JUNE,2013 Ph.D. in Engineering Physics MEHMET YAKUP HACIİBRAHİMOĞLU

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APPLICATION OF GROUP THEORY TO QUANTUM INFORMATION AND QUANTUM COMPUTATION

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MEHMET YAKUP HACIİBRAHİMOĞLU

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Application of Group Theory to

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Supervisor

Prof. Dr. Ramazan KOÇ

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Mehmet Yakup Hacıibrahimoğlu

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Name of the student: Mehmet Yakup Hacıibrahimoğlu

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Assoc.Prof.Dr. Metin BEDIR

Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof.Dr. A.Necmeddin YAZICI

Head of Department

This is to certify that we have read this thesis and that in our consensus/majority opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science/Doctor of Philosophy.

Prof.Dr. Ramazan KOÇ

Supervisor

Examining Committee Members	Signature
Prof. Dr Ramazan KOÇ	....
Prof. Dr Hayriye TÜTÜNCÜLER	....
Assoc.Prof.Dr. Gültekin ÇELİK	....
Prof.Dr. A.Necmeddin YAZICI	....
Assoc.Prof.Dr. Metin BEDİR	....

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Mehmet Yakup HACIİBRAHİMOĞLU

ABSTRACT

APPLICATION OF GROUP THEORY TO QUANTUM INFORMATION AND QUANTUM COMPUTATION

HACIİBRAHİMOĞLU, MEHMET YAKUP
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The method of group theory leads to the very simple and elegant description of the synthesis of quantum information and quantum computation processes. For example reversible quantum logic gates can be analyzed by using group theory. It is known that quantum entanglement can be also formulated by using group theory. The algorithms for quantum computations can be constructed in the framework of hidden subgroup. The Quantum Fourier Transform which can be used to solve some logarithm problems also closely related to the hidden subgroup and finite groups. There is also an interesting link between SU(2) symmetry and quantum beam splitter transformation, as well as n-qudit systems and quantum codes. We also found some fingerprints of quantum entanglements- in exceptional groups such as E_6, E_7 and E_8-generating by Quantum Gates and that concisely related with Clifford Groups, Coxeter Groups. During realization of Quantum Gates it is very important establishing connection between Coxeter Groups and qubit symmetries (e.g. unitary transformation). We would like to mention here that formalism of quantum information and quantum computation lies in the clever use of group theory. We also point out that quantum information and quantum computation theory are young sciences but they are very fast moving and very popular because of their fascinating beauty. Consequently, all this group theoretical analysis would be worth to solve the complicated problems in quantum information and computation theory.

Key Words: Quantum Information, Quantum Computing, Group theory, Symmetry, Quaternions, Octonions

ÖZ

GRUP TEORİNİN KUVANTUM ENFORMASYON VE KUVANTUM HESAPLAMASINA UYGULANMASI

HACIİBRAHİMOĞLU, MEHMET YAKUP

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67 Sayfa

Grup teori metodu kuvantum enformasyon ve kuvantum hesaplama işlemlerine basit ve şık açıklamalasıyla yön vermektedir. Örneğin tersine çevrilebilir kuvantum mantık kapıları grup teori kullanılarak analiz edilebilir. Kuvantum entanglemenın (karışıklığının) grup teoriyle formule edilebileceği bilinmektedir. Kuvantum hesaplama için bazı logaritmik mes'elelerde kullanılabilen algoritmalar saklı alt gruplar temelinde tertip edilebilirler,kuvantum foriyer dönüşüm saklı alt gruplarda ve sonlu gruplarla yakın ilşki içindedir. Ayrıca SU(2) simmetri ile kuvantum ışın ayracı, n-qudit sistemleri ve kuvantum kodları arasında ilginç bağ vardır. Kuvantum entanglementın (karışıklığının) parmak izlerine $A_4, B_4, D_4, H_4, F_4, A_8, B_8, D_8$in yanında E_6, E_7, E_8 gibi istisnai gruplarda ki kliford ve Coxeter gruplarıyla yakın ilşki içinde olan bunlar kuvantum kapıları tarafından meydana getirilmişlerdir. Kuvantum kapıları oluşurken Coxeter gruplar ile qubit simetrilerinin bağlantısını tespitte bulunmak ehemmiyetlidir. (örneğin aynı, tekdüze, dönüşüm). Burada kuvantum enformasyon ve kuvantum hesaplamanın grup teorinin akıllı kullanılmasında yattığını ifade etmek isteriz. Ayrıca kuvantum enformasyon ve kuvantum hesaplama genç bilim alanları olduğunuda işaret ediyoruz. Fakat etkileyici güzelliğinden cazibesinden dolayı hızla hareket etmekte ve gelişmektedirler. Netice olarak bütün teoretik grup analizleri kuvantum enformasyon ve kuvantum hesaplama terindeki problemleri çözmek için değerlidir.

Anahtar kelimeler: Kuvantum enformasyon, Kuvantum hesaplama, Gurup teori, Simetri, Kuaterniyon, Oktoniyon

Bu çalışma
gözünü
kırpmadan
bizler için
canlarını
ortaya koyan
müstesna insanlarımız
GÜVENLİK GÜÇLERİMİZE,
ŞEHİD ve GAZİLERİMİZE
ithaf edilmiştir.

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CHAPTER 1

INTRODUCTION

The new source of ***power*** is not the Money in the hands of a few but the ***Information*** in the hands of many said John Naisbitt (born in 1929) American author who is the first person introducing the word "globalization" to the world. Without any kind of discussion we can say that it's worth mentioning in this day and age "Information is the most valuable thing of all around the world".

In modern times we –mankind- have reached to the very nearly maximum manipulation of Information in almost every classical way. However, whole developments in conventional way are not enough to calculate and estimate more complexity in many scientific faces of crucial areas (e.g. in Medicine, Biology, Satellites, Space flight, Weather-forecasting, artificial intelligence etc.). That`s why the new approach or a new aspect through a new window with the manipulative effect to the aimed Information has taken further steps exponentially since 1980s. First Richard Feynman and then some other scientists [1-9] have mentioned the possibility of manipulation using Quantum Mechanical principles at subatomic level that is actually about the information storage in a state of quantum system based on Quantum Mechanics principle but especially after English scientist Deutsch who is the founder of the first universal Quantum Computer. If one is dealing with Quantum Computer subject, it should be namely considered the interdisciplinary of Physics, Mathematics, and Electronics & Computer Science.

Classical computers which we all use are still dominant devices in our daily life. They are also beginning part of Information age at these moments. The development of electronics in classical way shows us the technical difficulties of the best equipments based on transistors in a near future. Because the limited speed of charge carriers in Semiconductors force us to build the logic gates so closely causing heat producing and dissipation problem extremely. According to Moore's Law the number of transistors on integrated circuits twice as much in size in an 18-24 months period. That means with ca. 2020 the size speed of transistors reaches to atomic size and number of transistors up to 36 billion for classical computers. At this critical point the quantum effects must be taken into consideration. Classically a bit (charged particle) is defined as the smallest representation information and for this it is used base 2 or binary system. A bit in operated states at classical computers may exist in one or two states namely 0 or 1 which is deterministic and the measurement does not change the states of bits. But if we have to make the consideration about quantum effects in this case this bit – called quantum bit or qubit (actually quantum particle which we can call as quantum computer memory) can be at 0 position, or 1 or superposition of its possible classical states. This is the fascinating side of quantum mechanics phenomena. Unlike classical bit circumstances, qubit measurement is probabilistic that means the superposition collapses to one of its states. Using parallel data process is another interesting side of quantum computation.

In quantum information and quantum computation processes we can always be confronted with the group theory calculations. Because to work out and to uncover how the construction types of the unsolved structures there is a strong correlation between them, e.g. reflections, translations, rotations or reversible logic gates so these give more information on our system. That's why we are on the right track of this strong relation. There are of course more important classical groups for example general complex linear group, general real linear groups, orthogonal groups and unitary groups. Especially the finite Coxeter groups are beneficial tools to analyze the reflection and symmetry structures under these types in this work.

Coxeter groups have appeared in various fields of mathematics and physics. Their affine extension related to Lie algebra and found application in various fields of the physics. Recently, various models have been presented for quasicrystals and integrable systems. Classifications of the finite Coxeter groups correspond to the finite Euclidean reflection groups were completed [10-12]. These belong to four infinite series, $A_n, B_n, D_n \ and \ I_2(n)$ and exceptional types, $H_3, H_4, F_4, E_6, E_7 \ and \ E_8$. Coxeter groups include the symmetry groups of the Platonic solids as well as the Weyl groups of simple Lie algebras. Subset of these groups consists of various non crystallographic groups and they describe symmetries of lattices in two, three and four dimensions [13]. Although mathematical structures behind the Coxeter groups have been explored, their connections to entanglement, quantum information and quantum computation have not yet been fully explored. The main source of the quantum information theory [14] is quantum entanglement. It takes place at the center of the quantum information processing. If two or more quantum system (qubits) cannot be written as product of superposition of states of corresponding system, the quantum system is said to be entangled. Entangled states can be generated by Quantum gates that operate on qubits and perform unitary transformation. The unitary transformation can be modeled by Clifford Group [15-17]. Although Clifford unitaries more appropriate such a model, Coxeter Groups [18] are also important components of realizations of quantum gates. In this report we will show that Coxeter reflection groups form an important basis for quantum computation. Therefore it is important to establish relation between Coxeter groups and qubit symmetries. To support our approach, let us mention that, any unitary reflection group can be associated to Braid groups [19] and these groups provide anionic symmetries and also qubit symmetries. In this report we show there is a new bridge between Coxeter groups and quantum information theory.

It is well known that the link between different fields leads to a new insights in that areas. As we mentioned, the analogy between quantum information theory and Clifford groups is well known. In this report we establish the analogy between Coxeter groups and 2^2 and 2^3quantum systems. In this connection quaternions and octonions play a curicual role. Unitary representation of Coxeter group generators can be obtained by constructing roots system of Coxeter diagram interms of quaternions or octonions.

Recent studies show that there is an analogy between entropy of black holes in string theory and entanglement in quantum information theory [20]. This relation were studied in the context of exceptional groups $E_6\ and\ E_7$, describes biparite entanglements of three qubits and three parity entanglement of seven qubits respectively [21-22]. In this study we have obtained unitary representation of the largest crystallographic group E_8 [23-26] and we discussed its relationships to three qubit entanglement.

This study is organized as follows. *Chapter 1* is about introduction. In *Chapter 2* we introduce the properties of Group Theory. In *Chapter 3* quaternions and octonions are introduced. *Chapter 4* is about Coxeter groups and unitary representations of Coxeter group generators. The necessary equations to obtain Cartan matrices, reflection matrices are given in this section. *Chapter 5* is devoted to constructions of unitary matrix representations of Coxeter-Weyl groups $A_4, B_4, D_4, H_4, F_4, A_8, B_8, D_8, and\ E_8$. *Chapter 6* is the last part for conclusion in which the final general statement is discussed the important related points between group theory and Quantum Information theory.

CHAPTER 2

GROUP THEORY

2.1 Introduction

In this part we want to give brief information about fundamental notions of group theory besides the description of physical states symmetries (e.g., in Lie groups) play a decisive role for the simplification of the physics laws. There are many application areas of group theory in different disciplines and in their subsections e.g., classification of eigenfunction, energy levels of atomic, molecular, solid-state and nuclear systems, examining the different phase transition types in solids [27]. Especially group theory has many advantages in studying of quantum information and computation. In addition to those examples algorithms based on hidden subgroup problems consisting order-finding, period-finding or factoring can be also counted as examples of group theory applications.

2.1.2 Definition

G is a group, non empty, of elements x_1, x_2, x_3 if the following conditions are satisfied:

a-) Closure: The product of any two elements of group G gives the element of the same group G again.

$$x_1 \cdot x_2 \in G \; for \; all \; x_1, x_2 \in G \tag{1}$$

b-) Associativity: The associative law is as follows,

$$(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3) \quad for \; all \; x_1, x_2, x_3 \in G \tag{2}$$

c-) Identity: The group G consists of a unit element *e* calling identity element so that the product of e with any group element leaves that element unchanged

$$\forall\, x \in G, \qquad x \cdot e = e \cdot x = x \tag{3}$$

d-) Inverse: The group G also consists of an inverse element x^{-1} for any element of group so that

$$x \cdot x^{-1} = x^{-1} \cdot x = e \tag{4}$$

if the number of elements of group G is finite then this group G is also finite. The order of a finite group G is the number of elements which is in G, denoted as $|G|$. A Group is said to be **Abelian** (***commutative***) if

$$x_1 x_2 = x_2 x_1 \quad for\ all\ \ x_1, x_2\ \in G \tag{5}$$

For example 1 in multiplication or 0 in addition operation commutes with every element in G and element x always permutes with element x^{-1} demanded as in d. For the operation of addition the sets of integers, rational, real and complex numbers and for that of multiplication the sets of non-zero rational, real and complex numbers can be mentioned as Abelian groups.

If G is a group, and *H* is a subset of *G* then it can be said *H* is a subgroup of *G* if the following conditions are satisfied:

- the identity element of *G* is an element of *H*;
- the product of any two elements of *H* is itself an element of *H*;
- the inverse of any element of *H* is itself an element of *H*.

If *H* is a subgroup of a finite group *G* then |*H*| divides |*G*|. Also if *H* is subgroup of *G*, then *H* is known as a normal subgroup if

$$x^{-1} H x \ = H \qquad\qquad for\ all\ \ x\ \in G \tag{6}$$

The subgroups play important roles in understanding the structure of the group. The conjugacy class G_y of an element y in group G is

$$G_y \equiv \{x^{-1}yx\}\mathrm{I}\; x \;\in G \tag{7}$$

Pauli group on *n* qubits is an interesting example of not Abelian group. For a single qubit, the Pauli group is defined to consist of all the Pauli matrices, with multiplicative factors $\pm 1, \pm i$ allowed by the definition:

$$G_1 \equiv \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\} \tag{8}$$

The set of matrices forms a group under the operation of matrix multiplication. One cannot omit the multiplicative factors $\pm I, \pm i$ because these are included is to ensure that G_1 is closed under multiplication, and thus forms a legitimate group. The general Pauli group on n qubits is defined to consist of all n-fold tensor products of Pauli matrices and again one allows multiplicative factors of $\pm I, \pm i$ [14].

2.2 Generators

The set of elements $x_1, x_2, \ldots, x_m$ in a group *G* is called *generators* if every element of *G* can be written as a product of elements from the list $x_1, x_2, \ldots, x_m$ and we write $G = \langle x_1, x_2, \ldots, x_m \rangle$. For example, $G_1 = \langle X, Z, iI \rangle$, since every element in *G* can be written as a product of *X*, *Z* and *iI*. On the other hand, $\langle X \rangle = \{I, X\}$, a subgroup of G_1, since not every element of G_1 can be written as a power of *X*. Using generators has the great advantage about describing groups so that they provide a compact means of describing the group. Suppose *G* has size |*G*|. Then it is pretty easy to show that there is a set of *log*(|G|) generators generating *G*. To see this, suppose $x_1, x_2, \ldots, x_m$ is a set of elements in a group *G*, and *x* is not an element of $\langle x_1, x_2, \ldots, x_m \rangle$. Let $f \in \langle x_1, x_2, \ldots, x_m \rangle$. Then $fg \notin \langle x_1, x_2, \ldots, x_m \rangle$, since if it were then we would have $g = f^{-1}fg \in \langle x_1, x_2, \ldots, x_m \rangle$, which we know is false by assumption. Thus for each element $f \in \langle x_1, x_2, \ldots, x_m \rangle$ there is an element *f g* which is in $x_1, x_2, \ldots, x_m, g$ but not in $x_1, x_2, \ldots, x_m$. Thus adding the generator *g* to $x_1, x_2, \ldots, x_m$ doubles (or more) the size of the group being generated, from which we conclude that *G* must have a set of generators containing at most *log*(|*G*|) elements [14].

2.3 Cyclic Groups

a is an element of cyclic group G such that any element $x \in G$ can be expressed as a^n for some integer n, a as *a generator* of G, then we write $G = \langle a \rangle$. *A cyclic subgroup H* generated by $x \epsilon G$ is the group formed by $\{e, x, x^2, \dots, x^{r-1}$, where r is the order of x. That is, $H = \langle x \rangle$.Every subgroup of a cyclic group is cyclic [14].

2.4 Cosets

For H a subgroup of G, the right coset of H in G determined by $g \in G$ is the set $Hg \equiv \{ hg \text{ I } h \in H\}$. The left coset is described similarly. Many times whether a coset is a 'right' or 'left' coset is implied by context. In the case of a group like $Z_n$where the group operation is addition, it is traditional to write cosets of a subgroup H as $g \oplus H, for\ g \in Z_n$. Elements of a particular coset gH are known as coset representatives of that coset [14].

2.5 Represantations

A matrix group is a set of matrices in $M_n \to n \times n\ complex\ matrix$ satisfying properties of group under multiplication. A representation ρ of a group G can be defined as a function mapping G to a matrix group, preserving group multiplication

Particularly $g \in G$ia mapped to $\rho(g) \in M$ such that $g_1 g_2 = g_3 \Rightarrow \rho(g_1)\rho(g_2) = \rho(g_3)$. If the map is many to one so called homomorphism; if it is one to one then isomorphism. A representation ρ mapping into M_n has dimension $d_\rho = n$. The representations defined before are also referred to as being matrix representations [14].

CHAPTER 3

QUATERNIONS AND OCTONIONS

3.1 Quaternions

Historically Quaternions were conceived by Sir William Rowan Hamilton in 1843 during his famous walk through *Broome Bridge in Dublin, Ireland* [29]. This attempt was to generalize complex numbers in some way that would be applicable to three dimensional space (extending the complex numbers). Because complex numbers could be expressed in plane (2-D) system at that time but not in space system (3-D). According to Hamilton's introduction that was totally surprise to mathematical community, his algebra violated the commutativity law for multiplication which was impossible having an otherwise consistent algebra for which this fundamental property of the algebra of real numbers did not hold [30]. Nowadays quaternions find uses in many areas of science e.g. in mathematics, physics, Information technology, Topology (theoretical and applied) calculations etc. involving three dimensional rotations.

3.2 Properties of Quaternions

In honor of Hamilton, mathematicians denote the quaternions by **H**. The set **H** of all quaternions is a vector space over the all real numbers with dimension 4 and their multiplication is associative distributing over vector addition but not commutative. Let's take a short look at the properties of quaternions. **H** has three operations: addition, scalar multiplication, and quaternion multiplication [31]. Let's define a typical quaternion vector
(***Vector par*t** + *scalar part*) as $[\boldsymbol{v},r]$, with $\boldsymbol{v}=(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$, real numbers r with quaternion $[0,r]$, and vectors $v \in R^3$ with quaternion $[\boldsymbol{v},0]$ so here some basic forms

$$q = [\boldsymbol{v}, r], \qquad ; \boldsymbol{v} \in R^3, \quad r \in R,$$

$$q = [\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, r], \qquad ; \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, r \in R,$$

$$q = e_1\boldsymbol{x} + e_2\boldsymbol{y} + e_3\boldsymbol{z} + r \qquad ; \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, r \in R \tag{9}$$

There is also a relation between e_1, e_2 *and* e_3

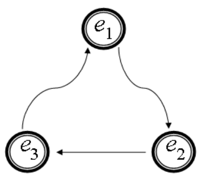
Multiplication of these basis elements is as follows

$$e_0 = 1, \qquad {e_1}^2 = {e_2}^2 = {e_3}^2 = e_1\, e_2\, e_3 = -1 = -\, e_0 \tag{10}$$

Right multiplying both sides by e_3

$$-1 = e_1\, e_2\, e_3 \to -\, e_3 = e_1\, e_2\, e_3 e_3 \to (-1)\, e_3 = e_1\, e_2\, (e_3)^2 = e_1\, e_2(-1) \to e_3 = e_1\, e_2 \tag{11}$$

All the other products can be determined by similar methods that result in



$$\begin{aligned} e_1\, e_2 &= e_3\ , & e_2 e_1 &= -e_3, \\ e_2\, e_3 &= e_1\ , & e_3\, e_2 &= -e_1 \\ e_3 e_1 &= e_2\ , & e_1 e_3 &= -e_2 \end{aligned} \tag{12}$$

As a table:

Table 1. Multiplication table of quaternions

x	***1***	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$
1	e_0	e_1	e_2	e_3
$\boldsymbol{e_1}$	e_1	$-e_0$	e_3	$-e_2$
$\boldsymbol{e_2}$	e_2	$-e_3$	$-e_0$	e_1
$\boldsymbol{e_3}$	e_3	e_2	$-e_1$	$-e_0$

Addition and subtraction of quaternions as follows:

$$If\ q_1 = e_1x_1 + e_2\,y_1 + e_3z_1 + r_1 \quad and \quad q_2 = e_1x_2 + e_2\,y_2 + e_3z_2 + r_2$$

$$q_1 \pm q_2 = (\,e_1x_1 + e_2\,y_1 + e_3z_1 + r_1 \pm e_1x_2 + e_2\,y_2 + e_3z_2 + r_2)$$

$$= (\,r_1 \pm r_2) + (\,x_1 \pm x_2)e_1 + (\,y_1 \pm y_2)e_2 + (z_1 \pm z_2)e_3 \qquad (13)$$

Multiplication of quaternions is not commutative as seen from below

$$q_1q_2 = (v_1, r_1)(v_2, r_2)$$

$$= (e_1x_1 + e_2\,y_1 + e_3z_1 + r_1\ )(\,e_1x_2 + e_2\,y_2 + e_3z_2 + r_2)$$

$$= (\,r_1r_2 - x_1x_2 - y_1y_2 - z_1z_2) +$$

$$(\,r_1x_2 + x_1r_2 + y_1z_2 - z_1y_2)e_1 +$$

$$(r_1y_2 - x_1z_2 + y_1r_2 + z_1x_2)e_2 + \qquad (14)$$

$$(r_1z_2 + x_1y_2 - y_1x_2 + z_1r_2\,)e_3$$

The multiplicative inverse of a quaternion q is denoted as q^{-1} so that $qq^{-1} = q^{-1}q\ where\ q^{-1} = q^*/N(q)$

Conjugate of quaternion and its properties are

$$q = (e_1x + e_2y + e_3z + r)^* \rightarrow q^* = r - e_1x - e_2y - e_3z$$

$$(q^*\,)^* = q \quad and \quad (q_1q_2)^* = q_2{}^*q_1{}^* \qquad (15)$$

The norm of a quaternion and of properties are given as

$$N(q) = N(e_1x + e_2y + e_3z + r) = r^2 + x^2 + y^2 + z^2 \qquad (16)$$

$$N(q^*) = N(q), \quad and \quad N(q_1q_2) = N(q_1)N(q_2) \qquad (17)$$

The magnitude of q is $\|q\| = \sqrt{(r^2 + x^2 + y^2 + z^2)}$ and $\|q\|^2$ is its norm as above given.

Matrix representations of quaternions:

Quaternions can also be represented as matrices like complex numbers. Quaternion addition and multiplication correspond to matrix addition and multiplication. Using 2x2 complex matrices the quaternion *H = w+ix+jy+zk* can be expressed

$$H = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$$

$$where \qquad \alpha = \omega + ix, \qquad \beta = y + iz \tag{18}$$

$$H = \begin{bmatrix} \omega + ix & y + iz \\ -y + iz & \omega - ix \end{bmatrix}$$

$$= \omega \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] + x \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right] + y \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] + z \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right]$$

$$= \omega U + xI + yJ + zK$$

So

$$U \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I \equiv \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

$$where\ U\ is\ identity\ matrix$$

$$J \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad K \equiv \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{19}$$

They are so related with the Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ combined with Identity matrix. From above the given explicit definitions we can derive relations as follows $I^2 = J^2 = K^2 = U$ this is like in 1 dimensional expression. A linear combination of basis quaternions with integer coefficients is sometimes called a Hamiltonian integer [32]. Similarly using 4x4 real matrices, that same quaternion can be written as

$$\begin{bmatrix} w & x & y & z \\ -x & w & -z & y \\ -y & z & w & -x \\ -z & -y & x & w \end{bmatrix} = w\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I} + x\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{H} + y\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}}_{J} + z\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}}_{K} \quad (20)$$

So the above 4x4 matrix can be written in this form Q = wI + xH + yJ + zK where I,H,J and K are 4x4 real matrices. The rules of quaternion addition and multiplication will follow from those of matrix addition and multiplication provided that the matrices H, J, K satisfy the 'Hamiltonian conditions'.

$HH = -I, JJ = -I, KK = -I, \quad HJ = K, JK = H, KH = J, \quad JH = -K, KJ = -H, HK = -J$ [33].

3.3 Octonions

The octonions –eight dimensional algebra or octaves- were discovered in 1843 by *John T. Graves* inspired by his friend *William Hamilton*'s discovery of quaternions. They were also discovered independently by Arthur Cayley [34] and therefore sometimes referred to as **Cayley numbers** or **Cayley algebra**. The octonions, **O**, are a normed division algebra over the real numbers. These are the one of the four algebras [Real numbers **R**, complex numbers **C**, and the quaternions **H**]. **O** are the largest algebra with eight dimensions, double the number of the quaternions from which they are an extension and a relative of complex numbers. They are noncommutative and nonassociative.

3.3.1 Properties of Octonions

The complex numbers can be defined by taking a pair of real numbers. In a similar manner quaternions can be described by taking a pair of complex numbers and in a chain rule the octonions can be defined as pairs of quaternions. The octonions can be thought of 8-tuples / octuple of real numbers. Every octonion is a real linear combination of the unit octonions: $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ every octonion x can be written in the form: $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$ with real coefficients $\{x_i\}$. Let x_0 and x_1 be two numbers so $[x_0\ ,\ x_1]$ is a complex number C that can be expressed as $C = [x_0, x_1] = x_0e_0 + x_1e_1 \ where\ e_0 = 1$,

$e_1^2 = 1$. Actually e_1 is our usual imaginary number i. So taking a pair of two complex numbers $\mathrm{C}_1 = [\mathrm{x}_0, \mathrm{x}_1] = \mathrm{e}_0\mathrm{x}_0 + \mathrm{e}_1\mathrm{x}_1$ and $\mathrm{C}_2 = [\mathrm{x}_3, \mathrm{x}_2] = \mathrm{x}_3 + \mathrm{e}_1\mathrm{x}_2$ after combining this pair we have quaternion $\mathrm{q} = [\mathrm{C}_1, \mathrm{C}_2] = \mathrm{x}_0 + \mathrm{e}_1\mathrm{x}_1 + \mathrm{e}_3(\mathrm{x}_3 + \mathrm{e}_1\mathrm{x}_2) = \mathrm{x}_0 + \mathrm{e}_1\mathrm{x}_1 + \mathrm{e}_2\mathrm{x}_2 + \mathrm{e}_3\,\mathrm{x}_3$ In addition taking a pair of this quaternion once as explained above as a result we get octonion equation following: $\boldsymbol{O} = [Q_1, Q_2] = x_0 + e_1x_1 + e_2x_2 + e_3\,x_3 + e_7(e_1x_4 + e_2x_5 + e_3x_6 + x_7)$ this is in compact form $\sum_{i=0}^{7} x_i e_i$ with the definition in figure 1. As we have mentioned every octonion $\boldsymbol{O}$ can be written with unit octonions in the form:

$\boldsymbol{O} = \{x_0, x_1, \ldots, x_7) = x_0e_0 + e_1x_1 + e_2x_2 + \cdots + e_7x_7$ with real coefficients $\{x_i\}$ which are added / subtracted same as quaternions and multiplied according to $e_i, e_j = -\delta_{ij}e_0 + \ \varepsilon_{ijk}e_k$ where $\varepsilon_{ijk}$is completely antisymmetric tensor and takes value +1 when $ijk = (123), (145), (176), (246), (257), (347), (365)$ and $e_ie_0 = e_0e_i = e_i;\quad e_0e_0 = e_0$ with e_0 identity element (the scalar element), and $i, j, k = 1,2, \ldots ,6,7$ so the product of each term can be given by multiplication table of the unit octonions [35-38]:

Table 2. Multiplication table of unit octonions

x	**e0**	**e1**	**e2**	**e3**	**e4**	**e5**	**e6**	**e7**
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The product of unit octonions can be remembered with the aid of Fano plane diagram -with seven points and seven lines (the circle through 1,2 and 3 is considered a line- as given below, which represents the multiplication table of Cayley and Graves:

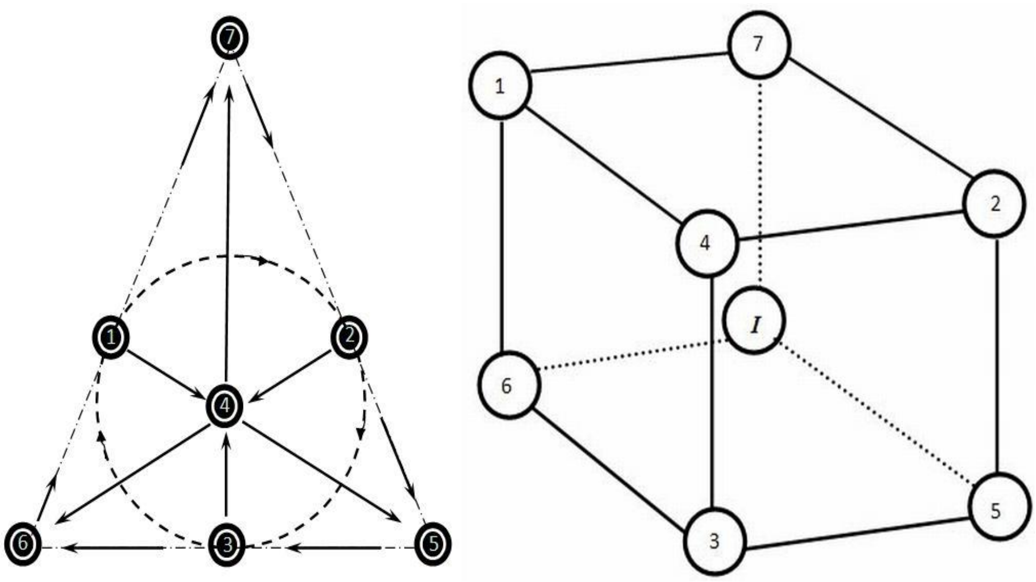


Figure 1. Fano plane, product of unit octonions and Lines in Fano Plane corresponding to planes through the origin in cube

Seven points are corresponding to seven Standard basis elements of Im (**O**). Each pair of distinct points lies on a unique line and each line runs through exactly three points. Let (a, b, c) be an ordered triple of points lying on a given line with the order specified by the direction of the arrow. Then multiplication is given:
ab = c and ba = -c together with the cyclic permutations. These rules are in harmony with

- 1 is the multiplicative identity
- $e_1, e_2, ..., e_7$ are square roots of -1 [$e_i^2 = -1$ for each point in the diagram]

Completely defines the multiplicative structure of the octonions. Each of the seven lines generates a subalgebra of ***O*** isomorphic to the quaternions ***H***
Definition of conjugate elements of octonions is $o^* = x_0 - x_i e_i \quad where \;\; i = 1,2, \dots ,7$

$$o^* = x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 - x_5e_5 - x_6e_6 - x_7e_7 \qquad (21)$$

Then the scalar and vector parts of octonions can be given as

$\boldsymbol{Sc}(o) = 1/2\,(o{+}o)^* = x_0 e_0 = x_0 \;\; where\; e_0 = 1$ and the vector part of that is

$$\boldsymbol{Vec}(o) = \frac{o - o^*}{2} = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \qquad (22)$$

So the scalar product is defined as $\langle o_1 o_2 \rangle = 1/2\,(o_1 o_2^* + o_2 o_1^*) = 1/2\,(o_1^* o_2 + o_2^* o_1)$ in termes of octonionic units this definition can be written as $\langle e_i, e_j \rangle = 1/2\,(e_i^* e_j + e_j^* e_i) = 1/2\,(e_i e_j^* + e_j e_i^*) = \delta_{ij}$. The set of all purely imaginary octonions span a 7 dimension subspace of $\boldsymbol{O}$ denoted Im ($\boldsymbol{O}$**)**. Conjugation of octonions satisfies the equation

$$o^* = -\frac{1}{6}[x + (e_1 x)e_1 + (e_2 x)e_2 + (e_3 x)e_3 + (e_4 x)e_4 + (e_5 x)e_5 + (e_6 x)e_6 + (e_7 x)e_7 \qquad (23)$$

The product of an octonion with its conjugate $oo^* = o^* o$ is always a nonnegative real number:

$$oo^* = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 \qquad (24)$$

Using this the norm of an octonion can be defined, as $\|o\| = \sqrt{o^* o}$ This norm – is its scalar product with itself- agrees with the standard Euclidian norm on $\mathbf{R}^{\mathbf{8}}$ the existence of inverses for every nonzero element of $\boldsymbol{O}$. The inverse of $o \neq 0$ is given by $o^{-1} = o^*/\|o\|^2$ It satisfies $oo^{-1} = o^{-1}o = 1$ Octonionic multiplication is neither commutative:

$$e_i e_j = -e_j e_i \neq e_j e_i\,, \qquad if\; i,j \neq 0 \qquad (25)$$

nor associative:

$$(e_i e_j)e_k = -e_i(e_j e_k) \neq e_i(e_j e_k)\,, \qquad if\; i,j,k \neq 0 \qquad (26)$$

In terms of octonionic units, we can write commutator $[e_i, e_j]$ of two octonions in the form $[e_i, e_j] = e_i e_j - e_j e_i$ from equations

$$\boldsymbol{O} = (x_1, x_2, \ldots, x_7) = e_0 x_0 + e_1 x_1 + \cdots + e_7 x_7 \qquad (27)$$

and

$$e_i e_j = -\delta_{ij}\, e_0 + \psi_{ijk}\, e_k \qquad (28a)$$

a relation can be shown as $[e_0, e_i] = 0$ and $[e_i, e_j] = 2\,\psi_{ijk}\,e_k \quad (i, j, k = 1,2, \dots ,7)$ this equation implies as above mentioned also that the octonions are non-commutative like quaternions defined. The associator of three octonions is: $[e_i, e_j, e_k] = (e_i, e_j)e_k - e_i(e_j e_k)$. The associator of octonions can also be written by defining a completely antisymmetric 4-index object ϕ_{ijkl} so octonions-associator becomes $[e_i, e_j, e_k] = 2\phi_{ijkl} e_l$. The values of $\phi_{ijkl}$are determined by using the definition of associator of three octonions and Table 2. But the diagrammatic multiplication table is conveniently more useful than multiplication Table 2. Figure 2 describes diagrammatic representation of the octonionic multiplication table that is obtained from a circle with 7 points representing the imaginary octonionic units and a triangle forming the associative group of three octonionic units. With the rotating triangle we can get the whole multiplication table. It gives us for example $e_1 e_2 = e_3$, $e_2 e_4 = e_6$, $e_4 e_3 = e_5$, $e_3 e_6 = e_7$ etc.

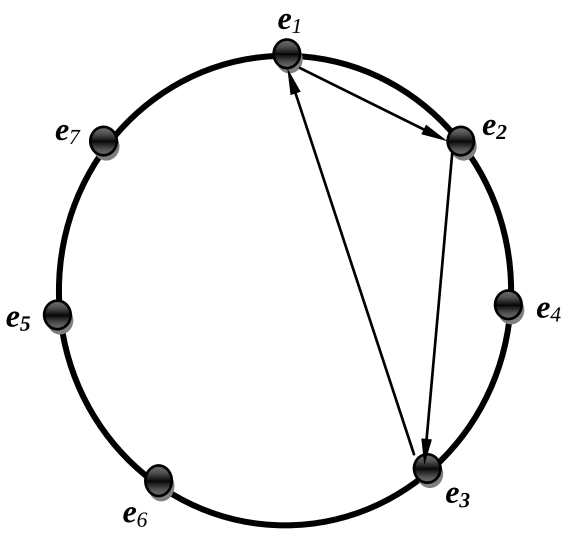


Figure 2. Diagrammatic representation of octonionic multiplication table

The other diagrammatic representation relating the associators of three octonions can be read in a similar manner from Figure 3 by rotating the triangle. For example the triangle defined by the vertices $[e_1, e_2, e_4]$ and the lines joining the vertices of the triangle to e_5 form an associator. If we rotate the quadrilateral in figure 4 by $2\pi/7$ we can obtain the associators

$$[e_1, e_4, e_2] = 2e_5, \qquad [e_2, e_3, e_4] = 2e_7,$$

$$[e_4, e_6, e_3] = 2e_1, \qquad [e_3, e_5, e_6] = 2e_2,$$

$$[e_6, e_7, e_1] = 2e_4, \qquad [e_5, e_1, e_7] = 2e_3,$$

$$[e_7, e_2, e_1] = 2e_6 \tag{28b}$$

The quaternions in physics are usually introduces as Pauli σ-matrices yielding a realization of the algebra of quaternions in connection with the rotation group in 3-dimensions. The product rule for quaternions can be obtained from octonionic multiplication table then the quaternionic unit e_i satisfies the relations

$$e_i e_j = -\delta_{ij} e_0 + \psi_{ijk} e_k \,, \qquad (i, j, k = 1,2,3) \tag{29}$$

Where δ_{ij} and ψ_{ijk} are the Kronecker and Levi-Civita symbols, respectively [24-27].

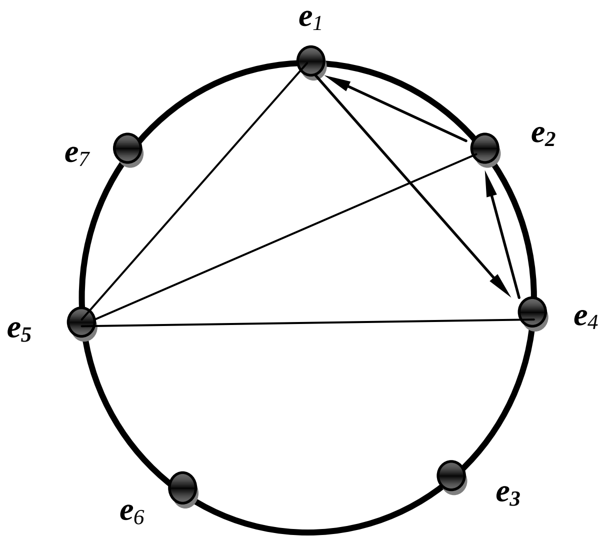


Figure 3. Diagrammatic representations of associator $[e_i, e_j, e_k]$ of three octonions.

CHAPTER 4

COXETER GROUPS

AND

UNITARY REPRESENTATIONS OF COXETER GROUP GENERATORS

4.1 Introduction

The Coxeter groups, *named after* ***Harold Scott MacDonald Coxeter***, describe the symmetry of the polyhedra and higher dimensional polytopes. Coxeter groups have intensively connecting with reflection groups that means they are actually abstractions of reflection groups proved by Coxeter [8] and are also the finite Euclidian reflection groups (Weyl groups are the best examples). They appear in various fields of mathematics and physics. Their Affine extension related to Lie algebra and found application in various fields of the physics and they are related to the presentation of groups.

In this section we will obtain unitary representations of the Coxeter group generators by the aid of Mathematica program. Here a Mathematica program to obtain unitary matrix generators of Coxeter groups is presented. There are various software packages and programs including computational tools for Lie algebra and group theory. Most of them provide an environment for computation involving facilities for manipulating roots, reflections, representations and irreducible characters of finite Coxeter groups, weights and characters of irreducible representations of semi-simple Lie algebras tensor product decompositions.

The program presented here is not a package program. It is a simple program that draws Coxeter diagram, calculate Cartan and reflection matrices, generates corresponding reflection group and transform the matrix generators into unitary basis by using a given transformation matrix that is obtained by representing roots system of the group by quaternions or octonions. The program is presented by using

Mathematica whose flexibility and intuition provide a great advantage to user. The resultant generators presented by this program are used for description of symmetries of multiple qubit systems. Codes of the program are given in the appendix. Before we explain the parts of the program step by step firstly we want to keep a watching brief over general properties of Coxeter groups and their relates.

4.2 Definition:

A Coxeter matrix of rank n is a symmetric $n \times n$ matrix $M = (m_{ij})$ with non-negative integer entries such that $m_{ij} = 1\ \ for\ all\ \ i \in [n] = \{1,2,\dots,n\}\ and\ m_{ij} \geq 2\ \ for\ i \neq j$. The meaning of $m_{ij} = 1$ is actually $(r_i r_i)^1 = r_i^2 = 1\ \ for\ all\ i$ .. What about the $m_{ij} \geq 2$? Here we can say that the generators r_i, r_j are commute. If $m_{ij} = \infty$ then there is no relation between $r_i\ and\ r_j$. The definition of coxeter group associated with the Coxeter matrix M is the $W(M)$being abstract group with R a set of generators of $W\ R = \{r_i \mathrm{I} i = 1,2,\dots,n\}$ and relations $(r_i r_j)^{m_{ij}} = 1\ for\ all\ i,j \in [n]$. The pair $(W(M), R)$ is a Coxeter system. So every Coxeter matrix is the matrix of Coxeter system due to generators and relations as in definitions given in a group.

If we are required to give some examples, the followings are for Coxeter diagram (or Coxeter graph or Coxeter Dynkin diagram) in Figure 4

Graph vertices $\to$ *R*

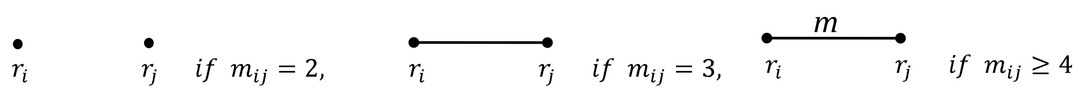


Figure 4. General illustration of Coxeter diagram

Where $r_i\ and\ r_j$ are vertices. If $m_{ij} = 2$ then there is no putting edge but if it is equal to 3 then we can assign an edge and if it is equal to four or more than that we mark m on the edge between vertices. So these numerically labeled edges method (branches) is called Coxeter-Dynkin diagram (Coxeter diagram or Coxeter graph) as mentioned above. During studying the Coxeter groups one of the powerful instruments is the root system (or root vectors). Dynkin has achieved that the simple roots can be represented by two dimensional diagrams so that the complete set of root vectors can be obtained as well as lengths of roots and angles of these respectively. The configuration of vectors – root system- in Euclidian space (**S**) is the basic of Lie algebras (here the root system is finite set Θ having non-zero vectors – called roots-). For an r-element Lie algebra there are the most r roots besides the degeneracies of the roots may exist. The root vectors fulfill the following properties:

- ✧ The roots span **S**.
- ✧ If α is a root vector, then -α is also root vector.
- ✧ For $\forall\, \alpha \in \Theta$ the set Θ is closed under reflection through the hyperplane perpendicular to α
- ✧ If α and β are root vectors then $\langle \alpha, \beta \rangle = 2\alpha \frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer
- ✧ Two root vectors α and β give the third one $\gamma = \beta - 2\alpha \frac{(\alpha,\beta)}{(\alpha,\alpha)}$ which is also element of the set Θ.

According to the definition of scalar product angle θ between two roots α and β in terms of Cosin(θ) is $\mathrm{Cos}(\theta) = \frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)(\beta,\beta)}}$ or $\mathrm{Cos}^2(\theta) = \frac{(\alpha,\beta)(\alpha,\beta)}{(\alpha,\alpha)(\alpha,\beta)} = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, or\ 1$ Thus for the different angle values between α and β roots we get 0^o, 90^o, 120^o, $135^o, or\ 150^o$. Related to the roots Weyl group is the result of the reflections through hyper planes which is orthogonal to the elements of S. It is a finite reflection group or finite Coxeter group. If we want to construct the root diagram – graphical representation of root vectors in an r-dimensional space. Due to the representation difficulty in three dimensions as an example the root diagram of SU(3) is discussed. The roots for SU(3) are as follows:

$$(1,0); \left(1/2\,, \sqrt{3}/2\right); \left(1/2\,, -\sqrt{3}/2\right);\ (-1,0); \left(-1/2\,, -\sqrt{3}/2\right); \left(-\,1/2\,, \sqrt{3}/2\right);$$

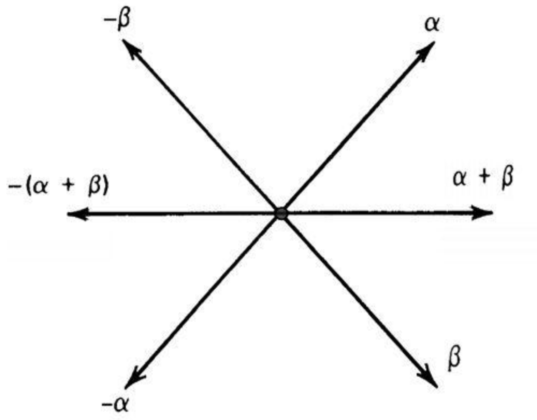
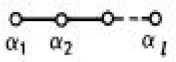
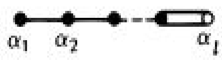
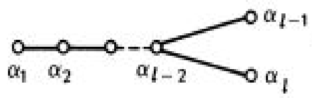
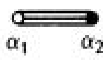
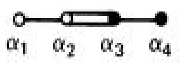
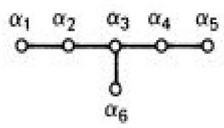
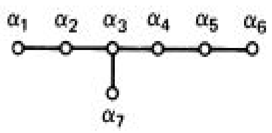
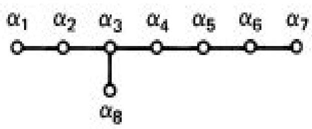


Figure 5. Root diagram of SU(3)

It is very difficult to represent a diagram as in Figure 4 for the Lie Algebra of rank $r > 2$. Therefore Eugene Dynkin has developed a method to draw a diagram in which the diagram is a kind of graph with some edges single or double or triple lines. Each simple root is illustrated by a small circle for the long edge the circle is white **o** otherwise it is **black filled** ● . The angle between any two roots is marked by the number of lines joining the circles. These circles are connected with one, two or three lines depending on the angles between the simple roots (angle values are for example 120^0, 135^0 and 150^0). If two roots are perpendicular to each other it is not needed to draw line. All Lie algebras of simple compact groups have been classified by Cartan (1933). According to the Cartan's classification no other diagrams are possible. Dynkin diagrams have a strong relation to the Coxeter diagrams of finite Coxeter Groups. Dynkin diagrams and root structures of the classical Lie groups are given in Table 3 [39].

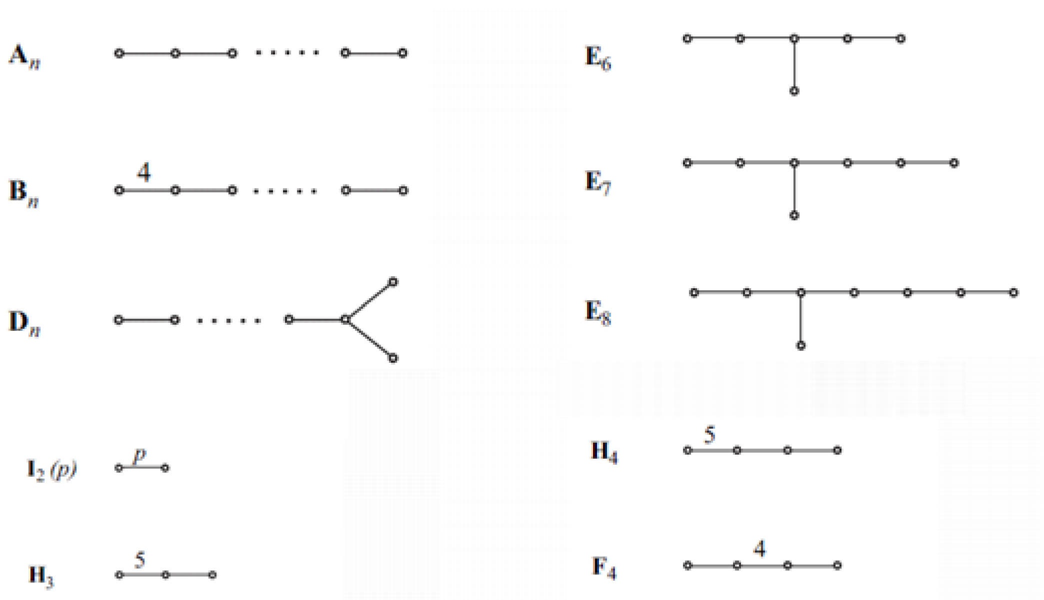
Table 3. Dynkin diagrams and root structures of the classical Lie algebras

Order	Cartan label	Group label	Dynkin diagram	Roots
$l(l+2)$	A_l	$SU(l+1)$		$\mathbf{e}_i-\mathbf{e}_j\ (i,j=1,\ldots,l+1)$
$l(2l+1)$ $l\geqslant 2$	B_l	$SO(2l+1)$		$\pm\mathbf{e}_i$ and $\pm\mathbf{e}_i\mathbf{e}_j(i,j=1,\ldots,l)$
$l(2l+1)$ $l\geqslant 3$	C_l	$Sp(2l)$		$\pm 2\mathbf{e}_i$ and $\pm\mathbf{e}_i\pm\mathbf{e}_j(i,j=1,\ldots,l)$
$l(2l-1)$ $l\geqslant 4$	D_l	$SO(2l)$		$\pm\mathbf{e}_i\pm\mathbf{e}_j(i,j=1,\ldots,l)$
14	G_2	G_2		$\mathbf{e}_i-\mathbf{e}_j(i,j=1,2,3;\ i\neq j)$ $\pm 2\mathbf{e}_i\mp\mathbf{e}_j\mp\mathbf{e}_k\,(i,j,k=1,2,3+b,i\neq j\neq k)$
52	F_4	F_4		As for B_4 plus the 16 roots $\frac{1}{2}(\pm\mathbf{e}_1\pm\mathbf{e}_2\pm\mathbf{e}_3\pm\mathbf{e}_4)$
78	E_6	E_6		As for A_5 plus the roots $\pm\sqrt{2}\,\mathbf{e}_7$ and $\frac{1}{2}(\pm\mathbf{e}_1\pm\mathbf{e}_2\pm\mathbf{e}_3\pm\mathbf{e}_4\pm\mathbf{e}_5\pm\mathbf{e}_6)\pm\mathbf{e}_7/\sqrt{2}$ (three signs + and three − in first fraction)
133	E_7	E_7		As for A_7, plus the roots $\frac{1}{2}(\pm\mathbf{e}_1\pm\mathbf{e}_2\pm\mathbf{e}_3\pm\mathbf{e}_4\pm\mathbf{e}_5\pm\mathbf{e}_6\pm\mathbf{e}_7\pm\mathbf{e}_8)$ with four signs + and four −.
248	E_8	E_8		As for D_8, plus the roots $\frac{1}{2}(\pm\mathbf{e}_1\pm\mathbf{e}_2\pm\mathbf{e}_3\pm\mathbf{e}_4\pm\mathbf{e}_5\pm\mathbf{e}_6\pm\mathbf{e}_7\pm\mathbf{e}_8)$ with the number of + signs even

4.3 Construction of Coxeter - Dynkin Diagrams

It is well known that a Coxeter–Dynkin diagram [38] is a graph collection of reflecting hyperplanes. Each graph consists of nodes (mirrors) and branches. Branches connect nodes together with edges labeled with a rational number m, representing dihedral angle π/m. When $m = 2$ the angle is $\pi/2$ and the mirrors have no interaction (the branch can be omitted from the diagram). If $m = 3$ a branch is unlabeled representing an angle $\pi/3$. If the nodes are parallel then their branch labeled with "∞". Following table 4 shows Coxeter–Dynkin diagrams for the fundamental finite Coxeter groups.

Table 4. Coxeter–Dynkin diagrams for the fundamental finite Coxeter groups.



In order to obtain diagram of a Coxeter group, number of nodes and branches should be given as input. If the nodes connected together as in A_n or B_n, the input will be in the form:

$$nodes = \{\{1,2\} \to m_1, \{2,3\} \to m_2, \ldots, \{n-1, n\} \to m_n\} \tag{30}$$

If the nodes are connected as in D_n or E_n then the last node and its placement should be given additionally. As an example for E_7 group input and additional input will be in the form:

$$\begin{gathered} input \to \{\{1,2\} \to 3, \{2,3\} \to 3, \{3,4\} \to 3, \{4,5\} \to 3, \{5,6\} \to 3\} \\ addinput \to \{\{3,7\} \to 3\} \\ addcoor \to \{\{3,0\}, \{3,1\}\}; \end{gathered} \tag{31}$$

where $\{\{x_i, y_i\}, \{x_f, y_f\}\}$ are beginning and end points of the additional rot in $\{x, y\}$ plane.

4.4 Cartan Matrix

The diagram carries information about the Cartan matrices. The code given in the appendix is used to calculate elements of the Cartan Matrices. The Cartan matrix of a simple Lie algebra is the matrix whose elements are the scalar products of roots (α_i, α_j), such that

$$c_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} = 2\frac{\|\alpha_j\|}{\|\alpha_i\|} cos\theta \tag{32}$$

where (α_i, α_j) is scalar product of two root vectors and θ is the angle between two vector. Indeed the angle between $\alpha_i \; and \; \alpha_j$ is $\frac{\pi}{m}$ plane spanning of $\alpha_i \; and \; \alpha_j$ is through the rotation $\pi - \frac{\pi}{m}$. It is obvious that the diagonal elements of the matrix are all equal to 2. The possible values of off-diagonal elements are: $0, \pm 1, \pm 2, \pm 3$. Cartan matrix is not necessarily be symmetric but if $c_{ij} \neq 0 \; then \; c_{ji} \neq 0$. The properties of Cartan matrix is summarized as:

1. No edge connection between the roots. They are orthogonal and $c_{ij} = 0$.

2. Single edge connection between the roots. The roots are not orthogonal and

$c_{ij} = c_{ji} = -1.$

3. A double edge connection from $\alpha_i \ to \ \alpha_j$. The weight of the root $\|\alpha_i\|^2 = 2\|\alpha_j\|^2$.

4. A triple edge connection from $\alpha_i \ to \ \alpha_j$. The weight of the root $\|\alpha_i\|^2 = 3\|\alpha_j\|^2$.

As an example consider Cartan matrix of A_4. Its Coxeter diagram is given in the figure 6.

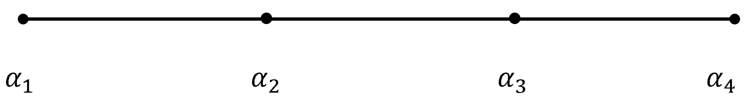


Figure 6. Coxeter diagram of A_4.

All of the roots connected with a single edge. The elements of the Cartan matrix are

$$\begin{gathered} c_{11} = c_{22} = c_{33} = c_{44} = 2; \\ c_{12} = c_{21} = -1; c_{34} = c_{43} = -1, \end{gathered} \tag{33}$$

and the other elements are zero. We can write:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{34}$$

Consider Cartan matrix of B_4. In this case the last two roots connected together by double edge. The angle between the last two roots is $\frac{\pi}{4}$.

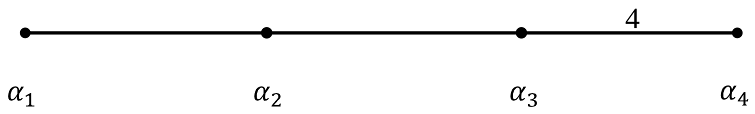


Figure 7. Coxeter diagram of B_4.

The elements can be calculated by using the equation $c_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} = 2\frac{\|\alpha_j\|}{\|\alpha_i\|}cos\theta$ and above definitions. The diagonal elemets are equal to 2. The other elements are:

$$c_{12} = c_{21} = -1; c_{23} = c_{32} = -1$$

$$c_{34} = 2\frac{\|\alpha_4\|}{\|\alpha_3\|}cos\theta = \frac{2}{\sqrt{2}}\cos\left(\frac{3\pi}{4}\right) = -1; \tag{35}$$

$$c_{43} = 2\frac{\|\alpha_3\|}{\|\alpha_4\|}cos\theta = 2\sqrt{2}\cos\left(\frac{3\pi}{4}\right) = -2,$$

The other elements of the matrix are zero. Then its Cartan matrix is given by:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix} \tag{36}$$

4.5 Weights ω_i

The concept of weight vectors plays an important role in the construction of unitary matrix generators study. Weight vectors appear as a factor in the lattice vector. The simple roots and weight vectors can be written as linear combinations of each other as follows:

$$\omega_i = (C^{-1})_{ij}\alpha_j, or\ \alpha_i = (C)_{ij}\omega_j$$

$$(\omega_i, \omega_j) = (C^{-1})_{ij}\ , or\ (\omega_i, \alpha_j) = \delta_{ij} \tag{37}$$

4.6 Reflection Matrices

Reflection matrices of the corresponding Coxeter group are also their generators. The calculation based on transformation of vectors, $\Lambda = a_i \omega_i$, such that $r_i \Lambda = \Lambda - (\alpha_i, \Lambda)\alpha_i$, where r_i is reflection matrix. Using the relations,

$$(\omega_i, \alpha_j) = \delta_{ij};\ \omega_i = (C^{-1})_{ij}\alpha_j;\ \alpha_i = (C)_{ij}\omega_j \tag{38}$$

one can easily calculate the matrices. As an example 1st reflection matrix of A_4 can be obtained as follows:

$$r_1\omega_1 = \omega_1 - (\omega_1, \alpha_1)\alpha_1 = \omega_1 - (2\omega_1 - \omega_2) = -\omega_1 + \omega_2 \tag{39}$$
$$r_1\omega_2 = \omega_2; r_1\omega_3 = \omega_3;\ r_1\omega_4 = \omega_4.$$

It can be written in the matrix form:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{40}$$

4.7 Generation of Group

We also presented a code to generate finite groups using matrix generators. We would like to mention here that generation of matrix group cannot be directly obtained in Mathematica. The following simple code safely generates a matrix group. We have generated matrix group of E_6 Coxeter group of order 51840 to test the program. Group elements and conjugacy classes of the group can also be obtained by using this program.

Generation of group based on projection of vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}\}$ under the action of group generators. We obtain all possible projections of an arbitrary vector then group elements can be obtained using projected vectors. Then the group elements can be obtained by representing coefficients of the vectors, in the matrix form. The program also determine conjugate classes of the by using the definition $C_k = g_i g_k g_i^{-1}$. One can obtain projection of a specific vector without generation the group and then project them on Coxeter plane.

4.8 Construction of Unitary matrices

The reflection matrices obtained in the previous section is not unitary. In order to obtain unitary matrix generators we represent roots by quaternions or octonions by following the method developed by Koca [23-28]. In the next section we obtain unitary representations of the reflection matrices of $A_4, B_4, D_4, H_4, F_4, A_8, B_8, D_8$, $and\ E_8$.

CHAPTER 5

COXETER GROUPS (A_4, B_4, D_4, F_4, H_4, A_8, B_8, D_8, AND E_8) AND THEIR RELATION WITH QUANTUM INFORMATION

5.1 Quantum gates

Before we start to the constructions of unitary matrix representations of Coxeter Weyl groups we should give crucial properties of single and two-qubit quantum gates. Because these are the cornerstones of this study. As we have mentioned in chapter 1 a unit of quantum information is called qubit -which is also analogue of classical bit- in Hilbert space ($H^{\otimes 2}$). The state of qubit is the result of canonical bases combination $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ so $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$ where α and β are the complex numbers. The expression of $|\psi\rangle$ as a vector is actually one-qubit representation in $H^{\otimes 2}$ vector space. After measurement the value of this vector state either we have $|0\rangle$ with probability $\lfloor\alpha\rfloor^2$ telling us the probabilistic value of $|\psi\rangle$ at the state $|0\rangle$ or at the other state $|1\rangle$ with probability $\lfloor\beta\rfloor^2$ or combination of $|0\rangle$ and $|1\rangle$ states with the summation of both $\lfloor\alpha\rfloor^2$ and $\lfloor\beta\rfloor^2$ -called superposition of bases- but this must be equal to one without doubt.

Now we can introduce the quantum gates. There are many of quantum gates in quantum computer like in classical computer's. The most useful and easy understanding and conceivable are single and double-qubit gates. The quantum gates acting on single qubits can be defined by two by two matrices. These quantum gates have unitary matrix-characteristic which means they have the following essential properties:

- U is unitary iff its inverse is equal to its conjugate transpose : $U^{-1} = U^{\dagger}$
- $U^{\dagger}$ and U^{-1} are unitary.
- $U^{\dagger} = U^{-1}$
- $UU^{\dagger} = \mathbf{1}$

- ✧ $|\det(U)| = 1$
- ✧ The columns [rows] of U form an orthonormal set of vectors.
- ✧ For a fixed row $\sum_{j=1}^{2^n} \left|U_{ij}\right|^2 = 1$ and for fixed column $\sum_{i=1}^{2^n} \left|U_{ij}\right|^2 = 1$
- ✧ $U = e^{i\boldsymbol{H}}$where $\boldsymbol{H}$ is an hermitian matrix, $\boldsymbol{H}=\boldsymbol{H}^{\dagger}$

As we know the single qubit quantum gates are represented by two by two unitary matrices.

They are identified and given in Table 5 by Pauli matrices $\sigma_1 = \sigma_x, \sigma_2 = \sigma_y, \sigma_3 = \sigma_z$ and rotation matrices $R_x,\ R_y\ and\ R_z$ in Table 6 respectively:

Table 5. Representation of Pauli matrices $\sigma_x, \sigma_y, \sigma_z$

$\sigma_1 = \sigma_x = \boldsymbol{X}\ (or\ \boldsymbol{NOT}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_x \sigma_x = {\sigma_x}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}$
$\sigma_2 = \sigma_y = \boldsymbol{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_y \sigma_y = {\sigma_y}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}$
$\sigma_3 = \sigma_z = \boldsymbol{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\sigma_z \sigma_z = {\sigma_z}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}$
${\sigma_x}^2 = {\sigma_y}^2 = {\sigma_x}^2 = \mathbf{1}$	
$\sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$	$\sigma_y \sigma_z = -\sigma_z \sigma_y = i\sigma_x$
$\sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y$	$\sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$

Table 6. Rotation matrices $R_x,\ R_y\ and\ R_z$

$R_x(\alpha) = Exp\left[-\frac{i\alpha}{2}\sigma_x\right] = \begin{pmatrix} \sum_{n=0,2,4,\dots}^{\infty}\left(\frac{\left(\frac{i\alpha}{2}\right)^n}{n!}\right) & \sum_{n=1,3,5,\dots}^{\infty}\left(\frac{\left(\frac{i\alpha}{2}\right)^n}{n!}\right) \\ \sum_{n=1,3,5,\dots}^{\infty}\left(\frac{\left(\frac{i\alpha}{2}\right)^n}{n!}\right) & \sum_{n=0,2,4,\dots}^{\infty}\left(\frac{\left(\frac{i\alpha}{2}\right)^n}{n!}\right) \end{pmatrix}$ $= \begin{pmatrix} \cos\frac{\alpha}{2} & -i\sin\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} = \cos\frac{\alpha}{2}\,\boldsymbol{I} - i\sin\frac{\alpha}{2}\,\boldsymbol{X}$
$R_y(\beta) = Exp\left[-\frac{i\beta}{2}\sigma_y\right] = \begin{pmatrix} \sum_{n=0,2,4,\dots}^{\infty}\left(\frac{\left(\frac{i\beta}{2}\right)^n}{n!}\right) & -i\sum_{n=1,3,5,\dots}^{\infty}\left(\frac{\left(\frac{i\beta}{2}\right)^n}{n!}\right) \\ i\sum_{n=1,3,5,\dots}^{\infty}\left(\frac{\left(\frac{i\beta}{2}\right)^n}{n!}\right) & \sum_{n=0,2,4,\dots}^{\infty}\left(\frac{\left(\frac{i\beta}{2}\right)^n}{n!}\right) \end{pmatrix}$ $= \begin{pmatrix} cos\frac{\beta}{2} & -sin\frac{\beta}{2} \\ sin\frac{\beta}{2} & cos\frac{\beta}{2} \end{pmatrix} = cos\frac{\beta}{2}\,\boldsymbol{I} - i\ sin\frac{\beta}{2}\,\boldsymbol{Y}$
$R_z(\gamma) = Exp\left[-\frac{i\gamma}{2}\sigma_z\right] = \begin{pmatrix} \sum_{n=0,2,4,\dots}^{\infty}\left(\frac{\left(\frac{-i\gamma}{2}\right)^n}{n!}\right) & 0 \\ 0 & \sum_{n=0,2,4,\dots}^{\infty}\left(\frac{\left(\frac{i\gamma}{2}\right)^n}{n!}\right) \end{pmatrix}$ $= \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \cos\frac{\gamma}{2}\,\boldsymbol{I} - i\sin\frac{\gamma}{2}\,\boldsymbol{Z}$

The Pauli matrices correspond to the Pauli Gates $\boldsymbol{X}$ ($\boldsymbol{NOT}$), $\boldsymbol{Y}$ and $\boldsymbol{Z}$. The rotation matrices $R_x,\ R_y\ and\ R_z$ are the standard Bloch sphere rotations. One can also construct some useful additional gates using linear combinations of the gates as shown in table 7. There are also very helpful gates Hadamard Gate $\boldsymbol{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\boldsymbol{S} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, $\boldsymbol{T} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{\pi}{4}} \end{pmatrix}$ and Phase gate $\boldsymbol{\phi} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = e^{-i\frac{\phi}{2}}\begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix}$. $\boldsymbol{NOT}$, $\sqrt{\boldsymbol{NOT}}$ and Hadamard Gate can be obtained from Rotation gates such as:

Table 7. Some examples of Gate derivation from Rotational matrices

$\boldsymbol{NOT} \equiv R_x(\pi) \cdot Ph(\pi/2)$
$\boldsymbol{NOT} \equiv R_y(\pi) \cdot R_z(\pi) \cdot Ph(\pi/2)$
$\sqrt{\boldsymbol{NOT}} \equiv R_x(\pi/2) \cdot Ph(\pi/4)$
$\sqrt{\boldsymbol{NOT}} \equiv R_z(-\pi/2) \cdot R_y\,(\pi/2) \cdot R_z(\pi/2) \cdot Ph(\pi/4)$
$\boldsymbol{H} \equiv R_x(\pi) \cdot R_y\,(\pi/2) \cdot Ph(\pi/2)$ $\boldsymbol{H} \equiv R_y\,(\pi/2) \cdot R_z(\pi) \cdot Ph(\pi/2)$

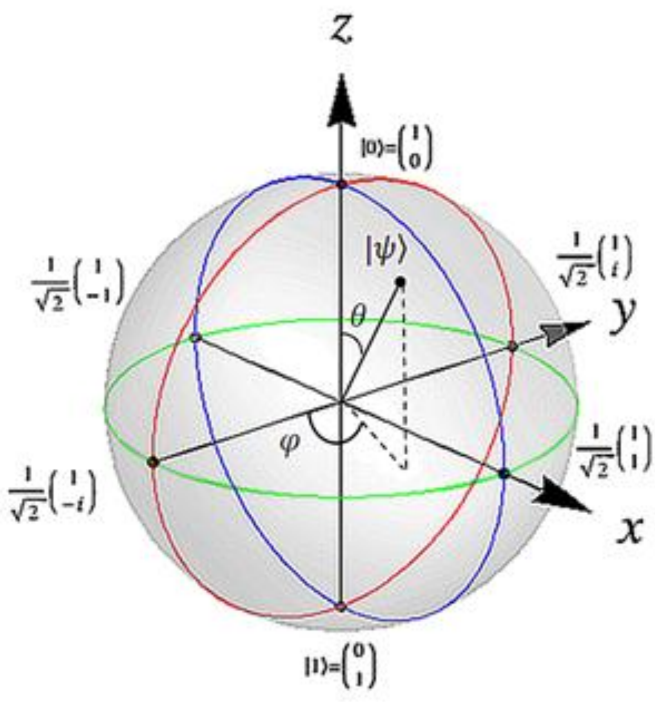


Figure 8. Bloch sphere *[permitted by Brad Rubin]*

Here we want to illustrate some other examples in addition to table 8 in details to give to the reader clear aspect about these wonder gates.

Table 8. Some examples of gates acting on qubits

$\boldsymbol{X}\,[\alpha\|0\rangle + \beta\|1\rangle] \rightarrow [\beta\|0\rangle + \alpha\|1\rangle]$
$\boldsymbol{Y}\,[\alpha\|0\rangle + \beta\|1\rangle] \rightarrow \boldsymbol{i}\,[-\beta\|0\rangle + \alpha\|1\rangle]$
$\boldsymbol{Z}\,\frac{1}{\sqrt{2}}[\|0\rangle + \|1\rangle] \rightarrow \frac{1}{\sqrt{2}}[\|0\rangle - \|1\rangle]$
$\boldsymbol{\varphi}\,\frac{1}{\sqrt{2}}[\|0\rangle + \|1\rangle] \rightarrow \frac{1}{\sqrt{2}}[\|0\rangle + e^{i\varphi}\|1\rangle]$
$\boldsymbol{H}\|0\rangle \equiv \boldsymbol{H}\otimes\|0\rangle \rightarrow \frac{1}{\sqrt{2}}[\|0\rangle + \|1\rangle]$
$\boldsymbol{HH} \equiv \boldsymbol{H}\otimes\boldsymbol{H} \rightarrow \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

These gates act on single qubits $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ in the quantum circuits and output of the circuit can be obtained by acting matrix representation of the gate on input. In a similar manner we introduce some of those mostly used two-qubit gates and their matrix representations

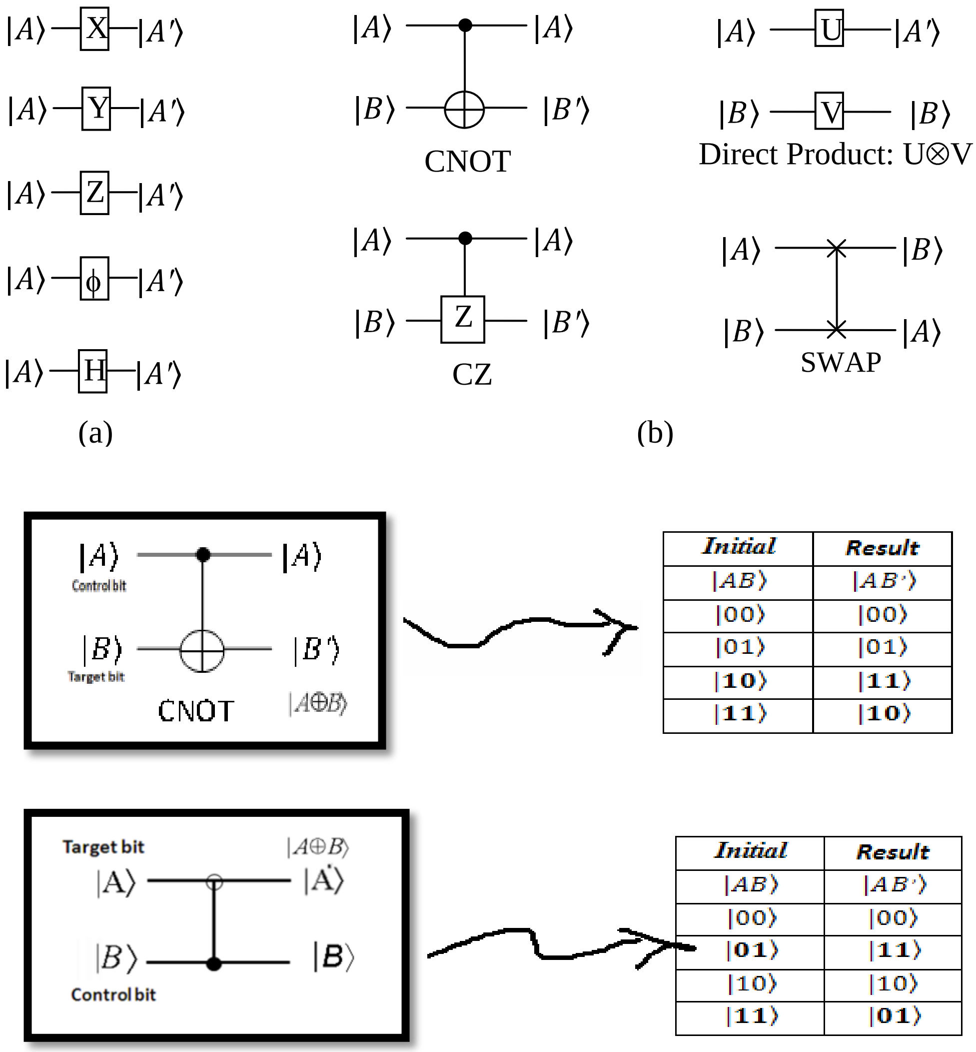
$$\mathrm{SWAP} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}, \quad \mathrm{CNOT} = \sigma_0 \oplus \sigma_1 = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix},$$

$$\mathrm{CZ} = \sigma_0 \oplus \sigma_3 = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & -1\end{pmatrix}, \quad \mathrm{CPHASE} = \sigma_0 \oplus \boldsymbol{S} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & i\end{pmatrix} \tag{41}$$

where CNOT gate is known as controlled NOT gate, CZ is controlled Z gate and CPHASE is the controlled PHASE gate. Parallel connections of gates are associated to the direct product of their matrix representations. If U and V are arbitrary 2×2 unitary gates, their parallel connection in a circuit is defined by $U \otimes V$. Circuit diagram of various gates are illustrated in figure 6. The two-qubit states can be represented by a set of orthogonal unit vectors in the 4-dimensional Euclidean space:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{42}$$

where $|ij\rangle$ is the tensor product of two states and can also be so represented $|ij\rangle = |i\rangle \otimes |j\rangle$. Output of a two-qubit circuit can be obtained by the product of two-qubit input with matrix representation of the gate. Icons of the gates from table 5 and eq. 41 are illustrated in figure 9. Note that Controlled-NOT (CNOT) and SWAP gates play a dominant role in the quantum information and quantum computation theories.



Initial	Result
$\lvert AB\rangle$	$\lvert AB'\rangle$
$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 01\rangle$
$\lvert 10\rangle$	$\lvert 11\rangle$
$\lvert 11\rangle$	$\lvert 10\rangle$

Initial	Result
$\lvert AB\rangle$	$\lvert AB'\rangle$
$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 11\rangle$
$\lvert 10\rangle$	$\lvert 10\rangle$
$\lvert 11\rangle$	$\lvert 01\rangle$

(c) Transform examples of $CNOT_{4x4}$ Gates

Figure 9. Icon of quantum gates.

(a) Single qubit gates. X- gate sometimes is displayed by an icon "⊕".

(b) Two-qubit gates. CZ is Controlled-Z gate.

(c) Transform example of $CNOT_{4x4}$ Gate

5.2 Quantum Entanglement

We are here looking for the relationship between entanglement and Coxeter groups. That's why Entanglement is very important centre of attention [40-42] and is one of the most striking phenomena of quantum information theory because it has been the core of many applications in quantum computing [43], quantum cryptography [44], quantum teleportation, as well as philosophically oriented discussions concerning quantum theory.

Mathematical description of entangled state is straightforward. If a vector state $|\psi\rangle$ is not a product vector state $|\psi_A\rangle\otimes|\psi_B\rangle$ it is called entangled since they are linked and correlated with each other). As an example Bell's singlet state- *Bell state can be defined as the maximally entangled state of two qubits* - $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is entangled because it cannot be written as a direct product of two states. For an entangled state we introduce a density operator related to entropy measure of the state, which is defined by [42]

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{43}$$

where p_i is probability density. The density operator representing a pure state is $Tr(\rho^2) = 1$, otherwise it is a mixed state. A consistent method to quantify the entanglement of pure states is provided by the entropy measure of entanglement,

$$E(\psi) = S(\rho_A) = S(\rho_B) \tag{44}$$

where

$$S(\rho_i) = Tr(\rho_i)\log_2 \rho_i = -\sum_i \lambda_i \log_2 |\lambda_i| \tag{45}$$

where $S(\rho)$ is the von Neumann entropy and λ_i represent eigenvalues of the reduced density matrix ρ_i. For separable state entropy measure $S(\rho) = 0$. Note that the value of E changes from 0 to 1 where 0 corresponds separable (unentangled) state and 1 corresponds to maximally entangled state. In order to obtain the entropy measure of the entanglement we define the density operator ρ_A as

$$\rho_A = Tr_B(|\psi_i\rangle\langle\psi_i|) = Tr_B \sum_{i,j} |ij\rangle\langle ij| = \sum_{i,j} |ij\rangle Tr\langle ji| = \sum_{i,j} |i\rangle\langle i|\langle j|j\rangle \tag{46}$$

This describes probability of the locally measured state A. Implication of the partial trace operation is that local measurements of quantum systems A and B that cannot give any information about preparation of the state.

5.3 Coxeter group A_4

The Coxeter diagram of A_4 is depicted in the figure.

Figure 10. Coxeter diagram of A_4

The group can be generated by the reflection matrices obtained by the method given in previous sections 4.4., 4.6 respectively. These matrices are:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (47)$$

and generate a group of order $(4+1)! = 120$. It is obvious that the generators are not unitary. Koca proposed a method in order to define root system of the diagram by quaternions, such that

$$\alpha_1 = -1; \alpha_2 = \frac{1}{2}(1 + e_1 + e_2 + e_3);\ \alpha_3 = -e_1; \alpha_4 = \frac{1}{2}(e_1 - \sigma e_2 - \tau e_3) \quad (48)$$

where $\sigma = \frac{1}{2}\left(1 - \sqrt{5}\right)$ *and* $\tau = \frac{1}{2}\left(1 + \sqrt{5}\right)$. This representation can be done by the matrix:

$$T = \frac{1}{2}\begin{pmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & -\sigma & -\tau \end{pmatrix} \quad (49)$$

Similarity transformation of the reflection matrices by T, such that $T^{-1}RT$, where R is reflection matrix, gives that:

$$r_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, r_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix},$$

(50)

$$r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, r_4 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & \sigma & \tau \\ 0 & \sigma & \tau & 1 \\ 0 & \tau & 1 & \sigma \end{pmatrix}$$

These generators acting on the 2 qubit system,

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle. \tag{51}$$

The qubits can be represented by column matrices

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix};\ |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\ ;\ |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix};\ |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{52}$$

As an example then action of the generators on the state reads as follows:

$$r_2|00\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle) \tag{53}$$

This produces an entangled state. Because the result cannot be written in the form:

$$\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle) = (a_1|0\rangle + a_2|1\rangle)(b_1|0\rangle + b_2|1\rangle) \tag{54}$$

Action of $r_1\ and\ r_2$ on two qubits cannot produce an entangled state. The last reflection matrix is different from the known matrices of quantum gates.

For this group we also presented an alternative transformation on unitary bases by using the transformation matrix:

$$T = \frac{1}{2}\begin{pmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & -1+i & -1-i \\ 0 & -1+i & 1-i & 0 \end{pmatrix} \tag{55}$$

we obtain

$$r_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, r_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

$$r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad r_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{56}$$

with this transformation, first two matrices remain the same. The matrices $r_3 and\, r_4$ represent matrix realizations of the CONTROLLED-Y gate and SWAP gate.

5.4 Coxeter group $\boldsymbol{B_4}$

The Coxeter diagram of B_4 is depicted in the figure.

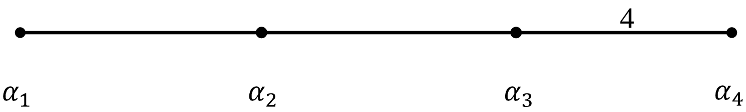


Figure 11. Coxeter diagram of B_4.

The group can be generated by the reflection matrices obtained by the method given in previous sections 4.4., 4.6 and 5.3 respectively. These reflection matrices are:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \sqrt{2} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (57)$$

and generate a group of order $2^4 4! = 384$. In order to obtain unitary representations of the generators of the group we define the transformation matrix, obtained from quaternionic representation of the group. This matrix is given by

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \quad (58)$$

Similarity transformation of the reflection matrices by T, such that $T^{-1}RT$, where R is reflection matrix, gives that:

$$r_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(59)

$$r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, r_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

These generators acting on the 2 qubit system, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. It is obvious that the generators corresponding to matrix representations of the well known quantum gates. Last 3 matrices are represented by SWAP, CNOT and CONTROLLED-Z gates.

5.5 Coxeter group F_4

The Coxeter diagram of F_4 is depicted in the figure.

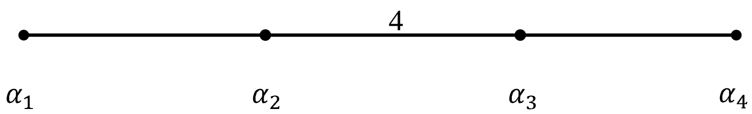


Figure 12. Coxeter diagram of F_4

The group can be generated by the reflection matrices obtained by the method given in previous sections 4.4., 4.6 and 5.3 respectively. These matrices are:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (60)$$

and generate a group of order 1152. In order to obtain unitary representations of the generators of the group we define the transformation matrix, obtained from quaternionic representation of the group. This matrix is given by

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \quad (61)$$

Similarity transformation of the reflection matrices by T, such that $T^{-1}RT$, where R is reflection matrix, gives that:

$$r_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, r_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{62}$$

These generators acting on the 2 qubit system, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. It is obvious that the generators corresponding to matrix representations of the well known quantum gates. Last 3 matrices are represented by SWAP, CNOT and CONTROLLED-Z gates again.

5.6 Coxeter group H_4

The Coxeter diagram of H_4 is given in the figure.

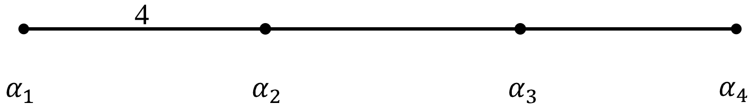


Figure 13. Coxeter diagram of H_4

The group can be generated by the reflection matrices obtained by the method given in previous sections 4.4., 4.6 and 5.3 respectively. These matrices are:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ \tau & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \tau & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (63)$$

and generate a group of order 14400. In order to obtain unitary representations of the generators of the group we define the transformation matrix, obtained from quaternionic representation of the group. This matrix is given by

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & \frac{\tau}{2} & \frac{1}{2} & \frac{\sigma}{2} \\ 0 & 0 & -1 & 0 \\ \frac{\sigma}{2} & 0 & \frac{1}{2} & \frac{\tau}{2} \end{pmatrix} \quad (64)$$

Similarity transformation of the reflection matrices by T, such that $T^{-1}RT$, where R is reflection matrix, gives that:

$$r_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, r_2 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \sigma & -\tau & 1 \\ 0 & -\tau & 1 & -\sigma \\ 0 & 1 & \sigma & \tau \end{pmatrix},$$

$$r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, r_4 = \frac{1}{2}\begin{pmatrix} \tau & 0 & \sigma & 1 \\ 0 & 2 & 0 & 0 \\ -\sigma & 0 & 1 & -\tau \\ 1 & 0 & -\tau & \sigma \end{pmatrix} \quad (65)$$

The $r_1 and\, r_3$ acting on the 2 qubit-system, $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ generators corresponding to matrix representations of the well known quantum gates however $r_2 and\, r_4$ are not well known representations. They must be analyzed with deeper steps to get the best generator representations.

5.7 Coxeter group D_4

The Coxeter diagram of D_4 is depicted in the figure.

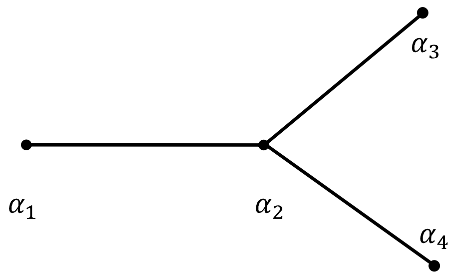


Figure 14. Coxeter diagram of D_4

The group can be generated by the reflection matrices obtained by the method given in previous sections 4.4., 4.6 and 5.3 respectively. These matrices are:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (66)$$

and generate a group of order $2^3 4! = 192$. In order to obtain unitary representations of the generators of the group we define the transformation matrix, obtained from quaternionic representation of the group. This matrix is given by

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \quad (67)$$

Similarity transformation of the reflection matrices by T, such that $T^{-1}RT$, where R is reflection matrix, gives that:

$$r_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$r_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, r_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{68}$$

These generators acting on the 2-qubit systems, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. It is obvious that the generators corresponding to matrix representations of the well known quantum gates. The matrices are represented by SWAP, CONTROLLED gates. Until here the study is about obtaining the unitary representations of Coxeter-Weyl groups A_4, B_4, D_4, F_4, and H_4. In the short term, most of generators of these groups, obtained by us, are useful for 2-qubit systems in quantum information theory as unitary representations. Now we want to analyze the behavior of A_8, B_8, D_8 and E_8 if it is possible to use their generators acting on 3-qubit systems.

5.8 Coxeter group A_8

The Coxeter diagram of A_8 is depicted in the figure.

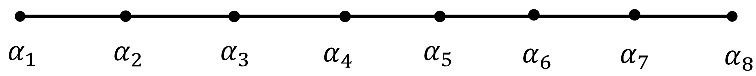


Figure 15. Coxeter diagram of A_8

The group can be generated by 8 reflection matrices. They are not given here. The reflection matrices generate a group of order $(8+1)! = 362880$. In order to obtain unitary representations of the generators of the group we define the transformation matrix, obtained from quaternionic representation of the group. This matrix is given by

$$T = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \end{pmatrix} \tag{69}$$

Similarity transformation of the reflection matrices by T, such that $T^{-1}RT$, where R is reflection matrix, gives that:

$$r_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$r_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, r_4 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$r_5 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, r_6 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$r_7 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}, r_8 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \tag{70}$$

These generators acting on the 3 qubit system,

$$|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |01, |111\rangle \tag{71}$$

As seen $r_1, r_2\ r_3 and\ r_7$ are the applicable generators which can be expressed in terms of universal quantum gates. The rest generators have for the first step less systematic analogy with qubit systems.

5.9 Coxeter group B_8

The Coxeter diagram of B_8 is depicted in the figure.

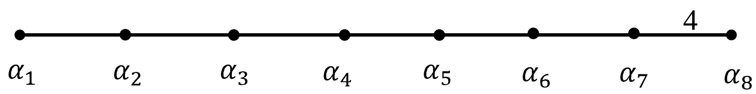


Figure 16. Coxeter diagram of B_8

The group can be generated by 8 reflection matrices. They are not given here. The matrices generate a group of order $2^8 8! = 10321920$. In order to obtain unitary representations of the generators of the group we define the transformation matrix, obtained from quaternionic representation of the group. This matrix is given by

$$T = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & \sqrt{2} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{72}$$

Similarity transformation of the reflection matrices by T, such that $T^{-1}RT$, where R is reflection matrix, gives that:

$$\begin{gathered}
r_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
r_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, r_4 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \\
r_5 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, r_6 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \\
r_7 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}, r_8 = \frac{1}{4}\begin{pmatrix} 3 & 0 & 0 & -1 & 2 & -1 & 0 & 1 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & 2 & -1 & 0 & 1 \\ 2 & 0 & 0 & 2 & 0 & 2 & 0 & -2 \\ -1 & 0 & 0 & -1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 1 & -2 & 1 & 0 & 3 \end{pmatrix}
\end{gathered} \tag{73}$$

These generators acting on the 3 qubit system,

$$|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle \tag{74}$$

Here is also similar case like in section 5.8. The generators as $r_1, r_2\ r_3 and\ r_7$ are the applicable generators expressing in terms of universal quantum gates. The rest generators have less systematic analogy with qubit systems for the first step.

5.10 Coxeter group D_8

The Coxeter diagram of D_8 is depicted in the figure.

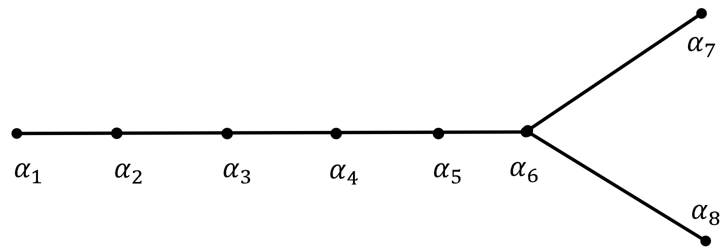


Figure 17. Coxeter diagram of D_8

The group can be generated by 8 reflection matrices. They are not given here. The matrices generate a group of order $2^{8-1}8! = 5160960$. In order to obtain unitary representations of the generators of the group we define the transformation matrix, obtained from quaternionic representation of the group. This matrix is given by

$$T = \frac{1}{2}\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \tag{75}$$

Similarity transformation of the reflection matrices by T, such that $T^{-1}RT$, where R is reflection matrix, gives that:

$$r_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$r_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, r_4 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$r_5 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, r_6 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$(76)$$

$$r_7 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}, r_8 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

These generators acting on the 3 qubit system,

$$|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle \qquad (77)$$

Here we can also say the same opinion like in A_8 Coxeter group generators.

5.11 Coxeter group E_8

The Coxeter diagram of E_8 is depicted in the figure.

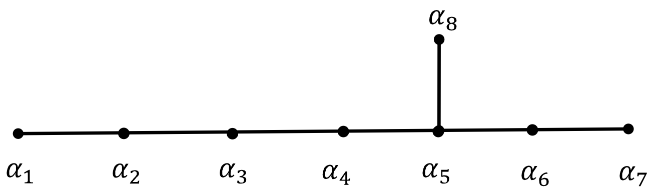


Figure 18. Coxeter diagram of E_8

The group can be generated by 8 reflection matrices. They are not given here. The matrices generate a group of order 696729600. In order to obtain unitary representations of the generators of the group we define the transformation matrix, obtained from quaternionic representation of the group. This matrix is given by

$$T = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 \end{pmatrix} \tag{78}$$

Similarity transformation of the reflection matrices by T, such that $T^{-1}RT$, where R is reflection matrix, gives that:

$$r_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, r_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$r_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, r_4 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$r_5 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, r_6 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (79)$$

$$r_7 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}, r_8 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

These generators acting on the 3 qubit system,

$$|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, 110\rangle, |111\rangle \quad (80)$$

Until now we looked for a possible applying way of the Coxeter-Weyl groups A_8, B_8, D_8 and E_8 to the 3-qubit systems with the aid of their generators to get a clear picture of well known quantum gates. With a panoramic view we can state that because of the obtained not unitary matrices of A_8, B_8, D_8 and E_8 Coxeter-Weyl groups we used the transformation matrices, origin from Koca-method (quaternionic approximation). Even if the transformation matrices of those groups are rather different, amazing thing is, the generators $r_1, r_2, r_3, r_4, r_5, r_6$ $and\, r_7$ from each of Coxeter groups are identical. But the last r_8 generators of those groups show us different characteristics. If one has a look at individual generator of whole 8x8 groups then state that

- The r_1 and r_2 are clearly related to the quantum gates.
- May be subgroups of r_3 and r_7 can be analyzed after decomposed to get a open connection door to the entanglement world.

The behavior of Coxeter-Weyl groups A_4, B_4 , $D_{4,}$ F_4 and H_4 (rank 4) acting on 2-qubit systems is rather clear than the A_8, B_8, D_8 and E_8 groups (rank 8) acting on 3-qubit system. They have directed binding to the maximal and non maximal entangled gates due to our calculations that means the relations to the prominent gates are visible. But for the A_8, B_8, D_8 and E_8 groups even though those found generator values it is needed further more complex detailed study.

CHAPTER 6

CONCLUSION

From the beginning it was already in mind that symmetry study (group theory) and quantum computer (quantum information and quantum computation) have absolutely led the way to a strong mutual relation. The paths, which we followed, coming through the principles of group theory (e.g. reverse processes at operational logic gates, Quantum Fourier transformation etc.) have clearly indicated that the unitary matrix representation of the groups explains these interrelations in a pretty mess. The quaternionic transformation of the usual reflection matrices are the basis for these representations which is of course having exact connections to the various universal quantum gates and entangled states produced by unitary matrix representation so that the construction of any of quantum circuit is perfect.

As seen from the calculations given in chapter 4 partially and in chapter 5 it is proven that the Coxeter-Weyl groups have an important role and are very useful in two- and three-qubit quantum information systems in quantum information theory. It is shown that these groups associated with maximal and minimal entangler gates. It is also important to note that the generators of these groups are related with the prominent quantum gates, such as CNOT, SWAP etc. and produce Bell states. Constructing unitary matrix representations of the groups, based on quaternionic transformation of the usual reflection matrices is performed with the aid of methods of this study. The von Neumann entropy of each reduced density matrix is calculated. It is shown that these unitary matrix representations are naturally related to various universal quantum gates and they produce entangled states. The canonical decomposition of generators in terms of fundamental gate representations is given to construct a quantum circuit.

As a result mastering the quantum world will take a bit duration while there will be many complex approximations to build the perfect quantum computer. In near future nobody expects a personal quantum computer. Because the duty of quantum computer is widely useful in the area of searching, calculating the complex higher order variables in biology, weather casting, medicine rather than individual tasks. At the same time the scientists are trying to get clear position for the near future about the probabilistic view of quantum information and quantum computation (e.g. with 2014 the scientists from china and Europe-Austria are going to try to send and to receive the entangled information through the space with the aid of satellites). That means the storage, coding, transmitting, hiding, protecting and evaluating you can use and practice in steps through all principles of quantum mechanics world and magical mathematical instrumental side – one of them is group theory study in these areas-. The dream that we want to see the world as the web of information is nonstop aim. However it seems hard but not difficult to write the historical story one step forward about civilization as before the mankind has already what achieved.

APPENDIX

Description of Problem:

Coxeter groups have many applications in different branches of physics, including molecular physics, particle physics, crystallography. Study of connection between Coxeter groups and quantum information theory has great importance. In order to obtain relation between Coxeter groups and quantum information theory, it is worth to construct unitary representation of the Coxeter group generators to transform qubits of the quantum system.

Programming Method:

The program is implemented by using Mathematica . Current features of the program are:

- ✧ Sketch of Coxeter - Dynkin diagrams.
- ✧ Calculation of Cartan and reflection matrices based on Coxeter-Dynkin diagram.
- ✧ After defining roots of the corresponding diagram interms of quaternions or octonions, reflection matrices are transformed into unitary basis.
- ✧ The program also generates Weyl (Reflection) groups.

APPENDIX A

Construction of Coxeter - Dynkin Diagrams, Cartan and Reflection Matrices, For E8

```
nvalues=8;
diagname=E1nvalues;
nodenum=nvalues;
nodes=Table[{i,i+1}→3,{i,1,nvalues-2}];
addnodes={{5,8}→3};
addcoor={{5,0},{5,1}};
nodeconnection=Table[If[i==j,{i,j}→0,
     {i,j}→2],{i,1,nodenum-
Length[addnodes]},{j,i,nodenum-
Length[addnodes]}]//Flatten;
Do[

nodeconnection=Cases[nodeconnection,Except[nodes[[i,1]]→
2]],{i,1,Length[nodes]}];
nodeconnection=Join[nodeconnection,nodes]//Flatten;
cdiagram=Sort[Cases[Table[
     If[nodeconnection[[i,2]]==
0||nodeconnection[[i,2]]==2,sα,


nodeconnection[[i,1,1]]•-•nodeconnection[[i,1,2]]],{i,1,Le

ngth[nodeconnection]}],Except[sα]]];
diffmvalues=Cases[Table[
    If[nodeconnection[[i,2]]==
0||nodeconnection[[i,2]]==2||nodeconnection[[i,2]]==3,sα,


nodeconnection[[i,1,1]]•-•nodeconnection[[i,1,2]]→nodeconn
```

```
ection[[i,2]],],{i,1,Length[nodeconnection]}],Except[sα]]
;
addnodeconnection=Table[If[i==j,{i,j}→0,
     {i,j}→2],{i,1,nodenum},{j,i,nodenum}]//Flatten;
Do[

addnodeconnection=Cases[addnodeconnection,Except[addnodes
[[i,1]]→2]],{i,1,Length[addnodes]}];
addnodeconnection=Join[addnodeconnection,addnodes]//Flatt
en;
cdiagadd=Cases[Table[
    If[addnodeconnection[[i,2]]==
0||addnodeconnection[[i,2]]==2,sα,addnodeconnection[[i,1,
1]]•-•addnodeconnection[[i,1,2]]],{i,1,Length[addnodeconne

ction]}],Except[sα]];
diffmvaluesadd=Cases[Table[
    If[addnodeconnection[[i,2]]==
0||addnodeconnection[[i,2]]==
2||addnodeconnection[[i,2]]==3,sα,

addnodeconnection[[i,1,1]]•-•addnodeconnection[[i,1,2]]→ad

dnodeconnection[[i,2]]],{i,1,Length[addnodeconnection]}],
Except[sα]];
Print[Text[Style["Coxeter Dynkin Diagram of
",Purple,Bold,12]], diagname];
If[cdiagadd=={},

Show[Graph[cdiagram,EdgeLabels→diffmvalues,VertexStyle→R
ed,VertexSize→0.1,VertexLabels→Table[i→Placed[αi,Below],
{i,1,nodenum-Length[addnodes]}],
   VertexCoordinates→Table[{i,0},{i,nodenum-
Length[addnodes]}]]],

Show[Graph[cdiagram,EdgeLabels→diffmvalues,VertexStyle→R
```

```
ed,VertexSize→0.1,VertexLabels→Table[i→Placed[αi,Below],
{i,1,nodenum-Length[addnodes]}],
   VertexCoordinates→Table[{i,0},{i,nodenum-
Length[addnodes]}]],

Graph[cdiagadd,EdgeLabels→diffmvaluesadd,VertexStyle→Red
,VertexSize→0.1,VertexLabels→Table[i→Placed[αi,Above],{i
,nodenum-Length[addnodes]+1,nodenum}],
   VertexCoordinates→addcoor]]]
```

APPENDIX B

Determination of Cartan and Reflection Matrices

```
cmat=Join[nodeconnection,addnodeconnection];cmat=Table[If
[cmat[[i,1]][[1]]==cmat[[i,1]][[2]],cmat[[i,1]]→2,cmat[[i
,1]]→-2Cos[Pi/cmat[[i,2]]]],{i,1,Length[cmat]}];
cartmat=Table[{i,j},{i,1,nodenum},{j,1,nodenum}];
Do[cartmat[[j,i]]=cartmat[[i,j]],{i,1,nodenum},{j,1,noden
um}];
cartmat=(cartmat/.cmat);
Print[Text[Style["Cartan Matrix of
",Purple,Bold,12]],diagname, "  is constructed"]
Print[Text[Style["Type cartmat to print Cartan
Matrix",Purple,Bold,12]]]
salp=Table[αi,{i,1,nodenum}];
somega=Table[ωi,{i,1,nodenum}];
salpomega=Table[salp[[i]]→(cartmat.somega)[[i]],{i,1,nod
enum}]
reflist=Table[ωj-KroneckerDelta[i,j]αi,
    {i,1,nodenum},{j,1,nodenum}]/.salpomega;
Do[r[k]=Table[Coefficient[reflist[[k]][[i]],ωj],{j,1,node
num},
    {i,1,nodenum}],{k,1,nodenum}];
Print[Text[Style["Reflection Matrices r[k] of
",Blue,Bold,12]], diagname,
 Text[Style["  is constructed",Blue,Bold,12]]]
Print[Text[Style["Type r[k] to print kth matrix
",Purple,Blue,12]]]
Do[gnr[i]=r[i],{i,1,nvalues}];numgen=nvalues;numvec=nvalu
es;
avec=Table[a_i,{i,1,numvec}]/.{a_1→1,a_2→0,a_3→0,a_4→0,a_5→0,a_6→0
,a_7→0,a_8→0};
```

```
groupset={avec};newset={};generated=groupset;
While[True,If[Length[generated]== 0,Break[]];

newset=Partition[Table[Expand[gnr[i].generated[[j]]],{i,1
,numgen},{j,1,Length[generated]}]//Flatten,numvec];genera
ted=Union[Complement[newset,groupset]];
  groupset=Union[Join[groupset,newset]]];
Print[ diagname,Text[Style["   Vector set of order
",Purple,Bold,12]] ,
  Length[groupset],Text[Style[" is generated
",Purple,Bold,12]]];
```

APPENDIX C

Unitary Transformation

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$$\mathrm{Do}[\mathrm{rp}[i] = \mathrm{Inverse}[T].r[i].T, \{i, 1, \mathrm{nvalues}\}]$$

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PERSONAL INFORMATION

Name and Surname	:	Mehmet Yakup HACIİBRAHİMOĞLU
Nationality	:	german
Birth place and date	:	Afşin, 1972
Phone number	:	+49 6062 910 410 / +90 507 92 39 522
Email	:	emailtoyakup@gmail.com

EDUCATION

		Graduate School	**Year**
Master Sc.	:	Eng. of Physics / Gaziantep University	1997
Bachelor Sc.	:	Eng. of Physics / Gaziantep University	1995
High School	:	Kahramanmaraş Lisesi	1988

WORK EXPERIENCE

	Place	**Enrollment**
2010-Present	Gaziantep-Türkiye	Ph.D.Student
2009-2010	Dieburg-Darmstadt-Germany	Teacher
2004-2009	Erbach-Dieburg-Germany	Ph.D.Student /Teacher

PUBLICATIONS

From this study: *Coxeter Groups A_4, B_4 and D_4 for two-qubit systems* (2013) published in Pranama- Journal of Physics

FOREIGN LANGUAGE

German, English, Turkish,
[French and Arabic (basic levels each) **]**