# RIESZ ALGEBRA VALUED BANACH-STONE THEOREMS

by

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# ABSTRACT

## RIESZ ALGEBRA VALUED BANACH-STONE THEOREMS

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Let $X$, $Y$ be compact Hausdorff spaces and let $E$, $F$ be both Banach lattices and Riesz algebras. The main result of this thesis is following: If $F$ has no zero-divisor and there exists a Riesz algebraic isomorphism $T : C(X,E) \to C(Y,F)$ such that $Tf$ has no zero if $f$ has none, then $X$ is homeomorphic to $Y$ and $E$ is Riesz algebraically isomorphic to $F$. This result is taken from the paper of Banach-Stone theorems and Riesz algebras.

This thesis consists of four chapters. In chapter 1, it is given some necessary definitions in topology, which is used in other chapters. In section 2, we present Riesz spaces, Riesz homomorphisms on $C(X)$ spaces and Riesz algebras which given some properties. In section 3 is devoted to the proofs of the versions of the Banach-Stone theorem. Finally, in section 4, it is proved that under some certain conditions $X$ and $Y$ are homeomorphic, $E$ and $F$ are Riesz algebraically isomorphic when $C(X,E)$ and $C(Y,F)$ are Riesz algebraically isomorphic.

**Keywords**: Riesz space; Riesz homomorphism; Banach lattice; Riesz algebra; Riesz algebraic isomorphism; Support; Banach-Stone theorem.

# ÖZET

## RIESZ ALGEBRA VALUED BANACH-STONE THEOREMS

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$X$, $Y$ kompakt Hausdorff uzayları, $E$, $F$ hem Banach örgüleri hem de Riesz cebirleri olsun. Bu tezin temel sonucu şudur: Eğer $F$'nin sıfır böleni yoksa ve $T : C(X,E) \to C(Y,F)$ bir Riesz cebirsel izomorfizma ise öyleki $f$ sıfırdan farklı iken $Tf$ de sıfırdan farklıdır, $X$ homeomorfiktir $Y$'ye ve $E$ Riesz cebirsel izomorftur $F$'ye. Bu sonuç Banach-Stone teoremleri ve Riesz cebirleri makalesinden alınmıştır.

Bu tez dört bölümden oluşmaktadır. 1. bölümde diğer bölümlerde kullanılmak üzere bazı gerekli tanımlamalar verilmiştir. 2. bölümde ise Riesz uzayları, $C(X)$ uzayları üzerindeki Riesz homomorfizmaları ve bazı özellikleri verilen Riesz cebirleri verilmiştir. 3. bölümde Banach-Stone teoreminin versiyonlarının ispatlarına yer verilmiştir. Son olarak 4. bölümde belli koşullar altında $C(X,E)$ ve $C(Y,F)$ Riesz cebirsel izomorfik olduğunda $X$ ve $Y$ homeomorfik, $E$ ve $F$ Riesz cebirsel izomorfik olduğu ispatlanmıştır.

**Anahtar Kelimeler**: Riesz uzayı; Riesz homomorfizma; Banach örgü; Riesz cebir; Riesz cebirsel izomorfizma; Support; Banach-Stone teorem.

*To my family and my friends*

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# CHAPTER 1

# INTRODUCTION AND PRELIMINARIES

In this chapter, we give some necessary definitions in topology, which is used in other chapters.

## 1.1 Some Basic Concepts in Topology

In this section we introduce some basic facts of topological spaces.

**Definition 1.1.** A *sequence* denoted by

$$(s_1, s_2, \ldots), \quad (s_n : n \in \mathbb{N}) \;\; or \;\; (s_n)$$

is a function whose domain is $\mathbb{N} = \{1, 2, 3, \ldots\}$, i.e., a sequence assigns a point $s_n$ to each positive integer $n \in \mathbb{N}$. The image $(s_n)$ of $n \in \mathbb{N}$ is called the $n$th *term* of the sequence.

**Example 1.1.** The sequences

$$(s_n) = (1, 3, 5), \quad (t_n) = (\frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \frac{1}{16}, \ldots), \quad (u_n) = (1, 0, 1, 0, \ldots)$$

can be defined, respectively, by the formulas

$$(s_n) = 2n - 1, \quad (t_n) = (-1)^n/2^n, \quad (u_n) = \frac{1}{2}(1 + (-1)^{n+1}) = \begin{cases} 1, & \text{if n is odd,} \\ 0, & \text{if n is even.} \end{cases}$$

**Definition 1.2.** A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $x_0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \ \ implies \ \ |f(x) - f(x_0)| < \epsilon.$$

The function $f$ is a *continuous function* if it is continuous at every point.

**Definition 1.3.** Let $X$ be a non-empty set. A class $\mathcal{T}$ of subsets of $X$ is a *topology* on $X$ if and only if $\mathcal{T}$ satisfies the following axioms;

*i*) $X$ and $\emptyset$ belong to $\mathcal{T}$,

*ii*) The union of any number of sets in $\mathcal{T}$ belong to $\mathcal{T}$,

*iii*) The intersection of any two sets in $\mathcal{T}$ belong to $\mathcal{T}$.

The members of $\mathcal{T}$ are then called *open sets*, and $X$ together with $\mathcal{T}$, i.e., the pair $(X, \mathcal{T})$ is called a *topological space*.

**Example 1.2.** Consider the following classes of subsets of $X = \{a, b, c, d, e\}$.

$$\begin{aligned}
\mathcal{T}_1 &= \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}, \\
\mathcal{T}_2 &= \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, \}\}, \\
\mathcal{T}_3 &= \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}.
\end{aligned}$$

Observe that $\mathcal{T}_1$ is a topology on $X$ since it satisfies necessary three axioms. But $\mathcal{T}_2$ is not a topology on $X$ since the union $\{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\}$ of two members of $\mathcal{T}_2$ does not belong to $\mathcal{T}_2$. Also, $\mathcal{T}_3$ is not a topology on $X$ since the intersection $\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\}$ of two sets in $\mathcal{T}_3$ does not belong to $\mathcal{T}_3$.

**Definition 1.4.** A sequence $(a_1, a_2, \ldots)$ of points in a topological space $X$ *converges* to a point $b \in X$, or $b$ is the limit of the sequence $(a_n)$, denoted by

$$\lim_{n \to \infty} a_n = b, \quad \lim a_n = b, \quad or \ \ a_n \to b$$

if and only if for each open set $U$ containing $b$ there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$n > n_0 \ \ implies \ \ a_n \in U$$

that is, if $U$ contains almost all i.e., all except the a finite number, of the terms of the sequence.

**Definition 1.5.** Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is said to be *continuous* if for each open subset $H$ of Y, the set $f^{-1}(H)$ is an open subset of $X$.

**Definition 1.6.** Two topological space $X$ and $Y$ are called *homeomorphic* if there exists a bijective function $f : X \to Y$ such that $f$ and $f^{-1}$ are continuous. The function $f$ is called a *homeomorphism*.

**Example 1.3.** Let $X = (-1, 1)$. The function $f : X \to \mathbb{R}$ defined by $f(x) = \tan \frac{1}{2}\pi x$ is one-one, onto and continuous. Furthermore, the inverse function $f^{-1}$ is also continuous. Hence the real line $\mathbb{R}$ and the open interval $(-1, 1)$ are homeomorphic.

**Definition 1.7.** Let $X$ be a non-empty set and $d$ be a real-valued function defined on $X \times X$ such that for $a, b, c \in X$:

*i*) $d(a, b) \geq 0$ and $d(a, a) = 0$,

*ii*) $d(a, b) = d(b, a)$ (*Symmetry*),

*iii*) $d(a, c) \leq d(a, b) + d(b, c)$ (Triangle Inequality),

*iv*) If $a \neq b$, then $d(a, b) > 0$.

Then $d$ is said to be a *metric* on $X$, $(X, d)$ is called a *metric space* and $d(a, b)$ is referred to as the *distance* between $a$ and $b$.

**Definition 1.8.** Let $(X, d)$ be a metric space. A sequence $(x_n)$ of points of $X$ is said to be a *Cauchy sequence* in $(X, d)$ if it has the property that given $\epsilon > 0$, there is an integer $n_0$ such that

$$d(x_n, x_m) < \epsilon \ \ whenever \ \ n, m \geq n_0.$$

The metric space $(X, d)$ is said to be *complete* if every Cauchy sequence in $X$ converges.

**Definition 1.9.** A metric space $(X, d)$ is *isometric* to a metric space $(Y, e)$ if and only if there exists a one-one, onto function $f : X \to Y$ which preserves distances, i.e., for all $a, b \in X$,

$$d(a, b) = e(f(a), f(b)).$$

Observe that the relation "$(X, d)$ is isometric to $(Y, e)$" is an equivalence relation in any collection of metric spaces.

**Definition 1.10.** Let $V$ be a real linear vector space. A function which assigns to each vector $v \in V$ the real number $||v||$ is a *norm* on $V$ if and only if it satisfies, for all $v, w \in V$ and $k \in \mathbb{R}$, the following axioms:

*i*) $||v|| \geq 0$,

*ii*) $||v|| = 0$ if and only if $v = 0$,

*iii*) $||v + w|| \leq ||v|| + ||w||$,

*iv*) $||kv|| = |k|||v||$.

A linear space $V$ together with a norm is called a *normed space*. The real number $||v||$ is called the *norm* of the vector $v$.

**Definition 1.11.** A set $A$ is called *countable* if there exists an injective function $f$ from $A$ to the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. If $f$ is also surjective and therefore bijective then $A$ is called *countably infinite*.

**Definition 1.12.** A topological space $X$ is said to have a *countable basis at x* if there is a countable collection $\mathcal{A}$ of neighborhoods of $x$ such that each neighborhood of $x$ contains at least one of the elements of $\mathcal{A}$. A space that has a countable basis at each of its points is said to satisfy the *first countability axiom*, or to be *first countable*.

**Definition 1.13.** A topological space $X$ is called a *Hausdorff space* if for each pair $a, b$ of distinct points of $X$, there exist neighborhoods $G$ and $H$ of $a$ and $b$, respectively, that are disjoint.

**Definition 1.14.** A topological space $X$ is *completely regular* if $A$ is a closed subset of $X$ and $x_0 \in X \backslash A$, then there exists a continuous function $f : X \to [0,1]$ such that $f(x_0) = 0$ and $f(A) = \{1\}$.

**Definition 1.15.** A collection $\mathcal{A}$ of subsets of a topological space $X$ is said to *cover* $X$, or to be a *covering* of $X$, if the union of the elements of $\mathcal{A}$ is equal to $X$. It is called an *open covering* of $X$ if its elements are open subsets of $X$.

**Definition 1.16.** A topological space $X$ is said to be *compact* if every open covering $\mathcal{A}$ of $X$ contains a finite subcollection that also covers $X$.

**Definition 1.17.** A topological space $X$ is said to be *countably compact* if every countable open covering of $X$ contains a finite subcollection that covers $X$.

**Example 1.4.** *i*) Every bounded closed interval $A = [a, b]$ is countably compact.

*ii*) The open interval $A = (0, 1)$ is not countably compact. For consider the infinite subset $B = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ of $A = (0, 1)$. Observe that $B$ has exactly one limit point which is 0 and that 0 does not belong to $A$. Hence $A$ is not countably compact.

**Definition 1.18.** A topological space $X$ is said to be *locally compact at $x$* if there is some compact subspace $C$ of $X$ that contains a neighborhood of $x$. If $X$ is locally compact at each of its points, $X$ is said simply to be *locally compact*.

Note that a compact space is automatically locally compact.

**Definition 1.19.** A function has *compact support* if it is zero outside of a compact set. Alternatively, one can say that a function has *compact support* if its support is a compact set.

# CHAPTER 2

# RIESZ SPACES

In this chapter, we introduce some basic facts of Riesz spaces, we present linear maps between Riesz spaces and we define Riesz algebras.

## 2.1 Partially Ordered Sets

Let $X$ be a non-empty set and $\leq$ be a relation on $X$ such that $\leq \subseteq X \times X$. Let us write $x \leq y$ whenever $(x, y) \in \leq$. If the followings are satisfying then the relation $\leq$ is said to be *partial ordering* on $X$,

*i*) $x \leq x$ for all $x \in X$ (*Reflexive*),

*ii*) $x \leq y$ and $y \leq z$ implies $x \leq z$ (*Transitive*),

*iii*) $x \leq y$ and $y \leq x$ implies $x = y$ (*Anti-symmetric*).

Elements $x, y$ of $X$ for which either $x \leq y$ or $y \geq x$ holds are said to be *comparable*; if neither $x \leq y$ nor $y \geq x$ holds, then $x$ and $y$ are said to be *incomparable*. If every two elements of $X$ are comparable, the partial ordering is called *a linear ordering*. The other extreme case is that every two different elements of $X$ are incomparable, and so the partial ordering states now only that $x \leq y$ holds if and only if $x = y$.

**Example 2.1.** *i*) The real numbers $\mathbb{R}$ is the familiar linear ordering.

*ii*) Let $S$ be a non-empty set and let $X$ consist of all subsets of $S$, partially ordered by inclusion, i.e., if $A$ and $B$ are points of $X$ (in other word, $A$ and $B$ are subsets of $S$), then $A \leq B$ if and only if $A \subseteq B$. In general, $X$ contains many incomparable pairs of points, certainly if $S$ is an infinite set. Note that the empty subset of $S$ is a point of $X$.

If $X$ is partially ordered and $Y$ is a non-empty subset of $X$, then $Y$ is partially ordered in a naturel manner by the partial ordering which $Y$ inherits from $X$. If the inherited partial ordering in $Y$ is a linear ordering, then $Y$ is said to be a *chain* in $X$.

**Example 2.2.** Let $L = \mathbb{R}^2$ with the coordinatewise ordering, i.e., if $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then $x \leq y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$ hold in $\mathbb{R}$. There are many incomparable points, such as for example $(0, 1)$ and $(1, 0)$. Note that every straight line through two different comparable points is a chain.

If $X$ is partial ordered, $Y$ is a non-empty subset of $X$, and $x_0 \in X$ satisfies $x_0 \geq y$ for every $y \in Y$, then $x_0$ is called an *upper bound* of $Y$. If $x_0$ is an upper bound of $Y$ such that $x_0 \leq x_0'$ for any other upper bound $x_0'$ of $Y$, then $x_0$ is called a *least upper bound* or *supremum* of $Y$. In this case $x_0$ is uniquely determined. Indeed, if both $x_0$ and $x_0'$ are suprema of $Y$ then $x_0 \leq x_0'$ and $x_0' \leq x_0$, and so $x_0 = x_0'$. If $x_0$ is the supremum of $Y$, this will be denoted by $x_0 = \sup Y$ or by $x_0 = \sup(y : y \in Y)$. The notations of *lower bound* and *greatest lower bound* or *infimum* are defined similarly. Notation: $x_0 = \inf(y : y \in Y)$ if $x_0$ is the infimum of $Y$.

The element $x_0$ of the partial ordered set $X$ is called *maximal element* if it follows from $x \in X$ and $x_0 \leq x$ that $x_0 = x$ (observe that this is not the same as requiring $x_0 \geq x$ holds for every $x \in X$). If there exists an element $x_0 \in X$ such that $x_0 \geq x$ holds for every $x \in X$, then $x_0$ is called the *largest element* of $X$, and in this case $x_0$ is also a maximal element. In the converse direction, if $x_0$ is the only maximal element of the partially ordered set $X$, then $x_0$ is not necessarily the largest element of $X$. Similar remarks hold for *minimal elements* and the possibly existing *smallest element*.

**Example 2.3.** *i*) Let $X$ be the closed unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$ with coordinatewise partial ordering. All points $(x, y)$ with $x \geq 0$, $y \geq 0$, $x^2 + y^2 = 1$ are maximal elements of $X$ and $X$ does not have a largest element.

(*ii*) Let $X$ be the union of the open unit disc $\{(x, y) : x^2 + y^2 < 1\}$ and the point $(1, 0)$, again with coordinatewise ordering. The point $(1, 0)$ is the only maximal element, but it is not the largest element.

**Definition 2.1.** An ordered set $(L, \leq)$ is called a *lattice* if any two elements $x, y \in L$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$.

If the lattice $X$ has a smallest element, this element is called the zero element and denoted by $\theta$. If $X$ has a largest element, this is called the unit element and denoted by $e$. Every finite lattice has a zero element and a largest element. The lattice consists of one element only if and only if $\theta = e$. If $X$ is a lattice with $\theta$ and $e$ and if the elements $x$ and $x'$ satisfy $x \wedge x' = \theta$ and $x \vee x' = e$, then $x'$ is called a complement of $x$. Then $x$ is a complement of $x'$. In a distributive lattice with $\theta$ and $e$ complement are unique.

**Example 2.4.** Let $S$ be a non-empty set and let $X$ consists of all subsets of $S$ partially ordered by inclusion. Then $\emptyset$ and $S$ are the smallest and largest of elements of $X$. Furthermore, $X$ is a lattice with $\sup(A, B) = A \cup B$ and $\inf(A, B) = A \cap B$.

**Example 2.5.** *i*) Define a relation on $\{-1, 0, 1\}$ as $0 < -1$ and $0 < 1$. Typically this example is drawn as follows:

-1 1

0

Certainly $a \leq a$. Whenever $a \neq b$ either $a < b$ or $b < a$ exclusively; thus, the relation is antisymmetric by the contrapositive. Finally $a \leq b$ and $b \leq c$ implies either $a = b$ or $b = c$ hence $a \leq c$ thus verifying the relation is a partial ordering.

However the ordering does not produce a lattice since $\{-1, 1\}$ has no upper bounds and thus no least upper bound.

*ii*) We fix a non-empty set $X$. Suppose that $X$ is infinite. Let $S$ consist of all finite subsets $A \subset X$ such that the cardinality of $A$ is even. Of course $S$ is a partially ordered set, but it fails to be a lattice.

## 2.2 Ordered Vector Spaces and Riesz Spaces

The real linear space $L$ is called an *ordered vector space* if $L$ is partially ordered in such a manner that the partial ordering is compatible with the algebraic structure of $L$ i.e.,

*i*) $f \leq g$ implies $f + h \leq g + h$ for all $h \in L$,

*ii*) $f \geq 0$ implies $af \geq 0$ for all real number $a \geq 0$.

**Definition 2.2.** The real linear space $L$ is called a *Riesz space* if $L$ is partially ordered in such a manner that

*i*) $L$ is lattice,

*ii*) $L$ is an ordered vector space.

**Definition 2.3.** If $L$ is an ordered vector space, and the element $f$ of $L$ has the property that $\sup(f, 0)$ exists, we will write

$$f^+ = \sup(f, 0), \quad f^- = \sup(-f, 0), \quad |f| = \sup(f, -f)$$

Note already that

$$\inf(f, 0) = -\sup(-f, 0) = -f^-.$$

**Definition 2.4.** If $L$ is a Riesz space, and the collection of every $u \in L$ satisfying $u \geq 0$ is called the *positive cone* of $L$. Elements in the positive cone are called *positive elements*. Notation

$$L^+ = \{u \in L : u \geq 0\}.$$

**Example 2.6.** *i*) Let $\mathbb{R}^n$ ($n \geq 1$) be the real linear space of every real $n-tuples$ $f = (f_1, \ldots, f_n)$ with the coordinatewise addition and multiplication by real numbers. If we define that $f \leq g$ means that $f_k \leq g_k$ holds for $1 \leq k \leq n$, then $\mathbb{R}^n$ is a Riesz space with respect to the thus introduced partial ordering.

*ii*) The partial ordering as defined in (*i*) above is not the only manner to make $\mathbb{R}^n$

($n \geq 2$) into a Riesz space. There exists also the lexicographical ordering, as follows. For $R^2$, let $f \leq g$ for $f = (f_1, f_2)$ and $g = (g_1, g_2)$ if either $f_1 < g_1$ or $f_1 = g_1$, $f_2 \leq g_2$. The ordering is now linear ordering, and so $R^2$ is a Riesz space with respect to the lexicographical ordering. Similarly for $n > 2$.

*iii*) If $L$ is the real linear space of all the finite valued functions $f$ on the arbitrary non-empty point set $X$ with pointwise addition and multiplication by real constants, then $L$ is a Riesz space with respect to the partial ordering introduced by defining that $f \leq g$ means that $f(x) \leq g(x)$ holds for every $x \in X$.

The linear subspace of all real bounded functions on $X$, with the same partial ordering, is a Riesz space by itself.

If $X$ consists of a finite number of points, say $n$ points, we obtain essentially the example in part *i*), i.e., the space $\mathbb{R}^n$ with the coordinatewise ordering. If $X$ consists of a countably infinite number of points, we obtain sequence space $(s)$ of all real sequences and the subspace $l_\infty$ of all bounded real sequences.

**Example 2.7.** *i*) Let $X$ be a non-empty set and let $B(X)$ be the collection of all bounded real valued functions defined on $X$. It is a simple and well-known fact that $B(X)$ is a vector space which is a partially ordered by the positive cone $B(X)^+ = \{f \in B(X) : f(t) \geq 0 \text{ for every } t \in X\}$. Thus $f \geq g$ holds if and only if $f - g \in B(X)^+$. Obviously, $(f \vee g)(t) = \max\{f(t), g(t)\}$ and $(f \wedge g)(t) = \min\{f(t), g(t)\}$ for every $t \in X$ and $f, g \in B(X)$. This shows that $B(X)$ is a Riesz space.

*ii*) Assume that $X$ is infinite and assume that $\mathcal{L}$ is the collection of subsets of $X$ as in the previous example. If $E$ is the linear subspace of $B(X)$ generated by the characteristic functions of the sets in $\mathcal{L}$, then $E$ is an ordered vector space but fails to be a Riesz space.

**Example 2.8.** If $X$ is a topological space and $C(X)$ is the real linear space of all real continuous functions on $X$, then $C(X)$ is a Riesz space with respect to the partial ordering introduced by defining that $f \leq g$ means that $f(x) \leq g(x)$ holds for every $x \in X$.

The linear subspace of all real bounded continuous functions on $X$, with the same partial ordering, is a Riesz Space by itself. If is locally compact space, then the linear subspace of all real continuous functions on $X$ with a compact carrier (also called compact support) is a Riesz space by itself.

**Definition 2.5.** The Riesz Spaces $E$ is said to be *Archimedean* if

$$\inf(n^{-1}u : n = 1, 2, \ldots) = 0$$

holds for every $u \in E^+$.

There exists non-Archimedean Riesz spaces. As an example, let $E = \mathbb{R}^2$ with the lexicographical ordering. The element $(0, 1)$ in $E$ is a lower bound of the sequence $\{(n^{-1}, n^{-1}) : n = 1, 2, \ldots\}$. Hence, $u = (1, 1)$ does not satisfies the condition that $\inf(n^{-1}u : n = 1, 2, \ldots) = 0$. Actually, the sequence of all $n^{-1}u$ does not have a infimum at all this case.

**Theorem 2.1.** *[3] Let E be a Riesz Space. The space E is Archimedean if and only if, given u and v in $E^+$ such that $0 \leq nv \leq u$ for $n = 1, 2, \ldots,$ it follows that $v = 0$. In other words, for any $v \geq 0$, $v \neq 0$, the sequence $(nv : n = 1, 2, \ldots)$ is not bounded above.*

**Proof.** Let $E$ be Archimedean, and assume that $0 \leq nv \leq u$ for $n = 1, 2, \ldots$. Then $0 \leq v \leq n^{-1}u$ for $n = 1, 2, \ldots$. Since $\inf_n(n^{-1}u) = 0$, it follows that $v = 0$. Conversely, assume that $0 \leq nv \leq u$ for $n = 1, 2, \ldots$ implies $v = 0$. We have to prove that $\inf_n(n^{-1}u_0) = 0$ for any $u_0 \in E^+$. It is sufficient to this end that any lower bound $w$ of the sequence $(n^{-1}u_0)$ satisfies $w \leq 0$. Now, if $w$ is such a lower bound, then $v = w \vee 0$ is still a lower bound of the sequence, so $0 \leq nv \leq u_0$ for $n = 1, 2, \ldots$. It follows by hypothesis that $v = 0$, i.e., $w \vee 0 = 0$, and so $w \leq 0$. □

**Example 2.9.** *i*) The space $\mathbb{R}^n$, with usual coordinatewise ordering, is Archimedean.

*ii*) The space $E$ of all real functions on a non-empty point set $X$ (with the algebraic operations and the ordering pointwise) is Archimedean because for any $u \in E^+$ the sequence $(n^{-1}u : n = 1, 2, \ldots)$ converges pointwise to zero as $n$ tends to infinity. The same holds for the space $C(X)$ of all real continuous functions on a topological space $X$.

## 2.3 Linear Maps Between Riesz Spaces

Let $V$ and $W$ be vector spaces. A function $T : V \to W$ is said to be a *linear map* if for any two vectors $x$ and $y$ in $V$ and any scalar $a \in \mathbb{R}$, the following two conditions are satisfied:

*i*) $T(x + y) = T(x) + T(y)$,

*ii*) $T(ax) = aT(x)$.

**Definition 2.6.** The linear mapping $T$ of $L$ into $M$ is called a *Riesz homomorphism* whenever $Tf \wedge Tg = 0$ holds for every pair of elements $f, g \in L$ satisfying $f \wedge g = 0$.

**Theorem 2.2.** *The linear mapping $T$ of $L$ into $M$ is a Riesz homomorphism if and only if*

$$T(f \wedge g) = Tf \wedge Tg$$

*holds for every $f, g \in L$. Equivalently, $T$ is a Riesz homomorphism if and only if*

$$T(f \vee g) = Tf \vee Tg$$

*holds for every $f, g \in L$.*

**Proof.** For the first part it will be sufficient to prove that a Riesz homomorphism $T$ satisfies

$$T(f \wedge g) = Tf \wedge Tg$$

for all $f, g \in L$. To this end, let $f, g \in L$ be given. Setting $h = f \wedge g$, we have

$$(f - h) \wedge (g - h) = 0 \ \ \textit{and} \ \ T(f - h) \wedge T(g - h) = 0, \ \ \textit{so}$$

$$(Tf - Th) \wedge (Tg - Th) = 0 \ \ \textit{by hypothesis}, \ \ \textit{i.e.},$$

$$Tf \wedge Tg = Th = T(f \wedge g).$$

For the second part $f \vee g$ follows by observing that

$$\begin{aligned}
f \vee g &= f + g - (f \wedge g), \\
T(f \vee g) &= T(f + g - (f \wedge g)), \\
T(f \vee g) &= Tf + Tg - T(f \wedge g), \\
T(f \vee g) &= Tf + Tg - (Tf \wedge Tg), \\
T(f \vee g) &= Tf \vee Tg.
\end{aligned}$$

In the next theorem we collect some properties of a Riesz homomorphism. □

**Theorem 2.3.** *[4] Let $T$ be a Riesz homomorphism of $L$ into $M$.*

*i) $f \leq g$ in $L$ implies $Tf \leq Tg$ in $M$. In particular, $T$ is a positive linear mapping, i.e., $T$ maps $L^+$ into $M^+$.*

*ii) We have*

$$Tf^+ = (Tf)^+, \;\; Tf^- = (Tf)^-, \;\; T|f| = |Tf|.$$

**Proof.** *i*) Let $f \leq g$, so $f = f \wedge g$. It follows that

$$Tf = T(f \wedge g) = Tf \wedge Tg, \;\; by\ Theorem\ 2.2,$$

which implies that $Tf \leq Tg$.

*ii*) We have

$$Tf^+ = T(f \vee 0) = Tf \vee T(0) = Tf \vee 0 = (Tf)^+,$$

$$Tf^- = T(-f \vee 0) = T(-f) \vee T(0) = -Tf \vee 0 = (Tf)^-,$$

$$T|f| = T(f \vee -f) = Tf \vee T(-f) = Tf \vee -Tf = |Tf|.$$

□

**Definition 2.7.** The Riesz spaces $L$ and $M$ are called *Riesz isomorphic* if there exists a Riesz homomorphism $T$ of $L$ onto $M$ such that $T$ is a one-one mapping. Evidently, the inverse mapping $T^{-1}$ is a one-one Riesz homomorphism of $M$ onto $L$. The mapping $T$ is now called a *Riesz isomorphism* of $L$ onto $M$, and evidently $T^{-1}$ is then a *Riesz isomorphism* of $M$ onto $L$ (notation $L \cong M$).

Any Riesz homomorphism $T$ of $L$ into $M$ is a positive mapping (i.e., $f \geq 0$ in $L$ implies $Tf \geq 0$ in $M$). Hence, any Riesz isomorphism of $L$ onto $M$ is a one-one positive linear mapping of $L$ onto $M$. It is not true, however, that every one-one positive linear mapping of $L$ onto $M$ is already a Riesz isomorphism. The following theorem and example will clarify this.

**Theorem 2.4.** *[4] The one-one positive linear mapping $T$ of $L$ onto $M$ is a Riesz isomorphism if and only if $T^{-1}$ is also positive.*

**Proof.** Assume that $T^{-1}$ is also positive. We have to show that

$$T(f \vee g) = Tf \vee Tg$$

holds for every $f, g \in L$. Since $T$ is positive, it follows from $(f \vee g) \geq f$ and $(f \vee g) \geq g$ that

$$T(f \vee g) \geq Tf \vee Tg.$$

Since $T^{-1}$ is positive, it follows from $Tf \vee Tg \geq Tf$ and $Tf \vee Tg \geq Tg$ that

$$T^{-1}(Tf \vee Tg) \geq (f \vee g),$$

and so

$$Tf \vee Tg \geq T(f \vee g).$$

□

**Example 2.10.** Let L be two-dimensional plane with the ordinary coordinatewise ordering (i.e., $(x_1, x_2) \leq (y_1, y_2)$ whenever $x_1 \leq y_1$ and $x_2 \leq y_2$), and let $M$ be the same plane with the lexicographic ordering (i.e., $(x_1, x_2) \leq (y_1, y_2)$ whenever $x_1 < y_1$ or $x_1 = y_1, x_2 \leq y_2$). The identity mapping of $L$ onto $M$ is a one-one positive linear mapping, but the inverse mapping is not positive.

The spaces $L$ and $M$ in this example are not Riesz isomorphic, i.e., not only that the identity mapping is not a Riesz isomorphism, but it is true that there exists no Riesz isomorphism of $L$ onto $M$ at all. Indeed, any one-one linear mapping of $M$ onto $L$ transforms the halfplane $\{(y_1, y_2) : y_1 > 0\}$ into a halfplane, so there must be positive elements of $M$ that are transformed into non-positive elements of $L$. Another proof is by observing that $L$ is Archimedean and $M$ is not.

## 2.4 Riesz Homomorphism On $C(X)$ Spaces

In this section, we specially consider the Riesz homomorphisms on $C(X)$ spaces. We present a survey of several definitions, examples and theorems on $C(X)$ spaces which will be Riesz homomorphism.

Let $X$ be a compact Hausdorff space. We recall that $C(X)$ denotes the vector spaces of all continuous real-valued functions on $X$. Let $f, g \in C(X)$ and under pointwise ordering,

$$f \leq g \;\; if \;\; f(x) \leq g(x) \;\; for\ all\ x \in X,$$

$C(X)$ is an Archimedean Riesz space. For all $x \in X$, we have

$$(f \vee g)(x) = f(x) \vee g(x) \;\; for\ all\ f, g \in C(X),$$

$$(f \wedge g)(x) = f(x) \wedge g(x) \;\; for\ all\ f, g \in C(X),$$

$$|f|(x) = |f(x)| \;\; for\ all\ f \in C(X).$$

$C(X)$ also carries a natural norm, the supremum-norm $\|\cdot\|_\infty$, defined by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|, \quad f \in C(X).$$

**Lemma 2.1.** *(URYSOHN'S LEMMA) Let $X$ be a compact Housdorff space, $K \subset X$ be a closed a subspace and $x_0 \in X \setminus K$ then there exists $f \in C(X)$ such that $f(x_0) = 1$ and $f(K) = 0$. In particular, if $x \neq y$ then there exists $f \in C(X)$ such that*

$$f(x) \neq f(y).$$

**Theorem 2.5.** *Let $X$ be a compact Hausdorff space. If $T : C(X) \to \mathbb{R}$ be a Riesz homomorphism then there exists $x_0 \in X$ such that*

$$Tf = f(x_0)$$

*for each $f \in C(X)$.*

**Proof.** Suppose that such $x_0$ does not exist. In this case for each $x_0 \in X$ there exists $f_x \in C(X)$ such that $Tf_x \neq f_x(x_0)$. For each $x \in X$, set $g_x = |f_x - Tf_x \cdot \mathbf{1}_X|$,

$$\begin{aligned} g_x(x_0) &= |f_x - Tf_x \cdot \mathbf{1}_X|(x_0), \quad for\ all\ x_0 \in X, \\ g_x(x_0) &= |f_x(x_0) - Tf_x \cdot \mathbf{1}_X(x_0)|, \quad for\ all\ x_0 \in X, \\ g_x(x_0) &= |f_x(x_0) - Tf_x|, \quad for\ all\ x_0 \in X. \end{aligned}$$

It follows by the hypothesis that for each $x \in X$, $g_x(x_0) > 0$. Since for each $x \in X$, $g_x$ is continuous function and $g_x(x_0) > 0$, it follows from there exists an open neighborhood $O_x \subset X$ such that for each $x \in O_x$, $g_x(x) > 0$. We can write as an open covering of $X$ that $X = \bigcup_{x \in X} O_x$. Since $X$ is compact, it follows from $X = \bigcup_{i=1,\ldots,n} O_{x_i}$ for some $x_1, \ldots, x_n$. Then $g = \sum_{i=1}^{n} g_{x_i}$ for each $x \in X$, there exists $i$ such that $g_{x_i}(x) > 0$ for every $x \in O_{x_i}$. We have

$$g(x) = \sum_{i=1}^{n} g_{x_i}(x) > 0, \quad for\ all\ x \in X.$$

Since $g$ is continuous function and $X$ is compact then there exists $\epsilon > 0$ such that

$g > \epsilon \cdot \mathbf{1}_X$. Let $T\mathbf{1}_X = 1$,

$$\begin{aligned} Tg &= T(\sum_{i=1}^{n} g_{x_i}), \\ Tg &= Tg_{x_1} + Tg_{x_2} + \ldots + Tg_{x_n}. \end{aligned}$$

We conclude that

$$\begin{aligned} Tg_{x_1} &= T(|f_{x_1} - Tf_{x_1} \cdot \mathbf{1}_X|), \\ &= |Tf_{x_1} - T(Tf_{x_1} \cdot \mathbf{1}_X)|, \\ &= |Tf_{x_1} - Tf_{x_1} \cdot T\mathbf{1}_X|, \\ &= |Tf_{x_1} - Tf_{x_1}| = 0. \end{aligned}$$

Similarly $Tg_{x_2} = 0$, $Tg_{x_3} = 0, \ldots, Tg_{x_n} = 0$. Hence

$$Tg = 0.$$

We know that $g > \epsilon \cdot \mathbf{1}_X \Rightarrow Tg \geq T(\epsilon \cdot \mathbf{1}_X) = \epsilon \cdot T\mathbf{1}_X = \epsilon$. Hence $Tg = 0 = \epsilon > 0$. It is a contradiction. Therefore there exists $x_0 \in X$ such that $Tf = fx_0$ for every $f \in C(X)$. Let $s \in X$ with $s \neq x_0$. By Urysohn's lemma if $s \neq x_0$ then there exists $f_k \in C(X)$ such that $f_k(x_0) \neq f_k(s)$. Therefore there exists unique $x_0 \in X$ such that $Tf = fx_0$ for each $f \in C(X)$. □

**Corollary 2.1.** *Let $X$ and $Y$ be compact Hausdorff spaces and $T$ be a Riesz homomorphism of $C(X)$ into $C(Y)$ such that $T\mathbf{1}_X = \mathbf{1}_Y$. Then there exists a unique continuous map $\sigma : Y \to X$ such that*

$$Tf = f \circ \sigma, \quad for\ all\ f \in C(X).$$

**Proof.** For all $y \in Y$, by applying Theorem 2.5 to the map $f \mapsto Tf(y)$ *for all* $f \in C(X)$, we see that there exists a unique element $\sigma(y)$ of $X$ such that $Tf(y) = f(\sigma(y))$

for all $f \in C(X)$. Thus we obtain a $\sigma : Y \to X$ with the property

$$Tf = f \circ \sigma, \quad for\ all\ f \in C(X).$$

Now we must prove that $\sigma$ is continuous. Let $U \subset X$ be open and let $b \in \sigma^{-1}(U) \subset Y$. By Urysohn's lemma there exists an $f \in C(X)$ such that $f(\sigma(b)) = 1$ while $f$ vanishes on $X \backslash U$. Setting $g = Tf$ we have $g \in C(Y)$, $g(b) = f(\sigma(b)) = 1$ and $g$ vanishes on $Y \backslash \sigma^{-1}(U)$. Now $\{y \in Y : g(y) > 0\}$ is open in $Y$ and $b \in \{y \in Y : g(y) > 0\} \subset \sigma^{-1}(U)$. Hence, $\sigma^{-1}(U)$ is open in $X$ and therefore $\sigma$ is continuous. □

## 2.5 Riesz Algebras

A Riesz space $L$ (which is Archimedean) is called a *Riesz algebra* if there exists an associative multiplication in $L$ with the usual algebraic property that $u, v \in L^+$ implies $uv \in L^+$.

**Definition 2.8.** An algebra (over $\mathbb{R}$) is a vector space $V$ provided with a multiplication $V \times V \to V$ such that for all $f \in V$ the maps $g \to fg$ and $g \to gf$ are linear.

**Example 2.11.** For every topological space $X$, $C(X)$ is an algebra under pointwise operations.

**Definition 2.9.** Two algebras, $V$ and $W$, are said to be *isomorphic* if there exists a linear bijection $T : V \to W$ such that $T(fg) = (Tf)(Tg)$ for all $f, g \in V$.

**Example 2.12.** Let $T : V \to V$ be a linear map such that $Tg = fg$ for all $f \in V$. Then

$$\begin{aligned} T(g_1 + g_2) &= f(g_1 + g_2), \\ Tg_1 + Tg_2 &= fg_1 + fg_2. \end{aligned}$$

**Proposition 2.1.** *In a Riesz algebra L the following statements hold.*

*i) If $f \leq g$ and $0 \leq u \in L$, then $uf \leq ug$. In particular $u^2 \geq 0$ whenever $u \geq 0$.*

*ii) If $0 \leq u \leq v$ and $0 \leq p \leq q$ in $L$, then $up \leq vq$*

*iii) If* $0 \leq u \in L$*, then* $u(f \vee g) \geq (uf) \vee (ug)$, $(f \vee g)u \geq (fu) \vee (gu)$, $u(f \wedge g) \leq (uf) \wedge (ug)$ *and* $(f \wedge g)u \leq (fu) \wedge (gu)$ *for every* $f, g \in L$.

*(iv)* $|fg| \leq |f| \cdot |g|$, $(fg)^+ \leq f^+g^+ + f^-g^-$ *and* $(fg)^- \leq f^+g^- + f^-g^+$ *for every* $f, g \in L$.

**Proof.** *i)* If $f \leq g$, then $g - f \geq 0$ and so $u(g - f) \geq 0$, i.e. $ug - uf \geq 0$. Hence $uf \leq ug$. Similarly we find $fu \leq gu$.

*ii)* If $0 \leq p \leq q$, then $up \leq uq$ and $vp \leq vq$, $up \leq vp \leq vq$. Hence $up \leq vq$.

*iii)* It follows from $f \leq f \vee g$ that $uf \leq u(f \vee g)$, and analogously we find $ug \leq u(f \vee g)$. Therefore $(uf) \vee (ug) \leq u(f \vee g)$. The other inequalities are proved same way.

*(iv)* For $f, g \in L$ we have $fg = (f^+g^+ + f^-g^-) - (f^+g^- + f^-g^+)$. Now it follows from $fg \leq f^+g^+ + f^-g^-$ and $0 \leq f^+g^+ + f^-g^-$ that $(fg)^+ \leq f^+g^+ + f^-g^-$. Analogously $(fg)^- \leq f^+g^- + f^-g^+$. Using these inequalities we get $|fg| = (fg)^+ + (fg)^- \leq (f^+g^+ + f^-g^-) + f^+g^- + f^-g^+ = (f^+ + f^-)(g^+ + g^-) = |f| \cdot |g|$. □

**Example 2.13.** Let $L = \mathbb{R}^2$ be a space with the coordinatewise addition, scaler multiplication by real numbers and coordinatewise order such that $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ for every $(x_1, y_1) \in L$ and $\alpha(x, y) = (\alpha x, \alpha y)$ for every $(x, y) \in L$ and $\alpha \in \mathbb{R}$. Let we define $\odot$ such that $(x, y) \odot (x', y') := (xx' - yy', yx' + xy')$. We know that $(x, y) \leq (x', y')$ implies $x \leq x'$ and $y \leq y'$. Obviously $L$ is algebra and $(0, 1) \odot (0, 1) = (-1, 0) \leq (0, 0)$. We see that $(-1, 0) \notin L^+$ so $L$ is not Riesz algebra.

# CHAPTER 3

# VERSIONS OF THE BANACH-STONE THEOREM

In this chapter we benefited from the book of Isometries on Banach Spaces [5]. An isometry, of course, is a transformation which preserves the distance between elements of a space. When Banach showed that every linear isometry on the space of continuous real valued functions on a compact metric space must transform a continuous function $x(t)$ into a continuous function $y(t)$ satisfying

$$y(t) = h(t)x(\sigma(t))$$

where $|h(t)| \equiv 1$ and $\sigma$ is a homeomorphism, he was establishing a canonical form characterization which fits in an astonishing number of cases. In this chapter we are interested primarily in just such explicit descriptions of isometries.

## 3.1 Banach's Version

**Definition 3.1.** A *Banach space* is a vector space $X$ over the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ which is equipped with a norm and which is complete with respect to that norm.

Suppose that $X$ is a compact metric space. By $C(X)$ we mean the Banach space of continuous real valued functions defined on $X$ with the supremum norm. We begin with Banach's characterization of a peak point for a function $f$ in relation to the existence of directional derivatives of the norm at $f$.

**Lemma 3.1.** *[5] Let $f \in C(X)$ and $x_0 \in X$. In order that*

$$|f(x_0)| > |f(x)|) \; for \; each \; x \in X \; with \; x \notin x_0, \tag{3.1}$$

*it is necessary and sufficient that*

$$\lim_{t\to 0}\frac{\|f+tg\|-\|f\|}{t} \tag{3.2}$$

*exists for each* $g \in C(X)$.

*Moreover, if f satisfies (3.1), we have*

$$\lim_{t\to 0}\frac{\|f+tg\|-\|f\|}{t} = g(x_0)\,\mathrm{sgn}\,f(x_0) \tag{3.3}$$

*for each* $g \in C(X)$.

**Proof.** Let us first show that (3.2) is necessary. If (3.1) holds, then $\|f\| = |f(x_0)|$. Now given $g \in C(X)$ and a real number $t$, $f + tg$ is continuous on $X$ and so attains its maximum absolute value at some $x_t \in X$. Therefore,

$$|f(x_0) + tg(x_0)| - |f(x_0)| \le \|f + tg\| - \|f\| = |f(x_t) + tg(x_t)| - |f(x_0)|. \tag{3.4}$$

Also, we have

$$|f(x_0) + tg(x_0)| \le |f(x_t) - tg(x_t)|$$

and a little manipulation yields the inequality

$$0 \le |f(x_0)| - |f(x_t)| \le |t||g(x_0)| + |t||g(x_t)| \le 2|t|\|g\|.$$

It now follows that $\lim_{t\to 0}|f(x_t)| = |f(x_0)|$ and the compactness of $X$ allows us to conclude that

$$\lim_{t\to 0} x_t = x_0. \tag{3.5}$$

Now let us first suppose that $f(x_0) < 0$. By virtue of the fact that $x_t \to x_0$, we may choose $t$ so small that

$$|f(x_0) + tg(x_0)| - |f(x_0)| = -f(x_0) - tg(x_0) + f(x_0) = -tg(x_0)$$

and

$$|f(x_t) + tg(x_t)| - |f(x_t)| = -f(x_t) - tg(x_t) + f(x_t) = -tg(x_t)$$

From these two statements and (3.4) we see that

$$-tf(x_0) \leq \|f + tg\| - \|f\| \leq -tg(x_t)$$

for sufficiently small t which combined with (3.5) and the continuity of $g$ leads to

$$\lim_{t\to 0} \frac{\|f + tg\| - \|f\|}{t} = -g(x_0).$$

The case where $f(x_0) > 0$ can be treated in a similar manner to establish that

$$\lim_{t\to 0} \frac{\|f + tg\| - \|f\|}{t} = g(x_0).$$

This completes the proof of the necessity of the existence of the limit in (3.2) and shows that (3.3) must hold.

For the sufficiency let us assume that $x_0, x_1 \in X$ with $x_0 \neq x_1$ and

$$\|f\| = |f(x_0)| = |f(x_1)| > |f(x)|$$

for all $x \in X$. If $f(x_0) < 0$, define $g(x) = -d(x, x_1)$ where $d$ denotes the metric on $X$. Then

$$\|f + tg\| - \|f\| \geq |f(x_0) + tg(x_0)| - |f(x_0)| = -f(x_0) - tg(x_0) + f(x_0) = td(x_0, x_1)$$

for all sufficiently small $t$. We conclude that

$$\liminf_{t\to 0^+} \frac{\|f + tg\| - \|f\|}{t} \geq d(x_0, x_1) > 0. \tag{3.6}$$

However,

$$\|f + tg\| - \|f\| \geq |f(x_1) + td(x_1, x_1)| - |f(x_1)| = 0.$$

for all $h$, whereby we must have

$$\liminf_{t \to 0^-} \frac{\|f + tg\| - \|f\|}{t} \leq 0. \tag{3.7}$$

The inequalities (3.6) and (3.7) show that the limit (3.2) cannot exist.

For the case where $f(x_0) > 0$, we define $f(x) = d(x, x_1)$ and give a similar argument. □

We now state and prove the theorem of Banach for surjective isometries on $C(X)$ spaces.

**Theorem 3.1.** *(Banach)[5] If $X$ and $Y$ are compact metric spaces then for the spaces of real continuous functions $C(X)$ and $C(Y)$ to be isometrically isomorphic it is necessary and sufficient that $X$ and $Y$ be homeomorphic. In this case, an isometric isomorphism $T$ from $C(X)$ onto $C(Y)$ must be given by*

$$Tf(t) = h(t)f(\sigma(t)) \;\; for\ t \in Y, \tag{3.8}$$

*where $\sigma$ is a homeomorphism from $Y$ onto $X$ and $h$ is a real valued unimodular function on $Y$.*

**Proof.** It is easy to see that if $\sigma$ is a homeomorphism from $Y$ onto $X$, then a transformation $U$ defined by (3.8) is an isometric isomorphism of $C(X)$ onto $C(Y)$ and thus the sufficiency of the condition is clear.

For the necessity, let $T$ be a linear isometry from $C(X)$ onto $C(Y)$, suppose $x_0 \in X$ and let $f \in C(X)$ be such that $|f(x_0)| > |f(x)|$ for all $x \in X$. By Lemma 3.1,

$$\lim_{r \to 0} \frac{\|f + rg\| - \|f\|}{r} = g(x_0)\,\mathrm{sgn}\, f(x_0)$$

must exist for every $g \in C(X)$. Since $T$ is an isometry,

$$g(x_0)\operatorname{sgn} f(x_0) = \lim_{r\to 0} \frac{\|f + rg\| - \|f\|}{r} = \lim_{r\to 0} \frac{\|Tf + rTg\| - \|Tf\|}{r} \tag{3.9}$$

and we apply the lemma again to conclude that there is some $y_0 \in Y$ such that

$$|Tf(y_0)| > |Tf(y)| \;\; for\ all\ y \in Y\ with y \neq y_0$$

Furthermore, we may conclude from (3.9) and (3.3) that

$$g(x_0)\operatorname{sgn} f(x_0) = Tg(y_0)\operatorname{sgn} Tf(y_0).$$

If we let $h(y_0) = \operatorname{sgn} f(x_0)Tf(y_0)$, then $|h(y_0)| = 1$ and we get

$$Ug(y_0) = h(y_0)g(x_0) \;\; for\ each\ g \in C(X). \tag{3.10}$$

Let us define $\omega$ from $X$ to $Y$ by $\omega(x_0) = y_0$. Now $\omega$ is injective, for if $\omega(x_1) = (x_2)$, then by (3.10) we have $|g(x_1)| = |g(x_2)|$ for all $g \in C(X)$ and so $x_1 = x_2$. To see that $\omega$ is surjective, let $y_0 \in Y$ and define $q$ on $Y$ by

$$q(y) = \frac{1}{1 + d(y, y_0)}$$

where $d$ denotes the metric on $Y$. If $f = T^{-1}q$, then by (3.10)

$$|f(x)| = \frac{1}{1 + d(\omega(x), y_0)}$$

for each $x \in X$. Since $\|f\| = \|q\| = 1$, there exists $x_0 \in X$ such that $|f(x_0)| = 1$. Therefore $\frac{1}{1+d(\omega(x),y_0)} = 1$ which implies that $y_0 = \omega(x_0)$.

Finally, suppose that $(x_n)$ is a sequence in $X$ converging to $x_0$, $g \in C(Y)$ and $Tf = g$. Since $|g(\omega(x_n))| = |f(x_n)|$ for each $n$ and $f$ is continuous, we have $|f(x_n)| \to |f(x_0)|$ so that $|g(\omega(x_n))| \to |g(\omega(x_0))|$ for every $g \in C(Y)$. By choosing $g$ defined by $g(y) = d(y, \omega(x_0))$, we obtain

$$d(\omega(x_n, x_0)) = |g(\omega(x_n))| \to |g(\omega(x_0))| = 0$$

from which we conclude that $\omega(x_n) \to \omega(x_0)$. This shows that $\omega$ is continuous and since $X$ is compact and $Y$ is Hausdorff, $\omega$ must in fact be a homeomorphism. Hence $X$ and $Y$ are homeomorphic and if we let $\sigma = \omega^{-1}$, we get from (3.10) the characterization of $U$ given by (3.8). □

**Corollary 3.1.** *[5] If $T$ is a surjective linear isometry on $C(X)$ where $X$ is compact and metric, then*

$$Tf(t) = h(t)f(\sigma(t))$$

*where $|h(t)| = 1$ and $\sigma$ is a homeomorphism of $X$ onto itself.*

## 3.2 Stone's Version

We gave Banach's original proof of the characterization of linear isometries from $C(X, \mathbb{R})$ to $C(Y, \mathbb{R})$ where $X$ and $Y$ are compact metric spaces. Stone improved the result in 1937 by giving the same characterization (Theorem 3.1) where $X$ and $Y$ are compact Hausdorff spaces.

**Theorem 3.2.** *(Stone) If $X$ and $Y$ are compact Hausdorff spaces and $T$ is an isometric isomorphism of $C(X, \mathbb{R})$ onto $C(Y, \mathbb{R})$, then there is a homeomorphism $\sigma$ from $Y$ onto $X$ and continuous unimodular function $h$ on $Y$ such that for each $f \in C(X)$,*

$$Tf(y) = h(y)f(\sigma(y)) \quad for\ y \in Y. \tag{3.11}$$

**Proof.** For $x \in X$, let $X(x) = \{f \in C(X) : |f(x)| = \|f\|\}$. Then $X(x)$ contains all the real constant functions and if $f_1, f_2, \ldots, f_n$ are in $X(x)$ then for

$$g = \sum_{i=1}^{n} f_i \operatorname{sgn} f_i,$$

we see that $g \in X(x)$ and $\|g\| = \sum \|f_i\|$.

If $x$ is any fixed element in $X$, let $F(f)$ denote the set of all $y$ in $Y$ such that $|Tf(y)| = \|Tf\|$ where $f \in X(x)$. Given $f_1, f_2, \ldots, f_n$ in $X(x)$ and $g = \sum f_i \operatorname{sgn} f_i$ as above, there must exist a $y \in Y$ such that $|Tg(y)| = \|Tg\|$. However,

$$\|Tg\| = |Tg(y)| \leq \sum |f_i(y)| \leq \sum \|Tf_i\| = \sum \|f_i\| = \|g\| = \|Tg\|$$

and we conclude that $|Tf_i(y)| = \|UTf_i\|$ for each $i = 1, \ldots, n$. Therefore, the closed sets $F(f)$ (for $f \in X(x)$) have the finite intersection property and since $Y$ is compact there is at least one $y \in Y$ common to all. Thus $Tf \in Y(y)$ for every $f \in X(x)$, and $U$ maps $X(x)$ into $Y(y)$ for some $y$. In the same way, $T^{-1}$ maps $Y(y)$ into $X(r)$ for some $r$. In fact, we must have $x = r$ and so there is a one-to-one map $x \to y = \omega(x)$ which establishes a one-to-one correspondance between the sets $X(x)$ and $Y(y)$. The sets $\{x \in X : |f(x)| = \|f\|\}$ correspond to $\{y \in Y : |Tf(y)| = \|Tf\|\}$ as do their complements which are shown to form a basis for the topologies of $X$ and $Y$, respectively.

If we let $\sigma = \omega^{-1}$ and $h = U1$, then (3.11) follows after certain manipulations, which we omit. □

## 3.3 Eilenberg's Version

As we have seen in both Banach's proof and Stone's proof, the desired homeomorphism arises by identifying points where given functions achieve their norms. In Banach's proof, the identification of so-called peak functions involves the existence of the directional derivative of the norm function. Eilenberg suggested another way to identify these functions.

**Definition 3.2.** A function $f$ in the space $C_b(Y)$ of bounded continuous functions on a topological space $Y$ is a *peak function* with $y_0 \in Y$ as *peak* if

$$|f(y)| < |f(y_0)| \;\; for\ all\ y \in Y\ with\ y \neq y_0.$$

**Definition 3.3.** If $X$ is a Banach space and $x \in X$ with $||x|| \leq 1$, then the *star of x* (denoted by $\mathrm{st}(x)$) is the set of all $s \in X$ with $||s|| \leq 1$ such that $||x + s|| = 2$.

Clearly, $x \in \mathrm{st}(x)$ if and only if $||x|| = 1$ and in fact any element of a star set must have norm exactly one.

**Lemma 3.2.** *[5] Suppose $Y$ is countably compact with $f, g \in C(Y)$ and $||f|| \leq 1$, $||g|| \leq 1$ (the functions may be either real or complex valued). Then $f \in \mathrm{st}(g)$ if and only if there is $y_0 \in Y$ such that $f(y_0) = g(y_0)$ and*

$$|f(y_0)| = |g(y_0)| = 1.$$

**Proof.** The sufficiency of the condition is obvious. For the necessity, the countable compactness guarantees to $y_0 \in Y$ so that

$$2 = ||f + g|| = |f(y_0) + g(y_0)|.$$

□

The result follows from some elementary complex arithmetic.

The next theorem gives a simple characterization of peak functions.

**Theorem 3.3.** *[5] Let $Y$ be countably compact, completely regular, and $C(Y)$ the space of continuous (real or complex) functions on $Y$. A function $f \in C(Y)$ with $||f|| = 1$ is a peak function if and only if $\mathrm{st}(f)$ is convex.*

**Proof.** If $f$ is a peak function, there exists $y_0 \in Y$ such that $|f(y_0)| = 1$ and $|f(y)| < 1$ for $y \neq y_0$. From Lemma 3.2, it is clear that $st(f) = \{g \in C(Y) : ||g|| = 1 \;\; and \;\; g(y_0) = f(y_0)\}$. This set is obviously convex.

Now suppose that $f$ is not a peak function. Then $|f|$ must attain its maximum value

1 (by the countable compactness) and it must occur for at least two distinct points $y_1, y_2 \in Y$. Since $Y$ is completely regular, there are disjoint open sets $G_1, G_2$ containing $y_1, y_2$, respectively, as well as continuous norm-one functions $g_1, g_2$ with $g_i(y_1) = f(y_i)$ and $g_i(y) = 0$ for $y \in Y \setminus G_i$, $i = l, 2$. Then

$$\|f + g_1\| \geq |f(y_1) + g_1(y_1)| = 2$$

so that $\|f+g_1\| = 2$; similarly $\|f+g_2\| = 2$. However, $\|g_1+g_2\| = 1$ by the construction of $g_1$ and $g_2$ so that both are in $st(f)$ while $\frac{1}{2}(g_1 + g_2)$ is not. Therefore, $st(f)$ is not convex. □

**Theorem 3.4.** *(Eilenberg)[5] Suppose X and Y are completely regular, countably compact spaces satisfying the first axiom of countability. Let T be an isometric isomorphism of $C(X, \mathbb{C})$ onto $C(Y, \mathbb{C})$. Then there is a homeomorphism $\sigma$ of Y onto X and a unimodular function $h \in C(Y, \mathbb{C})$ such that*

$$Tf(y) = h(y)f(\sigma(y)) \;\; for\ all\ y \in Y. \tag{3.12}$$

**Proof.** Let $x_0 \in X$. Since $Y$ satisfies the first axiom, there is a countable base $\{G_n\}$ of open neighborhoods of $x_0$ and since $Y$ is completely regular, $\{x_0\} = \cap G_n$. It follows that there is a peak function $f$ which peaks at $x_0$. By Theorem 3.3, st($f$) is convex and since $T$ is an isometry, st($Tf$) = $T$(st($f$)) is convex as well so that $Tf$ is a peak function. Therefore, there is some $y_0 \in Y$ such that $Tf$ peaks at $y_0$.

It is important to note that if $g$ is any norm-one function which attains its norm at $x_0$, then we can let $v = \frac{f(x_0)}{g(x_0)}g$ to get a function which is in st($f$). It follows that $Tv \in$ st($Tf$) from which we conclude that $Tv(y_0) = Tf(y_0)$ where $|Tf(y_0)| = 1$ (Lemma 3.2). Hence $|Tg(y_0)| = 1$ as well which shows that the function $\omega$ which pairs $x_0$ with $y_0$ is well defined. We will see that it is a homeomorphism.

If $\omega(x_1) = \omega(x_2)$ and $x_1 \neq x_2$, then there are peak functions $f_1, f_2$ which peak at $x_1$ and $x_2$, respectively. Since $Tf_1$ and $Tf_2$ both peak at $\omega(x_1) = \omega(x_2)$, application of $T^{-1}$ would imply that $f_1$ and $f_2$ must both peak at $x_1$ which is not true. It is also straightforward to show that $\omega$ is surjective, using the fact the $T^{-1}$ is an isometry.

For the continuity, let $x_0 \in X$ and suppose $G$ is a neighborhood of $\omega(x_0)$. Since $Y$ is completely regular, we may choose $g \in C(Y)$ with $\|g\| = 1$ such that $g(\omega(x_0)) = 0$ and $g(y) = 1$ for all $y \in Y \setminus G$. Let $f \in C(X)$ be such that $Uf = g$. Then $N = \{x \in X : |f(x)| < \|f\|\}$ is an open neighborhood of $x_0$ and $\omega(N) \subseteq G$ since $f$ and $Uf = g$ must peak at points which correspond under $\omega$.

It remains to show now that $T$ satisfies (3.12). From what we have already shown, we must have $|T\mathbf{1}(y)| = 1$ for every $y$. Let $h = T\mathbf{1}$ and $V = \bar{h}T$. Then $V(\mathbf{1}) = 1$, and we will show that if $v \in C(X)$ with $v(x) = 0$, then $Vv(\omega(x)) = 0$. The canonical form (3.12) follows from this since for every $x \in X$ and $f \in C(X)$, $v = f - f(x)$ vanishes at $x$ and $x_0$

$$0 = Vv(\omega(x)) = Vf(\omega(x)) - f(x).$$

If we write $\sigma(y) = \omega^{-1}(y)$ and replace $V$ by $\bar{h}T$, we get (3.12).

Hence, to complete the proof we suppose first that $v$ is a non-negative function in $C(X)$ with $v(x_0) = 0$, $v(x) \neq 0$ for $x \neq x_0$, and $\|v\| = 1$. With these assumptions, $1 - v$ is a peak function which peaks at $x_0$ and therefore $V(l - v) = 1 - V(v)$ is a peak function which peaks at $\omega(x_0)$. Now $1-v$ is in st$(\mathbf{1})$ so that $1-V(v) \in$ st$(V\mathbf{1}) =$ st$(1)$. By Theorem 3.3, there is some $y_0 \in Y$ at which $V(l-v)(y_0) = 1$. However, $|V(1-v)(\omega(x_0))| = 1$ and since $V(l - v)$ is peak function, we must conclude that $y_0 = \omega(x_0)$. Thus

$$1 = V(1 - v)(\omega(x_0)) = 1 - V(v)(\omega(x_0))$$

and we have $Vv(\omega(x_0)) = 0$. If $\|v\| \neq 1$ we replace $v$ by $\frac{v}{\|v\|}$ and get the same result.

If $f$ is any non-negative function with $f(x_0) = 0$ and $v$ is as above, then $v + f$ satisfies the hypotheses above so that $V(v + f)(\omega(x_0)) = 0$ and we get $Vf(\omega(x_0)) = 0$. By the linearity of $V$ we can extend the result, first to all real functions by using the decomposition into positive and negative parts, and finally to complex functions using the real and imaginary parts. □

## 3.4 $C_0(X)$ Space Version

We consider here the space $C_0(X)$, of all continuous functions vanishing at infinity on a locally compact space $X$, endowed with the usual supremum norm.

**Definition 3.4.** For a locally compact Hausdorff space $X$, we denote by $C_0(X)$ the Banach space of all continuous real-valued functions defined on $X$ which vanish at infinity, equipped with a usual supremum norm. In case $X$ is compact, we write $C(X)$ instead of $C_0(X)$.

**Definition 3.5.** If $X$ is a locally compact Hausdorff space and $M$ is a linear subspace of $C_0(X)$, the *Choquet boundary* for $M$ (denoted by $ch(M)$) is the set of all $x$ in $X$ such that $\omega_x$ is an extreme point of $B(M^*)$.

**Definition 3.6.** Let $X$ be a locally compact Hausdorff space. An element $x_0 \in X$ is said to be a *strong boundary point* of a subspace $M$ of $C_0(X)$ if for each neighborhood $U$ of $x_0$, and each $\epsilon > 0$, there exists $f \in M$ such that $1 = f(x) = \|f\|$, and $|f(x)| < \epsilon$ for all $x \in X \setminus U$. The set $\psi(M)$ of strong boundary points for $M$ is called the *strong boundary* of $M$. A subspace $M$ of $C_0(X)$ is said to be *extremely regular* if $\psi(M) = X$, and $M$ is *extremely* $C-$*regular* if $ch(M) \subset \psi(M)$.
Note that an extremely C-regular subspace $M$ must separate the points of $ch(M)$

**Theorem 3.5.** *[5] i) Let $X, Y$ be locally compact Hausdorff spaces and suppose $M$ is a closed subspace of $C_0(X)$ which separates the points of its Choquet Boundary. If $T$ is a linear isometry of $M$ onto a subspace $N$ of $C_0(Y)$, then there is a function $h$ from $ch(N)$ into the unit circle and a function $\sigma$ from $ch(N)$ onto $ch(M)$ such that*

$$Tf(t) = h(t)f(\sigma(t)) \quad for\ all\ f \in M\ and\ t \in ch(N). \tag{3.13}$$

*Furthermore, the functions $h, \sigma$ are continuous at each $t \in ch(N)$ for which $\sigma(t) \in \psi(M)$. In particular, if $M$ is extremely C-regular, then $h$ and $\sigma$ are continuous on $ch(N)$.*

*ii*) *If X is compact and M is any subspace of* $C(X)$ *which separates points of X and contains the constant functions, then the conclusion of the previous part holds. In this case, h is defined and continuous on all of Y and there is a continuous function* $\phi$ *from Y into* $S(M^*)$ *such that*

$$\overline{h(t)}Tf(t) = \phi(t)(f) \ \ for\ all\ t \in Y. \tag{3.14}$$

**Corollary 3.2.** *[5] Suppose* $M, N$ *are completely C-regular subspaces of* $C_0(X)$ *and* $C_0(Y)$*, respectively, and suppose T is a linear isometry from M onto N. Then* $Tf(t) = h(t)f(\sigma(t))$ *for all* $f \in M$ *and* $t \in ch(N)$ *holds where h is continuous on* $ch(N)$ *into the unit circle of C and* $\sigma$ *is a homeomorphism from* $ch(N)$ *onto* $ch(M)$*.*

**Theorem 3.6.** *(*$C_0(X)$ *Space)[5] Let X and Y be locally compact spaces. If T is a linear isometry from* $C_0(X)$ *onto* $C_0(Y)$*, then X and Y are homeomorphic. Furthermore* $Tf(y) = h(y)f(\sigma(y))$ *for all* $y \in Y$ *where h is continuous on Y such that* $|h(y)| = 1$ *for all* $y \in Y$ *and* $\sigma$ *is a homeomorphism of Y onto X.*

**Proof.** In this case, the Choquet boundaries of $M$ and $N$ are $X$ and $Y$, respectively, and they are homeomorphic by Corollary 3.2. □

# CHAPTER 4

# BANACH LATTICE VALUED BANACH-STONE THEOREM

In this chapter, we first give some necessary definition and theorems in section 4.1. Throughout the following section 4.2 and section 4.3, $X$ and $Y$ are always compact Hausdorff spaces, and $E$, $F$ are always both Banach lattices and Riesz algebras. We prove that if $F$ has no zero-divisor and $T : C(X, E) \to F$ is a non-trivial Riesz algebraic homomorphism, then $T$ has a unique support. In Section 4.3, by using a method similar to that in [10], we show that if $F$ has no zero-divisor and there is a Riesz algebraic isomorphism $T : C(X, E) \to C(Y, F)$ such that $Tf$ has no zero if $f$ has none, then $X$ is homeomorphic to $Y$ and $E$ is Riesz algebraically isomorphic to $F$. Also, an example is given to show that this conclusion fails if either $F$ has a zero-divisor or $f$ has no zero but $Tf$ has a zero. Finally, by using the same example, we point out some mistakes in [8]. For unmentioned basic facts and terminology, refer to [2, 4, 13].

## 4.1 Normed Riesz Spaces and Banach Lattices

Let $L$ be a (real) Riesz space, equipped with a norm. The norm in $L$ is called a *Riesz norm* if $|f| \leq |g|$ in $L$ implies $\|f\| \leq \|g\|$. Note that this implies that for any $f \in L$ the elements $f$ and $|f|$ have the same norm. Any Riesz space equipped with a Riesz norm, is called a *normed Riesz space*.

**Definition 4.1.** If the normed Riesz space $L$ is a norm complete (i.e., if every norm Cauchy sequence has a norm limit), $L$ is called a *Banach lattice*. As a first result, observe immediately that any normed Riesz space is Archimedean. Indeed, if $0 \leq nu \leq v$ for $n = 1, 2, \ldots \in L$, then $\|u\| \leq n^{-1}\|v\|$ for $n = 1, 2 \ldots$ so $\|u\| = 0$ *i.e.*, $u = 0$.

Also, note already that if $f_n$ converges in norm to $f$, then $|f_n|$ converges in norm to $|f|$ (since $|\,|f| - |f_n|\,| \leq |f - f_n|$).

Let $X$ be compact Hausdorff space, $E$ be both Banach lattice and Riesz algebras. Let $C(X, E)$ denotes the space of all continuous functions from $X$ into $E$ endowed with the pointwise order,

$$f \leq g \Leftrightarrow f(x) \leq g(x) \;\; for \; all \; x \in X.$$

For each $f \in C(X, E)$, the norm $\|f\|$ of $f$ is defined by

$$\|f\| = \sup\{\|f(x)\| : x \in X\}.$$

Since $E$ is a ordered vector space, it follows that for each $x \in X$, $f(x) \leq g(x)$ implies $f(x) + h(x) \leq g(x) + h(x)$ in $E$ for all $f, g, h \in C(X, E)$ and also $f(x) \geq 0$ implies $\alpha f(x) \geq 0$ in $E$, for every real number $\alpha \geq 0$ and for all $f \in C(X, E)$. Then $f \leq g$ implies $f + h \leq g + h$ for all $f, g, h \in C(X, E)$ and $\alpha f \geq 0$ in $C(X, E)$, for every real number $\alpha \geq 0$ by pointwise addition and scaler multiplication. Hence $C(X, E)$ endowed with pointwise order is an ordered vector space.

Since $E$ is a lattice it follows from for any two elements $f(x), g(x) \in E$ that $f(x) \vee g(x)$ and $f(x) \wedge g(x)$ exist. Then $C(X, E)$ endowed with the pointwise order, $f \vee g$ and $f \wedge g$ exist in $C(X, E)$ for each $f, g \in C(X, E)$ which implies that $C(X, E)$ is a lattice. Hence $C(X, E)$ is a Riesz space.

For each $f, g \in C(X, E)$, by the pointwise order, $|f| \leq |g|$ implies $|f(x)| \leq |g(x)|$ for all $x \in X$. Since $E$ is normed Riesz space then $|f(x)| \leq |g(x)|$ for all $x \in X$ in $E$ implies $\|f(x)\| \leq \|g(x)\|$. It follows from the norm on $C(X, E)$ defined by $\|f\| = \sup\{\|f(x)\| : x \in X\}$ that $\|f(x)\| \leq \|g(x)\|$ for all $x \in X$. By the definition of the norm of $f$ on $C(X, E)$, $|f| \leq |g|$ implies $\|f\| \leq \|g\|$, therefore $C(X, E)$ with this norm is a normed Riesz space.

Now, we will show that $C(X, E)$ with this norm is a Banach lattice. Let $(f_n)$ be a norm Cauchy sequence in $C(X, E)$. Then, it is obvious that $(f_n(x))$ is a Cauchy sequence in $E$ for each $x \in X$. Since $E$ is complete, for each $x \in X$ there exists $f(x) \in E$ such that $f_n(x) \to f(x)$ in $E$. Hence we have defined a function $f : X \to E$. We will show that $f_n \to f$ in $C(X,E)$. First we will show that $f$ is continuous. Let $\epsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\| < \frac{\epsilon}{3}$$

for all $n, m \geq n_0$. Let $x_0 \in X$ be given. Since $f_{n_0}$ is continuous, then there exists $\delta > 0$ such that

$$\|x - x_0\| < \delta \;\; implies \;\; \|f_{n_0}(x) - f_{n_0}(x_0)\| < \frac{\epsilon}{3}$$

We obtain that

$$\begin{aligned}
\|f(x) - f(x_0)\| &= \|f(x) - f_{n_0}(x) + f_{n_0}(x) - f_{n_0}(x_0) + f_{n_0}(x_0) - f(x_0)\| \\
\|f(x) - f(x_0)\| &\leq \lim_{m\to\infty} \|f_m(x) - f_{n_0}(x)\| + \|f_{n_0}(x) - f_{n_0}(x_0)\| + \lim_{m\to\infty} \|f_{n_0}(x_0) - f_m(x_0)\| \\
\|f(x) - f(x_0)\| &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Hence $f$ is continuous. Now it remain to show that $f_n \to f$ in $C(X,E)$. Let $\epsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\| < \epsilon \;\; for\ all \;\; n, m \geq n_0.$$

To this end, we have

$$\begin{aligned}
& \|f_n(x) - f_m(x)\| \leq \|f_n - f_m\| < \epsilon \;\; for\ all \;\; n, m \geq n_0, \\
\Rightarrow \quad & \lim_{m\to\infty} \|f_n(x) - f_m(x)\| < \epsilon \;\; for\ all \;\; n, m \geq n_0, \;\; for\ all \;\; x \in X, \\
\Rightarrow \quad & \|f_n(x) - f(x)\| < \epsilon \;\; for\ all \;\; n \geq n_0, \;\; for\ all \;\; x \in X, \\
\Rightarrow \quad & \sup_{x\in X} \|f_n(x) - f(x)\| < \epsilon \;\; for\ all \;\; n \geq n_0, \\
\Rightarrow \quad & \|f_n - f\| < \epsilon \;\; for\ all \;\; n \geq n_0.
\end{aligned}$$

Hence $f_n \to f$ in $C(X,E)$. Therefore $C(X,E)$ with this norm is a Banach lattice. Moreover, for each pair $f, g \in C(X,E)$, define $f, g \in C(X,E)$ by the pointwise multiplication, i.e., $fg(x) = f(x)g(x)$ for all $x \in X$. Since $E$ is Riesz algebra, it follows that for every $f(x), g(x) \in E^+$ implies $f(x)g(x) \geq 0$ in $E$ and by the pointwise order, i.e., $f \geq 0$ implies $f(x) \geq 0$ for all $x \in X$. Then $C(X, E)$ with this multiplication is a Riesz algebra.

**Example 4.1.** Let $E$ be a normed Riesz space of all real continuous functions $f$ on $[0, 1]$ with norm $\|f\| = \int_0^1 |f(x)|dx$ and let the sequence $(f_n)$ in $E$ be such that $0 \leq (f_n(x)) \leq 1$ on $[0, 1]$ for all $n$, and furthermore

$$f_1(1/2) = 1 \textit{ and } \|f_1\| = 2^{-1},$$

$$f_2(1/3) = f_2(2/3) = 1 \textit{ and } \|f_2\| = 2^{-2},$$

generally

$$f_n = 1 \textit{ in } 1/(n+1), \ldots, n/(n+1) \textit{ and } \|f_n\| = 2^{-n}.$$

We see that $(f_n)$ converges to zero in norm, but there is no subsequence converging to zero in order. Hence $E$ is a normed Riesz space but not a Banach lattice.

**Theorem 4.1.** *(Banach-Stone)[2] Let $X$ and $Y$ be compact Hausdorff spaces. If $C(X)$ and $C(Y)$ are Riesz isomorphic, then $X$ and $Y$ are homeomorphic.*

**Proof.** Let $T$ be a Riesz isomorphism of $C(X)$ onto $C(Y)$, let $u = T\mathbf{1}_X$ and take $v \in C(X)$ such that $Tv = \mathbf{1}_Y$. There exists a positive number $c$ for which $v \leq c\mathbf{1}_X$. Then $\mathbf{1}_Y = Tv \leq cT\mathbf{1}_X = cu$, so $u(y) > 0$ for every $y \in Y$. The formula

$$Hf(y) = \frac{Tf(y)}{u(y)} \ \textit{ for all } f \in C(X), \ \textit{ for all } y \in Y$$

can be used to define a Riesz isomorphism $H$ of $C(X)$ onto $C(Y)$ such that $H\mathbf{1}_X = \mathbf{1}_Y$. By the corollary 2.1 there exists continuous $\sigma_1 : Y \to X$ and $\sigma_2 : X \to Y$ with the properties

$$Hf = f \circ \sigma_1 \ \textit{ for all } f \in C(X)$$

and

$$H^{-1}g = g \circ \sigma_2 \ \textit{ for all } g \in C(Y).$$

If $x \in X$, then for every $f \in C(X)$, $f(x) = H^{-1}Hf(x) = (f \circ (\sigma_1 \circ \sigma_2))(x)$, so (by Urysohn's Lemma) $x = (\sigma_1 \circ \sigma_2)(x)$. Similarly, $y = (\sigma_2 \circ \sigma_1)(y)$ for each $y \in Y$. Thus, $\sigma_1$ and $\sigma_2$ are each other's inverses. Then $X$ and $Y$ are homeomorphic. □

**Theorem 4.2.** *[2] Let X and Y be compact Hausdorff spaces. If $C(X)$ and $C(Y)$ are algebraically isomorphic, then X and Y are homeomorphic.*

**Proof.** Let $T$ be a linear bijection $C(X) \to C(Y)$ that preserves the multiplication. An element of $C(X)$ or of $C(Y)$ is $\geq 0$ if and only if it is square. Hence, for $f \in C(X)$ we have $f \geq 0$ if and only if $Tf \geq 0$. Therefore $T$ is Riesz isomorphism. Now apply theorem 4.1, then $X$ and $Y$ are homeomorphic. □

**Theorem 4.3.** *[2] Let X and Y be compact Hausdorff spaces. If $C(X)$ and $C(Y)$ are linearly isometric, then X and Y are homeomorphic.*

**Proof.** Let $T$ be a linear bijection $C(X) \to C(Y)$ that preserves the norm. We first make a linear isometry $H$ of $C(X)$ onto $C(Y)$ for which $H\mathbf{1}_X = \mathbf{1}_Y$. Let $u = T\mathbf{1}_X$. Then $\|u\|_\infty = 1$, so $-\mathbf{1}_Y \leq u \leq \mathbf{1}_Y$. Set $h = \frac{1}{2}(u^2 - \mathbf{1}_Y)$. Then we have $u + h = \frac{1}{2}(u + \mathbf{1}_Y)^2 - \mathbf{1}_Y$, hence $\|u + h\|_\infty \leq 1$ and $\|\mathbf{1}_X + T^{-1}(h)\|_\infty = \|T^{-1}(u + h)\|_\infty \leq 1$. Similarly, $u - h = \mathbf{1}_Y - \frac{1}{2}(u - \mathbf{1}_Y)^2$, $\|u - h\|_\infty \leq 1$ and $\|\mathbf{1}_X - T^{-1}(h)\|_\infty \leq 1$. However, if $\|\mathbf{1}_X + T^{-1}(h)\|_\infty \leq 1$ and $\|\mathbf{1}_X - T^{-1}(h)\|_\infty \leq 1$, then $T^{-1}(h) = 0$, so $h = 0$, $u^2 = \mathbf{1}_Y$ and the only values taken by $u$ are 1 and $-1$. Then we can define a linear isometry $H$ of $C(X)$ onto $C(Y)$ by

$$Hf(y) = \frac{Tf(y)}{u(y)} \quad for\ all\ f \in C(X),\ for\ all\ y \in Y$$

and we obtain $H\mathbf{1}_X = \mathbf{1}_Y$. For $f \in C(X)$ or $f \in C(Y)$ one has $f \geq 0$ if and only if

$$\|f - \|f\|_\infty \mathbf{1}_X\|_\infty \leq \|f\|_\infty$$

Hence, for $f \in C(X)$ one has $f \geq 0$ if and only if $Hf \geq 0$. Apparently, $H$ is a Riesz isomorphism and we can apply theorem 4.1 then $X$ and $Y$ are homeomorphic. □

We see that if the spaces of the of the real-valued continuous function $C(X)$ and

$C(Y)$ are linearly isometric then $X$ and $Y$ are homeomorphic. The conclusion is still valid if $C(X)$ and $C(Y)$ are algebraically isomorphic or Riesz isomorphic. If we weaken the compactness of $X$ and $Y$ to realcompactness, then $C(X)$ and $C(Y)$ being algebraically isomorphic or Riesz isomorphic implies that $X$ and $Y$ are homeomorphic [6, 9, 12]. Furthermore, if we weaken the algebraic isomorphism or Riesz isomorphism to the biseparating map, the authors in [11] proved that if there exists a biseparating map $T$ from $C(X)$ onto $C(Y)$ then $vX$ is homeomorphic to $vY$ (where $vX$ and $vY$ are respectively the realcompactification of completely regular spaces $X$ and $Y$).

**Definition 4.2.** Let $X$ and $Y$ be compact Hausdorff space, $E$ and $F$ be both Banach lattices and Riesz algebras. Let $C(X, E)$ denotes the space of all continuous functions from $X$ into $E$ endowed with the pointwise order,

$$f \leq g \Leftrightarrow f(x) \leq g(x) \;\; for\ all\ x \in X.$$

For $f \in C(X, E)$, the function $f$ is said to *have no zero* if $f(x) \neq 0$ for every $x \in X$. A non-zero element $a \in F$ is called a $left\ zero\!-\!divisor$ (respectively $right\ zero\!-\!divisor$) if $ab = 0$ (respectively $ba = 0$) for some $b \neq 0$ in $F$. Then $F$ is said to *have no* $zero - divisor$ if it has neither left nor right zero-divisors.

**Definition 4.3.** Let $X$ and $Y$ be compact Housdorff spaces, $E$ and $F$ be both Banach lattices and Riesz algebras. A linear map $H : C(X, E) \rightarrow C(Y, F)$ is called $separating$ if $f(x)g(x) = 0$ for all $x \in X$ implies $Hf(y)Hg(y) = 0$ for all $y \in Y$. A linear bijective map $H : C(X, E) \rightarrow C(Y, F)$ is called $biseparating$ if both $H$ and $H^{-1}$ are separating.

**Theorem 4.4.** *[8] For compact Hausdorff spaces $X$, $Y$ and Banach spaces $E$, $F$, a linear isometry $H : C(X, E) \rightarrow C(Y, F)$ is a strong Banach-Stone map (i.e., there exist a homeomorphism $g : Y \rightarrow X$ and a map $J : Y \rightarrow L(E, F)$ such that $Jy$ is an isometry and $Hf(y) = Jy[f(g(y))]$ for each $y$ in $Y$ and each $f$ in $C(X, E)$) if and only if $H$ is biseparating.*

**Theorem 4.5.** *[10] For compact Hausdorff spaces $X$, $Y$ and a Banach lattice $E$, if there exists a Riesz isomorphism $T : C(X, E) \to C(Y, \mathbb{R})$ such that $Tf$ has no zero if $f$ has none, then $X$ is homeomorphic to $Y$ and $E$ is Riesz isomorphic to $\mathbb{R}$.*

The following two sections are motivated by the following two problems. First, since there is no algebraic structure defined in a general Banach space, it is meaningless to talk about the notion of biseparating maps in Theorem 4.4. Second, it is impossible to extend $\mathbb{R}$ in Theorem 4.5 to a general Banach lattice $F$ without extra conditions. Both these two problems suggest that more structures are needed in order to obtain a general vector (or lattice) version of the Banach-Stone theorem.

## 4.2 Supports for Riesz algebraic homomorphisms

Let $X$ and $Y$ be two compact Hausdorff spaces and let $E$, $F$ be both Banach lattices and Riesz algebras. The constant function $\mathbf{1}_X$ on $X$ is defined by $\mathbf{1}_X(x) = 1$ for all $x \in X$. For any $h \in C(X)$ and any $u \in E$, let $h \otimes u \in C(X, E)$ be defined by $(h \otimes u)(x) = h(x)u$ for each $x \in X$.

**Definition 4.4.** Let $X$, $Y$ be compact Hausdorff spaces, and let $E$, $F$ be both Banach lattices and Riesz algebras. An element $a$ in $X$ is said to be *support* for a Riesz algebra homomorphism $T : C(X, E) \to F$ if $Tf = T(\mathbf{1}_X \otimes f(a))$ for all $f \in C(X, E)$.

**Theorem 4.6.** *Let $X$ be compact Hausdorff space, and let $E$, $F$ be both Banach lattices and Riesz algebras. If $F$ has no zero-divisor and $T : C(X, E) \to F$ is a non-trivial Riesz algebraic homomorphism, then $T$ has a unique support.*

**Proof.** For this Riesz algebraic homomorphism $T : C(X, E) \to F$, let

$$A = \{a \in X : Tf = T(\mathbf{1}_X \otimes f(a)) \; for \; all \; f \in C(X, E)\}.$$

It is clear that each element in $A$ is a support for the Riesz algebraic homomorphism $T$. We shall show that $A$ is a singleton.

First, suppose that $A = \emptyset$. Then for every point $a \in X$ there exists an $f_a \in C(X, E)$ such that $Tf_a \neq T(\mathbf{1}_X \otimes f(a))$. Let $g_a = |f_a - \mathbf{1}_X \otimes f_a(a)|$. Then $g_a \in C(X, E)$, $g_a(a) = 0$ and $Tg_a > 0$. Since $T$ is a positive operator between two Banach lattices $C(X, E)$ and $F$, by [7], (Proposition 1.3.5), $T$ is norm-continuous. Thus, there exists an $\epsilon > 0$ such that the open ball $B(g_a, \epsilon) \subseteq C(X, E)\backslash T^{-1}(0)$. Since the function $x \mapsto \|g_a(x)\|$ is continuous on $X$ and $\|g_a(a)\| = 0$, there are open neighborhoods $U_a$ and $V_a$ of $a$ such that $a \in clU_a \subseteq V_a$ and $\|g_a(x)\| < \frac{\epsilon}{2}$ for all $x \in V_a$. By Urysohn's lemma, there exists a continuous function $h_a : X \to [0, 1]$ such that $h_a(clU_a) = \{0\}$ and $h_a(X\backslash V_a) = 1$. Define $h_a g_a \in C(X, E)$ by letting $(h_a g_a)(x) = h_a(x)g_a(x)$ for all $x \in X$. It follows from

$$\|g_a - h_a g_a\| = \sup_{x\in X} \|g_a(x) - (h_a g_a)(x)\| = \sup_{x\in V_a} \|g_a(x) - (h_a g_a)(x)\| \leq \frac{\epsilon}{2}$$

that $h_a g_a \in B(g_a, \epsilon)$, and hence $T(h_a g_a) > 0$. By the compactness of $X$, the open cover $\{U_a : a \in X\}$ of $X$ has a finite subcover $\{U_{a_1}, U_{a_2}, \ldots, U_{a_n}\}$. Let

$$hg = (h_{a_1} g_{a_1})(h_{a_2} g_{a_2}) \ldots (h_{a_n} g_{a_n}).$$

Then $hg \equiv 0$ on $X$. But, $T(h_{a_i} g_{a_i}) > 0$ for each $i = 1, 2, \ldots, n$, and $F$ has no zero-divisor, which implies that

$$T(hg) = T(h_{a_1} g_{a_1})T(h_{a_2} g_{a_2}) \ldots T(h_{a_n} g_{a_n}) \neq 0.$$

A contradiction occurs. Therefore, we deduce $A \neq \emptyset$.

Next we show that $A$ contains exactly one element. Suppose that $A$ contains at least two distinct points. Let $a_1, a_2 \in A$ with $a_1 \neq a_2$. Then there exists closed neighborhood $V_i$ of $a_i$ and open neighborhood $U_i$ of $a_i$ $(i = 1, 2,)$ such that $V_i \subseteq U_i$ $(i = 1, 2)$ and $U_1 \cap U_2 = \emptyset$. Again, by Urysohn's lemma, for each $i = 1, 2$ there exists a continuous function $f_i : X \to [0, 1]$ such that

$$f_i(V_i) = \{0\} \textit{ and } f_i(X\backslash U_i) = \{1\}.$$

Since $T$ is non-trivial, by the definition of a support, there must exist some $u \in E$ such that $u \neq 0$ and $T(\mathbf{1}_X \otimes u) \neq 0$. Observe that $(f_i \otimes u)(V_i) = \{0\}$ $(i = 1, 2)$. Then $T(f_i \otimes u) = T(\mathbf{1}_X \otimes (f_i \otimes u)(a_i)) = 0$ $(i = 1, 2)$. It follows from $f_1 \vee f_2 = \mathbf{1}_X$ that $T(\mathbf{1}_X \otimes u) = T((f_1 \vee f_2) \otimes u) = T(f_1 \otimes u \vee f_2 \otimes u) = T(f_1 \otimes u) \vee T(f_2 \otimes u) = 0$, which is also contradiction. Hence, we have shown that $T$ has a unique support. □

## 4.3 A Banach-Stone theorem

In this section, by applying a method similar to that in [10], we shall establish a Banach-Stone theorem for Riesz algebras. Let $X$, $Y$ be compact Hausdorff spaces, and let $E$, $F$ be both Banach lattices and Riesz algebras. For each $y \in Y$, let $\hat{y} : C(Y, F) \to F$ be the evaluation map defined by $\hat{y}(g) = g(y)$ for all $g \in C(Y, F)$. It is easy to see that if $T : C(X, E) \to C(Y, F)$ is a Riesz algebraic homomorphism, then for each $y \in Y$, $\hat{y} \circ T : C(X, E) \to F$ is also a Riesz algebraic homomorphism. For each $u \in E$, let

$$\hat{T}(y)(u) = \hat{y} \circ T(\mathbf{1}_X \otimes u).$$

Then $\hat{T}(y)$ is a linear operator from $E$ into $F$ for any $y \in Y$. If $T : C(X, E) \to C(Y, F)$ is a Riesz algebraic homomorphism such that $Tf$ has no zero if $f$ has none, then $\hat{y} \circ T$ is a Riesz algebraic homomorphism from $C(X, E)$ into $F$ for any $y \in Y$ such that $\hat{y} \circ T(1_X \otimes e) \neq 0$ for all $e \neq 0$. Furthermore, if $F$ has no zero-divisor, then it follows from Theorem 4.6 that for each $y \in Y$, $\hat{y} \circ T$ has a unique support $x_y \in X$ such that

$$\hat{y} \circ Tf = \hat{y} \circ T(\mathbf{1}_X \otimes f(x_y))$$

for all $f \in C(X, E)$. In this case, we have the following definition.

**Definition 4.5.** Let $X$, $Y$ and $T$ be the same as above. Assume that $F$ has no zero-divisor. The map $\sigma : Y \to X$ defined by $\sigma(y) = x_y$ for each $y \in Y$ (where $x_y$ is the unique support of $\hat{y} \circ T$) is called the *associate map* of $T$.

**Lemma 4.1.** *Let $X$, $Y$ be compact Hausdorff spaces, and let $E$, $F$ be both Banach lattices and Riesz algebras. If $F$ has no zero-divisor and $T : C(X, E) \to C(Y, F)$ is a surjective Riesz algebraic homomorphism such that $Tf$ has no zero if $f$ has none. Then $\hat{T}(y)$ is a Riesz algebraic isomorphism of $E$ onto $F$ for each $y \in Y$.*

**Proof.** Let $y \in Y$. It is obvious that $\hat{T}(y) : E \to F$ is a Riesz algebraic homomorphism. Thus, we only need to prove that $\hat{T}(y)$ is bijective. To do this, select any point $w \in E$ with $w \neq 0$. Since $\hat{T}(y)(w) = \hat{y} \circ T(\mathbf{1}_X \otimes w) = T(\mathbf{1}_X \otimes w)(y) \neq 0$, then $\hat{T}(y)$ is injective. Now, let $v \in F$. Since $T$ is surjective, then there exists an $f \in C(X, E)$ such that $Tf = v\mathbf{1}_Y$. Let $u = f \circ \sigma(y)$. It follows that

$$\hat{T}(y)(u) = \hat{y} \circ T(\mathbf{1}_X \otimes u) = \hat{y} \circ T(\mathbf{1}_X \otimes f(x_y)) = \hat{y} \circ Tf = \hat{y} \circ v\mathbf{1}_Y = v\mathbf{1}_Y(y) = v,$$

which implies that $\hat{T}(y)$ is surjective. □

**Lemma 4.2.** *Let $X$, $Y$ be compact Hausdorff spaces, and let $E$, $F$ be both Banach lattices and Riesz algebras. If $F$ has no zero-divisor and $T : C(X, E) \to C(Y, F)$ is a surjective Riesz algebraic homomorphism such that $Tf$ has no zero if $f$ has none, then the associate map $\sigma : Y \to X$ of $T$ is continuous.*

**Proof.** Let $U \subseteq X$ be an open set and $y_0 \in \sigma^{-1}(U)$. By Urysohn's lemma, there exists a continuous function $h : X \to [0, 1]$ satisfying

$$h(\sigma(y_0)) = 1 \text{ and } h(X \backslash U) = \{0\}.$$

Choose $u \in E$ such that $u \neq 0$. Then $(\mathbf{1}_X \otimes u)(x) \neq 0$ for every $x \in X$. By hypothesis, $T(\mathbf{1}_X \otimes u)(y_0) \neq 0$. Let $g = T(h \otimes u)$. Then

$$g(y_0) = T(h \otimes u)(y_0) = \hat{y}_0 \circ T(\mathbf{1}_X \otimes u) \neq 0.$$

Let $V = \{y \in Y : \|g(y)\| > 0\}$. Then $V \subseteq Y$ is an open subset and $y_0 \in V$. Next, we show $V \subseteq \sigma^{-1}(U)$. Suppose $y \notin \sigma^{-1}(U)$. Then $\sigma(y) \notin U$, and so $h(\sigma(y)) = 0$. We have

$$g(y) = T(h \otimes u)(y) = \hat{y} \circ T(\mathbf{1}_X \otimes ((h \otimes u) \circ \sigma(y))) = 0.$$

This implies that $y \notin V$. Therefore, $\sigma$ is continuous. □

Now, we are ready to present the main theorem of this thesis.

**Theorem 4.7.** *Let $X, Y$ be compact Hausdorff spaces, and let $E, F$ be both Banach lattices and Riesz algebras. If $F$ has no zero-divisor and $T : C(X, E) \to C(Y, F)$ is a Riesz algebraic isomorphism such that $Tf$ has no zero if $f$ has none, then $X$ and $Y$ are homeomorphic, $E$ and $F$ are Riesz algebraically isomorphic. More precisely, for any $y \in Y$, $\hat{T}(y)$ is a Riesz algebraic isomorphism from $E$ onto $F$ and the associate map $\sigma$ of $T$ is a homeomorphism from $Y$ onto $X$.*

**Proof.** By Lemma 4.1, for any $y \in Y$, $\hat{T}(y)$ is a Riesz algebraic isomorphism from $E$ onto $F$. Let $\sigma : Y \to X$ be the associate map of $T$. We first show that $\sigma$ is surjective. If there is some $x_0 \in X$ such that $\sigma(y) \neq x_0$ for all $y \in Y$, then for each $y \in Y$, there exists an open neighborhood $U_y$ of $\sigma(y)$ such that $x_0 \notin clU_y$. By Lemma 4.2, $\sigma$ is continuous, thus $\{\sigma^{-1}(U_y) : y \in Y\}$ is an open cover of $Y$. By the compactness of the space $Y$, there is a finite subset $\{y_1, y_2, \ldots, y_n\}$ of $Y$ such that $Y = \bigcup_{i=1}^{n} \sigma^{-1}(U_{y_i})$. Since $x_0 \notin clU_{y_i}$, there is a continuous function $f_i : X \to [0, 1]$ satisfying $f_i(x_0) = 1$ and $f_i(clU_{y_i}) = \{0\}$ $(1 \leq i \leq n)$. Let $f = \bigwedge_{i=1}^{n} f_i$. Then $f(clU_{y_i}) = \{0\}$ for each $1 \leq i \leq n$ and $f(x_0) = 1$. Take $u \in E$ with $u \neq 0$ and let $h = f \otimes u$. Now consider $Th$. For any $y \in Y$, $\sigma(y) \in U_{y_i}$ for some $1 \leq i \leq n$. Then

$$Th(y) = \hat{y} \circ Th = \hat{y} \circ T(\mathbf{1}_X \otimes h \circ \sigma(y)) = \hat{T}(y)(h \circ \sigma(y)) = \hat{T}(y)(0) = 0.$$

Thus, $Th = 0$. Since $T$ is an isomorphism, $h \equiv 0$. But $h(x_0) = u \neq 0$, which is a contradiction.

Next we show that $T^{-1}g$ has no zero if $g$ has none. Assume that $g \in C(Y, F)$ has no zero, but $f = T^{-1}g$ has a zero. Then there is some point $x_0 \in X$ such that $f(x_0) = 0$.

Since $\sigma$ is surjective, we can take $y_0 \in Y$ with $\sigma(y_0) = x_0$. It follows that

$$g(y_0) = Tf(y_0) = \hat{y}_0 \circ Tf = \hat{y}_0 \circ T(\mathbf{1}_X \otimes f \circ \sigma(y_0)) = \hat{T}(y_0)(f \circ \sigma(y_0)) = 0.$$

However, $g(y_0) \neq 0$, since $g$ has no zero. This also leads to a contradiction.

Let $\omega : X \to Y$ be the associate map of $T^{-1}$. To complete the proof, it suffices to show that $\mathbf{I}_X = \sigma \circ \omega$ and $\mathbf{I}_Y = \omega \circ \sigma$, where $\mathbf{I}_X, \mathbf{I}_Y$ are the identity maps on $X$ and $Y$, respectively. Suppose that there exists some point $z \in X$ with $z \neq \sigma \circ \omega(z)$. Then there exists a function $f \in C(X, E)$ such that $f(z) = 0$ and $f(\sigma \circ \omega(z)) \neq 0$. Let $k = \omega(z)$ then

$$\hat{T}(k)(f \circ \sigma(k)) = \hat{k} \circ T(\mathbf{1}_X \otimes f \circ \sigma(k)) = \hat{k} \circ Tf = Tf(k)$$

and

$$\widehat{T^{-1}}(z) \circ Tf(k) = \hat{z} \circ T^{-1}(\mathbf{1}_Y \otimes Tf(k)) = \hat{z} \circ T^{-1}(Tf) = T^{-1}(Tf)(z) = f(z) = 0.$$

Then

$$\widehat{T^{-1}}(z) \circ \hat{T}(\omega(z))(f(\sigma \circ \omega(z))) = T^{-1}(Tf)(z) = f(z) = 0.$$

By Lemma 4.1, $\widehat{T^{-1}}(z)$ and $\hat{T}(\omega(z))$ are Riesz algebraic isomorphisms between $E$ and $F$, and $f(\sigma \circ \omega(z)) \neq 0$. Then, we have

$$\widehat{T^{-1}}(z) \circ \hat{T}(\omega(z))(f(\sigma \circ \omega(z))) \neq 0,$$

which is a contradiction. Similarly, we can show $\mathbf{I}_Y = \omega \circ \sigma$. Thus the associate map $\sigma$ of $T$ is a homeomorphism from $Y$ onto $X$. □

The following example (similar to [8], (Example 4.2)) shows that for a Riesz algebraic isomorphism $T : C(X, E) \to C(Y, F)$, $X$ and $Y$ may not be homeomorphic if either $F$ has a zero-divisor or $f$ has no zero but $Tf$ has zero.

**Example 4.2.** Let $X = \{0\}$, $E = \mathbb{R}^2$ with sup norm and the product of two vectors $(a, b)$ and $(c, d)$ in $E$ be defined by $(a, b)(c, d) = (ac, bd)$, $Y = \{0, 1\}$ equipped the discrete topology and $F = \mathbb{R}$. Then $C(X, \mathbb{R}^2)$ is both Riesz algebraically and isometrically isomorphic to $\mathbb{R}^2$. Define a surjection $T : C(X, \mathbb{R}^2) \to C(Y, \mathbb{R})$ by

$$T(a,b)(y) = \begin{cases} a, & \text{if y=0,} \\ b, & \text{if y=1,} \end{cases}$$

for all $(a,b) \in \mathbb{R}^2$. It is obvious that $T$ is a linear isometry. Obviously, $T$ is a Riesz isomorphism. Since for any $(a,b)$ and $(c,d)$ in $C(X,E)$,

$$T[(a,b)(c,d)](y) = \begin{cases} ac, & \text{if y=0,} \\ bd, & \text{if y=1,} \end{cases} = T(a,b)(y)T(c,d)(y) = [T(a,b)T(c,d)](y),$$

then $T$ is a Riesz algebraic isomorphism. Note that $(a,0)$ is not a zero element whenever $a \neq 0$, but $T(a,0)$ has a zero. Thus, $T$ does not satisfy the condition of Theorem 4.7. On the other hand, $T^{-1} : C(Y,F) \to C(X,E)$ is also a Riesz algebraic isomorphism. For any $f \in C(Y,F)$, $T^{-1}f = (f(0), f(1))$. For any $a \neq 0$ and $d \neq 0$, both $(a,0)$ and $(0,d)$ are not zero elements. But the product $(a,0)(0,d) = (0,0)$. Therefore, $T^{-1}$ does not satisfy the condition of Theorem 4.7 either. Finally, it is evident that $X$ and $Y$ are not homeomorphic, $E$ and $F$ are not Riesz algebraically isomorphic.

**Remark 4.1.** Note that $X, Y, E$ and $F$ in Example 4.2 are the same as Example 4.2 in [8]. From Example 4.2, we can see that Example 4.2 in [8] as well as Theorem 4.1 in [8] are not correct. That is, the Riesz algebraic isomorphism $T$, which is also a linear isometry, is biseparating, but $X$ and $Y$ are not homeomorphic. In fact, for any $(a,b)$ and $(c,d)$ in $C(X,E)$, if the product $(a,b)(c,d) = (ac, bd) = 0$, then

$$T(a,b)T(c,d)(y) = T(a,b)(y)T(c,d)(y) = \begin{cases} ac = 0, & \text{if y=0,} \\ bd = 0, & \text{if y=1.} \end{cases}$$

Therefore, $T(a,b)T(c,d) = 0$. This implies that $T$ is separating. On the other hand, for any $f$ and $g$ in $C(Y,F)$, there exist unique $(a,b)$ and $(c,d)$ in $C(X,E)$ such that

$T(a,b) = f$ and $T(c,d) = g$. If $fg = 0$, then

$$ac = T(a,b)(0)T(c,d)(0) = T(a,b)T(c,d)(0) = fg(0) = 0,$$

and similarly $bd = fg(1) = 0$. Therefore, $T^{-1}(f)T^{-1}(g) = (a,b)(c,d) = (ac,bd) = 0$. This implies that $T^{-1}$ is separating.

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