

**ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE**  
**ENGINEERING AND TECHNOLOGY**

**LARGE DEFLECTION AND ELASTO-PLASTIC ANALYSIS OF MINDLIN  
PLATES WITH MIXED FEM**

**Ph.D. THESIS**

**Akif KUTLU**

**Department of Civil Engineering**

**Structural Engineering Programme**

**JUNE 2013**



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**İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ**

**MINDLIN PLAKLARININ KARIŞIK SONLU ELEMANLAR YÖNTEMİ İLE  
BÜYÜK ÇÖKME VE ELASTO-PLASTİK ANALİZİ**

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*To the memories of Muhibe (Kutlu) Sirt,  
Mehmet Kutlu and Nihat Kutlu,*



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## **ABBREVIATIONS**

|             |                                        |
|-------------|----------------------------------------|
| <b>CPT</b>  | : Classical Plate Theory               |
| <b>FE</b>   | : Finite Element                       |
| <b>FSDT</b> | : First-order Shear Deformation Theory |
| <b>HR</b>   | : Hellinger–Reissner                   |



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# **LARGE DEFLECTION AND ELASTO–PLASTIC ANALYSIS OF MINDLIN PLATES WITH MIXED FEM**

## **SUMMARY**

Understanding the behavior of structural elements under external loads is an important task for the engineering design process. According to the general opinion finite element methods are confirmed as the most suitable tool for engineering analysis. Development of robust, reliable and efficient elements has always been a desirable objective for researchers.

Plate elements are commonly used structural elements in civil, marine, mechanical and aeronautics engineering. The behavior of these flat elements has been investigated by many researchers in many aspects.

This dissertation is concerned with the large deflection and elasto–plastic analysis of Mindlin plates in the framework of a two field mixed variational formulation. By adopting the formulation, innovative solution strategies are proposed for mixed finite element procedure with monolithic approach. The idea can be outlined as follows. Finite element procedures based on mixed formulations can be concluded by two main approaches. The most common approach performs a static condensation procedure to eliminate stress type field variables from finite element equations and obtains the final equations only in terms of displacements. This study deviates from this approach by preserving the stress type field variables during solution procedure as well. As a result of this idea stresses or stress resultants are obtained at the nodes of finite elements directly. In the plate analysis stresses and strains at a node through the thickness can be calculated from stress resultants with some matrix operations. This procedure makes all the state variables to be continuously distributed over the plate domain.

In this study large deflection bending responses based on von Karman kinematics implemented to Mindlin's first order shear deformation assumptions are applied to laminated composite plates. Variational principle of Hellinger–Reissner is used to obtain the first variation of functional of the problem. Non–linear finite element equations are linearized by means of the incremental formulation. Elasto–plastic analysis of isotropic Mindlin plates were conducted with layered approach in order to detect the spread of the plasticity through the plate thickness. A 3–D full implicit algorithm (backward Euler) is employed for the numerical integration of constitutive equations.  $C^0$  quadrilateral elements are generated according to a  $2 \times 2$  Gauss integration scheme. Numerical problem is solved with Newton–Raphson iteration method. For both geometrically and materially non–linear problems, the proposed solution strategies are verified by solving the problems existing in the literature. It is observed that the presented formulation can predict the stresses in laminated plates very accurately and simulate the elasto–plastic behavior of plates successfully.



## MINDLIN PLAĞINDA KARIŞIK SONLU ELEMANLAR YÖNTEMİ İLE BÜYÜK ÇÖKME VE ELASTO-PLASTİK EĞİLME ANALİZİ

### ÖZET

Yapısal elemanların dış yük altındaki davranışlarının belirlenmesi mühendislik tasarım sürecinde önemli bir adımdır. Genel kanaat, özellikle kesin sonucun bulunmasında ciddi zorlukların olması durumunda sonlu elemanlar yönteminin mühendislik analizleri için en uygun çözüm tarzı olduğudur. Bu sebeple kararlı, güvenilir ve kabiliyetli sonlu eleman çözüm yöntemleri geliştirmek birçok araştırmacı için önemli bir hedef olmuştur.

Plak sistemler, inşaat, makina, gemi, uzay mühendisliği gibi yapısal sistemlerle uğraşan mühendislik alanları için sıklıkla tercih edilen taşıyıcı elemanlardır. Bu sebeple, plak elemanların mekanik etkiler altındaki davranışları bir çok açıdan ele alınmış ve üzerinde araştırmalar yapılmıştır.

Bu tez kapsamında, Mindlin plaklarının büyük çökme davranışı ve elasto-plastik analizi iki alanlı bir karışık sonlu eleman formülasyonu çerçevesinde ele alınmıştır. Formülasyon, tarif edilen problemlere uygulanırken aynı zamanda bazı yenilikçi çözüm stratejileri önerilmiştir. Karışık (mixed) yapıdaki sonlu eleman formülasyonlarında genel olarak iki çözüm yaklaşımı izlenebilir. En çok karşılaşılan yaklaşım, sistem veya eleman denklemleri üzerinde bir statik kondensasyon gerçekleştirerek gerilme tipindeki bölge değişkenlerini sistem denklemlerinden uzaklaştırmak ve yalnızca yerdeğiştirmelere bağlı olarak sonlu eleman denklemlerini elde etmektir. Bu sayede elde edilen denklemler mevcut bir yerdeğiştirme tabanlı sonlu eleman programına kolaylıkla monte edilebilmekte ve çözüm süreci devam ettirilebilmektedir. Karışık sonlu eleman çözümlerinde diğer bir yaklaşım ise, bu çalışmada sunulan yekpare (monolithic) karışık sonlu eleman çözüm yaklaşımı olup, gerilme tipindeki bölge değişkenleri tüm çözüm boyunda denklemler içinde yer almaktadır. Bu düşünce ile hareket edildiğinde gerilme tipindeki bölge değişkenleri sonlu eleman düğüm noktalarında doğrudan elde edilebilmekte, ve bunlar üzerinden şekil değiştirmeler uygun matris operasyonları ile hesaplanabilmektedir. Tercih edilen bu yaklaşım sayesinde, şekil değiştirme gibi normalde yerdeğiştirme alanının gradyanı olarak hesaplanan bölge değişkenlerinin belirlenmesinde sayısal türev alma gerekliliği ortadan kalkmakta ve olası sayısal hatalardan da kaçınılmaktadır. Yerdeğiştirme ve gerilme tipindeki bölge değişkenlerinin düğüm noktalarında doğrudan elde edilmesinin çözüme en önemli katkısı bu değişkenlerin incelenen plak ortamında süreklilik gösterecek biçimde hesaplanabiliyor olmalarıdır. Bunun doğal sonucu olarak gerilme bileşenlerinden üretilen şekil değiştirmeler de çözüm bölgesi boyunca sürekli olur.

Bu çalışma kapsamında tabakalı kompozit Mindlin plaklarının büyük çökme altında statik davranışı incelenmiştir. Plak bölgesinde şekil değiştirme alanı von Kármán kinematik ilişkileri üstünden yazılmıştır. Laminat plak çözümü için ortotrop tabakaların birleştirilmesiyle oluşturulan plak kesiti düşünülmüş, ve üç boyutlu

problem eşdeğer tek tabaka (equivalent single layer) yaklaşımı kullanılarak iki boyutlu plak alan denklemleri ile hareket edilmiştir. Doğrusal olmayan denge denklemleri virtüel iş prensibi kullanılarak türetilmiştir. Hellinger–Reissner prensibi hakkında bilgi verilmiş ve ilgili fonksiyonelin birinci varyasyonu bu prensibe bağlı olarak elde edilmiştir. Doğrusal olmayan sonlu eleman denklemleri artımsal formülasyon kullanılarak doğrusallaştırılmıştır. Böylece teğet sistem matrisi ve iç kuvvet matrisi açık yapıda elde edilmiştir. Tezin içinde bunlara ait tüm alt matrisler detaylı olarak sunulmuşlardır. Elde edilen fonksiyonelde bölge değişkenlerinin en yüksek birinci derece türevleri bulunduğu için şekil fonksiyonlarında  $C^0$  sürekliliğinin bulunması yeterli olmaktadır. Bu sebeple sonlu eleman formülasyonunda basit doğrusal interpolasyon fonksiyonları kullanılabilmiştir. Sonlu eleman matrisleri oluşturulurken dörtgen izoparametrik elemanlar üzerinde  $2 \times 2$  Gauss şeması kullanılarak sayısal integraller hesaplanmıştır. Doğrusal olmayan denklem takımı Newton–Raphson ardışık yaklaşım algoritması kullanılarak çözülmüştür.

Plak düzlemi içindeki gerilmeler kompiyans matrisleri kullanılarak doğrudan iç kuvvetler cinsinden hesaplanmıştır. Düzlem dışı kayma gerilmeleri eşdeğer kesit yöntemi kullanarak hesaplandığı için kesitte sabit kayma açısı varsayımının getirdiği olumsuzluk ortadan kaldırılmıştır. Daha sonra bünye bağıntıları üstünden gerilmeler kullanılarak şekil değiştirmeler elde edilmiştir.

Mindlin plağının geometrik doğrusal elasto–plastik analizinde tabakalı kesit yaklaşımı takip edilmiş ve kesitte plastisitenin yayılışı incelenebilmiştir. Elasto–plastik plak eğilme analizinde kompozit plak için geliştirilen denklem ve program rutinlerinin geometrik doğrusal terim içeren parçaları kullanılmıştır. Elasto–plastik bünye denklemlerinin sayısal integrasyonu ve bir sonlu eleman çözümüne nasıl dahil edileceği ayrıntılı olarak anlatılmıştır. Mevcut çalışmayla karışık sonlu eleman formülasyonuna uygun yeni bir çözüm algoritması önerilmiş olup, yaklaşımın en belirgin üstünlüğü gerilme ve şekil değiştirme tipindeki büyüklüklerin artımları Gauss noktaları yerine düğüm noktalarında doğrudan veya bünye denklemleriyle hesaplanıyor olmasıdır. Elasto–plastik bünye denklemleri Gauss noktaları yerine düğüm noktalarında integre edilmektedir. Böylece hem daha az noktada bünye denklemleri sağlatılmakta hem de alan denklemlerinin yanısıra malzeme özellikleri bakımından da çözüm bölgesi boyunca bir süreklilik elde edilebilmektedir.

İzotrop tabakalardan oluştuğu düşünülen plakta elasto–plastik davranışı yansıtmak üzere üç boyutlu gerilme durumunu gözetken bir algoritma ile bünye denklemleri kapalı olarak (implicit) integre edilmiştir. Yük artımı ve Newton–Raphon ardışık yaklaşım şeması ile doğrusal olmayan problemin çözümü tamamlanmıştır.

Sunulan formülasyon ve önerilen çözüm yöntemi için FORTRAN programlama dili kullanılarak bir sonlu eleman programı geliştirilmiş, ve çeşitli çözümler yapılmıştır. Büyük çökme problemi için ilk doğrulama örnekleri izotrop dikdörtgen ve daire geometrilik plaklar seçilmiş ve karşılaştırma çözümleriyle çok yakın sonuçlar elde edilmiştir. Daha sonra ortotrop ve tabakalı kompozit plaklarda da karşılaştırma örnekleri çözülmüş ve gerilme hesaplarının doğruluğu da araştırılmıştır. Plak eleman iki boyutlu olarak modellenmesine rağmen, karışık sonlu eleman formülasyonunun sağladığı avantajlar sayesinde üç boyutlu gerilme hali oldukça gerçekçi olarak elde edilmiş ve gerilme sonuçları kesin veya analitik sonuçlara çok yakın olarak bulunabilmiştir. Elasto–plastik örnekler literatürdeki dikdörtgen plak çözümleri ile

doğrulanmıştır. Bu çözümlerde hem ideal elasto–plastik hem de pekleşen elasto–plastik örnekleri üzerinde durulmuştur. Literatürdeki örneklerde çok fazla üzerinde durulmayan yük adım sayısının ve tabaka sayısının elasto–plastik davranış üzerindeki etkisi detaylı olarak irdelenmiştir. Önerilen çözüm yöntemi ile stratejinin ve üretilen sonlu eleman programının doğrusal olmayan plak problemlerinde güvenle kullanılabileceği düşünülmektedir.



## **1. INTRODUCTION**

This dissertation is about geometrically and materially non-linear Mindlin plates and solution algorithm with mixed finite element formulation. Geometrical non-linearity accounting large deflection assumption is combined with the orthotropic composite laminates. Material non-linearity is handled in the sense of distributed elasto-plasticity by means of the layered approach. A finite element computer program is developed using FORTRAN codes. Verification examples are solved and some precise analyses are performed.

### **1.1 Overview**

Plates are very common structural elements in mechanical, aerospace, civil, nuclear and marine engineering [1]. An accurate and economical design of plate structures requires a appropriate solution procedure. Closed form solutions exist only for some special cases of external loading, boundary conditions and geometries even for linear analysis [2]. Most of the actual engineering problems have to concern with challenging problems beyond the linear limits.

Many numerical formulations have been developed to overcome the difficulties met in scientific and engineering problems. Finite element methods have been the primary actor in the field of engineering applications where mechanical problems mostly faced in a non-linear manner. Several finite element solution procedures have been formulated in order to serve in the direction of the requirements of the engineers.

Mixed finite element formulations have been successfully applied to various types of engineering problems [3–7]. Non-linear behavior of beam/column-framed structures in particular has been intensively investigated with mixed formulations in recent decades [8–13]. Advantages/disadvantages of displacement type and mixed type finite elements, on the basis of variational formulations, are still under discussion and the main subject of many researches [14–16].

## **1.2 Literature Survey**

Among the other structural elements, plate structures have also attracted many researchers and have been the primary topic of many studies [17–19]. Related to the subject of the current thesis, the vast literature was narrowed and summarized in two main titles in the following section. A detailed survey was conducted on the topics and it has been presented in a chronological order in order to reflect the improvements and new aspects of the research area.

### **1.2.1 Large deflection analysis of laminated composite plates**

Although many of the geometrically non-linear plate problems were solved by numerical solution strategies, it was possible to solve some problems analytically covering limited configurations. The first closed form solution of the large deflection bending problem of laminated plates was provided by Pao [20] for simple bending conditions. Bennett [21] discussed the effect of bending–stretching coupling terms on the large amplitude oscillation response of simply supported angle ply laminated plates and concluded that the Berger [22] approximation does not yield a remarkable simplification for the non-linear problem. The problem of large amplitude free vibration for thin unsymmetrically laminated plates with elastic in-plane restraint and fully clamped edges was attacked by Bert [23]. Chia [24] presented an approximate solution for the large deflection bending problem of the rectangular laminated plates having clamped edge conditions and subjected to uniform loading. In an early study related to the large deflection bending behavior of symmetrically laminated plates Zaghoul and Kennedy [25] gave a finite difference solution with results from numerical method and experimental observations. The paper of Prabhakara and Chia [26] dealt with unsymmetrically laminated rectangular plates exposed to combined transverse and in-plane loads by means of finite deflection analysis. Chandra and Basava Raju [27] evaluated free flexural vibrations of cross-ply laminated plates in large amplitude regime depending on an assumed mode shape. Large deflection bending results of unsymmetric thin laminated composite plates were given by Chia and Prabhakara [28] formulating a perturbation technique for the non-linear problem. Chandra [29] solved the non-linear flexural vibration of cross-ply laminated plates by applying Galerkin's method where the amplitude of vibration was employed as the perturbation parameter. Large deflection bending

analysis of unsymmetric angle-ply simply supported rectangular plates subjected to edge bending moment loading was undertaken by Prabhakara [30]. Chia and Prabhakara [31] proceeded an analytical investigation of large amplitude free vibration of laminated thin rectangular plates where harmonic balance method was applied by the use of double Fourier series. The very famous book of Chia [32] is concerned with the large deflection, post-buckling and non-linear vibration of thin elastic plates where classical plate theory for laminates were also covered. Reddy and Chao [33] extended the plate finite element in [34] by adopting von Kármán strains to the deformation field of the first order laminated composite plate theory. Large deflection bending and large amplitude free vibration characteristics of laminated composite plates were computed by Reddy and Chao [35] using a linear and quadratic rectangular plate element considering the first order shear deformation theory. Circular plates made of two materials (glass-steel or glass-aluminium) were investigated by Turvey [36] using the dynamic relaxation method in the large deflection regime. Chia [37] discussed the effect of elastically restrained edges on the large deflection behavior of laminated composite thin plates by utilizing an analytical approach. A mixed finite element formulation was applied by Putcha and Reddy [38] to study large deflection behavior of laminated composite plates with strain field obtained from a refined higher order theory assumption. Gorji [39] treated non-linear terms of large deflection equations as a set of equivalent loads and in-plane forces to analyze bending of shear deformable laminated composite plates. Singh and Sadasiva Rao [40] computed the large deflection response of laminated composite plates with a displacement based finite element procedure where the full terms of Green strain beyond von Kármán strains were included in the non-linear analysis. They stated that taking the additional terms into consideration makes an influence on deflections around 13% for moderately thick plates. Chia [41] summarized the basics of elastic laminated composite plate's geometrically non-linear behavior from the aspects of large deflection bending, postbuckling, non-linear vibration etc. A broad range of literature related to the topics mentioned above was presented in the publication. Stiffened composite structures were modeled by Liao and Reddy [42] using a continuum based shell element for the investigation of their geometrically non-linear responses. An improved version of generalized laminated plate theory accounting for the large deflection kinematics was applied by Barbero and Reddy [43] to develop its associated finite element formulation. Large deflection bending and large amplitude

transient analysis of a higher order laminated composite plate was concerned by Chang and Huang [44] with a finite element solution procedure. Singh et al. [45] observed the influence of the additional non-linear terms related to the bending–stretching coupling on the large deflection responses of laminated composite plates. Linear and geometrically non-linear analysis of laminated composite plates was carried out by Kant and Kommineni [46] using a  $C^0$  continuity finite element and adopting a higher order shear deformation theory. Singh et al. [47] predicted the displacements and stresses in a laminated composite plate exhibiting large deflection with a four noded finite element based on a higher order theory. Savithri and Varadan [48] gave results for the large deflection bending of laminated composite plates according to a higher order shear deformation theory by using the Galerkin method. Carrera and Villani [49] employed finite element method for the large deflection and postbuckling analysis of moderate thick laminated composite plates. Argyris and Tenek [50] implemented natural mode method to investigate the linear and non-linear bending behavior of laminated composite plates. Singh et al. [51] reported a four-noded rectangular plate finite element having  $C^1$  continuity and 14 degrees of freedom per node for the bending analysis of laminated composite plates based on a higher order shear theory. Chattopadhyay et al. [52] improved their isoparametric first order shear deformable stiffened plate finite element [53] to handle the large deflection problem. For the geometrically non-linear analysis of plates and shell elements, Liu and Surana [54] reported a p-version 3-D finite element where displacements were interpolated in a piecewise hierarchical manner. Ganapathi et al. [55] derived a  $C^0$  membrane–plate quadrilateral finite element to investigate geometrically non-linear static, dynamic and buckling responses of laminated plates obeying Reissner–Mindlin plate theory. Large deflection of orthotropic sandwich composite plates was investigated by Riber [56] through an analytical approach. Liu et al. [57] obtained an analytical solution for large deflection problem of symmetrically laminated rectilinearly orthotropic clamped elliptical plates under uniform pressure by considering transverse shear. Non-linear behavior of a broad family of structural elements such as beams, frames, trusses, plates, shells and composites was investigated by Sathyamoorthy [58] in a theoretical and practical sense. A huge literature on the topic can be found at the end of each related chapter of this book. Günay and Erdem [59] improved the famous Heterosis element to get better performance for the linear and geometrically non-linear analysis of laminated

plates. Higher order shear deformation theory of laminated plates is incorporated with finite strip method by Akhras et al. [60] for the large deflection analysis of composite plates. Shen [61] conducted a large deflection analysis for thin plates resting on Pasternak type elastic foundation and subjected to lateral pressure and in-plane edge loading combinations by using a perturbation technique. A perturbation finite element approach was performed by Cheng et al. [62] to investigate the large deflection deformations of laminated composite plates with a discrete-layer shear deformation theory incorporating Mindlin assumptions in each layer. A mixed variational approach for the large deflection analysis of laminated plates was formulated by Fares [63] where a refined first order shear theory was adopted. Tan et al. [64] suggested a new geometrically non-linear laminated plate theory where normal stress and strain distribution can be evaluated. Shukla and Nath [65] derived an analytical solution for the large deflection bending of laminated moderately thick laminated composite plates configured by various boundary conditions. Yang and Zhang [66] examined the large deflection and postbuckling behavior of geometrically imperfect laminated rectangular thin plates exposed to combined transverse and in-plane loadings and interacting with elastic foundation by a semi-analytical approach where a perturbation technique, differential quadrature approximation and Galerkin method were employed. Large deflection bending behavior of thin laminated rectangular plates resting on non-linear elastic foundation was investigated by Renganathan et al. [67] using Galerkin method. Sita Thankam et al. [68] employed a coupled displacement field covering transverse shear to obtain a shear locking free rectangular laminated plate finite element exhibiting large deflection. Tanrıöver and Şenocak [69] conducted a Galerkin solution procedure for the large deflection analyses of unsymmetrically laminated composite plates in thin plates regime. Liew et al. [70] used the kernel particle approximation to derive a mesh-free Ritz method to analyze large deflection bending of laminated composite first-order shear deformable plates. Zhang and Kim [71] formulated two quadrilateral displacement based laminated plate finite elements depending on the von Kármán large deflection kinematics and on the first-order shear deformation theory. By adopting Timoshenko's laminated composite beam functions Zhang and Yang [72] developed a family of shear locking free triangular and quadrilateral flat plate/shell finite elements capable to analyze linear and geometrically non-linear static problems based on first order shear deformation theory. Shufrin et al. [73]

proposed a semi-analytical solution for the large deflection analysis of plates depending on the classical plate theory and subjected to a general class of load distribution under various in-plane and out-of-plane boundary conditions. The paper of Dash and Singh [74] is concerned with the development of a  $C^0$  isoparametric finite element for the geometrically non-linear analysis of laminated composite plates having full non-linear terms of Green-Lagrange strains in the framework of a higher order shear deformation theory. Baltacıoğlu et al. [75] adopted discrete singular convolution method to solve non-linear bending equations of a first order shear deformable laminated composite plate. In a recent study, laminated anisotropic plates governed by higher order shear deformation theory and von Kármán kinematics were investigated through a meshfree method by Singh and Shukla [76]. Very recently, an effort was also made by Kapoor and Kapania [77] to deal with NURBS finite element formulation for laminated composite plates exhibiting large deflection bending and obeying to first order shear deformation assumptions.

### **1.2.2 Elasto-plastic analysis of plates**

Structures can show high level performance due to plastic design philosophy and the structural system can benefit from extreme loading conditions when compared to an elastically designed system. Fundamentals of the plastic design process of structures can be found in the classical book of Hodge [78] in an elementary form.

Application and development of appropriate numerical solution algorithms are very important in solid mechanics, and sometimes they are unavoidable, especially if geometric and/or materially non-linearity exists in the problem. This is also the case for the elasto-plastic analysis of plates. Early attempts (e.g. [79,80,81]) regarding plastic analysis of plates used to concern with the upper and lower boundary limits of plates. Tekinalp [82] analyzed elasto-plastic built-in circular plates subjected to uniform loading. Tresca criterion was adopted in this plastic analysis and bending moments and deflections of the incompressible plate were investigated. Freiburger and Tekinalp [83] dealt with the minimum weight design of the thin circular homogeneous and sandwich plates. A limit analysis is conducted in that study where the plate material was obeying von Mises's criterion and the plate was subjected to a rotationally symmetric loading. Adaptation of computer usage to scientific and engineering analysis made it possible to handle more detailed and complex analyses.

A hybrid finite element approximation is used by Barnard and Sharman [84] to obtain elasto-plastic responses of plates considering spread of plasticity through the thickness. A plastic hinge theory with isotropic hardening was employed by Eggers and Kröplin [85] to analyze thin plates according to a mixed finite element formulation. Large deformation of elasto-plastic plates are covered in the scope of the study. Elastic-viscoplastic response of Mindlin plates were modeled by Dinis and Owen [86] with finite element formulation where stress resultant yield criterion was implemented. Some efforts were also made by Kuo-Kuang [87] to model the inelastic behavior of automobile components with a geometrically and materially non-linear plate finite element formulation. Time dependent plastic deflection of thin plates were investigated by Morjaria and Mukherjee [88] depending on the combination of boundary and finite element methodologies. A detailed treatment of elasto-plastic analysis of structures can be found at Owen and Hinton [89]. Beyond theoretical explanations, detailed finite element formulations were presented with corresponding computer program subroutines in open form. General geometric and material non-linear problems covering plate and shell elements were solved by Bathe and Bolourchi [90] in a comprehensive way. The semiloof thin shell element was extended to formulate geometrically and materially non-linear behavior by Martins and Owen [91]. A semiloof finite element was formulated by Dinis and Owen [92] to incorporate large deflection and elasto-viscoplastic analysis for thin plates and shells in terms of generalized stresses. A semiloof finite element solution was performed by Owen and Figueiras [93] with the combination of stress resultant anisotropic plasticity and thin shell theory. In addition to plate and shell type structures, they also analyzed aerofoil wind turbine blades to show the applicability of the numerical approach. Owen and Figueiras [94] treated the elasto-plastic bending behavior of orthotropic laminated shells with a layered plasticity approach. Reddy and Mitchell [95] have applied an extended kinematic minimum principle for the finite element approximation of Mindlin plates to study elasto-plastic behavior with the reflection of the spread of plasticity through the plate thickness. Club sandwich elements were employed by Wempner and Chao-Meng [96] to investigate the propagation of yielding through the thickness using elastic perfectly-plastic model where constant stress resultants were assumed over the plate elements. Moshaiov and Vorus [97] reported a boundary element method for elasto-plastic analysis of thin plates via the initial plastic moments. A boundary element formulation was presented by Song and

Mukherjee [98] where elasto–viscoplastic analyses were conducted for Kirchhoff plates containing cutouts and having various boundary conditions. Hu–Washizu variational principle was incorporated to stress resultant plasticity by Papadopoulos and Taylor [99] for the elasto–plastic bending analysis of Reissner–Mindlin plates. Daye and Toridis [100] developed a mixed finite element solution procedure based on Hellinger–Reissner to estimate static and dynamic elasto–plastic responses of shear deformable plate and shallow shell structures considering geometrical non–linearity. In the procedure, stress resultants and displacements were defined at different locations (nodes) and the authors preferred monolithic solution strategy. They adopted stress resultant plasticity with tangent stiffness approach for static analysis and initial stiffness approach for dynamic analysis. Papadopoulos and Taylor [101] employed a mixed triangular shear deformable plate element based on Hu–Washizu variational principle to investigate the elasto–viscoplastic plates depending on the stress resultant plasticity formulation. Ibrahimbegović and Frey [102] examined stress resultant plasticity of shear deformable plates using a four noded plate finite element with assumed strains. They used a single scalar equation to update material state and solved perfectly plastic plates with different shapes and under point and uniform loadings. Auricchio and Taylor [103] implemented a new definition of stress resultant plasticity to the finite element formulation of elasto–plastic shear deformable plates to take into account the diffusion of material non–linearity through plate thickness. They compared the improved method with 3–D model results and classical elasto–plastic plate model results on some structures and reported the advantages and accuracy of the approach. Croce et al. [104] conducted a mixed finite element simulation for elasto–plastic plate analysis with first order shear deformation theory and stress resultant plasticity, but after some mathematical manipulations they used only translations and rotations at nodes as field variables. Belinha and Dinis [105] derived an element free Galerkin method to analyze elasto–plastic Reissner–Mindlin plates with linear hardening by considering the spread of plastic zone through thickness adopting layered approach. Pioneering application of boundary element method to combination of geometric and material non–linearity in shear deformable plates was performed by Supriyono and Aliabadi [106]. Supriyono and Aliabadi [107] obtained boundary element integrals of a shear deformable plate undergoing large deflection with stress resultant perfect plasticity. A boundary element formulation for elasto–plastic analysis of plates was proposed

by Waidemam and Venturini [108] considering geometrical non-linearity in the regime of large deflection with small strain. von Mises type plasticity with linear isotropic hardening in plane stress assumptions is adopted for thin plate formulation. Very recently a geometrically linear shear deformable shell boundary element was formulated by Dirgantara and Aliabadi [109] comprising stress resultant elasto-plasticity.

### 1.3 Objectives and Scope

The main objective of current dissertation is to develop and implement a new mixed finite element solution algorithm for the Mindlin plates having geometric and material non-linearities. For this purpose, large deflection bending of laminated composite plates and elasto-plastic behavior of isotropic Mindlin plates are investigated in the framework of a two field mixed variational formulation.

Conventional approach in mixed solution procedures requires performing a static condensation and evaluating finite element equations only with the related displacement terms. Then, a suitable displacement based finite element program can undertake the rest of the procedure. Unlike almost all its counterparts (by means of mixed variational formulations) in the literature, this study prefers the monolithic solution strategy where the stress type field variables are preserved during the whole numerical process beside displacement type field variables. This choice provides some conveniences and benefits in the numerical formulation from many aspects. Firstly, all the field variables (including displacement and stress type) are interpolated independently, which relaxes the necessity of satisfying further continuity conditions. Therefore, a  $C^0$  continuity is enough for the compatibility requirements of the interpolation functions. Stress type field variables at the finite element nodes (e.g. forces and force couples in a plate formulation) are obtained from direct solution of the problem. A continuous distribution of the stress type field variables is achieved over the plate continuum automatically. However, in the displacement type formulations, stress type variables are determined at the Gauss points and from the displacement field with an error prone postprocess where numerical derivation operations are performed. The stress type variables at nodal points are calculated from the average values of neighbouring Gauss points. In the scope of this study since strains and stresses are computed from stress resultants with

the use of compliance matrices, a continuous strain and stress fields are issued. Another advantage of the mixed formulation is that, it is shear lock free.

The content of this dissertation is organized into four chapters as described below.

Chapter 2 covers all the principles, mathematical formulations and numerical procedures of the study. Hellinger–Reissner principle and Mindlin plate assumptions are summarized in the beginning of the chapter. Large deflection field equations of moderately thick laminated composite plate is presented by means of Mindlin assumptions and von Kármán strains. Two field functionals are obtained according to Hellinger–Reissner principle. Non-linear finite element equations are linearized with incremental formulation. Iterative solution procedures with the details of finite element matrices are given next. Calculation of stresses and strains are explained in detail. Basics of elasto-plastic analysis are recalled. A 3-D full implicit integration algorithm for elasto-plastic constitutive equations is implemented. An innovative mixed formulation and solution procedure for elasto-plastic analysis is proposed in this chapter. The method is adapted to the plate problem considering the spread of plasticity by employing the layered approach. A material model with linear or non-linear hardening and obeying von Mises yield criterion is considered.

Chapter 3 collects all the solved problems and presents all the obtained numerical results gathered from the application of the proposed formulations. Isotropic rectangular and circular plate's analyses are conducted for large deflection analysis and results are compared with existing studies. Large deflection responses of composite plates and the accuracy of the stresses are examined next. Elasto-plastic plate analysis is conducted for both perfectly plastic and linear hardening cases.

Concluding remarks are summarized in the Chapter 4 and directions for future research are addressed.

## 2. FORMULATION

### 2.1 Plates

Plates are flat structural elements bounded with two parallel surfaces. The distance between those two mentioned surfaces is called as the thickness ( $h$ ) of the plate. A plane structural member must have a considerably small thickness compared to other two dimensions to be mechanically modeled as a plate element. Plates are defined by their middle plane, which divides the thickness of the member at every point. Middle surface, which is flat (plane) before deformation, becomes a curved shape after lateral loading which is called as middle surface of the plate (Figure 2.1).

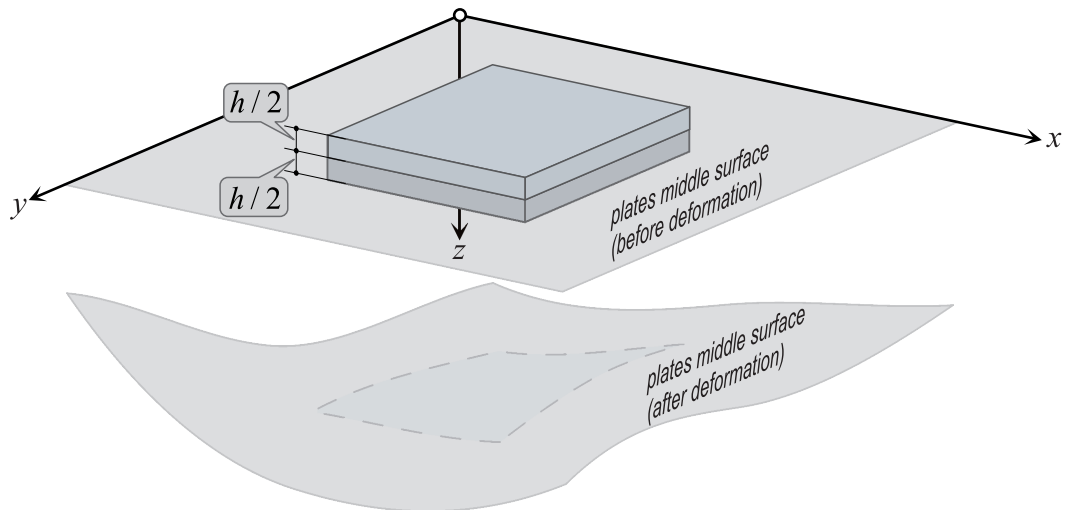


**Figure 2.1 :** Plates middle surface and definition of global coordinates.

#### 2.1.1 Classical plate theory

Classical plate theory (CPT) deals with thin plates and generally known as Kirchhoff or thin plate theory. Although there is not a certain restriction a plate is defined as thin if the plates thickness to other characteristic dimension of plate ratio ( $h/L$ ) is less than the value 0.1 [1]. The assumptions according to model are listed below.

- 1) Straight cross sections perpendicular to middle plane before deformation remain straight after deformation.
- 2) No deformation occurs along the thickness of the plate.

- 3) Plane cross sections perpendicular to middle plane before deformation remain perpendicular to middle surface after deformation thus transverse shear deformation is zero.

### 2.1.2 Moderately thick plate theory

Moderately thick plate theory is also known as first-order shear deformation plate theory (FSDT) or Mindlin plate theory. The only difference from classical plate theories assumptions is that plane cross sections do not remain perpendicular to middle surface after deformation [110]. Normal of the section deviates from middle surface with a constant and independent rotation angle (Figure 2.2).

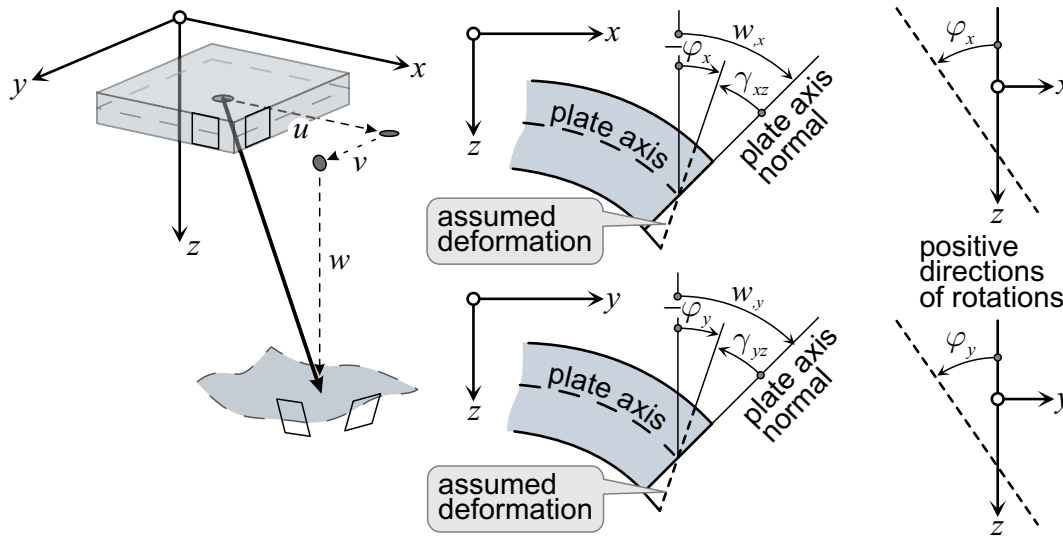


**Figure 2.2 :** Displacements and rotations in Mindlin plate theory.

## 2.2 Hellinger–Reissner Mixed Principle

A two field functional of a mechanical system can be obtained by using Hellinger–Reissner principle. Stresses and displacement are independent field variables of the Hellinger–Reissner functional [111,112]. Definition of Hellinger–Reissner functional requires the fulfillment of two basic assumptions [113].

- a) External loads must be conservative
- b) Material behavior must be hyperelastic

The general idea in energy principles states the variational principle as,

$$\delta \Pi = \delta (\Pi_{\text{int}} + \Pi_{\text{ext}}) = 0 \quad (2.1)$$

where the total strain energy  $\Pi_{\text{int}}$  is calculated as follows,

$$\Pi_{\text{int}} = \int_V W(\varepsilon) dV \quad (2.2)$$

Here,  $W(\varepsilon)$  is the strain energy density (see Figure 2.3 (a)) and  $V$  corresponds the volume of the undeformed geometry of the system.

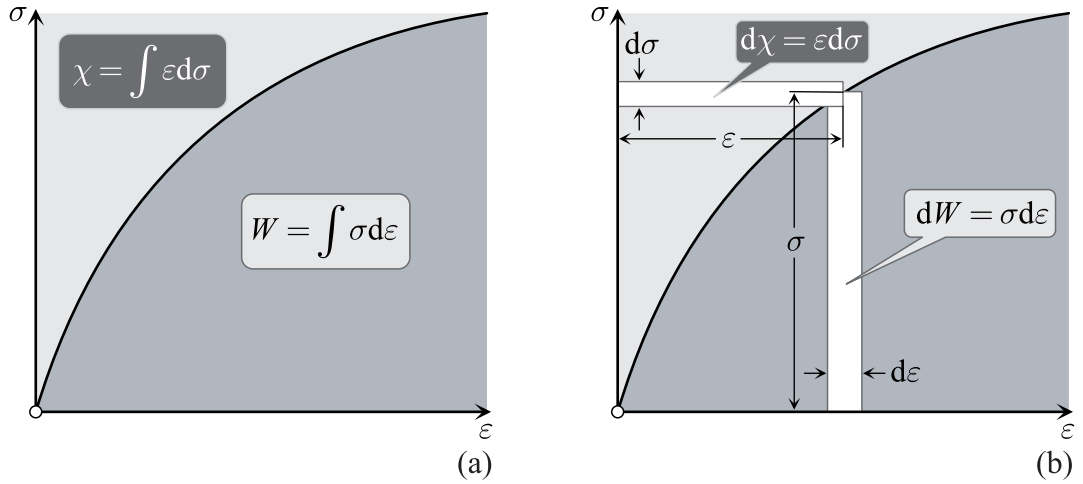


**Figure 2.3 :** Energy density definitions: (a) Strain and complementary energy. (b) Differentials of strain and complementary energy.

Stresses and strains are related to each other according to strain energy equation [114],

$$\sigma = \frac{\partial W(\varepsilon)}{\partial \varepsilon} \quad (2.3)$$

Similar approach can be followed to obtain the relation between strain and stress from complementary energy density ( $\chi(\sigma)$ , see Figure 2.3).

$$\varepsilon = \frac{\partial \chi(\sigma)}{\partial \sigma} \quad (2.4)$$

Hellinger–Reissner principle defines the total strain energy with the help of complementary energy as follows (Figure 2.3),

$$\Pi_{\text{int}} = \int_V W dV = \int_V (\sigma \varepsilon^u - \chi(\sigma)) dV \quad (2.5)$$

Here the superscript  $u$  symbolizes that  $\varepsilon^u$  is defined in terms of displacement components. Taking the first variation of the energy potential, gives the stationarity of the Hellinger–Reissner functional as,

$$\delta \Pi_{HR} = \delta (\Pi_{\text{int}}(u, s) + \Pi_{\text{ext}}(u)) = 0 \quad (2.6)$$

According to definition total strain energy is a function of both displacements ( $u$ ) and stresses ( $s$ ), whereas external energy  $\Pi_{\text{ext}}$  is just a function of displacements ( $u$ ) only. Extending the terms in (2.6) yields,

$$\delta \Pi_{HR} = \frac{\partial \Pi_{\text{int}}(u, s)}{\partial u} \delta u + \frac{\partial \Pi_{\text{int}}(u, s)}{\partial s} \delta s + \frac{\partial \Pi_{\text{ext}}(u)}{\partial u} \delta u = 0 \quad (2.7)$$

Using the terms defined in (2.5) yields,

$$\delta \Pi_{HR} = \int_V \left( \sigma \delta \varepsilon^u + \delta \sigma \varepsilon^u - \frac{\partial \chi(\sigma)}{\partial \sigma} \delta \sigma \right) dV + \delta \Pi_{\text{ext}}(u) = 0 \quad (2.8)$$

After generalizing with matrix notation variation of external energy may be expressed as,

$$\delta \Pi_{\text{ext}} = \int_V (-\mathbf{q}^T \delta \mathbf{u}) dV - \int_{\Gamma} \hat{\mathbf{t}}^T \delta \mathbf{u} d\Gamma = 0 \quad (2.9)$$

Here  $\mathbf{q}$  is the external load and  $\hat{\mathbf{t}}$  is the traction vector at the boundary ( $\Gamma$ ) of the plate. Referring to the relation (2.4) and organizing (2.8) the first variation of Hellinger–Reissner functional is obtained as follows,

$$\begin{aligned} \delta \Pi_{HR} = & \int_V (\varepsilon^u - \varepsilon^\sigma)^T \delta \sigma^\sigma dV + \int_V \left( (\sigma^\sigma)^T \delta \varepsilon^u - \mathbf{q}^T \delta \mathbf{u} \right) dV \\ & - \int_{\Gamma} \hat{\mathbf{t}}^T \delta \mathbf{u} d\Gamma = 0 \end{aligned} \quad (2.10)$$

The expression (2.10) can be investigated mainly in two parts. The first part is represented by  $\delta_u \Pi_{HR}$  and is,

$$\delta_u \Pi_{HR} = \int_V \left( (\boldsymbol{\sigma}^s)^T \delta \boldsymbol{\varepsilon}^u - \mathbf{q}^T \delta \mathbf{u} \right) dV \quad (2.11)$$

The equation (2.11) refers to the Principle of Virtual Work, which is a description for the weak form of the equilibrium equations.

The second part of (2.10) is  $\delta_s \Pi_{HR}$ ;

$$\delta_s \Pi_{HR} = \int_V \left( \boldsymbol{\varepsilon}^u - \boldsymbol{\varepsilon}^s \right)^T \delta \boldsymbol{\sigma}^s dV \quad (2.12)$$

It can be concluded that (2.12) is the representation of the compatibility relations (strain–displacement) in weak form.

As stated before, the Hellinger–Reissner functional stands on two basic assumptions. However, obtained weak forms of the equilibrium equations, and compatibility conditions allow the principle to be used for other type of materials and loading cases [113].

### 2.3 Large Deflection Analysis of the Laminated Composite Mindlin Plate

If the plate's middle plane is symbolized by  $\Omega$  then the whole plate domain is described by  $\Omega \times (-h/2, h/2)$  where. Boundary of the plate at the level of middle plane is denoted by  $\Gamma$ .

Under the assumptions of Mindlin plate theory, displacement field of the plate is given as [115],

$$u^*(x, y, z) = u(x, y) + z\varphi_x \quad (2.13)$$

$$v^*(x, y, z) = v(x, y) + z\varphi_y \quad (2.14)$$

$$w^*(x, y, z) = w(x, y) \quad (2.15)$$

Where,  $u^*(x,y,z)$ ,  $v^*(x,y,z)$ ,  $w^*(x,y,z)$  are displacements of any point in plate domain and  $u(x,y)$ ,  $v(x,y)$ ,  $w(x,y)$  are displacements of any point in plates middle plane.  $\varphi_x$  and  $\varphi_y$  are the rotations of the cross sections having normal directions  $x$  and  $y$ , respectively (Figure 2.2).

### 2.3.1 Kinematic relations

Non-linear strains in a general body in terms of displacements are defined as,

$$E_{xx} = u_{,x}^* + \frac{1}{2} \left[ (u_{,x}^*)^2 + (v_{,x}^*)^2 + (w_{,x}^*)^2 \right] \quad (2.16)$$

$$E_{yy} = v_{,y}^* + \frac{1}{2} \left[ (u_{,y}^*)^2 + (v_{,y}^*)^2 + (w_{,y}^*)^2 \right] \quad (2.17)$$

$$E_{zz} = w_{,z}^* + \frac{1}{2} \left[ (u_{,z}^*)^2 + (v_{,z}^*)^2 + (w_{,z}^*)^2 \right] \quad (2.18)$$

$$E_{xy} = \frac{1}{2} (u_{,y}^* + v_{,x}^* + u_{,x}^* u_{,y}^* + v_{,x}^* v_{,y}^* + w_{,x}^* w_{,y}^*) \quad (2.19)$$

$$E_{xz} = \frac{1}{2} (u_{,z}^* + w_{,x}^* + u_{,x}^* u_{,z}^* + v_{,x}^* v_{,z}^* + w_{,x}^* w_{,z}^*) \quad (2.20)$$

$$E_{yz} = \frac{1}{2} (v_{,z}^* + w_{,y}^* + u_{,y}^* u_{,z}^* + v_{,y}^* v_{,z}^* + w_{,y}^* w_{,z}^*) \quad (2.21)$$

von Kármán deformation field is a special case of general non-linear description, where some terms eliminated from equations. In a plate structure deflections ( $w$ ) are relatively bigger than in-plane displacements. Therefore, the contribution of deflection related non-linear terms are considered in the formulation. Eliminating other non-linear terms yields von Kármán strains where the corresponding theory is widely called as large deflection. According to definition, von Kármán strains are,

$$\mathcal{E}_{xx}^* = u_{,x}^* + \frac{1}{2} (w_{,x}^*)^2 \quad (2.22)$$

$$\mathcal{E}_{yy}^* = v_{,y}^* + \frac{1}{2} (w_{,y}^*)^2 \quad (2.23)$$

$$\varepsilon_{zz}^* = \cancel{w_{,z}^*} + \frac{1}{2} \left( \cancel{w_{,z}^*}^2 \right) = 0 \quad (2.24)$$

$$\varepsilon_{xy}^* = \frac{1}{2} \left( u_{,y}^* + v_{,x}^* + w_{,x}^* w_{,y}^* \right) \quad (2.25)$$

$$\varepsilon_{xz}^* = \frac{1}{2} \left( u_{,z}^* + w_{,x}^* + \cancel{w_{,x}^* w_{,z}^*} \right) = \frac{1}{2} \left( u_{,z}^* + w_{,x}^* \right) \quad (2.26)$$

$$\varepsilon_{yz}^* = \frac{1}{2} \left( v_{,z}^* + w_{,y}^* + \cancel{w_{,y}^* w_{,z}^*} \right) = \frac{1}{2} \left( v_{,z}^* + w_{,y}^* \right) \quad (2.27)$$

### 2.3.1.1 von Kármán strains of Mindlin plate

In order to obtain von Kármán strains in Mindlin plate assumptions, displacement field in terms of mid-plane displacements of Mindlin plate ((2.13)–(2.15)) are inserted to equations (2.22)–(2.27),

$$\varepsilon_{xx}^* = [u + z\varphi_x]_{,x} + \frac{1}{2} (w_{,x})^2 = u_{,x} + \frac{1}{2} (w_{,x})^2 + z\varphi_{x,x} \quad (2.28)$$

$$\left. \begin{aligned} \varepsilon_{yy}^* &= [v + z\varphi_y]_{,y} + \frac{1}{2} (w_{,y})^2 \\ &= v_{,y} + \frac{1}{2} (w_{,y})^2 + z\varphi_{y,y} \end{aligned} \right\} \quad (2.29)$$

$$\varepsilon_{zz}^* = 0 \quad (2.30)$$

$$\left. \begin{aligned} \varepsilon_{xy}^* &= \frac{1}{2} \left( [u + z\varphi_x]_{,y} + [v + z\varphi_y]_{,x} + w_{,x} w_{,y} \right) \\ &= \frac{1}{2} \left( [u_{,y} + z\varphi_{x,y}] + [v_{,x} + z\varphi_{y,x}] + w_{,x} w_{,y} \right) \\ &= \frac{1}{2} \left[ u_{,y} + v_{,x} + w_{,x} w_{,y} + z(\varphi_{x,y} + \varphi_{y,x}) \right] \end{aligned} \right\} \quad (2.31)$$

$$\left. \begin{aligned} \varepsilon_{xz}^* &= \frac{1}{2} \left( [u + z\varphi_x]_{,z} + w_{,x} + \cancel{w_{,x} w_{,z}} \right) \\ &= \frac{1}{2} \left( \cancel{u_{,z}} + \varphi_x + w_{,x} \right) \\ &= \frac{1}{2} (w_{,x} + \varphi_x) \end{aligned} \right\} \quad (2.32)$$

$$\left. \begin{aligned} \varepsilon_{yz}^* &= \frac{1}{2} \left( \left[ v + z\varphi_y \right]_{,z} + w_{,y} + w_{,y}w_{,z} \right) \\ &= \frac{1}{2} \left( v_{,z} + \varphi_y + w_{,y} \right) \\ &= \frac{1}{2} \left( \cancel{v}_{,z} + \varphi_y + w_{,y} \right) \\ &= \frac{1}{2} \left( w_{,y} + \varphi_y \right) \end{aligned} \right\} \quad (2.33)$$

### 2.3.1.2 Engineering strains in vector form

For the convenient vector description engineering shear strains ( $\gamma_{ij}^*$ ) are employed instead of shear strains ( $\varepsilon_{ij}^*$ ,  $i \neq j$ ). Curvature terms ( $\kappa_{ij}$ ) for Mindlin plate can be defined as (according to (2.28)–(2.33)),

$$\left\{ \begin{array}{c} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \\ \kappa_{xz} \\ \kappa_{yz} \end{array} \right\} = \left\{ \begin{array}{c} \varphi_{x,x} \\ \varphi_{y,y} \\ \varphi_{x,y} + \varphi_{y,x} \\ 0 \\ 0 \end{array} \right\} \quad (2.34)$$

The strains of the mid-plane are obtained as,

$$\left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{array} \right\} = \left\{ \begin{array}{c} u_{,x} + \frac{1}{2}(w_{,x})^2 \\ v_{,y} + \frac{1}{2}(w_{,y})^2 \\ u_{,y} + v_{,x} + w_{,x}w_{,y} \\ w_{,x} + \varphi_x \\ w_{,y} + \varphi_y \end{array} \right\} \quad (2.35)$$

Then the strains in general plate domain can be described as a combination of mid-plane strains and plate curvatures as,

$$\left\{ \begin{array}{c} \varepsilon_{xx}^* \\ \varepsilon_{yy}^* \\ \gamma_{xy}^* \\ \gamma_{xz}^* \\ \gamma_{yz}^* \end{array} \right\} = \left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{array} \right\} + z \left\{ \begin{array}{c} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \\ \kappa_{xz} \\ \kappa_{yz} \end{array} \right\} \quad (2.36)$$

### 2.3.2 Constitutive relations

Constitutive relations of a composite Mindlin plate having layers made of orthotropic linear elastic materials are going to be presented. For orthotropic material principal axes (1,2) are thought to lie parallel to  $x, y$  plane of the plate's global coordinate system.

#### 2.3.2.1 Constitutive equations of a single layer

Stresses in an orthotropic material with principal axes (1,2) (Figure 2.4 (a)) are related to strains at that point according to the following constitutive relation,

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & k_s Q_{77} & 0 \\ 0 & 0 & 0 & 0 & k_s Q_{88} \end{bmatrix} \begin{Bmatrix} \epsilon_{11}^* \\ \epsilon_{22}^* \\ \gamma_{12}^* \\ \gamma_{13}^* \\ \gamma_{23}^* \end{Bmatrix} \quad (2.37)$$

The stiffness parameters of orthotropic material are calculated from engineering constants as  $Q_{11} = E_1 / (1 - \nu_{12}\nu_{21})$ ,  $Q_{22} = E_2 / (1 - \nu_{12}\nu_{21})$ ,  $Q_{12} = \nu_{12}E_2 / (1 - \nu_{12}\nu_{21})$  or  $Q_{12} = \nu_{21}E_1 / (1 - \nu_{12}\nu_{21})$ ,  $Q_{66} = G_{12}$ ,  $Q_{77} = G_{13}$ ,  $Q_{88} = G_{23}$ . Where  $E_1$  and  $E_2$  are elasticity modulus of orthotropic material in principle axes,  $G_{ij}$  and  $\nu_{ij}$  are shear modulus and Poisson's ratio respectively [116]. Shear correction factor  $k_s$  depends on many parameters but in this study, it is chosen as  $5/6$  constant.

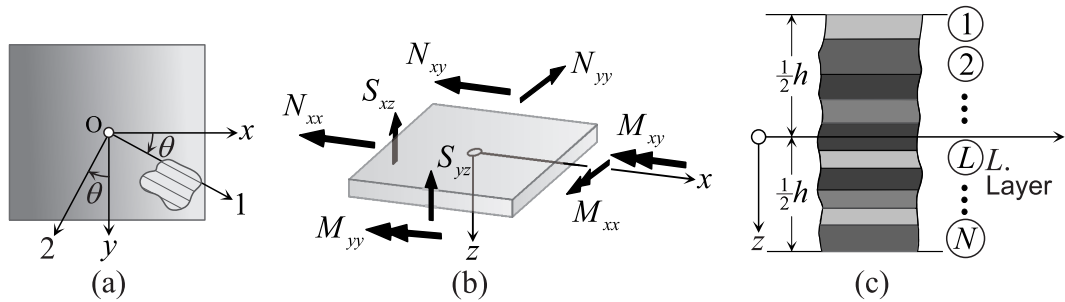


**Figure 2.4 :** (a)Plate and material coordinates. (b)Positive directions of stress resultants. (c)Lamination of the plate.

If there exists an angle  $\theta$  (Figure 2.4 (a)) between the material coordinates (1,2) and the plates global coordinates ( $x, y$ ), it is necessary to perform a transformation between the stresses and strain components. Before going further, some definitions

are made for the clarity. The stress vectors in plate global coordinates and in material principal directions will be symbolized as  $\{\sigma\}_p$  and  $\{\sigma\}_m$ , respectively. Similarly strains in plate global coordinates and material principal axes will be presented by  $\{\varepsilon^*\}_p$  and  $\{\varepsilon^*\}_m$ , respectively. According to these definitions (2.37) can be rewritten as,

$$\{\sigma\}_m = [Q] \{\varepsilon\}_m \quad (2.38)$$

Stresses in plates global coordinates are related to stresses in material axes with the following transformation matrix,

$$[T]_\sigma = \begin{bmatrix} c^2 & s^2 & 2sc & 0 & 0 \\ s^2 & c^2 & -2sc & 0 & 0 \\ -sc & sc & c^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & -s & c \end{bmatrix} \quad (2.39)$$

and the relation,

$$\{\sigma\}_m = [T]_\sigma \{\sigma\}_p \quad (2.40)$$

Here  $s$  and  $c$  symbolizes  $\sin\theta$  and  $\cos\theta$  respectively. Taking the inverse of transformation operator yields,

$$\{\sigma\}_p = [T]_\sigma^{-1} \{\sigma\}_m \quad (2.41)$$

which is in open form,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc & 0 & 0 \\ s^2 & c^2 & 2sc & 0 & 0 \\ sc & -sc & c^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & s & c \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} \quad (2.42)$$

A similar approach must be applied to the strain components as well. The strain transformation matrix,

$$[T]_{\varepsilon} = \begin{bmatrix} c^2 & s^2 & sc & 0 & 0 \\ s^2 & c^2 & -sc & 0 & 0 \\ -2sc & 2sc & c^2-s^2 & 0 & 0 \\ 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & -s & c \end{bmatrix} \quad (2.43)$$

is used to relate strains in material and plates global coordinate directions to each other,

$$\{\varepsilon^*\}_m = [T]_{\varepsilon} \{\varepsilon^*\}_p \quad (2.44)$$

The inverse relation similar to stress deformation is,

$$\{\varepsilon^*\}_p = [T]_{\varepsilon}^{-1} \{\varepsilon^*\}_m \quad (2.45)$$

which is in detail,

$$\begin{Bmatrix} \varepsilon_{xx}^* \\ \varepsilon_{yy}^* \\ \gamma_{xy}^* \\ \gamma_{xz}^* \\ \gamma_{yz}^* \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -sc & 0 & 0 \\ s^2 & c^2 & sc & 0 & 0 \\ 2sc & -2sc & c^2-s^2 & 0 & 0 \\ 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & s & c \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^* \\ \varepsilon_{22}^* \\ \gamma_{12}^* \\ \gamma_{13}^* \\ \gamma_{23}^* \end{Bmatrix} \quad (2.46)$$

At this point the aim is to relate the strains in global coordinates to the stresses in global coordinates for an orthotropic layer having an material orientation  $\theta$ . If  $\{\sigma\}_m$  from (2.38) is inserted to (2.41) one obtains,

$$\{\sigma\}_p = [T]_{\sigma}^{-1} [Q] \{\varepsilon^*\}_m \quad (2.47)$$

After inserting  $\{\varepsilon^*\}_m$  from (2.44) into (2.47) the final expression is obtained as,

$$\{\sigma\}_p = [T]_{\sigma}^{-1} [Q] [T]_{\varepsilon} \{\varepsilon^*\}_p \quad (2.48)$$

It can be easily recognized that  $[T]_{\varepsilon} = [T]_{\sigma}^{-T}$ . A further arrangement of above equation gives,

$$\{\sigma\}_p = [\bar{Q}]\{\varepsilon^*\}_p \quad (2.49)$$

with the definition of transformed stiffness matrix,

$$[\bar{Q}] = [T]_\sigma^{-1} [Q] [T]_\varepsilon \quad (2.50)$$

### 2.3.2.2 Multi-layer theory

In the case of a laminated Mindlin plate composed of  $N$  layers ( $L$ ), membrane forces ( $N_{xx}, N_{yy}, N_{xy}$ ), bending moments, ( $M_{xx}, M_{yy}$ ), torsional moment ( $M_{xy}$ ) and shear forces ( $S_{xz}, S_{yz}$ ) (Figure 2.4 (b)) are calculated through the integration of stresses and their moments through the thickness of the cross section of the plate [117]. For the plate structures the formulation is based on the mid-plane displacements and the stress resultants [118]. The development of the plate theory requires that the strains at a point to be defined in terms of mid-plane deformations (strains and curvatures) of the plate [119].

#### Calculation of membrane forces

Resultants of normal stresses at cross section give the membrane forces.

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz = \sum_{L=1}^N \int_{z_{L-1}}^{z_L} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}_L dz \quad (2.51)$$

where stress components can be written in terms of deformations for linear elastic material as in,

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \sum_{L=1}^N \int_{z_{L-1}}^{z_L} [\bar{Q}]_L \left\{ \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \right\} dz \quad (2.52)$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \sum_{L=1}^N [\bar{Q}]_L \left\{ \int_{z_{L-1}}^{z_L} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} dz + \int_{z_{L-1}}^{z_L} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} z dz \right\} \quad (2.53)$$

If we collect integral terms in matrices, one gets the relations,

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} + [B] \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \Rightarrow \left. \begin{aligned} A_{ij} &= \sum_{L=1}^N (\bar{Q}_{ij})_L (z_L - z_{L-1}) \\ B_{ij} &= \frac{1}{2} \sum_{L=1}^N (\bar{Q}_{ij})_L (z_L^2 - z_{L-1}^2) \end{aligned} \right\} \quad (2.54)$$

### Calculation of moments

Integrating moments of stress component through the cross section with respect to middle plane yields the moment components,

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z dz = \sum_{L=1}^N \int_{z_{L-1}}^{z_L} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}_L z dz \quad (2.55)$$

where stress components can be written in terms of deformations as in,

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \sum_{L=1}^N \int_{z_{L-1}}^{z_L} [\bar{Q}]_L \left\{ \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \right\} z dz \quad (2.56)$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \sum_{L=1}^N [\bar{Q}]_L \left\{ \int_{z_{L-1}}^{z_L} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} z dz + \int_{z_{L-1}}^{z_L} \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} z^2 dz \right\} \quad (2.57)$$

If we collect integral terms in matrices, one gets the relations,

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = [B] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} + [D] \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \Rightarrow \left. \begin{aligned} B_{ij} &= \frac{1}{2} \sum_{L=1}^N (\bar{Q}_{ij})_L (z_L^2 - z_{L-1}^2) \\ D_{ij} &= \frac{1}{3} \sum_{L=1}^N (\bar{Q}_{ij})_L (z_L^3 - z_{L-1}^3) \end{aligned} \right\} \quad (2.58)$$

### Calculation of transverse forces

Integrating stress components, which are in transverse directions through the cross section, yields the transverse forces,

$$\begin{Bmatrix} S_{xz} \\ S_{yz} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz = \sum_{L=1}^N \int_{z_{L-1}}^{z_L} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}_L dz \quad (2.59)$$

where stress components can be written in terms of deformations as in,

$$\begin{Bmatrix} S_{xz} \\ S_{yz} \end{Bmatrix} = \sum_{L=1}^N \int_{z_{L-1}}^{z_L} [\bar{Q}]_L \left\{ \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xz} \\ \kappa_{yz} \end{Bmatrix} \right\} dz \quad (2.60)$$

$$\begin{Bmatrix} S_{xz} \\ S_{yz} \end{Bmatrix} = \sum_{L=1}^N [\bar{Q}]_L \int_{z_{L-1}}^{z_L} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} z dz \quad (2.61)$$

Here  $\kappa_{xz}$  and  $\kappa_{yz}$  are zero terms in plate formulation. If we collect integral terms in matrices, one gets the relations,

$$\begin{Bmatrix} S_{xz} \\ S_{yz} \end{Bmatrix} = [C] \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \Rightarrow C_{ij} = \sum_{L=1}^N (\bar{Q}_{ij})_L k_s (z_L - z_{L-1}) \quad (2.62)$$

### 2.3.2.3 Displacement–internal force relations

Constitutive equations obtained in previous section can be written in matrix form as,

$$\{N\} = [A]\{\varepsilon\} + [B]\{\kappa\} \quad (2.63)$$

$$\{M\} = [B]\{\varepsilon\} + [D]\{\kappa\} \quad (2.64)$$

Multiplying (2.63) with  $[A]^{-1}$  yields,

$$[A]^{-1}\{N\} = [A]^{-1}[A]\{\varepsilon\} + [A]^{-1}[B]\{\kappa\} \quad (2.65)$$

$$\{\varepsilon\} = [A]^{-1}\{N\} - [A]^{-1}[B]\{\kappa\} \quad (2.66)$$

Inserting  $\{\varepsilon\}$  into (2.64) results with,

$$\{M\} = [B]\left([A]^{-1}\{N\} - [A]^{-1}[B]\{\kappa\}\right) + [D]\{\kappa\} \quad (2.67)$$

$$\{M\} = [B][A]^{-1}\{N\} + ([D] - [B][A]^{-1}[B])\{\kappa\} \quad (2.68)$$

It is now possible to obtain the curvature as,

$$\{\kappa\} = ([D] - [B][A]^{-1}[B])^{-1}\{M\} - ([D] - [B][A]^{-1}[B])^{-1}[B][A]^{-1}\{N\} \quad (2.69)$$

After inserting curvature into (2.66) one obtains,

$$\{\varepsilon\} = [A]^{-1}\{N\} - [A]^{-1}[B] \begin{pmatrix} ([D] - [B][A]^{-1}[B])^{-1}\{M\} \\ -([D] - [B][A]^{-1}[B])^{-1}[B][A]^{-1}\{N\} \end{pmatrix} \quad (2.70)$$

$$\begin{aligned} \{\varepsilon\} = & ([A]^{-1} + [A]^{-1}[B]([D] - [B][A]^{-1}[B])^{-1}[B][A]^{-1})\{N\} \\ & - [A]^{-1}[B]([D] - [B][A]^{-1}[B])^{-1}\{M\} \end{aligned} \quad (2.71)$$

With the definition of matrices

$$[A'] = [A]^{-1} + [A]^{-1}[B]([D] - [B][A]^{-1}[B])^{-1}[B][A]^{-1} \quad (2.72)$$

$$[B'] = -([D] - [B][A]^{-1}[B])^{-1}[B][A]^{-1} \quad (2.73)$$

$$[D'] = ([D] - [B][A]^{-1}[B])^{-1} \quad (2.74)$$

strains are calculated in compliance form as,

$$\{\varepsilon\} = [A']\{N\} + [B']\{M\} \quad (2.75)$$

$$\{\kappa\} = [B']\{N\} + [D']\{M\} \quad (2.76)$$

The compliance matrix is actually the inverse of the stiffness matrix, which is,

$$\begin{bmatrix} [A'] & [B'] \\ [B'] & [D'] \end{bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix}^{-1} \quad (2.77)$$

Similar approach can be easily applied for the transverse shear forces,

$$\{C\}=[C]\{\gamma\} \Rightarrow \{\gamma\}=[C]^{-1}\{N\} \quad (2.78)$$

$$\{\gamma\}=[C']\{S\} \Rightarrow [C']=[C]^{-1} \quad (2.79)$$

#### 2.3.2.4 Explicit form of constitutive relations

Using the kinematic relations and the constitutive equations described above, field variables of laminated composite Mindlin plate with von Kármán deformations are presented as follows,

$$\begin{aligned} \varepsilon_{xx} &= u_{,x} + \frac{1}{2}(w_{,x})^2 \\ &= A'_{11}N_{xx} + A'_{12}N_{yy} + A'_{13}N_{xy} + B'_{11}M_{xx} + B'_{12}M_{yy} + B'_{13}M_{xy} \end{aligned} \quad (2.80)$$

$$\begin{aligned} \varepsilon_{yy} &= v_{,y} + \frac{1}{2}(w_{,y})^2 \\ &= A'_{21}N_{xx} + A'_{22}N_{yy} + A'_{23}N_{xy} + B'_{21}M_{xx} + B'_{22}M_{yy} + B'_{23}M_{xy} \end{aligned} \quad (2.81)$$

$$\begin{aligned} \gamma_{xy} &= u_{,y} + v_{,x} + w_{,x}w_{,y} \\ &= A'_{31}N_{xx} + A'_{32}N_{yy} + A'_{33}N_{xy} + B'_{31}M_{xx} + B'_{32}M_{yy} + B'_{33}M_{xy} \end{aligned} \quad (2.82)$$

$$\begin{aligned} \kappa_{xx} &= \varphi_{x,x} \\ &= B'_{11}N_{xx} + B'_{12}N_{yy} + B'_{13}N_{xy} + D'_{11}M_{xx} + D'_{12}M_{yy} + D'_{13}M_{xy} \end{aligned} \quad (2.83)$$

$$\begin{aligned} \kappa_{yy} &= \varphi_{y,y} \\ &= B'_{21}N_{xx} + B'_{22}N_{yy} + B'_{23}N_{xy} + D'_{21}M_{xx} + D'_{22}M_{yy} + D'_{23}M_{xy} \end{aligned} \quad (2.84)$$

$$\begin{aligned} \kappa_{xy} &= \varphi_{x,y} + \varphi_{y,x} \\ &= B'_{31}N_{xx} + B'_{32}N_{yy} + B'_{33}N_{xy} + D'_{31}M_{xx} + D'_{32}M_{yy} + D'_{33}M_{xy} \end{aligned} \quad (2.85)$$

$$\gamma_{xz} = w_{,x} + \varphi_x = C'_{11}S_{xz} + C'_{12}S_{yz} \quad (2.86)$$

$$\gamma_{yx} = w_{,y} + \varphi_y = C'_{21}S_{xz} + C'_{22}S_{yz} \quad (2.87)$$

### 2.3.3 Equilibrium equations

#### 2.3.3.1 Virtual work

Weak formulation of the Mindlin plate with von Kármán deformations will be obtained by adopting the virtual displacement principle. The principle says that;

$$0 = \delta W \equiv (\delta U + \delta V) \quad (2.88)$$

Here  $U$  presents deformation energy,  $V$  presents the work done by the forces acting on the plate and  $W$  is the sum of entire energy. Virtual deformation energy of plate domain  $\Omega$  is,

$$\begin{aligned} \delta U &= \int_{\Omega} \int_{-h/2}^{h/2} (\sigma_{xx} \delta \varepsilon_{xx}^* + \sigma_{yy} \delta \varepsilon_{yy}^* + \sigma_{xy} \delta \gamma_{xy}^* + \sigma_{xz} \delta \gamma_{xz}^* + \sigma_{yz} \delta \gamma_{yz}^*) dz dx dy \\ \delta U &= \int_{\Omega} \int_{-h/2}^{h/2} (\sigma_{xx} (\delta \varepsilon_{xx} + z \delta \kappa_{xx}) + \sigma_{yy} (\delta \varepsilon_{yy} + z \delta \kappa_{yy})) dz dx dy \\ &\quad + \int_{\Omega} \int_{-h/2}^{h/2} (\sigma_{xy} (\delta \gamma_{xy} + z \delta \kappa_{xy}) + \sigma_{xz} \delta \gamma_{xz} + \sigma_{yz} \delta \gamma_{yz}) dz dx dy \quad (2.89) \\ \delta U &= \int_{\Omega} (N_{xx} \delta \varepsilon_{xx} + M_{xx} \delta \kappa_{xx} + N_{yy} \delta \varepsilon_{yy} + M_{yy} \delta \kappa_{yy} + N_{xy} \delta \gamma_{xy}) dx dy \\ &\quad + \int_{\Omega} (M_{xy} \delta \kappa_{xy} + Q_{xz} \delta \gamma_{xz} + Q_{yz} \delta \gamma_{yz}) dx dy \end{aligned}$$

The contribution of domain and boundary loads on the virtual work can be calculated as,

$$\begin{aligned} \delta V &= - \left\{ \int_{\Omega} q(x, y) \delta w dx dy + \oint_{\Gamma} \int_{-h/2}^{h/2} [\sigma_{mn} \delta u_n^* + \sigma_{ns} \delta u_s^* + \sigma_{nz} \delta w] dz ds \right. \\ \delta V &= - \int_{\Omega} q \delta w dx dy \\ &\quad - \oint_{\Gamma} \int_{-h/2}^{h/2} [\sigma_{mn} (\delta u_n - z \delta w_{,n}) + \sigma_{ns} (\delta u_s - z \delta w_{,s}) + \sigma_{nz} \delta w] dz ds \quad (2.90) \\ \delta V &= - \int_{\Omega} q \delta w dx dy \\ &\quad - \oint_{\Gamma} [N_{nn} \delta u_n - M_{nn} \delta w_{,n} + N_{ns} \delta u_s - M_{ns} \delta w_{,s} + S_n \delta w] ds \end{aligned}$$

Here  $n$  is the direction normal of the boundary of the plate. Tangential direction  $s$  generates a right handed coordinate system with directions  $n$  and  $z$ . Normal and tangent membrane forces  $(N_{nn}, N_{ns})$ , moment components  $(M_{nn}, M_{ns})$ , and shear forces  $(S_n)$  are calculated from the integration of stresses (or stress moments) as,

$$\begin{Bmatrix} N_{nn} \\ N_{ns} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{nn} \\ \sigma_{ns} \end{Bmatrix} dz \quad ; \quad \begin{Bmatrix} M_{nn} \\ M_{ns} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{nn} \\ \sigma_{ns} \end{Bmatrix} z dz \quad ; \quad S_n = \int_{-h/2}^{h/2} \sigma_{nz} dz \quad (2.91)$$

Stress components  $(\sigma_{nn}, \sigma_{ns})$  at the boundaries  $(\Gamma)$  of the plate can be calculated in terms of plates global coordinates with the definition of a transformation procedure,

$$\begin{Bmatrix} \sigma_{nn} \\ \sigma_{ns} \end{Bmatrix} = \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \quad (2.92)$$

Here  $n_x$  and  $n_y$  are the direction cosines of the normals of the boundary.

### 2.3.3.2 Weak formulation

Deformation field components of the Mindlin plate is symbolized here with  $\mathbf{e}$ ,

$$\mathbf{e} = [\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy} \quad \gamma_{xz} \quad \gamma_{yz} \quad \kappa_{xx} \quad \kappa_{yy} \quad \kappa_{xy}]^T \quad (2.93)$$

which is in terms of displacements,

$$\mathbf{e}^u = \begin{Bmatrix} u_{,x} + \frac{1}{2}(w_{,x})^2 \\ v_{,y} + \frac{1}{2}(w_{,y})^2 \\ u_{,y} + v_{,x} + w_{,x}w_{,y} \\ w_{,x} + \varphi_x \\ w_{,y} + \varphi_y \\ \varphi_{x,x} \\ \varphi_{y,y} \\ \varphi_{x,y} + \varphi_{y,x} \end{Bmatrix} \quad (2.94)$$

With the application of variational principles, the virtual counterpart of deformations in terms of virtual displacements obtained as,

$$\delta \mathbf{e}^u = \begin{bmatrix} \delta u_{,x} + w_{,x} \delta w_{,x} \\ \delta v_{,y} + w_{,y} \delta w_{,y} \\ \delta u_{,y} + \delta v_{,x} + \delta w_{,x} w_{,y} + w_{,x} \delta w_{,y} \\ \delta w_{,x} + \delta \varphi_x \\ \delta w_{,y} + \delta \varphi_y \\ \delta \varphi_{x,x} \\ \delta \varphi_{y,y} \\ \delta \varphi_{x,y} + \delta \varphi_{y,x} \end{bmatrix} \quad (2.95)$$

Inserting virtual deformation and work relations (2.89) and (2.90) into (2.88) yields the weak formulation of the mechanical system.

$$\begin{aligned} \delta W = & \int_{\Omega} (N_{xx} \delta \varepsilon_{xx} + M_{xx} \delta \kappa_{xx} + N_{yy} \delta \varepsilon_{yy} + M_{yy} \delta \kappa_{yy} + N_{xy} \delta \gamma_{xy}) dx dy \\ & + \int_{\Omega} (M_{xy} \delta \kappa_{xy} + S_{xz} \delta \gamma_{xz} + S_{yz} \delta \gamma_{yz}) dx dy - \int_{\Omega} q \delta w dx dy \\ & - \oint_{\Gamma} (N_{nn} \delta u_n - M_{nn} \delta w_{,n} + N_{ns} \delta u_s - M_{ns} \delta w_{,s} + S_n \delta w) ds = 0 \end{aligned} \quad (2.96)$$

After inserting virtual deformation field from (2.95) one obtains (2.96) as,

$$\begin{aligned} \delta W = & \int_{\Omega} [N_{xx} (\delta u_{,x} + w_{,x} \delta w_{,x}) + N_{yy} (\delta v_{,y} + w_{,y} \delta w_{,y})] dx dy \\ & + \int_{\Omega} [N_{xy} (\delta u_{,y} + \delta v_{,x} + \delta w_{,x} w_{,y} + w_{,x} \delta w_{,y})] dx dy \\ & + \int_{\Omega} [M_{xx} (\delta \varphi_{x,x}) + M_{yy} (\delta \varphi_{y,y}) + M_{xy} (\delta \varphi_{x,y} + \delta \varphi_{y,x})] dx dy \\ & + \int_{\Omega} [S_{xz} (\delta w_{,x} + \delta \varphi_x) + S_{yz} (\delta w_{,y} + \delta \varphi_y) - q \delta w] dx dy \\ & - \oint_{\Gamma} (N_{nn} \delta u_n + N_{ns} \delta u_s - M_{nn} \delta w_{,n} - M_{ns} \delta w_{,s} + S_n \delta w) ds = 0 \end{aligned} \quad (2.97)$$

### 2.3.3.3 Equilibrium equations

Integration by parts technique has to be applied to reduce differentiation terms in virtual displacements given in weak formulation (2.97) in order to obtain equilibrium equation of Mindlin plate with von Kármán deformations. Integration by parts formula says that,

$$\int u dv = -\int v du + uv \quad (2.98)$$

Application of integration by parts formula to the terms yields,

$$\int_{\Omega} N_{xx} \delta u_{,x} dx dy = -\int_{\Omega} N_{xx,x} \delta u dx dy + \oint_{\Gamma} N_{xx} n_x \delta u ds \quad (2.99)$$

$$\int_{\Omega} N_{xx} w_{,x} \delta w_{,x} dx dy = -\int_{\Omega} (N_{xx} w_{,x})_{,x} \delta w dx dy + \oint_{\Gamma} N_{xx} w_{,x} n_x \delta w ds \quad (2.100)$$

$$\int_{\Omega} N_{yy} \delta v_{,y} dx dy = -\int_{\Omega} N_{yy,y} \delta v dx dy + \oint_{\Gamma} N_{yy} n_y \delta v ds \quad (2.101)$$

$$\int_{\Omega} N_{yy} w_{,y} \delta w_{,y} dx dy = -\int_{\Omega} (N_{yy} w_{,y})_{,y} \delta w dx dy + \oint_{\Gamma} N_{yy} w_{,y} n_y \delta w ds \quad (2.102)$$

$$\int_{\Omega} N_{xy} \delta u_{,y} dx dy = -\int_{\Omega} N_{xy,y} \delta u dx dy + \oint_{\Gamma} N_{xy} n_y \delta u ds \quad (2.103)$$

$$\int_{\Omega} N_{xy} \delta v_{,x} dx dy = -\int_{\Omega} N_{xy,x} \delta v dx dy + \oint_{\Gamma} N_{xy} n_x \delta v ds \quad (2.104)$$

$$\int_{\Omega} N_{xy} w_{,y} \delta w_{,x} dx dy = -\int_{\Omega} (N_{xy} w_{,y})_{,x} \delta w dx dy + \oint_{\Gamma} N_{xy} w_{,y} n_x \delta w ds \quad (2.105)$$

$$\int_{\Omega} N_{xy} w_{,x} \delta w_{,y} dx dy = -\int_{\Omega} (N_{xy} w_{,x})_{,y} \delta w dx dy + \oint_{\Gamma} N_{xy} w_{,x} n_y \delta w ds \quad (2.106)$$

$$\int_{\Omega} M_{xx} \delta \varphi_{x,x} dx dy = -\int_{\Omega} M_{xx,x} \delta \varphi_x dx dy + \oint_{\Gamma} M_{xx} n_x \delta \varphi_x ds \quad (2.107)$$

$$\int_{\Omega} M_{yy} \delta \varphi_{y,y} dx dy = -\int_{\Omega} M_{yy,y} \delta \varphi_y dx dy + \oint_{\Gamma} M_{yy} n_y \delta \varphi_y ds \quad (2.108)$$

$$\int_{\Omega} M_{xy} \delta \varphi_{x,y} dx dy = -\int_{\Omega} M_{xy,y} \delta \varphi_x dx dy + \oint_{\Gamma} M_{xy} n_y \delta \varphi_x ds \quad (2.109)$$

$$\int_{\Omega} M_{xy} \delta \varphi_{y,x} dx dy = -\int_{\Omega} M_{xy,x} \delta \varphi_y dx dy + \oint_{\Gamma} M_{xy} n_x \delta \varphi_y ds \quad (2.110)$$

$$\int_{\Omega} S_{xz} \delta w_{,x} dx dy = -\int_{\Omega} S_{xz,x} \delta w dx dy + \oint_{\Gamma} S_{xz} n_x \delta w ds \quad (2.111)$$

$$\int_{\Omega} S_{xz} \delta \varphi_x dx dy = \int_{\Omega} S_{xz} \delta \varphi_x dx dy \quad (2.112)$$

$$\int_{\Omega} S_{yz} \delta w_{,y} dx dy = -\int_{\Omega} S_{yz,y} \delta w dx dy + \oint_{\Gamma} S_{yz} n_y \delta w ds \quad (2.113)$$

$$\int_{\Omega} S_{yz} \delta \varphi_y dx dy = \int_{\Omega} S_{yz} \delta \varphi_y dx dy \quad (2.114)$$

Inserting corresponding integrals into weak formulation, one gets the domain terms as;

$$\left. \begin{aligned} & -\int_{\Omega} (N_{xx,x} + N_{xy,y}) \delta u - \int_{\Omega} (N_{xy,x} + N_{yy,y}) \delta v - \int_{\Omega} (S_{xz,x} + S_{yz,y}) \delta w \\ & - \int_{\Omega} \left[ (N_{xx} w_{,x} + N_{xy} w_{,y})_{,x} + (N_{xy} w_{,x} + N_{yy} w_{,y})_{,y} \right] \delta w \\ & - \int_{\Omega} (M_{xx,x} + M_{xy,y} - S_{xz}) \delta \varphi_x - \int_{\Omega} (M_{yy,y} + M_{xy,x} - S_{yz}) \delta \varphi_y \\ & - \int_{\Omega} q \delta w = 0 \end{aligned} \right\} \quad (2.115)$$

The boundary terms can be collected together as,

$$\left. \begin{aligned} & + \oint_{\Gamma} (N_{xx} n_x + N_{xy} n_y) \delta u ds + \oint_{\Gamma} (N_{xy} n_x + N_{yy} n_y) \delta v ds \\ & + \oint_{\Gamma} \left[ (N_{xx} w_{,x} + N_{xy} w_{,y} + S_{xz}) n_x + (N_{xy} w_{,x} + N_{yy} w_{,y} + S_{yz}) n_y \right] \delta w ds \\ & + \oint_{\Gamma} (M_{xx} n_x + M_{xy} n_y) \delta \varphi_x ds + \oint_{\Gamma} (M_{yy} n_y + M_{xy} n_x) \delta \varphi_y ds \\ & - \oint_{\Gamma} (N_{nn} \delta u_n + N_{ns} \delta u_s - M_{nn} \delta w_{,n} - M_{ns} \delta w_{,s} + S_n \delta w) ds \end{aligned} \right\} \quad (2.116)$$

Equilibrium equations will be determined by equating coefficients of virtual displacements to zero in plates domain terms.

$$\delta u \neq 0 \Rightarrow N_{xx,x} + N_{xy,y} = 0 \quad (2.117)$$

$$\delta v \neq 0 \Rightarrow N_{xy,x} + N_{yy,y} = 0 \quad (2.118)$$

$$\delta w \neq 0 \Rightarrow \left. \begin{aligned} & (N_{xx} w_{,x} + N_{xy} w_{,y})_{,x} + (N_{xy} w_{,x} + N_{yy} w_{,y})_{,y} \\ & + S_{xz,x} + S_{yz,y} = 0 \end{aligned} \right\} \quad (2.119)$$

$$\delta \varphi_x \neq 0 \Rightarrow M_{xx,x} + M_{xy,y} - S_{xz} = 0 \quad (2.120)$$

$$\delta \varphi_y \neq 0 \Rightarrow M_{yy,y} + M_{xy,x} - S_{yz} = 0 \quad (2.121)$$

### 2.3.4 Mixed FEM formulation depending on Hellinger–Reissner Principle

Terms obtained from field equations and boundary conditions must be inserted in the following equation to get the first variation of Hellinger–Reissner functional ([120]).

$$\begin{aligned} \delta \Pi_{HR} = & \int_V (\boldsymbol{\varepsilon}^u - \boldsymbol{\varepsilon}^\sigma)^T \delta \boldsymbol{\sigma}^\sigma dV + \int_V \left( (\boldsymbol{\sigma}^\sigma)^T \delta \boldsymbol{\varepsilon}^u - \mathbf{q}^T \delta \mathbf{u} \right) dV \\ & - \int_\Gamma \hat{\mathbf{t}}^T \delta \mathbf{u} d\Gamma = 0 \end{aligned} \quad (2.122)$$

where  $\hat{\mathbf{t}}$  is the traction vector at the boundary ( $\Gamma$ ) of the plate. After inserting the field equations of laminated composite Mindlin plate with large deflection terms into (2.122), the variational equations in mixed form are obtained as follows,

$$\begin{aligned} & \int_\Omega \left[ u_{,x} + \frac{1}{2} (w_{,x})^2 \right] \delta N_{xx} d\Omega \\ & + \int_\Omega \left[ -A'_{11} N_{xx} - A'_{12} N_{yy} - A'_{13} N_{xy} - B'_{11} M_{xx} - B'_{12} M_{yy} - B'_{13} M_{xy} \right] \delta N_{xx} d\Omega \\ & + \int_\Omega \left[ v_{,y} + \frac{1}{2} (w_{,y})^2 - A'_{21} N_{xx} - A'_{22} N_{yy} - A'_{23} N_{xy} \right] \delta N_{yy} d\Omega \\ & + \int_\Omega \left[ -B'_{21} M_{xx} - B'_{22} M_{yy} - B'_{23} M_{xy} \right] \delta N_{yy} d\Omega \\ & + \int_\Omega \left[ u_{,y} + v_{,x} + w_{,x} w_{,y} - A'_{31} N_{xx} - A'_{32} N_{yy} - A'_{33} N_{xy} \right] \delta N_{xy} d\Omega \\ & + \int_\Omega \left[ -B'_{31} M_{xx} - B'_{32} M_{yy} - B'_{33} M_{xy} \right] \delta N_{xy} d\Omega \\ & + \int_\Omega \left[ \varphi_{x,x} - B'_{11} N_{xx} - B'_{12} N_{yy} - B'_{13} N_{xy} \right] \delta M_{xx} d\Omega \\ & + \int_\Omega \left[ -D'_{11} M_{xx} - D'_{12} M_{yy} - D'_{13} M_{xy} \right] \delta M_{xx} d\Omega \\ & + \int_\Omega \left[ \varphi_{y,y} - B'_{21} N_{xx} - B'_{22} N_{yy} - B'_{23} N_{xy} \right] \delta M_{yy} d\Omega \\ & + \int_\Omega \left[ -D'_{21} M_{xx} - D'_{22} M_{yy} - D'_{23} M_{xy} \right] \delta M_{yy} d\Omega \\ & + \int_\Omega \left[ \varphi_{x,y} + \varphi_{y,x} - B'_{31} N_{xx} - B'_{32} N_{yy} - B'_{33} N_{xy} \right] \delta M_{xy} d\Omega \\ & + \int_\Omega \left[ -D'_{31} M_{xx} - D'_{32} M_{yy} - D'_{33} M_{xy} \right] \delta M_{xy} d\Omega \\ & + \int_\Omega \left[ w_{,x} + \varphi_x - C'_{11} S_{xz} - C'_{12} S_{yz} \right] \delta S_{xz} d\Omega \\ & + \int_\Omega \left[ w_{,y} + \varphi_y - C'_{21} S_{xz} - C'_{22} S_{yz} \right] \delta S_{yz} d\Omega \\ & - \int_\Omega \delta u q_x d\Omega - \int_\Omega \delta v q_y d\Omega - \int_\Omega \delta w q_z d\Omega - \int_\Omega \delta \varphi_x m_x d\Omega \\ & - \int_\Omega \delta \varphi_y m_y d\Omega - \int_\Omega \left[ N_{xx,x} + N_{xy,y} \right] \delta u d\Omega - \int_\Omega \left[ N_{xy,x} + N_{yy,y} \right] \delta v d\Omega \\ & - \int_\Omega \left[ \left( N_{xx} w_{,x} + N_{xy} w_{,y} \right)_{,x} + \left( N_{xy} w_{,x} + N_{yy} w_{,y} \right)_{,y} + S_{xz,x} + S_{yz,y} \right] \delta w d\Omega \\ & - \int_\Omega \left[ M_{xx,x} + M_{xy,y} - S_{xz} \right] \delta \varphi_x d\Omega - \int_\Omega \left[ M_{yy,y} + M_{xy,x} - S_{yz} \right] \delta \varphi_y d\Omega \\ & - \int_\Gamma \hat{t}_j \delta u_j d\Gamma = 0 \end{aligned} \quad (2.123)$$

Integration by parts technique is applied to the first variation in order to proceed into a mixed finite element formulation which requires  $C^0$  continuity where bi-linear shape functions are used to discretize the plate domain  $\Omega$ .

$$\begin{aligned}
& \int_{\Omega} \left[ u_{,x} + \frac{1}{2} (w_{,x})^2 \right] \delta N_{xx} d\Omega \\
& + \int_{\Omega} \left[ -A'_{11} N_{xx} - A'_{12} N_{yy} - A'_{13} N_{xy} - B'_{11} M_{xx} - B'_{12} M_{yy} - B'_{13} M_{xy} \right] \delta N_{xx} d\Omega \\
& + \int_{\Omega} \left[ v_{,y} + \frac{1}{2} (w_{,y})^2 \right] \delta N_{yy} d\Omega \\
& + \int_{\Omega} \left[ -A'_{21} N_{xx} - A'_{22} N_{yy} - A'_{23} N_{xy} - B'_{21} M_{xx} - B'_{22} M_{yy} - B'_{23} M_{xy} \right] \delta N_{yy} d\Omega \\
& + \int_{\Omega} \left[ u_{,y} + v_{,x} + w_{,x} w_{,y} \right] \delta N_{xy} d\Omega \\
& + \int_{\Omega} \left[ -A'_{31} N_{xx} - A'_{32} N_{yy} - A'_{33} N_{xy} - B'_{31} M_{xx} - B'_{32} M_{yy} - B'_{33} M_{xy} \right] \delta N_{xy} d\Omega \\
& + \int_{\Omega} [\varphi_{x,x}] \delta M_{xx} d\Omega \\
& + \int_{\Omega} \left[ -B'_{11} N_{xx} - B'_{12} N_{yy} - B'_{13} N_{xy} - D'_{11} M_{xx} - D'_{12} M_{yy} - D'_{13} M_{xy} \right] \delta M_{xx} d\Omega \\
& + \int_{\Omega} [\varphi_{y,y}] \delta M_{yy} d\Omega \\
& + \int_{\Omega} \left[ -B'_{21} N_{xx} - B'_{22} N_{yy} - B'_{23} N_{xy} - D'_{21} M_{xx} - D'_{22} M_{yy} - D'_{23} M_{xy} \right] \delta M_{yy} d\Omega \\
& + \int_{\Omega} [\varphi_{x,y} + \varphi_{y,x}] \delta M_{xy} d\Omega \\
& + \int_{\Omega} \left[ -B'_{31} N_{xx} - B'_{32} N_{yy} - B'_{33} N_{xy} - D'_{31} M_{xx} - D'_{32} M_{yy} - D'_{33} M_{xy} \right] \delta M_{xy} d\Omega \\
& + \int_{\Omega} \left[ w_{,x} + \varphi_x - C'_{11} S_{xz} - C'_{12} S_{yz} \right] \delta S_{xz} d\Omega \\
& + \int_{\Omega} \left[ w_{,y} + \varphi_y - C'_{21} S_{xz} - C'_{22} S_{yz} \right] \delta S_{yz} d\Omega - \int_{\Omega} \delta u q_x d\Omega \\
& - \int_{\Omega} \delta v q_y d\Omega - \int_{\Omega} \delta w q_z d\Omega - \int_{\Omega} \delta \varphi_x m_x d\Omega - \int_{\Omega} \delta \varphi_y m_y d\Omega \\
& + \int_{\Omega} \left[ N_{xx} \delta u_{,x} + N_{xy} \delta u_{,y} \right] d\Omega + \int_{\Omega} \left[ N_{xy} \delta v_{,x} + N_{yy} \delta v_{,y} \right] d\Omega \\
& + \int_{\Omega} \left[ (N_{xx} w_{,x} + N_{xy} w_{,y} + S_{xz}) \delta w_{,x} + (N_{xy} w_{,x} + N_{yy} w_{,y} + S_{yz}) \delta w_{,y} \right] d\Omega \\
& + \int_{\Omega} \left[ M_{xx} \delta \varphi_{x,x} + M_{xy} \delta \varphi_{x,y} + S_{xz} \delta \varphi_x \right] d\Omega \\
& + \int_{\Omega} \left[ M_{xy} \delta \varphi_{y,x} + M_{yy} \delta \varphi_{y,y} + Q_{yz} \delta \varphi_y \right] d\Omega - \int_{\Gamma} \hat{t}_j \delta u_j d\Gamma = 0
\end{aligned} \tag{2.124}$$

## 2.3.5 Solution of geometrically non-linear problem with mixed formulation

### 2.3.5.1 Non-linear equations

Non-linear equations of mixed finite element formulation are obtained from the first variation of the Hellinger–Reissner functional (2.124). Equating the coefficients of variational terms in (2.124) yields the following collection of non-linear finite element equations.

$$\begin{aligned}\delta N_{xx} \neq 0 \Rightarrow & \int_{\Omega} \delta N_{xx} \left( u_{,x} + \frac{1}{2} (w_{,x})^2 \right) d\Omega \\ & + \int_{\Omega} \delta N_{xx} \left( -A'_{11} N_{xx} - A'_{12} N_{yy} - A'_{13} N_{xy} \right) d\Omega \\ & + \int_{\Omega} \delta N_{xx} \left( -B'_{11} M_{xx} - B'_{12} M_{yy} - B'_{13} M_{xy} \right) d\Omega = 0\end{aligned}\quad (2.125)$$

$$\begin{aligned}\delta N_{yy} \neq 0 \Rightarrow & \int_{\Omega} \delta N_{yy} \left( v_{,y} + \frac{1}{2} (w_{,y})^2 \right) d\Omega \\ & + \int_{\Omega} \delta N_{yy} \left( -A'_{21} N_{xx} - A'_{22} N_{yy} - A'_{23} N_{xy} \right) d\Omega \\ & + \int_{\Omega} \delta N_{yy} \left( -B'_{21} M_{xx} - B'_{22} M_{yy} - B'_{23} M_{xy} \right) d\Omega = 0\end{aligned}\quad (2.126)$$

$$\begin{aligned}\delta N_{xy} \neq 0 \Rightarrow & \int_{\Omega} \delta N_{xy} \left( u_{,y} + v_{,x} + w_{,x} w_{,y} \right) d\Omega \\ & + \int_{\Omega} \delta N_{xy} \left( -A'_{31} N_{xx} - A'_{32} N_{yy} - A'_{33} N_{xy} \right) d\Omega \\ & + \int_{\Omega} \delta N_{xy} \left( -B'_{31} M_{xx} - B'_{32} M_{yy} - B'_{33} M_{xy} \right) d\Omega = 0\end{aligned}\quad (2.127)$$

$$\begin{aligned}\delta M_{xx} \neq 0 \Rightarrow & \int_{\Omega} \delta M_{xx} \varphi_{x,x} d\Omega \\ & + \int_{\Omega} \delta M_{xx} \left( -B'_{11} N_{xx} - B'_{12} N_{yy} - B'_{13} N_{xy} \right) d\Omega \\ & + \int_{\Omega} \delta M_{xx} \left( -D'_{11} M_{xx} - D'_{12} M_{yy} - D'_{13} M_{xy} \right) d\Omega = 0\end{aligned}\quad (2.128)$$

$$\begin{aligned}\delta M_{yy} \neq 0 \Rightarrow & \int_{\Omega} \delta M_{yy} \varphi_{y,y} d\Omega \\ & + \int_{\Omega} \delta M_{yy} \left( -B'_{21} N_{xx} - B'_{22} N_{yy} - B'_{23} N_{xy} \right) d\Omega \\ & + \int_{\Omega} \delta M_{yy} \left( -D'_{21} M_{xx} - D'_{22} M_{yy} - D'_{23} M_{xy} \right) d\Omega = 0\end{aligned}\quad (2.129)$$

$$\begin{aligned}
\delta M_{xy} \neq 0 \Rightarrow & \int_{\Omega} \delta M_{xy} (\varphi_{x,y} + \varphi_{y,x}) d\Omega \\
& + \int_{\Omega} \delta M_{xy} (-B'_{31} N_{xx} - B'_{32} N_{yy} - B'_{33} N_{xy}) d\Omega \\
& + \int_{\Omega} \delta M_{xy} (-D'_{31} M_{xx} - D'_{32} M_{yy} - D'_{33} M_{xy}) d\Omega = 0
\end{aligned} \tag{2.130}$$

$$\begin{aligned}
\delta S_{xz} \neq 0 \Rightarrow & \int_{\Omega} \delta S_{xz} w_{,x} d\Omega \\
& + \int_{\Omega} \delta S_{xz} \varphi_x d\Omega + \int_{\Omega} \delta S_{xz} (-C'_{11} S_{xz} - C'_{12} S_{yz}) d\Omega = 0
\end{aligned} \tag{2.131}$$

$$\begin{aligned}
\delta S_{yz} \neq 0 \Rightarrow & \int_{\Omega} \delta S_{yz} w_{,y} d\Omega \\
& + \int_{\Omega} \delta S_{yz} \varphi_y d\Omega + \int_{\Omega} \delta S_{yz} (-C'_{21} S_{xz} - C'_{22} S_{yz}) d\Omega = 0
\end{aligned} \tag{2.132}$$

$$\delta u \neq 0 \Rightarrow \int_{\Omega} \delta u_{,x} N_{xx} d\Omega + \int_{\Omega} \delta u_{,y} N_{xy} d\Omega - \int_{\Omega} \delta u q_x d\Omega = 0 \tag{2.133}$$

$$\delta v \neq 0 \Rightarrow \int_{\Omega} \delta v_{,x} N_{xy} d\Omega + \int_{\Omega} \delta v_{,y} N_{yy} d\Omega - \int_{\Omega} \delta v q_y d\Omega = 0 \tag{2.134}$$

$$\begin{aligned}
\delta w \neq 0 \Rightarrow & \int_{\Omega} \delta w_{,x} (N_{xx} w_{,x} + N_{xy} w_{,y} + S_{xz}) d\Omega \\
& + \int_{\Omega} \delta w_{,y} (N_{xy} w_{,x} + N_{yy} w_{,y} + S_{yz}) d\Omega - \int_{\Omega} \delta w q_z d\Omega = 0
\end{aligned} \tag{2.135}$$

$$\begin{aligned}
\delta \varphi_x \neq 0 \Rightarrow & \int_{\Omega} \delta \varphi_{x,x} M_{xx} d\Omega + \int_{\Omega} \delta \varphi_{x,y} M_{xy} d\Omega \\
& + \int_{\Omega} \delta \varphi_x S_{xz} d\Omega - \int_{\Omega} \delta \varphi_x m_x d\Omega = 0
\end{aligned} \tag{2.136}$$

$$\begin{aligned}
\delta \varphi_y \neq 0 \Rightarrow & \int_{\Omega} \delta \varphi_{y,x} M_{xy} d\Omega + \int_{\Omega} \delta \varphi_{y,y} M_{yy} d\Omega \\
& + \int_{\Omega} \delta \varphi_y S_{yz} d\Omega - \int_{\Omega} \delta \varphi_y m_y d\Omega = 0
\end{aligned} \tag{2.137}$$

### 2.3.5.2 Linearization with incremental formulation

The non-linear equations obtained in Section 2.3.5.1 has to be linearized for an iterative solution approach. Incremental formulation [121] is used in the context of this study in order to transform non-linear large deflection equations of laminated composite plate to a suitable form for an iterative algorithm based on a mixed finite element procedure. Incremental formulation defines the final value  $\overline{(\dots)}$  of the

variables from accumulation of their initial value (...) and an increment of them ( $\dots^+$ ) as  $\overline{(\dots)} = (\dots) + (\dots)^+$  [122], (e.g.  $\bar{\mathbf{X}} = \mathbf{X} + \mathbf{X}^+$ ). During this process, variation is converted to a special variation  $\bar{\delta}$  which is applied only on the increments [123]. Substituting the variables with their incremental forms and ignoring the third and higher order terms, the mixed FEM formulation can be expressed in linearized form.

### Linearization of $\delta N_{xx}$ terms

Inserting incremental forms of field variables into non-linear equation (2.125) yields,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} \left( N_{xx} + N_{xx}^+ \right) \left( u_{,x} + u_{,x}^+ \right) d\Omega + \int_{\Omega} \bar{\delta} \left( N_{xx} + N_{xx}^+ \right) \frac{1}{2} \left( w_{,x} + w_{,x}^+ \right)^2 d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( N_{xx} + N_{xx}^+ \right) \left[ -A'_{11} \left( N_{xx} + N_{xx}^+ \right) - A'_{12} \left( N_{yy} + N_{yy}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( N_{xx} + N_{xx}^+ \right) \left[ -A'_{13} \left( N_{xy} + N_{xy}^+ \right) - B'_{11} \left( M_{xx} + M_{xx}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( N_{xx} + N_{xx}^+ \right) \left[ -B'_{12} \left( M_{yy} + M_{yy}^+ \right) - B'_{13} \left( M_{xy} + M_{xy}^+ \right) \right] d\Omega = 0
\end{aligned} \tag{2.138}$$

If the variation is applied to the increments only, then (2.138) becomes,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} N_{xx}^+ \left( u_{,x} + u_{,x}^+ \right) d\Omega + \int_{\Omega} \bar{\delta} N_{xx}^+ \frac{1}{2} \left( w_{,x} + w_{,x}^+ \right)^2 d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ \left[ -A'_{11} \left( N_{xx} + N_{xx}^+ \right) - A'_{12} \left( N_{yy} + N_{yy}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ \left[ -A'_{13} \left( N_{xy} + N_{xy}^+ \right) - B'_{11} \left( M_{xx} + M_{xx}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ \left[ -B'_{12} \left( M_{yy} + M_{yy}^+ \right) - B'_{13} \left( M_{xy} + M_{xy}^+ \right) \right] d\Omega = 0
\end{aligned} \tag{2.139}$$

Arrangind the terms in (2.139) results with the following equation,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} N_{xx}^+ u_{,x}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{xx}^+ u_{,x}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{xx}^+ \frac{1}{2} (w_{,x}^+)^2 d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ \frac{1}{2} (w_{,x}^+)^2 d\Omega + \int_{\Omega} \bar{\delta} N_{xx}^+ w_{,x}^+ w_{,x}^+ d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ (-A'_{11} N_{xx} - A'_{12} N_{yy} - A'_{13} N_{xy}) d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ (-B'_{11} M_{xx} - B'_{12} M_{yy} - B'_{13} M_{xy}) d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ \left( -A'_{11} N_{xx}^+ - A'_{12} N_{yy}^+ - A'_{13} N_{xy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ \left( -B'_{11} M_{xx}^+ - B'_{12} M_{yy}^+ - B'_{13} M_{xy}^+ \right) d\Omega
\end{aligned} \tag{2.140}$$

After neglecting higher order terms, one obtains the correction expression as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} N_{xx}^+ u_{,x}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{xx}^+ \frac{1}{2} (w_{,x}^+)^2 d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ (-A'_{11} N_{xx} - A'_{12} N_{yy} - A'_{13} N_{xy}) d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ (-B'_{11} M_{xx} - B'_{12} M_{yy} - B'_{13} M_{xy}) d\Omega
\end{aligned} \tag{2.141}$$

and the tangent system matrix terms as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} N_{xx}^+ u_{,x}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{xx}^+ w_{,x}^+ w_{,x}^+ d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ \left( -A'_{11} N_{xx}^+ - A'_{12} N_{yy}^+ - A'_{13} N_{xy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{xx}^+ \left( -B'_{11} M_{xx}^+ - B'_{12} M_{yy}^+ - B'_{13} M_{xy}^+ \right) d\Omega
\end{aligned} \tag{2.142}$$

### Linearization of $\delta N_{yy}$ terms

Inserting incremental forms of field variables into non-linear equation (2.126) yields,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} \left( N_{yy} + N_{yy}^+ \right) \left( v_{,y} + v_{,y}^+ \right) d\Omega + \int_{\Omega} \bar{\delta} \left( N_{yy} + N_{yy}^+ \right)^{\frac{1}{2}} \left( w_{,y} + w_{,y}^+ \right)^2 d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( N_{yy} + N_{yy}^+ \right) \left[ -A'_{21} \left( N_{xx} + N_{xx}^+ \right) - A'_{22} \left( N_{yy} + N_{yy}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( N_{yy} + N_{yy}^+ \right) \left[ -A'_{23} \left( N_{xy} + N_{xy}^+ \right) - B'_{21} \left( M_{xx} + M_{xx}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( N_{yy} + N_{yy}^+ \right) \left[ -B'_{22} \left( M_{yy} + M_{yy}^+ \right) - B'_{23} \left( M_{xy} + M_{xy}^+ \right) \right] d\Omega = 0
\end{aligned} \tag{2.143}$$

If the variation is applied to the increments only, (2.143) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} N_{yy}^+ v_{,y} d\Omega + \int_{\Omega} \bar{\delta} N_{yy}^+ v_{,y}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{yy}^+ \frac{1}{2} (w_{,y})^2 d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{yy}^+ \frac{1}{2} (w_{,y}^+)^2 d\Omega + \int_{\Omega} \bar{\delta} N_{yy}^+ w_{,y} w_{,y}^+ d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{yy}^+ \left( -A'_{21} N_{xx} - A'_{22} N_{yy} - A'_{23} N_{xy} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{yy}^+ \left( -B'_{21} M_{xx} - B'_{22} M_{yy} - B'_{23} M_{xy} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{yy}^+ \left( -A'_{21} N_{xx}^+ - A'_{22} N_{yy}^+ - A'_{23} N_{xy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{yy}^+ \left( -B'_{21} M_{xx}^+ - B'_{22} M_{yy}^+ - B'_{23} M_{xy}^+ \right) d\Omega = 0
\end{aligned} \tag{2.144}$$

After neglecting higher order terms, one obtains the correction expression as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} N_{yy}^+ v_{,y} d\Omega + \int_{\Omega} \bar{\delta} N_{yy}^+ \frac{1}{2} (w_{,y})^2 d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{yy}^+ \left( -A'_{21} N_{xx} - A'_{22} N_{yy} - A'_{23} N_{xy} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} N_{yy}^+ \left( -B'_{21} M_{xx} - B'_{22} M_{yy} - B'_{23} M_{xy} \right) d\Omega
\end{aligned} \tag{2.145}$$

and the tangent system matrix terms as,

$$\begin{aligned}
&= \int_{\Omega} \bar{\delta} N_{yy}^+ v_{,y}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{yy}^+ w_{,y}^+ w_{,y}^+ d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{yy}^+ \left( -A'_{21} N_{xx}^+ - A'_{22} N_{yy}^+ - A'_{23} N_{xy}^+ \right) d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{yy}^+ \left( -B'_{21} M_{xx}^+ - B'_{22} M_{yy}^+ - B'_{23} M_{xy}^+ \right) d\Omega
\end{aligned} \tag{2.146}$$

### Linearization of $\delta N_{xy}$ terms

Inserting incremental forms of field variables into non-linear equation (2.127) yields,

$$\begin{aligned}
&\int_{\Omega} \bar{\delta} \left( N_{xy} + N_{xy}^+ \right) \left( u_{,y} + u_{,y}^+ \right) d\Omega + \int_{\Omega} \bar{\delta} \left( N_{xy} + N_{xy}^+ \right) \left( v_{,x} + v_{,x}^+ \right) d\Omega \\
&+ \int_{\Omega} \bar{\delta} \left( N_{xy} + N_{xy}^+ \right) \left( w_{,x} + w_{,x}^+ \right) \left( w_{,y} + w_{,y}^+ \right) d\Omega \\
&+ \int_{\Omega} \bar{\delta} \left( N_{xy} + N_{xy}^+ \right) \left[ -A'_{31} \left( N_{xx} + N_{xx}^+ \right) - A'_{32} \left( N_{yy} + N_{yy}^+ \right) \right] d\Omega \\
&+ \int_{\Omega} \bar{\delta} \left( N_{xy} + N_{xy}^+ \right) \left[ -A'_{33} \left( N_{xy} + N_{xy}^+ \right) - B'_{31} \left( M_{xx} + M_{xx}^+ \right) \right] d\Omega \\
&+ \int_{\Omega} \bar{\delta} \left( N_{xy} + N_{xy}^+ \right) \left[ -B'_{32} \left( M_{yy} + M_{yy}^+ \right) - B'_{33} \left( M_{xy} + M_{xy}^+ \right) \right] d\Omega = 0
\end{aligned} \tag{2.147}$$

If the variation is applied to the increments only, (2.147) becomes when arranged,

$$\begin{aligned}
&\int_{\Omega} \bar{\delta} N_{xy}^+ u_{,y} d\Omega + \int_{\Omega} \bar{\delta} N_{xy}^+ u_{,y}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{xy}^+ v_{,x} d\Omega + \int_{\Omega} \bar{\delta} N_{xy}^+ v_{,x}^+ d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ w_{,x} w_{,y} d\Omega + \int_{\Omega} \bar{\delta} N_{xy}^+ w_{,x}^+ w_{,y}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{xy}^+ w_{,x}^+ w_{,y} d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ w_{,x}^+ w_{,y}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{xy}^+ \left( -A'_{31} N_{xx} - A'_{32} N_{yy} - A'_{33} N_{xy} \right) d\Omega \\
&\int_{\Omega} \bar{\delta} N_{xy}^+ \left( -B'_{31} M_{xx} - B'_{32} M_{yy} - B'_{33} M_{xy} \right) d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ \left( -A'_{31} N_{xx}^+ - A'_{32} N_{yy}^+ - A'_{33} N_{xy}^+ \right) d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ \left( -B'_{31} M_{xx}^+ - B'_{32} M_{yy}^+ - B'_{33} M_{xy}^+ \right) d\Omega = 0
\end{aligned} \tag{2.148}$$

After neglecting higher order terms, one obtains the correction expression as,

$$\begin{aligned}
&= \int_{\Omega} \bar{\delta} N_{xy}^+ u_{,y} d\Omega + \int_{\Omega} \bar{\delta} N_{xy}^+ v_{,x} d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ w_{,x} w_{,y} d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ (-A'_{31} N_{xx} - A'_{32} N_{yy} - A'_{33} N_{xy}) d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ (-B'_{31} M_{xx} - B'_{32} M_{yy} - B'_{33} M_{xy}) d\Omega
\end{aligned} \tag{2.149}$$

and the tangent system matrix terms as,

$$\begin{aligned}
&= \int_{\Omega} \bar{\delta} N_{xy}^+ u_{,y}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{xy}^+ v_{,x}^+ d\Omega + \int_{\Omega} \bar{\delta} N_{xy}^+ w_{,x}^+ w_{,y}^+ d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ w_{,x}^+ w_{,y}^+ d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ \left( -A'_{31} N_{xx}^+ - A'_{32} N_{yy}^+ - A'_{33} N_{xy}^+ \right) d\Omega \\
&+ \int_{\Omega} \bar{\delta} N_{xy}^+ \left( -B'_{31} M_{xx}^+ - B'_{32} M_{yy}^+ - B'_{33} M_{xy}^+ \right) d\Omega
\end{aligned} \tag{2.150}$$

### Linearization of $\delta M_{xx}$ terms

Inserting incremental forms of field variables into non-linear equation (2.128) yields,

$$\begin{aligned}
&\int_{\Omega} \bar{\delta} \left( M_{xx} + M_{xx}^+ \right) \left( \phi_{x,x} + \phi_{x,x}^+ \right) d\Omega \\
&+ \int_{\Omega} \bar{\delta} \left( M_{xx} + M_{xx}^+ \right) \left[ -B'_{11} \left( N_{xx} + N_{xx}^+ \right) - B'_{12} \left( N_{yy} + N_{yy}^+ \right) \right] d\Omega \\
&+ \int_{\Omega} \bar{\delta} \left( M_{xx} + M_{xx}^+ \right) \left[ -B'_{13} \left( N_{xy} + N_{xy}^+ \right) - D'_{11} \left( M_{xx} + M_{xx}^+ \right) \right] d\Omega \\
&+ \int_{\Omega} \bar{\delta} \left( M_{xx} + M_{xx}^+ \right) \left[ -D'_{12} \left( M_{yy} + M_{yy}^+ \right) - D'_{13} \left( M_{xy} + M_{xy}^+ \right) \right] d\Omega = 0
\end{aligned} \tag{2.151}$$

If the variation is applied to the increments only, (2.151) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} M_{xx}^+ \varphi_{x,x}^+ d\Omega + \int_{\Omega} \delta M_{xx}^+ \varphi_{x,x}^+ d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xx}^+ \left( -B'_{11} N_{xx} - B'_{12} N_{yy} - B'_{13} N_{xy} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xx}^+ \left( -D'_{11} M_{xx} - D'_{12} M_{yy} - D'_{13} M_{xy} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xx}^+ \left( -B'_{11} N_{xx}^+ - B'_{12} N_{yy}^+ - B'_{13} N_{xy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xx}^+ \left( -D'_{11} M_{xx}^+ - D'_{12} M_{yy}^+ - D'_{13} M_{xy}^+ \right) d\Omega = 0
\end{aligned} \tag{2.152}$$

After neglecting higher order terms, one obtains the correction expression as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} M_{xx}^+ \varphi_{x,x}^+ d\Omega + \int_{\Omega} \bar{\delta} M_{xx}^+ \left( -B'_{11} N_{xx} - B'_{12} N_{yy} - B'_{13} N_{xy} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xx}^+ \left( -D'_{11} M_{xx} - D'_{12} M_{yy} - D'_{13} M_{xy} \right) d\Omega
\end{aligned} \tag{2.153}$$

and the tangent system matrix terms as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} M_{xx}^+ \varphi_{x,x}^+ d\Omega + \int_{\Omega} \bar{\delta} M_{xx}^+ \left( -B'_{11} N_{xx}^+ - B'_{12} N_{yy}^+ - B'_{13} N_{xy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xx}^+ \left( -D'_{11} M_{xx}^+ - D'_{12} M_{yy}^+ - D'_{13} M_{xy}^+ \right) d\Omega
\end{aligned} \tag{2.154}$$

### Linearization of $\delta M_{yy}$ terms

Inserting incremental forms of field variables into non-linear equation (2.129) yields,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} \left( M_{yy} + M_{yy}^+ \right) \left( \varphi_{y,y} + \varphi_{y,y}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( M_{yy} + M_{yy}^+ \right) \left[ -B'_{21} \left( M_{yy} + M_{yy}^+ \right) - B'_{22} \left( M_{yy} + M_{yy}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( M_{yy} + M_{yy}^+ \right) \left[ -B'_{23} \left( M_{yy} + M_{yy}^+ \right) - D'_{21} \left( M_{yy} + M_{yy}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( M_{yy} + M_{yy}^+ \right) \left[ -D'_{22} \left( M_{yy} + M_{yy}^+ \right) - D'_{23} \left( M_{yy} + M_{yy}^+ \right) \right] d\Omega = 0
\end{aligned} \tag{2.155}$$

If the variation is applied to the increments only, (2.155) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} M_{yy}^+ \varphi_{y,y}^+ d\Omega + \int_{\Omega} \bar{\delta} M_{yy}^+ \varphi_{y,y}^+ d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{yy}^+ \left( -B'_{21} N_{xx} - B'_{22} N_{yy} - B'_{23} N_{xy} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{yy}^+ \left( -D'_{21} M_{xx} - D'_{22} M_{yy} - D'_{23} M_{xy} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{yy}^+ \left( -B'_{21} N_{xx}^+ - B'_{22} N_{yy}^+ - B'_{23} N_{xy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{yy}^+ \left( -D'_{21} M_{xx}^+ - D'_{22} M_{yy}^+ - D'_{23} M_{xy}^+ \right) d\Omega = 0
\end{aligned} \tag{2.156}$$

After neglecting higher order terms, one obtains the correction expression as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} M_{yy}^+ \varphi_{y,y}^+ d\Omega + \int_{\Omega} \bar{\delta} M_{yy}^+ \left( -B'_{21} N_{xx} - B'_{22} N_{yy} - B'_{23} N_{xy} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{yy}^+ \left( -D'_{21} M_{xx} - D'_{22} M_{yy} - D'_{23} M_{xy} \right) d\Omega
\end{aligned} \tag{2.157}$$

and the tangent system matrix terms as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} M_{yy}^+ \varphi_{y,y}^+ d\Omega + \int_{\Omega} \bar{\delta} M_{yy}^+ \left( -B'_{21} N_{xx}^+ - B'_{22} N_{yy}^+ - B'_{23} N_{xy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{yy}^+ \left( -D'_{21} M_{xx}^+ - D'_{22} M_{yy}^+ - D'_{23} M_{xy}^+ \right) d\Omega
\end{aligned} \tag{2.158}$$

### Linearization of $\delta M_{xy}$ terms

Inserting incremental forms of field variables into non-linear equation (2.130) yields,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} \left( M_{xy} + M_{xy}^+ \right) \left( \varphi_{x,y} + \varphi_{x,y}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( M_{xy} + M_{xy}^+ \right) \left( \varphi_{y,x} + \varphi_{y,x}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( M_{xy} + M_{xy}^+ \right) \left[ -B'_{31} \left( N_{xx} + N_{xx}^+ \right) - B'_{32} \left( N_{yy} + N_{yy}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( M_{xy} + M_{xy}^+ \right) \left[ -B'_{33} \left( N_{xy} + N_{xy}^+ \right) - D'_{31} \left( M_{xx} + M_{xx}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( M_{xy} + M_{xy}^+ \right) \left[ -D'_{32} \left( M_{yy} + M_{yy}^+ \right) - D'_{33} \left( M_{xy} + M_{xy}^+ \right) \right] d\Omega = 0
\end{aligned} \tag{2.159}$$

If the variation is applied to the increments only, (2.159) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} M_{xy}^+ \varphi_{x,y} d\Omega + \int_{\Omega} \bar{\delta} M_{xy}^+ \varphi_{x,y}^+ d\Omega + \int_{\Omega} \bar{\delta} M_{xy}^+ \varphi_{y,x} d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xy}^+ \varphi_{y,x}^+ d\Omega + \int_{\Omega} \bar{\delta} M_{xy}^+ (-B'_{31} N_{xx} - B'_{32} N_{yy} - B'_{33} N_{xy}) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xy}^+ (-D'_{31} M_{xx} - D'_{32} M_{yy} - D'_{33} M_{xy}) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xy}^+ (-B'_{31} N_{xx}^+ - B'_{32} N_{yy}^+ - B'_{33} N_{xy}^+) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xy}^+ (-D'_{31} M_{xx}^+ - D'_{32} M_{yy}^+ - D'_{33} M_{xy}^+) d\Omega = 0
\end{aligned} \tag{2.160}$$

After neglecting higher order terms, one obtains the correction expression as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} M_{xy}^+ \varphi_{x,y} d\Omega + \int_{\Omega} \bar{\delta} M_{xy}^+ \varphi_{y,x} d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xy}^+ (-B'_{31} N_{xx} - B'_{32} N_{yy} - B'_{33} N_{xy}) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xy}^+ (-D'_{31} M_{xx} - D'_{32} M_{yy} - D'_{33} M_{xy}) d\Omega
\end{aligned} \tag{2.161}$$

and the tangent system matrix terms as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} M_{xy}^+ \varphi_{x,y}^+ d\Omega + \int_{\Omega} \bar{\delta} M_{xy}^+ \varphi_{y,x}^+ d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xy}^+ (-B'_{31} N_{xx}^+ - B'_{32} N_{yy}^+ - B'_{33} N_{xy}^+) d\Omega \\
& + \int_{\Omega} \bar{\delta} M_{xy}^+ (-D'_{31} M_{xx}^+ - D'_{32} M_{yy}^+ - D'_{33} M_{xy}^+) d\Omega
\end{aligned} \tag{2.162}$$

### Linearization of $\delta S_{xz}$ terms

Inserting incremental forms of field variables into non-linear equation (2.131) yields,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} (S_{xz} + S_{xz}^+) (w_{,x} + w_{,x}^+) d\Omega + \int_{\Omega} \bar{\delta} (S_{xz} + S_{xz}^+) (\varphi_x + \varphi_x^+) d\Omega \\
& + \int_{\Omega} \bar{\delta} (S_{xz} + S_{xz}^+) \left[ -C'_{11} (S_{xz} + S_{xz}^+) - C'_{12} (S_{yz} + S_{yz}^+) \right] d\Omega = 0
\end{aligned} \tag{2.163}$$

If the variation is applied to the increments only, (2.163) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} S_{xz}^+ w_{,x}^+ d\Omega + \int_{\Omega} \bar{\delta} S_{xz}^+ w_{,x}^+ d\Omega + \int_{\Omega} \bar{\delta} S_{xz}^+ \varphi_x^+ d\Omega + \int_{\Omega} \bar{\delta} S_{xz}^+ \varphi_x^+ d\Omega \\
& + \int_{\Omega} \bar{\delta} S_{xz}^+ (-C'_{11} S_{xz} - C'_{12} S_{yz}) d\Omega + \int_{\Omega} \bar{\delta} S_{xz}^+ (-C'_{11} S_{xz}^+ - C'_{12} S_{yz}^+) d\Omega = 0
\end{aligned} \tag{2.164}$$

After neglecting higher order terms, one obtains the correction expression as,

$$= \int_{\Omega} \bar{\delta} S_{xz}^+ w_{,x}^+ d\Omega + \int_{\Omega} \bar{\delta} S_{xz}^+ \varphi_x^+ d\Omega + \int_{\Omega} \bar{\delta} S_{xz}^+ (-C'_{11} S_{xz} - C'_{12} S_{yz}) d\Omega \tag{2.165}$$

and the tangent system matrix terms as,

$$= \int_{\Omega} \bar{\delta} S_{xz}^+ w_{,x}^+ d\Omega + \int_{\Omega} \bar{\delta} S_{xz}^+ \varphi_x^+ d\Omega + \int_{\Omega} \bar{\delta} S_{xz}^+ (-C'_{11} S_{xz}^+ - C'_{12} S_{yz}^+) d\Omega \tag{2.166}$$

### Linearization of $\delta S_{yz}$ terms

Inserting incremental forms of field variables into non-linear equation (2.132) yields,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} (S_{yz} + S_{yz}^+) (w_{,y} + w_{,y}^+) d\Omega + \int_{\Omega} \bar{\delta} (S_{yz} + S_{yz}^+) (\varphi_y + \varphi_y^+) d\Omega \\
& + \int_{\Omega} \bar{\delta} (S_{yz} + S_{yz}^+) \left[ -C'_{21} (S_{xz} + S_{xz}^+) S_{xz} - C'_{22} (S_{yz} + S_{yz}^+) \right] d\Omega = 0
\end{aligned} \tag{2.167}$$

If the variation is applied to the increments only, (2.167) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} S_{yz}^+ w_{,y}^+ d\Omega + \int_{\Omega} \bar{\delta} S_{yz}^+ w_{,y}^+ d\Omega \\
& + \int_{\Omega} \bar{\delta} S_{yz}^+ \varphi_y^+ d\Omega + \int_{\Omega} \bar{\delta} S_{yz}^+ \varphi_y^+ d\Omega \\
& + \int_{\Omega} \bar{\delta} S_{yz}^+ (-C'_{21} S_{xz} - C'_{22} S_{yz}) d\Omega \\
& + \int_{\Omega} \bar{\delta} S_{yz}^+ (-C'_{21} S_{xz}^+ - C'_{22} S_{yz}^+) d\Omega = 0
\end{aligned} \tag{2.168}$$

After neglecting higher order terms, one obtains the correction expression as,

$$= \int_{\Omega} \bar{\delta} S_{yz}^+ w_{,y}^+ d\Omega + \int_{\Omega} \bar{\delta} S_{yz}^+ \varphi_y^+ d\Omega + \int_{\Omega} \bar{\delta} S_{yz}^+ (-C'_{21} S_{xz} - C'_{22} S_{yz}) d\Omega \tag{2.169}$$

and the tangent system matrix terms as,

$$= \int_{\Omega} \bar{\delta} S_{yz}^+ w_{,y}^+ d\Omega + \int_{\Omega} \bar{\delta} S_{yz}^+ \varphi_y^+ d\Omega + \int_{\Omega} \bar{\delta} S_{yz}^+ \left( -C'_{21} S_{xz}^+ - C'_{22} S_{yz}^+ \right) d\Omega \quad (2.170)$$

### Linearization of $\delta u$ terms

Inserting incremental forms of field variables into non-linear equation (2.133) yields,

$$\begin{aligned} & \int_{\Omega} \bar{\delta} \left( u_{,x} + u_{,x}^+ \right) \left( N_{xx} + N_{xx}^+ \right) d\Omega + \int_{\Omega} \bar{\delta} \left( u_{,y} + u_{,y}^+ \right) \left( N_{xy} + N_{xy}^+ \right) d\Omega \\ & - \int_{\Omega} \bar{\delta} \left( u + u^+ \right) q_x d\Omega = 0 \end{aligned} \quad (2.171)$$

If the variation is applied to the increments only, (2.171) becomes when arranged,

$$\begin{aligned} & \int_{\Omega} \bar{\delta} u_{,x}^+ N_{xx} d\Omega + \int_{\Omega} \bar{\delta} u_{,x}^+ N_{xx}^+ d\Omega + \int_{\Omega} \bar{\delta} u_{,y}^+ N_{xy} d\Omega \\ & + \int_{\Omega} \bar{\delta} u_{,y}^+ N_{xy}^+ d\Omega - \int_{\Omega} \bar{\delta} u^+ q_x d\Omega \end{aligned} \quad (2.172)$$

After neglecting higher order terms, one obtains the correction expression as,

$$= \int_{\Omega} \bar{\delta} u_{,x}^+ N_{xx} d\Omega + \int_{\Omega} \bar{\delta} u_{,y}^+ N_{xy} d\Omega - \int_{\Omega} \bar{\delta} u^+ q_x d\Omega \quad (2.173)$$

and the tangent system matrix terms as,

$$= \int_{\Omega} \bar{\delta} u_{,x}^+ N_{xx}^+ d\Omega + \int_{\Omega} \bar{\delta} u_{,y}^+ N_{xy}^+ d\Omega \quad (2.174)$$

### Linearization of $\delta v$ terms

Inserting incremental forms of field variables into non-linear equation (2.134) yields,

$$\begin{aligned} & \int_{\Omega} \bar{\delta} \left( v_{,x} + v_{,x}^+ \right) \left( N_{xy} + N_{xy}^+ \right) d\Omega + \int_{\Omega} \bar{\delta} \left( v_{,y} + v_{,y}^+ \right) \left( N_{yy} + N_{yy}^+ \right) d\Omega \\ & - \int_{\Omega} \bar{\delta} \left( v + v^+ \right) q_y d\Omega = 0 \end{aligned} \quad (2.175)$$

If the variation is applied to the increments only, (2.175) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} v_{,x}^+ N_{xy} d\Omega + \int_{\Omega} \bar{\delta} v_{,x}^+ N_{xy}^+ d\Omega + \int_{\Omega} \bar{\delta} v_{,y}^+ N_{yy} d\Omega \\
& + \int_{\Omega} \bar{\delta} v_{,y}^+ N_{yy}^+ d\Omega - \int_{\Omega} \bar{\delta} v q_y d\Omega = 0
\end{aligned} \tag{2.176}$$

After neglecting higher order terms, one obtains the correction expression as,

$$= \int_{\Omega} \bar{\delta} v_{,x}^+ N_{xy} d\Omega + \int_{\Omega} \bar{\delta} v_{,y}^+ N_{yy} d\Omega - \int_{\Omega} \bar{\delta} v q_y d\Omega \tag{2.177}$$

and the tangent system matrix terms as,

$$= \int_{\Omega} \bar{\delta} v_{,x}^+ N_{xy}^+ d\Omega + \int_{\Omega} \bar{\delta} v_{,y}^+ N_{yy}^+ d\Omega \tag{2.178}$$

### Linearization of $\delta w$ terms

Inserting incremental forms of field variables into non-linear equation (2.135) yields,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} \left( w_{,x} + w_{,x}^+ \right) \left[ \left( N_{xx} + N_{xx}^+ \right) \left( w_{,x} + w_{,x}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( w_{,x} + w_{,x}^+ \right) \left[ \left( N_{xy} + N_{xy}^+ \right) \left( w_{,y} + w_{,y}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( w_{,x} + w_{,x}^+ \right) \left( Q_{xz} + Q_{xz}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( w_{,y} + w_{,y}^+ \right) \left[ \left( N_{xy} + N_{xy}^+ \right) \left( w_{,x} + w_{,x}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( w_{,y} + w_{,y}^+ \right) \left[ \left( N_{yy} + N_{yy}^+ \right) \left( w_{,y} + w_{,y}^+ \right) \right] d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( w_{,y} + w_{,y}^+ \right) \left( Q_{yz} + Q_{yz}^+ \right) d\Omega \\
& - \int_{\Omega} \bar{\delta} \left( w + w^+ \right) q_z d\Omega = 0
\end{aligned} \tag{2.179}$$

If the variation is applied to the increments only, (2.179) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} w_{,x}^+ \left( N_{xx} w_{,x} + N_{xx}^+ w_{,x}^+ + N_{xx}^+ w_{,x}^+ + N_{xx}^+ w_{,x}^+ + N_{xy} w_{,y} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} w_{,y}^+ \left( N_{xy} w_{,y} + N_{xy}^+ w_{,y}^+ + N_{xy}^+ w_{,y}^+ + N_{xy}^+ w_{,y}^+ + S_{xz} + S_{xz}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} w_{,y}^+ \left( N_{xy} w_{,x} + N_{xy}^+ w_{,x}^+ + N_{xy}^+ w_{,x}^+ + N_{xy}^+ w_{,x}^+ + N_{yy} w_{,y} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} w_{,y}^+ \left( N_{yy} w_{,y} + N_{yy}^+ w_{,y}^+ + N_{yy}^+ w_{,y}^+ + N_{yy}^+ w_{,y}^+ + S_{yz} + S_{yz}^+ \right) d\Omega \\
& - \int_{\Omega} \bar{\delta} w q_z d\Omega = 0
\end{aligned} \tag{2.180}$$

After neglecting higher order terms, one obtains the correction expression as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} w_{,x}^+ \left( N_{xx} w_{,x} + N_{xy} w_{,y} + S_{xz} \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} w_{,y}^+ \left( N_{xy} w_{,x} + N_{yy} w_{,y} + S_{yz} \right) d\Omega - \int_{\Omega} \bar{\delta} w q_z d\Omega
\end{aligned} \tag{2.181}$$

and the tangent system matrix terms as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} w_{,x}^+ \left( N_{xx}^+ w_{,x}^+ + N_{xx}^+ w_{,x}^+ + N_{xy}^+ w_{,y}^+ + N_{xy}^+ w_{,y}^+ + S_{xz}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} w_{,y}^+ \left( N_{xy}^+ w_{,x}^+ + N_{xy}^+ w_{,x}^+ + N_{yy}^+ w_{,y}^+ + N_{yy}^+ w_{,y}^+ + S_{yz}^+ \right) d\Omega
\end{aligned} \tag{2.182}$$

### Linearization of $\delta\varphi_x$ terms

Inserting incremental forms of field variables into non-linear equation (2.136) yields,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} \left( \varphi_{x,x} + \varphi_{x,x}^+ \right) \left( M_{xx} + M_{xx}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( \varphi_{x,y} + \varphi_{x,y}^+ \right) \left( M_{xy} + M_{xy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( \varphi_x + \varphi_x^+ \right) \left( S_{xz} + S_{xz}^+ \right) d\Omega - \int_{\Omega} \bar{\delta} \left( \varphi_x + \varphi_x^+ \right) m_x d\Omega = 0
\end{aligned} \tag{2.183}$$

If the variation is applied to the increments only, (2.183) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} \varphi_{x,x}^+ M_{xx} d\Omega + \int_{\Omega} \bar{\delta} \varphi_{x,x}^+ M_{xx}^+ d\Omega + \int_{\Omega} \bar{\delta} \varphi_{x,y}^+ M_{xy} d\Omega \\
& + \int_{\Omega} \bar{\delta} \varphi_{x,y}^+ M_{xy}^+ d\Omega + \int_{\Omega} \bar{\delta} \varphi_x^+ S_{xz} d\Omega \\
& + \int_{\Omega} \bar{\delta} \varphi_x^+ S_{xz}^+ d\Omega - \int_{\Omega} \bar{\delta} \varphi_x^+ m_x d\Omega = 0
\end{aligned} \tag{2.184}$$

After neglecting higher order terms, one obtains the correction expression as,

$$\begin{aligned}
& = \int_{\Omega} \bar{\delta} \varphi_{x,x}^+ M_{xx} d\Omega + \int_{\Omega} \bar{\delta} \varphi_{x,y}^+ M_{xy} d\Omega \\
& + \int_{\Omega} \bar{\delta} \varphi_x^+ S_{xz} d\Omega - \int_{\Omega} \bar{\delta} \varphi_x^+ m_x d\Omega
\end{aligned} \tag{2.185}$$

and the tangent system matrix terms as,

$$= \int_{\Omega} \bar{\delta} \varphi_{x,x}^+ M_{xx}^+ d\Omega + \int_{\Omega} \bar{\delta} \varphi_{x,y}^+ M_{xy}^+ d\Omega + \int_{\Omega} \bar{\delta} \varphi_x^+ S_{xz}^+ d\Omega \tag{2.186}$$

### Linearization of $\delta\varphi_y$ terms

Inserting incremental forms of field variables into non-linear equation (2.137) yields,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} \left( \varphi_{y,x} + \varphi_{y,x}^+ \right) \left( M_{xy} + M_{xy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( \varphi_{y,y} + \varphi_{y,y}^+ \right) \left( M_{yy} + M_{yy}^+ \right) d\Omega \\
& + \int_{\Omega} \bar{\delta} \left( \varphi_y + \varphi_y^+ \right) \left( S_{yz} + S_{yz}^+ \right) d\Omega - \int_{\Omega} \bar{\delta} \left( \varphi_y + \varphi_y^+ \right) m_y d\Omega = 0
\end{aligned} \tag{2.187}$$

If the variation is applied to the increments only, (2.187) becomes when arranged,

$$\begin{aligned}
& \int_{\Omega} \bar{\delta} \varphi_{y,x}^+ M_{xy} d\Omega + \int_{\Omega} \bar{\delta} \varphi_{y,x}^+ M_{xy}^+ d\Omega + \int_{\Omega} \bar{\delta} \varphi_{y,y}^+ M_{yy} d\Omega \\
& + \int_{\Omega} \bar{\delta} \varphi_{y,y}^+ M_{yy}^+ d\Omega + \int_{\Omega} \bar{\delta} \varphi_y^+ S_{yz} d\Omega \\
& + \int_{\Omega} \bar{\delta} \varphi_y^+ S_{yz}^+ d\Omega - \int_{\Omega} \bar{\delta} \varphi_y^+ m_y d\Omega = 0
\end{aligned} \tag{2.188}$$

After neglecting higher order terms, one obtains the correction expression as,

$$\begin{aligned}
&= \int_{\Omega} \bar{\delta} \phi_{y,x}^+ M_{xy} d\Omega + \int_{\Omega} \bar{\delta} \phi_{y,y}^+ M_{yy} d\Omega \\
&+ \int_{\Omega} \bar{\delta} \phi_y^+ S_{yz} d\Omega - \int_{\Omega} \bar{\delta} \phi_y^+ m_y d\Omega
\end{aligned} \tag{2.189}$$

and the tangent system matrix terms as,

$$\begin{aligned}
&= \int_{\Omega} \bar{\delta} \phi_{y,x}^+ M_{xy}^+ d\Omega \\
&+ \int_{\Omega} \bar{\delta} \phi_{y,y}^+ M_{yy}^+ d\Omega + \int_{\Omega} \bar{\delta} \phi_y^+ S_{yz}^+ d\Omega
\end{aligned} \tag{2.190}$$

### 2.3.5.3 Finite element matrices

Global form of the finite element system matrices presented for the non-linear problem is as follows,

$$(\mathbf{K}^{++} + \mathbf{K}^{nl})^+ \mathbf{X} = \mathbf{F} - \mathbf{K}^+ \tag{2.191}$$

Here  $\mathbf{K}^{++}$  is linear part of the system matrix has the following submatrices which are constructed by the terms in linear parts of system matrix expressions presented in section 2.3.5.2.

$$\mathbf{K}^{++} = \begin{bmatrix} \mathbf{K}_{NN}^{++} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{NU}^{++} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{MM}^{++} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{MR}^{++} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{SS}^{++} & \mathbf{K}_{SU}^{++} & \mathbf{K}_{SR}^{++} \\ \mathbf{K}_{UN}^{++} & \mathbf{0} & \mathbf{K}_{US}^{++} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{RM}^{++} & \mathbf{K}_{RS}^{++} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{2.192}$$

$\mathbf{K}^{nl}$  in (2.191) is the system matrix that contains the non-linear terms due to geometrical non-linearity. It has the following submatrices, which are produced through the non-linear terms existing in the system matrix expressions gathered in section 2.3.5.2,

$${}^{++}\mathbf{K}^{nl} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & {}^{++}\mathbf{K}_{NU}^{nl} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^{++}\mathbf{K}_{UN}^{nl} & \mathbf{0} & \mathbf{0} & {}^{++}\mathbf{K}_{UU}^{nl} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (2.193)$$

${}^+\mathbf{K}$  in (2.191) is the correction vector and its submatrices are established by the correction expressions obtained in section 2.3.5.2.

$${}^+\mathbf{K} = \left\{ {}^+\mathbf{K}_N \quad {}^+\mathbf{K}_M \quad {}^+\mathbf{K}_S \quad {}^+\mathbf{K}_U \quad {}^+\mathbf{K}_R \right\}^T \quad (2.194)$$

The external load vector  ${}^+\mathbf{F}$  has only displacement related submatrices. To investigate the problems containing external distributed moment loads, rotary related submatrices in external load vector has to be constructed.

$${}^+\mathbf{F} = \left\{ \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad {}^+\mathbf{F}_U \quad \mathbf{0} \right\}^T \quad (2.195)$$

The unknown vector  $\mathbf{X}$  has both displacement and stress resultant type variables in it.

$$\mathbf{X} = \{\mathbf{N} \quad \mathbf{M} \quad \mathbf{S} \quad \mathbf{U} \quad \mathbf{R}\}^T \quad (2.196)$$

In equation (2.196),  $\mathbf{N}(N_{xx}, N_{yy}, N_{xy})$  contains member force variables,  $\mathbf{M}(M_{xx}, M_{yy}, M_{xy})$  contains bending and twisting moment variables,  $\mathbf{S}(S_{xz}, S_{yz})$  contains shear force variables,  $\mathbf{U}(u, v, w)$  contains translation type displacement variables and finally  $\mathbf{R}(\phi_x, \phi_y)$  contains rotation type displacement variables.

#### 2.3.5.4 Iterative solution procedure

The system of equations presented in section 2.3.5.3 are solved through an iterative solution procedure. To achieve a rapid convergence, the Newton–Raphson procedure is adopted where the tangent system matrix is used at every iteration step [124].

According to the scheme of the adopted method the system of equations is presented as,

$$\left( \overset{++}{\mathbf{K}} + \overset{++}{\mathbf{K}}^{nl(i-1)} \right) \overset{+}{\mathbf{X}}^{(i)} = \overset{+}{\mathbf{F}} - \overset{+}{\mathbf{K}}^{(i-1)} \quad (2.197)$$

where  $i$  presents the number of each iteration step. At an individual iteration step the increment of field variables is calculated from

$$\overset{+}{\mathbf{X}}^{(i)} = \left( \overset{++}{\mathbf{K}} + \overset{++}{\mathbf{K}}^{nl(i-1)} \right)^{-1} \left( \overset{+}{\mathbf{F}} - \overset{+}{\mathbf{K}}^{(i-1)} \right) \quad (2.198)$$

and the unknowns are calculated at every step after an update procedure as follows,

$$\bar{\mathbf{X}}^{(i)} = \mathbf{X}^{(i-1)} + \overset{+}{\mathbf{X}}^{(i)} \quad (2.199)$$

The iteration lasts until the second paranthesis in (2.198) becomes zero (with a predetermined tolerance [125]). This is the case where the external force vector becomes equal to the internal system vector.

### 2.3.5.5 Finite element submatrices in explicit form

Linear part of the system matrix contains the following submatrices given in detail,

$$\overset{++}{\mathbf{K}}_{NN} = \int_{\Omega} \begin{bmatrix} -\bar{\delta} \overset{+}{N}_{xx} A'_{11} \overset{+}{N}_{xx} & -\bar{\delta} \overset{+}{N}_{xx} A'_{12} \overset{+}{N}_{yy} & -\bar{\delta} \overset{+}{N}_{xx} A'_{13} \overset{+}{N}_{xy} \\ -\bar{\delta} \overset{+}{N}_{yy} A'_{21} \overset{+}{N}_{xx} & -\bar{\delta} \overset{+}{N}_{yy} A'_{22} \overset{+}{N}_{yy} & -\bar{\delta} \overset{+}{N}_{yy} A'_{23} \overset{+}{N}_{xy} \\ -\bar{\delta} \overset{+}{N}_{xy} A'_{31} \overset{+}{N}_{xx} & -\bar{\delta} \overset{+}{N}_{xy} A'_{32} \overset{+}{N}_{yy} & -\bar{\delta} \overset{+}{N}_{xy} A'_{33} \overset{+}{N}_{xy} \end{bmatrix} d\Omega \quad (2.200)$$

$$\overset{++}{\mathbf{K}}_{NU} = \int_{\Omega} \begin{bmatrix} \bar{\delta} \overset{+}{N}_{xx} u_{,x} & 0 & \bar{\delta} \overset{+}{N}_{xx} w_{,x}^i w_{,x} \\ 0 & \bar{\delta} \overset{+}{N}_{yy} v_{,y} & \bar{\delta} \overset{+}{N}_{yy} w_{,y}^i w_{,y} \\ \bar{\delta} \overset{+}{N}_{xy} u_{,y} & \bar{\delta} \overset{+}{N}_{xy} v_{,x} & \left( \bar{\delta} \overset{+}{N}_{xy} w_{,x}^i w_{,y} + \bar{\delta} \overset{+}{N}_{xy} w_{,y}^i w_{,x} \right) \end{bmatrix} d\Omega \quad (2.201)$$

$$\mathbf{K}_{MM}^{++} = \int_{\Omega} \begin{bmatrix} -\bar{\delta} M_{xx}^+ D'_{11} M_{xx}^+ & -\bar{\delta} M_{xx}^+ D'_{12} M_{yy}^+ & -\bar{\delta} M_{xx}^+ D'_{13} M_{xy}^+ \\ -\bar{\delta} M_{yy}^+ D'_{21} M_{xx}^+ & -\bar{\delta} M_{yy}^+ D'_{22} M_{yy}^+ & -\bar{\delta} M_{yy}^+ D'_{23} M_{xy}^+ \\ -\bar{\delta} M_{xy}^+ D'_{31} M_{xx}^+ & -\bar{\delta} M_{xy}^+ D'_{32} M_{yy}^+ & -\bar{\delta} M_{xy}^+ D'_{33} M_{xy}^+ \end{bmatrix} d\Omega \quad (2.202)$$

$$\mathbf{K}_{MR}^{++} = \int_{\Omega} \begin{bmatrix} \bar{\delta} M_{xx}^+ \phi_{x,x}^+ & 0 \\ 0 & \bar{\delta} M_{yy}^+ \phi_{y,y}^+ \\ \bar{\delta} M_{xy}^+ \phi_{x,y}^+ & \bar{\delta} M_{xy}^+ \phi_{y,x}^+ \end{bmatrix} d\Omega \quad (2.203)$$

$$\mathbf{K}_{SS}^{++} = \int_{\Omega} \begin{bmatrix} -\bar{\delta} S_{xz}^+ C'_{11} S_{xz}^+ & -\bar{\delta} S_{xz}^+ C'_{12} S_{yz}^+ \\ -\bar{\delta} S_{yz}^+ C'_{21} S_{xz}^+ & -\bar{\delta} S_{yz}^+ C'_{22} S_{yz}^+ \end{bmatrix} d\Omega \quad (2.204)$$

$$\mathbf{K}_{SU}^{++} = \int_{\Omega} \begin{bmatrix} 0 & 0 & \bar{\delta} S_{xz}^+ w_{,x}^+ \\ 0 & 0 & \bar{\delta} S_{yz}^+ w_{,y}^+ \end{bmatrix} d\Omega \quad (2.205)$$

$$\mathbf{K}_{SR}^{++} = \int_{\Omega} \begin{bmatrix} \bar{\delta} S_{xz}^+ \phi_x^+ & 0 \\ 0 & \bar{\delta} S_{yz}^+ \phi_y^+ \end{bmatrix} d\Omega \quad (2.206)$$

$$\mathbf{K}_{UN}^{++} = \int_{\Omega} \begin{bmatrix} \bar{\delta} u_{,x}^+ N_{xx}^+ & 0 & \bar{\delta} u_{,y}^+ N_{xy}^+ \\ 0 & \bar{\delta} v_{,y}^+ N_{yy}^+ & \bar{\delta} v_{,x}^+ N_{xy}^+ \\ 0 & 0 & 0 \end{bmatrix} d\Omega \quad (2.207)$$

$$\mathbf{K}_{US}^{++} = \int_{\Omega} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \bar{\delta} w_{,x}^+ S_{xz}^+ & \bar{\delta} w_{,y}^+ S_{yz}^+ \end{bmatrix} d\Omega \quad (2.208)$$

$$\mathbf{K}_{RM}^{++} = \int_{\Omega} \begin{bmatrix} \bar{\delta} \phi_{x,x}^+ M_{xx}^+ & 0 & \bar{\delta} \phi_{x,y}^+ M_{xy}^+ \\ 0 & \bar{\delta} \phi_{y,y}^+ M_{yy}^+ & \bar{\delta} \phi_{y,x}^+ M_{xy}^+ \end{bmatrix} d\Omega \quad (2.209)$$

$$\mathbf{K}_{RS}^{++} = \int_{\Omega} \begin{bmatrix} \bar{\delta} \phi_x^+ S_{xz}^+ & 0 \\ 0 & \bar{\delta} \phi_y^+ S_{yz}^+ \end{bmatrix} d\Omega \quad (2.210)$$

Non-linear part of the system matrix contains the following submatrices given in detail,

$$\mathbf{K}_{NU}^{++nl} = \int_{\Omega} \begin{bmatrix} 0 & 0 & \bar{\delta} N_{xx}^+ w_{,x}^j w_{,x}^+ \\ 0 & 0 & \bar{\delta} N_{yy}^+ w_{,y}^j w_{,y}^+ \\ 0 & 0 & \bar{\delta} N_{xy}^+ w_{,x}^j w_{,y}^+ + \bar{\delta} N_{xy}^+ w_{,y}^j w_{,x}^+ \end{bmatrix} d\Omega \quad (2.211)$$

$$\mathbf{K}_{UN}^{++nl} = \int_{\Omega} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{\delta} w_{,x}^+ w_{,x}^j N_{xx}^+ & \bar{\delta} w_{,y}^+ w_{,y}^j N_{yy}^+ & \left( \bar{\delta} w_{,x}^+ w_{,y}^j N_{xy}^+ + \bar{\delta} w_{,y}^+ w_{,x}^j N_{xy}^+ \right) \end{bmatrix} d\Omega \quad (2.212)$$

$$\mathbf{K}_{NU}^{++} = \int_{\Omega} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left( \bar{\delta} w_{,x}^+ N_{xx}^i w_{,x}^+ + \bar{\delta} w_{,x}^+ N_{xy}^i w_{,y}^+ \right. \\ & & \left. + \bar{\delta} w_{,y}^+ N_{xy}^i w_{,x}^+ + \bar{\delta} w_{,y}^+ N_{yy}^i w_{,y}^+ \right) \end{bmatrix} d\Omega \quad (2.213)$$

Correction vector contains the following subvectors given in detail,

$$\mathbf{K}_N^+ = \int_{\Omega} \left\{ \begin{array}{l} \bar{\delta} N_{xx}^+ \left( u_{,x}^i + \frac{1}{2} (w_{,x}^j)^2 - A'_{11} N_{xx}^i - A'_{12} N_{yy}^i - A'_{13} N_{xy}^i \right) \\ \bar{\delta} N_{yy}^+ \left( v_{,y}^i + \frac{1}{2} (w_{,y}^j)^2 - A'_{21} N_{xx}^i - A'_{22} N_{yy}^i - A'_{23} N_{xy}^i \right) \\ \bar{\delta} N_{xy}^+ \left( u_{,y}^i + v_{,x}^i + w_{,x}^j w_{,y}^j - A'_{31} N_{xx}^i - A'_{32} N_{yy}^i \right. \\ \left. - A'_{33} N_{xy}^i - B'_{31} M_{xx}^i - B'_{32} M_{yy}^i - B'_{33} M_{xy}^i \right) \end{array} \right\} d\Omega \quad (2.214)$$

$$\begin{aligned}
\mathbf{K}_M^+ = \int_{\Omega} \left\{ \begin{array}{l} \left( \bar{\delta} M_{xx}^+ (\phi_{x,x}^i - B'_{11} N_{xx}^i - B'_{12} N_{yy}^i - B'_{13} N_{xy}^i) \right) \\ \left( + \bar{\delta} M_{xx}^+ (-D'_{11} M_{xx}^i - D'_{12} M_{yy}^i - D'_{13} M_{xy}^i) \right) \\ \left( \bar{\delta} M_{yy}^+ (\phi_{y,y}^i - B'_{21} N_{xx}^i - B'_{22} N_{yy}^i - B'_{23} N_{xy}^i) \right) \\ \left( + \bar{\delta} M_{yy}^+ (-D'_{21} M_{xx}^i - D'_{22} M_{yy}^i - D'_{23} M_{xy}^i) \right) \\ \left( \bar{\delta} M_{xy}^+ (\phi_{x,y}^i + \phi_{y,x}^i - B'_{31} N_{xx}^i - B'_{32} N_{yy}^i - B'_{33} N_{xy}^i) \right) \\ \left( + \bar{\delta} M_{xy}^+ (-D'_{31} M_{xx}^i - D'_{32} M_{yy}^i - D'_{33} M_{xy}^i) \right) \end{array} \right\} d\Omega \quad (2.215)
\end{aligned}$$

$$\mathbf{K}_S^+ = \int_{\Omega} \left\{ \begin{array}{l} \bar{\delta} S_{xz}^+ (w_{,x}^i + \phi_x^i - C'_{11} S_{xz}^i - C'_{12} S_{yz}^i) \\ \bar{\delta} S_{yz}^+ (w_{,y}^i + \phi_y^i - C'_{21} S_{xz}^i - C'_{22} S_{yz}^i) \end{array} \right\} d\Omega \quad (2.216)$$

$$\mathbf{K}_U^+ = \int_{\Omega} \left\{ \begin{array}{l} \bar{\delta} u_{,x}^+ N_{xx}^i + \bar{\delta} u_{,y}^+ N_{xy}^i \\ \bar{\delta} v_{,x}^+ N_{xy}^i + \bar{\delta} v_{,y}^+ N_{yy}^i \\ \left( \bar{\delta} w_{,x}^+ (N_{xx}^i w_{,x}^i + N_{xy}^i w_{,y}^i + S_{xz}^i) \right) \\ \left( + \bar{\delta} w_{,y}^+ (N_{xy}^i w_{,x}^i + N_{yy}^i w_{,y}^i + S_{yz}^i) \right) \end{array} \right\} d\Omega \quad (2.217)$$

$$\mathbf{K}_R^+ = \int_{\Omega} \left\{ \begin{array}{l} \bar{\delta} \phi_{x,x}^+ M_{xx}^i + \bar{\delta} \phi_{x,y}^+ M_{xy}^i + \bar{\delta} \phi_x^+ S_{xz}^i \\ \bar{\delta} \phi_{y,x}^+ M_{xy}^i + \bar{\delta} \phi_{y,y}^+ M_{yy}^i + \bar{\delta} \phi_y^+ S_{yz}^i \end{array} \right\} d\Omega \quad (2.218)$$

The external load vector for the Mindlin plate is given as follows,

$$\mathbf{F}_U^+ = \int_{\Omega} \left\{ \begin{array}{l} \bar{\delta} u q_x^+ \\ \bar{\delta} v q_y^+ \\ \bar{\delta} w q_z^+ \end{array} \right\} d\Omega \quad (2.219)$$

### 2.3.5.6 Discretization of plate domain with finite elements

Generated finite element matrices require  $C^0$  type continuity. As a consequence the plate domain is discretized by quadrilateral isoparametric elements having one node

at each corner. According to the two field mixed formulation each node has 13 DOF and each element has a total of  $4 \times 13$  DOF. Bilinear shape functions are adopted to interpolate field variables. Numerical integrations are performed on master elements according to Gauss quadrature with  $2 \times 2$  scheme [126,127].

### 2.3.6 Postprocess of strains and stresses

Presented mixed finite element procedure considers displacements as well as internal forces and force couples as local unknowns. Therefore, internal forces and moments are directly acquired as outputs of solution without any post process unlike displacement based finite element procedures. As described before (see section 2.3.2.4), deformations of the plate's middle plane (strains and curvatures) can be calculated from partial derivatives of displacements or from constitutive relations by using internal forces and force couples with material matrices. Here in-plane stresses are calculated by using constitutive relations. Due to the Mindlin plate's deformation field assumptions transverse stresses through the cross-section cannot be calculated from constitutive relations. Instead of this, in this study they are calculated by using a method called equivalent section [128,129].

#### 2.3.6.1 Calculation of in-plane strains and stresses

Once internal forces and force couples are calculated at the nodes, strains at mid-plane and in-plane curvatures of the plate can be calculated directly by a matrix manipulation. According to the equations (2.75) and (2.76), deformations of the middle-plane are calculated as;

$$\{\epsilon\} = [A']\{N\} + [B']\{M\} \quad (2.220)$$

$$\{\kappa\} = [B']\{N\} + [D']\{M\} \quad (2.221)$$

Using the mid-plane deformation (where  $z = 0$ ), in-plane strains at any point  $z$  on the cross section can be expressed in terms of Mindlin assumptions (2.36),

$$\begin{Bmatrix} \varepsilon_{xx}^* \\ \varepsilon_{yy}^* \\ \gamma_{xy}^* \\ \gamma_{xz}^* \\ \gamma_{yz}^* \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \\ \kappa_{xz} \\ \kappa_{yz} \end{Bmatrix} \quad (2.222)$$

After obtaining central deformations of the plate, in-plane stresses at any  $z$  coordinate of a layer with number  $L$  is calculated using relation (2.49) as;

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [\bar{Q}]_{L(1,3 \times 1,3)} \left( \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \right) \quad (2.223)$$

It must be pointed out that during the calculation of strains and stresses, no differentiation operation is proceeded which can actually cause some significant errors in numerical analysis.

### 2.3.6.2 Calculation of transverse strains and stresses

When dealing with the analysis of composite laminates, accurate calculation of transverse shear strains and stresses is significantly important. Having a considerably equivalent transverse shear modulus compared to the effective elastic modulus in their fiber directions, fiber reinforced laminated composites are quite sensitive to the transverse stresses. Assuming a constant shear stress along the laminate thickness by the Mindlin plate theory yields an inexact stress distribution. Without changing the plate kinematics, a correction factor (shear correction factor) is adopted to improve the accuracy of shear prediction. Although the shear forces are calculated accurately with the use of shear correction factor, the shear stresses are still uniform through the cross-section, which leads wrong prediction of transverse stresses through the laminate. To avoid this deficiency, transformed section method (equivalent section method [128,129]) can be adopted to the problem. This method assumes a continuous distribution of the shear strains and uses effective section properties to determine the transverse stress distribution through the plate thickness. According to the transformed section method, as applied to the composite beam sections, layers composed of several different materials are transformed into an equivalent section,

and classical shear stress calculation procedure is applied to this new section (Figure 2.5) without changing reactions acting on the cross section of the plate.

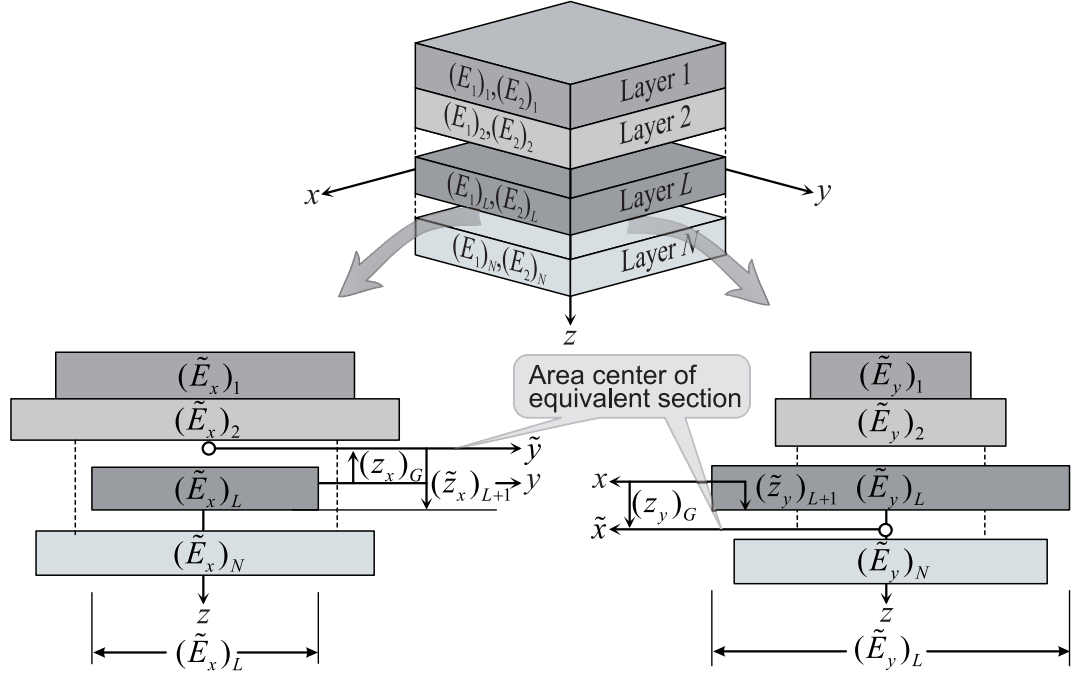


**Figure 2.5 :** Equivalent cross sections with normals in  $x$  and  $y$  directions.

Equivalent modulus of elasticity values  $\tilde{E}_x$ ,  $\tilde{E}_y$  of sections with normal directions  $x$  and  $y$ , respectively, are calculated for each layer  $L$  from;

$$\left. \begin{aligned} (\tilde{E}_x)_L &= (E_1)_L \cos^2 \theta_L + (E_2)_L \sin^2 \theta_L \\ (\tilde{E}_y)_L &= (E_2)_L \cos^2 \theta_L + (E_1)_L \sin^2 \theta_L \end{aligned} \right\} \quad (2.224)$$

In both sections, each lamina has a new width equals to equivalent modulus of elasticity in corresponding direction. In transformed sections, center of area coordinates  $(z_x)_G$  and  $(z_y)_G$  are calculated;

$$(z_x)_G = \frac{\sum_{L=1}^N h_L (\tilde{E}_x)_L (z_{L+1} + z_L) / 2}{\sum_L h_L (\tilde{E}_x)_L}, \quad (z_y)_G = \frac{\sum_{L=1}^N h_L (\tilde{E}_y)_L (z_{L+1} + z_L) / 2}{\sum_L h_L (\tilde{E}_y)_L} \quad (2.225)$$

where thickness of each lamina  $L$  is  $h_L = (z_{L+1} - z_L)$ . Flexure formula is now applied to the transformed sections with new center of areas. Second moments of areas;

$$\left. \begin{aligned} \tilde{I}_x &= \sum_{L=1}^N \left[ \frac{h_L^3}{12} + h_L \left( \frac{(\tilde{z}_x)_{L+1} + (\tilde{z}_x)_L}{2} \right)^2 \right] (\tilde{E}_x)_L \\ \tilde{I}_y &= \sum_{L=1}^N \left[ \frac{h_L^3}{12} + h_L \left( \frac{(\tilde{z}_y)_{L+1} + (\tilde{z}_y)_L}{2} \right)^2 \right] (\tilde{E}_y)_L \end{aligned} \right\} \quad (2.226)$$

are calculated at each sections with respect to the axis crossing center of areas of transformed sections. New coordinates  $\tilde{z}_x$  and  $\tilde{z}_y$  are defined for calculations in transformed sections at the center of area as  $\tilde{z}_x = z - (z_x)_G$  and  $\tilde{z}_y = z - (z_y)_G$ . Static moment of the area under  $\tilde{z}$  coordinate of layer boundary ( $K$ ), where shear stress is calculated, can be determined by the formula;

$$\left. \begin{aligned} (\tilde{S}_x)_K &= \sum_{L=1}^K h_L (\tilde{E}_x)_L \frac{(\tilde{z}_x)_L + (\tilde{z}_x)_{L+1}}{2} \\ (\tilde{S}_y)_K &= \sum_{L=1}^K h_L (\tilde{E}_y)_L \frac{(\tilde{z}_y)_L + (\tilde{z}_y)_{L+1}}{2} \end{aligned} \right\} \quad (2.227)$$

Finally, transverse stresses are calculated from expressions;

$$\left. \begin{aligned} \sigma_{xz}(z) &= -\frac{(\tilde{S}_x)_K S_{xz}}{\tilde{I}_x} \\ \sigma_{yz}(z) &= -\frac{(\tilde{S}_y)_K S_{yz}}{\tilde{I}_y} \end{aligned} \right\} \quad (2.228)$$

Once the shear stresses are computed, then using simple shear constitutive relations, shear strains can also be calculated.

$$\left. \begin{aligned} \gamma_{xz}(z) &= \frac{\sigma_{xz}(z)}{(G_{xz})_L} \\ \gamma_{yz}(z) &= \frac{\sigma_{yz}(z)}{(G_{yz})_L} \end{aligned} \right\} \quad (2.229)$$

## 2.4 Elasto–Plastic Analysis of Mindlin Plates with Mixed FEM

In this part of dissertation, a new mixed finite element solution procedure for the elasto–plastic analysis will be presented. Afterwards the formulation will be adapted

to the elasto–plastic analysis of Mindlin plates with a new solution algorithm. Outline of this section is as follows. Solution of general boundary value problems incorporating plasticity will be introduced after presenting fundamentals of elasto–plastic constitutive relations. Proposed mixed FEM formulation will be described and compared with general displacement type FEM formulations. Finally, the adaptation of proposed algorithm to Mindlin plates will be described.

#### **2.4.1 Introduction to elasto–plastic analysis**

It is well known that stress–strain relations are linear under certain limits depending on the material’s physical properties. Strain (and/or strain rate) dependent parameters of materials such as moduli, viscosity, conductivity etc. govern the physical non–linear behavior [124]. A significant number of materials such as metals, concrete and geo–materials behave in an elasto–plastic manner even for small deformation regimes [130]. Plastic flow as an irreversible stage is defined through plastic strains (permanent deformations) and some other material parameters (hardening parameters). Fundamental concepts of elasto–plastic material behavior can be found in classical books e.g. [131,132].

#### **2.4.2 Mathematical description of plasticity**

In the simplest definition, plasticity theory describes the behavior of solids that may contain permanent deformations when unloaded from a loading configuration. It must be pointed out that, in the context of plasticity theory permanent deformations are independent of rate of loading (rate–independent plasticity).

##### **2.4.2.1 Decomposition of strain tensor**

Infinitesimal strain theory allows total strain  $\boldsymbol{\varepsilon}$  to be decomposed into two main parts (see Figure 2.6 (a)) as follows;

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \quad (2.230)$$

where  $\boldsymbol{\varepsilon}^e$  and  $\boldsymbol{\varepsilon}^p$  define the elastic and plastic part of strain tensor, respectively. Here  $\boldsymbol{\varepsilon}$  is an independent variable and  $\boldsymbol{\varepsilon}^p$  is calculated from loading history through the flow rule. According to relation (2.230) elastic strain is calculated as given below.

$$\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p \quad (2.231)$$

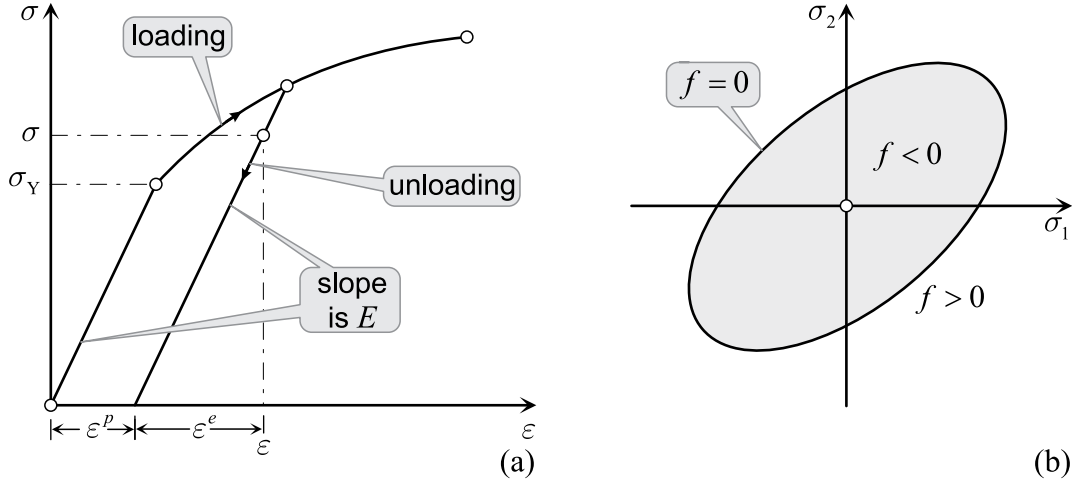


**Figure 2.6 :** Basic definitions in plasticity: (a) Split of strain in uni-axial case. (b) Admissible stresses and yield function.

#### 2.4.2.2 Stress response

In plasticity theory stress is defined in terms of elastic modulus tensor  $\mathbb{C}$  and the elastic strain  $\boldsymbol{\varepsilon}^e$  (observe uni-axial case from Figure 2.6 (a)),

$$\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \quad (2.232)$$

Because the relations (2.231) and (2.232) are local, it can be concluded that total strain is the gradient of the displacement field, whereas the elastic strain is generally not the gradient of the displacement field [133].

#### 2.4.2.3 Yield function and elastic domain

Admissible stresses in case of elasto-plastic behavior are defined in terms of yield function and referred as yield criterion (Figure 2.6 (b)). The constrain of the stress space regarding yield function  $f$  is;

$$f(\boldsymbol{\sigma}, \mathbf{q}) \leq 0 \quad (2.233)$$

The yield function has stresses ( $\boldsymbol{\sigma}$ ) and internal variables ( $\mathbf{q}$ ) as its parameters. The boundary ( $f=0$ ) of yield function is yield surface and outside of this surface is a space for non-admissible stresses. As long as the stress level lies inside the

admissible stress space, deformation stays in elastic strain limits. On the other hand if stress lies on the boundary, elastic unloading or plastic yielding occurs.

#### 2.4.2.4 Plastic flow rule and hardening law. Loading/unloading conditions

The irreversible mechanism of associative plasticity introduces the rate form of plastic strain and internal variables  $\mathbf{q}$ , as follows,

$$\left. \begin{aligned} \dot{\boldsymbol{\varepsilon}}^p &= \dot{\gamma} \frac{\partial f}{\partial \boldsymbol{\sigma}} \\ \dot{\mathbf{q}} &= -\dot{\gamma} \mathbb{H} : \frac{\partial f}{\partial \mathbf{q}} \end{aligned} \right\} \quad (2.234)$$

where  $\mathbb{H}$  is the matrix of generalized plastic moduli where hardening parameters are stored. Here  $\dot{\gamma} \geq 0$  parameter refers to the material slip (magnitude of plastic flow) in the stress direction and called as consistency parameter (or slip rate or plastic multiplier). Due to definition of yield function and consistency parameter Kuhn–Tucker complementarity conditions are collected as,

$$\dot{\gamma} \geq 0, \quad f(\boldsymbol{\sigma}, \mathbf{q}) \leq 0, \quad \dot{\gamma} f(\boldsymbol{\sigma}, \mathbf{q}) = 0 \quad (2.235)$$

Another condition is called as consistency requirement,

$$\dot{\gamma} \dot{f}(\boldsymbol{\sigma}, \mathbf{q}) = 0 \quad (2.236)$$

The summary of conditions and possible loading condition are presented as follows [133],

$$\left. \begin{aligned} f < 0 & \quad \dot{\gamma} = 0 \quad (\text{elastic}) \\ f = 0 & \quad \left\{ \begin{aligned} \dot{f} < 0 & \quad \dot{\gamma} = 0 \quad (\text{elastic unloading}) \\ \dot{f} = 0 & \quad \left\{ \begin{aligned} \dot{\gamma} = 0 & \quad (\text{neutral loading}) \\ \dot{\gamma} > 0 & \quad (\text{plastic loading}) \end{aligned} \right. \end{aligned} \right\} \end{aligned} \right\} \quad (2.237)$$

#### 2.4.3 Implementation of elasto–plastic boundary value problems

General idea in the numerical solution procedure of elasto–plastic boundary value problems will be mentioned briefly. The newly proposed mixed finite element formulation for the solution of general elasto–plastic problems will be presented in

detail. Finally, the implementation of the proposed algorithm to the Mindlin plates will be described.

#### 2.4.3.1 General idea in materially non-linear solutions

The solution of non-linear problems in solid mechanics depends on an iteration-based procedure, where discretized momentum balance equations are tried to be satisfied with a prediscrbed convergency. Basic steps for materially non-linear problems are [133],

1. The unbalanced momentum equations are used to obtain incremental motion (global step), wherefrom the incremental strain history is calculated with kinematic relations.
2. New values of the state variables  $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \mathbf{q})$  are calculated locally according to constitutive equations for current strain state (total strains are freezed).
3. The equilibrium of discretized momentum balance is checked for converge of computed stresses and if is violated the iteration keeps going on with step 1.

Non-linear solution progress can be controlled by the displacement increment or load increment or with an arc-length according to iteration scheme. In the context of this study load controlled iteration will be considered.

#### 2.4.3.2 Incremental formulation for path-dependent materials

As prescribed at section 2.4.2 , constitutive equations of material can be path-dependent. That means the stress tensor cannot be derived directly from the instantaneous value of the total strain. Instead, stress is a function of strain history until the instantaneous time. Even for static problems, a pseudo-time parameter ( $t$ ) is presented to reflect the historical reference of parameters. The appropriate procedure is then to integrate the rate form of constitutive equations through the time interval  $[t_n, t_{n+1}]$ . For the solution of this initial value problem, the set of parameters  $(\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_n^p, \boldsymbol{\alpha}_n)$  must be well known at time step  $t_n$ . According to a given increment of strain as  $\Delta\boldsymbol{\varepsilon}_n$  (due to unbalanced momentum) final values of parameters  $(\boldsymbol{\varepsilon}_{n+1}, \boldsymbol{\varepsilon}_{n+1}^p, \boldsymbol{\alpha}_{n+1})$  at  $t_{n+1}$  are obtained through the numerical constitutive integration algorithms [133,134].

### 2.4.3.3 Displacement type FEM solutions of elasto–plastic problems

Applying energy principles and discretization of system to finite elements result with the following global equation for non–linear problem [135].

$$\mathbf{f}^{\text{int}}(\mathbf{u}_{n+1}) = \mathbf{f}_{n+1}^{\text{ext}} \quad (2.238)$$

Here, the purpose is to determine displacements  $\mathbf{u}_{n+1}$  at the time  $t_{n+1}$  from given non–linear equation above. In the solution algorithm, equilibrium states are obtained at discrete points ( $n=1,2,\dots,NT$ ) rather than a continuous equilibrium path (Figure 2.7). At every step of increment, iteration starts from a known equilibrium state with displacement  $\mathbf{u}_n$  and corresponding internal forces  $\mathbf{f}(\mathbf{u}_n)$ . With an appropriate algorithm the displacements at  $\mathbf{u}_{n+1}$  are obtained for the external load  $\mathbf{f}_{n+1}^{\text{ext}}$ . Newton–Raphson method is one of the most preferred algorithm for non–linear systems (Figure 2.5).

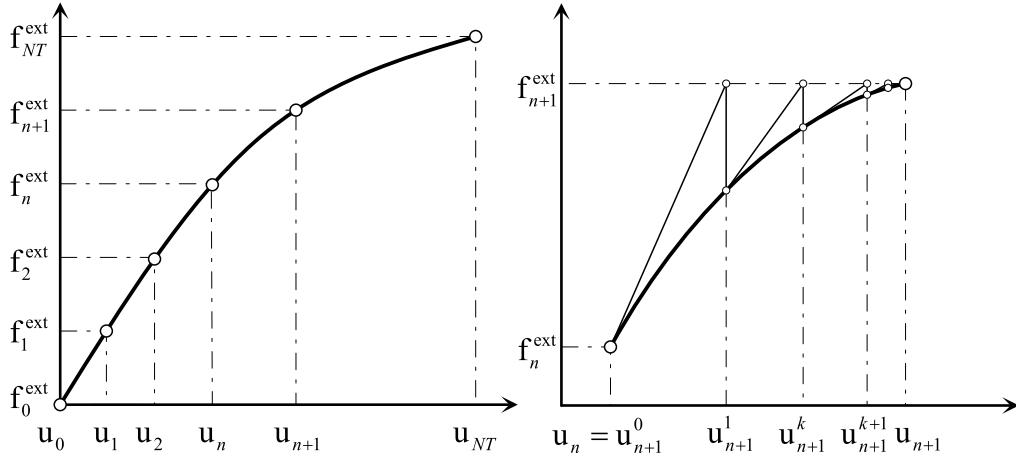


**Figure 2.7 :** Load increment iteration with Newton–Raphson method.

Multi–dimensional Taylor expansion of (2.238) with linear terms gives the approximated internal forces of the displacement state  $\mathbf{u}_{n+1}^{k+1}$  in the neighbourhood of point  $\mathbf{u}_{n+1}^k$  with the iteration index  $k$ .

$$\mathbf{f}^{\text{int}}(\mathbf{u}_{n+1}^{k+1}) = \mathbf{f}^{\text{int}}(\mathbf{u}_{n+1}^k) + \frac{\partial \mathbf{f}_{\text{int}}(\mathbf{u}_{n+1}^k)}{\partial \mathbf{u}_{n+1}^k} (\mathbf{u}_{n+1}^{k+1} - \mathbf{u}_{n+1}^k) \quad (2.239)$$

Displacement increment is calculated after an iteration step as;

$$\Delta \mathbf{u} = (\mathbf{u}_{n+1}^{k+1} - \mathbf{u}_{n+1}^k) \quad (2.240)$$

The partial derivative in (2.239) is nothing more than the tangential stiffness matrix at the state  $k$ ,

$$\mathbf{K}_t = \frac{\partial \mathbf{f}_{\text{int}}(\mathbf{u}_{n+1}^k)}{\partial \mathbf{u}_{n+1}^k} \quad (2.241)$$

According to the Newton–Raphson algorithm, the increment is calculated as,

$$\Delta \mathbf{u} = \mathbf{K}_t^{-1}(\mathbf{u}_{n+1}^k)(\mathbf{f}_{n+1}^{\text{ext}} - \mathbf{f}^{\text{int}}(\mathbf{u}_{n+1}^k)) \quad (2.242)$$

and the displacements are updated as,

$$\mathbf{u}_{n+1}^{k+1} = \mathbf{u}_{n+1}^k + \Delta \mathbf{u} \quad (2.243)$$

until the convergence of the solution achieved.

A typical finite element solution algorithm for elasto–plastic analysis is presented at Figure 2.8. Local integration of constitutive equations are made at Gauss points over finite elements. Total strains are calculated from the gradient of displacements, which can cause some numerical errors due to the nature of numerical differentiation regarding to an approximate solution. Details of the given algorithm can be found at textbooks [133] and [134]

#### 2.4.4 Numerical integration of elasto–plastic constitutive equations

The initial value problem of constitutive equation was mentioned in section 2.4.3.2. It is also one of the basic and crucial steps of general solution of elasto–plastic problems and corresponds to the 2. step as defined in section 2.4.3.1. Details of the solution algorithm and the numerical schemes will be presented in this section.

##### 2.4.4.1 Definition of the initial value problem

For a given strain increment  $\Delta \boldsymbol{\varepsilon}$  at a point  $\mathbf{p}$  in a body, evolution of the plastic strain and internal variables are sought regarding to the history of the strain. Initial values of the problem  $(\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_n^p, \boldsymbol{\alpha}_n)$  are well known at pseudo–time  $t_n$ . The purpose here is

to find the updated values ( $\boldsymbol{\varepsilon}_{n+1}^p$ ,  $\boldsymbol{\alpha}_{n+1}$ ) at corresponding time  $t_{n+1}$  according to the known strain increment  $\Delta\boldsymbol{\varepsilon}$  where total strain is calculated as  $\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_n + \Delta\boldsymbol{\varepsilon}$ .

```

01. Loop for load ( $n = 0, NT - 1$ ):
02.   Generate external load:  $\mathbf{f}_{n+1}^{\text{ext}}$ 
03.   Set initial displacement for iteration:  $\mathbf{u}_{n+1}^0 = \mathbf{u}_n$ 
04.   Loop for iteration ( $k = 0, \text{konv}$ ):
05.     Compute tangent matrices:  $\mathbf{C}_{\text{tan}}$ 
06.     Compute tangent stiffness matrix:  $\mathbf{K}_t(\mathbf{u}_{n+1}^k)$ 
07.     Compute displacement increment:  $\Delta\mathbf{u} = \mathbf{K}_t^{-1}(\mathbf{f}_{n+1}^{\text{ext}} - \mathbf{f}_{n+1}^{(k)\text{int}}(\boldsymbol{\sigma}_{n+1}^k))$ 
08.     Update displacements:  $\mathbf{u}_{n+1}^{k+1} = \mathbf{u}_{n+1}^k + \Delta\mathbf{u}$ 
09.     Loop for Gauss points ( $gp = 1, \text{ngp}$ ):
10.       Update total strains:  $\boldsymbol{\varepsilon}_{n+1}^{k+1}(\mathbf{u}_{n+1}^{k+1})$ 
11.       Call constitutive integration algorithms and calculate
12.       stresses and other state variables:  $\boldsymbol{\sigma}_{n+1}^{k+1}$ ,  $\boldsymbol{\alpha}_{n+1}^{k+1}$ ,  $\boldsymbol{\varepsilon}_{n+1}^{(k+1)p}$ 
13.       Compute internal force vector:  $\mathbf{f}_{n+1}^{(k+1)\text{int}}(\boldsymbol{\sigma}_{n+1}^{k+1})$ 
14.       Check for convergence:

```

**Figure 2.8 :** Solution algorithm for materially non-linear general finite elements.

#### 2.4.4.2 Algorithmic form of the elasto-plastic state and evolution equations

Numerical solution of the equation system (2.234) under given conditions and constraints (2.235) requires them to be defined in an algebraic form. Several numerical integration schemes can be used for this purpose. Backward-Euler (implicit or fully implicit) integration scheme is an optimal choice especially for von Mises plasticity and large time steps by providing accuracy [136] and unconditional stability [133]. A concise explanation of Backward-Euler approximation ([137,138]) can be given as follows. From a known value ( $y(t_n)$  or simply  $y_n$ ) of function  $y(t)$  to a desired value ( $y(t_{n+1})$  or simply  $y_{n+1}$ ) can be reached with the following equation (see Figure 2.9 (a)),

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} \dot{y} dt = y_n + \int_{t_n}^{t_{n+1}} f(y(t), t) dt \quad (2.244)$$

With the adoption of Backward-Euler scheme the integral in the equation (2.244) is approximated as (Figure 2.9 (b)),

$$y_{n+1} = y_n + f_{n+1}(t_{n+1} - t_n) = y_n + f_{n+1}\Delta t \quad (2.245)$$

For the sake of simplicity a new parameter  $\Delta\gamma$  will be introduced for pseudo-time derived problem as,

$$\Delta\gamma_{n+1} = \dot{\gamma}_{n+1}\Delta t \quad (2.246)$$

According to the predefined Backward-Euler scheme, the algorithmic forms of equations (2.234) are obtained as [139],

$$\left. \begin{aligned} \boldsymbol{\varepsilon}_{n+1}^p &= \boldsymbol{\varepsilon}_n^p + \Delta\gamma_{n+1} \frac{\partial f_{n+1}}{\partial \boldsymbol{\sigma}_{n+1}} \\ \mathbf{q}_{n+1} &= \mathbf{q}_n - \Delta\gamma_{n+1} \mathbb{H} : \frac{\partial f_{n+1}}{\partial \mathbf{q}_{n+1}} \end{aligned} \right\} \quad (2.247)$$

For the completeness Kuhn-Tucker conditions (2.235) are also defined in algorithmic form as,

$$\Delta\gamma_{n+1} \geq 0, \quad f_{n+1} \leq 0, \quad \Delta\gamma_{n+1} f_{n+1} = 0 \quad (2.248)$$

#### 2.4.4.3 Elastic trial step

First step of the numerical integration of constitutive equations starts with the fully elastic assumption of the strain increment. At this trial step, for the given strain increment  $\Delta\boldsymbol{\varepsilon}$ , trial stress tensor  $\boldsymbol{\sigma}_{n+1}^{trial}$  is calculated through the converged values from previous time step,

$$\boldsymbol{\sigma}_{n+1}^{trial} = \mathbb{C} : (\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p) \quad (2.249)$$

Violation of the yield condition is tested next according to the calculated trial stress. If the stress is admissible then the step  $n + 1$  is elastic and there occurs no plastic slip and there will be no change in the plastic strain and the internal variables. If the stress is out of yield surface then it has to be projected on the yield surface using one of the projection methods. In summary,

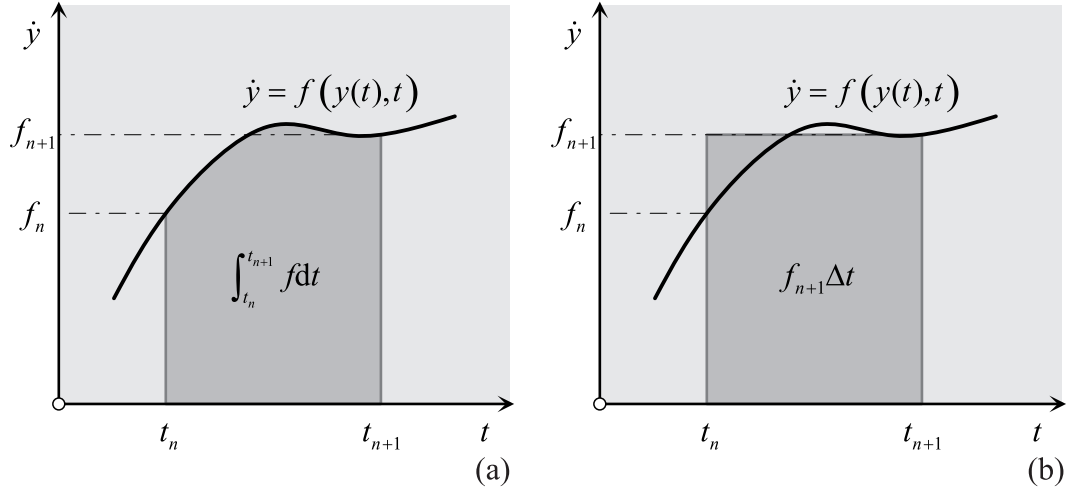


**Figure 2.9 :** Integral of derivative functions: (a)Exact inegration. (b)Backward Euler approximation.

$$\left. \begin{aligned}
 f_{n+1}^{trial} &= f_{n+1}^{trial}(\boldsymbol{\sigma}_{n+1}^{trial}, \mathbf{q}_n) \\
 f_{n+1}^{trial} < 0 &\Rightarrow \text{step } n+1 \text{ is elastic} \Rightarrow \boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p, \boldsymbol{\alpha}_{n+1}^p = \boldsymbol{\alpha}_n^p \\
 f_{n+1}^{trial} \geq 0 &\Rightarrow \text{step } n+1 \text{ is plastic} \Rightarrow \text{run projection}
 \end{aligned} \right\} \quad (2.250)$$

#### 2.4.4.4 Closest point projection algorithm

As the yield condition is violated elastic predictor stresses should be projected on the yield surface. Plastic step causes to the following equations,

$$\left. \begin{aligned}
 f_{n+1} &= 0 \quad , \quad \Delta\gamma_{n+1} > 0 \\
 \Delta\boldsymbol{\varepsilon}_{n+1}^p &\neq 0 \quad , \quad \Delta\boldsymbol{\alpha}_{n+1}^p \neq 0
 \end{aligned} \right\} \quad (2.251)$$

The non-linear algebraic equations (2.247) must be solved next under the restrictions of (2.248) by adopting Newton-Raphson iterative scheme. Iterative solution procedure is initialized with the following values of variables,

$$\left. \begin{aligned}
 i &= 0 \\
 \Delta\gamma_{n+1}^{(0)} &= 0 \\
 \boldsymbol{\varepsilon}_{n+1}^{p(0)} &= \boldsymbol{\varepsilon}_n^p \\
 \boldsymbol{\alpha}_{n+1}^{(0)} &= \boldsymbol{\alpha}_n
 \end{aligned} \right\} \quad (2.252)$$

Residuals of the algebraic equations (2.247) at the  $i^{th}$  iteration can be expressed as follows,

$$\mathbf{R}_{n+1}^{(i)} = \begin{Bmatrix} -\boldsymbol{\varepsilon}_{n+1}^{p(i)} + \boldsymbol{\varepsilon}_n^p + \Delta\boldsymbol{\gamma}_{n+1}^{(i)} \left( \frac{\partial f}{\partial \boldsymbol{\sigma}} \right)_{n+1}^{(i)} \\ -\boldsymbol{\alpha}_{n+1}^{(i)} + \boldsymbol{\alpha}_n + \Delta\boldsymbol{\gamma}_{n+1}^{(i)} \left( \frac{\partial f}{\partial \mathbf{q}} \right)_{n+1}^{(i)} \end{Bmatrix} \stackrel{?}{=} \mathbf{0} \quad (2.253)$$

where,

$$\mathbf{q}_{n+1}^{(i)} = -\mathbb{H} : \boldsymbol{\alpha}_{n+1}^{(i)} \quad (2.254)$$

It is worth noting that the stress at  $i^{th}$  step is calculated from,

$$\boldsymbol{\sigma}_{n+1}^{(i)} = \mathbb{C} : (\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_{n+1}^{p(i)}) \quad (2.255)$$

Non-linear equations in (2.253) has to be linearized to find the roots according to the Newton-Raphson algorithm [139],

$$\mathbf{R}_{n+1}^{(i)}(\mathbf{x}^{(i)}) + \left[ \frac{\partial \mathbf{R}_{n+1}}{\partial \mathbf{x}} \right]_i \Delta \mathbf{x} = \mathbf{0} \quad (2.256)$$

wherefrom  $\Delta \mathbf{x}$  can be calculated as follows,

$$\Delta \mathbf{x} = - \left[ \frac{\partial \mathbf{R}_{n+1}}{\partial \mathbf{x}} \right]_i^{-1} \mathbf{R}_{n+1}^{(i)}(\mathbf{x}^{(i)}) \quad (2.257)$$

and then the variables are updated,

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} + \Delta \mathbf{x} = \mathbf{x}^{(i)} - \left[ \frac{\partial \mathbf{R}_{n+1}}{\partial \mathbf{x}} \right]_i^{-1} \mathbf{R}_{n+1}^{(i)}(\mathbf{x}^{(i)}) \quad (2.258)$$

Linearization of the equations in (2.253) yields,

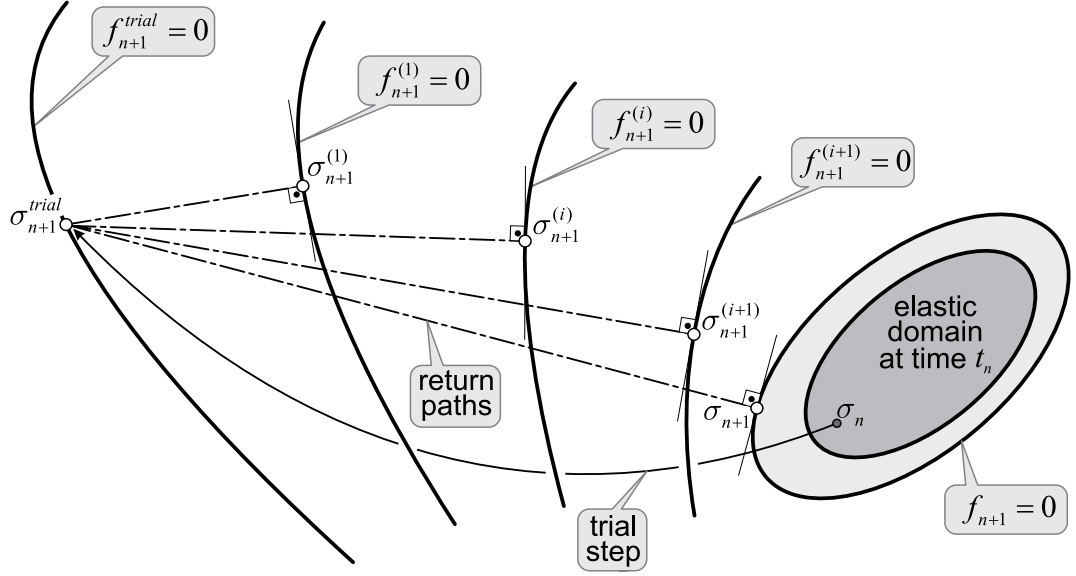


**Figure 2.10 :** Geometric interpretation of the projection algorithm.

$$\left[ \frac{\partial \mathbf{R}_{n+1}}{\partial \mathbf{x}} \right]_i \Delta \mathbf{x} = \begin{cases} -d\boldsymbol{\varepsilon}_{n+1}^{p(i)} + d\Delta\gamma_{n+1}^{(i)} \left( \frac{\partial f}{\partial \boldsymbol{\sigma}} \right)_{n+1}^{(i)} + \Delta\gamma_{n+1}^{(i)} \left( \frac{\partial f^2}{\partial \boldsymbol{\sigma}^2} \right)_{n+1}^{(i)} : d\boldsymbol{\sigma}_{n+1}^{(i)} + \Delta\gamma_{n+1}^{(i)} \left( \frac{\partial f^2}{\partial \boldsymbol{\sigma} \partial \mathbf{q}} \right)_{n+1}^{(i)} : d\mathbf{q}_{n+1}^{(i)} \\ -d\boldsymbol{\alpha}_{n+1}^{(i)} + d\Delta\gamma_{n+1}^{(i)} \left( \frac{\partial f}{\partial \mathbf{q}} \right)_{n+1}^{(i)} + \Delta\gamma_{n+1}^{(i)} \left( \frac{\partial f^2}{\partial \mathbf{q} \partial \boldsymbol{\sigma}} \right)_{n+1}^{(i)} : d\boldsymbol{\sigma}_{n+1}^{(i)} + \Delta\gamma_{n+1}^{(i)} \left( \frac{\partial f^2}{\partial \mathbf{q}^2} \right)_{n+1}^{(i)} : d\mathbf{q}_{n+1}^{(i)} \end{cases} \quad (2.259)$$

Partial derivatives can be symbolized in a much more compact form with a comma ',' in the subscript (...), indicating partial derivatives with respect to the variables inserted after the comma ( $\partial y / \partial x = y_{,x}$ ). According to this approach the equation (2.259) can be rearranged in a much more compact form as,

$$\left[ \mathbf{R}_{n+1,x}^{(i)} \right]_{n+1}^{(i)} \Delta \mathbf{x} = \begin{cases} -d\boldsymbol{\varepsilon}_{n+1}^{p(i)} + (d\Delta\gamma f_{,\boldsymbol{\sigma}})_{n+1}^{(i)} + (\Delta\gamma f_{,\boldsymbol{\sigma}\boldsymbol{\sigma}} : d\boldsymbol{\sigma})_{n+1}^{(i)} + (\Delta\gamma f_{,\boldsymbol{\sigma}\mathbf{q}} : d\mathbf{q})_{n+1}^{(i)} \\ -d\boldsymbol{\alpha}_{n+1}^{(i)} + (d\Delta\gamma f_{,\mathbf{q}})_{n+1}^{(i)} + (\Delta\gamma f_{,\mathbf{q}\boldsymbol{\sigma}} : d\boldsymbol{\sigma})_{n+1}^{(i)} + (\Delta\gamma f_{,\mathbf{q}\mathbf{q}} : d\mathbf{q})_{n+1}^{(i)} \end{cases} \quad (2.260)$$

Incremental stress and internal variable tensors in (2.260) can be obtained from equations (2.255) and (2.254), respectively,

$$d\boldsymbol{\sigma}_{n+1}^{(i)} = \mathbb{C} : (-d\boldsymbol{\varepsilon}_{n+1}^{p(i)}) \quad (2.261)$$

$$\mathbf{d}\mathbf{q}_{n+1}^{(i)} = -\mathbb{H} : \mathbf{d}\boldsymbol{\alpha}_{n+1}^{(i)} \quad (2.262)$$

After inserting (2.261) and (2.262) into (2.260) and establishing (2.256) yields,

$$\begin{aligned} & \mathbf{R}_{n+1}^{\varepsilon(i)} - \mathbf{d}\boldsymbol{\varepsilon}_{n+1}^{p(i)} - \left( \Delta \gamma f_{,\sigma\sigma} \right)_{n+1}^{(i)} : \mathbb{C} : \mathbf{d}\boldsymbol{\varepsilon}_{n+1}^{p(i)} \\ & - \left( \Delta \gamma f_{,\sigma\mathbf{q}} \right)_{n+1}^{(i)} : \mathbb{H} : \mathbf{d}\boldsymbol{\alpha}_{n+1}^{(i)} + \left( \mathbf{d}\Delta \gamma f_{,\sigma} \right)_{n+1}^{(i)} = \mathbf{0} \end{aligned} \quad (2.263)$$

$$\begin{aligned} & \mathbf{R}_{n+1}^{\alpha(i)} - \mathbf{d}\boldsymbol{\alpha}_{n+1}^{(i)} - \left( \Delta \gamma f_{,\mathbf{q}\sigma} \right)_{n+1}^{(i)} : \mathbb{C} : \mathbf{d}\boldsymbol{\varepsilon}_{n+1}^{p(i)} \\ & - \left( \Delta \gamma f_{,\mathbf{q}\mathbf{q}} \right)_{n+1}^{(i)} : \mathbb{H} : \mathbf{d}\boldsymbol{\alpha}_{n+1}^{(i)} + \left( \mathbf{d}\Delta \gamma f_{,\mathbf{q}} \right)_{n+1}^{(i)} = \mathbf{0} \end{aligned} \quad (2.264)$$

Arranging (2.263) and (2.264) to be suitable for a matrix description comes with,

$$\mathbf{R}_{n+1}^{(i)} + \mathbb{A}^{-1} \boldsymbol{\Delta} + \Delta \Delta \gamma^{(i)} \boldsymbol{\nabla} = \mathbf{0} \quad (2.265)$$

where,

$$\mathbb{A}^{-1} = \begin{bmatrix} \mathbb{I} + \left( \Delta \gamma f_{,\sigma\sigma} \right)_{n+1}^{(i)} : \mathbb{C} & \left( \Delta \gamma f_{,\sigma\mathbf{q}} \right)_{n+1}^{(i)} : \mathbb{H} \\ \left( \Delta \gamma f_{,\mathbf{q}\sigma} \right)_{n+1}^{(i)} : \mathbb{C} & \mathbb{I} + \left( \Delta \gamma f_{,\mathbf{q}\mathbf{q}} \right)_{n+1}^{(i)} : \mathbb{H} \end{bmatrix} \quad (2.266)$$

Here  $\mathbb{I}$  symbolizes the fourth order unit tensor.  $\boldsymbol{\nabla}$  in (2.265) corresponds to,

$$\boldsymbol{\nabla} = \begin{Bmatrix} f_{,\sigma} \\ f_{,\mathbf{q}} \end{Bmatrix}_{n+1}^{(i)} \quad (2.267)$$

and  $\boldsymbol{\Delta}$  contains the incremental state variables as follows,

$$\boldsymbol{\Delta} = \begin{Bmatrix} -\mathbf{d}\boldsymbol{\varepsilon}^p \\ -\mathbf{d}\boldsymbol{\alpha} \end{Bmatrix}_{n+1}^{(i)} \quad (2.268)$$

It should be pointed out that  $\mathbf{d}\Delta \gamma$  is replaced with  $\Delta \Delta \gamma$  in (2.265). For the solution of equation system, the yield condition must be considered as well. If the yield condition is linearised as follows,

$$f_{n+1}^{(i)} + \left( f_{,\sigma} : \mathbf{d}\boldsymbol{\sigma} \right)_{n+1}^{(i)} + \left( f_{,\mathbf{q}} : \mathbf{d}\mathbf{q} \right)_{n+1}^{(i)} = 0 \quad (2.269)$$

using the relations (2.261) and (2.262) one obtains (2.269) as,

$$f_{n+1}^{(i)} - f_{n+1,\sigma}^{(i)} : \mathbb{C} : d\boldsymbol{\varepsilon}_{n+1}^{p(i)} - f_{n+1,q}^{(i)} : \mathbb{H} : d\boldsymbol{\alpha}_{n+1}^{(i)} = 0 \quad (2.270)$$

The equation (2.270) can be written in a more compact form as,

$$f_{n+1}^{(i)} + \nabla^T \begin{bmatrix} \mathbb{C} & \mathbf{0} \\ \mathbf{0} & \mathbb{H} \end{bmatrix} \Delta = 0 \quad (2.271)$$

With the definition of a new symbol  $\nabla_C$ ,

$$\nabla_C = \begin{Bmatrix} f_{n+1,\sigma}^{(i)} : \mathbb{C} \\ f_{n+1,q}^{(i)} : \mathbb{H} \end{Bmatrix} \quad (2.272)$$

equation (2.271) can be given in a much more compact form as,

$$f_{n+1}^{(i)} + \nabla_C^T \Delta = 0 \quad (2.273)$$

If the equation (2.265) is multiplied by  $\nabla_C^T \mathbb{A}$  the following equation is obtained.

$$\nabla_C^T \mathbb{A} \mathbf{R}_{n+1}^{(i)} + \nabla_C^T \Delta + \nabla_C^T \mathbb{A} \Delta \Delta \gamma^{(i)} \nabla = 0 \quad (2.274)$$

Using the (2.273) equation with (2.274) the increment of consistency parameter is calculated from,

$$\Delta \Delta \gamma^{(i)} = \frac{f_{n+1}^{(i)} - \nabla_C^T \mathbb{A} \mathbf{R}_{n+1}^{(i)}}{\nabla_C^T \mathbb{A} \nabla} \quad (2.275)$$

Once the increment of consistency parameter is calculated then the increment in state variables can also be calculated. Inserting  $\Delta \Delta \gamma$  from (2.275) into (2.265) and arranging terms yields,

$$\begin{Bmatrix} d\boldsymbol{\varepsilon}^p \\ d\boldsymbol{\alpha} \end{Bmatrix}_{n+1}^{(i)} = -\Delta = \mathbb{A} \left( \mathbf{R}_{n+1}^{(i)} + \Delta \Delta \gamma^{(i)} \nabla \right) \quad (2.276)$$

which can be given in an open form as,

$$\begin{Bmatrix} d\boldsymbol{\varepsilon}^p \\ d\boldsymbol{\alpha} \end{Bmatrix}_{n+1}^{(i)} = \mathbb{A} \left[ \mathbf{R} + \Delta\Delta\gamma \begin{Bmatrix} f_{,\sigma} \\ f_{,q} \end{Bmatrix} \right]_{n+1}^{(i)} \quad (2.277)$$

Finally, the state variables and consistency parameter are updated as below,

$$\left. \begin{aligned} \boldsymbol{\varepsilon}_{n+1}^{p(i+1)} &= \boldsymbol{\varepsilon}_{n+1}^{p(i)} + \Delta\boldsymbol{\varepsilon}_{n+1}^{p(i)} \\ \boldsymbol{\alpha}_{n+1}^{(i+1)} &= \boldsymbol{\alpha}_{n+1}^{(i)} + \Delta\boldsymbol{\alpha}_{n+1}^{(i)} \\ \Delta\gamma^{(i+1)} &= \Delta\gamma^{(i)} + \Delta\Delta\gamma^{(i)} \end{aligned} \right\} \quad (2.278)$$

#### 2.4.4.5 Consistent tangent elasto–plastic moduli

Elasto–plastic material moduli is defined as the tangent operator from implicit solution algorithm,

$$\mathbb{C}^{ep} = \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \quad (2.279)$$

It is worth noting that with the termination of iteration, differentials in (2.261) and (2.262) becomes (from (2.255) and (2.254))

$$\left. \begin{aligned} d\boldsymbol{\sigma}_{n+1} &= \mathbb{C} : (d\boldsymbol{\varepsilon}_{n+1} - d\boldsymbol{\varepsilon}_{n+1}^p) \\ d\mathbf{q}_{n+1}^{(i)} &= -\mathbb{H} : d\boldsymbol{\alpha}_{n+1}^{(i)} \end{aligned} \right\} \quad (2.280)$$

After the conclusion of return map algorithm the residual vector becomes zero ( $\mathbf{R}_{n+1} = \mathbf{0}$ ). It follows from (2.260)

$$-d\boldsymbol{\varepsilon}_{n+1}^p + (d\Delta\gamma f_{,\sigma})_{n+1} + (\Delta\gamma f_{,\sigma\sigma} : \mathbb{C} : (d\boldsymbol{\varepsilon} - d\boldsymbol{\varepsilon}^p))_{n+1} - (\Delta\gamma f_{,\sigma q} : \mathbb{H} : d\boldsymbol{\alpha})_{n+1} = \mathbf{0} \quad (2.281)$$

$$-d\boldsymbol{\alpha}_{n+1} + (d\Delta\gamma f_{,q})_{n+1} + (\Delta\gamma f_{,q\sigma} : \mathbb{C} : (d\boldsymbol{\varepsilon} - d\boldsymbol{\varepsilon}^p))_{n+1} - (\Delta\gamma f_{,qq} : \mathbb{H} : d\boldsymbol{\alpha})_{n+1} = \mathbf{0} \quad (2.282)$$

It is obvious that  $f_{n+1} = 0$  when the projection algorithm terminates. Therefore, from (2.269) one obtains,

$$\left(f_{,\sigma} : \mathbb{C} : (\mathbf{d}\boldsymbol{\varepsilon} - \mathbf{d}\boldsymbol{\varepsilon}^p)\right)_{n+1} - \left(f_{,\mathbf{q}} : \mathbb{H} : \mathbf{d}\boldsymbol{\alpha}\right)_{n+1} = 0 \quad (2.283)$$

which can be summarized as,

$$\nabla_c^T \bar{\Delta} = 0 \quad (2.284)$$

With the definition of the new symbol  $\bar{\Delta}$ ,

$$\bar{\Delta} = \begin{Bmatrix} \mathbf{d}\boldsymbol{\varepsilon} - \mathbf{d}\boldsymbol{\varepsilon}^p \\ -\mathbf{d}\boldsymbol{\alpha} \end{Bmatrix}_{n+1} \quad (2.285)$$

After adding  $\mathbf{d}\boldsymbol{\varepsilon}_{n+1} - \mathbf{d}\boldsymbol{\varepsilon}_{n+1}$  to the equation (2.281) a compact form can present (2.281) and (2.282) in combination as follows,

$$\mathbb{A}^{-1} \bar{\Delta} + \Delta\Delta\gamma \nabla = \begin{Bmatrix} \mathbf{d}\boldsymbol{\varepsilon}_{n+1} \\ \mathbf{0} \end{Bmatrix} \quad (2.286)$$

Multiplying the equation (2.286) with  $\nabla_c^T \mathbb{A}$  yields the following scalar equation (where  $\nabla_c^T \bar{\Delta}$  cancels from (2.284)),

$$\cancel{\nabla_c^T \bar{\Delta}} + \Delta\Delta\gamma \nabla_c^T \mathbb{A} \nabla = \nabla_c^T \mathbb{A} \begin{Bmatrix} \mathbf{d}\boldsymbol{\varepsilon}_{n+1} \\ \mathbf{0} \end{Bmatrix} \quad (2.287)$$

$\Delta\Delta\gamma$  can be obtained from the equation (2.287) as follows,

$$\Delta\Delta\gamma = \frac{\nabla_c^T \mathbb{A} \begin{Bmatrix} \mathbf{d}\boldsymbol{\varepsilon}_{n+1} \\ \mathbf{0} \end{Bmatrix}}{\nabla_c^T \mathbb{A} \nabla} \quad (2.288)$$

After inserting  $\Delta\Delta\gamma$  from (2.288) into (2.286) and multiplying (2.286) with  $\mathbb{A}$  and  $\square$  one obtains,

$$\square \bar{\Delta} + \square \mathbb{A} \nabla \frac{\nabla_c^T \mathbb{A} \begin{Bmatrix} \mathbf{d}\boldsymbol{\varepsilon}_{n+1} \\ \mathbf{0} \end{Bmatrix}}{\nabla_c^T \mathbb{A} \nabla} = \square \mathbb{A} \begin{Bmatrix} \mathbf{d}\boldsymbol{\varepsilon}_{n+1} \\ \mathbf{0} \end{Bmatrix} \quad (2.289)$$

where  $\square$  is denoting,

$$\square = \begin{bmatrix} \mathbb{C} & \mathbf{0} \\ \mathbf{0} & \mathbb{H} \end{bmatrix} \quad (2.290)$$

Inserting the first term in (2.289) from (2.280) with the combinations of (2.290) and (2.285) yields,

$$\begin{Bmatrix} d\boldsymbol{\sigma}_{n+1} \\ d\mathbf{q}_{n+1} \end{Bmatrix} + \square \mathbb{A} \nabla \frac{\nabla_C^T \mathbb{A} \begin{Bmatrix} d\boldsymbol{\varepsilon}_{n+1} \\ \mathbf{0} \end{Bmatrix}}{\nabla_C^T \mathbb{A} \nabla} = \square \mathbb{A} \begin{Bmatrix} d\boldsymbol{\varepsilon}_{n+1} \\ \mathbf{0} \end{Bmatrix} \quad (2.291)$$

According to the property  $(\mathbf{A} : \mathbf{B})\mathbf{C} = \mathbf{A} : (\mathbf{B} \otimes \mathbf{C})$  the equation (2.291) becomes,

$$\begin{Bmatrix} d\boldsymbol{\sigma}_{n+1} \\ d\mathbf{q}_{n+1} \end{Bmatrix} + \frac{(\square \mathbb{A} \nabla \otimes \nabla_C^T \mathbb{A}) \begin{Bmatrix} d\boldsymbol{\varepsilon}_{n+1} \\ \mathbf{0} \end{Bmatrix}}{\nabla_C^T \mathbb{A} \nabla} = \square \mathbb{A} \begin{Bmatrix} d\boldsymbol{\varepsilon}_{n+1} \\ \mathbf{0} \end{Bmatrix} \quad (2.292)$$

From the first row of vector equation above, the consistent elasto–plastic tangent moduli is obtained as follows,

$$\mathbb{C}^{ep} = \mathbb{C} : \mathbb{A}_{\varepsilon\varepsilon} - \frac{(\mathbb{C} : \mathbb{A}_{\varepsilon\varepsilon} : f_{,\sigma} + \mathbb{C} : \mathbb{A}_{\varepsilon\sigma} : f_{,q}) \otimes (f_{,\sigma} : \mathbb{C} : \mathbb{A}_{\varepsilon\varepsilon} + f_{,q} : \mathbb{H} : \mathbb{A}_{\sigma\varepsilon})}{\nabla_C^T \mathbb{A} \nabla} \quad (2.293)$$

## 2.4.5 Yield condition

Yield functions define a set of admissible stresses where a plastic yielding cannot occur. It is very common to represent yield functions in terms of hydrostatic and deviatoric parts of stress tensor ( $\boldsymbol{\sigma}$ ).

### 2.4.5.1 Stress and strain tensor in decomposed forms

The fundamental invariants (first, second and third, respectively) of a stress tensor  $\boldsymbol{\sigma}$  are defined as follows [140,141],

$$I_1 = \text{tr} \boldsymbol{\sigma} = \sigma_{ii} \quad (2.294)$$

$$I_2 = \frac{1}{2} [(\text{tr} \boldsymbol{\sigma}) - \text{tr} \boldsymbol{\sigma}^2] = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ij}) \quad (2.295)$$

$$I_3 = \det \boldsymbol{\sigma} = \det \sigma_{ij} = |\sigma_{ij}| \quad (2.296)$$

Then with the definition of hydrostatic stress (shear-free)  $p$  as

$$p = \frac{1}{3} I_1 \quad (2.297)$$

stress tensor can be split into hydrostatic and deviatoric parts as follows,

$$\sigma_{ij} = \frac{1}{3} I_1 \delta_{ij} + s_{ij} \quad (2.298)$$

where  $s_{ij}$  is the deviatoric stress tensor and  $\delta_{ij}$  is the Kronecker delta. Deviatoric stress tensor can be calculated from (2.298) as,

$$s_{ij} = \sigma_{ij} - \frac{1}{3} I_1 \delta_{ij} \quad (2.299)$$

The norm and the second invariant of the deviatoric stress are calculated as

$$\|\mathbf{s}\| = \sqrt{s_{ij} s_{ij}} \quad (2.300)$$

$$J_2 = \frac{1}{2} s_{ij} s_{ij} \quad (2.301)$$

Similar to the stress tensor, strain tensor can be decomposed into volumetric and deviatoric part also:

$$\varepsilon_{ij} = \frac{1}{3} \bar{I}_1 \delta_{ij} + e_{ij} \quad (2.302)$$

where  $\bar{I}_1 = \varepsilon_{ii}$  denotes first invariant of strain tensor and  $e_{ij}$  denotes deviatoric strain tensor. Constitutive relation between deviatoric stress and strain is given as follows [142]:

$$s_{ij} = 2Ge_{ij} \quad (2.303)$$

where  $G$  denotes shear modulus of material.

#### 2.4.5.2 von Mises yield condition

Similar to stress and strain tensors, the strain energy density can also be decomposed into volumetric part and deviatoric part, where the latter which is related with the strain energy density used to change the shape of the material. von Mises hypothesis states that, in a body at a point under any combination of stress state, yielding occurs if the distortion energy of the stress state reaches to the value that is associated with the energy required to change the shape of the material in a simple uniaxial tension test.

For any three dimensional stress state the deviatoric strain energy density is calculated as,

$$W_{dev} = \frac{1}{2} s_{ij} e_{ij} \quad (2.304)$$

Inserting  $e_{ij}$  from (2.303) into (2.304) yields,

$$W_{dev} = \frac{1}{2} s_{ij} \frac{s_{ij}}{2G} \quad (2.305)$$

which is equal to:

$$W_{dev} = \frac{J_2}{2G} \quad (2.306)$$

Lets consider the uniaxial test. For the case that applied stress  $\sigma$  reaches yielding stress  $\sigma_Y$  the hydrostatic pressure takes the value  $p = \sigma_Y / 3$ , therefore deviatoric stress tensor becomes:

$$(s_Y)_{uniaxial} = \begin{bmatrix} \sigma_Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{\sigma_Y}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\sigma_Y & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_Y & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_Y \end{bmatrix} \quad (2.307)$$

According to the definition of the deviatoric strain energy density and regarding the relation (2.303), for the uniaxial case at the onset of yielding, deviatoric strain energy  $(W_{dev})_{uniaxial}$  becomes,

$$\begin{aligned} (W_{dev})_{uniaxial} &= \frac{1}{2} (s_{ij} e_{ij})_{uniaxial} = \frac{1}{2} \left[ \frac{2\sigma_Y}{3} \frac{2\sigma_Y}{3 \times 2G} + 2 \times \left( -\frac{\sigma_Y}{3} \right) \left( -\frac{\sigma_Y}{3 \times 2G} \right) \right] \\ &= \frac{\sigma_Y^2}{6G} \end{aligned} \quad (2.308)$$

Equating to deviatoric strain energy densities from definitions (2.306) and (2.308) one yields,

$$3J_2 = \sigma_Y^2 \quad (2.309)$$

Then the yield function can be defined as,

$$f(\boldsymbol{\sigma}) = f(J_2) = J_2 - \frac{\sigma_Y^2}{3} \quad (2.310)$$

An alternative form of the yield function is presented as

$$f(\boldsymbol{\sigma}) = f(J_2) = \sqrt{J_2} - \frac{\sigma_Y}{\sqrt{3}} \quad (2.311)$$

In this description, the yield function is in the stress dimension, which causes to gradient of  $f$  and the consistency parameter be dimensionless. Regarding to Eq. (2.301) another form can be expressed:

$$f(\mathbf{s}) = \|\mathbf{s}\| - \sqrt{\frac{2}{3}} \sigma_Y = 0 \quad (2.312)$$

#### 2.4.5.3 Non-linear form of yield function

With the introduction of inner variable  $\alpha$  yield function with isotropic hardening can be defined as,

$$f(\mathbf{s}, \alpha) = \|\mathbf{s}\| - \sqrt{\frac{2}{3}} K(\alpha) \quad (2.313)$$

For the linear hardening case, hardening parameter is just given with following expression,

$$K(\alpha) = \sigma_Y + \bar{H}\alpha \quad (2.314)$$

#### 2.4.6 The proposed algorithm for two-field mixed FEM formulation

According to Hellinger–Reissner principle, equilibrium equations and constitutive relations of the system are satisfied in a weak form. Hellinger–Reissner principle can be classified as a two-field formulation where stresses  $\boldsymbol{\sigma}$  and displacements  $\mathbf{u}$  are fundamental variables of the functional. Unlike displacement based formulations, increment of stresses are also obtained besides increment of displacements (Figure 2.11).

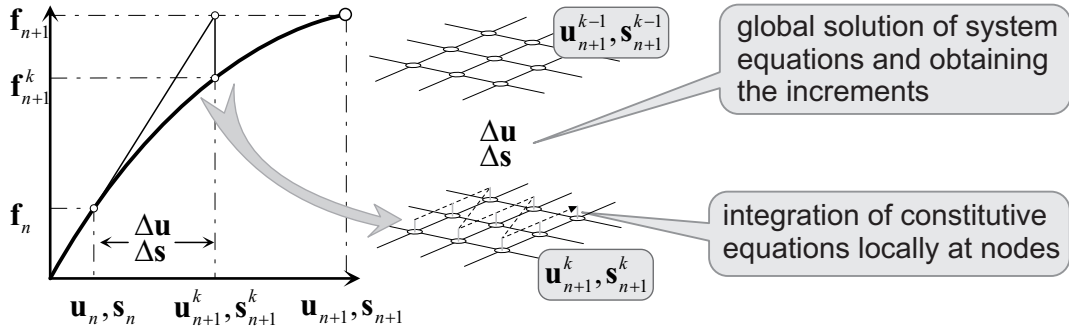


**Figure 2.11** : Overview of the proposed elasto–plastic algorithm.

Direct calculation of stress increments makes it possible to obtain also the strain increments  $\Delta \boldsymbol{\varepsilon}$  directly from constitutive relations instead of obtaining them from derivatives of displacement increments. Advantages of this approach can be summarized as follows.

1. Strain field is obtained through some matrix operations, over stresses and compliance matrix, instead of numerical differentiations. Using only matrix operations prevents the potential errors due to numerical differentiation.
2. Compared to the differentiation method, using matrix operations is more efficient regarding computation or calculation time.

3. At nodal points, since stress increments are calculated directly, strain increments are derived from them at each node. Therefore local constitutive integration processes can be conducted at the nodes instead of Gauss points. Then, total time for all local calculations are not more proportional to the number of total Gauss point, instead they are proportional to the number of total nodal points.
4. Performing local constitutive integrations at Gauss points causes a discontinuous material definition over finite elements, because, while a Gauss point becomes plastic after increment, the other nearby Gauss points can be still in elastic regime. Whereas in proposed algorithm integrating constitutive relations at nodal points, causes to a continuous material behavior over finite element domain. The material properties at Gauss points have to be interpolated from the material properties calculate at the nodes. It causes a continuous (relatively) distribution of material properties over the domain besides at each element.

The algorithm is summarized in pseudo-code and given in Figure 2.12.

#### 2.4.7 Adaptation of proposed algorithm to the Mindlin plates

The general two-field finite element formulation for elasto-plastic analysis algorithm given in section 2.4.4.2, will be now adopted to Mindlin plates. In the analysis of plates integrals of stresses  $\mathbf{s}$  (stress resultants) are main field variables instead of stresses  $\boldsymbol{\sigma}$ . Nodal displacements and rotations of Mindlin plate will be symbolized as  $\mathbf{u}$  and in-plane strains of mid-plane and curvatures will be symbolized as  $\mathbf{e}$ . Referring to the definition (2.10), the first variation of Hellinger-Reissner functional for Mindlin plate can be given as,

$$\delta\Pi_{HR} = \int_{\Omega} (\mathbf{e}^u - \mathbf{e}^s)^T \delta\mathbf{s} d\Omega + \int_{\Omega} \left( (\mathbf{s}^s)^T \delta\mathbf{e}^u - \mathbf{q}^T \delta\mathbf{u} \right) d\Omega = 0 \quad (2.315)$$

where the traction forces are ignored due to the clarity of the definition. Non-linear functional can be linearized about the state  $k$  as in the following form [143],

$$\mathcal{L}[\delta\Pi_{HR}] = \delta\Pi_{HR(k)} + \Delta\delta\Pi_{HR(k)} \quad (2.316)$$

Here  $\delta \Pi_{HR(k)}$  defines the state of the system at  $k$  and requires the state determination of the system with state variables  $(\mathbf{u}_k, \mathbf{s}_k)$ .

$$\delta \Pi_{HR(k)} = \delta \Pi_{HR}(\mathbf{u}_k, \mathbf{s}_k, \delta \mathbf{u}_k, \delta \mathbf{s}_k) \quad (2.317)$$

The increment of two fields  $(\Delta \mathbf{u}, \Delta \mathbf{s})$  will be determined from the calculation of directional derivative of the variation of the functional, where,

$$\Delta \delta \Pi_{HR(k)} = \frac{d}{d\alpha} [\delta \Pi_{HR}(\mathbf{u} + \alpha \mathbf{u}, \mathbf{s} + \alpha \mathbf{s}, \delta \mathbf{u}, \delta \mathbf{s})]_{(\mathbf{u}=\mathbf{u}_k, \mathbf{s}=\mathbf{s}_k)} \quad (2.318)$$

Considering only material non-linearity yields (2.318) as,

$$\begin{aligned} \Delta \delta \Pi_{HR(k)} &= \int_{\Omega} [\Delta \mathbf{e}^u \delta \mathbf{s} - \mathbf{D}_k^{-1} \Delta \mathbf{s} \delta \mathbf{s} + \Delta \mathbf{s} \delta \mathbf{e}^u] d\Omega \\ &= \mathbf{K}^{++} \end{aligned} \quad (2.319)$$

where  $\mathbf{D}_k^{-1}$  is the section compliance matrix at state  $k$  which is calculated as described in section 2.3.2.3 but with the integration of elasto-plastic tangent moduli  $\mathbb{C}^{ep}$  (instead of elastic moduli) of each layer through the thickness of the plate. Here,

$$\mathbf{D}_k^{-1} = \left. \frac{d\mathbf{e}^s}{d\mathbf{s}} \right|_k \quad (2.320)$$

To simulate the spread of the plastic behavior over plate thickness a layered approach [89] will be used. That means material constitutive equations will be integrated at each layer for each node.

For the geometrically linear problem, strain-displacement equations in state  $k$  vanishes so state determination is just related with the satisfaction of the equilibrium of the system,

$$\begin{aligned} \delta \Pi_{HR(k)} &= \int_{\Omega} (\mathbf{e}^{u_k} - \mathbf{e}^{s_k})^T \delta \mathbf{s} d\Omega + \int_{\Omega} \left( (\mathbf{s}^{s_k})^T \delta \mathbf{e}^{u^k} - \mathbf{q}^T \delta \mathbf{u} \right) d\Omega \\ &= \int_{\Omega} \left( (\mathbf{s}^{s_k})^T \delta \mathbf{e}^{u^k} - \mathbf{q}^T \delta \mathbf{u} \right) d\Omega \\ &= \mathbf{K}^{+k} - \mathbf{F}^+ \end{aligned} \quad (2.321)$$

Here  $\mathbf{s}^{s^k}$  is the integral of stress components at state  $k$  which are calculated through the local integration of constitutive equations as explained in section 2.4.4 .

#### 2.4.7.1 Matrix form of the problem

The proposed mixed finite element formulation for the elasto–plastic analysis of layered Mindlin plate has the following matrix form,

$$\mathbf{K}^{++(k-1)} \mathbf{X}^{+(k)} = \mathbf{F}^+ - \mathbf{K}^{+(k-1)} \quad (2.322)$$

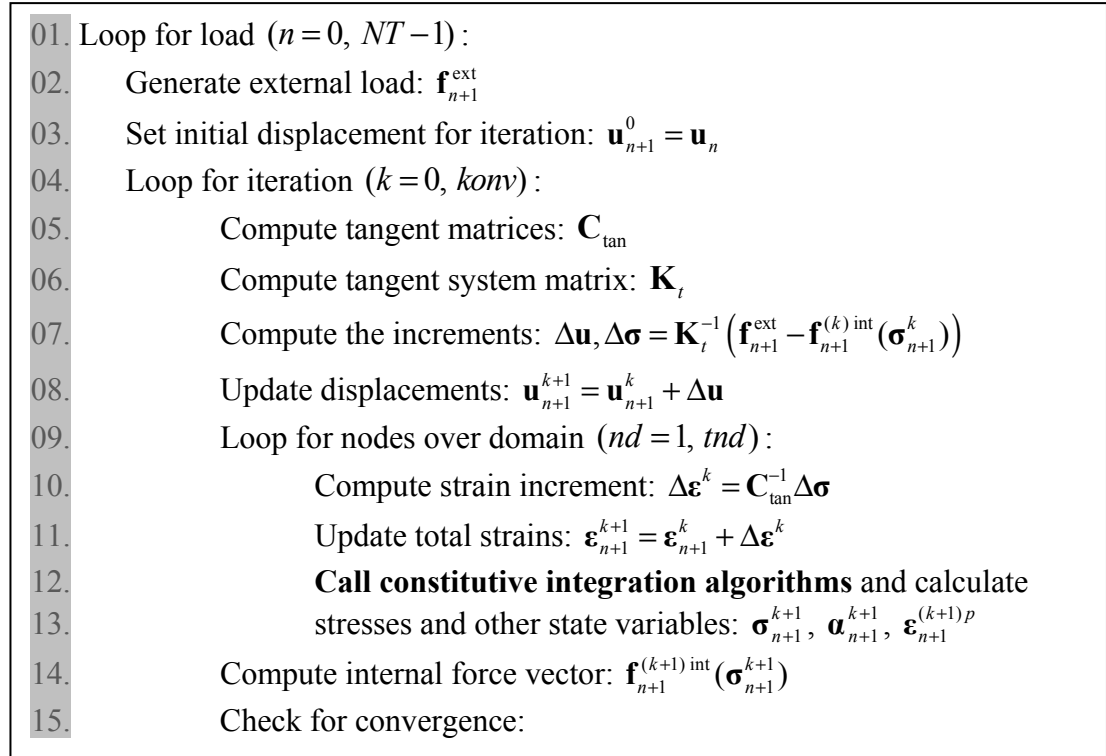


**Figure 2.12 :** Algorithm for materially non–linear two–field finite elements.

$\mathbf{K}^{++(k-1)}$  given in (2.322) refers to the tangent system matrix which is calculated as (2.192) but with the consideration of the elasto–plastic tangent moduli.  $\mathbf{F}^+$  is the external load vector and is same with (2.195). State determination or correction vector  $\mathbf{K}^+$  contains only the equilibrium related terms in (2.194), therefore it has the following non–zero sub–vectors in it,

$$\mathbf{K}^+ = \left\{ \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{K}_U^+ \quad \mathbf{K}_R^+ \right\}^T \quad (2.323)$$

Increment unknown vector is the same with (2.196) where,

$$\mathbf{X}^+ = \{\Delta \mathbf{s} \quad \Delta \mathbf{u}\}^T = \{\Delta \mathbf{N} \quad \Delta \mathbf{M} \quad \Delta \mathbf{S} \quad \Delta \mathbf{U} \quad \Delta \mathbf{R}\}^T \quad (2.324)$$

The summary of the proposed (global) algorithm in terms of pseudo-code is presented in Figure 2.13.

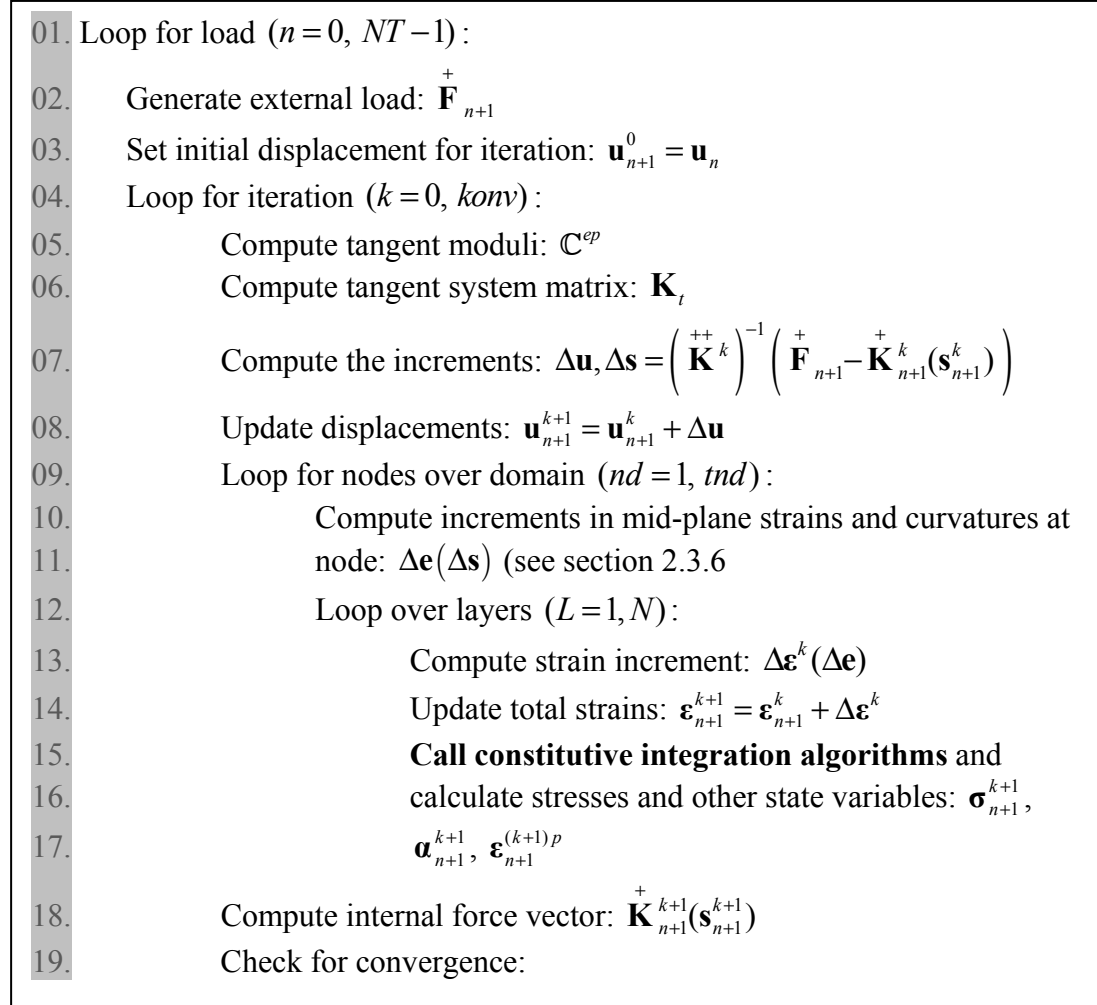


**Figure 2.13 :** Algorithm for materially non-linear Mindlin plates.

### 3. NUMERICAL RESULTS

The developed finite element formulation and the proposed solution algorithms are combined in a home made finite element software in FORTRAN programming language. Numerical values are stored with the double precision.

Plate analysis can be performed for various possible boundary conditions. In order to clarify the used edge conditions a list of figures presenting the constraints for corresponding boundary conditions and how they are called with a short name defined in the Figure 3.1. For example, SS1 and CC1 are corresponds of classical simply supported and clamped boundary conditions, respectively.

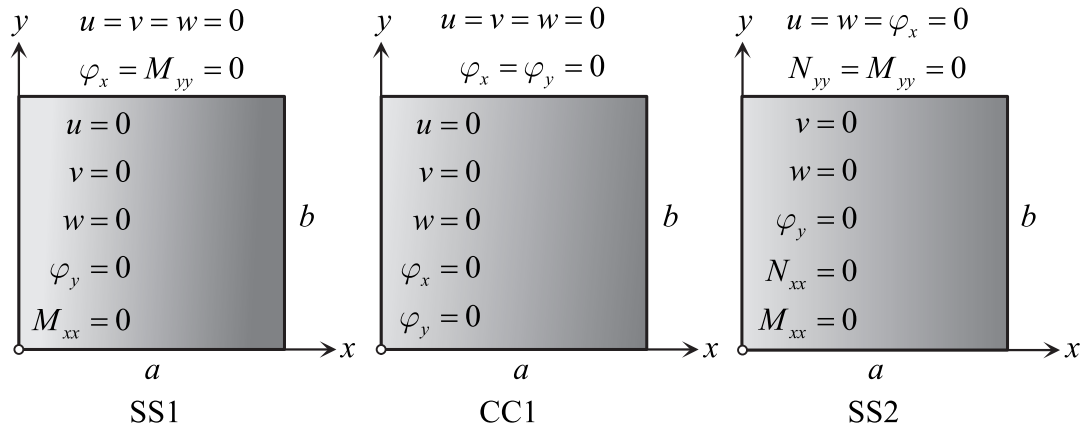


**Figure 3.1 :** Boundary conditions for Mindlin plate with in-plane terms.

#### 3.1 Linear and Large Deflection Analysis of Laminated Mindlin Plates

##### 3.1.1 Verification of mixed finite element formulation

###### 3.1.1.1 Large deflection of an isotropic thick rectangular plate

To verify the formulation some numerical computations are carried out for a rectangular plate under uniform transverse load and having SS1 boundary conditions. Obtained results are compared with a displacement type finite element solution presented in Reddy [110]. In this example plate dimensions are:  $a = b = 10\text{m}$ ,  $h = 1\text{m}$  and plates material properties are:  $E = 7.8 \times 10^6 \text{N/m}^2$  and  $\nu = 0.3$ . Three

type of meshes ( $6 \times 6$ ,  $12 \times 12$ ,  $18 \times 18$ ) are used to test the convergence of the formulation. Non-dimensional central deflection ( $\bar{w} = w/h$ ) versus non-dimensional load parameter  $\bar{q} = qa^4/(Eh^4)$  of the plate is depicted in Figure 3.2 (see Kutlu and Omurtag [144]). From figure, it can be seen that formulation gives good result even for loose meshes ( $6 \times 6$ ) and numerical results converge rapidly. For the whole load interval load-displacement curves of Reddy and present solution agree well with each other.

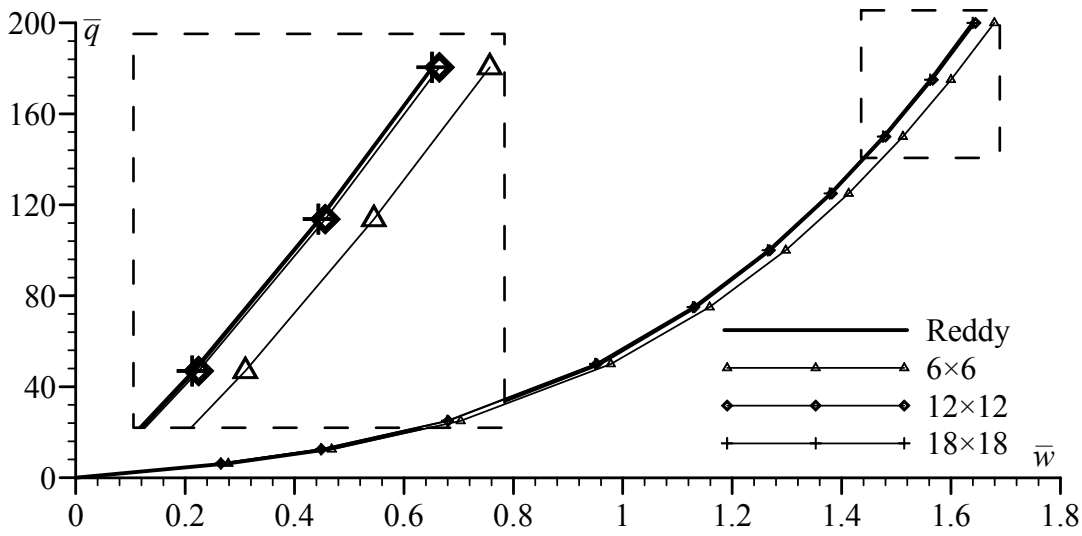


**Figure 3.2 :** Plot of central deflection versus load of a SS1 isotropic square plate under uniform loading.

### 3.1.1.2 Large deflection of an isotropic thin circular plate

Using the quadrilateral finite elements to discretize the plate domain allows to perform large deflection analysis of plates other than rectangular geometry (see Figure 3.3). To show the versatility of the proposed solution, thin clamped circular plates exhibiting large deflection presented in the study of Dumir [145] is considered in this example. Dumir employed orthogonal point collocation method to analyse non-linear problem in space domain. Investigated plate has the thickness to radius aspect ratio as  $h/a = 0.01$  and the Poisson's ratio as  $\nu = 0.25$ . Non-dimensional central deflections of the uniformly loaded thin circular plate obtained from mixed formulation and from the study of Dumir [145] are depicted in the Figure 3.4. Three different mesh configurations are used where 48, 192 and 768 elements are employed. According to the Figure 3.4 results of mixed solution procedure and results of Dumir [145] are in a good agreement. A steady convergence behavior can be observed from Figure 3.4 with the increasing element numbers. It must be pointed

out that the proposed mixed formulation automatically overcomes the shear–locking problem. Solutions for the improved version of this problem can be followed in the recent work of the author Kutlu and Omurtag [146], where elastic foundation–plate interaction was investigated.

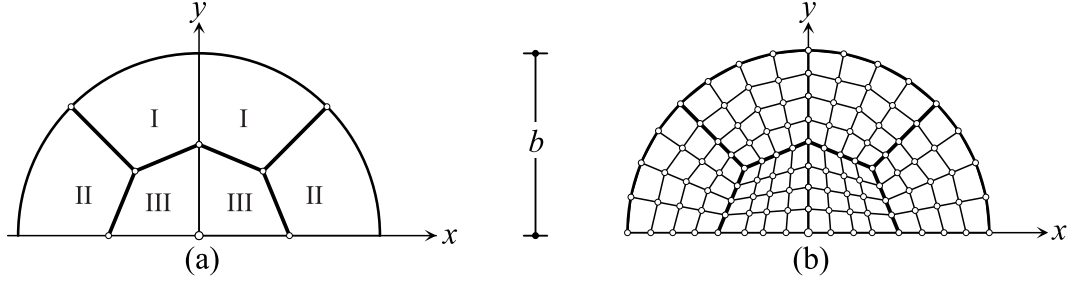


**Figure 3.3 :** a)Main parts used for mesh generation in a half domain of a circular plate. (b)Mesh of a half domain where each edge of main parts divided into 4 segments.

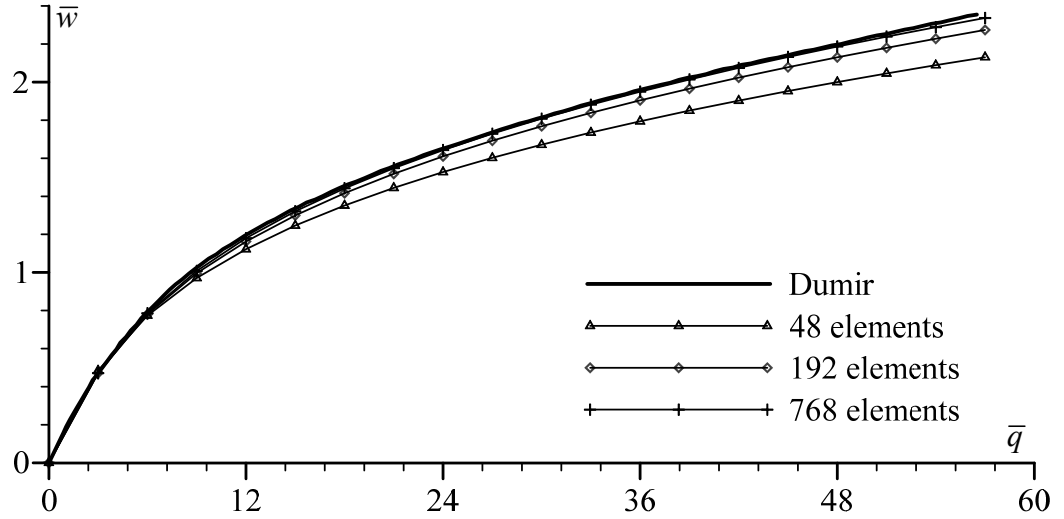


**Figure 3.4 :** Central deflection of a thin clamped circular plate.

### 3.1.1.3 Laminated clamped square plate exhibiting large deflection

This example deals with a square composite laminated plate with clamped edges (CC1) and subjected to lateral uniform loading. Large deflection bending solutions are made for three types of meshes ( $8 \times 8$ ,  $12 \times 12$ ,  $16 \times 16$ ) and the obtained results are compared with displacement type finite element results provided by Reddy [116]. Plate dimensions are  $a = b = 12\text{m}$ ,  $h = 0.3\text{m}$  and orthotropic material properties are  $E_1 = 40 \times 10^6 \text{N/m}^2$ ,  $E_2 = 10^6 \text{N/m}^2$ ,  $G_{12} = G_{13} = 0.6E_2$ ,  $G_{23} = 0.5E_2$  and  $\nu_{12} = 0.25$ . Laminae layup directions are  $(0^\circ / 90^\circ / 0^\circ / 90^\circ / 0^\circ / 90^\circ)$  for equal layer thickness through laminate and results are obtained for load step  $\Delta q = 200 \text{N/m}^2$ .

Central deflection of the plate versus uniform load value is depicted in Figure 3.5 (see also Kutlu and Omurtag [144]). It can be said that results are in a great agreement and for 16×16 mesh, they are almost identical.

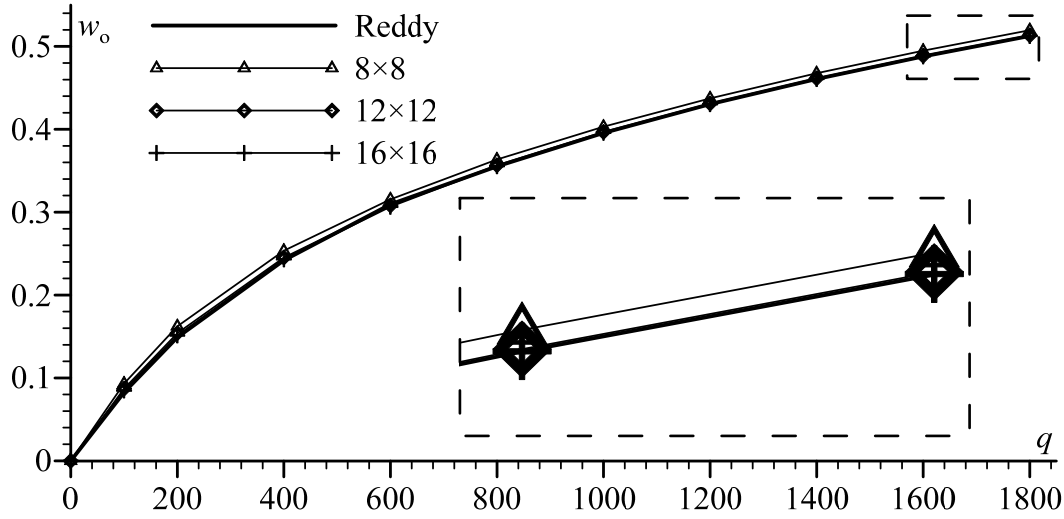


**Figure 3.5 :** Plot of central deflection versus uniform load of a clamped laminated square plate exhibiting large deflection.

### 3.1.2 Accuracy of the stress components

Composite laminates are preferred in engineering structures due to their superior high strength property and low unit weight. For a better evaluation of composite laminate's advantages, a more delicate design process is required. A precise and reliable design depends on accurate calculation of local quantities in a composite laminate. To identify the performance of presented method on computing these quantities some comparison studies are carried out. Firstly, linear problem is thought and stress components of mixed finite element are compared with exact, analytic and some numerical results. A number of solutions are also made for large deflection of composite laminated plates under transverse loads. According to convergence study 16×16 mesh is used unless stated otherwise.

#### 3.1.2.1 Stresses in a SS1 cross-ply laminated plate under sinusoidal load

Sinusoidally loaded cross-ply square plates with SS1 boundary condition are examined by means of geometrically linear stress analysis. Linear flexure of elastic plates with various side lengths to thickness ratios is considered. Laminations of the plates are symmetric according to the central plane where top and bottom layers have  $\theta = 0^\circ$  material direction. Thicknesses of the layers with the same material directions

are equal, where total thicknesses of layers with  $0^\circ$  and  $90^\circ$  material directions are also equal. According to definition a  $(0^\circ/90^\circ/0^\circ)$  laminate with material properties  $E_1/E_2=25$ ,  $G_{12}/E_2=0.5$ ,  $G_{13}=G_{12}$ ,  $G_{23}/E_2=0.2$ ,  $\nu_{12}=0.25$  studied. Sinusoidal load has the form  $q(x,y)=q_0 \sin(\pi x/a) \sin(\pi y/b)$  where  $q_0$  is a constant (see Figure 3.6).

$$q(x,y) = q_0 \sin(\pi x/a) \sin(\pi y/b)$$

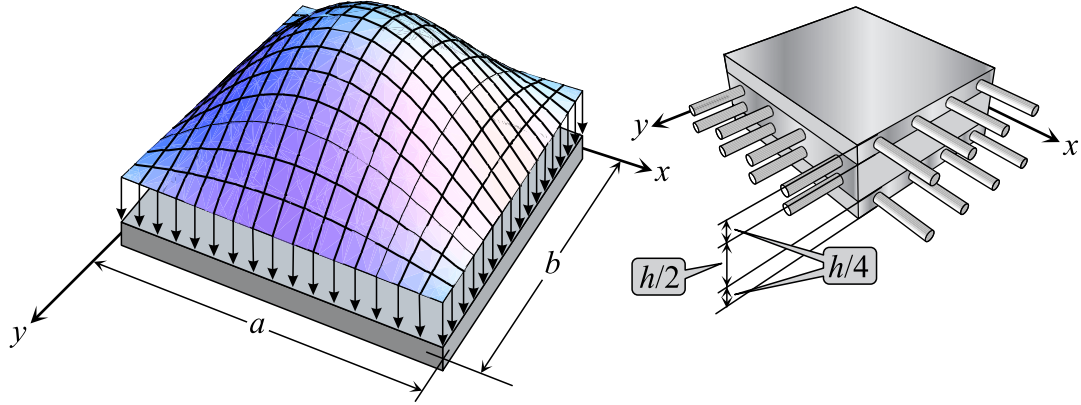


**Figure 3.6 :** Sinusoidally loaded three layered plate.

Parameters of the plates are given in non-dimensional form as:  $(\tilde{\sigma}_{xx}, \tilde{\sigma}_{yy}, \tilde{\sigma}_{xy}) = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})(h^2/q_0 a^2)$ ,  $(\tilde{\sigma}_{xz}, \tilde{\sigma}_{yz}) = (\sigma_{xz}, \sigma_{yz})(h/q_0 a)$ , and  $\tilde{w} = w\pi^4 \alpha h^3 / 12a^4 q_0$ , where  $\alpha = 4G_{12} + [E_1 + E_2(1+2\nu_{12})]/(1-\nu_{12}\nu_{21})$ . Arguments of the coordinates are  $(x, y, z/h)$  and stresses are computed at  $\tilde{\sigma}_{xx}(a/2, a/2, 1/2)$ ,  $\tilde{\sigma}_{yy}(a/2, a/2, 1/4)$  (at the 2. layer),  $\tilde{\sigma}_{xy}(0, 0, 1/2)$ ,  $\tilde{\sigma}_{xz}(0, a/2, 0)$ ,  $\tilde{\sigma}_{yz}(a/2, 0, 0)$  and deflections are computed at  $\tilde{w}(a/2, a/2, 0)$ . Results are compared with exact elasticity solution of Pagano and Hatfield [147], numerical results of Reddy [34], Panda and Natarajan [148] and Kam and Jan [149]. Table 3.1 shows non-dimensional in-plane and transverse stresses of a three-ply laminates with different side length to thickness ratios. Even for a very thick plate case  $a/h=10$  presented method gives better results than other finite element solutions except the layer-wise solution of Kam & Jan. As  $a/h$  increases presented results are getting more accurate compared to the other methods. Beyond the  $a/h=50$  values, difference ratios of exact elasticity solution and this study becomes in the order of thousandths. For example, errors in  $\tilde{\sigma}_{xx}$  for  $a/h=50$  and 100 are %0.74, %0.55 and errors in  $\tilde{\sigma}_{xz}$  are %0.92, %0.93 respectively.

**Table 3.1** : Stresses and deflections in 3–ply laminate.

| $a/h$ | Source            | $\tilde{\sigma}_{xx}$ | $\tilde{\sigma}_{yy}$ | $\tilde{\sigma}_{xy}$ | $\tilde{\sigma}_{xz}$ | $\tilde{\sigma}_{yz}$ | $\tilde{w}$ |
|-------|-------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-------------|
| 10    | this study        | 0.497                 | 0.360                 | -0.0241               | 0.314                 | 0.197                 | 1.598       |
|       | Pagano & Hatfield | 0.559                 | 0.401                 | -0.0275               | 0.301                 | 0.196                 | 1.709       |
|       | Reddy             | 0.484                 | 0.350                 | -0.0234               | -                     | -                     | 1.534       |
|       | Panda & Natarajan | 0.532                 | 0.307                 | -0.0250               | -                     | -                     | 1.448       |
|       | Mawenya & Davies  | 0.542                 | -                     | -0.0292               | -                     | -                     | 2.034       |
|       | Kam & Jan         | 0.561                 | 0.405                 | -0.0277               | -                     | -                     | 1.685       |
| 20    | this study        | 0.526                 | 0.295                 | -0.0220               | 0.329                 | 0.166                 | 1.184       |
|       | Pagano & Hatfield | 0.543                 | 0.308                 | -0.0230               | 0.328                 | 0.156                 | 1.189       |
|       | Reddy             | 0.511                 | 0.287                 | -0.0214               | -                     | -                     | 1.136       |
|       | Panda & Natarajan | 0.557                 | 0.307                 | -0.0231               | -                     | -                     | 1.114       |
|       | Mawenya & Davies  | 0.546                 | -                     | -0.0239               | -                     | -                     | 1.273       |
|       | Kam & Jan         | 0.554                 | 0.316                 | -0.0234               | -                     | -                     | 1.182       |
| 50    | this study        | 0.535                 | 0.273                 | -0.0214               | 0.334                 | 0.156                 | 1.063       |
|       | Pagano & Hatfield | 0.539                 | 0.276                 | -0.0216               | 0.337                 | 0.141                 | 1.031       |
|       | Reddy             | 0.520                 | 0.265                 | -0.0207               | -                     | -                     | 1.019       |
|       | Panda & Natarajan | 0.565                 | 0.287                 | -0.0225               | -                     | -                     | 1.016       |
|       | Mawenya & Davies  | 0.550                 | -                     | -0.0221               | -                     | -                     | 1.048       |
|       |                   |                       |                       |                       |                       |                       |             |
| 100   | this study        | 0.536                 | 0.270                 | -0.0212               | 0.335                 | 0.154                 | 1.046       |
|       | Pagano & Hatfield | 0.539                 | 0.271                 | -0.0214               | 0.339                 | 0.139                 | 1.008       |
|       | Reddy             | 0.523                 | 0.263                 | -0.0207               | -                     | -                     | 1.005       |
|       | Panda & Natarajan | 0.566                 | 0.284                 | -0.0223               | -                     | -                     | 1.003       |
|       | Mawenya & Davies  | 0.551                 | -                     | -0.0213               | -                     | -                     | 1.015       |
|       |                   |                       |                       |                       |                       |                       |             |

**3.1.2.2 Stresses in a SS2 square plate under lateral uniform load**

In this case, uniformly loaded cross–ply laminated square plates with SS2 boundary conditions are considered. Plates characteristic length to thickness ratio is varied as  $a/h=20,50$  and  $100$ . Plate parameters are  $E_1=25E_2$ ,  $G_{12}=G_{13}=0.5E_2$ ,  $G_{23}=0.2E_2$ ,  $\nu_{12}=0.25$  and shear correction factor is  $K=5/6$ .

A lamination with equal ply thickness in the  $(0^\circ,90^\circ,90^\circ,0)$  scheme is analyzed. Non–dimensional in–plane stresses and central deflection of the plates are compared with analytical FSDT plate results presented in [116]. Non–dimensional parameters of the plate are given in Table 3.2 as,  $\tilde{w}(a/2,a/2)=w(a/2,a/2)E_2h^3/qa^4$ ,  $\tilde{\sigma}_{xx}(a/2,a/2,h/2)=\sigma_{xx}(a/2,a/2,h/2)h^2/qa^2$ ,  $\tilde{\sigma}_{xy}(a,a,-h/2)=\sigma_{xy}h^2/qa^2$ ,  $\tilde{\sigma}_{yy}(a/2,a/2,h/4)=\sigma_{yy}h^2/qa^2$  (at 3. layer).

Presented mixed finite element solution and analytical solution are in grate agreement according to the Table 3.2. Biggest percent difference between presented

study and analytical method is 3 but much of the quantities are almost equal to each other where differences are in ten thousandths.

**Table 3.2 :** Local quantities of cross-ply square plate in uniform loading.

| $a/h$ |            | $\bar{w} \times 100$ | $\bar{\sigma}_{xx}$ | $\bar{\sigma}_{yy}$ | $\bar{\sigma}_{xy}$ |
|-------|------------|----------------------|---------------------|---------------------|---------------------|
| 10    | this study | 1.0266               | 0.7578              | 0.4962              | 0.0488              |
|       | Reddy      | 1.0250               | 0.7577              | 0.5006              | 0.0470              |
| 20    | this study | 0.7714               | 0.8050              | 0.3917              | 0.0434              |
|       | Reddy      | 0.7694               | 0.8045              | 0.3968              | 0.0420              |
| 100   | this study | 0.6854               | 0.8235              | 0.3502              | 0.0408              |
|       | Reddy      | 0.6833               | 0.8420              | 0.3558              | 0.0396              |

### 3.1.2.3 Stresses in a square orthotropic plate exhibiting large deflection

In this example, a square orthotropic plate exhibiting large deflection under uniform load with SS2 boundary condition is examined. Results of Turvey and Osman [150], and Reddy [151] are used for comparison. The material of the plate has properties:  $E_1/E_2 = 2.345$ ,  $G_{12}/E_2 = 0.289$ ,  $G_{13}/E_2 = 0.289$ ,  $G_{23}/E_2 = 0.289$ , and  $\nu_{12} = 0.32$ . Non-dimensional parameters of large deflection analysis are given in the form as:

$$\left. \begin{aligned} \bar{w} &= \frac{w}{h}, & \bar{\sigma}_{xx} &= \sigma_{xx} \left( \frac{a^2}{E_2 h^2} \right), & \bar{\sigma}_{yy} &= \sigma_{yy} \left( \frac{a^2}{E_2 h^2} \right) \\ \bar{\sigma}_{xy} &= \sigma_{xy} \left( \frac{a^2}{E_2 h^2} \right), & \bar{\sigma}_{xz} &= \sigma_{xz} \left( \frac{a^3}{E_2 h^3} \right), & \bar{\sigma}_{yz} &= \sigma_{yz} \left( \frac{a^3}{E_2 h^3} \right) \\ \bar{q} &= q \left( \frac{a^4}{E_2 h^4} \right) \end{aligned} \right\} \quad (3.1)$$

The stresses are calculated at locations  $\tilde{\sigma}_{xx}(a/2, a/2, h/2)$ ,  $\tilde{\sigma}_{yy}(a/2, a/2, h/2)$ ,  $\tilde{\sigma}_{xy}(0, 0, -h/2)$ ,  $\tilde{\sigma}_{yz}(a/2, 0, 0)$ ,  $\tilde{\sigma}_{xz}(0, a/2, 0)$  and displacement calculated at the centre of the plate. Parameters are given in Table 3.3. Table 3.3 reveals that in plane stresses are close to each other in both procedures. However, there is a slight difference between transverse stresses obtained from DR and mixed finite element solution. It can be concluded that deflection parameters computed by Reddy and present method are more close to each other compared to the Turvey and Osman.

**Table 3.3 :** Local quantities of SS2 square orthotropic plate exhibiting large deflection.

| $\tilde{q}$ |                  | $\tilde{w}$ | $\tilde{\sigma}_{xx}$ | $\tilde{\sigma}_{yy}$ | $\tilde{\sigma}_{xy}$ | $\tilde{\sigma}_{yz}$ | $\tilde{\sigma}_{xz}$ |
|-------------|------------------|-------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| 17.9        | this study       | 0.6598      | 8.5215                | 4.0378                | 2.2092                | 7.4484                | 10.8635               |
|             | Turvey and Osman | 0.5858      | 7.7623                | 3.8861                | 2.3345                | 5.7852                | 8.0985                |
|             | Reddy            | 0.58        |                       |                       |                       |                       |                       |
| 35.7        | this study       | 1.0008      | 13.6993               | 6.9689                | 4.4960                | 13.2763               | 18.0610               |
|             | Turvey and Osman | 0.9833      | 13.1180               | 6.7160                | 4.5644                | 11.0100               | 14.7420               |
|             | Reddy            | 1.01        |                       |                       |                       |                       |                       |
| 53.6        | this study       | 1.2962      | 17.6975               | 9.1317                | 6.5191                | 18.7514               | 24.7086               |
|             | Turvey and Osman | 1.2710      | 16.9200               | 8.8065                | 6.7040                | 15.8060               | 20.4570               |
|             | Reddy            | 1.30        |                       |                       |                       |                       |                       |
| 71.5        | this study       | 1.5288      | 20.7529               | 10.8447               | 8.4311                | 23.6974               | 30.5589               |
|             | Turvey and Osman | 1.4977      | 19.8560               | 10.4810               | 8.7962                | 20.2510               | 25.6490               |
|             | Reddy            | 1.56        |                       |                       |                       |                       |                       |
| 89.3        | this study       | 1.7220      | 23.2221               | 12.2810               | 10.2625               | 28.2199               | 35.8661               |
|             | Turvey and Osman | 1.6862      | 22.2550               | 11.8940               | 10.8600               | 24.3910               | 30.4350               |
|             | Reddy            | 1.74        |                       |                       |                       |                       |                       |

#### 3.1.2.4 Stresses in a cross-ply square plate exhibiting large deflection

This example gives deflection and stress components of a simply supported (SS1) cross-ply square plate under uniform load and exhibiting large deflection. Results obtained with mixed finite element are compared with shear deformable displacement based finite element solution of Reddy [116]. Material properties are in dimensional form:  $E_1 = 1.8282 \times 10^6 \text{ N/m}^2$ ,  $E_2 = 1.8315 \times 10^6 \text{ N/m}^2$ ,  $\nu_{12} = 0.2395$ ,  $G_{12} = G_{13} = G_{23} = 0.3125 \times 10^6 \text{ N/m}^2$ . Plates dimensions are  $h = 0.096 \text{ m}$  and  $a = b = 12 \text{ m}$ . Laminated plate is constituted from equal thickness plies with  $(0^\circ / 90^\circ / 90^\circ / 0^\circ)$  orientation. Calculations are made for a load increment of  $\Delta q = 0.2 \text{ N/m}^2$ . For  $A = 6.75 \text{ m}$ ,  $B = 11.25 \text{ m}$ ,  $\tilde{\sigma}_{yy} = \sigma_{yy}(A, A, -h/4)a^2 / E_2 h^2$  and  $\tilde{\sigma}_{xy} = \sigma_{xy}(B, B, -h/2)a^2 / E_2 h^2$  are non-dimensional stress parameters.  $\tilde{\sigma}_{yy}$  is calculated at the 2. Layer. Deflection  $w$  is calculated at the center of the plate.

It can be seen from the Table 3.4 that results obtained with presented method and results of Reddy agree each other very well. Compared to the stress components, displacement values are closer to each other. For example for the load parameter  $q = 1.40$  percent differences at  $w$ ,  $\tilde{\sigma}_{yy}$  and  $\tilde{\sigma}_{xy}$  parameters are %0.31, %2.33 and %2.33 respectively.

**Table 3.4 :** Large deflection stress components and central deflection of a SS1 square cross-ply plate.

| $q$  | 10w        |        | $\tilde{\sigma}_{yy}$ |         | $\tilde{\sigma}_{xy}$ |        |
|------|------------|--------|-----------------------|---------|-----------------------|--------|
|      | this study | Reddy  | this study            | Reddy   | this study            | Reddy  |
| 0.20 | 0.7341     | 0.7324 | -0.4128               | -0.4254 | 1.5087                | 1.4812 |
| 0.40 | 0.9973     | 0.9946 | 0.4265                | 0.3911  | 2.1828                | 2.1387 |
| 0.60 | 1.1711     | 1.1677 | 1.2800                | 1.2271  | 2.6714                | 2.6145 |
| 0.80 | 1.3050     | 1.3012 | 2.0985                | 2.0315  | 3.0686                | 3.0012 |
| 1.00 | 1.4158     | 1.4116 | 2.8799                | 2.8012  | 3.4086                | 3.3326 |
| 1.20 | 1.5113     | 1.5067 | 3.6280                | 3.5386  | 3.7087                | 3.6249 |
| 1.40 | 1.5958     | 1.5908 | 4.3470                | 4.2481  | 3.9787                | 3.8881 |
| 1.60 | 1.6720     | 1.6667 | 5.0406                | 4.9331  | 4.2252                | 4.1284 |
| 1.80 | 1.7416     | 1.7360 | 5.7120                | 5.5965  | 4.4525                | 4.3503 |
| 2.00 | 1.8059     | 1.8001 | 6.3637                | 6.2408  | 4.6638                | 4.5567 |
| 2.20 | 1.8658     | 1.8597 | 6.9979                | 6.8680  | 4.8616                | 4.7501 |
| 2.40 | 1.9220     | 1.9156 | 7.6163                | 7.4798  | 5.0476                | 4.9322 |

### 3.2 Elasto-plastic Analysis of Mindlin Plates

#### 3.2.1 Elasto-plastic plate with linear hardening

The presented formulation for the elasto-plastic analysis of Mindlin plates was adopted to the finite element code and a comparison study is presented here. A uniformly loaded simply supported square plate ( $6\text{m} \times 6\text{m}$ ) with thickness  $h = 0.2\text{m}$  is considered. Elastic modulus of the plate is  $E = 2.067 \times 10^8 \text{MPa}$  and yield stress is  $\sigma_Y = 206.7 \times 10^3 \text{kPa}$ , where tangent modulus is  $E_T = 2.067 \times 10^6 \text{MPa}$  after yield for linearly hardening material. Linear hardening parameter  $\bar{H}$  and  $E_T$  has the following relation [89],

$$\bar{H} = E_T / (1 - E_T / E) \quad (3.2)$$

For the comparison, the results from Belinha and Dinis [105] are used, where a layered approach was adopted with an amount of eight equal thickness layers through the plate thickness. In order to examine the performance of the proposed solution procedure, numerous solutions are carried out. Important parameters governing numerical results are investigated. Four different meshes as  $4 \times 4$ ,  $8 \times 8$ ,  $12 \times 12$ ,  $20 \times 20$ , four different total layer numbers as 4, 8, 12, 16 layers and four different load step numbers as 5, 10, 20, 40 load steps are considered.

Non-dimensional central deflection of the plate versus non-dimensional loading parameter are figured out with the definition of plastic moment  $M_p$  as,

$$M_p = \sigma_Y \frac{1}{4} \quad (3.3)$$

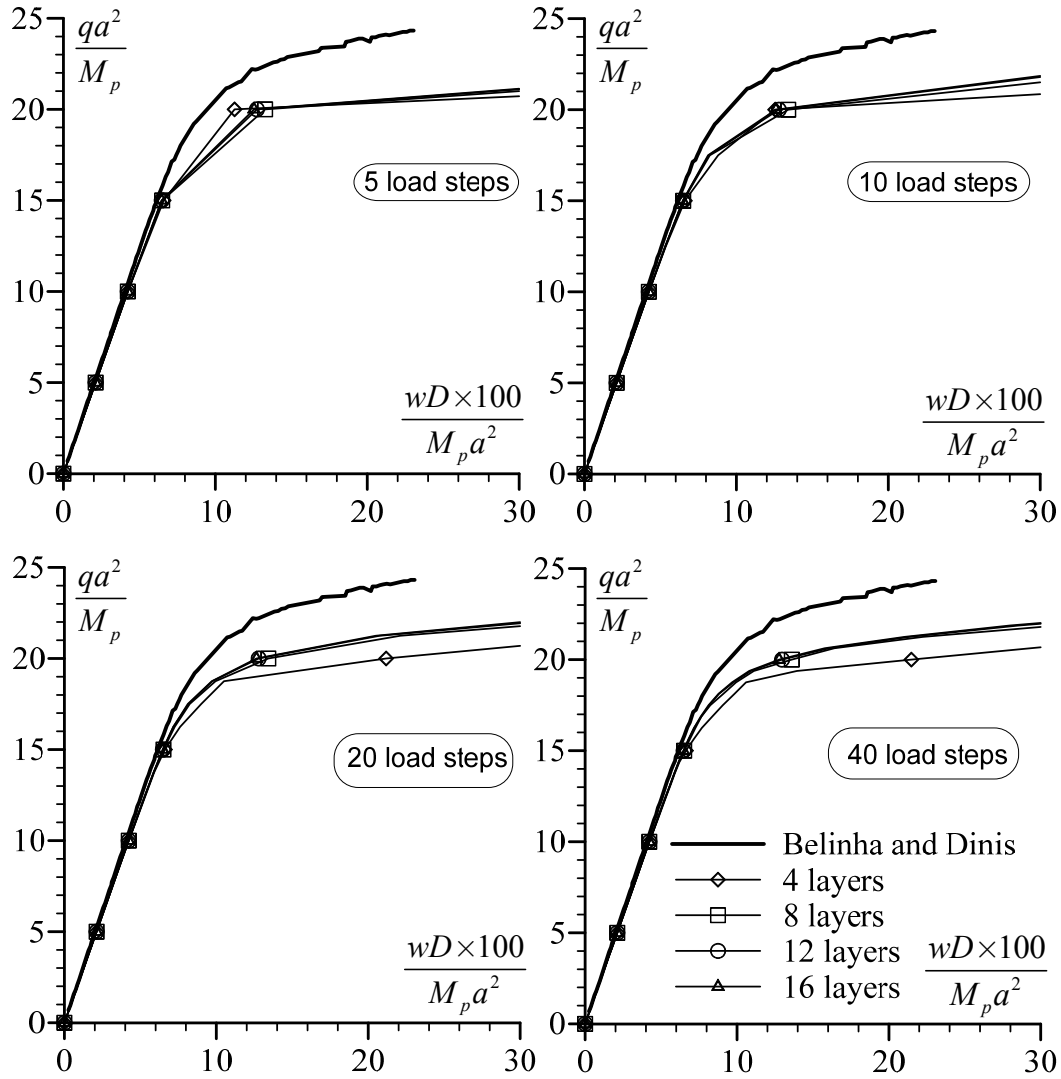


**Figure 3.7 :** Central deflection of a simply supported elasto-plastic square plate with  $4 \times 4$  mesh.

Figure 3.7 depicts the central deflection of the plate discretized by a  $4 \times 4$  mesh. Although  $4 \times 4$  can be thought of as a coarse mesh for a non-linear plate finite element analysis, the formulation and solution procedure can reflect the elasto-plastic behaviour of the plate. For small load steps, the number of the layers through the cross section of the plate does not have much effect on the deflection results. According to the Figure 3.7 behavior of the elasto-plastic plate with respect to the layer numbers differs from each other as load steps are increased. For 20 and 40 load

steps, it is obvious that a number of 4 layers through the plate thickness is remarkable not enough for a precise analysis. Specially for this problem with a thickness to side length ratio of  $h/a = 0.2/6$  at least 12 layers must be adapted to get a converged finite element solution.

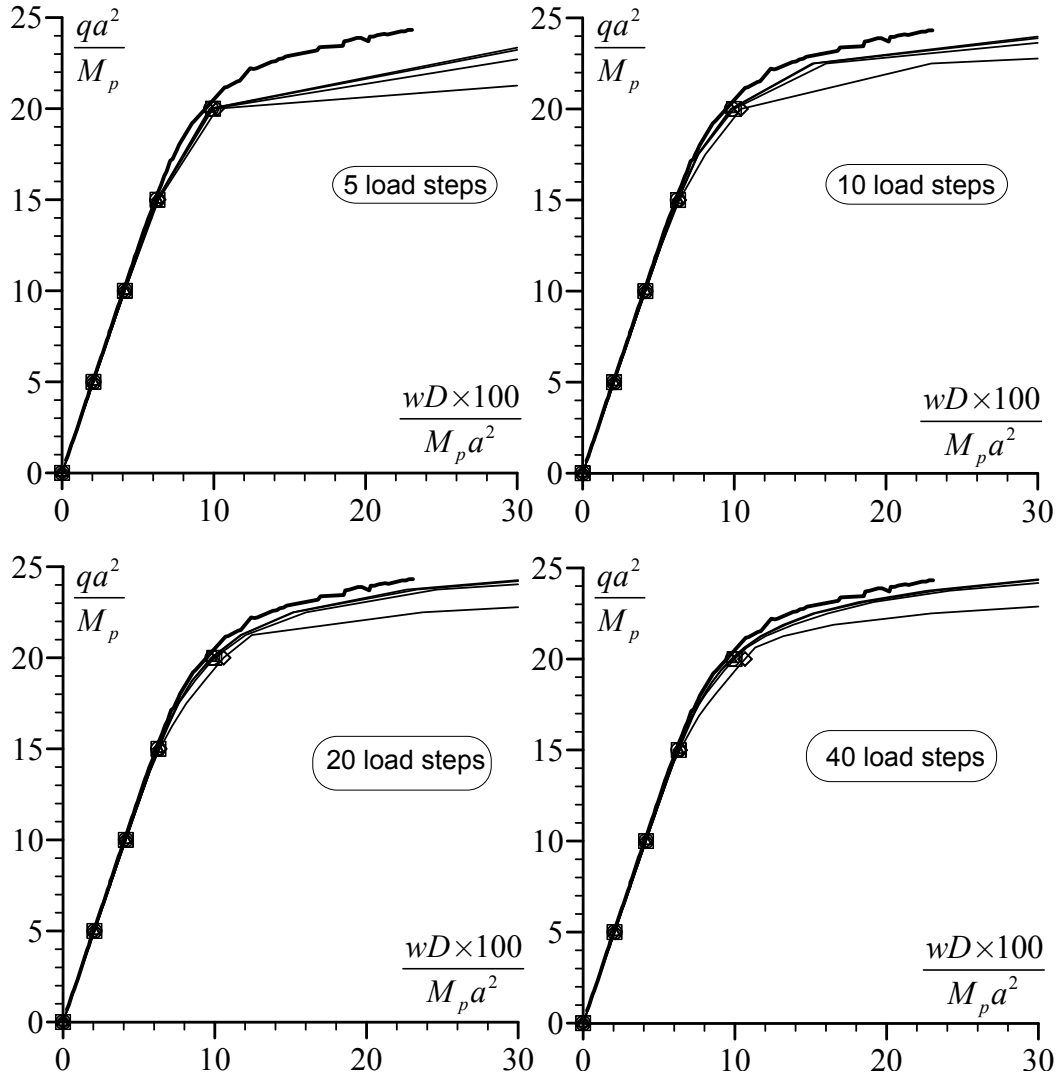


**Figure 3.8 :** Central deflection of a simply supported elasto-plastic square plate with  $8 \times 8$  mesh. — Belinha and Dinis, —◇— 4 layers, —□— 8 layers, —○— 12 layers, —△— 16 layers.

Central deflections calculated with a finer finite element mesh ( $8 \times 8$ ) are presented in Figure 3.8. For this mesh case, the importance of the number of layer on the reflecting non-linear behavior becomes apparent even for small load steps compared to the  $4 \times 4$  mesh case.

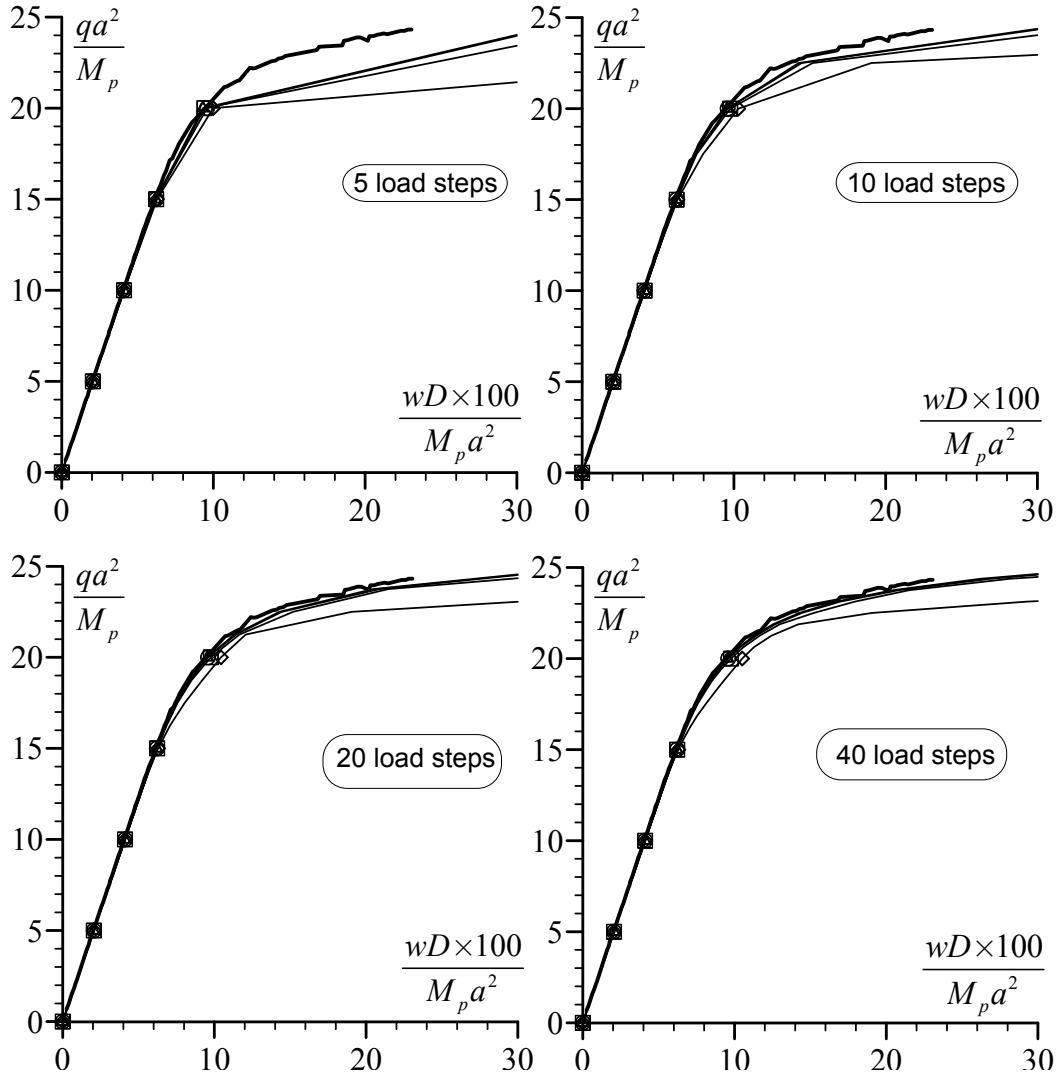


**Figure 3.9 :** Central deflection of a simply supported elasto–plastic square plate with  $12 \times 12$  mesh. — Belinha and Dinis, —◇— 4 layers, —□— 8 layers, —○— 12 layers, —▲— 16 layers.

Central deflections for the same plate with a  $12 \times 12$  mesh are plotted with respect to external load in Figure 3.9 to understand the mesh sensitivity of the non–linear behavior with a smooth transition from a mesh configuration to another. According to the Figure 3.9 it can be commented that 20 load steps and 12 layers provides a satisfactory convergence. Increasing the load steps from 20 to 40 just provides a smoother curve for the elasto–plastic problem. Finally a mesh composed of  $16 \times 16$  elements is considered and the results are presented in Figure 3.10. According to the Figure 3.10 it can be said that the 5 load steps and 4 layers through the thickness is not enough for a precise elasto–plastic analysis. It can be concluded that, a sufficient convergence can be achieved with an  $20 \times 20$  mesh, 12 layers and totally 20 load steps for the elasto–plastic plate analysis.

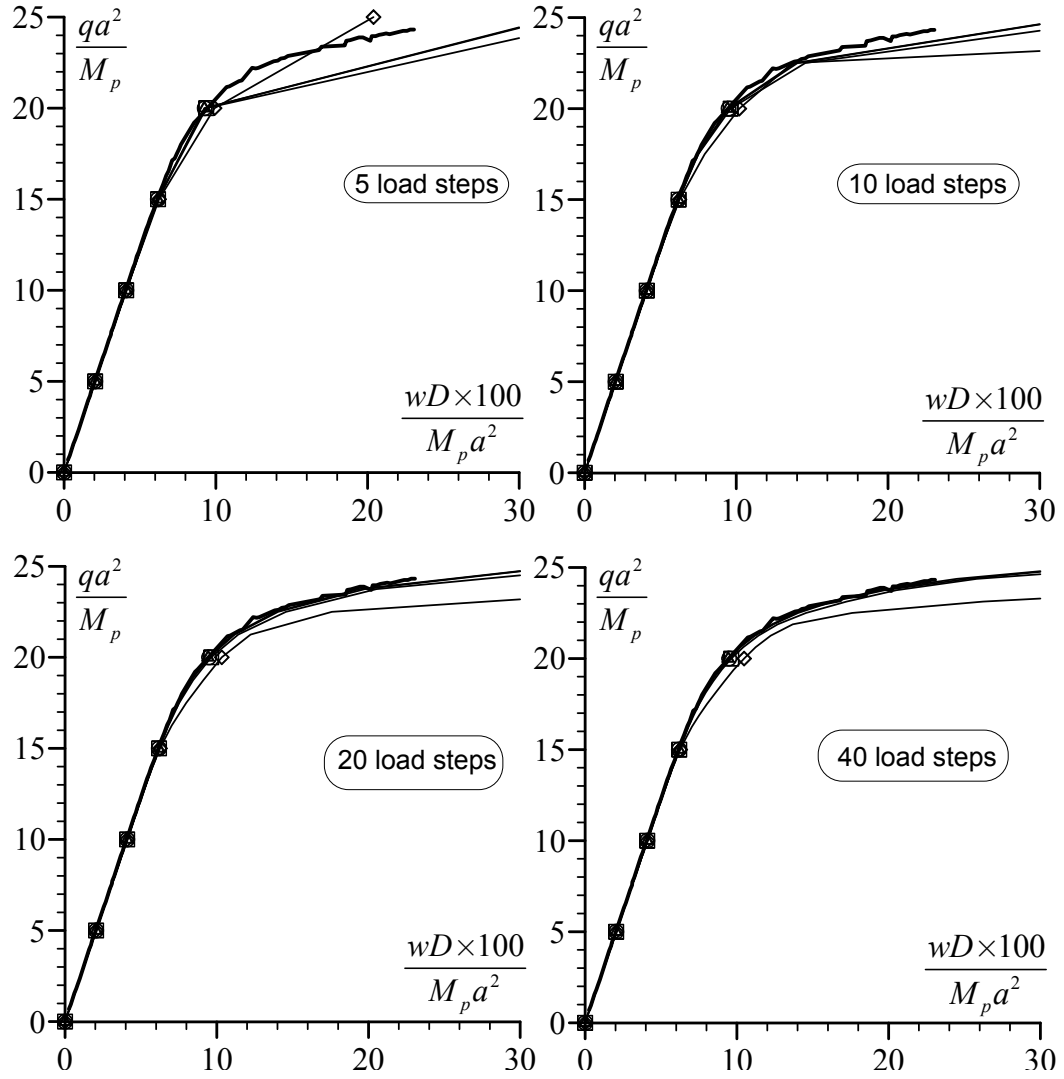


**Figure 3.10 :** Central deflection of a simply supported elasto-plastic square plate with  $20 \times 20$  mesh. — Belinha and Dinis, —◇— 4 layers, —□— 8 layers, —○— 12 layers, —△— 16 layers.

### 3.2.2 Elasto-plastic analysis of Mindlin plates with perfect plasticity

Another comparison and convergence study for elasto-plastic analysis of Mindlin plates is presented for perfect plasticity case. For this purpose the results obtained with the proposed solution procedure are given obtained for several finite element parameters and compared with the classical results given in Owen and Hinton [89] which are obtained from the famous heterosis elements. A simply supported and uniformly loaded square plate with non-dimensional parameters (given by [89])  $a=1$ ,  $E=10.92$ ,  $\nu=0.3$ ,  $h=0.01$  and  $\sigma_y=1600$ . Central deflections of the plate are provided by Owen and Hinton [89] for both layered and non-layered approach. Mixed finite element solutions are obtained for 4 layers, 8 layers, 12 layers and 16 layers. Non-dimensional central deflections with respect to non-dimensional

uniform loading parameters are presented for  $8 \times 8$  and  $20 \times 20$  finite element meshes in Figure 3.11 and Figure 3.12, respectively.

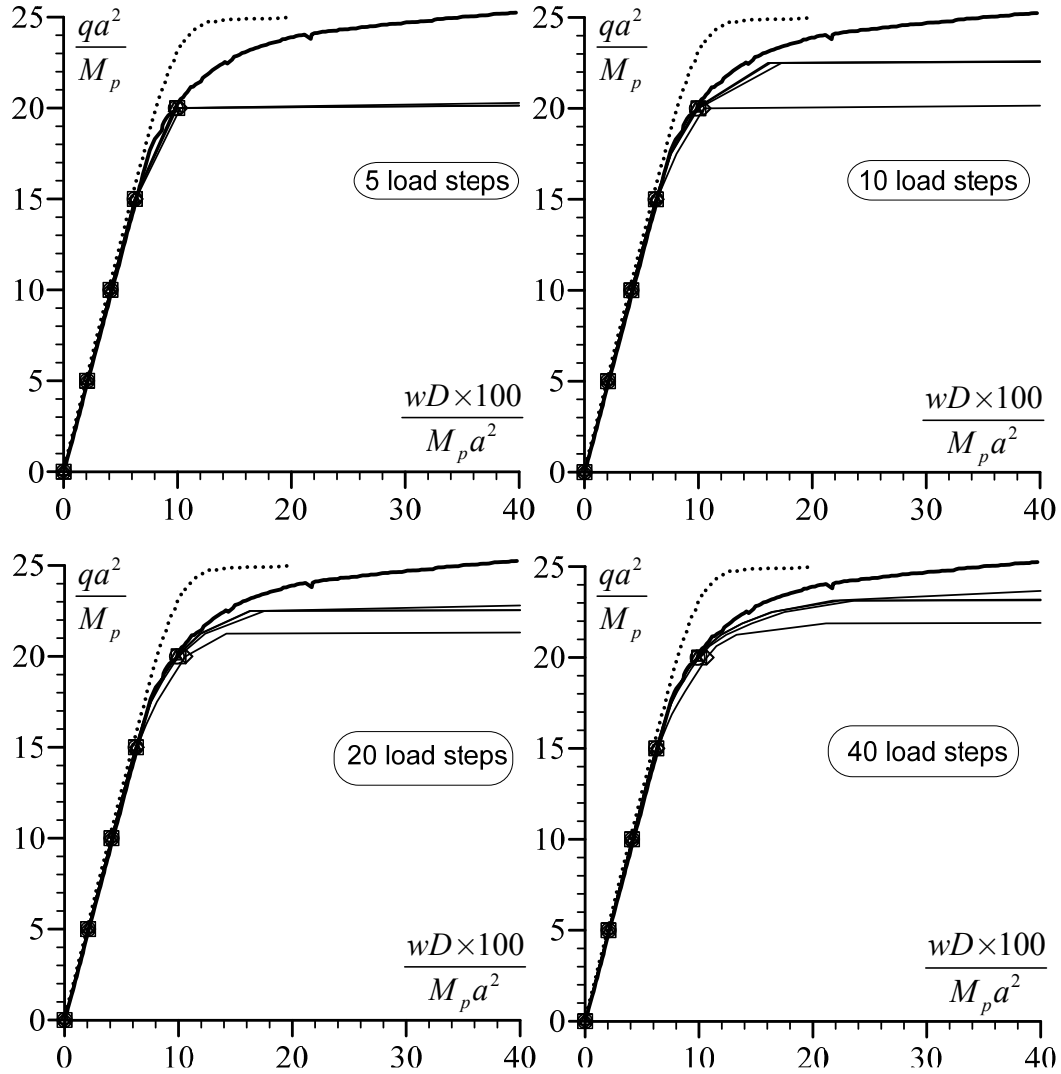


**Figure 3.11 :** Central deflection of a simply supported perfectly elasto-plastic square plate with  $8 \times 8$  mesh. — Owen and Hinton layered approach, ..... Owen and Hinton non-layered approach, —◇— 4 layers, —□— 8 layers, —○— 12 layers, —△— 16 layers.

Figure 3.11 indicates that the number of layers through the thickness of the plate does not effect the elasto-plastic behavior in the 5 load steps case. Even for a thin plate case ( $h/a = 0.01$ ), considering 4 layers on the cross section is not adequate for a precise analysis. Presented results obtained with proposed solution procedure is in a good aggrement with the results of Owen and Hinton [89] (see Figure 3.11 and Figure 3.12).

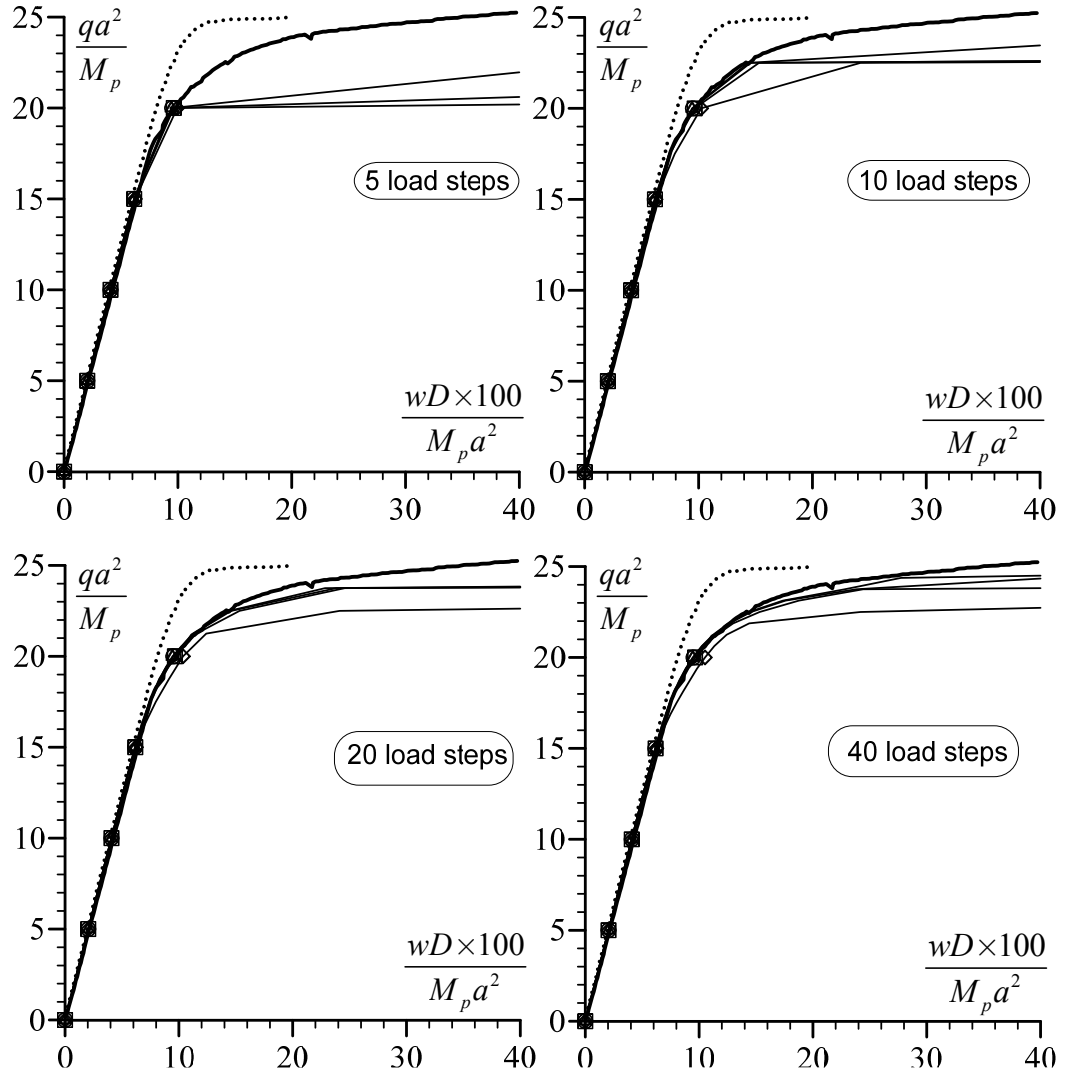


**Figure 3.12 :** Central deflection of a simply supported perfectly elasto-plastic square plate with  $20 \times 20$  mesh. —Owen and Hinton layered approach, .....Owen and Hinton non-layered approach, —◇— 4 layers, —□— 8 layers, —○— 12 layers, —△— 16 layers.



## **4. CONCLUSIONS**

Non-linear problems of laminated and isotropic Mindlin plates were investigated in this study by employing a two field variational principle with some newly proposed solution strategies. The content of the current dissertation is outlined in this chapter. Conclusions, coming next, are followed by the recommendations of the author about the possible future research.

### **4.1 Summary of The Work**

The main goal of this dissertation was to concern with the implementation of a two field mixed finite element formulation to the non-linear bending analysis of Mindlin plates. Large deflection responses of laminated composite plates and elasto-plastic behavior of isotropic plates were examined in the context of this study.

Mindlin deformation assumptions and von Kármán kinematics are used to handle the geometrically non-linear analysis of moderately thick laminated composite plates. Hellinger-Reissner variational principle is adapted to obtain finite element equations of the problem. Non-linear finite element equations are linearized according to the incremental formulation. Newton-Raphson iteration scheme is used to solve the finite element equation system. During this process, a monolithic solution strategy is preferred and stress type field variables are calculated directly, which made it possible to obtain stresses and strains continuously over plate domain from stress resultants with some matrix operations.

Elasto-plastic analysis of isotropic plates is conducted by employing a 3-D and full implicit integration algorithm. Layered approach is considered to simulate the spread of plasticity through the plate thickness. Material matrices and operations of laminated theory are utilized here to compute section properties. A new solution procedure is derived and applied to the elasto-plastic analysis of Mindlin plates. The proposed strategy keeps the stress type field variables during the whole solution procedure beside displacement type variables. Hence, this approach generates a

continuous stress and strain field through the physical and makes it possible to perform local constitutive integrations at the domain instead of Gauss points.

The proposed formulations are implemented to a homemade finite element program for the non-linear analysis of Mindlin plates. FORTRAN programming language is used and double precision is preferred in numerical computations. Several numerical results are presented to observe the applicability and accuracy of the developed formulations.

## 4.2 Conclusions

The proposed mixed finite element formulations and solution strategies were successfully applied to the target problems. Quadrilateral  $C^0$  plate finite elements were able to reflect large deflection and elasto-plastic behavior accurately. Even for loose meshes both geometric and material non-linear problem solutions show stable convergence in iterative procedures. There exists a good agreement with the present mixed finite element algorithm outcomes and results from literature for large deflection problem of isotropic plates having rectangular and circular geometries. According to the reported numerical results, the proposed formulation does not suffer from shear locking. Comparisons with benchmark examples from literature reveals that the formulation and numerical solution procedure can reflect the large deflection behavior of composite laminated plates successfully. Furthermore the developed finite elements can predict the stresses through the thickness of the composite Mindlin plate very accurately. For linear case, 3-D exact results are almost caught by the formulation in the limits of thin and moderately thick plate dimensions. Although plate formulation depends on first order shear deformation assumptions and linear interpolation of field variables it can compete with higher order elements.

Elasto-plastic analysis of isotropic Mindlin plate was conducted for perfectly-plastic and linear hardening cases. The effects of mesh configuration, number of layers through the plate section and number of total load steps are observed through the examples. All the governing parameters have significant influences on the numerical results. Increasing the total load step number causes to smoother curves for external load versus plate responses. It must be pointed out that using layered approach alters the behavior of the elasto-plastic plate compared to stress resultant approach. It can

be concluded that even for thin plates more than 4 layers are required to reflect the elasto–plastic behavior accurately.

### **4.3 Recommendations for Future Research**

Based on the current study several recommendations for the future research are outlined. Mindlin assumptions incorporated with mixed principle and excellent approximations for shear stresses at the cross section of the plate are obtained in the scope of this study. Higher order shear deformation theories should be employed and plate behavior with respect to different shear deformation assumptions should be examined. Geometrical non–linearity beyond large deflection formulation needs to be included in the plate analysis. Variational framework and the proposed solution strategies should be expanded for the dynamic and post–buckling analysis of structures. Most importantly, combination of geometrical and material non–linearity should be achieved. The proposed formulations are expected to handle elasto–plastic analysis of composite and other advanced materials such as functionally graded materials. A comprehensive investigation should be performed to discover advantages and weaknesses of the proposed model by implementing and comparing other finite element solution procedures.



## REFERENCES

- [1] **Szilard, R.** (2004). *Theories and Applications of Plate Analysis*. John Wiley and Sons.
- [2] **Timoshenko, S., & Woinowsky-Krieger, S.** (1959). *Theory of Plates and Shells*. McGraw-Hill.
- [3] **Aköz, A. Y., Omurtag, M. H., & Dogruoglu, A. N.** (1991). The mixed finite element formulation for three-dimensional bars. *International Journal of Solids and Structures*, 28(2), 225–234. doi:10.1016/0020-7683(91)90207-V
- [4] **Kutlu, A., Uğurlu, B., Omurtag, M. H., & Ergin, A.** (2012). Dynamic response of Mindlin plates resting on arbitrarily orthotropic Pasternak foundation and partially in contact with fluid. *Ocean Engineering*, 42(0), 112–125. doi:10.1016/j.oceaneng.2012.01.010
- [5] **Omurtag, M. H., & Aköz, A. Y.** (1993). A compatible cylindrical shell element for stiffened cylindrical shells in a mixed finite element formulation. *Computers & Structures*, 49(2), 363–370. doi:10.1016/0045-7949(93)90115-T
- [6] **Auricchio, F., & Sacco, E.** (1999). A mixed-enhanced finite-element for the analysis of laminated composite plates. *International Journal for Numerical Methods in Engineering*, 44(10), 1481–1504. doi:10.1002/(SICI)1097-0207(19990410)44:10<1481::AID-NME554>3.0.CO;2-Q
- [7] **Ayoub, A.** (2003). Mixed formulation of nonlinear beam on foundation elements. *Computers & Structures*, 81(7), 411–421. doi:10.1016/S0045-7949(03)00015-4
- [8] **Nukala, P. K. V. V., & White, D. W.** (2004). Variationally consistent state determination algorithms for nonlinear mixed beam finite elements. *Computer Methods in Applied Mechanics and Engineering*, 193(33-35), 3647–3666. doi:10.1016/j.cma.2004.01.027
- [9] **Alemdar, B. N.** (2001). *Distributed plasticity analysis of steel building structural systems* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (UMI No. 3032418)
- [10] **Ayoub, A.** (1999). *Mixed formulation for seismic analysis of composite steel-concrete frame structures* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database. (UMI No. 9966293)

- [11] **Hjelmstad, K. D., & Taciroglu, E.** (2002). Mixed methods and flexibility approaches for nonlinear frame analysis. *Journal of Constructional Steel Research*, 58(5-8), 967–993. doi:10.1016/S0143-974X(01)00100-6
- [12] **Taylor, R. L., Filippou, F. C., Saritas, A., & Auricchio, F.** (2003). A mixed finite element method for beam and frame problems. *Computational Mechanics*, 31, 192–203. doi:10.1007/s00466-003-0410-y
- [13] **Alsafadie, R., Hjiat, M., & Battini, J.-M.** (2010). Corotational mixed three-dimensional finite element for structural stability problems, *IV European Conference on Computational Mechanics*, Paris, France : May 16-21.
- [14] **Hjelmstad, K. D., & Taciroglu, E.** (2005). Variational basis of nonlinear flexibility methods for structural analysis of frames. *Journal of Engineering Mechanics*, 131(11), 1157–1169. doi:10.1061/(ASCE)0733-9399(2005)131:11(1157)
- [15] **Alsafadie, R., Hjiat, M., Somja, H., & Battini, J.-M.** (2011). A comparative study of displacement and mixed-based corotational finite element formulations for elasto-plastic three-dimensional beam analysis. *Engineering Computations*, 28(7), 939–982. doi:10.1108/02644401111165149
- [16] **Saritas, A., & Soydas, O.** (2012). Variational base and solution strategies for non-linear force-based beam finite elements. *International Journal of Non-Linear Mechanics*, 47(3), 54–64. doi:10.1016/j.ijnonlinmec.2012.01.003
- [17] **Reddy, J., & Arciniega, R.** (2004). Shear deformation plate and shell theories: From Stavsky to present. *Mechanics of Advanced Materials and Structures*, 11(6), 535–582. doi:10.1080/15376490490452777
- [18] **Liew, K. M., Xiang, Y., & Kitipornchai, S.** (1995). Research on thick plate vibration: a literature survey. *Journal of Sound and Vibration*, 180(1), 163–176. doi:10.1006/jsvi.1995.0072
- [19] **Zhang, Y. X., & Yang, C. H.** (2009). Recent developments in finite element analysis for laminated composite plates. *Composite Structures*, 88(1), 147–157. doi:10.1016/j.compstruct.2008.02.014
- [20] **Pao, Y. C.** (1970). Simple bending analysis of laminated plates by large-deflection theory. *Journal of Composite Materials*, 4(3), 380–389. doi:10.1177/002199837000400309
- [21] **Bennett, J. A.** (1971). Nonlinear vibration of simply supported angle ply laminated plates. *AIAA Journal*, 9(10), 1997–2003. doi:10.2514/3.50007
- [22] **Berger, H. M.** (1954). *A new approach to the analysis of large deflections of plates* (Doctoral dissertation). Retrieved from <http://thesis.library.caltech.edu/>
- [23] **Bert, C. W.** (1973). Nonlinear vibration of a rectangular plate arbitrarily laminated of anisotropic material. *Journal of Applied Mechanics*, 40(2), 452–458. doi:10.1115/1.3423005

- [24] **Chia, C. Y.** (1974). Large deflections of heterogeneous anisotropic rectangular plates. *International Journal of Solids and Structures*, 10(9), 965–976. doi:10.1016/0020-7683(74)90004-3
- [25] **Zaghloul, S. A., & Kennedy, J. B.** (1975). Nonlinear behavior of symmetrically laminated plates. *Journal of Applied Mechanics*, 42(1), 234–236. doi:10.1115/1.3423532
- [26] **Prabhakara, M. K., & Chia, C. Y.** (1975). Finite deflections of unsymmetrically layered anisotropic rectangular plates subjected to the combined action of transverse and in-plane loads. *Journal of Applied Mechanics*, 42(2), 517–518. doi:10.1115/1.3423623
- [27] **Chandra, R., & Basava Raju, B.** (1975). Large amplitude flexural vibration of cross ply laminated composite plates. *Fibre Science and Technology*, 8(4), 243–263. doi:10.1016/0015-0568(75)90018-4
- [28] **Chia, C. Y., & Prabhakara, M. K.** (1976). Large deflection of unsymmetric cross-ply and angle-ply plates. *Journal of Mechanical Engineering Science*, 18(4), 179–183. doi:10.1243/JMES\_JOUR\_1976\_018\_031\_02
- [29] **Chandra, R.** (1976). Large deflection vibration of cross-ply laminated plates with certain edge conditions. *Journal of Sound and Vibration*, 47(4), 509–514. doi:10.1016/0022-460X(76)90878-6
- [30] **Prabhakara, M. K.** (1977). Finite deflections of unsymmetric angle-ply anisotropic rectangular plate under edge moments. *Journal of Applied Mechanics*, 44(1), 171–172. doi:10.1115/1.3423988
- [31] **Chia, C. Y., & Prabhakara, M. K.** (1978). A general mode approach to nonlinear flexural vibrations of laminated rectangular plates. *Journal of Applied Mechanics*, 45(3), 623–628. doi:10.1115/1.3424372
- [32] **Chia, C.-Y.** (1980). *Nonlinear Analysis of Plates*. McGraw-Hill Inc.,US.
- [33] **Reddy, J. N., & Chao, W. C.** (1981). Non-linear bending of thick rectangular, laminated composite plates. *International Journal of Non-Linear Mechanics*, 16(3-4), 291–301. doi:10.1016/0020-7462(81)90042-1
- [34] **Reddy, J. N.** (1980). A penalty plate-bending element for the analysis of laminated anisotropic composite plates. *International Journal for Numerical Methods in Engineering*, 15(8), 1187–1206. doi:DOI: 10.1002/nme.1620150807
- [35] **Reddy, J. N., & Chao, W. C.** (1981). Large-deflection and large-amplitude free vibrations of laminated composite-material plates. *Computers & Structures*, 13(1–3), 341–347. doi:10.1016/0045-7949(81)90142-5
- [36] **Turvey, G. J.** (1982). Axisymmetric elastic large deflection analysis of composite circular plates. *Fibre Science and Technology*, 16(3), 191–217. doi:10.1016/0015-0568(82)90034-3
- [37] **Chia, P. C. Y.** (1984). Nonlinear bending of unsymmetric angle-ply plates with edges elastically restrained against rotation. *Acta Mechanica*, 53(3-4), 201–212. doi:10.1007/BF01177951

- [38] **Putcha, N. S., & Reddy, J. N.** (1986). A refined mixed shear flexible finite element for the nonlinear analysis of laminated plates. *Computers & Structures*, 22(4), 529–538. doi:10.1016/0045-7949(86)90002-7
- [39] **Gorji, M.** (1986). On large deflection of symmetric composite plates under static loading. *ARCHIVE: Proceedings of the Institution of Mechanical Engineers, Part C: Mechanical Engineering Science 1983-1988 (vols 197-202)*, 200(13), 13–19. doi:10.1243/PIME\_PROC\_1986\_200\_089\_02
- [40] **Singh, G., & Sadasiva Rao, Y. V. K.** (1987). Large deflection behaviour of thick composite plates. *Composite Structures*, 8(1), 13–29. doi:10.1016/0263-8223(87)90013-4
- [41] **Chia, C.-Y.** (1988). Geometrically nonlinear behavior of composite plates: A review. *Applied Mechanics Reviews*, 41(12), 439–451. doi:10.1115/1.3151873
- [42] **Liao, C.-L., & Reddy, J. N.** (1990). Analysis of anisotropic, stiffened composite laminates using a continuum-based shell element. *Computers & Structures*, 34(6), 805–815. doi:10.1016/0045-7949(90)90351-2
- [43] **Barbero, E. J., & Reddy, J. N.** (1990). Nonlinear analysis of composite laminates using a generalized laminated plate theory. *AIAA Journal*, 28(11), 1987–1994. doi:10.2514/3.10509
- [44] **Chang, J.-S., & Huang, Y.-P.** (1991). Geometrically nonlinear static and transiently dynamic behavior of laminated composite plates based on a higher order displacement field. *Composite Structures*, 18(4), 327–364. doi:10.1016/0263-8223(91)90003-H
- [45] **Singh, G., Venkateswara Rao, G., & Iyengar, N. G. R.** (1991). Some observations on the large deflection bending of rectangular antisymmetric cross-ply plates. *Composite Structures*, 18(1), 77–91. doi:10.1016/0263-8223(91)90015-Q
- [46] **Kant, T., & Kommineni, J. R.** (1992). C0 Finite element geometrically nonlinear analysis of fibre reinforced composite and sandwich laminates based on a higher-order theory. *Computers & Structures*, 45(3), 511–520. doi:10.1016/0045-7949(92)90436-4
- [47] **Singh, G., Rao, G. V., & Iyengar, N. G. R.** (1993). Large deflection of shear deformable composite plates using a simple higher-order theory. *Composites Engineering*, 3(6), 507–525. doi:10.1016/0961-9526(93)90049-P
- [48] **Savithri, S., & Varadan, T. K.** (1993). Large deflection analysis of laminated composite plates. *International Journal of Non-Linear Mechanics*, 28(1), 1–12. doi:10.1016/0020-7462(93)90002-3
- [49] **Carrera, E., & Villani, M.** (1994). Large deflections and stability FEM analysis of shear deformable compressed anisotropic flat panels. *Composite Structures*, 29(4), 433–444. doi:10.1016/0263-8223(94)90112-0

- [50] **Argyris, J., & Tenek, L.** (1994). Linear and geometrically nonlinear bending of isotropic and multilayered composite plates by the natural mode method. *Computer Methods in Applied Mechanics and Engineering*, 113(3–4), 207–251. doi:10.1016/0045-7825(94)90047-7
- [51] **Singh, G., Rao, G. V., & Iyengar, N. G. R.** (1994). Geometrically nonlinear flexural response characteristics of shear deformable unsymmetrically laminated plates. *Computers & Structures*, 53(1), 69–81. doi:10.1016/0045-7949(94)90131-7
- [52] **Chattopadhyay, B., Sinha, P. K., & Mukhopadhyay, M.** (1995). Geometrically nonlinear analysis of composite stiffened plates using finite elements. *Composite Structures*, 31(2), 107–118. doi:10.1016/0263-8223(95)00004-6
- [53] **Chattopadhyay, B., Sinha, P. K., & Mukhopadhyay, M.** (1992). Finite element free vibration analysis of eccentrically stiffened composite plates. *Journal of Reinforced Plastics and Composites*, 11(9), 1003–1034. doi:10.1177/073168449201100903
- [54] **Liu, J. H., & Surana, K. S.** (1995). Piecewise hierarchical p-version curved shell element for geometrically nonlinear behavior of laminated composite plates and shells. *Computers & Structures*, 55(1), 47–66. doi:10.1016/0045-7949(94)00506-X
- [55] **Ganapathi, M., Polit, O., & Touratier, M.** (1996). A C0 eight-node membrane-shear-bending element for geometrically non-linear (static and dynamic) analysis of laminates. *International Journal for Numerical Methods in Engineering*, 39(20), 3453–3474. doi:10.1002/(SICI)1097-0207(19961030)39:20<3453::AID-NME9>3.0.CO;2-7
- [56] **Riber, H. J.** (1997). Non-linear analytical solutions for laterally loaded sandwich plates. *Composite Structures*, 39(1-2), 63–83. doi:10.1016/S0263-8223(97)00000-7
- [57] **Liu, R., Xu, J., & Zhai, S.** (1997). Large-deflection bending of symmetrically laminated rectilinearly orthotropic elliptical plates including transverse shear. *Archive of Applied Mechanics*, 67(7), 507–520.
- [58] **Sathyamoorthy, M.** (1997). *Nonlinear Analysis of Structures*. CRC Press.
- [59] **Günay, E., & Erdem, A. Ü.** (1997). A new heterosis plate element for geometrically non-linear finite element analysis of laminated plates. *Computers & Structures*, 65(6), 819–828. doi:10.1016/S0045-7949(97)00005-9
- [60] **Akhras, G., Cheung, M. S., & Li, W.** (1998). Geometrically nonlinear finite strip analysis of laminated composite plates. *Composites Part B: Engineering*, 29(4), 489–495. doi:10.1016/S1359-8368(97)00038-3
- [61] **Shen, H.-S.** (1999). Large deflection of composite laminated plates under transverse and in-plane loads and resting on elastic foundations. *Composite Structures*, 45(2), 115–123. doi:10.1016/S0263-8223(99)00007-0

- [62] **Cheng, Q. H., Lok, T. S., & C. Xie, Z.** (1999). Geometrically non-linear analysis including shear deformation of composite laminates. *Thin-Walled Structures*, 35(1), 41–59. doi:10.1016/S0263-8231(99)00016-6
- [63] **Fares, M. E.** (1999). Non-linear bending analysis of composite laminated plates using a refined first-order theory. *Composite Structures*, 46(3), 257–266. doi:10.1016/S0263-8223(99)00062-8
- [64] **Tan, H.-F., Tian, Z.-H., & Du, X.-W.** (2000). A new geometrical nonlinear laminated theory for large deformation analysis. *International Journal of Solids and Structures*, 37(18), 2577–2589. doi:10.1016/S0020-7683(99)00011-6
- [65] **Shukla, K. K., & Nath, Y.** (2000). Nonlinear analysis of moderately thick laminated rectangular plates. *Journal of Engineering Mechanics*, 126(8), 831–838. doi:10.1061/(ASCE)0733-9399(2000)126:8(831)
- [66] **Yang, J., & Zhang, L.** (2000). Nonlinear analysis of imperfect laminated thin plates under transverse and in-plane loads and resting on an elastic foundation by a semi-analytical approach. *Thin-Walled Structures*, 38(3), 195–227. doi:10.1016/S0263-8231(00)00045-8
- [67] **Renganathan, K., Nageswara Rao, B., & Manoj, T.** (2002). Moderately large deflection of laminated thin rectangular plates. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 82(5), 352–360. doi:10.1002/1521-4001(200205)82:5<352::AID-ZAMM352>3.0.CO;2-H
- [68] **Sita Thankam, V., Singh, G., Venkateswara Rao, G., & Rath, A. K.** (2003). Shear flexible element based on coupled displacement field for large deflection analysis of laminated plates. *Computers & Structures*, 81(5), 309–320. doi:10.1016/S0045-7949(02)00450-9
- [69] **Tanrıöver, H., & Şenocak, E.** (2004). Large deflection analysis of unsymmetrically laminated composite plates: analytical-numerical type approach. *International Journal of Non-Linear Mechanics*, 39(8), 1385–1392. doi:10.1016/j.ijnonlinmec.2004.01.001
- [70] **Liew, K. M., Wang, J., Tan, M. J., & Rajendran, S.** (2004). Nonlinear analysis of laminated composite plates using the mesh-free kp-Ritz method based on FSDT. *Computer Methods in Applied Mechanics and Engineering*, 193(45–47), 4763–4779. doi:10.1016/j.cma.2004.03.013
- [71] **Zhang, Y. X., & Kim, K. S.** (2006). Geometrically nonlinear analysis of laminated composite plates by two new displacement-based quadrilateral plate elements. *Composite Structures*, 72(3), 301–310. doi:10.1016/j.compstruct.2005.01.001
- [72] **Zhang, Y. X., & Yang, C. H.** (2006). A family of simple and robust finite elements for linear and geometrically nonlinear analysis of laminated composite plates. *Composite Structures*, 75(1-4), 545–552. doi:10.1016/j.compstruct.2006.04.016

- [73] **Shufrin, I., Rabinovitch, O., & Eisenberger, M.** (2008). A semi-analytical approach for the non-linear large deflection analysis of laminated rectangular plates under general out-of-plane loading. *International Journal of Non-Linear Mechanics*, 43(4), 328–340. doi:10.1016/j.ijnonlinmec.2007.12.018
- [74] **Dash, P., & Singh, B. N.** (2010). Geometrically nonlinear bending analysis of laminated composite plate. *Communications in Nonlinear Science and Numerical Simulation*, 15(10), 3170–3181. doi:10.1016/j.cnsns.2009.11.017
- [75] **Baltacıoğlu, A. K., Akgöz, B., & Civalek, Ö.** (2010). Nonlinear static response of laminated composite plates by discrete singular convolution method. *Composite Structures*, 93(1), 153–161. doi:10.1016/j.compstruct.2010.06.005
- [76] **Singh, J., & Shukla, K. K.** (2012). Nonlinear flexural analysis of laminated composite plates using RBF based meshless method. *Composite Structures*, 94(5), 1714–1720. doi:10.1016/j.compstruct.2012.01.001
- [77] **Kapoor, H., & Kapania, R. K.** (2012). Geometrically nonlinear NURBS isogeometric finite element analysis of laminated composite plates. *Composite Structures*, 94(12), 3434–3447. doi:10.1016/j.compstruct.2012.04.028
- [78] **Hodge, P. G.** (1967). *Yapıların Plastik Analizi*. Arı Kitabevi.
- [79] **Hopkins, H. G., & Prager, W.** (1953). The load carrying capacities of circular plates. *Journal of the Mechanics and Physics of Solids*, 2(1), 1–13. doi:10.1016/0022-5096(53)90022-2
- [80] **Hopkins, H. G., & Wang, A. J.** (1955). Load-carrying capacities for circular plates of perfectly-plastic material with arbitrary yield condition. *Journal of the Mechanics and Physics of Solids*, 3(2), 117–129. doi:10.1016/0022-5096(55)90055-7
- [81] **Hodge, P. G., & Belytschko, T.** (1968). Numerical methods for the limit analysis of plates. *Journal of Applied Mechanics*, 35(4), 796. doi:10.1115/1.3601308
- [82] **Tekinalp, B.** (1957). Elastic-plastic bending of a built-in circular plate under a uniformly distributed load. *Journal of the Mechanics and Physics of Solids*, 5(2), 135–142. doi:10.1016/0022-5096(57)90057-1
- [83] **Freiberger, W., & Tekinalp, B.** (1956). Minimum weight design of circular plates. *Journal of the Mechanics and Physics of Solids*, 4(4), 294–299. doi:10.1016/0022-5096(56)90038-2
- [84] **Barnard, A. J., & Sharman, P. W.** (1976). The elasto-plastic analysis of plates using hybrid finite elements. *International Journal for Numerical Methods in Engineering*, 10(6), 1343–1356. doi:10.1002/nme.1620100613
- [85] **Eggers, H., & Kröplin, B.** (1978). Yielding of plates with hardening and large deformations. *International Journal for Numerical Methods in Engineering*, 12(5), 739–750. doi:10.1002/nme.1620120501

- [86] **Dinis, L. M. S., & Owen, D. R. J.** (1978). Elastic-viscoplastic analysis of plates by the finite element method. *Computers & Structures*, 8(2), 207–215. doi:10.1016/0045-7949(78)90024-X
- [87] **Kuo-Kuang, C.** (1979). A triangular plate finite element for large-displacement elastic-plastic analysis of automobile structural components. *Computers & Structures*, 10(1–2), 203–215. doi:10.1016/0045-7949(79)90088-9
- [88] **Morjaria, M., & Mukherjee, S.** (1980). Inelastic analysis of transverse deflection of plates by the boundary element method. *Journal of Applied Mechanics*, 47(2), 291. doi:10.1115/1.3153657
- [89] **Owen, D. R. J., & Hinton, E.** (1980). *Finite Elements in Plasticity: Theory and Practice*. Pineridge Press.
- [90] **Bathe, K., & Bolourchi, S.** (1980). A geometric and material nonlinear plate and shell element. *Computers & Structures*, 11(1–2), 23–48. doi:10.1016/0045-7949(80)90144-3
- [91] **Martins, R. A. F., & Owen, D. R. J.** (1981). Elastoplastic and geometrically nonlinear thin shell analysis by the semiloof element. *Computers & Structures*, 13(4), 505–513. doi:10.1016/0045-7949(81)90045-6
- [92] **Dinis, L. M. S., & Owen, D. R. J.** (1982). Elasto-viscoplastic and elasto-plastic large deformation analysis of thin plates and shells. *International Journal for Numerical Methods in Engineering*, 18(4), 591–607. doi:10.1002/nme.1620180409
- [93] **Owen, D. R. J., & Figueiras, J. A.** (1983). Elasto-plastic analysis of anisotropic plates and shells by the Semiloof element. *International Journal for Numerical Methods in Engineering*, 19(4), 521–539. doi:10.1002/nme.1620190406
- [94] **Owen, D. R. J., & Figueiras, J. A.** (1983). Anisotropic elasto-plastic finite element analysis of thick and thin plates and shells. *International Journal for Numerical Methods in Engineering*, 19(4), 541–566. doi:10.1002/nme.1620190407
- [95] **Reddy, B. D., & Mitchell, G. P.** (1983). The analysis of elastic-plastic plates: A quadratic programming problem and its solution by finite elements. *Computer Methods in Applied Mechanics and Engineering*, 41(2), 237–248. doi:10.1016/0045-7825(83)90008-7
- [96] **Wempner, G., & Chao-Meng, H.** (1984). A simple model of elastic-plastic plates. *International Journal of Solids and Structures*, 20(1), 77–80. doi:10.1016/0020-7683(84)90077-5
- [97] **Moshaiov, A., & Vorus, W. S.** (1986). Elasto-plastic plate bending analysis by a boundary element method with initial plastic moments. *International Journal of Solids and Structures*, 22(11), 1213–1229. doi:10.1016/0020-7683(86)90077-6
- [98] **Song, G.-S., & Mukherjee, S.** (1989). Boundary element method analysis of bending of inelastic plates with general boundary conditions. *Computational Mechanics*, 5(2-3), 104–112. doi:10.1007/BF01046479

- [99] **Papadopoulos, P., & Taylor, R. L.** (1990). Elasto-Plastic analysis of Reissner-Mindlin plates. *Applied Mechanics Reviews*, 43(5), S40–S50. doi:10.1115/1.3120846
- [100] **Daye, M. A., & Toridis, T. G.** (1991). Elasto-plastic algorithms for plates and shells under static and dynamic loads. *Computers & Structures*, 39(1–2), 195–205. doi:10.1016/0045-7949(91)90087-3
- [101] **Papadopoulos, P., & Taylor, R. L.** (1991). An analysis of inelastic Reissner-Mindlin plates. *Finite Elements in Analysis and Design*, 10(3), 221–233. doi:10.1016/0168-874X(91)90010-V
- [102] **Ibrahimbegović, A., & Frey, F.** (1993). An efficient implementation of stress resultant plasticity in analysis of Reissner-Mindlin plates. *International Journal for Numerical Methods in Engineering*, 36(2), 303–320. doi:10.1002/nme.1620360209
- [103] **Auricchio, F., & Taylor, R. L.** (1994). A generalized elastoplastic plate theory and its algorithmic implementation. *International Journal for Numerical Methods in Engineering*, 37(15), 2583–2608. doi:10.1002/nme.1620371506
- [104] **Croce, L. D., Venini, P., & Nascimbene, R.** (2003). Numerical simulation of an elastoplastic plate via mixed finite elements. *Journal of Engineering Mathematics*, 46(1), 69–86.
- [105] **Belinha, J., & Dinis, L. M. J. S.** (2006). Elasto-plastic analysis of plates by the element free Galerkin method. *Engineering Computations*, 23(5), 525–551. doi:10.1108/02644400610671126
- [106] **Supriyono, & Aliabadi, M. H.** (2006). Boundary element method for shear deformable plates with combined geometric and material nonlinearities. *Engineering Analysis with Boundary Elements*, 30(1), 31–42. doi:10.1016/j.enganabound.2005.08.005
- [107] **Supriyono, & Aliabadi, M. H.** (2007). Analysis of shear deformable plates with combined geometric and material nonlinearities by boundary element method. *International Journal of Solids and Structures*, 44(3–4), 1038–1059. doi:10.1016/j.ijsolstr.2006.06.004
- [108] **Waidemam, L., & Venturini, W. S.** (2009). A boundary element formulation for analysis of elastoplastic plates with geometrical nonlinearity. *Computational Mechanics*, 45(4), 335–347. doi:10.1007/s00466-009-0447-7
- [109] **Dirgantara, T., & Aliabadi, M. H.** (2012). Elastoplastic boundary element method for shear deformable shells. *Engineering Structures*, 45(0), 62–67. doi:10.1016/j.engstruct.2012.05.045
- [110] **Reddy, J. N.** (2006). *Theory and Analysis of Elastic Plates and Shells*. CRC Press.
- [111] **Hellinger, E.** (1914). Die allgemeinen Ansätze der Mechanik der Kontinua. In *Encyklopädie der Mathematischen Wissenschaften* (Vol. 4, pp. 602–694). Leipzig: Druck und Verlag Von B. G. Teubner.
- [112] **Reissner, E.** (1950). On a variational theorem in elasticity. *Journal of Mathematical Physics*, 29, 90–95.

- [113] **De Souza, R. M.** (2000). *Force-based finite element for large displacement inelastic analysis of frames* (Doctoral dissertation). Retrieved from <http://www.ufpa.br/nicae/>
- [114] **De Veubeke, O. C.** (2001). Displacement and equilibrium models in the finite element method by B. Fraeijs de Veubeke, Chapter 9, Pages 145–197 of *Stress Analysis*, Edited by O. C. Zienkiewicz and G. S. Holister, Published by John Wiley & Sons, 1965. *International Journal for Numerical Methods in Engineering*, 52(3), 287–342. doi:10.1002/nme.339
- [115] **Wang, C. M., Reddy, J. N., & Lee, K. H.** (2000). *Shear Deformable Beams and Plates: Relationships with Classical Solutions* (1st ed.). Elsevier Science.
- [116] **Reddy, J. N.** (2004). *Mechanics of Laminated Composite Plates and Shells*. CRC Press.
- [117] **Jones, R. M.** (1999). *Mechanics of Composite Materials*. Taylor & Francis.
- [118] **Ventsel, E., & Krauthammer, T.** (2001). *Thin Plates and Shells*. CRC Press.
- [119] **Mansfield, E. H.** (1989). *The Bending and Stretching of Plates*. Cambridge University Press.
- [120] **Aksoylar, C.** (2010). *Anlık basınç yükü etkisindeki kompozit plakların, doğrusal olmayan dinamik davranışının, sonlu elemanlarla çözümü* (Yayınlanmamış doktora tezi). İstanbul Teknik Üniversitesi, Fen Bilimleri Enstitüsü, İstanbul.
- [121] **Başar, Y., & Krätzig, W. B.** (1985). *Mechanik Der Flächentragwerke: Theorie, Berechnungsmethoden, Anwendungsbeispiele*. Wiesbaden: Vieweg & Sohn.
- [122] **Doğruoğlu, A. N., & Omurtag, M. H.** (2000). Stability analysis of composite-plate foundation interaction by mixed FEM. *Journal of Engineering Mechanics-ASCE*, 126(9), 928–936.
- [123] **Başar, Y., & Omurtag, M. H.** (2000). Free-vibration analysis of thin/thick laminated structures by layer-wise shell models. *Computers & Structures*, 74(4), 409–427. doi:10.1016/S0045-7949(99)00061-9
- [124] **Reddy, J. N.** (2004). *An Introduction to Nonlinear Finite Element Analysis*. Oxford University Press, USA.
- [125] **Bathe, K.-J.** (1996). *Finite Element Procedures*. Prentice Hall.
- [126] **Carey, G. F., & Oden, J. T.** (1983). *Finite Elements: A Second Course* (Vol. II). Prentice Hall.
- [127] **Kutlu, A.** (2007). *Keyfi doğrultuda ortotrop Pasternak zeminine oturan Mindlin plaklarının serbest titreşimlerinin karışık sonlu elemanlarla analizi* (Yayınlanmamış yüksek lisans tezi). İstanbul Teknik Üniversitesi, Fen Bilimleri Enstitüsü, İstanbul.
- [128] **Voyiadjis, G. Z., & Woelke, P.** (2008). *Elasto-Plastic and Damage Analysis of Plates and Shells*. Springer.

- [129] **Ugural, A. C., & Fenster, S. K.** (2003). *Advanced Strength and Applied Elasticity* (4th ed.). Prentice Hall.
- [130] **Wriggers, P.** (2008). *Nonlinear Finite Element Methods*. Springer.
- [131] **Kachanov, L. M.** (2004). *Fundamentals of the Theory of Plasticity*. Courier Dover Publications.
- [132] **Hill, R.** (1998). *The Mathematical Theory of Plasticity*. Oxford University Press, USA.
- [133] **Simo, J. C., & Hughes, T. J. R.** (1998). *Computational Inelasticity*. New York: Springer-Verlag.
- [134] **Neto, E. de S., Peric, D., & Owen, D.** (2009). *Computational Methods for Plasticity: Theory and Applications*. Wiley.
- [135] **Meschke, G.** (2011). *Advanced Finite Element Methods Part I - Geometrically Non-linear Finite Element Methods (Lecture Notes)*. Bochum: Institute for Structural Mechanics - Ruhr-University Bochum.
- [136] **Ortiz, M., & Popov, E. P.** (1985). Accuracy and stability of integration algorithms for elastoplastic constitutive relations. *International Journal for Numerical Methods in Engineering*, 21(9), 1561–1576. doi:10.1002/nme.1620210902
- [137] **Boyce, W. E., & DiPrima, R. C.** (2008). *Elementary Differential Equations and Boundary Value Problems*. Wiley.
- [138] **Ascher, U. M., Mattheij, R. M. M., & Russell, R. D.** (1987). *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*. Society for Industrial and Applied Mathematics.
- [139] **Meschke, G.** (2011). *Finite Element Method for Nonlinear Analysis of Inelastic Materials and Structures (Lecture Notes)*. Bochum: Institute for Structural Mechanics - Ruhr-University Bochum.
- [140] **Itskov, M.** (2009). *Tensor Algebra and Tensor Analysis for Engineers*. Springer.
- [141] **Sadd, M. H.** (2009). *Elasticity: Theory, Applications, and Numerics* (2nd ed.). Academic Press.
- [142] **LeMaitre, J.** (2001). *Handbook of Materials Behavior Models, Three-Volume Set, Volume 1-3: Nonlinear Models and Properties*. Academic Press.
- [143] **Hjelmstad, K. D., & Taciroglu, E.** (2003). Mixed variational methods for finite element analysis of geometrically non-linear, inelastic Bernoulli-Euler beams. *Communications in Numerical Methods in Engineering*, 19(10), 809–832. doi:10.1002/cnm.622
- [144] **Kutlu, A., & Omurtag, M. H.** (2013). Tabakalı kompozit Mindlin plağının karışık sonlu eleman yöntemi ile geometrik doğrusal olmayan analizi. *Sigma Mühendislik ve Fen Bilimleri Dergisi*, (3).
- [145] **Dumir, P. C.** (1987). Circular plates on Pasternak elastic foundations. *International Journal for Numerical and Analytical Methods in Geomechanics*, 11(1), 51–60.

- [146] **Kutlu, A., & Omurtag, M. H.** (2012). Large deflection bending analysis of elliptic plates on orthotropic elastic foundation with mixed finite element method. *International Journal of Mechanical Sciences*, 65(1), 64–74. doi:10.1016/j.ijmecsci.2012.09.004
- [147] **Pagano, N. J., & Hatfield, H. J.** (1972). Elastic behavior of multilayered bidirectional composites. *AIAA Journal*, 10(7), 931–933.
- [148] **Panda, S. C., & Natarajan, R.** (1979). Finite element analysis of laminated composite plates. *International Journal for Numerical Methods in Engineering*, 14(1), 69–79. doi:10.1002/nme.1620140106
- [149] **Kam, T. Y., & Jan, T. B.** (1995). First-ply failure analysis of laminated composite plates based on the layerwise linear displacement theory. *Composite Structures*, 32(1-4), 583–591.
- [150] **Turvey, G. J., & Osman, M. Y.** (1991). DR large deflection analysis of orthotropic mindlin plates with simply-supported and clamped-edge conditions. *Composites Engineering*, 1(4), 235–248. doi:10.1016/0961-9526(91)90036-R
- [151] **Reddy, J. N.** (1986). *A refined shear deformation theory for the analysis of laminated plates* (NASA No. NASA-CR-3955) (p. 36). NASA-CR-3955.

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### List of Publications and Patents:

- Uğurlu B., **Kutlu A.**, Ergin A., Omurtag M.H., Dynamics of a rectangular plate resting on an elastic foundation and partially in contact with a quiescent fluid, Journal of Sound and Vibration. 317 (2008) 308-328.
- **Kutlu A.**, Uğurlu B., Omurtag M.H., Ergin A., Dynamic response of Mindlin plates resting on arbitrarily orthotropic Pasternak foundation and partially in contact with fluid, Ocean Engineering. 42 (2012) 112-125.
- **Kutlu A.**, Omurtag M.H., Large deflection bending analysis of elliptic plates on orthotropic elastic foundation with mixed finite element method, International Journal of Mechanical Sciences. 65 (2012) 64–74.

### PUBLICATIONS/PRESENTATIONS ON THE THESIS

- **Kutlu, A.**, Omurtag, M. H., Tabakalı kompozit Mindlin plağının karışık sonlu eleman yöntemi ile geometrik doğrusal olmayan analizi, Sigma Mühendislik ve Fen Bilimleri Dergisi. 3 (2013).

