$ at $\mathcal{H}_S^{(T_1)}$ then as the above diagram is valid for the projection map it also coincides at V. On the other hand, a homomorphism $f_0 : \mathcal{H}_S^{(T_1)} \longrightarrow \mathcal{H}_S^{(T_2)}$ can be considered as the sum of endomorphisms of $\mathcal{H}_S$. Hence, it is sufficient to verify that all endomorphisms of the abelian group $\mathcal{H}_S$ which commute with $End_{\mathcal{H}}(\mathcal{H}_S)$ are defined by an element of $\mathcal{H}^\wedge$. In other words if the statement is true for $V_1 = V_2 = \mathcal{H}_s$ then it is true in general.

If $\phi \in End_{\mathbf{Z}}(\mathcal{H})$ commutes with $End_{\mathcal{H}}(\mathcal{H}_S)$ then $h^\wedge = (\phi(e))_{e \in I}$ is an element of $\mathcal{H}^\wedge$ coinciding with ϕ since it commutes right multiplication. □

Lemma 3.1.7. *$Z(\mathcal{H}^\wedge) \simeq Z(non-degenerate\ \mathcal{H}-modules)$.*

Proof. Let $\varphi : \mathcal{H}^\wedge \xrightarrow{\sim} End\ (\omega)$ in Lemma 3.1.6. We also have $i : Z(non-degenerate\ \mathcal{H}-modules) \hookrightarrow End\ (\omega)$. We will show that the images of $Z(\mathcal{H}^\wedge)$ and $Z(non-degenerate\ \mathcal{H}-modules)$ coincide in $End\ (\omega)$.

Since the multiplication on $\mathcal{H}^\wedge$ is continuous ([7]: ch. III, §3.4) and the embedding of $\mathcal{H}$ is dense in $\mathcal{H}^\wedge$ we know that

$$Z(\mathcal{H}^\wedge) = \{h^\wedge \in \mathcal{H}^\wedge \mid \forall h \in \mathcal{H},\ h^\wedge * h = h * h^\wedge\}.$$

By the proof of Lemma 3.1.6 an element of the center is determined by its component on $\mathcal{H}_S$. Then, the following equivalences conclude the proof :

$$\begin{aligned}(z_V)_V \in Z(non-degenerate\ \mathcal{H}-modules) \quad &\Leftrightarrow \quad z_{\mathcal{H}_S} \in End_{\mathcal{H}}(\mathcal{H}_S) \\ &\qquad and\ i((z_V)_V) \in \varphi(\mathcal{H}^\wedge) \\ &\Leftrightarrow \quad i((z_V)_V) \in \varphi(Z(\mathcal{H}^\wedge)).\end{aligned}$$

□

Corollary 3.1.8. *$Z(\mathcal{H}^\wedge) \simeq Z(Alg\ G)$.*

Now we return to the main result of this section. We will ask whether we can see the elements of the center of our category as the functions on the set of irreducible representations.

As G is σ-compact the irreducible G-modules are now at most countable dimensional complex vector spaces. Then Schur's Lemma is valid :

Lemma 3.1.9 (Schur's Lemma)**.** *If $z \in Z(Alg\ G)$,then for all irreducible algebraic representations (V, π) of G, z acts on V by a scalar $z(\pi)$ ([2] : 2.11).*

Proposition 3.1.10. *Let $G^\wedge$ denote the set of irreducible algebraic representations of G. The morphism of algebras*

$$Z(\mathcal{H}(G)^\wedge) \longrightarrow \ (\ functions\ on\ G^\wedge\ )$$

determined by Schur's Lemma is injective.

Proof. We will assume that it is an algebra homomorphism and we will only prove injectivity. We have $\forall\ h \neq 0\ \in\ \mathcal{H}(G),\quad \exists\ (V, \pi) \in G^\wedge : \pi(h) \neq 0$ ([2] : 2.12). Then, $\exists\{(V_i, \pi_i)\}_{i \in I} \subset G^\wedge : \mathcal{H}(G)_s \hookrightarrow \prod_{i \in I}(V_i, \pi_i)$. Therefore, for any $z \in Z(\mathcal{H}(G)^\wedge)$, if $\forall (V, \pi) \in G^\wedge\ \ z(\pi) = 0$ then $z.\mathcal{H}(G)_s = 0$; i.e. $z = 0$. □

3.2. Category Theory

In this section we will give some categorical tools to decompose our category into its smaller parts and to analyze the center of the category by means of the centers of these smaller parts. We begin by asserting a basic categorical fact. If $\mathcal{A}$ and $(\mathcal{A}_i)_{i \in I}$ are abelian categories such that $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$ then $Z(\mathcal{A}) = \prod Z(A_i)$.

Proposition 3.2.1. *For an abelian category $\mathcal{A}$, suppose that $\mathcal{A}$ admits coproducts on the index set I, and that all morphisms $f : X \longrightarrow \bigoplus_{i \in I} Y_i$ in it such that $\forall i \in I\ \ pr_i \circ f$ is nul, are also nul. Then to give a decomposition of $\mathcal{A}$ in a product of abelian categories as above, is equivalent to give full subcategories $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{A}$ such that:*

(i) For X in $\mathcal{A}_i$ and Y in $\mathcal{A}_j$, we have $Hom(X, Y) = 0$ if $i \neq j$.

(ii) All object X is a sum: $X = \bigoplus X_i$, with X_i in $\mathcal{A}_i$. ([5])

Proof. One way of the equivalence is obvious. For the converse, all we should note is that the canditate-isomorphism functor $F : \prod_{i\in I} \mathcal{A}_i \longrightarrow \mathcal{A}$ sending $< X_i \mid i \in I >$ to $\bigoplus_{i\in I} X_i$ is onto for arrows.

If f is an arrow in $\mathcal{A}$ then the image of

$$f' = \langle pr_i \circ f \circ \iota_i \rangle_{i\in I} \quad \in \prod_{i\in I} \mathcal{A}_i$$

under F must be f by using the conditions (a)&(b) in the construction of f', and by the condition on morphisms given at the beginning as far as $\forall i \in I \;\; pr_i \circ (f - F(f')) = 0$. $\square$

Corollary: $(Alg\ G)$ satisfies the conditions in Proposition 3.2.1. In fact it admits coproducts on any index set I and any morphism $f : X \longrightarrow \bigoplus_{i\in I} Y_i$ in it such that $\forall i \in I \;\; pr_i \circ f$ is nul, is itself nul.

3.3. Finite Representations

In this section we will define finite algebraic representations of an l-group. These representations will provide us with a primary decomposition for the categories which admit them. Later, using this result. we will generalize this decomposition to all l-groups.

For a representation (V, π) of G; let T denote the topological structure on V which has $(1 - e_K)V$ (cf. Proposition 3.1.3) as the fundamental system of neighborhoods around 0 (K's are compact open subgroups of G).

Lemma 3.3.1. *([1] : 2.40) The following conditions are equivalent:*

(i) The coefficients $< \zeta, gw >$ $(w \in W, \zeta \in W^{\sim})$ *are compactly supported functions on* G*.*

(ii) For all compact open subgroup K *and all* $w \in W$*, the function of* G *in* W *:* $g \mapsto e_K gw$ *is compactly supported.*

(iii) For all $w \in W$*, if* $g \to \infty$ *in* G *(w.r.t. the filter of complement of compacts for* T *above), we have* $gw \to 0$*.*

Proof. The equivalence of (a) and (b) are given in the reference. We will prove the equivalence of (b) and (c).

If (b) holds and $g \to \infty$ then for each compact set K there is a complement of a compact set C such that on it $gw \in (1 - e_K)V$. Hence, for all K; $(1 - e_K)V$ is in the filter in V whose base is the image of the filter of complement of compacts under the map $g \mapsto gw$. 0 being the limit point of the coarser filter whose base are sets $(1-e_K)V$, is also limit point of this map. Conversely, let K be a compact open subgroup, then for a complement of compact its image is in $(1 - e_K)V$. Then on this set the map $g \mapsto e_K gw$ vanishes. □

Let $(Alg\ G)_f$ denote the category of representations of G satisfying the above equivalent conditions (i), (ii) and (iii). We call them "finite representations of G".

Proposition 3.3.2. *For an irreducible representation* (V, π) *in* $(Alg\ G)_f$ *let* $(Alg\ G)(\pi)$ $::$ *the category of the representations that are a sum of the copies of* (V, π) *and let* $(Alg\ G)(out\ \pi)$ $::$ *the category of the representations without a sub-quotient isomorphic to* (V, π)*. Then,*

$$(Alg\ G) = (Alg\ G)(\pi) \times (Alg\ G)(out\ \pi).$$

Proof. (Sketch of ([2] : 2.44)) Let $\theta_\pi ::$ the character of π ([2] 2.17), (also cf. Appendix, Remark 10). Let $d_\pi ::$ the formal degree of π (a Haar measure on G) (ibid.). For the distribution $e(\pi) := \theta_\pi(g^{-1}).d_\pi$ (cf. [2] 2.43); for any representation (V, ρ)

$$(\rho(e(\pi))V, ker\rho(e(\pi))) \in (Alg\ G)(\pi) \times (Alg\ G)(out\ \pi)([2]\ :\ 2.43)$$

and

$$V = \rho(e(\pi))V \oplus ker\rho(e(\pi)) \qquad ([2]\ :\ 2.44).$$

Moreover, $\forall X \in (Alg\ G)(\pi)\ \forall Y \in (Alg\ G)(out\ \pi)\ Hom(X, Y) = 0$. The conclusion now follows by Proposition 3.2.1. $\square$

We also know that if $V \in (Alg\ G)_f$ and V is finitely generated then V is admissible. It will be used to generalize the previous proposition. A proof of it can be given by the following argument. Let us assume that these conditions are satisfied and V is finitely generated, or for simplicity, it is irreducible and let ξ be a generator. If for a sequence g_i $(i = 1, 2, \ldots)$ $e_K.g_i.\xi$ are linearly independent then for the functional $\tilde{\xi}$ on W satisfying $\left\langle \tilde{\xi}, \xi \right\rangle = \left\langle \tilde{\xi}, e_K.\xi \right\rangle$ and vanishing on the supplement in V^K of the space spanned by these vectors and taking the value i on $e_K.g_i.\xi$; the coefficient $\left\langle \tilde{\xi}, g.\xi \right\rangle$ is not compactly supported. Hence, our hypothesis is false and V^K is finite dimensional.

Proposition 3.3.3. *Let A denote a set of isomorphism classes of irreducible representations in $(Alg\ G)_f$. Let $(Alg\ G)$ $(out\ A)$ denote the category of representations π of G such that $JH(\pi) \cap A = \emptyset$. We have: if either A is finite or for each compact subgroup K of G there exist only finitely many (V, π) in A such that $V^K \neq 0$ then*

$$(Alg\ G) = (\prod_{\pi \in A} (AlgG)(\pi)) \times (Alg\ G)\ (out\ A).$$

Proof. The case A is finite is trivial, we need to consider the sum $\sum_{\pi \in A} e(\pi)$. We will prove the infinite case: Let W be an arbitrary representation of G. Let $A(K)$ be the finite subset of the elemets (V, π) in A such that $V^K \neq 0$. Then by finite case

$$W = \bigoplus_{\pi \in A(K)} W(\pi) \times W(out\ A(K)).$$

Also

$$W^K = \bigoplus_{\pi \in A(K)} W(\pi)^K \times W(out\ A(K))^K \simeq \bigoplus_{\pi \in A} W(\pi)^K \times W(out\ A(K))^K.$$

Here $W(out\ A(K))^K$ is independent of $A(K)$; i.e. for any finite A', $A(K) \subset A' \subset A$, $W(out\ A(K))^K \simeq W(out\ A')^K$.For each K, let $W(out\ A)^K := W(out\ A(K))^K$ and $W(out\ A) := \bigcup_K W(out\ A)^K$. Then

$$W = \bigcup_K W^K = \bigcup_K (\bigoplus_{\pi \in A} W(\pi)^K \times W(out\ A)^K) = \bigoplus_{\pi \in A} W(\pi) \times W(out\ A).$$

For X in $(\bigoplus_{\pi \in A}(Alg\ G)(\pi))$ and Y in $(Alg\ G)$ $(out\ A)$, we have $Hom(X, Y) = 0$. By Proposition 3.2.1. the conclusion follows. ∎

Let $\hat{G}_f$ denote the set of all isomorphism classes of irreducibles in $(Alg\ G)_f$. Then,

Corollary 3.3.4.

$$(Alg\ G)_f = \prod_{\pi \in \hat{G}_f} (Alg\ G)(\pi).$$

Proof. For an admissible representation (V, ρ) of G and for a compact subgroup K of G, $\exists$ only finitely many π in $\hat{G}_f$, $(\rho(e(\pi))V)^K \neq 0$. Then by the proof of Proposition 3.3.3, an admissible representation (V, ρ) of G can be decomposed

$$V = \bigoplus_{\pi \in \hat{G}_f} V(\pi) \times V(out\ \hat{G}_f).$$

The decomposition is compatible with direct limits; i.e. direct limit of decomposable representations are decomposable. (ZZZ)Also a G-module is direct limit of its finitely generated submodules(ZZZ). We above showed that finitely generated modules in $(Alg\ G)_f$ are admissible. Moreover, for a finitely generated $(V, \pi) \in (Alg\ G)_f$ $V(out\ \hat{G}_f) = 0$. The result follows. □

Corollary 3.3.5. *If for each compact subgroup K of G there exist only finitely many (V, π) in $\hat{G}_f$ such that $V^K \neq 0$ then*

$$(Alg\ G) = (Alg\ G)_f \times (Alg\ G)(out\ \hat{G}_f).$$

3.4. Quasi-Cuspidal Representations And Bernstein Decomposition

For some $r \in \mathbf{N}$, let $u :: G \to \mathbf{Z}^r$ a group homomorphism and $G^0 := ker\ (u)$. For all G under consideration let us assume that $ker(u|_{Z(G)})$ is compact and $coker(u|_{Z(G)})$ is finite. Let (V^0, π^0) be an irreducible representation in $(Alg\ G^0)_f$. Let $G^1 := \{g \in G \mid (\pi^0)^g = \pi^0\}$. Let A be the set of isomorphism classes of conjugates of (V^0, π^0). For $g_0 \in G^0$, the module homomorphism $v \mapsto \pi^0(g_0).v$ provides a module isomorphism. Hence, G^1 contains G^0 and the center of G. The assumption that u has finite cokernel and, thus, the isomorphism $u(G) \cong (G/(ker\ u))$ implies that G^1 has finite index and A is finite. Then $\forall (V, \pi) \in (Alg\ G)\ \exists (V', V'') \in (Alg\ G^0)(A) \times (Alg\ G^0)(out\ A)$:

$$V = V' \oplus V''.$$

The given decomposition $V = V' \oplus V''$ is stable under the action of G because the application of G on a conjugate gives a conjugate, and $e(\pi^0)\pi(g)v = \pi(g).e(\pi^0)v = 0$ for $v \in V''$ (cf. [2] : 2.43 c)). Such objects constitute two subcategories of $(Alg\ G)$, we denote them by $(Alg\ G)_A$ and $(Alg\ G)_{out\ A}$. Then we have:

$$(Alg\ G) = (Alg\ G)_A \times (Alg\ G)_{out\ A}.$$

Proposition 3.4.1. *Let $(Alg\ G^1)_{\pi^0}$: : the category of G^1-modules whose restriction to G^0 is a multiple of π^0. Then*

$$(Alg\ G)_A \simeq (Alg\ G^1)_{\pi^0}.$$

Proof. Let us define the functor $F : (Alg\ G)_A \longrightarrow (Alg\ G^1)_{\pi^0}$ by the rule

$$(W, \pi) \mapsto (W_1, \rho|_{G^1})$$

where W_1 is the subspace of W that is sum of sub-G^0-representations isomorphic to (V^0, π^0). By definition W_1 is stable under G_1. Conversely let us define the functor $G : (Alg\ G^1)_{\pi^0} \longrightarrow (Alg\ G)_A$ by the rule

$$(W, \rho) \mapsto ind(G, G^1, \rho).$$

A natural transformation from $G \circ F$ to the identity functor in $(Alg\ G)_A$ can be given by the following rule : if $g_1 \dots g_n$ are left coset representatives of G^1 then $f \in L(G, W_1) \mapsto \sum_{i=1}^n \pi(g_i^{-1}) f(g_i) \in W$. On the other hand, a natural transformation from $F \circ G$ to the identity functor in $(Alg\ G^1)_{\pi^0}$ sends f to $f(e)$. □

Let $\tilde{G}^1 := \{(g, P) \in G \times GL(V^0) \mid \forall h \in G^0 \quad P\pi^0(h)P^{-1} = \pi^0(ghg^{-1})\}$. Then $\tilde{G}^1$ is a group under componentwise multiplication. And using Schur's lemma one can show that $\tilde{G}^1$ is a central extension of G^1 by $\mathbf{C}^*$ and $G^0 \hookrightarrow \tilde{G}^1$:

$$\begin{array}{ccccc} & & G^0 & & \\ & & \Big\downarrow{\scriptstyle g\mapsto(g,\pi(g))} & & \\ \mathbf{C} & \xrightarrow[z\mapsto(e,z)]{} & \tilde{G}_1 & \xrightarrow[(g,P)\mapsto g]{} & G_1 \end{array}$$

Let $M_1 := u(G^1)$ and $\tilde{M}_1 := \tilde{G}^1/G^0$. Then, $\tilde{M}_1$ is a central extension of M_1 by $\mathbf{C}^*$. Let $\tilde{C}$ be the center of $\tilde{M}_1$; let us define

$$C := \{m \in M_1 \mid \exists\overline{(g, P)} \in \tilde{C} \quad m = u(g)\ \}.$$

Then, C contains $u(Z(G))$ and $\tilde{M}_1/\tilde{C}$ is finite by the first paragraph.

Proposition 3.4.2. *Let $\mathcal{B}$ be the category of representations of $\tilde{M}_1$ inducing $z \mapsto z^{-1}$ on $\mathbf{C}^*$. Then*

$$(Alg\ G)_A \simeq (Alg\ G^1)_{\pi^0} \simeq \mathcal{B}.$$

Proof. We have shown the first equivalence in Proposition 3.4.1. We will show that $(Alg\ G^1)_{\pi^0} \simeq \mathcal{B}$. Let us define the functor $F : \mathcal{B} \longrightarrow (Alg\ G^1)_{\pi^0}$ by the rule $(\rho, H) \mapsto (\pi, H \otimes_{\mathbf{C}} V_0)$. The action π of G^1 on this space is defined as follows: Let $(g, P) \in G_1$, $h \otimes v \in H \otimes_{\mathbf{C}} V_0$, then

$$\pi(g).(h \otimes v) = \rho(\overline{(g, P)}).h \otimes P.v$$

where $\overline{(g, P)}$ is the image of (g, P) in $\tilde{M}_1$. Conversely, let us define the functor G :

$(Alg\ G^1)_{\pi^0} \longrightarrow \mathcal{B}$ by the rule

$$(\rho, G_1, W) \mapsto (\pi, \tilde{M}_1, Hom_{G^0}(V_0, W))$$

where $(\pi(\overline{(g,P)}).f)(v) = \rho(g).f(P^{-1}.v)$ $(g \in G_1,\ f \in Hom_{G^0}(V_0, W),\ v \in V_0)$.

A natural isomorphism from $F \circ G$ to the identity functor sends $(f \otimes v_0) \in Hom(V_0, W) \otimes V_0$ to $f(v_0) \in W$. Conversely, a natural isomorphism from $G \circ F$ to the identity functor sends $f \in Hom(V_0, W \otimes V_0)$ to $k.w$ where $k \in \mathbf{C}$ such that $f(v) = k.w \otimes v \quad (\forall v \in V_0)$. □

A corollary to Proposition 3.4.2 is the following : $Z((Alg\ G)_A) \simeq Z(\mathcal{B})$.

We can classify the irreducible representations by means of their central characters. Below we will give this classification.

Proposition 3.4.3. *Let X denote a maximal subgroup of $\tilde{M}_1/\tilde{C}$ on which the form induced by the commutator application is trivial and X' the Pontryagin dual of X. Then, $\tilde{M}_1/\tilde{C} = X \times X'$.*

Proof. The induced form of commutator application on $\tilde{M}_1/\tilde{C}$ is

$$\left\langle \overline{(g_1, P_1)}\tilde{C}, \overline{(g_2, P_2)}\tilde{C} \right\rangle = \frac{pr_2([\overline{(g_1, P_1)}, \overline{(g_2, P_2)}])}{\left|pr_2([\overline{(g_1, P_1)}, \overline{(g_2, P_2)}])\right|}$$

where pr_2 is the projection to the second coordinate, the brackets [ ] denote the usual commutator application on $\tilde{M}_1$ and the divisor is the absolute value of the complex number coming from the projection map (the projection map gives a complex number, because the left coordinate of representatives is in G^0).

Let $H = \tilde{M}_l/\tilde{C}$. H is commutative and finite. The form $\langle\ \rangle$ defined above is bimultiplicative nondegenerate and alternating. Let X be a maximal subgroup of H such that the form is trivial on it. The Pontryagin dual X' of X and X are of the same order.

Let $\{x_1 \dots x_n\}$ be a set of generators of X. Let $x_i^\wedge$ denote the element of H such that $\langle x_i^\wedge, x_j \rangle = e^{(2\pi i : o(x_j))\delta_{ij}}$ (δ is the Kronecker delta, $o()$ denotes the order of the element and i in $2\pi i$ is the unit imaginary number). By the mentioned properties of the form we can show that such elements exist for each x_i. Then $\{x_1^\wedge \dots x_n^\wedge\}$ generate a subgroup isomorphic to H/X. Therefore, H/X has the same order as X and since H/X embeds into X' they are isomorphic; i.e.

$$\tilde{M}_l/\tilde{C} = X \times X'$$

$\square$

Let $\tilde{X}$: : the inverse image of X in $\tilde{M}_1$ under the quotient map. The commutator application is trivial on the image of $\tilde{X}$ means that the commutator is trivial on $\tilde{X}$; i.e. it is commutative. If it were not maximal, say $\tilde{X} < \tilde{Y}$ for some such subgroup Y of $\tilde{M}_1$, then its image Y would contain X and on it the commutator application would be trivial. Therefore $\tilde{X}$ is a maximal commutative subgroup.

Proposition 3.4.4. *For each character χ of $\tilde{C}$ such that $\chi(z) = z^{-1}$ on $\mathbf{C}^*$, $\exists$ up to isomorphism unique irreducible representation of $\tilde{M}_l$ with central character χ.*

Proof. **Existence:** Let $\{\tilde{x}_1 \dots \tilde{x}_n\}$ be a set of coset representatives of generators of X. Let $\{x'_1 \dots x'_n\}$ be a set of coset representatives of generators of X' such that $\langle [x'_i], [\tilde{x}_j] \rangle = e^{(2\pi i : o([\tilde{x}_j]))\delta_{ij}}$ (δ is the Kronecker delta, and $o()$ denotes the order of the element in X) (cf. Proposition 3.4.3).

For $\tilde{x}_1$, $\exists n \in \mathbf{N}\ \tilde{x}_1^n \in \tilde{C}$. $o(\tilde{x}_1)$: : the smallest n satisfying this. Then, let $ext(\chi, \tilde{x}_1)$ denote the extension of χ to the group generated as $\left\langle \{\tilde{x}_1\} \cup \tilde{C} \right\rangle$, that is induced by the law

$$ext(\chi, \tilde{x}_1)(\tilde{x}_1) := (\chi(\tilde{x}_1^{o(\tilde{x}_1)}))^{1/o(\tilde{x}_1)}$$

Continuing in this manner, we can extend χ to $\tilde{X}$. Let $\tilde{\chi}$ denote this extension of χ to $\tilde{X}$.

Let us define the complex vector space $L(\tilde{M}_1, \tilde{\chi}) := \{f : \tilde{M}_1 \longrightarrow \mathbf{C} \mid f(x.y) = \tilde{\chi}(x).f(y)\ \forall x \in \tilde{X}, y \in \tilde{M}_1\}$. This space together with the action of right translation by elements of $\tilde{M}_1$, denoted as $ind(\tilde{M}_1, \tilde{X}, \tilde{\chi})$, is a representation of the group $\tilde{M}_1$. $f_{(m_1 \ldots m_n)}$ (or simply $f_{(m_i)_i}$) : : $f_{(m_1 \ldots m_n)} \in L(\tilde{M}_1, \tilde{\chi})$, $f_{(m_i)_i}(x.((x'_1)^{m_1} \ldots (x'_n)^{m_n}))\ :=\ \tilde{\chi}(x)\ \ (\forall x \in \tilde{X})$ and it is defined to be 0 otherwise. These make a basis of the complex vector space $L(\tilde{M}_1, \tilde{\chi})$. Moreover, as a monogenous $\tilde{M}_1$-module $< f_{(1,0,\ldots,0)} >= L(\tilde{M}_1, \tilde{\chi})$.

Let $f := \sum_{(m_i)_i} c_{(m_i)_i} f_{(m_i)_i}$. Then $f - \tilde{\chi}^{-1}(\tilde{x}_i).ind(\tilde{M}_1, \tilde{X}, \tilde{\chi})(\tilde{x}_i).f$ has no component of the form $f_{(m_1, \ldots, m_{i-1}, 0, m_{i+1}, \ldots, m_n)}$. Continuing in this manner we obtain f' whose nonzero components are of the form

$$f_{(m_i)_i}\quad (\forall i\ m_i > 0).$$

Furthermore, $ind(\tilde{M}_1, \tilde{X}, \tilde{\chi})(x'_i).f' - \tilde{\chi}^{-1}(\tilde{x}_i).ind(\tilde{M}_1, \tilde{X}, \tilde{\chi})(\tilde{x}_i.x'_i)f'$ has no components for which $m_i \leq 1$. Continuing in this manner we can obtain $f_{(1,0,\ldots,0)}$. Therefore, $ind(\tilde{M}_1, \tilde{X}, \tilde{\chi})$ is an irreducible representation whose central character is χ.

Uniqueness: Let (V, π) : : an irreducible representation whose central character is χ. Let v_0 : : a generator of V. Then,The mapping $\pi(\tilde{x}_i.\tilde{c}.x'_j.)v_0 \mapsto \tilde{\chi}(\tilde{x}_i.\tilde{c})f_{x'_j}$ is an isomorphism
$(f_{x'_j} := f_{(0,\ldots,0,1,0,\ldots,0)}$). □

Corollary to Proposition 3.4.4 : Let π be an irreducible representation of $\tilde{M}_1$ such that $\pi(z) = z^{-1}\ \forall z \in \mathbf{C}^*$. Then τ is an irreducible representation with the same property $\Longleftrightarrow \exists\omega$ a character of $M_1 :\ \ \tau = \pi \otimes \omega$. The isomorphism class of $\pi \otimes \omega$ only depends on $\omega|C$.

Proof. Let (V, π) be an irreducible representation of $\tilde{M}_1$ with $z \mapsto z^{-1}$ and central character χ. Let (W, ρ) be an other irreducible representation with $z \mapsto z^{-1}$ and central character θ. Then $\chi^{-1}.\theta$ is 1 on $\mathbf{C}$; i.e. it is a character of C. As M is commutative, we can extend it to M, call this extension ω. Then, $\pi.\omega$ has the central character θ; i.e. it is isomorphic to ρ. The converse is trivial. $\square$

Let $\mathbf{C}[\tilde{M}_1; z^{-1}]$ denote the $\mathbf{C}$-algebra generated by the elements δ_m $(m \in \tilde{M}_1)$, with the relations $\delta_m.\delta_n := \delta_{m.n}(\forall m, n \in \tilde{M}_1)$ and $\delta_z := z^{-1}.\delta_e$. Any representation of $\tilde{M}_1$ where $z \in \mathbf{C}$ acts by z^{-1} is a $\mathbf{C}[\tilde{M}_1; z^{-1}]$-module.

Let $M := Z^n$, $T(\mathbf{C}) : = Hom(M, \mathbf{C}^*)$. As a group $T(\mathbf{C})$ is isomorphic to the nonsingular diagonal matrices $\mathbf{D}_n$ in $\mathbf{GL}(n)$. Let ϕ be this group isomorphism.

Fixing a basis $\{e_i\}$ of M, $T(\mathbf{C})$ is isomorphic to the nonsingular diagonal matrices $\mathbf{D}_n$ in $\mathbf{GL}(n)$ where n is the dimension of M. If $\oplus$ denotes the addition and $+$ the multiplication for the group ring $\mathbf{C}[M]$, then for an element $p = \bigoplus_j c_j.(\sum_i m_{ij}.e_i) \in \mathbf{C}[M]$ $(c_i \in \mathbf{C}, m_i \in \mathbf{Z})$ and the diagonal matrix $\delta = (\delta_{ii}) \in \mathbf{D}_n$; $p(\delta) = \sum_j c_j \prod_i \delta_{ii}^{m_{ij}}$. This shows how $\mathbf{C}[T(\mathbf{C})] = \mathbf{C}[M]$;i.e. an element of the algebra $\mathbf{C}(M)$ of M defines a polynomial function on $T(\mathbf{C})$.

If we endow $T(\mathbf{C})$ with the topological structure induced by ϕ, $(T(\mathbf{C}),\mathbf{C}(M))$ is a ringed space and ϕ is an isomorphism of algebraic varieties . Hence $(T(\mathbf{C}),\mathbf{C}(M))$ is an algebraic torus.(For definitions [4])

Let $F := C^{\perp} = \{x \in T(\mathbf{C}) \mid \forall c \in C \ x(c) = 1\} < T(\mathbf{C})$. F is a finite subgroup. Thus the quotient is a torus. Considering the restriction map we can see that it is isomorphic to the character torus on C.

Let $\mathbf{Irr}(\mathcal{B})$ denote the set of isomorphism classes of irreducible representations of $\tilde{M}_1$ with $z \mapsto z^{-1}$. Let us fix an element $[\pi] \in \mathbf{Irr}(\mathcal{B})$. Let us define

$$d :: (T/F)(\mathbf{C}) \longrightarrow \mathbf{Irr}(\mathcal{B})$$

by the rule

$$d([\chi]) := [\pi \otimes \chi] \qquad (\chi \in T(\mathbf{C})).$$

Then, d is well defined, 1-1 and onto; i.e. $\mathbf{Irr}(\mathcal{B})$ is a principally homogenous space under the action of $(T/F)(\mathbf{C})$(cf. Corollary to Proposition 3.4.4).

This endows $\mathbf{Irr}(\mathcal{B})$ with a structure of algebraic variety (cf. pg. 27): a function f is regular on $\mathbf{Irr}(\mathcal{B})$ iff the function $\chi \mapsto f(\pi\chi)$ over $T(\mathbf{C})$ is regular ([1] : 1.14). These f's then comes from regular functions on $(T/F)(\mathbf{C})$; i.e. a finite linear combination of the functions $\chi \mapsto \chi(c)$ $(c \in C\ )$. Let $\mathcal{O}(\mathbf{Irr}(\mathcal{B}))$ denote the regular functions on $\mathbf{Irr}(\mathcal{B})$.

Proposition 3.4.5. *If $z \in Z(\mathcal{B})$ then z acts as a scalar on irreducibles of $\mathcal{B}$. For each $[\pi] \in \mathbf{Irr}(\mathcal{B})$, let $z(\pi)$ denote this scalar. Then, for each z*

$$[\pi] \mapsto z(\pi)$$

is a well-defined function on $\mathbf{Irr}(\mathcal{B})$. Moreover

$$Z(\mathcal{B}) \simeq \mathcal{O}(\mathbf{Irr}(\mathcal{B}))$$

Proof. By the very construction of $\mathbf{C}[\tilde{M}_1; z^{-1}]$, we have

$$Z(\mathcal{B}) \simeq Z(\mathbf{C}[\tilde{M}_1; z^{-1}]) = \mathbf{C}[\tilde{C}; z^{-1}].$$

Considering this we map a regular function

$$\chi \mapsto \sum_{i=1}^{i=n} c_i.\chi(x_i) \qquad (c_i \in \mathbf{C}, x_i \in C)$$

to

$$\sum_i \delta_{(x_i, c_i^{-1})} \in \mathbf{C}[\tilde{C}; z^{-1}].$$

□

Proposition 3.4.6. $\mathbf{Irr}((Alg\ G)_A)$ *is an algebraic variety isomorphic to* $\mathbf{Irr}(\mathcal{B})$.

Proof. Let F be the functor $(Alg\ G)_A \sim \mathcal{B}$ in Proposition 3.4.2. F induces a 1-1 correspondance $\bar{F}$ between $\mathbf{Irr}((Alg\ G)_A)$ and $\mathbf{Irr}(\mathcal{B})$. Hence we can endow $\mathbf{Irr}((Alg\ G)_A)$ with a structure of algebraic variety using this $\bar{F}$ such that it is an isomorphism of algebraic varieties. □

Let $\mathcal{O}(\mathbf{Irr}((Alg\ G)_A))$ denote the algebra of regular functions on $\mathbf{Irr}((Alg\ G)_A)$ in this structure.

Theorem 3.4.7.

$$\mathcal{O}(\mathbf{Irr}((Alg\ G)_A)) \simeq Z((Alg\ G)_A)$$

Proof. We know by Schur's lemma that each element of $Z((Alg\ G)_A)$ defines a function on $\mathbf{Irr}((Alg\ G)_A)$. For a character χ of G that is 1 on G^0, i.e. for a character of M, for an irreducible π, $F(\pi.\chi) \simeq F(\pi).\chi$. Hence

$$\bar{F}([\pi.\chi]) = [F(\pi).\chi] \qquad (\forall \pi\ ,\ \forall \chi\ ).$$

Let $\phi : : \mathcal{O}(\mathbf{Irr}(\mathcal{B})) \simeq Z(\mathcal{B}) = \mathbf{C}[\tilde{C}; z^{-1}]$ (Proposition 3.4.5). For a function f on $\mathbf{Irr}((Alg\ G)_A)$,

$$f \in \mathcal{O}(\mathbf{Irr}((Alg\ G)_A)) \Leftrightarrow \phi(f \circ \bar{F}^{-1}) \in \mathbf{C}[\tilde{C}; z^{-1}];$$

i.e. by Proposition 3.4.6,

$$(\forall f \in \mathcal{O}[\mathbf{Irr}((Alg\ G)_A)])(\exists c \in \mathbf{C}[\tilde{C}; z^{-1}]) : \quad f = \phi^{-1}(c) \circ F$$

Then we fix π for $\mathbf{Irr}((Alg\ G)_A)$ and $F(\pi)$ for $\mathbf{Irr}(\mathcal{B})$ in accordance with pg. 28. Hence for f and c as above and for an irreducible $\pi.\chi \in \mathbf{Irr}((Alg\ G)_A)$,

$$f([\pi.\chi]) = \phi^{-1}(c)([F(\pi).\chi]) = \chi(c), \quad i.e.$$

f comes from an element of $Z((Alg\ G)_A)$.

Conversely, by a similar reasoning, for each element $z \in Z((Alg\ G)_A)$ we can find an element c in $\mathbf{C}[\tilde{C}; z^{-1}]$ such that

$$z(\pi.\chi) = \chi(c) \qquad (\forall \chi)$$

and by Proposition 3.4.5 each element of $\mathbf{C}[\tilde{C}; z^{-1}]$ gives a regular function on $\mathbf{Irr}((Alg\ G)_A)$. We can easily show that these maps are ring homomorphisms one the inverse of the other. $\square$

Proposition 3.4.8. *A representation* (V, π) *of* G *is said to be quasi-cuspidal if the following equivalent conditions are satisfied:*

(i) $(V, \pi|_{G^0}) \in (Alg\ G^0)_f$.

(ii) The matrix coefficients of V *are compact mod* $Z(G)$*; i.e. for some compact set* K *of* G *their support is contained in* $Z(G).K$

(iii) $\forall v \in V$*, if* $g \to \infty$ *in* G *w.r.t. the filter of complements of compacts mod* $Z(G)$*, then* $\pi(g).w \to 0$ *for the topology* T *in pg. 17 (cf. Lemma 3.6) ([2] : 3.21)*

Proof. (ii) $\Rightarrow$ (i) : G^0 is open, hence $V^{\sim}$ for G^0 or G are the same (the stabilizer is still open after restriction to G^0). So matrix coefficients for $\pi|G^0$ are just those of π restricted to G^0. Then let us note that $Z(G).K \cap G^0$ is compact ($u(K)$ is finite, so for only finitely many $z \in Z(G)$ $z.K \subset G^0$) ([4] : 3.21).

(iii) $\Rightarrow$ (ii) : For $\xi^{\sim} \in (V^K)^{\sim}$ and $\xi \in V$

$$\langle \xi^{\sim}, \pi(g).\xi \rangle = \langle \pi^{\sim}(e_K).\xi^{\sim}, \pi(g).\xi \rangle = \langle \xi^{\sim}, \pi(e_K).\pi(g).\xi \rangle\ ;\ i.e.$$

the matrix coefficient is compact mod $Z(G)$ ([2] : 3.21).

(i) $\Rightarrow$ (iii) : (i) $\Leftrightarrow$ (Lemma 3.6/(iii)). And the latter implies (iii) since then for any compact set K there is a $g_0 \in G^0 \cap (Z(G).K)^c$ such that $\pi(e_K).\pi(g_0).w = 0$. □

A representation (V, π) of G is said to be *cuspidal* if it is quasi-cuspidal and admissible. Let $(Alg\ G)_{qc}$ denote the category of quasi cuspidal representations of G; D the index of orbits of $T(\mathbf{C})$ acting by torsion on the set $\mathbf{Irr}((Alg\ G)_{qc})$ and $(Alg\ G)(D)$ the category of representations (V, π) of G for which $JH(\pi) \subset D$.

Proposition 3.4.9. $(Alg\ G)_{qc} = \prod_D (Alg\ G)(D)$ *(The formula 1.20.1 in [1])*

Proof. Considering Proposition 3.2.1 we will show that any quasi-cuspidal representation is a sum with summands from $(Alg\ G)_D$. Let $(V, \pi) \in (Alg\ G)_{qc}$. By corollary 3.3.4, we know that $(V, \pi|0) = \bigoplus_{\tau \in \hat{G}_f} V_\tau$. For an irreducible quasi-cuspidal representation ρ_D in an orbit D, its restriction to G^0 constitutes of the sum of conjugates of an irreducible ρ_D^0 in $(Alg\ G^0)_f$ and the same conjugates constitute all other irreducibles in D. Then, $V_D := \bigoplus_{[g] \in G/G^1} (V_{(\rho_D^0)^g})$ is a G-submodule which is a direct summand of V such that $JH(V_D) \subset D$. Therefore

$$V = \bigoplus_D [\bigoplus_{[g] \in G/G^1} (V_{(\rho_D^0)^g})] = \bigoplus_D V_D.$$

□

If, for all compact open subgroup K of G^0, G^0 has only a finite number of isomorphism classes of irreducible representations (V, π) in $(Alg\ G^0)_f$ such that $V^K \neq 0$ then by Proposition 3.3.3 (the formula 1.20.2,3 in [1]),

$$Alg\ G \ = \ (Alg\ G)_{qc} \times (Alg\ G)_{nqc}.$$

$\mathbf{Irr}(Alg\ G)_{qc}$ has the structure of an algebraic variety; it is as a scheme the disjoint union of the algebraic varieties D. It is a scheme by Glueing Lemma ([14] : 2.12)and its sheaf of regular functions consists of the families of regular functions on the summands. If $\mathcal{O}(\mathbf{Irr}(Alg\ G)_{qc})$ denotes the regular functions on $\mathbf{Irr}(Alg\ G)_{qc}$ then,

$$Z((Alg\ G)_{qc}) = \mathcal{O}(\mathbf{Irr}(Alg\ G)_{qc}).$$

Moreover, if the finiteness condition above is satisfied then

$$Z(Alg\ G) \ = \ Z((Alg\ G)_{qc}) \times Z((Alg\ G)_{nqc}).$$

4. PART II : THE CENTER OF NON-QUASI-CUSPIDAL COMPONENTS

4.1. General Setting For $GL(n, \mathbf{Q}_p)$

We will begin with some preliminaries from [2]. This section will be rather sketchy; in some paragraphs we will be contented by noting some standard facts.

The field of p-adic numbers $\mathbf{Q}_p$ is the completion of the field of rational numbers with respect to the p-adic valuation ([12] : ch. 1). The set of p-adic integers $\mathbf{Z_p}$ is the unique maximal compact subring of $\mathbf{Q}_p$ ([12] : ch.1,§4, Thm. 6). The sets $p^i.\mathbf{Z}_p$ ($i \in \mathbf{N}$) constitute a fundamental system of compact open neighborhoods around 0 ([12]: ibid.).

(From [2] : 3.1) Now let $G := GL(n, \mathbf{Q}_p)$. As the preimage of a continuous map (determinant) G is an open subset of the topological ring $M(n, \mathbf{Q}_p)$ whose topology is the product topology of $\mathbf{Q}_p$'s as above. Let $\Gamma := GL(n, \mathbf{Z}_p)$. Γ is the maximal compact subgroup of G. Moreover let $N_i := 1 + p^i.M(n, \mathbf{Z}_p)$ $(i = 0, 1, 2, \ldots)$, where $M(n, \mathbf{Z}_p)$ is the ring of $n \times n$ matrices with coefficients in $\mathbf{Z}_p$. We call them congruence subgroups of G. One can show that the family $N_1 \supset N_2 \supset \ldots$ forms a fundamental system of neighborhoods around 1 consisting of open compact subgroups; i.e. G is an l-group.

Moreover, as the space $M(n, \mathbf{Q}_p)$ is σ-compact (cf. [13] : VII.18) and $p^i.M(n, \mathbf{Z}_p)$ constitute a fundamental system of compact-open neighborhoods around 0, by previous paragraph G is also σ-compact. Also Bernstein-Zelevinski in [2] assert that G is unimodular ((ZZZ) [13] : VII.60, Examples (ZZZ)).

We call the N_i's congruence subgroups. We denote by $Z = Z_n$ the centre of G;

$$Z = \{\lambda.1 \mid \lambda \in \mathbf{Q}_p, \lambda \neq 0\}.$$

Let $\mathbf{N}_n$ denote the index set $\{1, 2, \ldots, n\}$. By a partition α of n we mean a partition of $\mathbf{N}_n$ into disjoint segments. The set of partitions of n is a partially ordered set with respect to the relation: $\beta < \alpha$ iff each segment in β is a subsegment of a segment in α.

Let $\alpha = (n_1, \ldots, n_r)$ be a partition of n. We set

$$G_\alpha = \prod_{i=1}^{r} G_{n_i}.$$

We assume that G_α is embedded in $G = G_{(n)}$. We call subgroups of the form $N_i(\alpha) = N_i \cap G_\alpha$, where N_i is the congruence subgroup of G, congruence subgroups of G_α. Without hesitation we can generalize the assertions above to G_α's with appropriate modifications.

Let B be the subgroup of upper triangular matrices in G, $U = U_n$ the subgroup of B consisting of matrices with 1 along the diagonal, and $D = D_n$ the subgroup of diagonal matrices. For each permutation ω of $\mathbf{N}_n$ we associate a matrix $w = (w_{ij})$, where $w_{ij} = \delta_{i,\omega(j)}$ and $\delta_{i,j}$ denotes the Kronecker delta([2] : 3.2). Let W denote the set of these matrices.

Let β be a partition of n. We denote by U_β the subgroup of U_n consisting of matrices $u = (u_{ij})$ for which $u_{ij} = 0$ if $i \neq j$ and i, j lie in the same segment of β. We set $P_\beta := G_\beta.U_\beta$ and $P_\beta(\alpha) := G_\beta.U_\beta(\alpha)$ where $U_\beta(\alpha) := U_\beta \cap G_\alpha$ $(\beta < \alpha)$.

When we consider G as an algebraic group over $\mathbf{Q}_p$ we have the following results:

D is a maximal torus of G. B is a Borel subgroup of G containing D. All parabolic subgroups of G containg D are of the form $P_\beta(\alpha)$. The other parabolic subgroups are conjugates of these. The unipotent radical of a parabolic subgroup $P_\beta(\alpha)$ is $U_\beta(\alpha)$ and G_β is a Levi subgroup of it. W is the Weyl group corresponding to the maximal torus D.

For a Levi subgroup G_γ of G, we obtain similar results as above; its parabolic subgroups containing D are of the form $P_\beta(\alpha)$ $(\beta < \alpha < \gamma)$.

Let $u : G \longrightarrow \mathbf{Z}$ be the homomorphism that assigns $|det(g)|_p$ to each element g of G. Let Z denote the center of G. Then, for this morphism:

- $G^0 \cap Z$ is compact
- G/G^0 is finite.

In other words, the conditions at the beginning of the section 3.4 are satisfied for G and u ([2] : 3.20). Moreover, let K be a congruence subgroup of G. Then G^0 has only finitely many non-isomorphic finite irreducible representations (V, ω) such that $V^K \neq 0$ ([2] : 4.14); i.e. the conditions of finiteness at the end of the section 3.4 are satisfied. Therefore

$$Alg\ G \ = \ (Alg\ G)_{qc} \times (Alg\ G)_{nqc}$$

and

$$Z(Alg\ G) \ = \ Z((Alg\ G)_{qc}) \times Z((Alg\ G)_{nqc}).$$

If we take G_α instead of G, $u : G \longrightarrow \mathbf{Z}^k$ defined by $(|det(g_i)|_p)_i$ where $\alpha = (n_1 \ldots n_k)$ and $g_i \in G_{n_i}$; we obtain the same conclusions.

In part I we described $Z((Alg\ G)_{qc})$ as the regular functions on the algebraic variety $\mathbf{Irr}(Alg\ G)_{qc}$ i.e.

$$Z((Alg\ G)_{qc}) = \mathcal{O}_{\mathbf{Irr}(Alg\ G)_{qc}} = \prod_D \mathcal{O}_D$$

where D's are orbits of $\mathbf{Irr}(Alg\ G)_{qc}$ for the action of $T(\mathbf{C})$ by torsion. We will now describe the center of the second component.

4.2. Jacquet Functor and Parabolic Induction

Let $P_\alpha(\beta) = G_\alpha.U_\alpha(\beta)$ be a parabolic subgroup of G_γ $(\alpha < \beta < \gamma)$ with corresponding unipotent radical and Levi subgroups.

Let $(V, \pi) \in Alg\ G_\gamma$, $V(U_\alpha(\beta)) := \langle \xi - u.\xi \mid \xi \in V,\ u \in U_\alpha(\beta)\rangle$ and $V_{U_\alpha(\beta)} := V/V(U_\alpha(\beta))$.

Let us define the functor $r^{G_\gamma}_{P_\alpha(\beta)} :: Alg\ G_\gamma \longrightarrow Alg\ G_\alpha$ by the rule

$$r^{G_\gamma}_{P_\alpha(\beta)}(V) = V_{U_\alpha(\beta)} \otimes_{\mathbf{C}} (\Delta_{G_\gamma}/\Delta_{P_\alpha(\beta)})^{-1/2}.$$

Let us define $i^{G_\gamma}_{P_\alpha(\beta)}$; the functor of induction modified by $(\Delta_{G_\gamma}/\Delta_{P_\alpha(\beta)})^{1/2}$. This means that for $(V, \rho) \in Alg\ G_\alpha$, $i^{G_\gamma}_{P_\alpha(\beta)}(V, \rho)$ is

$$(ind(G_\gamma, P_\alpha(\beta), (\Delta_{G_\gamma}/\Delta_{P_\alpha(\beta)})^{1/2}\rho)\ ,\ L(G_\gamma, (\Delta_{G_\gamma}/\Delta_{P_\alpha(\beta)})^{1/2}\rho));$$

i.e. the space of functions $f : G_\gamma \longrightarrow V$ satisfying:

(i) $f(hug) = \Delta^{1/2}(hu)\rho(h)f(g)$ for all $h \in G_\alpha$, $u \in U_\alpha(\beta)$, $g \in G_\gamma$. $(\Delta = \Delta_{G_\gamma}/\Delta_{P_\alpha(\beta)})$

(ii) There exists an open compact subgroup N of G_γ such that $f(g.g_0) = f(g)$ for all $g \in G_\gamma$ $g_0 \in N$

together with the action of G_γ by right translations. Then by a similar reasoning to the proof of Frobenius duality ([2] : 2.28) it follows that $r^{G_\gamma}_{P_\alpha(\beta)}$ is the left adjoint of the functor $i^{G_\gamma}_{P_\alpha(\beta)}$; that is, for $(V, \pi) \in Alg\ G_\gamma$ and $(W, \rho) \in Alg\ P_\alpha(\beta)$

$$Hom_{G_\gamma}(\pi, i^{G_\gamma}_{P_\alpha(\beta)}(\rho)) \simeq Hom_{P_\alpha(\beta)}(r^{G_\gamma}_{P_\alpha(\beta)}(\pi), \rho).$$

(V, π) in $Alg\ G_\gamma$ is quasi-cuspidal if and only if for any $P_\alpha(\beta) = G_\alpha.U_\alpha(\beta)$, a proper parabolic subgroup of G_γ $(\alpha < \beta < \gamma)$, we have $V_{U_\alpha(\beta)} = 0$ ([2] : 3.21 Harish-Chandra's Theorem).

Proposition 4.2.1. *For* $(V, \pi) \in Alg\ G_\gamma$,

$$V_{qc} = Ker(V \longrightarrow \prod_{\alpha<\beta<\neq\gamma} i^{G_\gamma}_{P_\alpha(\beta)} r^{G_\gamma}_{P_\alpha(\beta)}(V)).$$

Proof. The map sends $v \in V$ to $(g \mapsto [\pi(g).v + V(U_\alpha(\beta))] \otimes_{\mathbf{C}} 1)_{(\alpha<\beta<\neq\gamma)}$. This is a G_γ-module homomorphism. If $v \in V_{qc}$ then $v \in V(U_\alpha(\beta))$ for every parabolic subgroup by Harish-Chandra's Theorem. Hence it is in the kernel. Conversely, if v is in the kernel, then for any parabolic subgroup $P_\alpha(\beta) = G_\alpha.U_\alpha(\beta)$ of $G_{\gamma.}$, we have $v \in V(U_\alpha(\beta))$. The submodule generated by v is a quasi-cuspidal sub-representation of V by Harish-Chandra's Theorem. Hence $v \in V_{qc}$. □

Let $\alpha > \beta > \gamma$ be a sequence of partitions of n. The functors $r^{G_\alpha}_{P_\beta(\alpha)}$ and $i^{G_\alpha}_{P_\beta(\alpha)}$ are exact. Moreover, by ([2] : 3.13) we know that

$$i^{G_\alpha}_{P_\beta(\alpha)} \circ i^{G_\beta}_{P_\gamma(\beta)} = i^{G_\alpha}_{P_\gamma(\alpha)}$$

and

$$r^{G_\beta}_{P_\gamma(\beta)} \circ r^{G_\alpha}_{P_\beta(\alpha)} = r^{G_\alpha}_{P_\gamma(\alpha)}.$$

4.3. Bernstein Decomposition And Bernstein Center

Let G_α be a Levi subgroup of G_γ ($\alpha = \gamma$ is included). Let D be a connected component of the algebraic variety $\mathbf{Irr}(Alg\ G_\alpha)_{qc}$ (Section 3.4). We denote such couples by (G_α, D).

Lemma 4.3.1. *(This is Lemma 2.7. in [1]) Let (V,π) be in $(Alg\ G_\gamma)$. If for all couples (G_β, D) as in the above paragraph and for all parabolic subgroups $P_\beta(\alpha)$ of G_γ ($\beta < \alpha < \gamma$), $JH(r^{G_\gamma}_{P_\beta(\alpha)}(\pi)) \cap D = \emptyset$; then $V = 0$.*

Proof. If $V \neq 0$ then there exists a minimal Parabolic subgroup $P_\alpha \leq G_\gamma$ such that $r^{G_\gamma}_{P_\alpha}(V)) \neq 0$. Let G_α be Levi subgroup of P_α. Then, for $\alpha > \beta$, $r^{G_\alpha}_{P_\beta(\alpha)}(r^{G_\gamma}_{P_\alpha} V) = 0$. Hence, by Harish-Chandra's Theorem, $r^{G_\gamma}_{P_\alpha} V$ is in $(Alg\ G_\alpha)_{qc}$. Therefore, by part I, for some D, $JH((r^{G_\gamma}_{P_\alpha} V)) \cap D \neq \emptyset$. $\square$

Let P and Q be two parabolic subgroups of G_γ with Levi subgroups G_α and G_β and unipotent radicals U and V. The functor $F := r^{G_\gamma}_Q \circ i^{G_\gamma}_P : Alg\ G_\alpha \longrightarrow Alg\ G_\beta$ is an iterated extension of the functors $F_w := i^{G_\beta}_{wPw^{-1} \cap G_\beta} \circ r^{w.G_\alpha.w^{-1}}_{Q \cap w.G_\alpha.w^{-1}} \circ int(w)$ where w's are representatives of the double cosets $Q \setminus G/P$ and $int(w) : Alg\ P \longrightarrow Alg\ wPw^{-1}$ is the functor induced from the group homomorphism $w : P \longrightarrow wPw^{-1}$ which sends points to their conjugates by w.

The proof of the above statement is a little long, for details one can see ([3] : 2.11 and 5.2) where it deals the general case of reductive p-adic groups. Here we will not deal with the proof. Yet, to appreciate the intricateness of the assertion let us see what it corresponds to in a concrete example such as $G = GL(4, \mathbf{Q}_p)$: Let $\alpha := (3, 1)$ and $\beta := (2, 2)$, $P := P_\alpha$, $Q := P_\beta$, U and V be defined correspondingly. Let $w \in W$ be the matrix associated to the permutation (432) of $\mathbf{N}_n$. We have $G = QP \cup QwP$, $G_\beta \cap P = P_{(2,1,1)}(2, 2)$, $G_\beta \cap wPw^{-1} = P_{(1,1,2)}(2, 2)$ and $G_\alpha \cap Q = P_{(2,1,1)}(3, 1)$. Let $(V, \pi) \in Alg\ G_\alpha$. Then the above statements assert that

$$F(V, \pi) = ([(\Delta_G/\Delta_Q)^{-1/2}.\rho_1]|_{G_\beta}, L(G, [\Delta_G/\Delta_P]^{1/2}.\pi)_{U_{(3,1)}})$$

is glued from the modules

$$(ind(G_\beta, G_{(2,1,1)}, [\Delta_{G_\beta}/\Delta_{G_{(2,1,1)}}]^{1/2}\rho_2), L[G_\beta, [\Delta_{G_\beta}/\Delta_{G_{(2,1,1)}}]^{1/2}\rho_2])$$

and

$$(ind[G_\beta, G_{(1,1,2)}, (\Delta_{G_\beta}/\Delta_{G_{(1,1,2)}})^{1/2}\rho_3], L[G_\beta, (\Delta_{G_\beta}/\Delta_{G_{(1,1,2)}})^{1/2}\rho_3]);$$

where

$$\begin{aligned}
\rho_1 &= ind(G, P, [\Delta_G/\Delta_P]^{1/2}\pi) \\
\rho_2 &= (V_{U_{(2,1,1)}(3,1)}, (\pi \otimes [\Delta_{G_\alpha}/\Delta_{P_{(2,1,1)}(3,1)}]^{-1/2})|_{G_{(2,1,1)}}) \\
\rho_3 &= (V_{U_{(1,1,2)}(1,3)}, (int(w)(\pi) \otimes [\Delta_{G_{(1,3)}}/\Delta_{P_{(1,1,2)}(1,3)}]^{-1/2})|_{G_{(1,1,2)}}).
\end{aligned}$$

i.e. from the modules $F_e(V, \pi)$ and $F_w(V, \pi)$.

Let (G_{α_0}, D) be a couple as above. Then $(Alg\ G_\gamma)(G_{\alpha_0}, D)$ is defined to be the subcategory of $Alg\ G_\gamma?l?$ consisting of representations (W, ρ) satisfying the following equivalent conditions :

Proposition 4.3.2. *(This is Proposition-Definition 2.8 in [1]) TFAE:*

(i) $W \hookrightarrow \bigoplus_{\alpha_0 \leq \beta} i^{G_\gamma}_{P_{\alpha_0}(\beta)}(W_{P_{\alpha_0}(\beta)})$ *where* $W_{P_{\alpha_0}(\beta)}$ *is in* $(Alg\ G_{\alpha_0})(D)$.

(ii) W *is a subquotient of a sum as in i.*

(iii) Let (G_α, E) *be another couple which is not a conjugate of* (G_{α_0}, D) *and* $Q = G_\alpha.U_\alpha(\beta)$. *Then the component of* $r^{G_\gamma}_Q(W)$ *in* $(Alg\ G_\alpha)(E)$ *is null.*

(iv) Let $\alpha_0 \leq \theta \leq \gamma$. *Then* $r^{G_\gamma}_{P_{\alpha_0}(\theta)}(W)$ *is in* $\prod_E (Alg\ G_{\alpha_0})(E)$ *where* (G_{α_0}, E) *is conjugate to* (G_{α_0}, D). *Let* $(r^{G_\gamma}_{P_{\alpha_0}(\theta)}W)(D)$ *be the component of* $r^{G_\gamma}_{P_{\alpha_0}(\theta)}W$ *in* $(Alg\ G_{\alpha_0})(D)$. *Then the natural mapping*

$$\phi : W \longrightarrow \bigoplus_{(\alpha_0 \leq \beta)} i^{G_\gamma}_{P_{\alpha_0}(\beta)}(r^{G_\gamma}_{P_{\alpha_0}(\beta)}W)(D)$$

is injective.

Proof. Firstly let us notice that the natural mapping ϕ sends an element w of W to the tuple $(f_\beta)_{\alpha_0 \leq \beta}$ where $f_\beta : G_\gamma \longrightarrow W_{U_{\alpha_0}(\beta)} \otimes \mathbf{C}$ is in $L(G_\gamma, [\Delta_{G_\gamma}/\Delta_{P_{\alpha_0}(\beta)}]^{1/2}\rho|_{P_{\alpha_0}(\beta)})$ and is determined by the condition that $f_\beta(g) = (\rho(g)w + W(U_{\alpha_0}(\beta))) \otimes_{\mathbf{C}} 1$.

There is nothing to prove for the implication iv.⇒ i.⇒ ii. As to ii.⇒ iii., it is sufficient to deal the case W is of the form $i^{G_\gamma}_{P_{\alpha_0}(\beta_0)}X$ where X is in $(Alg\ G_{\alpha_0})(D)$; for, W is a subquotient of a sum as in i. Then, since X is quasi-cuspidal, $r^{G_\gamma}_{P_\alpha(\alpha_0)}X$ is null for $\alpha < \alpha_0$ (Harish-Chandra). Then, by the decomposition (iterated extensions) of $r^{G_\gamma}_{P_{\alpha_0}(\beta)} i^{G_\gamma}_{P_{\alpha_0}(\beta_0)}$ in functors $i \circ r$ it follows that $JH(r^{G_\gamma}_{P_{\alpha_0}(\beta)} i^{G_\gamma}_{P_{\alpha_0}(\beta_0)}X)$ consists of either representations in $(Alg\ G_\alpha)(E)$ for (G_α, E) conjugate to $(G_{\alpha_0,}, D)$ or representations induced from these. Then by Proposition 4.2.1 it follows iii.

Finally, we will prove iii. $\Rightarrow$ iv. Applying Lemma 4.3.1 to the component of $r^{G_\gamma}_{P_{\alpha_0}(\theta)}(W)$ in $(Alg\ G_\alpha)(non-conjugate\ to\ D)$ we have that $r^{G_\gamma}_{P_{\alpha_0}(\theta)}(W)$ is in $\prod_E(Alg\ G_{\alpha_0})(E)$ where (G_{α_0}, E) is conjugate to (G_{α_0}, D). Again by Lemma 4.3.1 $ker\ \phi = 0$; i.e. ϕ is injective. □

The following two results give the Bernstein decomposition for our case.

Lemma 4.3.3. *(Lemma 2.9 in [1]) If X is in $(Alg\ G_\gamma)(G_{\alpha_0}, D)$ and Y is in $(Alg\ G_\gamma)(G_{\beta_0}, E)$ and (G_{α_0}, D) is not conjugate to (G_{β_0}, E) then $Hom(X, Y) = 0$.*

Proof. Let $f \in Hom(X, Y)$. Then applying Proposition 4.3.1/iii to f(X) we have $r^{G}_{P_\beta(\beta_0)} f(X) = 0$ $(\beta < \beta_0)$. Hence, by Lemma 4.3.1 applied to G_{β_0} we have $f = 0$. □

Proposition 4.3.4 (Proposition 2.10 ibid.)**.**

$$Alg\ G_\gamma = \prod_{(G_\alpha, D)} (Alg\ G_\gamma)(G_\alpha, D). \quad (\alpha < \gamma)$$

where (G_α, D)'s are taken up to conjugation by elements of G_γ.

Proof. By Proposition 3.2.1, having asserted Lemma 4.3.3 we need only to prove that each object of $Alg\ G_\gamma$ can be written as a sum

$$\bigoplus X_{G_\alpha, D}$$

where $X_{G_\alpha,D}$ is in $(Alg\ G_\gamma)(G_\alpha, D)$. By induction we can assume that the proposition is valid for all Levi subgroups of G_γ. Let (W, π) be a representation of G_γ. We have $W = W_{qc} + W_{nqc}$ with

$$W_{qc}\ in\ (Alg\ G_\gamma)_{qc} = \bigoplus_D (Alg\ G_\gamma)(G_\gamma, D)$$

and W_{nqc} embeds in a sum of $i^{G_\gamma}_P(X_P)$. The hypothesis allows the decomposition of X_P. Then by consecutive induction described above the proposition follows. □

Now we are ready to describe Bernstein Center. Firstly, we give without proof the following theorem which states that we can see the elements of the center as regular functions.

Proposition 4.3.5. *(Proposition 2.11 in [1]) Let z be in the center of the abelian category $(Alg\ G)(G_\alpha, D)$. For a parabolic subgroup $P_\alpha(\beta)$ and π in D, z acts as a scalar in $i_P^G(\pi)$. This scalar $z(\pi)$ is independent of P, and the function $\pi \mapsto z(\pi)$ is a regular function on the algebraic variety D.*

Let $W(L, D)$ be the subgroup of $N_G(L)/L$ ($N_G(L)$ is the normalizer of L in G.) consisting of n such that D coincides to its conjugate by n. This finite group acts on D and the function $z(\pi)$ above is invariant by $W(L, D)$: this is a regular function on the quotient algebraic variety.

Thus, to each z in $Z(\mathcal{H}(G)^\wedge)$, we associate a collection of regular functions on $D/W(G_\alpha, D)$ for (G_α, D); i.e. a regular function on the variety that is disjoint sum of $D/W(G_\alpha, D)$'s.

Theorem 4.3.6. *(Bernstein Center) The construction above establishes an isomorphism between $Z(\mathcal{H}(G)^\wedge)$ and the ring of regular functions over $\coprod D/W(G_\alpha, D)$.*

Proof. By Proposition 4.3.3 and Proposition 3.2.1 it is sufficient for us to show that for each (G_α, D) the mapping $z \mapsto z(\pi)$ identifies the center of $(Alg\ G)(G_\alpha, D)$ to the ring of regular functions on $D/W(G_\alpha, D)$. If z_1 and z_2 agree on $D/W(G_\alpha, D)$ then they agree on $i_P^G(\pi)$ ($\pi \in D$). As any irreducible of $(Alg\ G)(G_\alpha, D)$ is a quotient of a $i_P^G(\pi)$ it follows that this mapping is one-to-one.

Let us prove surjectivity. Let z_α be a regular function on $D/W(G_\alpha, D)$. By Proposition 3.1 it corresponds to an element z_α of the center of $(Alg\ G_\alpha)(D)$ that is invariant by $W(G_\alpha, D)$. We will find an element z in the center of $(Alg\ G)(G_\alpha, D)$ such that for each induced representation $i_P^G X$ (X in $(Alg\ G_\alpha)(D)$, $P = P_\alpha(\beta)$ for some $\alpha < \beta$)it acts like i_P^G(the endomorphism of X defined by z_α) on it; i.e. the regular function $z(\pi)$ is equal to z_α.

Let us consider the injection $\phi : W \hookrightarrow \bigoplus_{\alpha \leq \beta} i^G_{P_\alpha(\beta)} r^G_{P_\alpha(\beta)}(W)$. The endomorphism $\bigoplus_P i^G_P(z_\alpha)$ induces on W the desired endomorphism z if $\phi(W)$ is stable under it. The set of such W's is stable by sums and sub-quotients. Hence, we can take W as $i^G_P(X)$ where X is in $(Alg\ G_\alpha)(D)$. In order to show that $\phi(i^G_P(X))$ is stable under $\bigoplus_P i^G_P(z_\alpha)$ we will more precisely show that (*) the endomorphism z_α on $r^G_Q i^G_P(X)(D)$ and the endomorphism on it induced by $r^G_Q i^G_P(z_\alpha)$ are identical.

The functor $r^G_Q i^G_P$ is glued from the functors $F_i : (X, \pi) \mapsto (X, \pi \circ int^{-1}_{\tilde{n_i}})$ where $\tilde{n_i}$ is a representative of $n_i \in W(G_\alpha, D)$ in $N_G(G_\alpha)$. By the assumption that z_L is invariant by $W(G_\alpha, D)$, z_L and $F_i(z_L)$ are identical on $F_i(X)$. Hence, considering Proposition 3.2.1., the proof is complete for X such that for each $n_1 \neq n_2$ in $W(G_\alpha, D)$ $Hom_{G_\alpha, End(X)}(\tilde{n_1}(X), \tilde{n_2}(X)) = 0$.

Bernstein continues his argument as follows which I did not fully grasp yet: The set of X in $(Alg\ G_\alpha)(D)$ satisfying (*) is stable by sums and sub-quotients. Hence in view of Appendix 1.18 it is sufficient to prove it for X of the form $(V \otimes B, \pi \otimes \chi_{un})$. We need to show that $Hom_{G_\alpha, B}(\tilde{n_1}(X), \tilde{n_2}(X)) = 0$. As for each congruence subgroup N_i, $\tilde{n_i}(X)^{N_i}$ is projective B-module of finite type. Hence, it is sufficient to prove that for a Zariski-dense of $\chi \in Hom(B, \mathbf{C})$ we have

$$Hom_{G_\alpha}(\tilde{n_1}(X) \otimes_{B,\chi} \mathbf{C}, \tilde{n_2}(X) \otimes_{B,\chi} \mathbf{C}) = 0.$$

We have $\tilde{n_i}(X) \otimes_{B,\chi} \mathbf{C} = \tilde{n_i}(X \otimes_{B,\chi} \mathbf{C}) = \tilde{n_i}(V \otimes \chi)$, hence it is sufficient to take χ such that $\tilde{n_1}(V \otimes \chi)$ is not isomorphic to $\tilde{n_2}(V \otimes \chi)$. That the statement is valid for most of χ results from the fact that $W(G_\alpha, D)$ acts faithfully on D. $\square$

APPENDIX A: THE FIRST PART OF J.N.BERNSTEIN'S ARTICLE (1984)

In this appendix, we will give a translation of the first part of the original article [1], together with some additional explanatory remarks. Below, the passages in the quotation environment refer to the passages translated from [1].

> Let us assume that k is a local non-archimedean field, and $\mathbf{G}$ is a reductive group over k. Let $\mathcal{H}$ be the algebra of locally constant functions on $\mathbf{G}$ with compact support. In this paper, we will determine the algebra of the 'multiplicators' of $\mathcal{H}$ in terms of the cuspidal representations of the Levi subgroups of the parabolic subgroups of$\mathbf{G}$. The proof consists of a study of the category of the algebraic (also said smooth) representations of $\mathbf{G}$. In part three of the article we will obtain as corollary various results of finitude.

> Let $(G, +, \tau)$ be a locally compact totally discontinuous group. Let us fix a field k of characteristic 0. Let $\mathcal{H}_k(G)$ be the algebra over k of the locally constant compactly supported measures over G with respect to the laws of composition addition and multiplication given by the usual addition and convolution of measures(denoted by the symbols + and * respectively); for $k =$**C** 'locally constant' means 'locally multiple of a Haar measure'; it is purely an algebraic notion which allows us to work on an arbitrary field k. If G has a compact open subgroup that is a pro-p-group, we could even take k as a $\mathbf{Z}[1 \setminus p]$-algebra. Not a little important: the reader would lose almost nothing if she assumed $k =$**C**, and in any case, we will suppose it after 1.8. .

Remark A.1. Let μ_G be a left Haar measure on G. Then, $S(G)$ is identified with $\mathcal{H}_{\mathbf{C}}(G)$ by the isomorphism $f \mapsto f.\mu_G$ ([2], 1.29.Corollory). Hence, 'locally constant' means 'locally multiple of a Haar measure' for $k = \mathbf{C}$.

The Haar measure on a locally compact and totally discontinuous group is constructed purely using algebraic structures and algebraic relations. Hence, for example, in complex case, proving the proposition that 'locally constant' means 'locally multiple of a Haar measure,' we considered **C** only qua a field of characteristic zero. Therefore, the proposition that 'locally constant' means 'locally multiple of a Haar measure' is also a meaningful and true statement for any field of characteristic zero.

Moreover, if N_0 is a compact open subgroup of G, then we have a neighborhood basis about the identity consisting of open normal subgroups of N_0. If N_0 is a pro-p-group, then taking it as a standard set of measure $1 \in k$ and measuring its open normal subgroups by their indices in N_0 we can construct a Haar measure on G -with coefficients in k- such that in construction we only use k qua a $\mathbf{Z}[1 \setminus p]$-algebra.]

1.1. Let us consider the following property of a ring $\mathcal{H}$ (which does not need to have an identity)

(Id.) For all finite family $(x_i : i = 1, \cdots, n)$ of elements of $\mathcal{H}$; there exists an idempotent e such that $e.x_i.e = x_i$ (i.e. such that $x_i \in e.\mathcal{H}.e = e.\mathcal{H} \cap \mathcal{H}.e$) for all i.

D. Flath call **C**-algebras verifying **(Id.)** as "Idempotented Algebras". Let us denote the set of idempotents of $\mathcal{H}$ by I. For $e, f \in I$, the following conditions are equivalent:

(i) $e\mathcal{H}e \subset f\mathcal{H}f, e \in f\mathcal{H}f$; and

(ii) $e = f.e.f$.

Remark A.2. Assume the condition (i). Then $e = fhf$ for some $h \in \mathcal{H}$. Then $fef = f(fhf)f = fhf = e$. Conversely, $e \in f\mathcal{H}f$ is immediate. Furthermore, for any $h \in \mathcal{H}$; $ehe = f(efhfe)f \in f\mathcal{H}f$.

For all compact open subgroup K of G; $p_K^*(\mu_K)$ is an idempotent e_K of $\mathcal{H}(G)$; where $p_K^* : (S_k(K))^* \longrightarrow (S_k(G))^*$ is the dual of the restriction mapping $p_K : S_k(G) \longrightarrow S_k(K)$ and μ_K is the normalized left Haar measure on K. In this way, we obtain enough idempotents to satisfy **(Id.)** : $\mathcal{H}$ verify **(Id.)**, and the set of the elements e_K is cofinal in the set I of idempotents of $\mathcal{H}(G)$.

For a k vector space V and a k-linear action of G on V; we say a representation (V, π) of G is *algebraic* if $\forall v \in V$; the stabilizer of v - $stab(v) := \{g \in G \mid \pi(g)v = v\}$-, is open. This notion is equivalent to that of $\mathcal{H}(G)$-module $non-degenerate$: such that $\mathcal{H}(G).V = V$; i.e. $\forall v \in V$; $\exists e \in I(G)$: $e.v = v$. The dictionnary is: for a compact open subgroup K of G; $v \in V$ is fixed by K if and only if $e_K.v = v$. Then, we have for $\lambda \in k, g \in G$; $(\lambda.\delta_g * e_K)v = \lambda\pi(g)v$ (δ is Dirac distribution). We denote the category of algebraic representations of G by the symbol $Alg\ G$. Henceforth, we will simply say *representation* instead of *algeraic representation*.

1.2 Let $\mathcal{H}$ be a ring satisfying **(Id.)**. For $h \in \mathcal{H}$; the endomorphism $v \mapsto h.v$ of the underlying abelian group to the non-degenerate $\mathcal{H}$-module V is functorial on V. If $\mathcal{H}$ has not an identity then the forgetful functor ω : $(non-degenerate\ modules) \longrightarrow (abelian\ groups)$ can have other endomorphisms. For example, $\mathcal{H}(G)$ has not an identity for non-discrete G, and for $g \in G$, $v \mapsto g.v$ is a functorial endomorphism of the underlying k vector space of the representation V.

For all idempotent e, we have $\mathcal{H} = \mathcal{H}e \oplus \mathcal{H}(1-e)$, where $\mathcal{H}(1-e) := \{h - he \mid h \in \mathcal{H}\}$ is the left-annulator of e. Let τ be the topology of $\mathcal{H}$ which has as the fundamental system of neighborhoods of 0: $\{\mathcal{H}(1-e) \mid e \in I\}$.The *completion* $\mathcal{H}^\wedge$ of $\mathcal{H}$ for τ is the limit projective over I of $\mathcal{H}e$'s, for the morphisms of transition $\mathcal{H}f \longrightarrow \mathcal{H}e$: $x \mapsto x.e$: an element $h^\wedge$ of $\mathcal{H}^\wedge = limproj\mathcal{H}e$ is a system of elements $h^\wedge(e) \in \mathcal{H}e$, with $h^\wedge(e) = h^\wedge(f).e$ for $e \ll f$. Taking $g \in I$ dominating e and f, we verify that $h^\wedge(e) = h^\wedge(f).e$ as $\mathcal{H}e \subset \mathcal{H}f$ (i.e. $e = ef$).

For all non-degenerate module V, equipped with the discrete topology; $\mathcal{H} \times V \longrightarrow V \ : \ (h, v) \mapsto hv$ is continuous, and $\mathcal{H}^\wedge$ acts on V by continuity : if $ev = v, h^\wedge.v = h^\wedge(e).v$.

Remark A.3. The endomorphism $v \mapsto h.v$ is functorial on V means that this endomorphism induces a natural transformation of the forgetful functor : $h : \omega \longrightarrow \omega$.

For non-discrete G there is no identity as the elements e_K are cofinal in the set I of idempotents of $\mathcal{H}(G)$. For $g \in G$, the Dirac distribution δ_g is not an idempotent in $\mathcal{H}(G)$, yet, it also induces such an endomorphism of the underlying k vector space of the representation V.

The ring $\mathcal{H}$ qua **Z**-module is equal to the internal direct sum of its submodules $\mathcal{H}e$ and $\mathcal{H}(1-e)$ as the canonical mapping ([6]: Ch.2, §1, no.6 and no.8) is a **Z**-module isomorphism (It is a **Z**-module homomorphism, surjectivity is clear and injectivity follows the fact that $\mathcal{H}e \cap \mathcal{H}(1-e) = \{0\}$).

Let $\mathcal{G}$ be the set of all translations of $\mathcal{H}(1-e)$ through the underlying abelian group of $\mathcal{H}$. Then τ is the topology generated by $\mathcal{G}$; i.e. the coarsest topology on the underlying set of $\mathcal{H}$ for which the sets of $\mathcal{G}$ are open ([7]). Moreover, τ is Haussdorff.

Lemma 1.2.1. We have $\mathcal{H}^\wedge \xrightarrow{\sim} End(forgetful\ functor)$

Let $\mathcal{H}_S$ be the $\mathcal{H} - module$ $\mathcal{H}$, the action being the left multiplication. As all non-degenerate $\mathcal{H} - module$ is quotient of a multiple of $\mathcal{H}_S$ it is sufficient to verify that all endomorphisms of the abelian group $\mathcal{H}_S$ which commute with $End_{\mathcal{H}}(\mathcal{H}_S)$ are defined by an element of $\mathcal{H}^\wedge$. More precisely:

Lemma 1.2.2. $\mathcal{H}^\wedge$ embeds in $End_{\mathbf{Z}}(\mathcal{H})$. Its image is the closure of the set of left multiplications for the topology of simple convergence (the topology of pointwise convergence). It is also the commutant of the right multiplications.

The identity $h^\wedge(e) = h^\wedge.e$ shows that $\mathcal{H}^\wedge$ embeds in $End_{\mathbf{Z}}(\mathcal{H})$ and that the topology induced on it by the topology of simple convergence coincides with τ. As the composition is continuous for the topology of simple convergence, we have

$\mathcal{H}^\wedge \hookrightarrow \{$ *left multiplications* $\}^{-} \subset \{commutant\ of\ right\ multiplications\}$,
and the reader will verify that if ϕ is in the commutant, it is the image of $h^\wedge \in \mathcal{H}^\wedge$ defined by $h^\wedge(e) = \phi(e)$.

Remark A.4. For $h^\wedge \in \mathcal{H}^\wedge$, the endomorphism defined by $h^\wedge$ sends an element v in $\mathcal{H}$ to $h^\wedge(e).v$ where e is an idempotent that fixes v. The identity $h^\wedge(e) = h^\wedge.e$ shows that this map is well defined.

The sets of the form $\mathcal{H}^\wedge \cap O(U_i, a_i;\ \ i = 1 \dots n)$, where $O(U_i, a_i;\ \ i = 1 \dots n) = \{f \in End_{\mathbf{Z}}(\mathcal{H}) \mid f(a_i) \in U_i\}$ is an element of the neighborhood basis of 0 in $End_{\mathbf{Z}}(\mathcal{H})$ for the simple topology, constitute a fundamental system of neighborhoods of 0 in $\mathcal{H}^\wedge$ for the induced topology. Then, by the same identity $h^\wedge(e) = h^\wedge.e$, for an idempotent e that fixes all of a_i $(i = 1 \dots n)$, $\mathcal{H}^\wedge(1-e) \subset \mathcal{H}^\wedge \cap O(U_i, a_i;\ \ i = 1 \dots n)$. Hence, τ is finer than the induced topology. Conversely, let us consider $\mathcal{H}^\wedge(1-e)$. Then $\mathcal{H}^\wedge \cap O(\mathcal{H}(1-e),\ e) \subset \mathcal{H}^\wedge(1-e)$ as $\mathcal{H} = \mathcal{H}e \oplus \mathcal{H}(1-e)$ (1.2§2). Therefore, the induced topology and τ coincide.

We can take the elements of $\mathcal{H}^\wedge$ as limits of composition in $\{left\ \ multipli-cations\}$, hence they are in the closure of $\{$ *left multiplications* $\}$ with respect to the simple topology. By a similar reasoning, the boundary points, being the limit points of left multiplications and as the latter commute with the right multiplications it follows that $\{$ *left multiplications* $\}^{-} \subset \{commutant\ \ of\ right\ multiplications\}$.

Finally, if ϕ is in the commutant, then $h^\wedge = (\phi(e))_{e \in I}$ is in $\mathcal{H}^\wedge$, and it is identical with ϕ as an endomorphism. Therefore the last statement of the lemma follows.

1.3. Let us explain the multiplication in $\mathcal{H}^\wedge$. Let $h_1^\wedge$ and $h_2^\wedge$ be in $\mathcal{H}^\wedge$, and e in I. There exists f in I such that $h_2^\wedge(e) = f.h_2^\wedge(e)$, and $(h_1^\wedge.h_2^\wedge)(e)$ is equal to
$(h_1^\wedge.h_2^\wedge).e = h_1^\wedge(h_2^\wedge e) = h_1^\wedge h_2^\wedge(e) = h_1^\wedge f.h_2^\wedge(e) = h_1^\wedge(f).h_2^\wedge(e)$.
The multiplication is continuous.

1.4. Translation (of $\mathcal{H}(G)^\wedge = limproj\mathcal{H}(G)e_K$) . $\mathcal{H}(G)^\wedge$ is the space of distributions T over G such that for all compact open subgroup K of G, $T * e_K$ is compactly supported.

Our goal is , for a reductive group G over a local field, the calculation of the center of $\mathcal{H}(G)^\wedge$.

For $\mathcal{H}$ verifying **(Id.)**, the center Z of $\mathcal{H}(G)^\wedge$ is the commutant of $\mathcal{H}$ in $\mathcal{H}(G)^\wedge$ (by continuity of multiplication), and hence, 1.2.2 identifies it to the commutant of left and right multiplications in $End_{\mathbf{Z}}(\mathcal{H})$.

Lemma 1.5 (i) The center Z of $\mathcal{H}^\wedge$ is the ring of endomorphisms of the identity functor of the category of non-degenerate $\mathcal{H} - modules$.

(ii) It is the projective limit over I of the centers of $e\mathcal{H}e$.

In order that $z \in \mathcal{H}^\wedge = End(\omega)$ provides every non-degenerate module V with an module endomorphism of V, it is necessary and sufficient that it commutes with $\mathcal{H}$. This proves (i).

If $e\mathcal{H}e \subset f\mathcal{H}f$ and z is central in $f\mathcal{H}f$, we have $e.z = z.e = eze$, and if $x \in e\mathcal{H}e$, $[eze, x] = [z, x] = 0$: eze is central in $e\mathcal{H}e$. This gives a sense to (ii): the morphism of transition is $z \mapsto ze = eze$, and we need to show that $z \in \mathcal{H}^\wedge$ commutes with $\mathcal{H}$ if and only if ze is in the centre of $e\mathcal{H}e$ for all e in I. For sufficiency, we verify then that $ez = ze$: for all $f \gg e$, we have $(ez - ze)f = ezf - zfe = e(fzf) - (fzf)e$, and we use that fzf is central in $f\mathcal{H}f$. This achieved, z commutes with $e\mathcal{H}e$ if and only if $ze = eze$ is in the centre of $e\mathcal{H}e$, and we use that $\mathcal{H}$ is the union of $e\mathcal{H}e$.

1.6. Translation (of 1.5. (ii)).

The center of $\mathcal{H}(G)^\wedge$ is the space of the distributions T over G such that for all compact-open subgroup K of G, $T * e_K$ is of compact support, and in the center of the Hecke algebra $\mathcal{H}(G, K) := e_K * \mathcal{H}(G) * e_K$

1.7. Variation. The center of $\mathcal{H}(G)^\wedge$ is the space of the distributions T over G, invariant by conjugation, and such that for all compact-open subgroup K of G, $T * e_K$ is of compact support.

In fact, the invariance by conjugation of T is equivalent to that for all compactly supported locally constant measure (resp. distribution) we have $T * U = U * T$.

Remark A.5. A distribution is said to be invariant by conjugation if $\delta_g * T * \delta_{g^{-1}} = T$ for all $g \in G$.

If T is in the center of $\mathcal{H}(G)^\wedge$, then for $h \in \mathcal{H}(G)$ and f a Schwartz function on G;

$$\begin{aligned}
\langle T, f * h\rangle &= T * (f * h)^\vee(\mathbf{1}_G) \\
&= ((T * h^\vee) * f^\vee)(\mathbf{1}_G) \\
&= ((h^\vee * T) * f^\vee)(\mathbf{1}_G) \\
&= (f^\vee * (h^\vee * T))(\mathbf{1}_G) \\
&= \langle T, h * f\rangle
\end{aligned}$$

For any Schwartz function f and $g \in G$ there is an idempotent e_K of $\mathcal{H}(G)$ such that $e_K * f * e_K = f$, $e_K * (\delta_g * f) = \delta_g * f$, $(f * \delta_g) * e_K = f * \delta_g$. Then, taking $h = e_K * \delta_g * e_K$ above, we have $< T, e_K * \delta_g * e_K * f >=< T, f * e_K * \delta_g * e_K >$, or again, $< T, \delta_g * f >=< T, f * \delta_g >$ ([9]). This last equation is equivalent to the the invariance by conjugation of T .

Conversely, let $h \in \mathcal{H}(G)$. For some compact open subgroup sufficiently small K, $h \in e_K * \mathcal{H}(G) * e_K$. Thus, it can be written as a linear combination of the distributions of the form $e_K * \delta_g * e_K$. As T is invariant by conjugation, it follows that for any Schwartz function f; $< T, e_K * \delta_g * e_K * f >=< T, f * e_K * \delta_g * e_K >$, hence $< T, f * h >=< T, h * f >$. Hence, T is in the centre of $\mathcal{H}(G)$ by 1.5.

1.8. If k' is an extension of k then $\mathcal{H}_{k'}(G) = \mathcal{H}_k(G) \otimes_k k'$. For a compact-open subgroup K, the $k' - algebra$ $e_K \mathcal{H}_{k'}(G) e_K$, and its center, are deduced from the k-algebra $e_K \mathcal{H}_k(G) e_K$, and its center by extension of scalars. This and 1.5 (ii), control the dependence on k of the center of $\mathcal{H}(G)^\wedge$. In what follows we will make the assumption

1.8.1 $k = \mathbf{C}$ and G is countable at infinity ($\sigma - compact$).

Remark A.6. Let us define the universal mapping $\phi : \mathcal{H}_k(G) \times k' \longrightarrow \mathcal{H}_{k'}(G)$ by $\phi(h, x) = h.x$ where $x \in k'$, $h \in \mathcal{H}_k(G)$, and also let us note that $\mathcal{H}_k(G)$ can be considered as a subalgebra of $\mathcal{H}_{k'}(G)$.

Let f be a $\mathbf{Z} - bilinear$ mapping from $\mathcal{H}_k(G) \times k'$ to a $k - algebra$ E, satisfying that for all $x \in \mathcal{H}_k(G)$, $y \in k'$ and $\lambda \in k$ $f(x\lambda, y) = f(x, \lambda y)$. Then, the mapping $\tilde{f}$ sending generators $(1/\mu_G(K))\chi_{gK}\mu_G$ of $\mathcal{H}_{k'}(G)$, K a compact subgroup of G, to $f((1/\mu_G(K))\chi_{gK}\mu_G, 1)$, as $(1/\mu_G(K))\chi_{gK}\mu_G \in \mathcal{H}_k(G)$, induces the unique algebra homomorphism satisfying $\tilde{f} \circ \phi = f$. Hence, considering the uniqueness of the solution to the universal mapping problem of tensor products of algebras, $\mathcal{H}_{k'}(G) = \mathcal{H}_k(G) \otimes_k k'$. Similar considerations would account for other statements of the paragraph.

Then, the algebras $\mathcal{H}(G, K)$ are of countable dimension and Schur's lemma is valid ([2] 2.11). In particular, if z is in the center of $\mathcal{H}(G)^\wedge$, for all irreducible algebraic representation (V, π) of G, z acts on V by a scalar $z(\pi)$, and in this way, we associate to z a function on the set $G^\wedge$ of irreducible algebraic representations of G. The morphism of algebras

$$(1.8.2) \quad Z \longrightarrow (\text{ functions on } \mathrm{G}^\wedge)$$

is injective : all representations of G is quotient of a multiple of $\mathcal{H}(G)_s$. For all $h \neq 0$ in $\mathcal{H}(G)$, there exists (V, π), irreducible, such that $\pi(h) \neq 0$ ([2] 2.12). The representation $\mathcal{H}(G)_s$ thus imbeds in a product of irreducible representations, and z is determined by the function $z(\pi)$. Our determination of Z will essentially be a description of its image by (1.8.2).

Remark A.7. We assumed that G is σ-compact. The elements of $\mathcal{H}(G,K)$ are K-invariant distributions in $\mathcal{H}(G)$. Let $\{g_n\}_{n\in\mathbf{N}}$ be a collection of double coset representatives of $K\setminus G/K$. By the assumption of σ-compactness, they are countably many. For a subset X of G let χ_X : : the characteristic function of X. We argue that $\{e_K * \delta_{g_n} * e_K \mid n\in\mathbf{N}\}$ is a basis of $\mathcal{H}(G,K)$. They are linearly independent as their supports are disjoint. Let $T\in\mathcal{H}(G)$. Then

$$e_K * T * e_K = \sum\left\{\int \chi_{K.g_n.K} d(e_K * T * e_K)\right\}.e_K * \delta_{g_n} * e_K$$

where χ denotes the characteristic function and the sum is finite as T is compactly supported.

Let $\prod_{i\in\mathcal{H}(G)}(\pi_i, V_i)$ be the product of irreducible modules over $\mathcal{H}(G)$, such that $\pi_h(h)\neq 0$ for all h in $\mathcal{H}(G)$. As these irreducible modules are monogenous, they are generated by a single element, say e_i. Then the module homomorphism $f:\mathcal{H}(G)_s \longrightarrow \prod_{i\in\mathcal{H}(G)}(\pi_i, V_i)$ which sends h to $(\pi_i(h)e_i)_{i\in\mathcal{H}(G)}$ is an embedding.

1.9. The center $Z(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the ring of endomorphisms of the identity functor of $\mathcal{A}$: the given of an endomorphism z_A for each object A such that for each morphism $f: A\to B$ we have $z_B\circ f = f\circ z_A$.

If the abelian category $\mathcal{A}$ is the product of abelian categories $(\mathcal{A}_i)_{i\in I}$ we have $Z(\mathcal{A}) = \prod Z(A_i)$. Let us suppose the category admits direct sums indexed by I, and that all morphism $f: X\longrightarrow \bigoplus Y_i$ such that all $pr_i\circ f$ are nul, is also nul. This is the case for (Alg G). Thus, to give a decomposition of $\mathcal{A}$ in a product as above amounts to give full sub-categories $\mathcal{A}_i$ of $\mathcal{A}$ $(i\in I)$ such that

(i) For X in $\mathcal{A}_i$ and Y in $\mathcal{A}_j$, we have $Hom(X,Y)=0$ if $i\neq j$.

(ii) All object X is a sum:

$$(1.9.1)\quad X = \bigoplus X_i,\ \text{with } \mathrm{X_i} \text{ in } \mathcal{A}_\mathrm{i}.$$

The center of $\mathcal{H}(G)$ is that of the category $(Alg\ G)$ (1.1) and (1.5.(i)). A first stage in the determinatin of the center of $\mathcal{H}(G)^\wedge$ will be the decomposition 2.10, implicit in [2], of $(Alg\ G)$ in product. The results of the decomposition to which devoted the end of this paragraph are the preliminaries to 2.10.

Remark A.8. 'Direct sum' means coproduct. Under the above conditions the functor $F : \prod_{i\in I} \mathcal{A}_i \longrightarrow \mathcal{A}$ sending $< X_i \mid i \in I >$ to $\bigoplus_{i\in I} X_i$ is an isomorphism.

1.10. Let W be a representation of G. Its *algebraic dual*, or simply *dual*, is the space $W^{\sim}$ of the linear forms fixed by an open subgroup of G. If we endow W with the linear topology for which the sets $(1 - e_K)W$ (K compact open subgroup) form a fundamental system of neighborhoods of 0, it is the topological dual :

$$W^{\sim} \;=\; lim\ ind(W/(1-e_K)W)^V = lim\ ind(W^K)^V.$$

Let's remind that the following conditions are equivalents ([2] 2.40) :

(i) The coefficients $< \zeta, gw >$ ($w \in W, \zeta \in W^{\sim}$) are compactly supported functions over G.

(ii) For all compact open subgroup K and all $w \in W$, the function of G in W : $g \mapsto e_K gw$ is of compact support.

If they are verified, and W is finitely generated, for example irreducible, W is *admissible* ([2] 2.40), i.e. W^K are finite dimensional. We can again reformulate the condition (b) :

(ii*) For all $w \in W$, if $g \to \infty$ in G (w.r.t. the filtre of complement of compacts), we have $gw \to 0$ (for the linear topolgy above).

We note $(Alg\ G)_f$ the category of representations of G verifying (i) and (ii).

Remark A.9. The 'algebraic dual' means the contagradient representation in [2]. For a continuous linear form f, for a compact neighborhood of 0 in $\mathbf{C}$, there is a compact open subgroup K of G such that the preimage of the neighborhood contains $(1-e_K)W$. So that, f is fixed by the open subgroup K. Hence, the algebraic dual, in this sense, is the topologic dual.

$W/(1 - e_K)$ is isomorphic to W^K and W is isomorphic to $lim\ ind(W^K)$ ([10] : II.31. Examples). Now the fact that the algebraic dual is the topologic dual, shows the equalities.

1.11. Let (V, π) be irreducible in $(Alg\ G)_f$, $(Alg\ G)(\pi)$ the category of the representations that are sum of the copies of $(V; \pi)$, and $(Alg\ G)(out\ \pi)$ that of the representations without a sub-quotient isomorphic to (V, π). Arguments parallel to those of the theory of finite groups gives

$$(1.11.1) \quad (Alg\ G) = (Alg\ G)(\pi) \times (Alg\ G)(out\ \pi)$$

([2] 2.44, in any case for unimodular G) where an idempotent $e(\pi)$ of the center of $\mathcal{H}(G)^\wedge$, takes the value 1 on $(Alg\ G)(\pi)$ and 0 over $(Alg\ G)(out\ \pi)$.

Let us suppose that G is unimodular. Only this case will interest us. The proof of (1.11.1) gives for the distribution $e(\pi)$ the following expression, which will not serve us. Let d_π be the formal degree of π (dependent on the Haar measure on G) and θ_π the character of π (a generalized function). We have $\bar{\theta}_\pi(g^{-1})$, for π is unitarizable; and

$$(1.11.2) \quad e(\pi) = \theta_\pi(g^{-1}).d_\pi.$$

Remark A.10. The parallelity seems to correspond to Maschke's Theorem in the theory of finite groups. I could not give an account for the general case of (1.11.1); i.e. without the assumption that G is unimodular and it seems that, reading this article, we will only need it for the sake of curiosity. In fact, as far as I see, in order to provide the expression (1.11.2) of the idempotent $e(\pi)$ we must assume that G is unimodular.

The proof for the unimodular case is given in [2] 2.44. where θ_π is the distribution acting on a Schwartz function f by the rule $f \mapsto tr(\pi(f\mu_G))$ and the formal degree comes from the scalar of an operator on the irreducible algebraic part of $G \times G-$ *module End V* (the constant c in 2.44). The dependence of formal degree on the choice of Haar measure comes from the fact that the above-mentioned operator is constructed using the isomorphism between Schwartz functions and Hecke Algebra.

1.11. Variation. Let A be a set of isomorphism classes of irreducible representations in $Alg\ G_f$, and $(Alg\ G)\ (out\ \pi)$ the category of representations without sub-quotient of the isomorphism class in A.

(i) If A is finite, we have

$$(1.11.3) \quad (Alg\ G) = (\bigoplus_{\pi \in A} (AlgG)(\pi)) \times (Alg\ G)\ (out\ A).$$

(ii) The same decomposition is valid for an infinite family, if for each compact-open subgroup K there are only finitely many (V, π) in A such that $V^K \neq 0$. It is necessary to verify 1.9.(ii). Thus, let W be a representation of G. Let $W(\pi)$ the part (1.11.1) of the composite W in $(Alg\ G)(\pi)(\pi \in A)$. For a compact-open subgroup K of G and a finite subset $A' \subset A$ that contains all (V, π) in A such that $V^K \neq 0$, the decomposition (1.9.1) deduced from (1.11.3) for $A' : W = \bigoplus_{\pi \in A'} W(\pi) \times W(outA')$ induces a decomposition

$$W^K = \bigoplus_{\pi \in A'} W(\pi)^K \times W(outA')^K = \bigoplus_{\pi \in A} W(\pi)^K \times W(out\ A')^K,$$

with $W(out\ A')^K$ independent of A'. Let us put $W(out\ A)^K := W(out\ A')^K$ (A' enough big, rel. K) and $W(out\ A) = \bigcup_K W(out\ A)^K$. By passing to the limit over smaller K's, we again have

$$W = \bigoplus_{\pi \in A} W(\pi) \times W(out\ A)$$

and this decomposition, functorial on W, gives (1.11.3).

(iii) Without the hypothese of finitude over A, an argument parallel to (ii) shows that all admissible representations admits a unique decomposition

$$W = \bigoplus_{\pi \in A} W(\pi) \times W(out\ A).$$

This decomposition is still valid for all direct limit of admissible representations, and in particular for all representations in $(Alg\ G)_f$ (let us recall that all finitely generated W in $(Alg\ G)_f$ is admissible). Let us take for A the set $\hat{G}_f$ of all isomorphism classes of irreducibles in $(Alg\ G)_f$, we find that

$$(1.11.4) \quad (Alg\ G)_f = \prod_{\pi \in \hat{G}_f} (Alg\ G)(\pi).$$

If $\hat{G}_f$ verifies the condition of finitude (b), we have also

$$(1.11.5) \quad (Alg\ G) = (Alg\ G)_f \times (Alg\ G)(out\ of \hat{G}_f).$$

Remark A.11. I won't give a further account for (i) and (ii). As to (iii) we may state two points. Firstly, we need to state that for a family of admissible representations (ZZZ)W_α; $lim_{\rightarrow} W_\alpha = lim_{\rightarrow} \bigoplus_{\pi \in A} W_\alpha(\pi) \times W_{alpha}(out\ \pi) = \bigoplus_{\pi \in A} lim_{\rightarrow}(W_\alpha(\pi)) \times lim_{\rightarrow}(W_{alpha}(out\ \pi))$(ZZZ). Secondly, since a $G - module$ is union of an increasing family of its finitely generated submodules, it is the direct limit of the right directed set of its finitely generated submodules. Hence, as the finitely generated submodules of a finite representation (i.e. $\in (Alg\ G)_f$) are admissible, the previous assertion leads to the desired result.

1.12 Let M be a finitely generated free $\mathbf{Z} - module$ and $u : G \longrightarrow M$ a group homomorphism. We suppose that the restricition of u to the center $Z(G)$ of G :$Z(G) \longrightarrow M$ has a compact kernel and finite cokernel. Let $G^0 := Ker(u)$ and (V^0, π^0) be an irreducible in $(Alg\ G^0)_f$. Let G^1 the subgroup of G consisting of g such that π^0 is isomorphic to its conjugate by g. It contains G^0 and the center of G, thus of finite index, and the set A of the isomorphism classes of conjugates of (V^0, π^0) is finite. For all representation (V, π) of G, (1.11.3) applied to the restriction of V to G^0 gives a decomposition $V = V' \oplus V''$, where V' is the sum of subrepresentations isomorphic to a conjugate of (V^0, π^0). This decomposition is stable under G, and gives a decomposition

$$(1.12.1) \quad (Alg\ G) = (Alg\ G)_A \times (Alg\ G)_{out\ A}.$$

We ask to describe the category $(Alg\ G)_A$.

(1.12.2) For W in $(Alg\ G)_A$, let W_l be the subspace of W sum of $sub - G^0 - representations$ of G^0 isomorphic to (V^0, π^0). It is stable under G^1. The functor $W \mapsto W_l$ is an equivalence of $(Alg\ G)_A$ with the category $(Alg\ G^1)_{\pi^0}$ of the representations of G^1 such that their restriction to G^0 is a multiple of (V^0, π^0). The inverse equivalence is the induction of G^1 to G.

(1.12.3) Let $\tilde{G}_l$ be the group of pairs (g, P) with $g \in G^1$ and $P : V^0 \tilde{\longrightarrow} V^0$ such that $P\pi^0(h)P^{-1} = \pi^0(ghg^{-1})$. This is a central extension of G^1 by $\mathbf{C}*$, and G^0 is isomorphic to a subgroup of $\tilde{G}_l$:

$$\begin{array}{ccccc} & & G^0 & & \\ & & \downarrow{\scriptstyle g \mapsto (g, \pi(g))} & & \\ \mathbf{C} & \xrightarrow[z \mapsto (e,z)]{} & \tilde{G}_l & \xrightarrow[(g,P) \mapsto g]{} & G_l \end{array}$$

The representation π^0 of G^0 embeds to $\tilde{G}_l$ by $(g, P) \mapsto P$.

The quotient $\tilde{M}_l$ of $\tilde{G}_l$ by the invariant subgroup G^0 is a central extension of the image M_l of G^1 in M by $\mathbf{C}^*$. The center $\tilde{C}$ of $\tilde{M}_l$ is the inverse image of a subgroup C of M_l that contains the image of the center of G, thus it is of finite index.

The functor $H \mapsto H \otimes V_0$ is an equivalence of categories between vectoriels and the representations of G^0 that are multiples of V_0. The inverse equivalence is $W \mapsto Hom(V_0, W)$. To give an action π of G^1 over $H \otimes V_0$, extending the given action of G^0, becomes to give an action ρ of $\tilde{M}_l$ over H, such that $z \in \mathbf{C}* \subset \tilde{M}_1$ acts by multiplication by z^{-1} : to make act the image of $(g, P) \in \tilde{G}_1$ in $\tilde{M}_1$ by Q such that $\pi(g) = Q \otimes P$:

$$\pi(g) = \rho(g, P) \otimes P.$$

This construction is an equivalence between $(Alg\ G^1)_{\pi^0}$ and the category $\mathcal{B}$ of the representations of $\tilde{M}_l$, inducing $z \mapsto z^{-1}$ over $\mathbf{C}*$. The equivalences $(Alg\ G)_A \sim (Alg\ G^1)_{\pi^0} \sim \mathcal{B}$ induce an isomorphism of the center $Z((Alg\ G)_A)$ with $Z(\mathcal{B})$. The category $\mathcal{B}$ is well known (the theory of Heisenberg Group) and, after recalling some facts about its structure, we will deduce from it a description of $Z((Alg\ G)_A)$.

1.13 The application of commutator induces a bimultiplicative, alternating, non-degenerate form on $\tilde{M}_1/\tilde{C} = M/C$: $< >$: $M/C \times M/C \longrightarrow U_1 \subset \mathbf{C}^*$. We know that any finite group H, admitting such a form $< >$ on it, can be written $H = X \times X'$ where X' is the Pontryagin dual of X and where $<(x, x'), (y, y')> = <x, y'> . <y, x'>^{-1}$. In particular, its ordre is a square number. Let X be a subgroup of M/C, maximal over subgroups on which the commutator is trivial. For example : corresponding to a decomposition like the above one. Its inverse image $\tilde{X}$ in $\tilde{M}_1$ is a maximal commutative subgroup.

We know that for each character χ of $\tilde{C}$, such that $\chi(z) = z^{-1}$ for $z \in \mathbf{C}^*$, there exists up to isomorphism a unique irreducible representation of $\tilde{M}_1$ of the central character χ. We obtain it by extending -it does not matter how - χ to $\tilde{\chi}$ and by inducing from $\tilde{\chi}$ to $\tilde{M}_1$. If π is an irreducible representation, with $\pi(z) = z^{-1}$ for $z \in \mathbf{C}^*$, the others are then of the form $\pi\omega$, ω a character of M_1, and the isomorphism class of $\pi\omega$ only depends on $\omega|C$.

Let us denote $\mathbf{C}[\tilde{M}_1; z^{-1}]$ the $\mathbf{C}$-algebra generated by the elements δ_m ($m \in \tilde{M}_1$), with the relations $\delta_m.\delta_n = \delta_{m.n}$ and $\delta_z = z^{-1}.\delta_e$. If the $\tilde{m}$'s form a embedding of M_1 in $\tilde{M}_1$, it admits as a vectorial basis the $\delta_{\tilde{m}}$. It becomes the same to give to give $\mathbf{C}[\tilde{M}_1; z^{-1}]$-module or a representation of $\tilde{M}_1$ where $z \in \mathbf{C}$ acts by z^{-1} : make δ_m to act like m. The algebra $\mathbf{C}[\tilde{M}_1; z^{-1}]$, having a unity, the center of the category of its modules is simply its center, to know $\mathbf{C}[\tilde{C}; z^{-1}]$: it is simply generated by the center of the group $\tilde{M}_1$. Let us translate in terms of representations.

Remark A.12. We see that $\tilde{M_1}/\tilde{C} = M_1/C$ by the map $[(g,P)]\tilde{C} \mapsto u(g)C$ where C is the image of $\tilde{C}$ in M_1 as in 1.12(§3). Is the equality in 1.13 a misprint? It does not need to be so if we interpret the symbol 'C' in 1.13 as denoting the subgroup of M whose inverse image under the natural embedding in M_1 is C-as in 1.12(§3).

1.14 Let T be the torus (in the sense of algebraic groups) of the group of characters of M. It has as points

$$T(\mathbf{C}) = Hom(M, \mathbf{C}^*)$$

and for the algebra of regular functions the algebra $\mathbf{C}[M]$ of the group M. For $M = \mathbf{Z}^n$, it is the algebra of Laurent polynomials $\mathbf{C}[t_1, t_1^{-1} \dots t_n, t_n^{-1}]$. Let $F \subset T(\mathbf{C})$ be the orthogonal of $C \subset M_1 \subset M$. The torus quotient T/F is the torus of the group of characters of C.

The set $Irr(\mathcal{B})$ of the classes of isomorphism of irreducible representations of $\tilde{M_1}$ with $z \mapsto z^{-1}$, is , as we see it in 1.13, the principal homogenous space under $(T/F)(\mathbf{C}) = Hom(M, \mathbf{C}^*)/F = Hom(C, \mathbf{C}^*)$: if (V, π) is irreducible in $\mathcal{B}$, any irreducible is isomorphic to $(V, \pi\chi)$, for $\chi \in Hom(M, \mathbf{C}^*)$ and $\pi\chi$ is isomorphic to $\pi\chi'$ if and only if $\chi|C = \chi'|C$. This endows $Irr(\mathcal{B})$ with a structure of algebraic variety, a function f on $Irr(\mathcal{B})$ is regular if and only if the function $\chi \mapsto f(\pi\chi)$ over $T(\mathbf{C})$ is regular. Then, by passing to the quotient, it becomes a regular function over T/F, i.e. a finite linear combination of functions $\chi \mapsto \chi(c)$ ($c \in C$).

An element z of the center of $\mathcal{B}$ furnishes a function $z(\pi)$ over $Irr(\mathcal{B})$: to (V, π) it attaches the scalar by which z acts on V. The description 1.13 of the center of $\mathcal{B}$: $Z(\mathcal{B}) = \mathbf{C}[\tilde{C}; z^{-1}]$ can be reformulated as follows : the application $z \mapsto$ (function $z(\pi)$) is an isomorphism of $Z(\mathcal{B})$ with the ring of regular functions over the algebraic variety $Irr(\mathcal{B})$.

The categorical equivalences of 1.12

$$(Alg\ G)_A \sim (Alg\ G^1)_{\pi^0} \sim \mathcal{B}$$

are compatible to torsion by a character of M. By translation we find that $Irr(Alg\ G)_A$ is a homogenous space under $T(\mathbf{C})$, and principally homogenous under $(T/F)(\mathbf{C})$, from which we have a structure of algebraic variety over $Irr(Alg\ G)_A$. Each element z of the center of $(Alg\ G)_A$ defines a function $z(\pi)$ on $Irr(Alg\ G)_A$: $\pi \mapsto$ *the scalar by which z acts*, and we have

Proposition 1.15 **The application $z \mapsto$ (*function* $z(\pi)$) is an isomorphism of the center of the abelian category $(Alg\ G)_A$ with the ring of regular functions over the algebraic variety $Irr(Alg\ G)_A$.**

Remark A.13. Fixing a basis $\{e_i\}$ of M, $T(\mathbf{C})$ is isomorphic to the nonsingular diagonal matrices $\mathbf{D}_n$ in $\mathbf{GL}(n)$ where n is the dimension of M. For an element $p = \bigoplus_j c_j.(\sum_i m_{ij}.e_i) \in \mathbf{C}[M]$ $(c_i \in \mathbf{C}, m_i \in \mathbf{Z})$ and the diagonal matrix $\delta = (\delta_{ii}) \in \mathbf{D}_n$; $p(\delta) = \sum_j c_j \prod_i \delta_{ii}^{m_{ij}}$. This shows how $\mathbf{C}[T(\mathbf{C})] = \mathbf{C}[M]$. F is a finite subgroup, thus the quotient is a torus and, considering the restriction map, we can see that it is isomorphic to the character torus on C.

As to the paragraph 2, besides the results of 1.13, I want to remind that the regular functions on T/F are coming from $\mathbf{C}[C]$ as in the case of M. Hence, the statement they are finite linear combinations of the functions $\chi \mapsto \chi(c)$ is a reformulation of this fact. Let us note that a function $\chi \mapsto \chi(c)$ $(c \in C$) gives a center element of $\mathcal{B}$, this explains that $z \mapsto$ (function $z(\pi)$) is onto.

If χ is a character of M, (V, π) an irreducible representation in $(Alg\ G)_A$, and $F : (Alg\ G)_A \sim (Alg\ G^1)_{\pi^0}$, $G : (Alg\ G^1)_{\pi^0} \sim \mathcal{B}$ are corresponding functors, then $(G \circ F)(V, \pi.\chi) = G(V_1, \pi|_{G_1}.\chi) = (Hom(V_0, V), \rho.\chi) = \chi.(G \circ F)(V, \pi)$ where ρ is defined as in 1.12 and (V_0, π^0) is the finite irreducible representation in $(Alg\ G^0)_f$ by which we define $(Alg\ G)_A$ in 1.12. The converse is similar. This may convince us that the equivalences are compatible with the torsion by a character of M. Then, we can follow the rest from the original text.

1.16 The fact that for an irreducible π in $(Alg\ G)_A$, and z in the center of $(Alg\ G)_A$, the function $\chi \mapsto z(\pi\chi)$ over $T(\mathbf{C}) = Hom(M, \mathbf{C}^*)$ is regular can also be deduced from the fact that $\pi\chi$ depends "*algebraically*" on χ. Let us explain what this means and how to use it.

Let S be an affine algebraic variety, with B as the ring of regular functions. We define an *algebraic family* of admissible representations of G, parametrized by S, being a $B-module$ V, endowed with an action of G (commuting that of B), flat and for which is verified the following property:

$(B - adm)$. For each compact open subgroup K of G, V^K is a finitely generated (finite type) $B - module$.

The conjunction of the conditions "*flat*" and $(B-adm)$ is equivalent to : each V^K is a projectif $B-module$ of finite type. If the condition $(B-adm)$ is verified, for all point $s \in S$, corresponding to a homomorphism $\sigma : b \mapsto b(s) : B \longrightarrow \mathbf{C}^*$, the representation $V_S := V \otimes_{B,\sigma} \mathbf{C}$ of C is admissible.

Let us show that, for an irreducible (V,π) in $(Alg\ G)_A$, the $(V,\pi\chi)$'s form an algebraic family, in the above sense. We take $B = \mathbf{C}[M]$, the ring of regular functions on T. We dispose a character "*universal*$''$ $\chi_{un} : M \longrightarrow B$, such that for $\chi \in T(\mathbf{C})$ corresponding to $x : B \longrightarrow \mathbf{C}^*$, we have $x\chi_{un} = \chi$. Let us make G to act on $V_B := V \otimes B$ by $\pi.\chi_{un}$. It is the algebraic family we are looking for : we have $(V_B, \pi.\chi_{un}) \otimes_{B,x} \mathbf{C} \simeq (V, \pi\chi)$.

Remark A.14. V^K is a flat $B-module$. Since B is Noetherian it is finitely presented ([11]). Hence, it is projective ([11]: Ch.2 §5.3 Corollary 2). Conversely, every projective module is a flat module ([11] : Ch.1 §2.4) and V being the direct limit of V^K's is flat by ([11]: Ch.1 §2.3 Proposition 2). Hence the equivalence between the conjunction and, finitude and projectivity of V^K's is explained.

For $V_0 := \{\ b.v\ \in V \mid b \in\ I_S(s)\ and\ v \in V\ \}$ the sequence $0 \longrightarrow V_0 \overset{i}{\hookrightarrow} V \overset{\phi}{\longrightarrow} V_s \longrightarrow 0$, where $\phi : v \mapsto v \otimes 1$, is an exact sequence of $G - modules$. Hence, for K, a compact open subgroup of G; the sequence $0 \longrightarrow V_0^K \overset{i}{\hookrightarrow} V^K \overset{\phi}{\longrightarrow} V_s^K \longrightarrow 0$ is also exact ([2] : 2.4). This shows that $V_s^K = V^K \otimes_{B,\sigma} \mathbf{C}$. The rest of the statement follows from this.

Fixing a basis $\{e_i\}$ of M as in 1.14, $\chi_{un}(\sum n_i.e_i) := \prod t_i^{n_i}$ gives a "universal" character as desired. V_B is a $B - module$ endowed with an action of G commuting that of B. V is flat as a complex vector space and B itself is a flat B module. Then $V_B = V \otimes B$ is a flat $B - module$ ([11] : ch.1 ,§2.7 Prop.8).

(ZZZ) Moreover, we know by Baire Category Theorem that G^0 is open, hence that u is locally constant and on a compact subgroup $\chi_{un} \circ u$ is trivial. This shows that $V_B^K = V^K \otimes B$.-(ZZZ). Then, as K is a subgroup of G^0; V^K is finite dimensional.

Lastly, the map $v \otimes b \otimes c = v \otimes 1 \otimes c.b(\chi) = c.b(\chi).v \otimes 1 \otimes 1 \mapsto c.b(\chi).v$ is a G-module isomorphism $(V_B, \pi.\chi_{un}) \otimes_{B,x} \mathbf{C} \simeq (V, \pi\chi)$.

Lemma 1.17 **Let S and B be as above and V an algebraic family of admissible representations of G, parametrized by S. Let us suppose that there exists a set Ξ of points of S : $\Xi \subset S(\mathbf{C})$ such that the maps $b \mapsto b(s)$ $(s \in \Xi)$ seperates $b \in B$ ($\Leftrightarrow$ S is reduced and Ξ is Zariski-dense) and that V_s are irreducible. Then, all $B - linear$ endomorphisms z of V that commute with the action of G is multiplication by a $b \in B$.**

Let K be an open compact subgroup. By hypothesis, V^K is a projectif $B-module$ of finite type. Let $z(K)$ be the endomorphism of V^K induced by z. Schur's lemma for V_s ($s \in \Xi$) implies that for all $\sigma : B \to \mathbf{C}$ corresponding to $s \in \Xi$, $z(K)$ acts as a scalar on $(V_B, \pi.\chi_{un}) \otimes_{B,x} \mathbf{C}$. It follows that $z(K)$ is the multiplication by a $b \in B$:

By localisation we may assume that the projective module V^K is free. The endomorphism $z(K)$ is then represented as a square matrix z_{ij}. The $\sigma((z_{ij}))$ are scalar matrices. As σ ($s \in \Xi$) seperate $b \in B$, (z_{ij}) is scalar : we have $z_{ii} = z_{jj}$ and $z_{ij} = 0$ for $i \neq j$. This,being valid for all K, proves the lemma.

We will again use the algebraic family $(V, \pi\chi)$ of representations of G introduced in 1.16. Formally : the representation $(V \otimes \mathbf{C}[M], \pi \otimes \chi_{un})$. It parametrizes all irreducibles in $(Alg\ G)_A$ (with repetitions if $F \neq 1$). Moreover,

Lemma 1.18 **All representations in** $(Alg\ G)_A$ **is quotient of a multiple of** $(V \otimes \mathbf{C}[M], \pi \otimes \chi_{un})$**.**

The representation $V \otimes \mathbf{C}[M]$ is induced from G^0 to G of the restriction of V to G^0, in the sense that the restriction to $V \hookrightarrow V \otimes \mathbf{C}[M] : v \mapsto v \otimes \delta_e = v \otimes 1$ gives an isomorphism

$$Hom_G(V \otimes \mathbf{C}[M], X) = Hom_{G_0}(V, X).$$

The lemma then results from the fact that X in $(Alg\ G)_A$ has a restriction to G^0 quotient of a sum of copies of $V|G^0$.

1.19.Let us suppose that G is unimodular, and $u : G \to M$ is surjectif. Let $n.\sum \pi_i$ be the decomposition of $\pi|G^0$ to irreducible representations, d_i the formal degree of π_i (the Haar measures on G^0, are already equal between them) and dg the Haar measure on G the restriction of which to G^0 is $n.\sum \pi_i$. Let $m \in M$ be orthogonal to F. We have (1.5(i), 1.9, (1.12.1))

$$Z(H(G)^\wedge) \sim Z(Alg\ G) \sim Z((Alg\ G)_A) \times Z((Alg\ G)_{out\ A});$$

let $z(\pi, m)$ have the image $\pi\chi \mapsto \chi(m)$ in $Z((Alg\ G)_A)(1.18)$ and
0 in $Z((Alg\ G)_{out\ A})$. We verify that, seen as a distribution on G (1.7) $z(\pi, m)$ is given by the formula

$$z(\pi, m) = (1/\#F)\theta_\pi(g^{-1})dg|u^{-1}(m) = (1/\#F)(\int \theta_{\pi\chi}(g^{-1})\chi(m)d\chi)dg,$$

where the integration is taken over the compact group of unitary characters of M, endowed with its normalised Haar measure.

1.20. A representation W of G is said to be **quasi-cuspidal** if the following equivalent conditions are verified.

(a) The coeficients of the restriction of W to G^0 are compactly supported.

(b) The coefficients of W are of compact support *mod* the center $Z(G)$ of G.

(c) For all $w \in W$, if $g \to \infty$ in G, w.r.t. the filter of complements of compacts *mod* $Z(G)$, then $gw \to 0$ in W, for the topology of $(1 - e_K)W$ (cf. 1.10. (b*)).

We have (a) $\Leftrightarrow$ (c) for G^0 (1.10) and (c) (for G^0) $\Rightarrow$ (b) $\Rightarrow$ (a). We say that W is **cuspidal** if it is quasi-cuspidal and admissible.

Let $(Alg\ G)_{qc}$ the category of quasi-cuspidal representations of G. For any orbit Ω of G in the set $((G^0)^\wedge)_f$ of the isomorphism classes of irreducibles in $(Alg\ G^0)_f$, let us provisionally note $(Alg\ G)_\Omega$ the category of representations of G the restriction of which to G^0 is the sum of representations in Ω. We deduce from (1.11.4) that

$$(Alg\ G)_{qc} = \prod_\Omega (Alg\ G)_\Omega.$$

By 1.13, the irreducibles in $(AlgG)_{qc}$, being irreducible in one of $(Alg\ G)_\Omega$, are admissible (i.e. quasi-cuspidal + irreducible $\Rightarrow$ cuspidal), and the orbits D of $T(\mathbf{C}) = Hom(M, \mathbf{C}^*)$ acting by torsion $\pi \to \pi\chi$ in the set $Irr((Alg\ G)_{qc})$ of the isomorphism classes of irreducibles in $(AlgG)_{qc}$, then correspond, bijectively ,to the orbits Ω : to the orbite $\{\pi\chi \mid \chi \in T(\mathbf{C})\}$ attach Ω such that π is in $(Alg\ G)_\Omega$: Ω is the set of isomorphism classes of constituants of $\pi|G^0$. We will write $(Alg\ G)(D)$ for $(Alg\ G)_\Omega$. It is the category of representations of G such that its sub-quotients are in D. We have

$$(1.20.1) \quad (Alg\ G)_{qc} = \prod_D (Alg\ G)(D)$$

. (The product is over the orbites of $T(\mathbf{C}) = Hom(M, \mathbf{C}^*)$ in $Irr((Alg\ G)_{qc})$.

If, for all compact open subgroup K of G^0, G^0 has only a finite number of isomorphism classes of irreducible representations (V, π) in $(Alg\ G^0)_f$ such that $V^K \neq 0$, we deduce from (1.11.5) a decomposition

$$(1.20.2) \quad (Alg\ G) = (Alg\ G)_{qc} \times (Alg\ G)_{nqc}.$$

Let us endow $Irr((Alg\ G)_{qc})$ with the structure of algebraic variety for which it is, as a scheme, disjoint union of the orbites of $T(\mathbf{C}) = Hom(M, \mathbf{C}^*)$, endowed with their structure 1.14. By 1.9 and 1.18, the application $z \mapsto z(\pi)$ which attach to $(V, \pi) \in Irr((Alg\ G)_{qc})$ the scalar by which z acts on V, identifies $Z((Alg\ G)_{qc})$ (i.e. by definition of disjoint union, regular on each conjunct of the union). If the finitude condition is satisfied, we have moreover by (1.20.2)

$$(1.20.3) \quad Z(Alg\ G) \ = \ Z((Alg\ G)_{qc}) \times Z((Alg\ G)_{nqc}).$$

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