DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

GENERALIZED INVERSES OF MAX-PLUS MATRICES AND THEIR ALGEBRAIC AND TOPOLOGICAL PROPERTIES

by
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İZMİR

GENERALIZED INVERSES OF MAX-PLUS MATRICES AND THEIR ALGEBRAIC AND TOPOLOGICAL PROPERTIES

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We have read the thesis entitled "**GENERALIZED INVERSES OF MAX-PLUS MATRICES AND THEIR ALGEBRAIC AND TOPOLOGICAL PROPERTIES** " completed by **BEYHAN YENİ** under supervision of **YRD. DOÇ. DR. AHMET Z. ÖZÇELİK** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

In this thesis firstly algebraic and topological structure of max-plus algebra is given. And then we also explain how to solve systems of max-plus equation. The techniques for find the eigenvalue and eigenvector in max-plus algebra are given. Additively computing way of invertible matrices and determinant in max-plus algebra are given. Lastly in our original research we address numerically reliable computation of generalized and Moore-Penrose inverse in max-plus algebra.

Keywords: Max-Plus algebra, eigenvalue and eigenvector, discrepancy matrix, generalized inverse,Moore-Penrose ınverse, determinant, reduced discrepancy matrix.

MAKSİMUM TOPLAMA MATRİSLERİNİN GENELLEŞTİRİLMİŞ TERSELERİ VE BUNLARIN CEBİRSEL VE TOPOLOJİK ÖZELLİKLERİ

ÖZ

Bu tezde ilk olarak max-plus cebrinin cebirsel ve topolojik özellikleri verilmiştir. Daha sonrasında max-plus cebrinde denklem sistemlerinin nasıl çözümleneceği gösterilmiştir. Özdeğer ve özvektör hesaplmaları yapılmıştır. Son olarak orjinal araştırmamızda max-plus cebrindeki matrislerin genelleştirilmiş ve Moore-Penrose tersleri hesaplanmıştır.

Anahtar kelimeler: Max-Plus cebri, özdeğerler ve özvektörler, fark matrisi, genelleştirilmiş ters, Moore-Penrose ters, determinant, indirgenmiş fark matrisi.

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CHAPTER ONE
INTRODUCTION

The max-plus algebra emerged in the late 1950's. Then this algebra was improved by Cuninghame-Green (Cuninghame-Green, 1979). Also max-pus algebra uses in many areas such as mathematical physics, algebraic geometry and combinatorics. Gaubert, Gondian and Minoux are among the researchers who have contributed to max-plus algebra.

On the other hand the generalized inverse was first introduced by Fredholm (Fredholm, 1903) on the generalized inverse of an integral operator. He called particular generalized inverse as pseudo inverse. The pseudo inverse was firstly qualified by Hurwitz (Hurwitz, 1912). The notion of the inverse of a singular matrix has been introduced by Moore in 1920 (Moore, 1920). But the systematical study has not been made until Penrose redefined the Moore Inverse (Penrose, 1955). Then Penrose showed that the Moore's Inverse has to satisfy the equations and it must be unique. This unique inverse is now called the Moore-Penrose Inverse.

Many of the theorems and techniques which we use in classical linear algebra have analogues in the max-plus algebra theory. The main goal in this thesis is to introduce the matrices in max-plus algebra, compute generalized inverses of matrices and also connect between generalized inverses of these matrices and the solutions of the systems of equations in max-plus algebra. We introduce the matrices in max-plus algebra and we give how we compute the generalized inverses of the matrices in max-plus algebra in the article Hanifa Zekraoui (2017). So we will present our original results in Chapter 6.

In the preparation of this thesis we use classical algebra, linear algebra, graph theory, and some mathematical mature. The structure of the thesis is the following order: In Chapter 2 we give the algebraic structure of the max-plus algebra with various properties. In Chapter 3 it is shown how we can solve the systems of equations. In Chapter 4 the required graph theory is introduced and computing of eigenvalues and eigenvectors are given. In the fifth chapter the invertible matrices and determinant in max-plus algebra are treated. In the last chapter our original research provides the computation of the generalized and Moore-Penrose inverses of any given matrices in max-plus algebra.

CHAPTER TWO
MAX-PLUS ALGEBRA

2.1 Algebraic Characteristics of Max-Plus Algebra

(Samuel Asante Gyamerah, 2016)

The max-plus algebra is an algebraic structure made up of real numbers and $\{-\infty\}$ where the traditional operations of multiplication is substituted by the operation of standard addition and addition is substituted by the operation of taking a maximum of real number. For all $\tau, \psi, \lambda \in \mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, the operations $\oplus, \otimes$ can be defined as

$$\tau \otimes \psi = \tau + \psi$$

and

$$\tau \oplus \psi = \max(\tau, \psi).$$

For example,

$$3 \otimes 2 = 3 + 2 = 5 = 2 + 3 = 2 \otimes 3$$

and

$$2 \oplus 7 = \max(2, 7) = 7 = \max(7, 2) = 7 \oplus 2.$$

Now let us examine the some properties of $\oplus$ and $\otimes$.

- Additive Identity : In max-plus algebra $\epsilon = -\infty$ is the additive identity; for any $\tau \in \mathbb{R}_{\max}$

$$\tau \oplus \epsilon = \max(-\infty, \tau) = \tau.$$

- Multiplicative Identity : In max-plus algebra $e = 0$ is the multiplicative identity;

$$\tau \otimes e = e \otimes \tau = \tau + 0 = \tau$$

for every $\tau \in \mathbb{R}_{\max}$.

- Multiplicative Inverse : If $\tau \neq e$ then $\exists$ a distinct ψ with $\tau \otimes \psi = e$.

- The distributive property also exist in max-plus algebra, that is

$$\begin{aligned}\tau \otimes (\psi \oplus \lambda) &= \tau + \max(\psi, \lambda) \\ &= \max(\tau + \psi, \tau + \lambda) \\ &= (\tau \otimes \psi) \oplus (\tau \otimes \psi).\end{aligned}$$

- $\tau \otimes (-\infty) = \tau + (-\infty) = (-\infty)$ (absorbing element) Hence the additive identity ϵ is absorbing under multiplication thus for $\tau \in \mathbb{R}_{\max}$; $-\infty \otimes \tau = -\infty = \tau \otimes (-\infty)$.

- The operation of taking a maximum is commutative and associative. Respectively; For any $\tau, \psi \in \mathbb{R}_{\max}$

$$\tau \oplus \psi = \max(\tau, \psi) = \max(\psi, \tau) = \psi \oplus \tau.$$

  And for any $\tau, \psi, \lambda \in \mathbb{R}_{\max}$

  $(\psi \oplus \lambda) \oplus \tau = \max(\max(\psi, \lambda), \tau) = \max(\psi, \max(\lambda, \tau)) = \psi \oplus (\lambda \oplus \tau)$.

  Note that $(\mathbb{R}_{\max}, \oplus)$ is an abelian semi-group because the operation taking maximum is commutative and associative but it is not group since $\tau \in \mathbb{R}_{\max}$ has not additive inverse except $\tau = -\infty$.

Definition 2.1.1. ((Farlow, 2009), Definition 2) For x $\in \mathbb{R}_{\max}$ and n $\in \mathbb{N}$ we define

$$x^{\otimes n} = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n \text{ times}}$$

In the max-plus algebra exponentiation reduces to conventional multiplication $x^{\otimes n} =$ n.x.

In max-plus algebra the exponent satisfies the following properties;

1. $x^{\otimes m} \oplus x^{\otimes n} = mx + nx = (m+n)x = x^{\otimes(m \otimes n)}$.
2. $(x^{\otimes m})^{\otimes n} = (m.x)^{\otimes n} = nmx = x^{\otimes(m^{\otimes n})}$.
3. $x^{\otimes 1} = 1x = x$.
4. $x^{\otimes m} \otimes y^{\otimes m} = mx \otimes my = m(x+y) = (x \otimes y)^{\otimes m}$.

Note that the operation $\oplus$ is idempotent that is $\tau \oplus \tau = \max(\tau, \tau) = \tau$.

Lemma 2.1.2. *((Farlow, 2009), Lemma 3) The idempotency of $\oplus$ in the max-plus semiring implies that for every $a \in \mathbb{R}_{\max}$ has no additive inverse except ϵ.*

Proof. Suppose $a \in \mathbb{R}_{\max}$ such that $a \neq \epsilon$ has a inverse with respect to $\oplus$. Let b be the inverse of a.Then we would have

$$a \oplus b = \epsilon.$$

If we add a two both left sides of the equation then we get

$$a \oplus (a \oplus b) = a \oplus \epsilon = a.$$

Using the associativity property and the idempotency property of $\oplus$;

$$a = a \oplus (a \oplus b) = (a \oplus a) \oplus b = a \oplus b = \epsilon$$

which is a contradiction to the assumption $a \neq \epsilon$. □

2.2 Matrices in Max-Plus Algebra

(Samuel Asante Gyamerah, 2016)

Also we can define matrices in max-plus algebra semi-ring. That is $A = \left[a_{ij}\right]_{m\times n}$, where $a_{ij} \in \mathbb{R}_{\max}$. For the $m \times m$ matrices $X = \left[\tau_{ij}\right]$, $Y = \left[\psi_{ij}\right]$ and $q \in \mathbb{R}_{\max}$, the new addition $\oplus$, new matrix multiplication $\otimes$ and the scalar multiplication operations are defined respectively as follows:

- *Addition:* $X \oplus Y = \left[\tau_{ij} \oplus \psi_{ij}\right] = \left[\max(\tau_{ij}, \psi_{ij})\right].$
- *Multiplication:* $X \otimes Y = \left[\bigoplus_{k=1}^{m} \tau_{ik} \otimes \psi_{kj}\right].$
- *Scalar Multiplication:* $q \otimes Z = \left[q \otimes \lambda_{ij}\right] = \left[q + \lambda_{ij}\right] = \left[\lambda_{ij} + q\right].$

Let give some numerical example for matrix operation.
Let $X = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $Y = \begin{bmatrix} 5 & 2 \\ 1 & -\infty \end{bmatrix}$.

Then we have

$$X \oplus Y = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \oplus \begin{bmatrix} 5 & 2 \\ 1 & -\infty \end{bmatrix} = \begin{bmatrix} 2 \oplus 5 & 1 \oplus 2 \\ 1 \oplus 1 & 3 \oplus -\infty \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}.$$

And

$$
\begin{aligned}
X \otimes Y &= \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \otimes \begin{bmatrix} 5 & 2 \\ 1 & -\infty \end{bmatrix} \\
&= \begin{bmatrix} (2 \otimes 5) \oplus (1 \otimes 1) & (2 \otimes 2) \oplus (1 \otimes -\infty) \\ (-1 \otimes 5) \otimes (3 \otimes 1) & (-1 \otimes 2) \oplus (3 \otimes -\infty) \end{bmatrix} \\
&= \begin{bmatrix} 7 \oplus 2 & 4 \oplus -\infty \\ 4 \oplus 4 & 1 \oplus -\infty \end{bmatrix} \\
&= \begin{bmatrix} 7 & 4 \\ 4 & 1 \end{bmatrix}.
\end{aligned}
$$

Since the zero element and the multiplicative identity of the Max-Plus algebra are $-\infty$ and 0, respectively, then the $m \times m$ zero matrix O_m and the $m \times m$ identity matrix I_m are defined as

$$
O_m = \begin{bmatrix} -\infty & -\infty & \dots & -\infty \\ -\infty & -\infty & \dots & -\infty \\ \vdots & \vdots & \ddots & \\ -\infty & -\infty & \dots & -\infty \end{bmatrix} \text{ and } I_m = \begin{bmatrix} 0 & -\infty & \dots & -\infty \\ -\infty & 0 & \dots & -\infty \\ \vdots & \vdots & \ddots & \\ -\infty & -\infty & \dots & 0 \end{bmatrix},
$$

respectively.

Theorem 2.2.1. *((Andersen, 2002), Theorem 1.1.2) Matrix multiplication in* $(\mathbb{R}_{\max}, \oplus, \otimes)$ *is associative but not necessarily commutative.*

Proof. Let $A,\ B,\ C$ be three matrices with size $m \times n$, $n \times p$ and $p \times q$ respectively with size from $\mathbb{R}_{\max}$.

The $i,\ l$ entry of $(AB)C$ is $\max_k \left(\left(\max_j (a_{ij} + b_{jk}) \right) + c_{kl} \right) = \max_{k,\ j}(a_{ij} + b_{jk} + c_{kl})$.
The $i,\ l$ entry of $A(BC)$ is $\max_j \left(a_{ij} + \left(\max_k (b_{jk} + c_{kl}) \right) \right) = \max_{j,\ k}(a_{ij} + b_{jk} + c_{kl})$.
Thus $(AB)C = A(BC)$, max-plus matrix multiplication is associative.
To show matrix multiplication is not commutative let us give the following counterexample.

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}. \qquad \square$$

2.2.1 Properties of The Transpose

Definition 2.2.2. Given a $m \times n$ matrix A, the transpose of A is the $n \times m$ and denoted by A^T whose columns are rows of A. That is if $A = (a_{ij})$ then $A^T = (a_{ji})$.

For example let take $A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 4 & 2 \\ 0 & 1 & -3 \end{bmatrix}$.Then we have $A^T = \begin{bmatrix} 1 & -2 & 0 \\ 5 & 4 & 1 \\ 3 & 2 & -3 \end{bmatrix}$.

Theorem 2.2.3. *Let $c \in \mathbb{R}_{\max}$ and A and B matrices of appropriate sizes, then the following properties hold.*

1. $(A^T)^T = A$.
2. $(A \oplus B)^T = A^T \oplus B^T$.
3. $(A \otimes B)^T = B^T \otimes A^T$.
4. $(c \otimes A)^T = c \otimes A^T$.

Proof. 1. Let $A = [a_{ij}]_{m \times n}$, where $a_{ij} \in \mathbb{R}_{max}$. Then $A^T = [a_{ji}]_{n \times m}$ and so $(A^T)^T = [a_{ij}]_{mxn} = A$.

2. Let us take $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. So we have $A \oplus B = [\max\{a_{ij}, b_{ij}\}]_{m\times n}$.Let write $A \oplus B = [c_{ij}]_{m\times n}$. Hence we have $(A \oplus B)^T = [c_{ji}]_{nxm}$. And also we have $A^T = [a_{ji}]_{n\times m}$ and $B^T = [b_{ji}]_{n\times m}$. So we obtain $A^T \oplus B^T = [\max\{a_{ji}, b_{ji}\}]_{n\times m} = [c_{ji}]_{n\times m}$. So $(A \oplus B)^T = A^T \oplus B^T$.

3. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$. Then $A^T = [e_{ij}]_{n\times m}$ and $B^T = [f_{ij}]_{p\times n}$, where $e_{ij} = a_{ji}$ and $f_{ij} = b_{ji}$. So we have, $A \otimes B = [c_{ij}]_{mxp}$, where $c_{ij} = \max\limits_{1\leqslant k\leqslant n}\{a_{ik} + b_{kj}\}$.
   Let $(A \otimes B)^T = G = [g_{ij}]_{p\times m}$, where $g_{ij} = c_{ji}$.

$$\begin{aligned} B^T \otimes A^T = H = [h_{ij}]_{pxm} \text{ such that } h_{ij} &= \max_{1\leqslant k\leqslant n}\{f_{ik} + e_{kj}\} \\ &= \max_{1\leqslant k\leqslant n}\{b_{ki} + a_{jk}\} \\ &= \max_{1\leqslant k\leqslant n}\{a_{jk} + b_{ki}\} \\ &= c_{ji} = g_{ij}. \end{aligned}$$

4. Let us take $A = [a_{ij}]_{m\times n}$ then;
   $c \otimes A = [c + a_{ij}]_{m\times n} \Rightarrow (c \otimes A)^T = [c + a_{ji}]_{n\times m} = c \otimes A^T$.

□

2.2.2 Properties of the Trace

Definition 2.2.4. Let A be a $n \times n$ square matrix with entries a_{ij}.The trace of A is denoted by tr(A) and defined as
tr$(A) = \sum\limits_{i=1}^{n} a_{ii}$ i.e sum of the diagonal element of A.

Theorem 2.2.5. *Let $c \in \mathbb{R}_{\max}$ A and B be any matrices with suitable sizes. Then the following properties hold.*

1. $tr(c \otimes A) = c \otimes tr(A)$.

2. $tr(A \otimes B) = tr(B \otimes A)$ *while* $A \otimes B \neq B \otimes A$.

Proof. 1.Let $A = [a_{ij}]_{n \times n}$ and $c \in \mathbb{R}_{max}$ then $tr(c \otimes A) = tr([c + a_{ij}]_{n \times n}) = \max_{1 \leqslant i \leqslant n}(\{c + a_{ii}\})$and $c \otimes tr(A) = c \otimes \max_{1 \leqslant i \leqslant n}\{a_{ii}\} = \max_{1 \leqslant i \leqslant n}\{c + a_{ii}\}$.

2.Let take $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$.
$A \otimes B = [c_{ij}]_{n \times n}$ where $c_{ij} = \max_{1 \leqslant k \leqslant n}(\{a_{ik} + b_{kj}\})$, $B \otimes A = [d_{ij}]_{n \times n}$ where $d_{ij} = \max_{1 \leqslant k \leqslant n}(\{b_{ik} + a_{kj}\})$.

$$
\begin{aligned}
\text{tr}(A \otimes B) &= \max_{1 \leqslant i \leqslant n}\{\max_{1 \leqslant k \leqslant n}\{a_{ik} + b_{ki}\}\} \\
&= \max_{1 \leqslant i \leqslant n}\{\max_{1 \leqslant k \leqslant n}\{b_{ki} + a_{ik}\}\} \\
&= \max_{1 \leqslant k \leqslant n}\{\max_{1 \leqslant i \leqslant n}\{b_{ik} + a_{ki}\}\} \\
&= \max_{1 \leqslant i \leqslant n}\{\max_{1 \leqslant k \leqslant n}\{b_{ik} + a_{ki}\}\} \\
&= \text{tr}(B \otimes A)
\end{aligned}
$$

□

Definition 2.2.6. An upper triangular $n \times n$ matrix U is defined as

$$
U_{ij} = \begin{cases} a_{ij}, & \text{for } i \leq j \\ 0, & \text{for } i > j \end{cases}
$$

Written explicitly $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$.

A lower triangular $m \times m$ matrix V is also defined as

$$V_{ij} = \begin{cases} b_{ij}, & \text{for } i \geq j \\ 0, & \text{for } i < j \end{cases}$$

Written explicitly $\begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & \cdots & b_{mm} \end{bmatrix}$.

Theorem 2.2.7. *If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ upper(or lower) triangular matrices so is $A \otimes B$.*

Proof. We have to show that $A \otimes B$ is upper triangular that is $A \otimes B = [g_{ij}], a_{ij} = b_{ij} = -\infty \; for \; i > j$.

For $i > j$ we want to show that $g_{ij} = -\infty$.Let $i > j$ then $g_{ij} = \max_{1 \leqslant k \leqslant n}\{a_{ik} + b_{kj}\}$.

We have three cases:

Case 1: $k \geqslant i$:

If $k \geqslant i$ then $k > j$.So $b_{kj} = -\infty$ since B is an upper triangular matrix.

Case 2: $i > k > j$:

Hence we have $a_{ik} = b_{kj} = -\infty$.

Case 3: $j \geqslant k$:

If $j > k$ then we have $i > j > k$. Hence we have $a_{ik} = -\infty$.

So by the three cases we obtain $g_{ij} = -\infty$ when $i > j$. This means that $A \otimes B$ is upper triangular. Similarly we can prove the same way this lemma for lower triangular. $\square$

CHAPTER THREE
SYSTEMS OF EQUATION IN MAX-PLUS ALGEBRA

3.1 System Solution In Max-Plus Algebra

In (Samuel Asante Gyamerah, 2016) it is given how we solve the system equations in max-plus algebra.

Let $B\vec{x} = \vec{\psi}$*, where B is a matrix and* ψ *and x are a vector of any suitable dimension. We can rewrite* $B\vec{x} = \vec{\psi}$ *into the following matrix equation:*

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{bmatrix}.$$

$$(b_{11} \otimes x_1) \oplus (b_{12} \otimes x_2) \oplus \cdots \oplus (b_{1n} \otimes x_n) = \psi_1$$

$$(b_{21} \otimes x_1) \oplus (b_{22} \otimes x_2) \oplus \cdots \oplus (b_{2n} \otimes x_n) = \psi_2$$

$$\vdots$$

$$(b_{m1} \otimes x_1) \oplus (b_{m2} \otimes x_2) \oplus \cdots \oplus (b_{mn} \otimes x_n) = \psi_m$$

.

If we continue then we obtain following system;

$$\max\{(b_{11} + x_1), (b_{12} + x_2), \cdots, (b_{1n} + x_n)\} = \psi_1$$

$$\max\{(b_{21} + x_1), (b_{22} + x_2), \cdots, (b_{2n} + x_n)\} = \psi_2$$

$$\vdots$$

$$\max\{(b_{m1}+x_1),(b_{m2}+x_2),\cdots,(b_{mn}+x_n)\}=\psi_m$$

.

Firstly, without loss of generality we consider the case that a solution exists and some of the entries of ψ is $-\infty$.

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_k \\ -\infty \\ \vdots \\ -\infty \end{bmatrix}.$$

$$\max\{(b_{11}+x_1),(b_{12}+x_2),\cdots,(b_{1n}+x_n)\}=\psi_1$$

$$\vdots$$

$$\max\{(b_{k1}+x_1),(b_{k2}+x_2),\cdots,(b_{kn}+x_n)\}=\psi_k$$

$$\max\{(b_{(k+1,1)}+x_1),(b_{(k+1,2)}+x_2),\cdots,(b_{(k+1,n)}+x_n)\}=-\infty$$

$$\vdots$$

$$\max\{(b_{m1}+x_1),(b_{m2}+x_2),\cdots,(b_{mn}+x_n)\}=-\infty$$

.

The finite part of B is assumed to be B_1 with dimension $k \times l$, that of ψ be

$$\psi' = \begin{bmatrix} \psi_1 & \cdots & \psi_k \end{bmatrix}'$$

and that of x be

$$x^{'} = \begin{bmatrix} x_1 & \cdots & x_l \end{bmatrix}^{'}.$$

It can be noted that if Bx=ψ *has a solution, then* $x_{k+1} = x_n = -\infty$ *, and B*$x^{'}$*=*$\psi^{'}$*. Thus Bx=* ψ *has a solution if and only if* $x^{'}$ *is a solution to* $B_1 x^{'} = \psi^{'}$ *and solutions to Bx=* ψ *are*

$$x^{'} = \begin{bmatrix} x^{'} & -\infty & \cdots & -\infty \end{bmatrix}^{'}.$$

The solvability of a system with infinite entries in ψ *can consequently be reduced to that of a system where all entries in* ψ *are finite. For that reason attention will be limited to systems Bx=*ψ *where all entries of* ψ *are finite. If there is to be a solution to the system of max-plus equations, then* $b_{ij} + x_j \leqslant \psi_i$*. To find a solution to the system, firstly consider each component of x separately.*
When x_1 *is considered for example if there is a solution to the system then* $b_{i1} + x_1 \leqslant \psi_i$ *for i=1,2,...,n.Thus* $x_1 \leqslant \psi_i - b_{i1}$ *for each i leads to the following system of upper bound on* x_1 *;*

$$x_1 \leqslant \psi_1 - b_{11}$$

$$x_1 \leqslant \psi_2 - b_{21}$$

$$\vdots$$

$$x_1 \leqslant \psi_m - b_{m1}.$$

If this system of inequalities has a solution, then it satisfies;

$$x_1 \leqslant \min\{(\psi_1 - b_{11}), (\psi_2 - b_{21}), \ldots, (\psi_m - b_{m1})\}$$

Similarly we can find for $x_2, \ldots, x_n$ giving the following system of inequalities :

$$x_1 \leqslant \min\{(\psi_1 - b_{11}), (\psi_2 - b_{21}), \ldots, (\psi_m - b_{m1})\}$$

$$x_2 \leqslant \min\{(\psi_1 - b_{12}), (\psi_2 - b_{22}), \ldots, (\psi_m - b_{m2})\}$$

$$\vdots$$

$$x_n \leqslant \min\{(\psi_1 - b_{1n}), (\psi_2 - b_{2n}), \ldots, (\psi_m - b_{mn})\}.$$

This leads to candidate for the solution to Bx=ψ which will be denoted by $x^{'}$.

$$x^{'} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where

$$x_1 \leqslant \min\{(\psi_1 - b_{11}), (\psi_1 - b_{21}), \ldots, (\psi_m - b_{m1})\}$$

$$x_2 \leqslant \min\{(\psi_1 - b_{12}), (\psi_2 - b_{22}), \ldots, (\psi_m - b_{m2})\}$$

$$\vdots$$

$$x_n \leqslant \min\{(\psi_1 - b_{1n}), (\psi_2 - b_{2n}), \ldots, (\psi_m - b_{mn})\}.$$

To simplify the process of solving a system of max-plus equations another matrix *can be introduced. The discrepancy matrix $D_{B,\psi}$ can be defined as follows*

$$\begin{bmatrix} \psi_1 - b_{11} & \psi_1 - b_{12} & \dots & \psi_1 - b_{1n} \\ \psi_2 - b_{21} & \psi_2 - b_{22} & \dots & \psi_2 - b_{2n} \\ \vdots & & & \\ \psi_n - b_{n1} & \psi_m - b_{n2} & \dots & \psi_m - b_{mn} \end{bmatrix}$$

Note that $D_{B,\psi}$ is simply a matrix with all the upper bounds of the x_i 's and that each x_i can be found by taking the minimum of the j-th column of $D_{B,\psi}$.

Another matrix is formed from $D_{B,\psi}$ called reduced discrepancy matrix, $R_{B,\psi}$: $R_{B,\psi} = r_{ij}$

where

$$r_{ij} = \begin{cases} 1, & \text{if } d_{ij} = \text{minimum of column } j \\ 0, & \text{otherwise} \end{cases}$$

Note that here we must know that if there exist a row of $R_{B,\psi}$ containing zeros (or no 1's) then there is no solution or else each row contains only one 1 and other entries 0 then there exist unique solution or if there exist any row of $R_{B,\psi}$ containing more than one 1 and other entries $R_{B,\psi}$ of which not all of them zero then there exist infinitely many solution.

Here we will give two cases $m = n$ and $m < n$.

3.1.1 The case of $m = n$

Example 3.1.1. (Max-Plus System with One Solution)

To solve Bx=ψ where

$$B = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \psi = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}.$$

We need the calculate the discrepancy matrix;

$$D_{B,\psi} = \begin{bmatrix} \psi_1 - b_{11} & \psi_1 - b_{12} & \psi_1 - b_{13} \\ \psi_2 - b_{21} & \psi_2 - b_{22} & \psi_2 - b_{23} \\ \psi_3 - b_{31} & \psi_3 - b_{32} & \psi_3 - b_{33} \end{bmatrix} = \begin{bmatrix} 0 & 6 & -3 \\ -2 & -5 & -1 \\ -6 & -3 & -1 \end{bmatrix}.$$

Also since $R_{B,\psi}$ equal to $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, the system has a unique solution.

If we take the minimum each column of $D_{B,\psi}$ the it gives us the solution:

$$x_1^{'} = \min(0, -2, -6) = -6 \tag{3.1}$$

$$x_2^{'} = \min(6, -5, -3) = -5 \tag{3.2}$$

$$x_3^{'} = \min(-3, -1, -1) = -3 \tag{3.3}$$

Hence the unique solution of the system is $x^{'} = (-6, -5, -3)^T$. So it can verify the system Bx= ψ by substituting it back in :

$$\begin{bmatrix} 1 & -5 & 4 \\ 0 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix} \otimes \begin{bmatrix} -6 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} (1 \otimes -6) \oplus (-5 \otimes -5) \oplus (4 \otimes -3) \\ (0 \otimes -6) \oplus (3 \otimes -5) \oplus (-1 \otimes -3) \\ (2 \otimes -6) \oplus (-1 \otimes -5) \oplus (-3 \otimes -3) \end{bmatrix}$$

$$= \begin{bmatrix} \max(-5, -10, 1) \\ \max(-6, -2, -4) \\ \max(-4, -6, -6) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}.$$

Example 3.1.2. (Max-Plus System with Infinitely Many Solution)

To solve Bx=ψ where

$$B = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \psi = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}.$$

We need the calculate the discrepancy matrix, $D_{B,\psi} = \begin{bmatrix} 4 & 4 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 2 \end{bmatrix}$. If we take the minimum each column of $D_{B,\psi}$ then it gives us the solution;

$$x_1^{'} = \min(4, 0, 0) = 0$$

$$x_2^{'} = \min(4, 1, 4) = 1$$

$$x_3^{'} = \min(1, 2, 2) = 1$$

$x^{'}$ is therefore solution to given matrix equation. It can be seen that there are other feasible solutions. Also since the form of reduced discrepancy matrix

$$R_{B,\psi} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

this system has infinitely many solution. So any other solution of the system is

$$\{x : x = (0, u, 1)^T \; where \; u \leq 1\}.$$

But in conventional algebra we have a unique solution

$$\begin{bmatrix} 1 & 1 & 4 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}.$$

Then we have;

$$x_1 + x_2 + 4x_3 = 5$$

$$2x_1 + x_2 = 2$$

$$3x_1 - x_2 + x_3 = 3$$

Since we have three unknown and three equation we have only one solution. We see that this not the same in max-plus algebra.

Example 3.1.3. (Max-Plus System with No Solution)

To solve Bx=ψ where

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ -1 & -2 & 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \psi = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

The discrepancy matrix $D_{B,\psi} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 1 & 2 \\ 7 & 8 & 6 \end{bmatrix}$.

If we take the minimum each column of $D_{B,\psi}$ then it gives us the solution;

$$x_1' = \min(2, 4, 7) = 2$$

$$x_2' = \min(3, 1, 8) = 1$$

$$x_3' = \min(3, 2, 6) = 2$$

$x' = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}^T$ is verified to see that it is not a solution.

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ -1 & -2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \max(4, 2, 3) \\ \max(2, 4, 4) \\ \max(1, -1, 2) \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} \neq \psi = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

We can also observe that from $R_{B,\psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

But if we solve the system in conventional algebra we have a unique solution;

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Then we have;

$$2x_1 + x_2 + x_3 = 4$$

$$3x_2 + 2x_3 = 4$$

$$-x_1 - 2x_2 = 6$$

Since we have three unknown and three equation we have only one solution. We see that this not the same in max-plus algebra.

Now, the above analysis explain that the method used works for all $n \times n$ system of equations:

Example 3.1.4. (Max-Plus System with Only One Solution)

To solve Bx=ψ where

$$B = \begin{bmatrix} 4 & -2 & -2 & -2 \\ -2 & 3 & -2 & -2 \\ -2 & -2 & 2 & -2 \\ -2 & -2 & -2 & 3 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \psi = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

The discrepancy matrix $D_{B,\psi} = \begin{bmatrix} -2 & 4 & 4 & 4 \\ 4 & -1 & 4 & 4 \\ 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & -1 \end{bmatrix}$.

If we take the minimum each column of $D_{B,\psi}$ then it gives us the solution;

$$x_1^{'} = \min(-2, 4, 4, 4) = -2$$

$$x_2^{'} = \min(4, -1, 4, 4) = -1$$

$$x_3^{'} = \min(4, 4, 0, 4) = 0$$

$$x_4^{'} = \min(4, 4, 4, -1) = -1$$

.

$x^{'} = \begin{bmatrix} -2 & -1 & 0 & -1 \end{bmatrix}^T$ the candidate solution to $Bx = \psi$.

The solution can be verified by substituting it back in

$$\begin{bmatrix} 4 & -2 & -2 & -2 \\ -2 & 3 & -2 & -2 \\ -2 & -2 & 2 & -2 \\ -2 & -2 & -2 & 3 \end{bmatrix} \otimes \begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \max(2, -3, -2, -3) \\ \max(-4, 2, -2, -3) \\ \max(-4, -3, 2, -3) \\ max(-4, -3, -2, 2) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

Because of the $R_{B,\psi}$ such that $R_{B,\psi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $x^{'}$ is unique solution to the matrix equation.

Example 3.1.5. (Max-Plus System with Infinitely Many Solution)

To solve Bx=ψ where

$$B = \begin{bmatrix} 3 & -1 & 1 & 1 \\ -1 & 2 & -1 & -1 \\ -1 & 0 & -2 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \psi = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

The discrepancy matrix $D_{B,\psi} = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & -1 \end{bmatrix}.$

If we take the minimum each column of $D_{B,\psi}$ then it gives us the solution;

$$x_1^{'} = \min(-2, 1, 0, -1) = -2$$

$$x_2^{'} = \min(2, -2, -1, -1) = -2$$

$$x_3^{'} = \min(0, 1, 1, 0) = 0$$

$$x_4^{'} = \min(0, 1, 0, -1) = -1$$

.

$x^{'} = \begin{bmatrix} -2 & -2 & 0 & -1 \end{bmatrix}^T$ can be verified by substituting it;

$$\begin{bmatrix} 3 & -1 & 1 & 1 \\ -1 & 2 & -1 & -1 \\ -1 & 0 & -2 & -1 \\ 1 & 1 & 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} -2 \\ -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \max(1, -3, 1, 0) \\ \max(-3, 0, -1, -2) \\ \max(-3, -2, -2, -2) \\ max(-1, -1, 0, -2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

So $x^{'}$ is a solution to the given matrix equation. But there are other solutions that also work. Also we can understand that from the matrix $R_{B,\psi}$ such that

$$R_{B,\psi} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Any x of the form $\{x : x = (u, -2, 0, 0)^T \text{where } u \leq -2\}$ is also a solution to the given matrix equation.

Example 3.1.6. (Max-Plus System with No Solution)

To solve Bx=ψ where

$$B = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ -2 & 2 & -2 & -2 \\ -2 & 4 & 2 & -2 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \psi = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -3 \end{bmatrix}.$$

The discrepancy matrix $D_{B,\psi} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -3 \\ 3 & -1 & 3 & 3 \\ -1 & -7 & -5 & -1 \end{bmatrix}.$

If we take the minimum each column of $D_{B,\psi}$ then it gives us the solution;

$$x_1^{'} = \min(1, 0, 3, -1) = -1$$

$$x_2^{'} = \min(1, -1, -1, -7) = -7$$

$$x_3^{'} = \text{min}(1, -1, 3, -5) = -5$$

$$x_4^{'} = \text{min}(1, -3, 3, -1) = -3$$

.

$x^{'} = \begin{bmatrix} -1 & -7 & -5 & -3 \end{bmatrix}^T$ is not verified the equation;

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ -2 & 2 & -2 & -2 \\ -2 & 4 & 2 & -2 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ -7 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} \text{max}(1, -5, -3, -1) \\ \text{max}(1, -4, -2, 2) \\ \text{max}(-3, -5, -7, -5) \\ \text{max}(-3, -3, -3, -5) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 2 \\ 1 \\ -3 \end{bmatrix}.$$

Also the reduced discrepancy matrix equal to

$$R_{B,\psi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

so the matrix equation has no solution.

3.1.2 The case of $m < n$

Example 3.1.7. (Max-Plus System with Infinite Solution)

To solve Bx=ψ where

$$B = \begin{bmatrix} 0 & -2 & 2 & 2 \\ 2 & 3 & 3 & -2 \\ 2 & 4 & 2 & 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \psi = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}.$$

The discrepancy matrix $D_{B,\psi} = \begin{bmatrix} 3 & 5 & 1 & 1 \\ 2 & 1 & 1 & 6 \\ 1 & -1 & 1 & 3 \end{bmatrix}$.

If we take the minimum each column of $D_{B,\psi}$ then it gives us the solution such that

$$x_1^{'} = \min(3, 2, 1) = 1$$

$$x_2^{'} = \min(5, 1, -1) = -1$$

$$x_3^{'} = \min(1, 1, 1) = 1$$

$$x_4^{'} = \min(1, 6, 3) = 1$$

.

Hence $x^{'} = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^T$ is solution to given matrix equation;

$$\begin{bmatrix} 0 & -2 & 2 & 2 \\ 2 & 3 & 3 & -2 \\ 2 & 4 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \max(1, -3, 3, 3) \\ \max(3, 2, 4, -1) \\ \max(3, 3, 3, 1) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}.$$

Also there exists another solution since$R_{B,\psi}$ of the form $R_{B,\psi} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

So any x of the form $\{x : x = (u, v, 1, s)^T$ where $u \leq 1,\ v \leq -1\ and\ s \leq 1\}$.

Example 3.1.8. (Max-Plus System with No Solutions)

To solve Bx=ψ where

$$B = \begin{bmatrix} -2 & 1 & 0 & 3 \\ -1 & 4 & 2 & -2 \\ -2 & 5 & -1 & 7 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \psi = \begin{bmatrix} 11 \\ 15 \\ 10 \end{bmatrix}.$$

The discrepancy matrix $D_{B,\psi} = \begin{bmatrix} 13 & 10 & 11 & 8 \\ 16 & 11 & 13 & 17 \\ 12 & 5 & 11 & 3 \end{bmatrix}$.

Hence $D_{B,\psi}$ gives the solution $x' = \begin{bmatrix} 12 & 5 & 11 & 13 \end{bmatrix}^T$.

But x' is not verified the system.That is;

$$\begin{bmatrix} -2 & 1 & 0 & 3 \\ -1 & 4 & 2 & -2 \\ -2 & 5 & -1 & 7 \end{bmatrix} \otimes \begin{bmatrix} 12 \\ 15 \\ 11 \\ 3 \end{bmatrix} = \begin{bmatrix} \max(10,6,11,6) \\ \max(11,9,13,1) \\ \max(10,10,10,10) \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \\ 10 \end{bmatrix} \neq \begin{bmatrix} 11 \\ 15 \\ -9 \end{bmatrix}.$$

Also we see that from reduced discrepancy matrix such that $R_{B,\psi} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

Since $R_{B,\psi}$ has a row which has all entries zero system has no solution.

CHAPTER FOUR
EIGENVALUES AND EIGENVECTORS IN MAX-PLUS ALGEBRA

4.1 Graph Theory

Before we examine eigenproblem in the max-plus algebra we review some terminology from the graph theory.

Definition 4.1.1. (Andersen, 2002) For $n \times n$ matrix A, we can define the digraph (or directed graph of A) as the graph with vertices 1,2,...,n where there is a directed arc from vertices i to j with weight a_{ij} if and only if $a_{ij} \neq -\infty$.
We should note that if $a_{ij} = -\infty$ there is no an arc from vertices i to j.

Definition 4.1.2. (Andersen, 2002) A path is a sequence of distinct vertices $i_1, i_2, \ldots, i_k$ such that there is an arc from i_j to i_{j+1} for j=1,2,...,k-1.

Also note that we can refer to weight of a path as the sum of the weights of the arcs that make up that path.

Definition 4.1.3. (Andersen, 2002) The graph D_A is strongly connected if there is a path from any vertex to any other vertex. Also , if D_A is strongly connected then we say the matrix A is irreducible.

For example, let us take the matrix A_1 such that

$$A_1 = \begin{bmatrix} 1 & -\infty & 3 \\ 3 & 2 & -\infty \\ -\infty & 0 & -\infty \end{bmatrix}.$$

We have;

$a_{11} = 1$ $\quad a_{21} = 3$ $\quad a_{31} = -\infty$

$a_{12} = -\infty$ $\quad a_{22} = 2$ $\quad a_{32} = 0$

$a_{13} = 3$ $\quad a_{23} = -\infty$ $\quad a_{33} = -\infty$

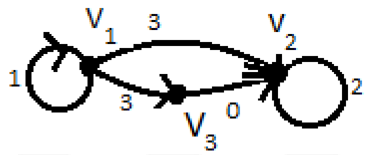

Figure 4.1 Graph of matrix A_1

Since D_{A_1} is strongly connected A_1 is irreducible matrix.

$$\text{For example } B_1 = \begin{bmatrix} 1 & -\infty & 4 \\ 2 & 1 & 0 \\ -\infty & -\infty & -\infty \end{bmatrix}.$$

We have;

$b_{11} = 1$ $\quad b_{21} = 2$ $\quad b_{31} = -\infty$

$b_{12} = -\infty$ $\quad b_{22} = 1$ $\quad b_{32} = -\infty$

$b_{13} = 4$ $\quad b_{23} = 0$ $\quad b_{33} = -\infty$

Then we obtain;

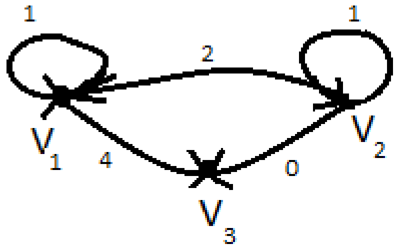

Figure 4.2 Graph of matrix B_1

Since the graph of B_1 is not strongly connected B_1 is not irreducible matrix.

Definition 4.1.4. (Andersen, 2002) A cycle σ is a sequence $i_1, i_2, \ldots, i_k$ of distinct vertices such that $i_1 \to i_2, i_2 \to i_3, \ldots, i_k \to i_1$ of adjacent arcs in the digraph that start and ends at the same vertex and does not travel through any other vertex more than once.
Also note that the number of arcs in a cycle is called the length.

For example in matrix A_1; $V_1 \to V_3 \to V_2 \to V_1$ has length 3.(Since the graph has 3 arcs.)

Definition 4.1.5. (Andersen, 2002) A loop is a cycle with length 1 or is on edge that connects a vertex to itself.

Definition 4.1.6. (Andersen, 2002) For a cycle σ described as a sequence of vertices such that $\sigma : i \to j \to \ldots \to i$ the sum of its arc weights divided by the length l_σ is called the mean and we show that $M(\sigma)$.
And, for a matrix A with distinct cycles $\sigma_1, ..., \sigma_n$ we define a maximum cycle mean by $\mu(A) = \max M(\sigma_i)$.
A graph contains only the cycles with the maximum cycle mean is called a critical graph.

Definition 4.1.7. ((Farlow, 2009), Definition 13) The weight of a path p from vertex i to j of length s is denoted by $\|p\|_w = \otimes_{k=1}^{s} a_{i_{k+1}i_k}$ where $i = i_1$ and $j = i_{s+1}$.

The ij^{th} entry in $A^{\otimes k}(A \in \mathbb{R}_{\max}^{n\times m})$ is the maximum weight of all paths of length k from vertex j to i in D_A. The entry is ϵ if no such paths exist. This is stated as the following theorem.

Theorem 4.1.8. *((Farlow, 2009), Theorem 1) Let $A \in \mathbb{R}_{\max}^{nxn}$.For all $k \geqslant 1$*

$$[A^{\otimes k}]_{ji} = \max\{|p|_w : p \in P(i,j,k)\}$$
$$[A^{\otimes k}]_{ji} = \epsilon \;\; if \;\; P(i,j,k) = \emptyset.$$

Note that P(i,j,k) is the set of paths from i to j of length k.

Definition 4.1.9. (Farlow, 2009) For $A \in \mathbb{R}_{\max}^{n\times n}$ let $A^+ = \bigoplus_{k=1}^{\infty} A^{\otimes k}$ and $A^* = E \oplus A^+ = \oplus_{k\geqslant 0} A^{\otimes k}$.
This says that $[A^+]_{ij}$ is the maximal weight of any path from vertex j to i.$[A^+]_{ij} = \epsilon$ if no such path exists. It is also possible that $[A^+]_{ij} = \infty$.In this case we will stay that $[A^+]_{ij}$ is undefined.

Lemma 4.1.10. *((Farlow, 2009), Lemma 5) Suppose that $A \in \mathbb{R}_{\max}^{n\times n}$ is such that the maximum average weight (maximum cycle mean) of any circuit in D_A is less then or equal to e then A^+ exists and is given by*

$$A^+ = A \oplus A^{\otimes 2} \oplus \ldots \oplus A^{\otimes n} = \bigoplus_{k=1}^{n} A^{\otimes k}.$$

4.2 Eigenvalues and Eigenvectors

Definition 4.2.1. ((Farlow, 2009), Definition 25) Let $A \in \mathbb{R}_{\max}^{n\times n}$ be a square matrix.If $\mu \in \mathbb{R}_{\max}$ is a scalar, V $\in \mathbb{R}_{\max}^{n}$ is a vector that contains at least one element not equal to ϵ and

$$A \otimes V = \mu \otimes V$$

then μ is called an eigenvalue of A and V is an associated eigenvector of A. Also we note that an eigenvector is called a finite eigenvector if it has all finite entries.

Lemma 4.2.2. *((Farlow, 2009), Lemma 14) ϵ is an eigenvalue of A if and only if A has a column of all ϵ entries.*

Proof. Let U be an eigenvector of A associated with eigenvalue $\lambda = \epsilon$.Define the set $I = \{i : U_i > \epsilon\}$ then $J = \{i : U_i = \epsilon\}$. Since λ is an eigenvalue we have $A \otimes U = \lambda \otimes U = -\infty$ so the following is true.

- For each $j \in J$, $\bigoplus_{i\in I} a_{ji} \otimes U_i = \epsilon \;\Rightarrow\; a_{ji} = \epsilon$ for all j.
- For each $i \in I$, $\bigoplus_{k\in I} a_{ik} \otimes U_k = \epsilon \;\Rightarrow a_{ik} = \epsilon$ for all i.

Hence we obtain A has column of all ϵ. So as a result of previous lemma we have the following corollary. □

Corollary 4.2.3. ((Farlow, 2009), Corollary 1) ϵ is not eigenvalue of A if A is irreducible.

Proof. For each i there exists j such that $a_{ji} > \epsilon$ since A is irreducible. So i^{th} column is not all ϵ. □

Lemma 4.2.4. *((Andersen, 2002), Lemma 1.3.1) Let A be $n \times n$ matrix with eigenpair (λ, x) and $c \in \mathbb{R}$. Then $(\lambda, c \otimes x)$ is also an eigenpair of A.*

Proof. Suppose (λ, x) is an eigenvector for the irreducible $n \times n$ matrix A. Then we have $A \otimes x = \lambda \otimes x$ multiplying both sides of the equation by the scalar and then using the commutativity of max-plus scalar multiplication and the associativity of max-plus matrix multiplication;

$$c \otimes A \otimes x = c \otimes \lambda \otimes x$$

$$(A \otimes c) \otimes x = \lambda \otimes c \otimes x$$

$$A \otimes (c \otimes x) = \lambda \otimes (c \otimes x).$$

□

Lemma 4.2.5. *((Farlow, 2009), Lemma 15) Any finite eigenvalue μ of square matrix A is the average value of some elementary circuit in D(A).*

Lemma 4.2.6. *((Andersen, 2002), Lemma 1.3.2) Let A be $n \times n$ matrix with at least one cycle, then*

a.) *A has finite eigenvalue k if and only if $-k \otimes A$ has eigenvalue 0.*

b.) *$\mu(A) = m$ where $\mu(A)$ is the maximum cycle mean if and only if $\mu(-m \otimes A) = 0$.*

Proof. **a.)** ($\Leftarrow$) Assume that $-k \otimes A$ has eigenvalue 0. Let (0,x) be the eigenpair $-k \otimes A$. Then we have following statements ;

$$\begin{bmatrix} a_{11}-k & a_{12}-k & \dots & a_{1n}-k \\ a_{21}-k & a_{22}-k & \dots & a_{2n}-k \\ \vdots & \vdots & \ddots & \\ a_{n1}-k & a_{n2}-k & \dots & a_{nn}-k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \otimes \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\max\{a_{11}-k+x_1, a_{12}-k+x_2, \dots, a_{1n}-k+x_n\} = x_1$$

$$\vdots$$

$$\max\{a_{n1} - k + x_1, a_{n2} - k + x_2, \ldots, a_{nn} - k + x_n\} = x_n.$$

Then

$$-k \otimes \max\{a_{11} + x_1, a_{12} + x_2, \ldots, a_{1n} + x_n\} = x_1$$

$$\vdots$$

$$-k \otimes \max\{a_{n1} + x_1, a_{n2} + x_2, \ldots, a_{nn} + x_n\} = x_n.$$

So we have

$$\max\{a_{11} + x_1, a_{12} + x_2, \ldots, a_{1n} + x_n\} = k \otimes x_1$$

$$\vdots$$

$$\max\{a_{n1} + x_1, a_{n2} + x_2, \ldots, a_{nn} + x_n\} = k \otimes x_n$$

Hence we obtain

$$A \otimes x = k \otimes x.$$

$(\Rightarrow)$ Assume that a has finite eigenvalue k that is $A \otimes x = k \otimes x$. If we add -k the both side; then we have $-k \otimes A \otimes x = -k \otimes (k \otimes x) \implies (-k \otimes A) \otimes x = 0 \otimes x$. Hence 0 is an eigenvalue of $-k \otimes A$.

b.) Let $\mu(A) = m$ and let $B = -m \otimes A$. Then the weight of any arc in D_B has been decreased by m from the corresponding arc in A, i.e, $b_{ij} = a_{ij} - m$. Let the cycle; $\sigma_A : a_{ij}, a_{jk}, \ldots, a_{*i}$ have the maximum cycle mean in A with length l.Then

$$\mu(A) = \frac{a_{ij} + a_{jk} + \ldots + a_{*i}}{l} = m.$$

Let the cycle $\sigma_B : b_{ij}, b_{jk}, \ldots, b_{*i}$ be the same cycle taken, only taken from B, so then

$$\begin{aligned}\mu(B) &= \frac{b_{ij} + b_{jk} + \ldots + b_{*i}}{l} \\ &= \frac{(a_{ij} - m) + (a_{jk} - m) + \ldots + (a_{*i} - m)}{l} \\ &= \frac{a_{ij} + a_{jk} + \ldots + a_{*i} - ml}{l} \\ &= \frac{a_{ij} + a_{jk} + \ldots + a_{*i}}{l} - \frac{ml}{l} \\ &= \mu(A) - m \\ &= m - m = 0.\end{aligned}$$

Hence we have as desired ; $\mu(-m \otimes A) = 0$.

□

Definition 4.2.7. ((Farlow, 2009), Definition 28) We know that if average weight of any circuit $P \in D(A)$ is maximal then P is called critical. We will show the critical graph of A by $D^c(A)$. Also the set of vertices in $D^c(A)$ are denoted by $V^c(A)$.

Definition 4.2.8. ((Farlow, 2009), Definition 30) Let λ be a finite eigenvalue of A. The matrix A_λ is defined by $[A_\lambda]_{ij} = a_{ij} - \lambda$.

Also note that e is an eigenvalue of A_λ. Let see that

$$[\lambda \otimes V]_j = [A \otimes V]_j$$

$$V_j = [A \otimes V]_j - \lambda$$

$$e \otimes v_j = [A - \lambda \otimes V]_j = [A_\lambda \otimes V]_j$$

.

Lemma 4.2.9. *((Farlow, 2009), Lemma 19) Let the communication graph D(A) of* $A \in \mathbb{R}_{\max}^{nxn}$ *have a finite maximal average circuit weight* λ*. Then* λ *is an eigenvalue of A and for any* $V \in V^c(A)$*, the column* $[A_\lambda^*]_v$ *is an eigenvector of A associated with* λ*.*

Theorem 4.2.10. *((Farlow, 2009), Theorem 8) If* $A \in \mathbb{R}_{\max}^{n\times n}$ *is an irreducible matrix the maximal average circuit is the unique eigenvalue.*

4.3 Examples of Eigenvalue and Eigenvector

Example 4.3.1. (Finding the Eigenvalue and Eigenvector for Irreducible Matrix)

Let

$$A_2 = \begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & 2 & 1 \\ 2 & 1 & -\infty \end{bmatrix}$$

$$a_{11} = -\infty \qquad a_{21} = -\infty \qquad a_{31} = 2$$

$$a_{12} = 1 \qquad a_{22} = 2 \qquad a_{32} = 1$$

$$a_{13} = -\infty \qquad a_{23} = 1 \qquad a_{33} = -\infty$$

Then we obtain the graph of A_2

There are three cycles

$$\sigma_1 : (a_{22}) \Rightarrow m(\sigma_1) = \frac{2}{1} = 2$$

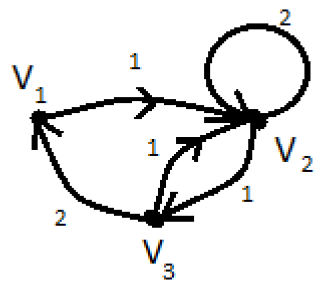

Figure 4.3 Graph of matrix A_2

$$\sigma_2 : (a_{23}, a_{32}) \Rightarrow m(\sigma_2) = \frac{1+1}{2} = 1$$

$$\sigma_3 : (a_{12}, a_{23}, a_{31}) \Rightarrow m(\sigma_3) = \frac{1+1+2}{3} = \frac{4}{3}$$

Since D_{A_2} is strongly connected graph, matrix of A_2 is irreducible. Hence we have a unique eigenvalue λ such that λ is an maximal cycle mean.
So $\lambda = \mu(A_2) = \max(m(\sigma_i)) = 2$.
To find the eigenvector for $\lambda = 2$, we must solve the equation $A_2 \otimes x = \lambda \otimes x$ that is ;

$$\begin{bmatrix} -\infty & 1 & -\infty \\ -\infty & 2 & 1 \\ 2 & 1 & -\infty \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We have

$$\max\{-\infty, 1 + x_2, -\infty\} = 2 + x_1$$

$$\max\{-\infty, 2 + x_2, 1 + x_3\} = 2 + x_2$$

$$\max\{2 + x_1, 1 + x_2, -\infty\} = 2 + x_3$$

If we choose $x_1 = -\infty$ then x_2 and x_3 must be $-\infty$. But eigenvector must have at least one entry not equal to $-\infty$, this is impossible. Since eigenvectors are only unique up to max-plus scalar multiplies, so we can choose x_1 any finite value. Let take $x_1 = 0$;

$$\max\{-\infty, 1 + x_2, -\infty\} = 2$$

$$\max\{-\infty, 2 + x_2, 1 + x_3\} = 2 + x_2$$

$$\max\{2, 1 + x_2, -\infty\} = 2 + x_3$$

$2 = 1 + x_2 \Rightarrow x_2 = 1$. And reduce the system two equations;

$$\max\{-\infty, 2, 1 + x_3\} = 2$$

$$\max\{2, 2, -\infty\} = 2 + x_3$$

Hence $x_3 = 0$.

Thus we have that for $\lambda = 2$, the eigenvector x can be any x of the form $x = c \otimes (0, 1, 0)^T$ where $c \in \mathbb{R}$.

Definition 4.3.2. (Farlow, 2009) If A is an irreducible matrix then the eigenspace of A is given by
$V_\lambda(A) = \{v \in \mathbb{R}^n_{\max} : V = \oplus_{i \in V^c(A)} a_i \otimes [A^*_\lambda]_i \; for \; a_i \in \mathbb{R}_{\max}\}$.

Theorem 4.3.3. *((Andersen, 2002), Theorem 1.3.5) Let A be an irreducible $n \times n$ matrix with eigenpair (λ, x). If the critical graph of A is strongly connected then the x is unique. (up to max-plus scalar multiplies)*

Proof. Assume that A be an irreducible $n \times n$ matrix and the critical digraph D of A is strongly connected.
Let x and y be eigenvector of A corresponding $\lambda = \mu(A)$ (maximum cycle mean)
Our aim: x=y
Since A is irreducible all entries of x and y must be finite. By using max-plus scaling, we may assume that $x_1 = y_1 = 0$. Let j be any vertex since D is strongly connected there exists a path in D from i to j:
$i = i_1 \to i_2 \to i_3 \to .... \to i_k = j$.

Since each arc of D lies on a cycle of mean $\mu(A)$, $a_{i_l i_{l+1}} + x_{i_{l+1}} = \lambda + x_{i_l}$ and $a_{i_l i_{l+1}} + y_{i_{l+1}} = \lambda + y_{i_l}$.

Hence $x_{i_{l+1}} - x_{i_l} = y_{i_{l+1}} - y_{i_l}$ for l=1,2,...,k-1.As $x_{i_1} = y_{i_1}, x_{i_2} - x_{i_1} = y_{i_2} - y_{i_1} \Rightarrow x_{i_2} = y_{i_2}$. Continuing in this way we obtain $x_{i_l} = y_{i_l}$ that is x=y. $\square$

Example 4.3.4. (Matrix with Multiple Eigenvalues)

Let

$$A_3 = \begin{bmatrix} 2 & 3 & -\infty \\ 5 & 1 & -\infty \\ -\infty & -\infty & 2 \end{bmatrix}$$

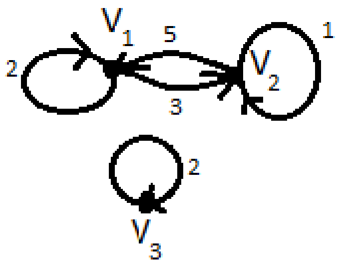

Figure 4.4 Graph of matrix A_3

Since D_{A_3} is not strongly connected A_3 has not irreducible matrix.

Now let find eigenvalue and eigenvector of this matrix.

By lemma 4.2.5 we know that any finite eigenvalue of square matrix A_3 is the average value of some elementary circuit in D_{A_3}. Since there exist only one elementary circuit such that

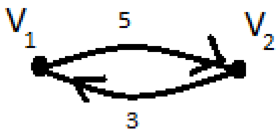

Figure 4.5 Graph of matrix A_3

So $\frac{3+5}{2} = 4$ is an eigenvalue for A_3 and also if we solve the system $A_3 \otimes x = \lambda \otimes x$,

$$\begin{bmatrix} 2 & 3 & -\infty \\ 5 & 1 & -\infty \\ -\infty & -\infty & 2 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\max\{2 + x_1, 3 + x_2, -\infty\} = \lambda + x_1 \tag{4.1}$$

$$\max\{5 + x_1, 1 + x_2, -\infty\} = \lambda + x_2 \tag{4.2}$$

$$\max\{-\infty, -\infty, 2 + x_3\} = \lambda + x_3. \tag{4.3}$$

By the equation (4.3) $2 + x_3 = \lambda + x_3 \Rightarrow \lambda_2 = 2$.
Now let find the eigenvectors with respect to $\lambda_1 = 4$ and $\lambda_2 = 2$.

For $\lambda = 4$ we have the following system

$$\max\{2 + x_1, 3 + x_2, -\infty\} = 4 + x_1$$
$$\max\{5 + x_1, 1 + x_2, -\infty\} = 4 + x_2$$
$$\max\{-\infty, -\infty, 2 + x_3\} = 4 + x_3.$$

So we obtain

$$3 + x_2 = 4 + x_1$$

$$5 + x_1 = 4 + x_2.$$

If we solve this two equation we obtain $x_2 - x_1 = 1$. Let choose $x_1 = 0$, hence $x_2 = 1$.

Also we must have $2 + x_3 = 4 + x_3 \Rightarrow 2 = 4$. But this is impossible. So x_3 must be equal to $-\infty$.

Hence the eigenspace of $\lambda = 4$,

$$V_{\lambda=4} = \left(c \otimes \begin{bmatrix} 0 \\ 1 \\ -\infty \end{bmatrix} : \ c \in \mathbb{R}_{\max} \right).$$

For $\lambda = 2$;

$$\max\{2 + x_1, 3 + x_2, -\infty\} = 2 + x_1 \tag{4.4}$$

$$\max\{5 + x_1, 1 + x_2, -\infty\} = 2 + x_2 \tag{4.5}$$

$$\max\{-\infty, -\infty, 2 + x_3\} = 2 + x_3 \tag{4.6}$$

By the equation (4.5), we should have $1 + x_2 = 2 + x_2$ because if $5 + x_1 = 2 + x_2$ then we obtain $x_1 - x_2 = -3$ but this contradicts by equation (4.4) since we have by the equation (4.4) $x_1 - x_2 > 1$.
But we have contradiction again since $1 \neq 2$.So x_2 must be $-\infty$, x_1 too.
Then by the equation (4.6) we have $2 + x_3 = 2 + x_3$. This equation satisfies each value of x_3.
Hence the eigenspace of $\lambda = 2$;

$$V_{\lambda=2} = \left(d \otimes \begin{bmatrix} -\infty \\ -\infty \\ 0 \end{bmatrix} : \ d \in \mathbb{R}_{\max} \right).$$

Example 4.3.5. (Irreducible Matrix with Multiple Eigenvectors)

Let

$$A_4 = \begin{bmatrix} 3 & 2 & -\infty \\ -\infty & -\infty & 3 \\ 1 & 3 & -\infty \end{bmatrix}$$

and

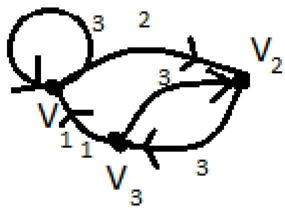

Figure 4.6 Graph of matrix A_4

Since the graph of A_4 is strongly connected A_4 has only one eigenvalue. Let find it: there exists three cycles

$$\sigma_1 : (a_{11}) \Rightarrow m(\sigma_1) = \frac{3}{1} = 3$$
$$\sigma_2 : (a_{12}, a_{23}, a_{31}) \Rightarrow m(\sigma_2) = \frac{2+3+1}{3} = 2$$
$$\sigma_3 : (a_{23}, a_{32}) \Rightarrow m(\sigma_3) = \frac{3+3}{2} = 3.$$

So the eigenvalue of A_4 ; $\lambda = \max(m(\sigma_i)) = 3$.

Critical graph of A_4 is not strongly connected.

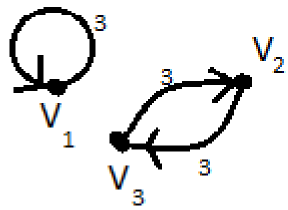

Figure 4.7 Graph of matrix A_4

So $\lambda = 3$ has not unique eigenvector. So let compute the eigenvectors for $\lambda = 3$. We have a system

$$\begin{bmatrix} 3 & 2 & -\infty \\ -\infty & -\infty & 3 \\ 1 & 3 & -\infty \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \otimes \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\max\{3 + x_1, 2 + x_2, -\infty\} = 3 + x_1 \tag{4.7}$$

$$\max\{-\infty, -\infty, 3 + x_3\} = 3 + x_2 \tag{4.8}$$

$$\max\{1 + x_1, 3 + x_2, -\infty\} = 3 + x_3. \tag{4.9}$$

By the equation (4.8) we must have $3 + x_3 = 3 + x_2 \Rightarrow x_2 = x_3$.

Any value of x_1, this equation holds. So let us take $x_1 = 0$. So by equation (4.7), $\max\{3, 2 + x_2, -\infty\} = 3 \Rightarrow 2 + x_2 \leq 3 \Rightarrow x_2 \leq 1 \; so \; x_3 \leq 1$. Hence the eigenspace of $\lambda = 3$,

$$V_{\lambda=3} = \left(\begin{bmatrix} a \\ a \\ 0 \end{bmatrix} : \; a \leq 1, \; a \in \mathbb{R}_{\max} \right).$$

CHAPTER FIVE

INVERSE MATRICES AND DETERMINANT IN MAX-PLUS ALGEBRA

5.1 Inverse Matrices In Max-Plus Algebra

In conventional algebra we know that not all matrices have inverses. Also we know that there is a relation between the existence of inverses of matrices and determinants of matrices. We will see some situations in the max-plus algebra. So let us give some definitions.

Definition 5.1.1. ((Farlow, 2009), Definition 17) A matrix $A \in R_{\max}^{nxn}$ is called invertible in the max-plus sense if there exists a matrix B such that $A \otimes B = E$ where E is an identity matrix and we write $A^{\otimes^{-1}} = B$. Note that here we called B right inverse of A. We will see that if A is invertible right and left inverse must be same.

To define the invertible matrices we have some basic definitions.

Definition 5.1.2. ((Farlow, 2009), Definition 18) A permutation matrix is a matrix in which each row and each column contains exactly one entry equal e and all other entries equal to ϵ.
If $\sigma : \{1, 2, ..., n\} \longmapsto \{1, 2, ..., n\}$ is a permutation we define the max-plus permutation matrix $P_\sigma = [p_{ij}]$ where

$$p_{ij} = \begin{cases} e, & \text{for } i = \sigma(j) \\ \epsilon, & \text{for } i \neq \sigma(j) \end{cases}$$

So that the j^{th} column of P_σ has e in the $\sigma(j)^{th}$ row.
Left multiplication by P_σ permutes the rows of a matrix, so that the i^{th} row of A appears as the $\sigma(i)^{th}$ row of $P_\sigma \otimes A$. .... (*)

For example, let us take the permutation $\sigma : \{1, 2, 3, 4, 5\} \longmapsto \{1, 2, 3, 4, 5\}$ such that

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{bmatrix}.$$

So we have

$$P_\sigma = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} \end{bmatrix} = \begin{bmatrix} -\infty & -\infty & -\infty & -\infty & 0 \\ 0 & -\infty & -\infty & -\infty & -\infty \\ -\infty & 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 & -\infty \\ -\infty & -\infty & 0 & -\infty & -\infty \end{bmatrix}.$$

let take $A = \begin{bmatrix} 1 & -\infty & 2 & 3 & 5 \\ -4 & 3 & 0 & -\infty & 4 \\ 1 & -1 & 3 & -\infty & 3 \\ 6 & 4 & 2 & 1 & -2 \\ 1 & 0 & -2 & 3 & 5 \end{bmatrix}.$

$$P_\sigma \otimes A = \begin{bmatrix} -\infty & -\infty & -\infty & -\infty & 0 \\ 0 & -\infty & -\infty & -\infty & -\infty \\ -\infty & 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 & -\infty \\ -\infty & -\infty & 0 & -\infty & -\infty \end{bmatrix} \otimes \begin{bmatrix} 1 & -\infty & 2 & 3 & 5 \\ -4 & 3 & 0 & -\infty & 4 \\ 1 & -1 & 3 & -\infty & 3 \\ 6 & 4 & 2 & 1 & -2 \\ 1 & 0 & -2 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 & 3 & 5 \\ 1 & -\infty & 2 & 3 & 5 \\ -4 & 3 & 0 & -\infty & 4 \\ 6 & 4 & 2 & 1 & -2 \\ 1 & -1 & 3 & -\infty & 3 \end{bmatrix}.$$

So we see that 1. row of A appears as the $\sigma(1) = 2^{th}$ row of $P_\sigma \otimes A$, 2. row of A appears as the $\sigma(2) = 3^{th}$ row of $P_\sigma \otimes A$, and continuing in this way, we see that 5. row of A appear as the $\sigma(5) = 1^{th}$ row of $P_\sigma \otimes A$.

Definition 5.1.3. ((Farlow, 2009), Definition 19) If $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}_{\max},\ \lambda_i \neq \epsilon$ we define the diagonal matrix:

$$D(\lambda_i) = \begin{bmatrix} \lambda_1 & \epsilon & \cdots & \epsilon \\ \epsilon & \lambda_2 & \cdots & \epsilon \\ \vdots & & & \\ \epsilon & \epsilon & \cdots & \lambda_n \end{bmatrix}.$$

Theorem 5.1.4. *((Farlow, 2009), Theorem 2) $A \in \mathbb{R}_{\max}^{n\times n}$ has right inverse if and only if there is a permutation σ and values $\lambda_i > \epsilon$, $i \in \{1, 2, ..., n\}$ such that $A = P_\sigma \otimes D(\lambda_i)$.*

Note that by the theorem we learn that invertible matrix is a permuted diagonal matrix in max-plus algebra.

Theorem 5.1.5. *((Farlow, 2009), Theorem 3) For $A, B \in \mathbb{R}_{\max}^{nxn}$ if $A \otimes B = E$ then $B \otimes A = E$ and B is uniquely determined by A.*

Proof. Let $A \otimes B = E$, i.e. B is right inverse of A. We must show that $B \otimes A = E$, i.e. B is left inverse of A.
By the above theorem we know that $A = P_\sigma \otimes D(\lambda_i)$ for some values $\lambda_i > \epsilon$ and permutation σ. Observe that $B^* = D(-\lambda_i) \otimes P_{\sigma^{-1}}$ is a left inverse of A. If $A \otimes B = E$, then

$$B^* \otimes (A \otimes B) = B^* \otimes E$$
$$(B^* \otimes A) \otimes B = B^*$$
$$B = B^*.$$

Hence B is uniquely determined and also a left inverse of A. □

Lemma 5.1.6. *((Farlow, 2009), Lemma 7) If $A \in \mathbb{R}_{\max}^{nxn}$ and $B \in \mathbb{R}_{\max}^{nxn}$ are invertible then $A \otimes B$ is invertible.*

Proof. By Theorem 5.1.4 we have $A = P_{\sigma_a} \otimes D(\lambda_i{}^a)$ and $B = D(\lambda_i{}^b) \otimes P_{\sigma_b}$ where σ_a and σ_b are permutations and $\lambda_i{}^a, \lambda_i{}^b > \epsilon$.
Then $A \otimes B = P_{\sigma_a} \otimes D(\lambda_i^a) \otimes D(\lambda_i{}^b) \otimes P_{\sigma_b}$.
Since the product of diagonal matrices is a diagonal matrix so we have

$$A \otimes B = P_{\sigma_a} \otimes D(\lambda_i{}^a \otimes \lambda_i{}^b) \otimes P_{\sigma_b}.$$

So it follows that $A \otimes B$ is a permuted diagonal matrix. Therefore $A \otimes B$ is invertible. □

5.2 Determinants in Max-Plus Algebra

The determinant of a square matrix $A = [a_{ij}]$ in conventional algebra is a number denoted by $|A|$ or $\det(A)$ and this number is defined as the following function of the matrix elements:

$$|A| = \det(A) = \sum_{\sigma \in P_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

where P_n denotes the set of all permutation of $1, 2, ..., n$ and $sgn(\sigma)$ is the sign of the permutation σ. The plus(minus) sign is taken if the permutation P_n is even(odd). But in max-plus algebra the determinant has no direct analogue because of the absence of additive inverses. Here two related quantities, the permanent of A and the determinant of A, partially take over the role of the determinant.

Definition 5.2.1. ((Farlow, 2009), Definition 20) For a matrix $A \in R_{\max}^{nxn}$, the permanent of A is defined to be

$$\text{perm}(A) = \bigoplus_{\sigma \in P_n} \bigotimes_{i=1}^{n} a_{i\sigma(i)}.$$

Lemma 5.2.2. *((Farlow, 2009), Lemma 8) If $A \in R_{\max}^{nxn}$ is invertible then* $\text{perm}(A) \neq \epsilon$.

Proof. In max-plus algebra an invertible matrix is a diagonal matrix times a permutation matrix. So if A is invertible the $\text{perm}(A)$ is just the max-plus product of the diagonal entries of the diagonal matrix. Since all the entries of diagonal matrix does not ϵ, $\text{perm}(A) \neq \epsilon$.

But we can not say, if $\text{perm}(A) \neq \epsilon$ then A is invertible.

For example let take $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$.

Note that here $\sigma_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $P_2 = \{\sigma_1, \sigma_2\}$.

$$
\begin{aligned}
\text{perm}(A) &= \bigoplus_{\sigma \in P_2} \bigotimes_{i=1}^{2} (a_{i\sigma(i)}) \\
&= \max\{(a_{1\sigma_1(1)} + a_{2\sigma_1(2)}), (a_{1\sigma_2(1)} + a_{2\sigma_2(2)})\} \\
&= \max\{(a_{12} + a_{21}), (a_{11} + a_{22})\} \\
&= \max\{4+2, 1+0\} \\
&= \max\{6, 1\} = 6 \neq \epsilon
\end{aligned}
$$

But we can not write $A = P_\sigma \otimes D(\lambda_i)$.

For example let us take $\sigma = \sigma_1$

$$
P_{\sigma_1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} -\infty & 0 \\ 0 & -\infty \end{bmatrix} \text{ where}
$$

$$
P_{ij} = \begin{cases} e, & \text{for } i = \sigma_1(j) \\ \epsilon, & \text{for } i \neq \sigma_1(j) \end{cases}
$$

.

But we can not find a diagonal matrix. So we can not write the A permuted diagonal matrix, that is, A is not invertible. □

Definition 5.2.3. (Farlow, 2009) For a matrix $A \in R_{\max}^{nxn}$ the dominant of A is defined to be

$$
\text{dom}(A) = \begin{cases} \text{highest exponent in } \det(z^A), & \text{if } \det(z^A) \neq 0 \\ \epsilon, & \text{if } \det(z^A) = 0 \end{cases}
$$

.

where z^A is the $n \times n$ matrix with entries $z^{a_{ij}}$ with z is variable.

We can replace z by e^s, given a matrix $A \in R_{\max}^{n\times n}$ the matrix e^{sA} has entries $e^{sa_{ij}}$ where $a_{ij} \in \mathbb{R}_{\max}$ are the entries in A and here $[e^{sA}]_{ij} = e^{sa_{ij}}$.

Definition 5.2.4. ((Farlow, 2009), Definition 22)

$$\text{dom}(A) = \begin{cases} \lim_{s\to\infty} \frac{1}{s} \ln|\det(e^{sA})| & , \text{if } \det(e^{sA}) \neq 0 \\ \epsilon & , \text{if } \det(e^{sA}) = 0 \end{cases}$$

.

Lemma 5.2.5. *((Farlow, 2009), Lemma* 9*)* dom$(A) \leq$ perm(A).

Proof. This is true when calculating the dominant. We can have cancelations which will not occur when calculating the permanent. Note that due to the cancelations the dom(A) can be ϵ. In order for it to be possible to have perm$(A) = \epsilon$ each column of A must have at least one entry equal to ϵ.

For example, let take us the matrix $A = \begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}$.

Now, let find the dom(A) and perm(A).

dom$(A) = \epsilon$since $\det(e^{sA}) = \begin{bmatrix} e^{5s} & e^{2s} \\ e^{6s} & e^{3s} \end{bmatrix} = e^{8s} - e^{8s} = 0$.

perm$(A) = \bigoplus_{\sigma\in P_2} \bigotimes_{i=1}^{2} a_i\sigma(i)$ where $P_2 = \{\sigma_1, \sigma_2\}$ with $\sigma_1 = \{(1)\}$, $\sigma_2 = \{(12), (21)\}$.

$$\begin{aligned} \text{perm}(A) &= \max\{a_1\sigma_1(1) + a_1\sigma_1(2), a_1\sigma_2(1) + a_2\sigma_2(2)\} \\ &= \max\{a_{11} + a_{22}, a_{12} + a_{21}\} \\ &= \max\{5+3, 6+2\} = 8. \end{aligned}$$

□

Lemma 5.2.6. *((Farlow, 2009), Lemma* 10*) If* $A \in \mathbb{R}_{\max}^{n\times n}$ *is invertible then* $\text{dom}(A) \neq \epsilon$*.*

Proof. Since A is invertible, A is a permuted diagonal matrix. We know

$$\text{dom}(A) = \begin{cases} \text{highest exponent in } \det(z^A), & \text{if } \det(z^A) \neq 0 \\ \epsilon, & \text{if } \det(z^A) = 0 \end{cases}$$

.

Since $\det(z^A) \neq 0$, $dom(A) \neq \epsilon$.

But we cant say if $\text{dom}(A) \neq \epsilon$ then A is invertible.

For example, let take $A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$.

Let find the $\text{dom}(A)$.

$$\text{dom}(A) = \begin{cases} \text{highest exponent in } \det(z^A), & \text{if } \det(z^A) \neq 0 \\ \epsilon, & \text{if } \det(z^A) = 0 \end{cases}$$

.

Then $z^A = \begin{bmatrix} z^2 & z^3 \\ z^1 & z^3 \end{bmatrix}$. So; $\det(z^A) = z^5 - z^4 \neq 0$.

So $\text{dom}(A) = 5$.

But A is not permuted diagonal matrix. So A is not invertible.But in conventional algebra we know that A is invertible if and only if $\det(A) \neq 0$. So the max-plus version of determinants and invertible matrices is not completely analogous to the conventional case. □

Lemma 5.2.7. *((Farlow, 2009), Lemma* 11*) If* $A \in \mathbb{R}_{\max}^{nxn}$ *is invertible, then* $\operatorname{dom}(A) = \operatorname{perm}(A)$.

Proof. By Lemma 5.2.2 and 5.2.6 the proof is clear. □

In conventional algebra, we know that for $A, B \in \mathbb{R}_{\max}^{nxn}$ $\det(AB) = \det(A)\det(B)$. But in max-plus algebra, $\operatorname{dom}(A \otimes B) \neq \operatorname{dom}(A) \otimes \operatorname{dom}(B)$ and $\operatorname{perm}(A \otimes B) \neq \operatorname{perm}(A) \otimes \operatorname{perm}(B)$.

For example, let us take the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$$\operatorname{perm}(A) = \bigoplus_{\sigma \in P_3} \bigotimes_{i=1}^{3} a_i \sigma_i \text{ where } P_3 = \{(1), (12), (13), (23), (123), (132)\}$$

$$= \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}.$$

$$
\begin{aligned}
\text{perm}(A) = \max\{&a_1\sigma_1(1) + a_2\sigma_1(2) + a_3\sigma_1(3), a_1\sigma_2(1) + a_2\sigma_2(2) + a_3\sigma_2(3), a_1\sigma_3(1) + a_2\sigma_3(2)\\
&a_1\sigma_3(1) + a_2\sigma_3(2) + a_3\sigma_3(3), a_1\sigma_4(1) + a_2\sigma_4(2) + a_3\sigma_4(3),\\
&a_1\sigma_5(1) + a_2\sigma_5(2) + a_3\sigma_5(3), a_1\sigma_6(1) + a_2\sigma_6(2) + a_3\sigma_6(3)\}\\
= \max\{&a_{11} + a_{22} + a_{33}, a_{12} + a_{21} + a_{33}, a_{11} + a_{23} + a_{32}, a_{13} + a_{22} + a_{31},\\
&a_{12} + a_{23} + a_{31}, a_{13} + a_{21} + a_{32}\}\\
= \max\{&1 + 1 + 1, 2 + 3 + 1, 1 + 2 + 1, 1 + 1 + 2, 2 + 2 + 2, 1 + 3 + 1\}\\
= \max\{&3, 6, 4, 4, 6, 5\}\\
= 6.&
\end{aligned}
$$

$$
\begin{aligned}
\text{perm}(B) = \bigoplus_{\sigma \in P_3} \bigotimes_{i=1}^{3} b_i\sigma_i \text{ where } P_3 &= \{(1), (12), (13), (23), (123), (132)\}\\
&= \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}.
\end{aligned}
$$

$$
\begin{aligned}
\text{perm}(B) = \max\{&b_1\sigma_1(1) + b_2\sigma_1(2) + b_3\sigma_1(3), b_1\sigma_2(1) + b_2\sigma_2(2) + b_3\sigma_2(3),\\
&b_1\sigma_3(1) + b_2\sigma_3(2) + b_3\sigma_3(3), b_1\sigma_4(1) + b_2\sigma_4(2) + b_3\sigma_4(3),\\
&b_1\sigma_5(1) + b_2\sigma_5(2) + b_3\sigma_5(3), b_1\sigma_6(1) + b_2\sigma_6(2) + b_3\sigma_6(3)\}\\
= \max\{&b_{11} + b_{22} + b_{33}, b_{12} + b_{21} + b_{33}, b_{11} + b_{23} + b_{32}, b_{13} + b_{22} + b_{31},\\
&b_{12} + b_{23} + b_{31}, b_{13} + b_{21} + b_{32}\}\\
= \max\{&2 + 3 + 1, 2 + 1 + 1, 2 + 1 + 1, 1 + 3 + 1, 2 + 1 + 1, 1 + 1 + 1\}\\
= \max\{&6, 4, 4, 5, 4, 3\}\\
= 6.&
\end{aligned}
$$

Next let find perm$(A \otimes B)$; $A \otimes B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ 5 & 5 & 4 \\ 4 & 4 & 3 \end{bmatrix}$.

$$\text{perm}(A \otimes B) = \bigoplus_{\sigma \in P_3} \bigotimes_{i=1}^{3} c_i \sigma_i \text{ where } P_3 = \{(1), (12), (13), (23), (123), (132)\}$$
$$= \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}.$$

$$\begin{aligned} \text{perm}(A \otimes B) &= \max\{3+5+3, 5+5+3, 3+4+4, 3+5+4, 5+4+4, 3+5+4\} \\ &= \{10, 13, 11, 11, 13, 9\} \\ &= 13. \end{aligned}$$

Hence we can see that perm$(A \otimes B) = 13 \neq$ perm$(A) \otimes$ perm$(B)(= 6 \otimes 6 = 12)$.

Now let show that dom$(A \otimes B) \neq$ dom$(A) \otimes$ dom(B).

$$\text{dom}(A) = \begin{cases} \text{highest exponent in } \det(z^A), & \text{if } \det(z^A) \neq 0 \\ \epsilon, & \text{if } \det(z^A) = 0 \end{cases}$$

.

$$Z^A = \begin{bmatrix} z^1 & z^2 & z^1 \\ z^3 & z^2 & z^2 \\ z^2 & z^1 & z^1 \end{bmatrix}.$$

$$\begin{aligned}\det(z^A) &= z(z^2 - z^3) - z^2(z^4 - z^4) + z(z^4 - z^3)\\ &= z^3 - z^4 - 0 + z^5 - z^4\\ &= z^5 - 2z^4 + z^3\end{aligned}$$

So $\text{dom}(A) = 5$.

$$\text{dom}(B) = \begin{cases} \text{highest exponent in } \det(z^B), & \text{if } \det(z^B) \neq 0\\ \epsilon, & \text{if } \det(z^B) = 0\end{cases}$$

.

$$Z^B = \begin{bmatrix} z^2 & z^2 & z^1\\ z^1 & z^3 & z^1\\ z^1 & z^1 & z^1\end{bmatrix}.$$

$$\begin{aligned}\det(z^B) &= z^2(z^4 - z^2) - z^2(z^2 - z^2) + z(z^2 - z^4)\\ &= z^6 - z^4 - 0 + z^3 - z^5\\ &= z^6 - z^5 - z^4 + z^3\end{aligned}$$

So $\text{dom}(B) = 6$.

$$A \otimes B = \begin{bmatrix} 3 & 5 & 3\\ 5 & 5 & 4\\ 4 & 4 & 3\end{bmatrix} \text{ then } Z^{A\otimes B} = \begin{bmatrix} z^3 & z^5 & z^3\\ z^5 & z^5 & z^4\\ z^4 & z^4 & z^3\end{bmatrix}.$$

$\det(z^{A\otimes B}) = z^3(z^8 - z^8) - z^5(z^8 - z^8) + z^3(z^9 - z^9) = 0.$

So $\text{dom}(A \otimes B) = \epsilon$.

Hence we obtain $\text{dom}(A \otimes B) = \epsilon \neq \text{dom}(A) \otimes \text{dom}(B)(= 5 \otimes 6 = 5 + 6 = 11)$.

CHAPTER SIX

GENERALIZED INVERSES OF MATRICES IN MAX-PLUS ALGEBRA

6.1 Generalized Inverse of Matrices In Max-Pus Algebra

In linear algebra, we have the matrix equation $Ax = y$ has a solution if and only if$y \in R(A)$. Therefore, it will be in the form $x = A^{-}y$, where A^{-} is a generalized inverse of A. There is a form for the general solution.(by replacing the chosen A^{-} ny the general formwhich gives all g-inverses and so all solutions) As the role of generalized inverses in Linear Algebra and optimization, the Max-Plus Algebra uses the operation of taking a maximum, thus making it an ideal candidate for mathematically describing problems in operations research. Most often, problems in Operations Research have been solved by the development of algorithmic procedures that lead to optimal solutions. Solving the matrix equation $Ax = y$ in Max -Plus-Algebra (when there is a solution) by using the concept of generalized inverses of matrices will be an original research. The aim of this paper is to study the generalized inverses of a matrix in Max-plus-Algebra and examine the analogy with some remarkable properties of them in classical Matrix Theory (in Linear Algebra). Because of the complexity of computation on matrices with high sizes, we will deal with the 2×2 and 3×3 matrices only. The paper is divided in three parts; the first one is destined for the 2 by 2 matrices while the second one treats the 3 by 3 matrices. The last part checks the existence of the Moore Penrose inverse. If the Moore Penrose inverse exists, then it is unique.This result can be found any literature for instance, one can see Zekraoui (2011) or prove it directly by the use of the four equations in the definition of the Moore-Penrose.

In Linear Algebra, a square matrix A is invertible if and only if there exists a matrix B satisfying $AB = BA = I$ or equivalently to det $A \neq 0$, otherwise, there exists what is called a generalized inverse of a matrix A, it is a matrix X satisfying $AXA = A$. it is often called a $\{1\}$- inverse. A $\{2\}$- inverse of A is a matrix X satisfying $XAX = X$. If the matrix X is both $\{1\}$- inverse and $\{2\}$- inverse, then it called a $\{1, 2\}$- inverse and often denoted by $A^{(1,2)}$. Using the analogy of this definition we define the generalized inverses of a matrix A in Max-Plus Algebra with respect to the notations above, i.e. $A \otimes X \otimes A = A$ and $X \otimes A \otimes X = X$.

The following example shows that an invertible matrix in classical matrix theory doesn't necessarily invertible in Max-Plus Algebra.

Example 6.1.1. Let $A = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$. If there exist $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $AB = I$, then we get the folowing system

$$\begin{aligned} \max(a, c-1) &= 0 \text{ and } \max(a+2, c) = -\infty \\ \max(b, d-1) &= -\infty \text{ and } \max(b+2, d) = 0. \end{aligned}$$

The first equation gives

$$(a = 0 \text{ and } c \leq 1 \text{ or } c = 1 \text{ and } a \leq 0) \text{ and } (a = c = -\infty)$$

which is impossible to hold. Then A^{-1} doesn't exist in Max-Plus Algebra, however $A^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & 0 \end{pmatrix}$ in classical matrix theory.

Example 6.1.2. Now, we try to see if a g-inverse of A in the previous example exists. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that $A \otimes B \otimes A = A$.

$$\begin{bmatrix} \max(\max(a, c-1), \max(b, d-1)+2) & \max(\max(a, c-1)-1, \max(b, d-1)) \\ \max(\max(a+2, c), \max(b+2, d)+2) & \max(\max(a+2, c)-1, \max(b+2, d)) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$$

Then, we get the following system :

$$\begin{aligned}
\max\left(\max\left(a, c-1\right), \max\left(b, d-1\right)+2\right) &= 0 \\
\max\left(\max\left(a+2, c\right), \max\left(b+2, d\right)+2\right) &= 2 \\
\max\left(\max\left(a, c-1\right)-1, \max\left(b, d-1\right)\right) &= -1 \\
\max\left(\max\left(a+2, c\right)-1, \max\left(b+2, d\right)\right) &= 0,
\end{aligned}$$

which is equivalent to the following system:

$$\begin{aligned}
\max\left(a, c-1\right) &\leq 0\,,\ \max\left(b, d-1\right) \leq -2, \\
\max\left(a+2, c\right) &\leq 2,\ \max\left(b+2, d\right) \leq 0 \\
\max\left(a, c-1\right) &\leq 0,\ \max\left(b, d-1\right) \leq -1, \\
\max\left(a+2, c\right) &\leq 1,\ \max\left(b+2, d\right) \leq 0.
\end{aligned} \tag{6.1}$$

From the first equation in (6.1), we get

$$a \leq 0, c \leq 1, b \leq -2, d \leq -1.$$

We remark that if we take, equality for a and c, then, the third inequality in (6.1) can't hold because we get max $(a+2, c) = 2 \neq 1$. So the solutions are

$$a < 0, c < 1, b \leq -2, d \leq -1,$$

which means that there are infinitely many g-inverses of $A = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$. However this matrix has only one inverse equals to A^{-1} in classical matrix theory.

6.1.1 The generalized inverses of 2 *by* 2 *matrix*

6.1.1.1 A $\{1\}$*-inverse of a square matrix*

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, such that $A \otimes X \otimes A = A$. Then, we get the following system

$$\begin{aligned} \max\left(\max\left(a + x_1, b + x_3\right) + a, \max\left(a + x_2, b + x_4\right) + c\right) &= a \\ \max\left(\max\left(a + x_1, b + x_3\right) + b, \max\left(a + x_2, b + x_4\right) + d\right) &= b \\ \max\left(\max\left(c + x_1, d + x_3\right) + a, \max\left(c + x_2, d + x_4\right) + c\right) &= c \\ \max\left(\max\left(c + x_1, d + x_3\right) + b, \max\left(c + x_2, d + x_4\right) + d\right) &= d \end{aligned} \quad (6.2)$$

.

Equations in system (6.2) respectively yield

$$\begin{aligned} x_1 &\leq -a, x_2 \leq -c, x_3 \leq -b, x_4 \leq a - b - c \\ x_1 &\leq -a, x_2 \leq b - d - a, x_3 \leq -b, x_4 \leq -d \\ x_1 &\leq -a, x_2 \leq -c, x_3 \leq c - a - d, x_4 \leq -d \\ x_1 &\leq d - b - c, x_2 \leq -c, x_3 \leq -b, x_4 \leq -d \end{aligned}$$

which give

$$\begin{aligned} x_1 &\leq \min\left(-a, d - b - c\right), x_2 \leq \min\left(-c, b - d - a\right), \\ x_3 &\leq \min\left(-b, c - a - d\right), x_4 \leq \min\left(-d, a - b - c\right) \end{aligned} \quad (6.3)$$

Thus, we can confirm the following proposition:

Proposition 6.1.3. *Every* 2 *by* 2 *matrix has infinitely many* $\{1\}$*-inverses in Max-Plus Algebra.*

6.1.1.2 A $\{2\}$*-inverse of a square matrix*

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, such that $X \otimes A \otimes X = X$. Then, we get the following system

$$\begin{aligned}
\max\left(\max\left(x_1 + a, x_2 + c\right) + x_1, \max\left(x_1 + b, x_2 + d\right) + x_3\right) &= x_1 \qquad (6.4)\\
\max\left(\max\left(x_1 + a, x_2 + c\right) + x_2, \max\left(x_1 + b, x_2 + d\right) + x_4\right) &= x_2\\
\max\left(\max\left(x_3 + a, x_4 + c\right) + x_1, \max\left(x_3 + b, x_4 + d\right) + x_3\right) &= x_3\\
\max\left(\max\left(x_3 + a, x_4 + c\right) + x_2, \max\left(x_3 + b, x_4 + d\right) + x_4\right) &= x_4
\end{aligned}$$

.

Equations in system (6.4) respectively yield

$$\begin{aligned}
x_1 &\leq -a, x_2 \leq -c, x_3 \leq -b\\
x_1 &\leq -a, x_2 \leq -c, x_4 \leq -d, x_4 \leq x_2 - x_1 - b\\
x_1 &\leq -a, x_4 \leq x_3 - x_1 - c, x_3 \leq -b, x_4 \leq -d\\
x_2 &\leq -c, x_3 \leq -b, x_2 + x_3 + a \leq x_4 \leq -d
\end{aligned}$$

.

which give

$$x_1 \leq -a, x_2 \leq -c, x_3 \leq -b, x_2 + x_3 + a \leq x_4 \leq \min\left(-d, x_2 - x_1 - b, x_3 - x_1 - c\right). \qquad (6.5)$$

Thus, we have the following proposition:

Proposition 6.1.4. *Every* 2 *by* 2 *matrix has infinitely many* $\{2\}$*-inverses in Max-Plus Algebra.*

If we examine relations in (6.3) and (6.5) , we remark that, which make a $\{1\}$-inverse not to be a $\{2\}$-inverse is its last entry. Hence, we have the following proposition:

Proposition 6.1.5. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$.
Then X *is a* $\{1, 2\}$*-inverse of* A *if and only if*

$$x_1 \leq -a,\ x_2 \leq -c,\ x_3 \leq -b \text{ and } x_2+x_3+a \leq x_4 \leq \min\left(-d, a-b-c, x_2-x_1-b, x_3-x_1-c\right). \tag{6.6}$$

If we examine the previous situation and the result in (6.6) , we find an analogy with the results in classical matrix theory: If we take a block matrix $M = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, such that A invertible, then a $\{1\}$-inverse of M is $\begin{pmatrix} A^{-1} & X \\ Y & Z \end{pmatrix}$, the such generalized inverse to be a $\{2\}$-inverse, is that the last block Z (entry for the case 2 by 2 matrix) has to satisfy

$$Z = YAX.$$

Now, comparing this result with x_4 in (6.6), we find the analogy:

$$x_3 + a + x_2 = x_2 + x_3 + a \leq x_4.$$

The last minimization of x_4 in (6.6) has exactly the analogy with Shur complement in classical matrix theory: If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, such that A is invertible of order $r = rank\,(M)$, then M can be represented in the form:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & CA^{-1}B \end{pmatrix} \begin{pmatrix} I_r & A^{-1}B \\ 0 & I \end{pmatrix}$$

(i.e. $D = CA^{-1}B$), then a $\{1,2\}$-inverse of M is

$$\begin{pmatrix} I_r & -A^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (CA^{-1}B)^{-} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ -CA^{-1} & I \end{pmatrix}$$

such that $(CA^{-1}B)^{-}$ represent a $\{1,2\}$-inverse of $CA^{-1}B$. In the case of 2 by 2 matrix, A, B and C are reals, then $(CA^{-1}B)^{-} = (CA^{-1}B)^{-1} = B^{-1}AC^{-1}$ which exactly analogues to $-b+a-c$. If A^{-1} does not exist in Max-Plus-Algebra, then we replace it by a $\{1\}$-inverse of A as this one always exists.

6.1.2 The generalized inverse of a 3 *by* 3 *matrix*

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$, such that $A \otimes X \otimes A = A$.

Then, the first column of this product is

$$\begin{aligned}
&\max\left(a_{11} + \max_{j=1}^{3}(a_{1j} + x_{j1}), a_{21} + \max_{j=1}^{3}(a_{1j} + x_{j2}), a_{31} + \max_{j=1}^{3}(a_{1j} + x_{j3})\right) \\
&\quad = \max_{i=1}^{3}\left(a_{i1} + \max_{j=1}^{3}(a_{1j} + x_{ji})\right) = a_{11} \\
&\max\left(a_{11} + \max_{j=1}^{3}(a_{2j} + x_{j1}), a_{21} + \max_{j=1}^{3}(a_{2j} + x_{j2}), a_{31} + \max_{j=1}^{3}(a_{2j} + x_{j3})\right) \\
&\quad = \max_{i=1}^{3}\left(a_{i1} + \max_{j=1}^{3}(a_{2j} + x_{ji})\right) = a_{21} \\
&\max\left(a_{11} + \max_{j=1}^{3}(a_{3j} + x_{j1}), a_{21} + \max_{j=1}^{3}(a_{3j} + x_{j2}), a_{31} + \max_{j=1}^{3}(a_{3j} + x_{j3})\right) \\
&\quad = \max_{i=1}^{3}\left(a_{i1} + \max_{j=1}^{3}(a_{3j} + x_{ji})\right) = a_{31}
\end{aligned}$$

.

By the same way, we obtain the remaining columns, which give the explicit form; for all k and l in the index set $\{1,2,3\}$

$$a_{kl} = \max_{i=1}^{3} \left(a_{il} + \max_{j=1}^{3} (a_{kj} + x_{ji}) \right). \tag{6.7}$$

From (6.7), for all k and l in the index set $\{1,2,3\}$, we get the solutions x_{ji} in the form

$$x_{ji} \leq \min (a_{kl} - a_{il} - a_{kj}) \text{ for all } j, i = 1, 2, 3.$$

Then, we have the following proposition

Proposition 6.1.6. *Every 3 by 3 matrix has infinitely many $\{1\}$-inverses in Max-Plus Algebra.*

6.1.3 The Moore Penrose Inverse

We recall that the moore Penrose inverse of a matrix A is the matrix denoted A^+ satisfying the four equations $A \otimes A^+ \otimes A = A$, $A^+ \otimes A \otimes A^+ = A^+$, $(A \otimes A^+)^* = A \otimes A^+$ and $(A^+ \otimes A)^* = A^+ \otimes A$. The existence of the Moore Penrose Inverse in Linear Algebra depends of the vector space we work in, If the vector space is over field of characteristic zero such $\mathbb{R}$ or $\mathbb{C}$, with an inner product, then the Moore Penrose inverse always exists. If the vector space is over a finite field, or without an inner product such Minkovski space, then there are conditions of existence. For this subject, one can refer to H.Zekraoui & Özel (2013), Fulton (1978).

For the reason of calculation, in this paper, we will look for the Moore Penrose inverse of 2 by 2 matrix, just to have an idea.

Let us use results in (6.6) of Proposition 6.1.5 and we examine the third and the fourth equations to extract our conditions.

Let $A^+ = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, such that $(A \otimes A^+)^* = A \otimes A^+$, then we get

$$\max(x_1 + c, x_3 + d) = \max(x_2 + a, x_4 + b),$$

which gives

$$x_1 - x_2 = a - c \text{ or, } x_1 - x_4 = b - c \text{ or, } x_2 - x_3 = d - a \text{ or, } x_3 - x_4 = b - d \quad (6.8)$$

Let in addition $(A^+ \otimes A)^* = A^+ \otimes A$, then we get

$$\max(x_1 + b, x_2 + d) = \max(x_3 + a, x_4 + c),$$

which gives

$$x_1 - x_3 = a - b \text{ or, } x_1 - x_4 = c - b \text{ or, } x_2 - x_3 = a - d \text{ or, } x_2 - x_4 = c - d \quad (6.9)$$

The existence of the Moore Penrose inverse of A depends on the existence of the solutions of the system of inequalities in (6.6), (6.8) and (6.9). Hence, by (6.6) and from (6.8), suppose that $x_1 - x_2 = a - c$, then we have

$$\begin{aligned} x_1 &\le -a \\ -c &\ge x_2 = x_1 - a + c \ge 2x_1 + c \end{aligned}$$

which gives

$$x_1 \le \min(-a, -c).$$

By (6.6) and from (6.9), suppose that $x_1 - x_3 = a - b$, then

$$\begin{aligned} x_1 &\le -a \\ -b &\ge x_3 = x_1 - a + b \ge 2x_1 + b \end{aligned}$$

which yields

$$x_1 \leq \min(-a, -b)$$

.

Hence,

$$x_1 \leq \min(-a, -b, -c).$$

By 6.6 and the previous assumptions, we get

$$2x_1 - a + b + c \leq x_4 \leq \min(-d, a - b - c, c - a - b, b - c - a).$$

Proposition 6.1.7. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$*, such that*

$$\begin{aligned} x_1 &\leq \min(-a, -b, -c), \\ x_2 &= x_1 - a + c \leq -c, \\ x_3 &= x_1 - a + b \leq -b, \\ 2x_1 - a + b + c &\leq x_4 \leq \min(-d, a - b - c, c - a - b, b - c - a), \end{aligned}$$

then $X = A^+$.

CHAPTER SEVEN
CONCLUSION

The aim of this thesis is to study the generalized inverses of a matrix in Max-plus algebra and examine the analogy with some remarkable properties of them in classical Matrix Theory (in Linear Algebra). Because of the complexity of computation on matrices with high sizes, we have dealt with the 2×2 and 3×3 matrices only. The original results are divided in three parts; the first one is destined for the 2 by 2 matrices while the second one treats the 3 by 3 matrices. The last part checks the existence of the Moore Penrose inverse. If the Moore Penrose inverse exists, then it is unique. But the relationship between generalized inverses of matrices and the solutions of systems of equations in the max-plus algebra is an open problem yet.

Additionally, in this thesis we discussed the following proved results:

1) Introduce algebraic operations on matrices in the max-plus algebra and discuss their algebraic properties on matrices in the max-plus algebra.

2) Give the methods how to solve systems of equations in the max-plus algebra.

3) Using a relation between the matrices and graphs we have introduced a procedure to calculate eigen-values and eigen-vectors.

4) We have given a result when a matrix in the max-plus algebra has an inverse.

5) We have introduced the notion of determinants of the matrices in the max-plus algebra.

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