GRÖBNER-SHIRSHOV BASES AND REDUCED FORMS FOR AFFINE WEYL GROUPS

by

UĞUR USTAOĞLU

THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE ABANT İZZET BAYSAL UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF MATHEMATICS

DECEMBER 2012

Approval of the Graduate School of Natural and Applied Sciences

Prof. Dr. Yaşar Dürüst
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Zafer Ercan
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Erol Yılmaz
Supervisor

Examining Committee Members

Prof. Dr. Cenap Özel	(AİBÜ)	
Assist. Prof. Dr. Seyhun Kesim	(BEÜN)	
Assist. Prof. Dr. Erol Yılmaz	(AİBÜ)	

ABSTRACT

GRÖBNER-SHIRSHOV BASES AND REDUCED FORMS FOR AFFINE WEYL GROUPS

USTAOĞLU, UĞUR
M.Sc., Department of Mathematics
Supervisor: Assist. Prof. Dr. Erol Yılmaz

December 2012, 52 pages

Gröbner-Shirshov basis and reduced forms for the elements in the affine Weyl group of type $\widetilde{A_n}$ are presented. Starting from the standard defining relations R of $\widetilde{A_n}$, a new set of relations R' is founded. Then, by using the algorithm of elimination of leading words with respect to the relations in R', all the forms in the affine Weyl group $\widetilde{A_n}$ are reduced to an explicit list of forms. Using combinatorial techniques, the number of forms in such a list is found to be equal to the number of elements in $\widetilde{A_n}$. Therefore, by the Composition-Diamond Lemma, the relations in R' are Gröbner-Shirshov basis for the affine Weyl group $\widetilde{A_n}$.

Keywords: Affine Weyl Groups, Composition-Diamond Lemma, Gröbner-Shirshov Basis, q-binomials and Basic Partitions.

ÖZET

AFFINE WEYL GRUPLARI İÇİN GRÖBNER-SHIRSHOV TABANLARI VE İNDİRGENMİŞ FORMLARI

UĞUR USTAOĞLU
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi: Yrd. Doç. Dr. Erol Yılmaz

Aralık 2012, 52 sayfa

$\widetilde{A_n}$ tipindeki affine Weyl grubu için Gröbner-Shirshov tabanları ve indirgenmiş formları gösterilmiştir. $\widetilde{A_n}$'nin standart tanımlanan bağıntıları olan R'den başlanarak, yeni bir bağıntı kümesi olan R' bulunur. Sonrasında, R' içindeki bağıntılara göre önde gelen kelimelerinin eliminasyonlarının algoritması kullanılarak, $\widetilde{A_n}$ tipi affine Weyl gruplarının içerisindeki tüm kelimelerin belirgin listeleri indirgenir. Kombinatoryal tekniği kullanılarak, listedeki gibi kelimelerin sayısı $\widetilde{A_n}$'nin elemanlarının sayısına eşit olması için bulunur. Böylece Composition-Dimanomd teoremi sayesinde, R' içerisindeki bağıntılar $\widetilde{A_n}$ tipindeki affine Weyl grubu için Gröbner-Shirshov tabanıdır.

Anahtar Kelimeler: Affine Weyl Grupları, Composition-Diamond Teoremi, Gröbner-Shirshov Tabanı, q-Binomialları ve Temel Bölümleri.

To my family and my supervisor

ACKNOWLEDGEMENTS

I would like to express sincere appreciation to my supervisor, Assist. Prof. Dr. Erol Yılmaz for his motivation, helpful discussions, encouragement, patience and constant guidance during this work. I would also like to thank other examining committee members for their suggestions.

I am grateful to express my appreciation to my family, relatives and friends for their continuous helps.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Gröbner and Gröbner-Shirshov bases theories are generating increasing interest because of its usefulness in providing computational tools and in giving algebraical structures which are applicable to a wide range of problems in mathematics, science, engineering, and computer science. In general, Gröbner-Shirshov bases theory is a powerful tool to solve the following classical problems.

(i) reduced form;

(ii) word problem;

(iii) rewriting system;

(iv) embedding theorems;

(v) extentions;

(vi) growth function...

The Gröbner basis theory for commutative algebras, introduced by Buchberger [13], provides a solution to the reduction problem for commutative algebras. More precisely, it gives an effective algorithm of computing a set of generators for a given ideal of a commutative ring which can be used to determine the reduced elements with respect to the relations given by the ideal. In [8], Bergman generalized the Gröbner basis theory to associative algebras by proving the Diamond Lemma. On the other hand, the parallel theory of Gröbner bases was developed for Lie algebras by Shirshov [15]. The key ingredient of the theory is called Composition Lemma which characterizes the leading terms of elements in the given ideal. In [2], Bokut noticed that Shirshov's method works for associative algebras as well, and for this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called the Gröbner-Shirshov basis theory.

In particular, the technique of Gröbner-Shirshov bases is proved to be very useful in the study of presentations of associative algebras, Lie algebras, semigroups, groups, etc. by generators and defining relations, see, for example, the book [4] by L. A. Bokut and G. Kukin, survey papers [[5], [6]] by L. A. Bokut and P. Kolesnikov, and [7] by L. A. Bokut and Yuqun Chen.

This thesis deals with Gröbner-Shirshov bases theory for affine Weyl groups. Gröbner-Shirshov bases and reduced form of the elements were already found for the Coxeter groups of type A_n, B_n and D_n in [1]. They also proposed a conjecture for the general form of Gröbner-Shirshov bases for all Coxeter groups. In [9], an example was given to show that the conjecture is not true in general. The Gröbner-Shirshov bases of the other finite Coxeter groups are given in [10] and [16]. This thesis is another example of finding Gröbner-Shirshov bases for groups, defined by generators and defining relations.

CHAPTER 2

RELATIONS AND WEYL GROUPS

2.1 Coxeter Groups

Let us define that W is a group and S is a set of distinguish generators of W of order 2. For s and $s^{'}$ belong to S. The order of the product $ss^{'}$ is denoted by $m_{s,s'} \in \{1, 2, \ldots\} \bigcup \{\infty\}$.

Definition 2.1. (Coxeter System) We define that the pair (W, S) is a Coxeter System if W has a presentation by generators and relations given by the set of generators S and the relations:

$$s^2 = 1 \quad for \quad s \in S,$$

$ss^{'}ss^{'} \ldots = s^{'}ss^{'}s \ldots$ which are braid relations for $s, s^{'} \in S$ such that $m_{s,s'}$ is finite. Afterwards, we can also say that W is a Coxeter Group.

The rank of the system is the cardinality of S.The matrix of the Coxeter system (W, S) is $(m_{s,s'})$ where $s, s^{'} \in S$; it has values in $\{1, 2, \ldots\} \bigcup \{\infty\}$.

Definition 2.2. (Coxeter Matrix) The coxeter matrix is $n \times n$ symmetric matrix with diagonal entries 1 and off-diagonal entries are greater than and equal to 2.

Example 2.1. $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 & 2 & 3 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 3 & 2 & 3 & 1 \end{pmatrix}$ are Coxeter Matrix.

Every Coxeter matrix is the matrix of a Coxeter system.

Definition 2.3. (Coxeter Graph) The coxeter graph of (W, S) is the graph with set of vertices S and edges $\{s, s^{'}\}$ when $m_{s,s'} > 2$. Furthermore, the edge is then labelled by $m_{s,s'}$.

Proposition 2.1. *If*

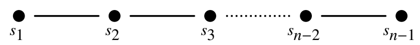
$$W = \prod_{\alpha} W_{\alpha}$$

then its Coxeter graph Γ_W *is the disjoint union*

$$\coprod_{\alpha} \Gamma_{W_{\alpha}}.$$

In particular, a Coxeter group W is irreducible if and only if Γ_W *is connected.*

Example 2.2. The symmetric group $\sigma_n = \sigma(\{1, 2, \ldots, n\})$. Let $s_i = (i, i + 1)$ and $S_{\sigma_n} = \{s_1, s_2, \ldots, s_{n-1}\}$. Then (σ_n, S_{σ_n}) is a Coxeter system of type A_{n-1} with graph



(The label is omitted when it is 3.) The rank of the system is $n - 1$.

Theorem 2.1. *[17] Let* (W, S) *be a Coxeter system,* $S^{'}$ *a subset of* S *and* $W^{'}$ *the subgroup of* W *generated by* $S^{'}$*. Then* $(W^{'}, S^{'})$ *is a Coexeter system with Coxeter matrix the submatrix of the Coxeter matrix of* (W, S) *given by* $S^{'}$*.*

Lemma 2.1. *[18] A Coxeter system is irreducible if its associated graph is connected. For instance, the system in example 2.2 is irreducible.*

If S is the disjoint union of two subsets S_1 and S_2, and no vertex of S_1 is connected to a vertex of S_2, then $W = W_1 \times W_2$, where W_i is the subgroup of W generated by S_i.

Definition 2.4. (Length Function) If (W, S) is a Coxeter system, the length of $w \in W$, denoted by $l(w)$, is the smallest integer m such that w is the product of m elements of S.

Definition 2.5. (Reduced Decomposition) If m equals $l(w)$, a decomposition $w = s_1 s_2 \dots s_m$ with $s_1, s_2, \dots, s_m \in S$ is reduced.

Proposition 2.2. *[19]*

(i) $l(w) = l(w^{-1})$. *[If $w = s_1 s_2 \dots s_p$, $w^{-1} = s_p s_{p-1} \dots s_1$ so $l(w^{-1}) \leq l(w)$, and similarly for w^{-1} in place of w]*

(ii) $l(w) = 1$ *if and only if* $w \in S$

(iii) $l(ww^{'}) \leq l(w) + l(w')$. *[If $w = s_1 s_2 \dots s_p$ and $w = s_1^{'} s_2^{'} \dots s_q^{'}$, then the product $ww^{'} = w = s_1 s_2 \dots s_p s_1^{'} s_2^{'} \dots s_q^{'}$ has length at most $p + q$]*

(iv) $l(ww^{'}) \geq l(w) - l(w^{'})$. *[Applying (iii) to pair $ww^{'}, (w^{'})^{-1}$, then use (i)]*

(v) $l(w) - 1 \leq l(ws) \leq l(w) + 1$ *for* $s \in S$ *and* $w \in W$. *[Use (iii) and (iv)]*

Theorem 2.2. *[18] Let $w = s_1 s_2 \dots s_m$ with $s_1, s_2, \dots, s_m \in S$. Then there is a subset $I = \{i_1 < i_2 < \dots < i_p$ of $\{1, 2, \dots, m\}$ with $p = l(w)$ elements such that $w = s_{i_1} s_{i_2} \dots s_{i_p}$.*

The above theorem is a result of the exchange lemma of [17]

Lemma 2.2. *[18] Let $w = s_1 s_2 \dots s_m$ be a reduced decomposition where $s_1, s_2, \dots, s_m$ belong to S. Let $s \in S$, then one of the following claim holds;*

(i) $l(sw) = l(w) + 1$ *and $s s_1 s_2 \dots s_m$ is a reduced decomposition of sw.*

(ii) $l(sw) = l(w) - 1$ *and $\exists j \in \{1, 2, \dots, m\}$ such that $s_1 \dots s_{j-1} s_{j+1} \dots s_m$ is a reduced decomposition of sw and $s s_1 \dots s_{j-1} s_{j+1} \dots s_m$ is reduced decomposition of w.*

2.2 Real Reflection Groups

Definition 2.6. (General Linear Group of a Vector Space ($GL(V)$)) If V is a vector space over $\mathbb{F}$, the general linear group of V, written as $GL(V)$ or $Aut(V)$, is the group of all automorphisms of V, i.e. the set of all bijective linear transformations $V \rightarrow V$, together with functional compositions as a group operation. If V has finite dimension n, then $GL(V)$ and $GL(n, \mathbb{F})$ are isomorphic. The isomorphism is not canonical; it depends on a choice of basis in V. Given a basis $(\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n)$ of V and an automorphism

$$T\mathbf{e}_k = \sum_{j=1}^{n} a_{jk}\mathbf{e}_j$$

for some constants a_{jk} in $\mathbb{F}$, the matrix corresponding to T is then just the matrix with entries given by the a_{jk}.

Let V be a finite dimensional real vector space. A reflection of V is an automorphism of order 2 whose set of fixed points is an hyperplane. A finite reflection group W in V is a finite subgroup of $GL(V)$ generated by reflections.

Definition 2.7. (Crystallographic) If there is $W - invariant\ Z - lattice$ of V, i.e, if there exists a free $Z - submodule\ L$ of V stable under W such that the canonical map $L \bigotimes_Z R \to V$ is an isomorphism, then the group W is called crystallographic.

Definition 2.8. (Chamber) Let A be the set of reflecting hyperplanes of W. Then $V - \bigcap_{H \in A} H$ is in general non-connected: its connected components are the chambers of W.

Theorem 2.3. *[17] The group W acts simply transitively on the set of chambers; the closure of chamber is a fundamental domain for the action of W on V.*

Theorem 2.4. *[17] The pair (W, S) is a Coxeter System.*

2.3 Coxeter Groups as Reflection Groups

We define a set S and a Coxeter matrix $M = (m_{s,s'})$ where $s, s^{'} \in S$. Let $V = \mathbb{R}^S$ and denote by $\{e_s\}$ where $s, s^{'} \in S$ its canonical basis. Let us define a bilinear form B_M on V by

$$B_M(e_s, e_{s'}) = -cos\frac{\pi}{m_{s,s'}}$$

and we know that $B_M(s_s, e_{s'}) = -cos\frac{\pi}{m_{s,s'}} = -cos\frac{\pi}{1} = 1$. Let ρ_s be the reflection in V given by

$$\begin{aligned} \rho_s(x) &= x - 2B_M(e_s, x)e_s \\ &= x + 2cos\frac{\pi}{m_{s,s'} e_s} \end{aligned}$$

One has $V = \mathbb{R}e_s \oplus H_s$, where H_s is the hyperplane orthogonal to e_s.

Let W be the group with set of generators S and relations

$$s^2 = 1$$

$ss^{'}ss^{'} \ldots = s^{'}ss^{'}s \ldots$ for those $s, s^{'} \in S$ such that $m_{s,s'} \neq \infty$.

Lemma 2.3. *For any $s \in S$, ρ_s is a reflection that preserves the bilinear form $B_M(.,.)$.*

Proof.

$$\begin{aligned} B_M(\rho_s(x), \rho_s(y)) &= B_M(x - 2B_M(e_s, x)e_s, y - 2B_M(e_s, y)e_s) \\ &= B_M(x, y) - 4B_M(x, e_s)B_M(y, e_s) + 4B_M(x, e_s)B_M(y, e_s) \\ &= B_M(x, y) \end{aligned}$$

□

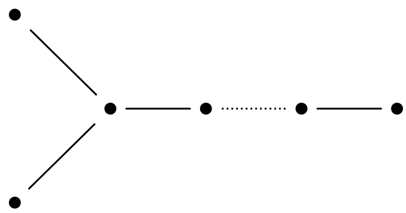
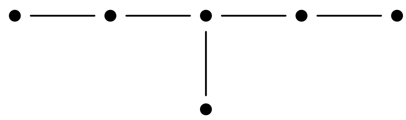
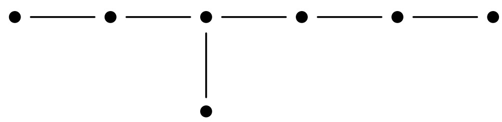
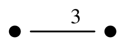
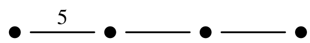
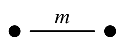
Theorem 2.5. *[17] The map $s \mapsto \rho_s$ extends to an injective morphism $W \to GL(V)$, the reflection representations of W. Furthermore, (W, S) is a Coxeter system.*

When S is finite, B_M is positive definite if and only if W is finite.

Theorem 2.6. *[18] The construction of real reflection groups and Coxeter groups as reflection groups give rise to inverse bijections between the set of conjugacy classes of finite subgroups of $GL_n(\mathbb{R})$ generated by reflections and the set of those rank n Coxeter matrices give rise to a finite group.*

2.4 Finite Coxeter Groups

The classification of Coxeter graphs causing irreducible finite Coxeter groups is the following([17])

A_n	
B_n	
D_n	
E_6	
E_7	
F_4	
$G_2 = I_2(6)$	
H_3	
H_4	
$I_2(m)$	

2.5 Root Systems and Weyl Groups

Definition 2.9. (Root System) Let V be a finite-dimensional real vector space, with the standard inner product denoted by $(.,.)$. A root system in V is a finite set ϕ of non-zero vectors (called roots) that satisfy the following properties:

(i) The roots span V

(ii) The only scalar multiplies of a root $\alpha \in \phi$ that belong to ϕ are α itself and $-\alpha$.

(iii) For every root $\alpha \in \phi$, the set ϕ is closed under reflection through the hyperplane perpendicular to α. That is, for any two roots α and β, the set ϕ contains the reflection of β.

$$\sigma_\alpha(\beta) = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \phi$$

(iv) (Integrality Condition) If α and β are roots in ϕ, then the projection of β onto the line through α is a half-integral multiple of α. That is,

$$<\beta,\alpha>= 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$$

The integral condition is known as a crystallographic root system.

In view of property (iii), the integrality condition is equivalent to stating that β and its reflection $\sigma_\alpha(\beta)$ differ by an integral multiple of α.

Note that the operator $< .,. >: \phi \times \phi \to \mathbb{Z}$ defined by property (iv) is not an inner product. It is not necessarily symmetric and is linear only in the first argument. The rank of a root system ϕ is the dimension of V.

The cosine of the angle between two roots is constrained to be a half-integral multiple of a square root of an integer:

$$\begin{aligned}
<\beta,\alpha><\alpha,\beta> &= 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}2\frac{(\beta,\alpha)}{(\beta,\beta)} \\
&= 2\frac{(\alpha,\beta)}{|\alpha|^2}2\frac{(\alpha,\beta)}{|\beta|^2} \\
&= 4\frac{(\alpha,\beta)^2}{|\alpha|^2|\beta|^2} \\
&= 4cos^2(\theta) \\
&= (2cos(\theta))^2 \in \mathbb{Z}
\end{aligned}$$

Since $2cos(\theta) \in [-2,2]$, condition (ii) says that no scalar multiples of α other than 1 and -1 can be roots which would correspond to 2α or -2α are out.

Definition 2.10. (Dual Root System and Coroot) If ϕ is a root system in V, the coroot α^V of a root α is defined by

$$\alpha^V = \frac{2}{(\alpha,\alpha)}\alpha$$

The set of coroots also forms a root system ϕ^V in V, called the dual root system (or sometimes inverse root system). By definition $(\alpha^V)^V = \alpha$, so that ϕ is the dual root system of ϕ^V. The lattice in V spanned by ϕ^V is called the coroot lattice.

Definition 2.11. (Alternative Definition of Root System) Let V be a finite dimensional real vector space ϕ a finite subset of V and ϕ^V a finite subset of V^* parameterized by $\phi : \phi \to \phi^V, \alpha \to \alpha^V$. Assume that

(i) $<\phi>= V$.

(ii) $\forall \alpha \in \phi$, we have $(\alpha^V, \alpha) = 2$ and the reflection $s_\alpha : V \to V$, $x \mapsto x - (\alpha^V, x)\alpha$ stabilizes ϕ.

(iii) We have $\alpha^V(\phi) \subset \mathbb{Z}$, $\forall \alpha \in \phi$.

(iv) $\forall \alpha \in \phi$, we have $2\alpha \notin \phi$

Then, ϕ is a root system in V

Definition 2.12. (Irreducible) If $\phi = \phi_1 \bigcup \phi_2$ and ϕ_i is a root system in V_i together with $\phi_i^V = \{\alpha^V\}_{\alpha\in\phi_i}$, the subspace of V generated by ϕ_i, for $i \in \{1, 2\}$, then we say that $\phi = \phi_1 \bigoplus \phi_2$. The root system ϕ is irreducible if it is non-empty and it is not the direct sum of two non-empty root systems.

Definition 2.13. (Weyl Group) The Weyl group of the root system ϕ is the subgroup of $GL(V)$ generated by the reflections s_α for $\alpha \in \phi$.

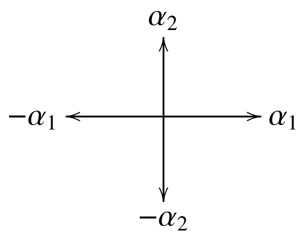
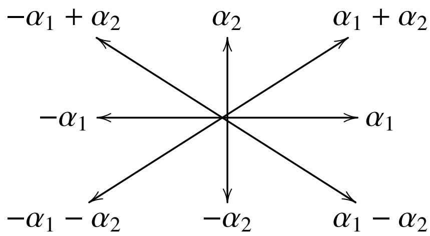
Note that W is a crystallographic finite reflection group with $Z - lattice$ the $Z - submodule$ of V generated by ϕ.

Proposition 2.3. *[18]Let W be crystallographic reflection group in a finite dimensional real vector space V. Then there is a root system ϕ in V with Weyl group W.*

The set $\Delta = \{\alpha_i\}_{1\leq i\leq n}$ is called a basis of ϕ. Let $\phi^+ = \{\alpha \in \phi | \alpha = \sum n_i\alpha_i, n_i \geq 0\}$ (the positive roots) and $\phi^- = \{\alpha \in \phi | \alpha = \sum n_i\alpha_i, n_i \leq 0\}$ (the negative roots).

Proposition 2.4. *[17] The set Δ is a basis of V and $\phi = \phi^+ \bigcup \phi^-$.*

The Cartan matrix is $(< \alpha, \beta^V >)$ where α and β belong to Δ. Let us define $S = \{s_\alpha\}$ where $\alpha \in \Delta$. Then, (W, S) is a Coxeter system.

<table>
<tr><td colspan="2">The rank 2 root systems</td></tr>
<tr><td>Type $A_1 \times A_2$</td><td>Type A_2</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Proposition 2.5. *[18] Let $w \in W$. The cardinality of $\phi^- \cap w(\phi^+)$ is the length $l(w)$*

The root system in V^* which is either an inverse to ϕ or a dual to ϕ is defined by the set ϕ^V. There is an isomorphism of groups

$$\begin{array}{ccc} W(\phi) & \rightarrow & W(\phi^V) \\ u & \mapsto & {}^t u^{-1} \end{array}$$

sending s_α on s_{α^V}. Through this isomorphism, $W(\phi)$ operators on V^*.

Remark 2.1. *[18] The root systems (V, ϕ) and (V^*, ϕ^*) are not isomorphic in general: for example, the root system of type C_n is the inverse of the root system of B_n, when $n \geq 3$, these root systems are not isomorphic.*

2.6 Affine Weyl Groups

Let us define that ϕ is a root system in a finite dimensional real vector space V. We set a subgroup of the group $Aff(V^*)$ of affine transformations of V^* as follows:

For $\alpha \in \phi$ and $k \in \mathbb{Z}$, let $H_{\alpha,k}$ be the affine hyperplane of V^* defined by

$$H_{\alpha,k} = \{x \in V^* | (\alpha, v) = k\}$$

Let $s_{\alpha,k}$ be the orthogonal reflection with respect to $H_{\alpha,k}$

$$s_{\alpha,k}(x) = x - ((\alpha, x) - k)\alpha^V$$

The affine Weyl group associated to ϕ is the subgroup W of $Aff(V^*)$ generated by $s_{\alpha,k}$, $\alpha \in \phi$, $k \in \mathbb{Z}$.

Let Q be the subgroup of $Aff(V^*)$ generated by the transformations by elements of ϕ^V.

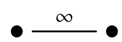
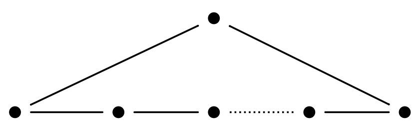
Proposition 2.6. *[17]We have $W = Q \rtimes W$.*

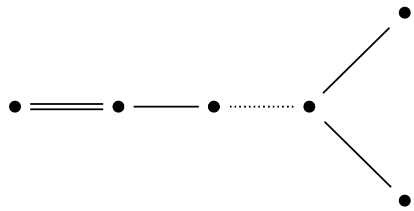
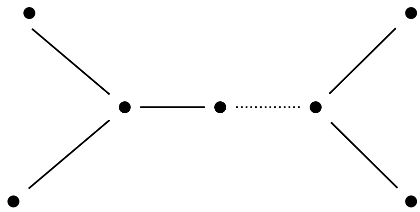
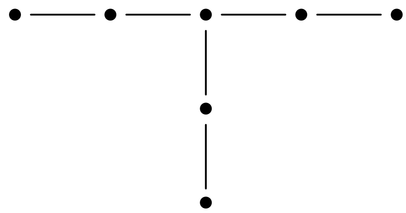
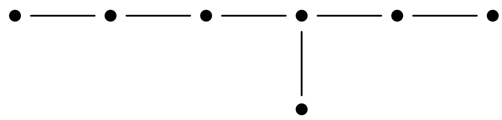
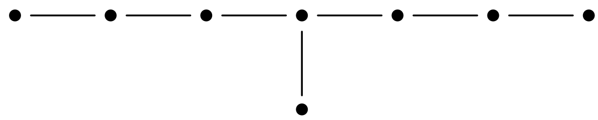
Definition 2.14. (Alcove) An alcove is connected component of $V^* - \bigcup_{\alpha\in\phi,k\in\mathbb{Z}} H_{\alpha,k}$ where the set $\{H_{\alpha,k}\}_{\alpha\in\phi,k\in\mathbb{Z}}$ is the set of reflecting hyperplanes of W.

Theorem 2.7. *[17] The group W acts simply transitively on the set of alcoves. The closure of an alcove is a fundamental domain for the action of W on V^*.*

Assume that C be a chamber for W. Afterwards, there is a unique alcove $A \subset C$ such that 0 is in $\bar{A}$, which is the closure of A. Suppose that $\tilde{S}$ be the set of reflections with respect to the walls of A. The pair $(W, \tilde{S})$ is a Coxeter group. There is a root $\tilde{\alpha} = \sum n_i\alpha_i$ in ϕ such that if β belongs to ϕ, $\beta = \sum m_i\alpha_i$, then $m_i \leq n_i$: $\tilde{\alpha}$ is the highest root if ϕ is irreducible. This root is orthogonal to the wall of A which does not contain 0 ([17]). The length of an element $w \in W$ is the minimal length of a gallery of alcoves from A to $w(A)$.

The classification of the irreducible affine Weyl groups (or of their Coxeter graphs) is the following [17];

The type with cases	The graphs
$\widetilde{A_1}$	
$\widetilde{A_n}$ $n \geq 2$	

The type with cases	The graphs
$\widetilde{B_n}$ $n \geq 3$	
$\widetilde{C_n}$ $n \geq 2$	
$\widetilde{D_n}$ $n \geq 4$	
$\widetilde{E_6}$	
$\widetilde{E_7}$	
$\widetilde{E_8}$	
$\widetilde{F_4}$	
$\widetilde{G_2}$	

Theorem 2.8. *(A formula for order of W)[19] If W is an irreducible Weyl group of rank n, then*

$$|w| = n!c_1c_2 \dots c_n f,$$

Coefficients of Highest Root and Index of Connection		
Type	*Coefficient of* $\widetilde{\alpha}$	f
A_n	$1, 1, \dots, 1$	$n + 1$
B_n	$1, 2, 2, \dots, 2$	2

Coefficients of Highest Root and Index of Connection		
Type	*Coefficient of* $\widetilde{\alpha}$	f
C_n	$2, 2, \ldots, 2, 1$	2
D_n	$1, 2, 2, \ldots, 2, 1, 1$	4
E_6	$1, 2, 2, 3, 2, 1$	3
E_7	$2, 2, 3, 4, 3, 2, 1$	2
E_8	$2, 3, 4, 6, 5, 4, 3, 2$	1
F_4	$2, 3, 4, 2$	1
G_2	$3, 2$	1

where f is the index of connection and the c_i are the coefficient of the highest root. $\widetilde{\alpha} = \sum c_i\alpha_i$ where $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

CHAPTER 3

GRÖBNER-SHIRSHOV BASES AND REDUCED FORMS FOR $\widetilde{A_n}$

3.1 Gröbner-Shirshov Bases for the Associative Algebras

Definition 3.1. (Linearly Ordered Set) A linearly ordered set (or a total ordered set) is partial ordering set $(S, \leq)$ which has the property of comparability

$$\forall r_i, r_j \in S \quad either \quad r_i \leq r_j \quad or \quad r_j \leq r_i$$

In other words, a linearly ordered set is a set S with a binary relation $\leq$ on it such that the followings hold for all r_i, r_j and r_k are in S.

- Reflexivity: $r_i \leq r_i$.
- Antisymmetry: if $r_i \leq r_j$ and $r_j \leq r_i$, then $r_i = r_j$.
- Transitivity: if $r_i \leq r_j$ and $r_j \leq r_k$, then $r_i \leq r_k$.
- Comparability: Either $r_i \leq r_j$ or $r_j \leq r_i$.

Definition 3.2. (Well Ordering) A well-ordered relation on a set S is a total order on S with the property that every nonempty subset of S has a least element in this ordering. The set S together with the well-order relation is then called a well-ordered.

Example 3.1. $\mathbb{N}$ is a well-ordering but $\mathbb{Z}$ is not a well-ordering because $\mathbb{Z}$ has no least element.

Definition 3.3. (Semigroup) A semigroup is a non-empty set R with a binary operation on "$*$" such that binary operation is associative

$$a * (b * c) = (a * b) * c.$$

Definition 3.4. (Monoid) If S is a monoid set together with a binary operation "$*$", then S satisfies the following three axioms;

(i) Closure: $\forall a, b \in S \quad a * b \in S$,

(ii) Associativity: $\forall a, b, c \in S \quad (a * b) * c = a * (b * c)$,

(iii) Identity Element: $\exists e \in S \; : \; \forall a \in S \; : \; e * a = a * e = a$.

A monoid is a semigroup with an identity element.

Example 3.2. Let us give 5 examples of monoid;

- $(\mathbb{Z}^{+}, +)$ is a semigroup but not a monoid since the identity 0 is not in the set.
- $(\mathbb{Z}^{+}, \cdot)$ is a monoid under multiplication since the multiplicative identity 1 is in the set $\mathbb{Z}^{+}$.
- Every group is a monoid and every abelian group is a commutative monoid.
- $\mathbb{N}$ forms a commutative monoid under addition (identity element 0), or under multiplication (identity element 1).
- Let S be a set. The set of all functions $S \to S$ forms a monoid under function composition. The identity is the identity function. It is also called full transformation monoid of S. If S is finite with n elements, the monoid of functions on S is finite with n^n elements.

Definition 3.5. (Free Monoid) The free monoid on a set S is the monoid whose elements are all the finite sequence of zero or more elements from S. It is usually denoted by S^*.

Suppose S is a linearly ordered set and $\Bbbk$ is a field. Let S^* be the free monoid generated by S. The elements of S^* are called words. The empty word is the identity which is denoted by 1. Let $\Bbbk < S >$ be free associative algebra over $\Bbbk$ generated by S defined by

$$\Bbbk < S > \quad = \quad \{\sum_{i=1}^{m} c_{\alpha_i} w_i, \quad c_{\alpha_i} \in \Bbbk \quad and \quad w_i \in S^*\}.$$

Definition 3.6. (Monomial Ordering) A well ordering $<$ on S^* is called monomial ordering if it agrees with left and right multiplications by words:

$$u > v \quad \Rightarrow \quad w_1 u w_2 > w_1 v w_2 \quad \forall w_1, w_2 \in S^*.$$

Example 3.3. A standard example of monomial ordering on S^* deglex ordering which first compare two words by length and then compare by lexicographical where S is well ordered set.

Definition 3.7. (Leading Word) Let $f = \alpha\overline{f} + \sum \alpha_i u_i \in \Bbbk < S >$ where $\alpha, \alpha_i \in \Bbbk$, $\overline{f} \in S^*$ and $u_i < \overline{f}$ for each i. Then we call $\overline{f}$ the leading word of f.

For a word $w \in S^*$, we denote the length of w by $|w|$.

Example 3.4. Let $S = \{r_0, r_1, r_2\}$ and $f = r_0r_1r_2 + r_0r_1r_0 + r_1r_2 + r_2$ using deglex order with $r_0 > r_1 > r_2$. Firstly we check lengths of the words.

$$\begin{aligned} w_1 = r_0r_1r_2 &\Rightarrow |w_1| = 3, \\ w_2 = r_0r_1r_0 &\Rightarrow |w_2| = 3, \\ w_3 = r_1r_2 &\Rightarrow |w_3| = 2, \\ w_4 = r_2 &\Rightarrow |w_4| = 1. \end{aligned}$$

Then, $|w_1| = |w_2| > |w_3| > |w_4|$ and secondly, using lexicographic order $w_2 > w_1 > w_3 > w_4$ because of $r_0 > r_1 > r_2$. $\overline{f} = w_2 = r_0r_1r_0$ is the leading word of f.

Definition 3.8. (Monic) If leading word of f, $\overline{f}$, has a coefficient 1, then f is called monic.

Definition 3.9. (Composition of Intersection) For two monic polynomials f and g in $\Bbbk < S >$ and a word w, the composition of intersection is defined by

$$(f,g)_w = fb - ag \quad w = \overline{f}b = a\overline{g}, \quad |\overline{f}| + |\overline{g}| > |w|.$$

Example 3.5. Let $f = r_0r_1r_2r_0 + 2r_0r_2r_0r_1 + r_0r_1 + r_1$ and $g = r_2r_0r_1r_2 + 2r_2r_1r_0r_1 + r_1r_2 + r_2$. $\overline{f} = r_0r_1r_2r_0$, $\overline{g} = r_2r_0r_1r_2$ and $w = r_0r_1r_2r_0r_1r_2 = \overline{f}(r_1r_2) = (r_0r_1)\overline{g}$ and $|\overline{f}| + |\overline{g}| = 4 + 4 = 8 > |w| = 6$. Then the composition of intersection is $(f,g)_w = f.(r_1r_2) - (r_0r_1).g = r_0r_1r_2r_0r_1r_2 + 2r_0r_2r_0r_1r_1r_2 + r_0r_1r_1r_2 + r_1r_1r_2 - r_0r_1r_2r_0r_1r_2 - 2r_0r_1r_2r_1r_0r_1 - r_0r_1r_1r_2 - r_0r_1r_2 = 2r_0r_2r_0r_1r_1r_2 + r_1r_1r_2 - 2r_0r_1r_2r_1r_0r_1 - r_0r_1r_2$

Definition 3.10. (Composition of Inclusion) For two monic polynomial f and g in $\Bbbk < S >$ and a word w, the composition of inclusion is defined by

$$(f,g)_w = f - agb \quad if \quad w = \overline{f} = a\overline{g}b.$$

Example 3.6. Let $f = r_0r_1r_2r_0r_1r_2 + r_2r_0r_1 + r_0r_1 + r_1$ and $g = r_1r_2r_0r_1 + r_2r_1r_0 + r_1r_2 + r_2$. $\overline{f} = r_0r_1r_2r_0r_1r_2$ and $\overline{g} = r_1r_2r_0r_1$. $\overline{f} = r_0r_1r_2r_0r_1 = (r_0)\overline{g}(r_2)$, then the composition of inclusion is $(f,g)_w = r_0r_1r_2r_0r_1r_2 + r_2r_0r_1 + r_0r_1 + r_1 - (r_0)(r_1r_2r_0r_1 + r_2r_1r_0 + r_1r_2 + r_2)(r_2) = -r_0r_2r_1r_0r_2 - r_0r_1r_2r_2 - r_0r_2r_2 + r_2r_0r_1 + r_0r_1 + r_1$.

Definition 3.11. (Elimination of Leading Words) In the event of the composition of including the transformation $f \mapsto (f,g)_w = f - agb$ is called elimination of the leading word (ELW) of g in f.

Example 3.7. If we use the previous Example 3.6, then the ELW is $f - agb = r_0r_1r_2r_0r_1r_2 + r_2r_0r_1 + r_0r_1 + r_1 - (r_0)(r_1r_2r_0r_1 + r_2r_1r_0 + r_1r_2 + r_2)(r_2) = -r_0r_2r_1r_0r_2 - r_0r_1r_2r_2 - r_0r_2r_2 + r_2r_0r_1 + r_0r_1 + r_1$.

Definition 3.12. (Trivial Relative) A composition $(f,g)_w$ is called trivial relative to some $R \subset \Bbbk < S >$ if $(f,g)_w = \sum \alpha_i a_i r_i b_i + r$ where $\alpha_i \in \Bbbk$, $r_i \in R$, $a_i, b_i \in S^*$ and $a_i\overline{r_i}b_i < w$. In particular, both $f \mapsto (f,g)_w = f - agb = 0$ and $r = 0$ of R, then $(f,g)_w$ is trivial relative to R.

Example 3.8. Let h_1, h_2 and h_3 belong to R. $h_1 = r_0r_1r_2r_0r_1r_2 + r_0r_2r_1r_2$, $h_2 = r_1r_2r_0r_1 + r_2r_1 + r_1$ and $h_3 = r_1$. Also, $\overline{h_1} = r_0(\overline{h_2})r_2$,

$$\begin{aligned}(h_1, h_2)_w &= h_1 - r_0(h_2)r_2 \\ &= r_0r_1r_2r_0r_1r_2 + r_0r_2r_1r_2 - r_0(r_1r_2r_0r_1 + r_2r_1 + r_1)r_2 \\ &= -r_0r_1r_2 + 0 \\ &= -r_0h_3r_2.\end{aligned}$$

where $r_0\overline{h_3}r_2 < w$ and $r = 0$.Then, $(h_1, h_2)_w$ is trivial relative to R.

Definition 3.13. (Gröbner-Shirshov Basis) A subset R of $\Bbbk < S >$ is called a Gröbner-Shirshov basis if any composition of polynomials from R is trivial relative to R.

The following lemma was first proved by Shirshov [15] for Lie algebras presented by generators and defining relations. He called it the composition lemma. A similar lemma for free associative algebras was formulated later by Bokut [1] and by Bergman [8] under the name "Diamond lemma" after celebrated Newman's Diamond lemma [11] for graphs. This kind of lemmata are now named as composition-diamond lemma. We will use Bokut's version of this lemma for free associative algebras. Similar ideas were independently discovered by Hironaka [12] for power series algebras and by Buchberger [13, 14] for polynomial algebras.

Lemma 3.1. *(Composition-Diamond Lemma for associative algebras) Let $\Bbbk$ be a field, $A = \Bbbk\langle S|R\rangle = \Bbbk\langle S\rangle/I(R)$ and $<$ a monomial ordering on S^*, where $I(R)$ is the ideal of $\Bbbk\langle S\rangle$ generated by R. Then the following statements are equivalent:*

(i) *R is a Gröbner-Shirshov basis.*

(ii) *$f \in I(R) \Rightarrow \overline{f} = a\overline{s}b$ for some $s \in R$ and $a, b \in S^*$.*

(iii) *The set of R-reduced forms*

$$Red(R) = \{w \in S^* | w \neq a\overline{s}b, a, b \in S^*, s \in R\}$$

is a $\mathbb{k}$*-linear basis for the algebra* $A = \mathbb{k}\langle S|R\rangle$.

For commutative polynomials, this lemma is known as Buchberger's Theorem for the Gröbner basis (see [13, 14]).

Definition 3.14. (Buhberger-Shirshov Algorithm) If $R \subset \mathbb{k}\langle S\rangle$ is not a Gröbner-Shirshov basis, then we reduce every nontrivial compositions to a polynomial relative to R and add this polynomial to the R. Having repeated this procedure (possibly infinitely many times), we obtain a Gröbner-Shirshov basis R^{comp}. The process is called the Buchberger-Shirshov algorithm.

If the set R consists of semigroup relations (i.e. $u - v$, where $u, v \in S^*$), then each nontrivial composition of polynomials from R has the same semigroup form. Hence, R^{comp} consists of semigroup relations too.

Let $A = smg\langle S|R\rangle$ be a semigroup presentation. Then $R \subset \mathbb{k}\langle S\rangle$ and we can obtain the Gröbner-Shirshov bases R^{comp}. The set R^{comp} does not depend on the field $\mathbb{k}$ and consists of semigroup relations. We will call R^{comp} the Gröbner-Shirshov bases for the semigroup A.

The main purpose of this thesis is to find a Gröbner-Shirshov basis and classify all reduced forms for the affine Weyl group $\widetilde{A}_n$. The strategy for solving the problem is as follows.

Let R be the set of polynomials of the defining relations of $\widetilde{A}_n$. Using Buchberger-Shirshov algorithm, we obtain a new set R' of polynomials including R. Then, by using the algorithm of elimination of leading words with respect to the polynomials in R', all the words in the group $\widetilde{A}_n$ are reduced to the explicit classes of words. After that, we compute the number of all reduced forms with respect to these classes by means of a generating function. This generating function turns out to be same with the well known Poincaré polynomial of the affine Weyl group $\widetilde{A}_n$. Therefore, by the Composition-Diamond Lemma the functions in R' form a Gröbner-Shirshov basis for the affine Weyl group $\widetilde{A}_n$. Furthermore, one can easily see that this basis is in fact a reduced Gröbner-Shirshov basis.

3.2 Gröbner-Shirshov Bases for $\widetilde{A_n}$

In this section we explicitly present Gröbner-Shirshov bases for the affine Weyl group of type $\widetilde{A_n}$.

Definition 3.15. The affine Weyl group $\widetilde{A_n}$ has a presentation with generators $S = \{r_0, r_1, \ldots, r_n\}$ and defining relations

$r_i r_i = 1 \quad 0 \le i \le n,$

$r_i r_j = r_j r_i \quad 0 \le i < j-1 < n$ and $(i, j) \ne (0, n)$,

$r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1} \quad 0 \le i \le n-1,$

$r_0 r_n r_0 = r_n r_0 r_n.$

Identifying each relation $u = v$ by a polynomial $u - v$, we define

$f_1^{(i)} = r_i r_i - 1 \quad 0 \le i \le n,$

$f_2^{(i,j)} = r_i r_j - r_j r_i \quad 0 \le i < j-1 < n$ and $(i, j) \ne (0, n)$,

$f_3^{(i)} = r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1} \quad 0 \le i \le n-1,$

$f_4 = r_0 r_n r_0 - r_n r_0 r_n.$

Let us define

$$r_{ij} = \begin{cases} r_i r_{i+1} \ldots r_j, & 0 \le i < j \le n; \\ r_i r_{i-1} \ldots r_j, & 0 < j < i \le n; \\ r_i, & i = j; \\ 1, & i = 1, j = 0; \\ 1, & i = n, j = n+1. \end{cases}$$

Lemma 3.2. *Let $R = \{f_1, f_2, f_3, f_4\}$. A Gröbner-Shirshov Basis of $\widetilde{A_n}$ with respect to deglex order with $r_0 > r_1 > \cdots > r_n$ contains the following polynomials.*

$g_1^{(i,j)} = r_{ij} r_i - r_{i+1} r_{ij} \qquad 0 \le i < j-1 < n$ *with* $(i, j) \ne (0, n)$,

$g_2 = r_{0n} r_0 r_n - r_1 r_{0n} r_0,$

$g_3^{(j,k)} = r_0 r_{nk} r_j - r_j r_0 r_{nk} \qquad (n > 3) \qquad 2 \le j < k-1 < n,$

$g_4^{(j)} = r_0 r_{nj} r_{j+1} - r_j r_0 r_{nj} \qquad 2 \le j < n,$

$g_5^{(k)} = r_0 r_{nk} r_0 - r_n r_0 r_{nk} \qquad 2 \le k < n,$

$g_6^{(k,l)} = r_0 r_{nk} r_{1l} r_{0l} - r_n r_0 r_{nk} r_{1l} r_{0,l-1} \qquad 1 \le l < n \quad , \quad 2 \le k \le n,$

$g_7^{(k,l)} = r_0 r_{nk} r_{1l} r_0 r_{nk} - r_1 r_0 r_{nk} r_{1l} r_0 r_{n,k+1} \qquad 1 \le l < k-1 < n,$

$g_8^{(k,l)} = r_0 r_{nk} r_{1l} r_0 r_{n,k-1} - r_1 r_0 r_{nk} r_{1l} r_0 r_{nk} \quad 3 \le k \le n \quad , \quad k-1 \le l \le n,$

$g_9^{(j,k,l)} = r_0 r_{nk} r_{1l} r_0 r_{nj} r_{1l} - r_n r_0 r_{nk} r_{1l} r_0 r_{nj} r_{1,l-1}$

$2 \le k \le n-1 \quad , \quad k+1 \le j \le n \quad , \quad 1 \le l \le j-2,$

$g_{10}^{(j,k,l)} = r_0 r_{nk} r_{1l} r_0 r_{nj} r_{1,l+1} - r_n r_0 r_{nk} r_{1l} r_0 r_{nj} r_{1l}$

$2 \le k \le n \quad , \quad k \le j \le n \quad , \quad j-1 \le l \le n-1.$

Proof. Let $R = \{f_1, f_2, f_3, f_4\}$. We apply the Buchberger-Shirshov algorithm to the R. We show every ELW in the below computations except ELW's of ${f_2}^{(i,j)}$, the commutativity relations. Notice that at this point we are not claiming that this is a Gröbner-Shirshov basis for $\widetilde{A}_n$.

$(f_3^{(i)}, f_2^{(i,i+2)}) = g_1^{(i,i+2)} \quad$ for $\quad i = 0, 1, 2, \ldots, n-2.$

$(g_1^{(i,j-1)}, f_2^{(i,j)}) = g_1^{(i,j)}$

for $\quad i = 0, 1, 2, \ldots, n-2 \quad , \quad j = i+3, i+4, \ldots, n \quad$ with $\quad (i, j) \neq (0, n).$

$(g_1^{(0,n-1)}, f_4) = g_2.$

$(f_2^{(0,j)}, f_2^{(j,n)}) = g_3^{(j,n)} \quad (n > 3) \quad$ for $\quad j = 2, 3, \ldots, n-2.$

$(g_3^{(j,k+1)}, f_2^{(j,k)}) = g_3^{(j,k)}$

$(n > 4) \quad$ for $\quad k = n-1, n-2, \ldots, 4 \quad , \quad j = 2, 3, \ldots, k-2.$

$(f_2^{(0,n-1)}, f_3^{(n-1)}) = g_4^{(n-1)}.$

$(g_3^{(j,j+2)}, f_3^{(j)}) = g_4^{(j)} \quad (n > 3) \quad$ for $\quad j = n-2, n-3, \ldots, 2.$

$(f_4, f_2^{(0,n-1)}) = g_5^{(n-1)}.$

$(g_5^{(k+1)}, f_2^{(0,k)}) = g_5^{(k)} \quad (n > 3) \quad$ for $\quad k = n-2, n-3, \ldots, 2.$

$(f_4, f_3^{(0)}) = g_6^{(n,1)}$

$(g_5^{(k)}, f_3^{(0)}) = g_6^{(k,1)} \quad$ for $\quad k = 2, 3, \ldots, n-1.$

$(g_6^{(k,l-1)}, f_3^{(l-1)}) = g_6^{(k,l)} \quad$ for $\quad k = 2, 3, \ldots, n \quad , \quad l = 2, 3, \ldots, n-1.$

$$
\begin{aligned}
(f_3^{(0)}, f_4) &= r_0 r_n r_1 r_0 r_n - r_1 r_0 r_n r_1 r_0 \\
&= g_7^{(n,1)}.
\end{aligned}
$$

$$(g_1^{(0,l)}, f_4) = g_7^{(n,l)} \quad (n > 3) \quad \text{for} \quad l = 2, 3, \ldots, n-2.$$

$$
\begin{aligned}
(g_7^{(k+1,l)}, g_4^{(k)}) &= r_0 r_{n,k+1} r_{1l} r_k r_0 r_{nk} - r_1 r_0 r_{n,k+1} r_{1l} r_0 r_{n,k+2} r_k r_{k+1} \\
&= g_7^{(k,l)} - r_1 r_0 r_{n,k+1} r_{1l} g_3^{(k,k+2)} r_{k+1} \\
&\quad (n > 3) \quad \text{for} \quad k = n-1, n-2, \ldots, 3 \quad , \quad l = 1, 2, \ldots, k-2.
\end{aligned}
$$

$$
\begin{aligned}
(g_7^{(k,k-2)}, g_4^{(k-1)}) &= r_0 r_{nk} r_{1,k-1} r_0 r_{n,k-1} - r_1 r_0 r_{nk} r_{1,k-2} r_0 r_{n,k+1} r_{k-1} r_k \\
&= g_8^{(k,k-1)} - r_1 r_0 r_{nk} r_{1,k-2} g_3^{(k-1,k+1)} r_k \quad (n > 3) \quad \text{for} \quad k = n-1, \ldots, 3.
\end{aligned}
$$

$$
\begin{aligned}
(g_7^{(n,n-2)}, g_4^{(n-1)}) &= r_0 r_n r_{1,n-1} r_0 r_{n,n-1} - r_1 r_0 r_n r_{1,n-2} r_0 r_{n-1} r_n \\
&= g_8^{(n,n-1)} - r_1 r_0 r_n r_{1,n-2} f_2^{(0,n-1)} r_n.
\end{aligned}
$$

$$(g_2, g_4^{(n-1)}) = r_{0,n-2} f_3^{(n-1)} r_0 r_n r_{n-1} - r_1 r_{0n} f_2^{(0,n-1)} r_n - r_1 r_{0,n-2} f_3^{(n-1)} r_0 r_n + g_8^{(n,n)}$$

$$
\begin{aligned}
(g_8^{(k+1,l)}, g_4^{(k-1)}) &= r_0 r_{n,k+1} r_{1,k-2} g_1^{(k-1,l)} r_0 r_{n,k-1} - r_1 r_0 r_{n,k+1} r_{1l} g_3^{(k-1,k+1)} r_k \\
&\quad - r_1 r_0 r_{n,k+1} r_{1,k-2} g_1^{(k-1,l)} r_0 r_{n,k+1} r_k + g_8^{(k,l)} \\
&\quad (n > 3) \quad \text{for} \quad k = 3, 4, \ldots, n-1 \quad , \quad l = k, k+1, \ldots, n.
\end{aligned}
$$

$$(g_6^{(k,1)}, f_2^{(1,n)}) = g_9^{(n,k,1)} \quad \text{for} \quad k = 2, 3, \ldots, n-1.$$

$$
\begin{aligned}
(g_9^{(j+1,k,1)}, f_2^{(1,j)}) &= g_9^{(j,k,1)} \\
&\quad (n > 3) \quad \text{for} \quad k = 2, 3, \ldots, n-2 \quad , \quad j = k+1, k+2, \ldots, n-1.
\end{aligned}
$$

$$
\begin{aligned}
(g_9^{(j,k,l-1)}, f_3^{(l-1)}) &= r_0 r_{nk} r_{1,l-1} r_0 r_{nj} r_{1,l-2} r_l r_{l-1} r_l - r_n r_0 r_{nk} r_{1,l-1} r_0 r_{nj} r_{1,l-2} r_l r_{l-1} \\
&= g_9^{(j,k,l)}
\end{aligned}
$$

$$(n > 3) \quad \text{for} \quad k = 2, 3, \ldots, n-1 \quad , \quad j = k+1, k+2, \ldots, n \quad , \quad l = 2, \ldots, j-2.$$

$$
\begin{aligned}
(g_9^{(j+1,k,j-1)}, f_3^{(j-1)}) &= r_0 r_{nk} r_{1,j-1} r_0 r_{n,j+1} r_{1,j-2} r_j r_{j-1} r_j - r_n r_0 r_{nk} r_{1,j-1} r_0 r_{n,j+1} r_{1,j-2} r_j r_{j-1} \\
&= g_{10}^{(j,k,j-1)}
\end{aligned}
$$

$$(n > 3) \quad \text{for} \quad k = 2, 3, \ldots, n-1 \quad , \quad j = k, k+1, \ldots, n-1.$$

$$
\begin{aligned}
(g_6^{(k,n-1)}, f_3^{(n-1)}) &= r_0 r_{nk} r_{1,n-1} r_{0,n-2} r_n r_{n-1} r_n - r_n r_0 r_{nk} r_{1,n-1} r_0 r_n r_{1,n-1} \\
&= g_{10}^{(n,k,n-1)} \quad (n > 3) \quad \text{for} \quad k = 2, 3, \ldots, n.
\end{aligned}
$$

$$
\begin{aligned}
(g_{10}^{(j,k,l-1)}, f_3^{(l)}) &= r_0 r_{nk} r_{1,l-1} r_0 r_{nj} r_{1,l-1} r_{l+1} r_l r_{l+1} - r_n r_0 r_{nk} r_{1,l-1} r_0 r_{nj} r_{1,l-1} r_l r_{l+1} r_l \\
&= r_0 r_{nk} r_{1,l-1} g_4^{(l)} r_{l-1,j} r_{1,l+1} - r_n r_0 r_{nk} r_{1,l-1} g_4^{(l)} r_{l-1,j} r_{1l} + g_{10}^{(j,k,l)} \\
&\quad \text{for} \quad k = 2, 3, \ldots, n-1 \quad , \quad j = k, k+1, \ldots, n-1 \quad , l = j-1, \ldots, n-1.
\end{aligned}
$$

In the above equation if $l = j$, then $r_{l-1,j}$ assumed to be the identity 1. □

Example 3.9. In this example, we examined Gröbner-Shirshov Basis of Affine Weyl Group $\widetilde{A_4}$ using the Lemma 3.2;

$$\underline{g_1^{(i,j)} = r_{ij}r_i - r_{i+1}r_{ij} \qquad 0 \le i < j-1 < 4 \quad \text{with} \quad (i,j) \ne (0,4)}$$

$$\begin{aligned}
(f_3^{(0)}, f_2^{(0,2)})_{r_{01}r_0r_2} &= f_3^{(0)}r_2 - r_{01}f_2^{(0,2)} \\
&= (r_0r_1r_0 - r_1r_0r_1)r_2 - r_0r_1(r_0r_2 - r_2r_0) \\
&= r_{02}r_0 - r_1r_{02} \\
&= g_1^{(0,2)}.
\end{aligned}$$

$$\begin{aligned}
(g_1^{(0,2)}, f_2^{(0,3)})_{r_{02}r_0r_3} &= g_1^{(0,2)}r_3 - r_{02}f_2^{(0,3)} \\
&= (r_{02}r_0 - r_1r_{02})r_3 - r_{02}(r_0r_3 - r_3r_0) \\
&= r_{03}r_0 - r_1r_{03} \\
&= g_1^{(0,3)}.
\end{aligned}$$

$$\begin{aligned}
(f_3^{(1)}, f_2^{(1,3)})_{r_{12}r_1r_3} &= f_3^{(1)}r_3 - r_{12}f_2^{(1,3)} \\
&= (r_1r_2r_1 - r_2r_1r_2)r_3 - r_{12}(r_1r_3 - r_3r_1) \\
&= r_{13}r_1 - r_2r_{13} \\
&= g_1^{(1,3)}.
\end{aligned}$$

$$\begin{aligned}
(g_1^{(1,3)}, f_2^{(1,4)})_{r_{13}r_1r_4} &= g_1^{(1,3)}r_4 - r_{13}f_2^{(1,4)} \\
&= (r_{13}r_1 - r_2r_{13})r_4 - r_{13}(r_1r_4) \\
&= r_{14}r_1 - r_2r_{14} \\
&= g_1^{(1,4)}.
\end{aligned}$$

$$\begin{aligned}
(f_3^{(2)}, f_2^{(2,4)})_{r_2r_{32}r_4} &= f_3^{(2)}r_4 - r_{23}f_2^{(2,4)} \\
&= (r_2r_3r_2 - r_3r_2r_3)r_4 - r_{23}(r_2r_4 - r_4r_2) \\
&= r_{24}r_2 - r_3r_{24} \\
&= g_1^{(2,4)}.
\end{aligned}$$

$\underline{g_2 = r_{04}r_0r_4 - r_1r_{04}r_0}$

$$\begin{aligned}(g_1^{(0,3)}, f_4)_{r_{03}r_0r_4r_0} &= g_1^{(0,3)}r_4r_0 - r_{03}f_4 \\ &= (r_{03}r_0 - r_1r_{03})r_4r_0 - r_{03}(r_0r_4r_0 - r_4r_0r_4) \\ &= r_{04}r_0r_4 - r_1r_{04}r_0 \\ &= g_2.\end{aligned}$$

$\underline{g_3^{(j,k)} = r_0r_{4k}r_j - r_jr_0r_{4k} \qquad 2 \leq j < k-1 < 4}$

$$\begin{aligned}(f_2^{(0,2)}, f_2^{(2,4)})_{r_0r_2r_4} &= f_2^{(0,2)}r_4 - r_0f_2^{(2,4)} \\ &= (r_0r_2 - r_2r_0)r_4 - r_0(r_2r_4 - r_4r_2) \\ &= r_0r_4r_2 - r_2r_0r_4 \\ &= g_3^{(2,4)}.\end{aligned}$$

$\underline{g_4^{(j)} = r_0r_{4j}r_{j+1} - r_jr_0r_{4j} \qquad 2 \leq j < 4}$

$$\begin{aligned}(f_2^{(0,3)}, f_3^{(3)})_{r_0r_{34}r_3} &= f_2^{(0,3)}r_{43} - r_0f_3^{(3)} \\ &= (r_0r_3 - r_3r_0)r_{43} - r_0(r_3r_4r_3 - r_4r_3r_4) \\ &= r_0r_{43}r_4 - r_3r_0r_{43} \\ &= g_4^{(3)}.\end{aligned}$$

$$\begin{aligned}(g_3^{(2,4)}, f_3^{(2)})_{r_0r_4r_{23}r_2} &= g_3^{(2,4)}r_{32} - r_0r_4f_3^{(2)} \\ &= (r_0r_4r_2 - r_2r_0r_4)r_{32} - r_0r_4(r_2r_3r_2 - r_3r_2r_3) \\ &= r_0r_{42}r_3 - r_2r_0r_{42} \\ &= g_4^{(2)}.\end{aligned}$$

$\underline{g_5^{(k)} = r_0r_{4k}r_0 - r_4r_0r_{4k} \qquad 2 \leq k < 4}$

$$(f_4, f_2^{(0,3)})_{r_0r_4r_0r_3} = f_4r_3 - r_0r_4f_2^{(0,3)}$$

$$= (r_0 r_4 r_0 - r_4 r_0 r_4) r_3 - r_0 r_4 (r_0 r_3 - r_3 r_0)$$

$$= r_0 r_{43} r_0 - r_4 r_0 r_{43}$$

$$= g_5^{(3)}.$$

$$(g_5^{(3)}, f_2^{(0,2)})_{r_0 r_{43} r_0 r_2} = g_5^{(3)} r_2 - r_0 r_{43} f_2^{(0,2)}$$

$$= (r_0 r_{43} r_0 - r_4 r_0 r_{43}) r_2 - r_0 r_{43} (r_0 r_2 - r_2 r_0)$$

$$= r_0 r_{42} r_0 - r_4 r_0 r_{42}$$

$$= g_5^{(2)}.$$

$g_6^{(k,l)} = r_0 r_{4k} r_{1l} r_{0l} - r_4 r_0 r_{4k} r_{1l} r_{0,l-1} \qquad 1 \leq l < 4,\ 2 \leq k \leq 4$

$$(f_4, f_3^{(0)})_{r_0 r_4 r_0 r_1 r_0} = f_4 r_1 r_0 - r_0 r_4 f_3^{(0)}$$

$$= (r_0 r_4 r_0 - r_4 r_0 r_4) r_1 r_0 - r_0 r_4 (r_0 r_1 r_0 - r_1 r_0 r_1)$$

$$= r_0 r_4 r_1 r_{01} - r_4 r_0 r_4 r_1 r_0$$

$$= g_6^{(4,1)}.$$

$$(g_6^{(4,1)}, f_3^{(1)})_{r_0 r_4 r_1 r_{02} r_1} = g_6^{(4,1)} r_2 r_1 - r_0 r_4 r_1 r_0 f_3^{(1)}$$

$$= (r_0 r_4 r_1 r_0 r_1 - r_4 r_0 r_4 r_1 r_0) r_2 r_1 - r_0 r_4 r_1 r_0 (r_1 r_2 r_1 - r_2 r_1 r_2)$$

$$= r_0 r_4 r_{12} r_{02} - r_4 r_0 r_4 r_{12} r_{01}$$

$$= g_6^{(4,2)}.$$

$$(g_6^{(4,2)}, f_3^{(2)})_{r_0 r_4 r_{12} r_{03} r_2} = g_6^{(4,2)} r_{32} - r_0 r_4 r_{12} r_{01} f_3^{(2)}$$

$$= (r_0 r_4 r_{12} r_{02} - r_4 r_0 r_4 r_{12} r_{01}) r_{32} - r_0 r_4 r_{12} r_{01} (r_2 r_3 r_2 - r_3 r_2 r_3)$$

$$= r_0 r_4 r_{13} r_{03} - r_4 r_0 r_4 r_{13} r_{02}$$

$$= g_6^{(4,3)}.$$

$$(g_5^{(3)}, f_3^{(0)})_{r_0 r_{43} r_{01} r_0} = g_5^{(3)} r_1 r_0 - r_0 r_{43} f_3^{(0)}$$

$$= (r_0 r_{43} r_0 - r_4 r_0 r_{43}) r_1 r_0 - r_0 r_{43} (r_0 r_1 r_0 - r_1 r_0 r_1)$$

$$= r_0 r_{43} r_1 r_{01} - r_4 r_0 r_{43} r_1 r_0$$

$$= g_6^{(3,1)}.$$

$$(g_6^{(3,1)}, f_3^{(1)})_{r_0r_{43}r_1r_{02}r_1} = g_6^{(3,1)}r_2r_1 - r_0r_{43}r_1r_0f_3^{(1)}$$

$$= (r_0r_{43}r_1r_0r_1 - r_4r_0r_{43}r_1r_0)r_2r_1 - r_0r_{43}r_1r_0(r_1r_2r_1 - r_2r_1r_2)$$

$$= r_0r_{43}r_{12}r_{02} - r_4r_0r_{43}r_{12}r_{01}$$

$$= g_6^{(3,2)}.$$

$$(g_6^{(3,2)}, f_3^{(2)})_{r_0r_{43}r_{12}r_{03}r_2} = g_6^{(3,2)}r_{32} - r_0r_{43}r_{12}r_{01}f_3^{(2)}$$

$$= (r_0r_{43}r_{12}r_{02} - r_4r_0r_{43}r_{12}r_{01})r_{32} - r_0r_{43}r_{12}r_0r_1(r_2r_3r_2 - r_3r_2r_3)$$

$$= r_0r_{43}r_{13}r_{03} - r_4r_0r_{43}r_{13}r_{02}$$

$$= g_6^{(3,3)}.$$

$$(g_5^{(2)}, f_3^{(0)})_{r_0r_{42}r_{01}r_0} = g_5^{(2)}r_1r_0 - r_0r_{42}f_3^{(0)}$$

$$= (r_0r_{42}r_0 - r_4r_0r_{42})r_1r_0 - r_0r_{42}(r_0r_1r_0 - r_1r_0r_1)$$

$$= r_0r_{42}r_1r_{01} - r_4r_0r_{42}r_1r_0$$

$$= g_6^{(2,1)}.$$

$$(g_6^{(2,1)}, f_3^{(1)})_{r_0r_{42}r_1r_{02}r_1} = g_6^{(2,1)}r_2r_1 - r_0r_{42}r_1r_0f_3^{(1)}$$

$$= (r_0r_{42}r_1r_{01} - r_4r_0r_{42}r_1r_0)r_2r_1 - r_0r_{42}r_1r_0(r_1r_2r_1 - r_2r_1r_2)$$

$$= r_0r_{42}r_{12}r_{02} - r_4r_0r_{42}r_{12}r_{01}$$

$$= g_6^{(2,2)}.$$

$$(g_6^{(2,2)}, f_3^{(2)})_{r_0r_{42}r_{12}r_{03}r_2} = g_6^{(2,2)}r_{32} - r_0r_{42}r_{12}r_0r_1f_3^{(2)}$$

$$= (r_0r_{42}r_{12}r_{02} - r_4r_0r_{42}r_{12}r_{01})r_{32} - r_0r_{42}r_{12}r_0r_1(r_2r_3r_2 - r_3r_2r_3)$$

$$= r_0r_{42}r_{13}r_{03} - r_4r_0r_{42}r_{13}r_{02}$$

$$= g_6^{(2,3)}.$$

$g_7^{(k,l)} = r_0r_{4k}r_{1l}r_0r_{4k} - r_1r_0r_{4k}r_{1l}r_0r_{4,k+1} \qquad 1 \le l < k-1 < 4$

$$(f_3^{(0)}, f_4)_{r_{01}r_0r_4r_0} = f_3^{(0)}r_4r_0 - r_{01}f_4$$

$$= (r_0r_1r_0 - r_1r_0r_1)r_4r_0 - r_{01}(r_0r_4r_0 - r_4r_0r_4)$$

$$= r_0r_4r_1r_0r_4 - r_1r_0r_4r_1r_0$$

$$= g_7^{(4,1)}.$$

$$\begin{aligned}
(g_1^{(0,2)}, f_4)_{r_{02}r_0r_4r_0} &= g_1^{(0,2)}r_4r_0 - r_{02}f_4 \\
&= (r_{02}r_0 - r_1r_{02})r_4r_0 - r_{02}(r_0r_4r_0 - r_4r_0r_4) \\
&= r_0r_4r_{12}r_0r_4 - r_1r_0r_4r_{12}r_0 \\
&= g_7^{(4,2)}.
\end{aligned}$$

$$\begin{aligned}
(g_7^{(4,1)}, g_4^{(3)})_{r_0r_4r_1r_0r_{43}r_4} &= g_7^{(4,1)}r_{34} - r_0r_4r_1g_4^{(3)} \\
&= (r_0r_4r_1r_0r_4 - r_1r_0r_4r_1r_0)r_{34} - r_0r_4r_1(r_0r_{43}r_4 - r_3r_0r_{43}) \\
&= r_0r_{43}r_1r_0r_{43} - r_1r_0r_{43}r_1r_0r_4 \\
&= g_7^{(3,1)}.
\end{aligned}$$

$$\underline{g_8^{(k,l)} = r_0r_{4k}r_{1l}r_0r_{4,k-1} - r_1r_0r_{4k}r_{1l}r_0r_{4k} \qquad 3 \le k \le 4, \qquad k-1 \le l \le 4}$$

$$\begin{aligned}
(g_7^{(4,2)}, g_4^{(3)})_{r_0r_4r_{12}r_0r_{43}r_4} &= g_7^{(4,2)}r_{34} - r_0r_4r_{12}g_4^{(3)} \\
&= (r_0r_4r_{12}r_0r_4 - r_1r_0r_4r_{12}r_0)r_{34} - r_0r_4r_{12}(r_0r_{43}r_4 - r_3r_0r_{43}) \\
&= r_0r_4r_{12}r_3r_0r_{43} - r_1r_0r_4r_{12}r_0r_{34} \\
&= (r_0r_4r_{13}r_0r_{43} - r_1r_0r_4r_{13}r_0r_4) - r_1r_0r_4r_{12}(r_0r_3 - r_3r_0)r_4 \\
&= g_8^{(4,3)} - r_1r_0r_4r_{12}f_2^{(0,3)}r_4.
\end{aligned}$$

$$\begin{aligned}
(g_2, g_4^{(3)})_{r_{04}r_0r_{43}r_4} &= g_2r_{34} - r_{04}g_4^{(3)} \\
&= (r_{04}r_0r_4 - r_0r_{04}r_0)r_{34} - r_{04}(r_0r_{43}r_4 - r_3r_0r_{43}) \\
&= r_{02}(r_3r_4r_3 - r_4r_3r_4)r_0r_{43} - r_1r_{04}(r_0r_3 - r_3r_0)r_4 \\
&\qquad + r_0r_1(r_2r_4 - r_4r_2)r_{34}r_0r_{43} - r_1r_0r_{12}(r_3r_4r_3 - r_4r_3r_4)r_0r_4 \\
&\qquad + r_0(r_1r_4 - r_4r_1)r_{24}r_0r_{43} - r_1r_0r_1(r_2r_4 - r_4r_2)r_{34}r_0r_4 \\
&\qquad - r_1r_0(r_1r_4 - r_4r_1)r_{24}r_0r_4 + r_0r_4r_{14}r_0r_{43} - r_1r_0r_4r_{14}r_0r_4 \\
&= r_{02}f_3^{(3)}r_0r_{43} - r_1r_{04}f_2^{(0,3)}r_4 + r_0r_1f_2^{(2,4)}r_{34}r_0r_{43} - r_1r_0r_{12}f_3^{(3)}r_0r_4 \\
&\qquad + r_0f_2^{(1,4)}r_{24}r_0r_{43} - r_1r_0r_1f_2(2,4)r_{34}r_0r_4 - r_1r_0f_2^{(1,4)}r_{24}r_0r_4 \\
&\qquad + g_8^{(4,4)}.
\end{aligned}$$

$$
\begin{aligned}
(g_7^{(3,1)}, g_4^{(2)})_{r_0r_{43}r_1r_0r_{42}r_3} &= g_7^{(3,1)}r_{23} - r_0r_{43}r_1g_4^{(2)} \\
&= (r_0r_{43}r_1r_0r_{43} - r_1r_0r_{43}r_1r_0r_4)r_{23} - r_0r_{43}r_1(r_0r_{42}r_3 - r_2r_0r_{43}) \\
&= (r_0r_{43}r_{12}r_0r_{42} - r_1r_0r_{43}r_{12}r_0r_{43}) - r_1r_0r_{43}r_1(r_0r_4r_2 - r_2r_0r_4)r_3 \\
&= g_8^{(3,2)} - r_1r_0r_{43}r_1g_3^{(2,4)}r_3.
\end{aligned}
$$

$$
\begin{aligned}
(g_8^{(4,4)}, g_4^{(2)})_{r_0r_4r_{13}r_0r_{42}r_3} &= g_8^{(4,4)}r_{23} - r_0r_4r_{13}g_4^{(2)} \\
&= (r_0r_4r_{14}r_0r_{43} - r_1r_0r_4r_{14}r_0r_4)r_{23} - r_0r_4r_{13}(r_0r_{42}r_3 - r_2r_0r_{42}) \\
&= r_0r_4r_1(r_{24}r_2 - r_3r_{24})r_0r_{42} - r_1r_0r_4r_{14}(r_0r_4r_2 - r_2r_0r_4) \\
&\qquad + r_0r_4(r_1r_3 - r_3r_1)r_{24}r_0r_{42} - r_1r_0r_4r_1(r_{24}r_2 - r_3r_{24})r_0r_{43} \\
&\qquad - r_1r_0r_4(r_1r_3 - r_3r_1)r_{24}r_0r_{43} + r_0r_{43}r_{14}r_0r_{42} \\
&\qquad - r_1r_0r_{43}r_{14}r_0r_{43} \\
&= r_0r_4r_1g_1^{(2,4)}r_0r_{42} - r_1r_0r_4r_{14}g_3^{(2,4)} + r_0r_4f_2^{(1,3)}r_{24}r_0r_{42} \\
&\qquad - r_1r_0r_4r_1g_1^{(2,4)}r_0r_{43} - r_1r_0r_4f_2^{(1,3)}r_{24}r_0r_{43} + g_8^{(3,4)}.
\end{aligned}
$$

$$
\begin{aligned}
(g_8^{(4,3)}, g_4^{(2)})_{r_0r_4r_{14}r_0r_{42}r_3} &= g_8^{(4,3)}r_{23} - r_0r_4r_{14}g_4^{(2)} \\
&= (r_0r_4r_{13}r_0r_{43} - r_1r_0r_4r_{13}r_0r_4)r_{23} - r_0r_4r_{14}(r_0r_{42}r_3 - r_2r_0r_{42}) \\
&= r_0r_4r_1(r_2r_3r_2 - r_3r_2r_3)r_0r_{42} - r_1r_0r_4r_{13}(r_0r_4r_2 - r_2r_0r_4)r_3 \\
&\qquad + r_0r_4(r_1r_3 - r_3r_1)r_{23}r_0r_{42} + r_0r_4(r_1r_3 - r_3r_1)r_{23}r_0r_{42} \\
&\qquad - r_1r_0r_4r_1(r_2r_3r_2 - r_3r_2r_3) + r_1r_0r_4(r_1r_3 - r_3r_1)r_{23}r_0r_{43} \\
&\qquad + r_0r_{43}r_{13}r_0r_{42} - r_1r_0r_{43}r_{13}r_0r_{43} \\
&= r_0r_4r_1g_1^{(2,3)}r_0r_{42} - r_1r_0r_4r_{13}g_3^{(2,4)}r_3 + r_0r_4f_2^{(1,3)}r_{23}r_0r_{42} \\
&\qquad - r_1r_0r_4r_1g_1^{(2,3)} + r_1r_0r_4f_2^{(1,3)}r_{23}r_0r_{43} + g_8^{(3,3)}.
\end{aligned}
$$

$g_9^{(j,k,l)} = r_0r_{4k}r_{1l}r_0r_{4j}r_{1l} - r_4r_0r_{4k}r_{1l}r_0r_{4j}r_{1,l-1} \quad 2 \le k \le 3, \quad k+1 \le j \le 4, 1 \le l \le j-2$

$$
\begin{aligned}
(g_6^{(3,1)}, f_2^{(1,4)})_{r_0r_{43}r_1r_{01}r_4} &= g_6^{(3,1)}r_4 - r_0r_{43}r_1r_0f_2^{(1,4)} \\
&= (r_0r_{43}r_1r_0r_1 - r_4r_0r_{43}r_1r_0)r_4 - r_0r_{43}r_1r_0(r_1r_4 - r_4r_1) \\
&= r_0r_{43}r_1r_0r_4r_1 - r_4r_0r_{43}r_1r_0r_4
\end{aligned}
$$

$$= g_9^{(4,3,1)}.$$

$$
\begin{aligned}
(g_9^{(4,3,1)}, f_3^{(1)})_{r_0r_{43}r_1r_0r_4r_{12}r_1} &= g_9^{(4,3,1)}r_2r_1 - r_0r_{43}r_1r_0r_4f_3^{(1)} \\
&= (r_0r_{43}r_1r_0r_4r_1 - r_4r_0r_{43}r_1r_0r_4)r_2r_1 \\
&\qquad - r_0r_{43}r_1r_0r_4(r_1r_2r_1 - r_2r_1r_2) \\
&= r_0r_{43}r_{12}r_0r_4r_{12} - r_4r_0r_{43}r_{12}r_0r_4r_{12} \\
&= g_9^{(4,3,2)}.
\end{aligned}
$$

$$
\begin{aligned}
(g_6^{(2,1)}, f_2^{(1,4)})_{r_0r_{42}r_1r_{01}r_4} &= g_6^{(2,1)}r_4 - r_0r_{42}r_1r_0f_2^{(1,4)} \\
&= (r_0r_{42}r_1r_{01} - r_4r_0r_{42}r_1r_0)r_4 - r_0r_{42}r_1r_0(r_1r_4 - r_4r_1) \\
&= r_0r_{42}r_1r_0r_4r_1 - r_4r_0r_{42}r_1r_0r_4 \\
&= g_9^{(4,2,1)}.
\end{aligned}
$$

$$
\begin{aligned}
(g_9^{(4,2,1)}, f_3^{(1)})_{r_0r_{42}r_1r_0r_4r_{12}r_1} &= g_9^{(4,2,1)}r_2r_1 - r_0r_{42}r_1r_0r_4f_3^{(1)} \\
&= (r_0r_{42}r_1r_0r_4r_1 - r_4r_0r_{42}r_1r_0r_4)r_2r_1 \\
&\qquad - r_0r_{42}r_1r_0r_4(r_1r_2r_1 - r_2r_1r_2) \\
&= r_0r_{42}r_{12}r_0r_4r_{12} - r_4r_0r_{42}r_{12}r_0r_4r_1 \\
&= g_9^{(4,2,2)}.
\end{aligned}
$$

$$
\begin{aligned}
(g_9^{(4,2,1)}, f_2^{(1,3)})_{r_0r_{42}r_1r_0r_{43}r_1r_3} &= g_9^{(4,2,1)}r_3 - r_0r_{42}r_1r_0r_{43}f_2^{(1,3)} \\
&= (r_0r_{42}r_1r_0r_4r_1 - r_4r_0r_{42}r_1r_0r_4)r_3 \\
&\qquad - r_0r_{42}r_1r_0r_{43}(r_1r_3 - r_3r_1) \\
&= r_0r_{42}r_1r_0r_{43}r_1 - r_4r_0r_{42}r_1r_0r_{43} \\
&= g_9^{(3,2,1)}.
\end{aligned}
$$

$$\underline{g_{10}^{(j,k,l)} = r_0r_{4k}r_{1l}r_0r_{4j}r_{1,l+1} - r_4r_0r_{4k}r_{1l}r_0r_{4j}r_{1l} \quad 2 \le k \le 4, \quad k \le j \le 4, j-1 \le l \le 3}$$

$$
\begin{aligned}
(g_6^{(4,3)}, f_3^{(3)})_{r_0r_4r_{13}r_{03}r_{43}} &= g_6^{(4,3)}r_{43} - r_0r_4r_{13}r_{02}f_3^{(3)} \\
&= (r_0r_4r_{13}r_{03} - r_4r_0r_4r_{13}r_{02})r_{43} - r_0r_4r_{13}r_{02}(r_3r_4r_3 - r_4r_3r_4) \\
&= r_0r_4r_{13}r_0r_4r_{14} - r_4r_0r_4r_{13}r_0r_4r_{13}
\end{aligned}
$$

$$
\begin{aligned}
&= g_{10}^{(4,4,3)}.\\
(g_6^{(3,3)}, f_3^{(3)})_{r_0r_{43}r_{13}r_{04}r_3} &= g_6^{(3,3)}r_{43} - r_0r_{43}r_{13}r_{02}f_3^{(3)}\\
&= (r_0r_{43}r_{13}r_{03} - r_4r_0r_{43}r_{13}r_{02})r_{43} - r_0r_{43}r_{13}r_{02}(r_3r_4r_3 - r_4r_3r_4)\\
&= r_0r_{43}r_{13}r_0r_4r_{14} - r_4r_0r_{43}r_{13}r_0r_4r_{13}\\
&= g_{10}^{(4,3,3)}.\\
(g_6^{(2,3)}, f_3^{(3)})_{r_0r_{42}r_{13}r_{04}r_3} &= g_6^{(2,3)}r_{43} - r_0r_{42}r_{13}r_{02}f_3^{(3)}\\
&= (r_0r_{42}r_{13}r_{03} - r_4r_0r_{42}r_{13}r_{02})r_{43} - r_0r_{42}r_{13}r_{02}(r_3r_4r_3 - r_4r_3r_4)\\
&= r_0r_{42}r_{13}r_0r_4r_{14} - r_4r_0r_{42}r_{13}r_0r_4r_{13}\\
&= g_{10}^{(4,2,3)}.\\
(g_9^{(4,3,2)}, f_3^{(2)})_{r_0r_{43}r_{12}r_0r_4r_{13}r_2} &= g_9^{(4,3,2)}r_{32} - r_0r_{43}r_{12}r_0r_4r_1f_3^{(2)}\\
&= (r_0r_{43}r_{12}r_0r_4r_{12} - r_4r_0r_{43}r_{12}r_0r_4r_{12})r_3r_2\\
&\qquad - r_0r_{43}r_{12}r_0r_4r_1(r_2r_3r_2 - r_3r_2r_3)\\
&= r_0r_{43}r_{12}r_0r_{43}r_{13} - r_4r_0r_{43}r_{12}r_0r_{43}r_{12}\\
&= g_{10}^{(3,3,2)}.\\
(g_{10}^{(3,3,2)}, f_3^{(3)})_{r_0r_{43}r_{12}r_0r_{43}r_{14}r_3} &= g_{10}^{(3,3,2)}r_{43} - r_0r_{43}r_{12}r_0r_{43}r_{12}f_3^{(3)}\\
&= (r_0r_{43}r_{12}r_0r_{43}r_{13} - r_4r_0r_{43}r_{12}r_0r_{43}r_{12})r_{43}\\
&\qquad - r_0r_{43}r_{12}r_0r_{43}r_{12}(r_3r_4r_3 - r_4r_3r_4)\\
&= r_0r_{43}r_{12}r_0r_{43}r_1(r_2r_4 - r_4r_2)r_{34}\\
&\qquad - r_4r_0r_{43}r_{12}r_0r_{43}r_1(r_2r_4 - r_4r_2)r_3\\
&\qquad + r_0r_{43}r_{12}r_0r_{43}(r_1r_4 - r_4r_1)r_{24}\\
&\qquad - r_4r_0r_{43}r_{12}r_0r_{43}(r_1r_4 - r_4r_1)\\
&\qquad + r_0r_{43}r_{12}(r_0r_{43}r_4 - r_3r_0r_{43})r_{14}\\
&\qquad - r_4r_0r_{43}r_{12}(r_0r_{43}r_4 - r_3r_0r_{43})r_{13}\\
&\qquad + r_0r_{43}r_{13}r_0r_{43}r_{14} - r_4r_0r_{43}r_{13}r_0r_{43}r_{13}\\
&= r_0r_{43}r_{12}r_0r_{43}r_1f_2^{(2,4)}r_{34} - r_4r_0r_{43}r_{12}r_0r_{43}r_1f_2^{(2,4)}r_3
\end{aligned}
$$

$$
\begin{aligned}
&+ r_0r_{43}r_{12}r_0r_{43}f_2^{(1,4)}r_{24} - r_4r_0r_{43}r_{12}r_0r_{43}f_2^{(1,4)} \\
&+ r_0r_{43}r_{12}g_4^{(3)}r_{14} - r_4r_0r_{43}r_{12}g_4^{(3)}r_{13} + g_{10}^{(3,3,3)}.
\end{aligned}
$$

$$
\begin{aligned}
(g_9^{(4,2,2)}, f_3^{(2)})_{r_0r_{42}r_{12}r_0r_4r_{13}r_2} &= g_9^{(4,2,2)}r_{32} - r_0r_{42}r_{12}r_0r_4r_1f_3^{(2)} \\
&= (r_0r_{42}r_{12}r_0r_4r_{12} - r_4r_0r_{42}r_{12}r_0r_4r_1)r_{32} \\
&\quad - r_0r_{42}r_{12}r_0r_4r_1(r_2r_3r_2 - r_3r_2r_3) \\
&= r_0r_{42}r_{12}r_0r_{43}r_{13} - r_4r_0r_{42}r_{12}r_0r_{43}r_{12} \\
&= g_{10}^{(3,2,2)}.
\end{aligned}
$$

$$
\begin{aligned}
(g_{10}^{(3,2,2)}, f_3^{(3)})_{r_0r_{42}r_{12}r_0r_{43}r_{14}r_3} &= g_{10}^{(3,2,2)}r_{43} - r_0r_{42}r_{12}r_0r_{43}r_{12}f_3^{(3)} \\
&= (r_0r_{42}r_{12}r_0r_{43}r_{13} - r_4r_0r_{42}r_{12}r_0r_{43}r_{12})r_{43} \\
&\quad - r_0r_{42}r_{12}r_0r_{43}r_{12}(r_3r_4r_3 - r_4r_3r_4) \\
&= r_0r_{42}r_{12}r_0r_{43}r_1(r_2r_4 - r_4r_2)r_{34} \\
&\quad - r_4r_0r_{42}r_{12}r_0r_{43}r_1(r_2r_4 - r_4r_2)r_3 \\
&\quad + r_0r_{42}r_{12}r_0r_{43}(r_1r_4 - r_4r_1)r_{24} \\
&\quad - r_4r_0r_{42}r_{12}r_0r_{43}(r_1r_4 - r_4r_1)r_{23} \\
&\quad + r_0r_{42}r_{12}(r_0r_{43}r_4 - r_3r_0r_{43})r_{14} \\
&\quad - r_4r_0r_{42}r_{12}(r_0r_{43}r_4 - r_3r_0r_{43})r_{13} \\
&\quad + r_0r_{42}r_{13}r_0r_4r_{14} - r_4r_0r_{42}r_{13}r_0r_4r_{13} \\
&= r_0r_{42}r_{12}r_0r_{43}r_1f_2^{(2,4)}r_{34} - r_4r_0r_{42}r_{12}r_0r_{43}r_1f_2^{(2,4)}r_3 \\
&\quad + r_0r_{42}r_{12}r_0r_{43}f_2^{(1,4)}r_{24} - r_4r_0r_{42}r_{12}r_0r_{43}f_2^{(1,4)}r_{23} \\
&\quad + r_0r_{42}r_{12}g_4^{(3)}r_{14} - r_4r_0r_{42}r_{12}g_4^{(3)}r_{13} + g_{10}^{(3,2,3)}.
\end{aligned}
$$

$$
\begin{aligned}
(g_9^{(3,2,1)}, f_3^{(1)})_{r_0r_{42}r_1r_0r_{43}r_{12}r_1} &= g_9^{(3,2,1)}r_2r_1 - r_0r_{42}r_1r_0r_{43}f_3^{(1)} \\
&= (r_0r_{42}r_1r_0r_{43}r_1 - r_4r_0r_{42}r_1r_0r_{43})r_2r_1 \\
&\quad - r_0r_{42}r_1r_0r_{43}(r_1r_2r_1 - r_2r_1r_2) \\
&= r_0r_{42}r_1r_0r_{42}r_{12} - r_4r_0r_{42}r_1r_0r_{42}r_1 \\
&= g_{10}^{(2,2,1)}.
\end{aligned}
$$

$$
\begin{aligned}
(g_{10}^{(2,2,1)}, f_3^{(2)})_{r_0r_{42}r_1r_0r_{42}r_{13}r_2} &= g_{10}^{(2,2,1)}r_{32} - r_0r_{42}r_1r_0r_{42}r_1f_3^{(2)} \\
&= (r_0r_{42}r_1r_0r_{42}r_{12} - r_4r_0r_{42}r_1r_0r_{42}r_1)r_{32} \\
&\qquad - r_0r_{42}r_1r_0r_{42}r_1(r_2r_3r_2 - r_3r_2r_3) \\
&= r_0r_{42}r_1r_0r_{42}(r_1r_3 - r_3r_1)r_{23} - r_4r_0r_{42}r_1r_0r_{42}(r_1r_3 - r_3r_1)r_2 \\
&\qquad + r_0r_{42}r_1(r_0r_{42}r_3 - r_2r_0r_{42})r_{13} \\
&\qquad - r_4r_0r_{42}r_1(r_0r_{42}r_3 - r_2r_0r_{42})r_{12} \\
&\qquad + r_0r_{42}r_{12}r_0r_{42}r_{13} - r_4r_0r_{42}r_{12}r_0r_{42}r_{12} \\
&= r_0r_{42}r_1r_0r_{42}f_2^{(1,3)}r_{23} - r_4r_0r_{42}r_1r_0r_{42}f_2^{(1,3)}r_2 + r_0r_{42}r_1g_4^{(2)}r_{13} \\
&\qquad - r_4r_0r_{42}r_1g_4^{(2)}r_{12} + g_{10}^{(2,2,2)}.
\end{aligned}
$$

$$
\begin{aligned}
(g_{10}^{(2,2,2)}, f_3^{(3)})_{r_0r_{42}r_{12}r_0r_{42}r_{14}r_3} &= g_{10}^{(2,2,2)}r_{43} - r_0r_{42}r_{12}r_0r_{42}r_{12}f_3^{(3)} \\
&= (r_0r_{42}r_{12}r_0r_{42}r_{13} - r_4r_0r_{42}r_{12}r_0r_{42}r_{12})r_{43} \\
&\qquad - r_0r_{42}r_{12}r_0r_{42}r_{12}(r_3r_4r_3 - r_4r_3r_4) \\
&= r_0r_{42}r_{12}r_0r_{42}r_1(r_2r_4 - r_4r_2)r_{34} \\
&\quad - r_4r_0r_{42}r_{12}r_0r_{42}r_1(r_2r_4 - r_4r_2)r_3 \\
&\quad + r_0r_{42}r_{12}r_0r_{42}(r_1r_4 - r_4r_1)r_{24} \\
&\quad - r_4r_0r_{42}r_{12}r_0r_{42}(r_1r_4 - r_4r_1)r_{23} \\
&\quad + r_0r_{42}r_{12}r_0r_{43}(r_2r_4 - r_4r_2)r_{14} \\
&\quad - r_4r_0r_{42}r_{12}r_0r_{43}(r_2r_4 - r_4r_2)r_{13} \\
&\quad + r_0r_{42}r_{13}r_0r_{42}r_{14} - r_4r_0r_{42}r_{13}r_0r_{42}r_{13} \\
&= r_0r_{42}r_{12}r_0r_{42}r_1f_2^{(2,4)}r_{34} - r_4r_0r_{42}r_{12}r_0r_{42}r_1f_2^{(2,4)}r_3 \\
&\qquad + r_0r_{42}r_{12}r_0r_{42}f_2^{(1,4)}r_{24} - r_4r_0r_{42}r_{12}r_0r_{42}f_2^{(1,4)}r_{23} \\
&\qquad + r_0r_{42}r_{12}r_0r_{43}f_2^{(2,4)}r_{14} - r_4r_0r_{42}r_{12}r_0r_{43}f_2^{(2,4)}r_{13} \\
&\qquad + r_0r_{42}r_{12}g_4^{(3)}r_2r_{14} - r_4r_0r_{42}r_{12}g_4^{(3)}r_2r_{13} + g_{10}^{(2,2,3)}.
\end{aligned}
$$

3.3 Reduced Forms for $\widetilde{A_n}$

In the previous section, we explicitly presented Gröbner-Shirshov bases for the affine Weyl group of type $\widetilde{A_n}$, and now we will classify all reduced forms for the affine Weyl group of type $\widetilde{A_n}$ in this section.

Definition 3.16. (Reduced Form) Let $R' = R \cup \{g_1, g_2, \ldots, g_{10}\}$. We want to find the properties of the elements of the set $Red(R') = \{w \in S^* | w \neq a\overline{f}b; a, b \in S^*, f \in R'\}$. If $w \in Red(R')$, then we call that it is a reduced form.

Notice that elements of R' not containing r_0 is in fact a Gröbner-Shirshov basis for the Coxeter group A_n. The following lemma is just another way of expressing of the Lemma 3.2 of [1].

Lemma 3.3. *Any reduced form not containing r_0 is in the format*

$$s = (r_{nj_n})^{\alpha_n}(r_{n-1,j_{n-1}})^{\alpha_{n-1}} \ldots (r_{2j_2})^{\alpha_2}(r_{1j_1})^{\alpha_1}$$

where $i \leq j_i \leq n$ and $\alpha_i \in \{0, 1\}$.

After investigation of leading words of the elements of R', we can claim the following results. For convenience, we write $r_0 r_{n,n+1} r_{1l}$ instead of r_{0l} and $r_0 r_{nk} r_{10}$ instead of $r_0 r_{nk}$.

Lemma 3.4. *The following words are reduced forms.*

(i) $w = r_0 r_{nk} r_{1l}$ *for* $2 \leq k \leq n+1,\ 0 \leq l \leq n$.

(ii) $(r_0 r_{nk} r_{1l})(r_0 r_{np} r_{1q}) = \begin{cases} (k < p) \wedge (l > q), & \text{if} \quad q - p < l - k < -1 \\ (k \leq p) \wedge (l > q), & \text{if} \quad (q - p < -1) \wedge (l - k \geq -1) \\ (k \leq p) \wedge (l \geq q), & \text{if} \quad l - k > q - p \geq -1 \end{cases}$

for $2 \leq k, p \leq n+1,\ 0 \leq l, q \leq n$.

Lemma 3.5. *Let w be a reduced form starting with r_i for $i = 1, 2, \ldots, n$ and let t be a reduced form starting with r_0. Then wt is also a reduced form.*

Proof. The result follows from the following observation. Any leading word starting with r_i for $i = 1, 2, \ldots, n$ in R' does not contain r_0. □

Lemma 3.6. *Let w_i be reduced forms in the format $r_0 r_{nk} r_{1l}$ where $2 \leq k \leq n+1$, $0 \leq l \leq n$ for $i = 1, 2, \ldots, j$. If $w_1 w_2 \ldots w_{j-1}$ and $w_{j-1} w_j$ are reduced, then $w_1 w_2 \ldots w_j$ is also reduced.*

Proof. We prove by induction. Let $w = w_1 w_2 w_3$. The only possible reducible subform of w is in the format $r w_2 s$ where r and s are proper subforms of w_1 and w_3, respectively. Since $w_2 w_3$ is a reduced form, $w_2 s$ is also reduced form starting with r_0. Since r is a proper subform of w_1, r is also a reduced form not containing r_0. By lemma 3.5, $r w_2 s$ is also reduced. Hence $w_1 w_2 w_3$ should be a reduced form.

Assume $w_1 w_2 \ldots w_{j-1}$ and $w_{j-1} w_j$ are reduced. So $w_2 w_3 \ldots w_{j-1}$ is also reduced. By inductive assumption $w_2 w_3 \ldots w_j$ should be reduced. Hence the only possible reducible subform of $w = w_1 w_2 \ldots w_j$ is in the format $r w_2 w_3 \ldots w_j$ where r is a proper subform of w_1. Since w_1 is reduced, r is also a reduced form not containing r_0. By lemma 3.5, $r w_2 w_3 \ldots w_j$ is also reduced. Therefore w should be a reduced form. □

Definition 3.17. In this area, we will give some definitions which are required.

(i) Let $b_i = a_{(p_i, q_i)} = r_0 r_{n, n-p_i+2} r_{1, q_i}$ where $0 \leq p_i, q_i \leq n$ and $p_i + q_i = i$ for $i = 1, 2, \ldots, 2n$.

(ii) Define

$$u = (b_{2n})^{m_{2n}} (b_{2n-1})^{m_{2n-1}} \ldots (b_{n+1})^{m_{n+1}}$$

where $p_{i-1} = p_i$ or $p_{i-1} = p_i + 1$ for $i = n+2, n+3, \ldots, 2n$. Furthermore, $m_i = 1$ when $p_{i+1} = p_i > p_{i-1}$ for $i = n+2, n+3, \ldots, 2n-1$. In particular $m_{n+1} = 1$ when $p_{n+2} = p_{n+1}$. Otherwise, $m_i \in \{0, 1\}$. Each b_i is called a component of the word u. If m_i can take only the value 1, then b_i is called a marked component of u. The other components are called unmarked components of u.

(iii) Let $(t_j) \subseteq \{0, 1, 2, \ldots, n\}$ be a decreasing subsequence such that $t_j - t_{j-1} > 1$.

Define

$$v = b_{t_1} b_{t_2} \cdots b_{t_s}$$

where $(p_{t_i}, q_{t_i}) \neq (0, n)$, $p_{t_i} > p_{t_{i-1}}$ and $q_{t_i} > q_{t_{i-1}}$ for $i = 2, 3, \ldots, s$.

(iv) Assume that u and v are the words defined as above except the value of m_{n+1}. Let $w = uv$ where $p_{n+1} \geq p_{t_1}$ and $q_{n+1} > q_{t_1}$. Furthermore, $m_{n+1} = 1$ when $p_{n+2} = p_{n+1} > p_{t_1}$. Otherwise, $m_{n+1} \in \{0, 1\}$.

After this point u, v and w will represent the words given by above definition.

Proposition 3.1. *The words u, v and w given by definition 3.17 are reduced.*

Proof. Let $n + 1 \leq j < i \leq 2n$ and

$$b_i = a_{(p_i q_i)} = r_0 r_{n,n-p_i+2} r_{1,q_i}$$

and

$$b_j = a_{(p_j q_j)} = r_0 r_{n,n-p_j+2} r_{1,q_j}.$$

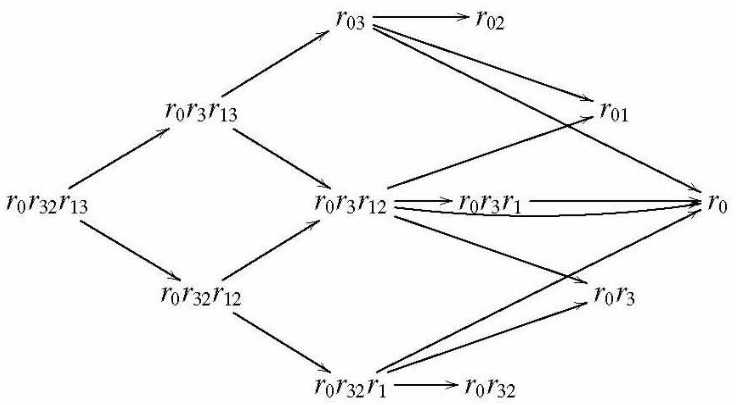
$q_j - (n - p_j + 2) = q_j + p_j - (n + 2) = j - (n + 2) \geq -1$. Similarly, $q_i - (n - p_i + 2) > -1$. By definition of the word u, $p_j \leq p_i$ and then $n - p_i + 2 \leq n - p_j + 2$. By lemma 3.4, $b_i b_j$ is a reduced form. Hence u is a reduced form. □

The figure 3.1 shows every possible reduced forms u, v and w for $n = 3$ and $n = 4$, respectively. Before we count these forms, let us explain the reason to introduce marked components. We can produce several words from a word u by taking exponents of unmarked components zero or one. The marked components are introduced to prevent a word being produced by two different u's or w's. The next example will explain that why we introduce the concept of marked component.

Example 3.10. Let

$$w_1 = (r_0 r_{42} r_{14})^{m_1} (r_0 r_{42} r_{13})^{m_2} (r_0 r_{43} r_{13})^{m_3} (r_0 r_4 r_{13})^{m_4} r_{01}$$

For $n = 3$



For $n = 4$

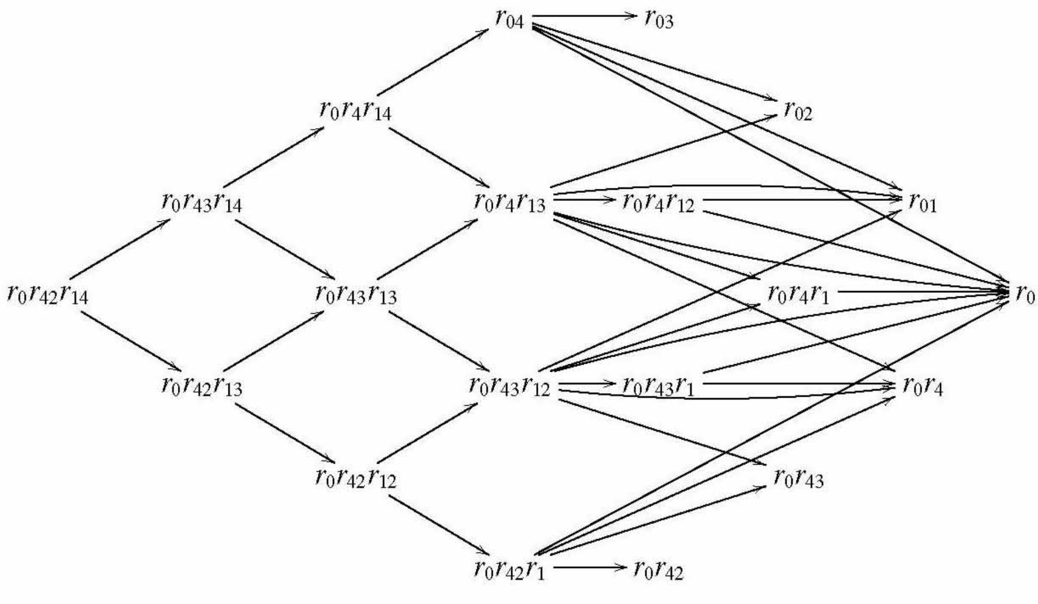


Figure 3.1: Graphics of $\widetilde{A_3}$ and $\widetilde{A_4}$, respectively.

and

$$w_2 = (r_0r_{42}r_{14})^{m_1}(r_0r_{43}r_{14})^{m_2}(r_0r_4r_{14})^{m_3}(r_0r_4r_{13})^{m_4}r_{01}.$$

If we take $m_2 = m_3 = 0$ and $m_1 = m_4 = 1$, then the word $(r_0r_{42}r_{14})(r_0r_4r_{13})r_{01}$ can be produced by using both w_1 and w_2. However, $r_0r_{42}r_{13}$ is a marked component in the word w_1. Hence $m_2 = 1$ in w_1. Therefore, the word $(r_0r_{42}r_{14})(r_0r_4r_{13})r_{01}$ can be produced by using only w_2.

Theorem 3.1. **(i)** *Let*

$$u_1 = (b_{2n})^{m_{2n}}(b_{2n-1})^{m_{2n-1}}\dots(b_{n+1})^{m_{n+1}}$$

and

$$u_2 = (\overline{b}_{2n})^{\overline{m}_{2n}}(\overline{b}_{2n-1})^{\overline{m}_{2n-1}}\dots(\overline{b}_{n+1})^{\overline{m}_{n+1}}$$

two different reduced forms given by definition 3.17. One of u_1 and u_2 has a marked component which is not a component of the other.

(ii) *Let v be a word given by definition 3.17. The above argument also valid for the words $w_1 = u_1v$ and $w_2 = u_2v$.*

Proof. **(i)** Since the only possibility for the $b_{2n} = \overline{b}_{2n} = a_{(n,n)}$, the first components of u_1 and u_2 are same. Let j be the largest integer such that $b_j \neq \overline{b}_j$. Assume that $b_{j+1} = \overline{b}_{j+1} = a_{(p,q)}$, $b_j = a_{(p,q-1)}$ and $\overline{b}_j = a_{(p-1,q)}$. There are two cases to consider. In the first case, there exists an index $j-k < n+1$ such that $b_{j-1} = a_{(p,q-2)}, b_{j-2} = a_{(p,q-3)}, \dots, b_{j-k} = a_{(p,q-k-1)}, b_{j-k-1} = a_{(p-1,q-k-1)}$. Hence b_{j-k} is a marked component of u_1. Since $\overline{b}_j = a_{p-1,q}$, $b_{j-k} \neq \overline{b}_{j-k}$. In the second case, $b_{j-1} = a_{(p,q-2)}, b_{j-2} = a_{(p,q-3)}, \dots, b_{n+2} = a_{(p,q+n-j+1)}, b_{n+1} = a_{(p,q+n-j)}$. Hence b_{n+1} is a marked component of u_1. Since $\overline{b}_j = a_{p-1,q}$, $b_{n+1} \neq \overline{b}_{n+1}$.

(ii) If $w_1 = u_1v$ and $w_2 = uv_2$, then the only case we have to consider $b_{j-1} = a_{(p,q-2)}$, $b_{j-2} = a_{(p,q-3)}, \dots$, $b_{n+2} = a_{p,q+n-j+1}$, $b_{n+1} = a_{p,q+n-j}$ and b_{n+1} is not a marked component of w_1. This is true provided that $p_{t_1} = p$. Let $\overline{b}_{n+1} = a_{\overline{p}_{n+1},\overline{q}_{n+1}}$. Since $\overline{b}_j = a_{p-1,q}$, $\overline{p}_{n+1} < p = p_{t_1}$ which is impossible. Hence b_{n+1} is a marked component and $\overline{b}_{n+1} \neq b_{n+1}$.

□

Corollary 3.1. *Let s be a word which is produced by a word u or $w = uv$ by taking certain exponents of unmarked components equal to one. Then this word u or $w = uv$ is unique.*

Proof. Assume that the word s produced by two different u_1 and u_2 (or $w_1 = u_1 v$ and $w_2 = u_2 v$). Then s should contain every marked components of both u_1 and u_2. However, this is impossible. Because one of the u_1 and u_2 has a marked component which is not component of the other. □

Example 3.11. We show the forms that may occur for $n = 4$.

$(r_0 r_{42} r_{14})^{n_1} (r_0 r_{43} r_{14})^{n_2} (r_0 r_4 r_{14})^{n_3} (r_{04})^{n_4}$	1, r_0, r_{01}, r_{02}, r_{03}.
$(r_0 r_{42} r_{14})^{n_1} (r_0 r_{43} r_{14})^{n_2} (r_0 r_4 r_{14})^{n_3} (r_0 r_4 r_{13})^{n_4}$	$1,\ n_4 = 1$; $r_0,\ n_4 = 1$; $r_{01},\ n_4 = 1$; $r_0 r_4$, $r_{02},\ n_4 = 1$; $r_0 r_4 r_1$, $r_0 r_4 r_{12}$, $r_0 r_4 r_1 r_0$, $r_0 r_4 r_{12} r_0$, $r_0 r_4 r_{12} r_{01}$.

$(r_0r_{42}r_{14})^{n_1}(r_0r_{43}r_{14})^{n_2}(r_0r_{43}r_{13})^{n_3}(r_0r_4r_{13})^{n_4}$	$1,\ n_3=1;$ $r_0,\ n_3=1;$ $r_{01},\ n_3=1;$ $r_0r_4,\ n_3=1;$ $r_{02},\ n_3=1;$ $r_0r_4r_1,\ n_3=1;$ $r_0r_4r_{12},\ n_3=1;$ $r_0r_4r_1r_0,\ n_3=1;$ $r_0r_4r_{12}r_0,\ n_3=1;$ $r_0r_4r_{12}r_{01},\ n_3=1.$
$(r_0r_{42}r_{14})^{n_1}(r_0r_{43}r_{14})^{n_2}(r_0r_{43}r_{13})^{n_3}(r_0r_{43}r_{12})^{n_4}$	$1,\ n_4=1;$ $r_0,\ n_4=1;$ $r_{01},\ n_4=1;$ $r_0r_4,\ n_4=1;$ $r_0r_4r_1,\ n_4=1;$ $r_0r_{43},$ $r_0r_{43}r_1,$ $r_0r_4r_1r_0,\ n_4=1;$ $r_0r_{43}r_1r_0,$ $r_0r_{43}r_1r_0r_4.$
$(r_0r_{42}r_{14})^{n_1}(r_0r_{42}r_{13})^{n_2}(r_0r_{43}r_{13})^{n_3}(r_0r_4r_{13})^{n_4}$	$1,\ n_2=1;$ $r_0,\ n_2=1;$ $r_{01},\ n_2=1;$ $r_0r_4,\ n_2=1;$ $r_{02},\ n_2=1;$ $r_0r_4r_1,\ n_2=1;$ $r_0r_4r_{12},\ n_2=1;$ $r_0r_4r_1r_0,\ n_2=1;$ $r_0r_4r_{12}r_0,\ n_2=1;$ $r_0r_{43}r_{12}r_{01},\ n_2=1.$

$(r_0r_{42}r_{14})^{n_1}(r_0r_{42}r_{13})^{n_2}(r_0r_{43}r_{13})^{n_3}(r_0r_{43}r_{12})^{n_4}$	$1,\ n_2 = n_4 = 1;$ $r_0,\ n_2 = n_4 = 1;$ $r_{01},\ n_2 = n_4 = 1;$ $r_0r_4,\ n_2 = n_4 = 1;$ $r_0r_4r_1,\ n_2 = n_4 = 1;$ $r_0r_{43},\ n_2 = n_4 = 1;$ $r_0r_{43}r_1,\ n_2 = n_4 = 1;$ $r_0r_4r_1r_0,\ n_2 = n_4 = 1;$ $r_0r_{43}r_1r_0,\ n_2 = n_4 = 1;$ $r_0r_{43}r_1r_0r_4,\ n_2 = n_4 = 1.$
$(r_0r_{42}r_{14})^{n_1}(r_0r_{42}r_{13})^{n_2}(r_0r_{42}r_{12})^{n_3}(r_0r_{43}r_{12})^{n_4}$	$1,\ n_3 = 1;$ $r_0,\ n_3 = 1;$ $r_{01},\ n_3 = 1;$ $r_0r_4,\ n_3 = 1;$ $r_0r_4r_1,\ n_3 = 1;$ $r_0r_{43},\ n_3 = 1;$ $r_0r_{43}r_1,\ n_3 = 1;$ $r_0r_4r_1r_0,\ n_3 = 1;$ $r_0r_{43}r_1r_0,\ n_3 = 1;$ $r_0r_{43}r_1r_0r_4,\ n_3 = 1.$
$(r_0r_{42}r_{14})^{n_1}(r_0r_{42}r_{13})^{n_2}(r_0r_{42}r_{12})^{n_3}(r_0r_{42}r_1)^{n_4}$	$1,\ n_4 = 1;$ $r_0,\ n_4 = 1;$ $r_0r_4,\ n_4 = 1;$ $r_0r_{43},\ n_4 = 1;$ $r_0r_{42}.$

3.4 Counting Reduced Forms

Theorem 3.2. *In a word $w = uv$, the indexes of the marked components of u together with indexes of v form a decreasing sequence (s_i) of the integers between 1 and $2n - 1$ such that $s_i - s_{i-1} > 1, q_{s_i} \neq n, p_{s_i} > p_{s_{i-1}}$ and $q_{s_i} > q_{s_{i-1}}$.*

Conversely, if (s_i) is a decreasing sequence satisfying these conditions, then a word $w = uv$ where the indexes of the marked components of u together with indexes of v is equal to sequence (s_i) can be constructed.

Proof. The first part of the theorem is easily follows from Definition 3.17. For the second part the word $w = uv$ can be constructed as follows:

The components of u between b_{2n} and b_{s_1} are

$$a_{(n,n)}, a_{(n-1,n)}, \ldots, a_{(p_{s_1},n)}, a_{(p_{s_1},n-1)}, \ldots, a_{(p_{s_1},q_{s_1})}.$$

The components of u between $b_{s_{k-1}}$ and b_{s_k} where $s_k \geq n + 1$ are

$$a_{(p_{s_{k-1}},q_{s_{k-1}})}, a_{(p_{s_{k-1}}-1,q_{s_{k-1}})}, \ldots, a_{(p_{s_k},q_{s_{k-1}})}, a_{(p_{s_k},q_{s_{k-1}}-1)}, \ldots, a_{(p_{s_k},q_{s_k})}.$$

If $s_k \neq n + 1$, then the components of u between b_{s_k} and b_{n+1} are

$$a_{(p_{s_k},q_{s_k})} a_{(p_{s_k}-1,q_{s_k})} \cdots a_{(p_l,q_{s_k})}$$

where $p_l + q_{s_k} = n + 1$. If s_k is the last term of the sequence (s_i), then $w = u$. Otherwise, $v = b_{s_{k+1}} \cdots b_{s_t}$ where s_t is the last term of the sequence (s_i) and $w = uv$. □

Let $w = uv$ and $\widetilde{w} = \widetilde{u}v$ where $\widetilde{u}$ is the marked components of u. We will find a correspondence between the words $\widetilde{w}$ and some special partitions of integers.

3.5 Integer Partitions

Definition 3.18. (Partition)If m is a positive integer, then a partition of m is a non-increasing sequence of positive integers $p_1, p_2, \ldots, p_k$ whose sum is m. Each p_i is called a part of the partition of m.

We let a function $p(n)$ denotes the number of partitions of the integer n.

Example 3.12. $p(7) = 15$ and partitions of 7 are

=7	=4+2+1	=3+1+1+1+1
=6+1	=4+1+1+1	=2+2+2+1
=5+2	=3+3+1	=2+2+1+1+1
=5+1+1	=3+2+2	=2+1+1+1+1+1
=4+3	=3+2+1+1	=1+1+1+1+1+1+1

We take $p(n) = 0$ for all negative values of n and $p(0) = 1$.

Integer partitions were first studied by Euler. For many years one of the most intriguing and difficult questions about them was determining the asymptotic properties of $p(n)$ as n got large. This question was finally answered quite completely by Hardy, Ramanujan, and Rademacher [[20],[21]]. An example of a problem in the theory of integer partitions that remains unsolved, despite a good deal of effort having been expended on it, is to find a simple criterion for deciding whether $p(n)$ is even or odd. Though values of p(n) have been computed for n into the billions, no pattern has been discovered to date. Many other interesting problems in the theory of partitions remain unsolved today.

Theorem 3.3. *[22] The number of partitions of the integer n whose largest part is k is equal to the number of partitions of n with k parts.*

We define a function $p(n, k)$ to be the number of partitions of n whose largest part is k (or equivalently, the number of partitions of n with k parts.)

We will now derive Euler's generating function for the sequence $\{p(n)\}_{n=0}^{\infty}$. In other words, we are looking for some nice form for the function which gives us $\sum_{n=0}^{\infty} p(n)x^n$. Consider

$$(1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + x^6 + \ldots)(1 + x^3 + x^6 + x^9 + \ldots)\ldots \qquad (*)$$

We claim that by expanding this product, we obtain the desired result, namely $\sum_{n=0}^{\infty} p(n)x^n$. It is important to understand why this is true because when we look at several variations, they will be derived in a similar manner. To illustrate, consider the coefficient of x^3. By choosing x from the first parenthesis, x^2 from the second, and 1 from the remaining parentheses, we obtain a contribution of 1 to the coefficient of x^3. Similarly, if we choose x^3 from the third parenthesis, and 1 from all others, we will obtain another contribution of 1 to the coefficient of x^3. So how does this relate to integer partitions?

Let the monomial chosen from the $i - th$ parenthesis $1 + x^i + x^{2i} + x^{3i} + \ldots$ in $(*)$ represent the number of times the part i appears in the partition. In particular, if we choose the monomial $x^{c_i i}$ from the $i - th$ parenthesis, then the value i will appear c_i times in the partition. Each selection of monomials makes one contribution to the coefficient of x^n and in general, each contribution must be of the form $x^{1c_1}x^{2c_2}x^{3c_3}\ldots = x^{c_1+2c_2+3c_3+\ldots}$. Thus the coefficient of x^n is the number of ways of writing $n = c_1 + 2c_2 + 3c_3 + \ldots$ where each $c_i \geq 0$. Notice that this is just another way to represent an integer partition. For example, the partition $25 = 6 + 4 + 4 + 3 + 2 + 2 + 2 + 1 + 1$ could be represented by $25 = 1(2) + 2(3) + 3(1) + 4(2) + 5(0) + 6(1)$. Thus, there is an one to one correspondence between choosing monomials whose product is x^n out of the parentheses in $(*)$ and the partitions of the integer n.

Now we turn back to the original product in $(*)$ and recognize that each term is a geometric series. The product can be written as:

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$

Theorem 3.4. *[22](EULER)*

$$\varepsilon(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n.$$

Let n be a positive integer. Here we identify any partition of $m = d_1 + d_2 + \ldots + d_k$ by the $n-$tuple $(d_1, d_2, \ldots, d_k, 0, 0, \ldots, 0)$ when $k \leq n$. The map forms $b_i = a_{(p_i,q_i)}$ and some specific partitions defined below.

Definition 3.19. (Basic Partition) Let $n \in \mathbb{Z}^+$. The n-tuples $(k, 1 \ldots, 1, 0, \ldots, 0)$ where number of 1's is l for $1 \leq k \leq n$, $1 \leq l \leq n-1$ are called basic partitions.

The basic partition $(k_1, 1, \ldots, 1, 0 \ldots, 0)$ is said to be connected to the basic partition $(k_2, 1, \ldots, 1, 0, \ldots, 0)$ if $k_1 > k_2$ and the number of 1's in the first one is greater than the number of 1's in the second one.

According to theorem 3.2, two basic partitions being connected means corresponding words are successive components of $\widetilde{w}$.

Theorem 3.5. *There is one to one correspondence between words $\widetilde{w}$ and the partitions in which there are at most n parts and in which no parts is larger than n.*

Proof. Since we identify each word $\widehat{w}$ with a sequence of connected basic partitions, we must find a correspondence between sequences of connected partitions and the partitions fit into a box of size $n \times n$. Let $a_1, a_2, \ldots, a_m$ be a sequence of connected partitions where $a_i = (k_i, \underbrace{1, \ldots, 1}_{l_i}, 0, \ldots, 0)$. Hence $k_i > k_j$ and $l_i > l_j$ for $1 \leq i < j \leq m$.

Define

$$\bigoplus_{i=1}^{m} a_i = \sum_{i=1}^{m} \sigma_1^{i-1}(a_i)$$

where $\sigma(p_1, p_2, \ldots, p_{n-1}, p_n) = (p_n, p_1, p_2, \ldots, p_{n-1})$. Then

$$\bigoplus_{i=1}^{m} a_i = (k_1, k_2 + 1, \ldots, k_m + m - 1, \underbrace{m, \ldots, m}_{l_m}, \underbrace{m-1, \ldots, m-1}_{l_{m-1}-(l_m+1)}, \ldots, \underbrace{1, \ldots, 1}_{l_1-(l_2+1)}, 0 \ldots, 0)$$

We prove the last equation by induction on m.

Let $m = 2$, Since $l_1 - 1 \geq l_2$,

$$
\begin{aligned}
a_1 \oplus a_2 &= (k_1, \underbrace{1, \dots, 1}_{l_1}, 0, \dots, 0) + (0, k_2, \underbrace{1, \dots, 1}_{l_2}, 0, \dots, 0) \\
&= (k_1, k_2 + 1, \underbrace{2, \dots, 2}_{l_2}, \underbrace{1, \dots, 1}_{l_1-(l_2+1)}, 0, \dots, 0)
\end{aligned}
$$

Let us assume that

$$
\bigoplus_{i=1}^{m-1} a_i = (k_1, k_2 + 1, \dots, k_{m-1} + m - 2, \underbrace{m-1, \dots, m-1}_{l_{m-1}}, \dots, \underbrace{1, \dots, 1}_{l_1-(l_2+1)}, 0, \dots, 0).
$$

Then we have

$$
\begin{aligned}
\bigoplus_{i=1}^{m} a_i &= \sum_{i=1}^{m} \sigma^{i-1}(a_i) \\
&= \sum_{i=1}^{m-1} \sigma^{i-1}(a_i) + \sigma^{m-1}(a_m) \\
&= \bigoplus_{i=1}^{m-1} a_i + \sigma^{m-1}(a_m) \\
&= (k_1, k_2 + 1, \dots, k_{m-1} + m - 2, \underbrace{m-1, \dots, m-1}_{l_{m-1}}, \dots, \underbrace{1, \dots, 1}_{l_1}, 0, \dots, 0) \\
&\quad + (\underbrace{0, 0, \dots, 0}_{m-1}, k_m, \underbrace{1, \dots, 1}_{l_m}, 0, \dots, 0) \\
&= (k_1, k_2 + 1, \dots, k_m + m - 1, \underbrace{m, \dots, m}_{l_m}, \dots, \underbrace{1, \dots, 1}_{l_1-(l_2+1)}, 0, \dots, 0).
\end{aligned}
$$

The last equality easily follows from the fact $l_{m-1} - 1 \geq l_m$. Since $n \geq k_1 \geq k_2 + 1 \geq \cdots > k_m + (m-1) \geq m$, the last line corresponds to a partition of $2n$ fits into a n by n box.

Conversely let $m = (m_1, m_2, \dots, m_n)$ be a partition where $n \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. If i_1 is the last index such that $m_{i_1} \neq 0$, then let $a_1 = (m_1, 1, \dots, 1, 0, \dots 0)$ where the last 1 in i_1-th position. Then let

$$
x = \sigma^{-1}(m - a_1) = (m_2 - 1, \dots, m_{i_1} - 1, 0, \dots, 0)
$$

and $a_2 = (m_2 - 1, \dots, 1, 0, \dots, 0)$ where the position of the last nonzero element in x and the position of last 1 in a_2 are same. Clearly, a_1 and a_2 are basic partitions and a_1

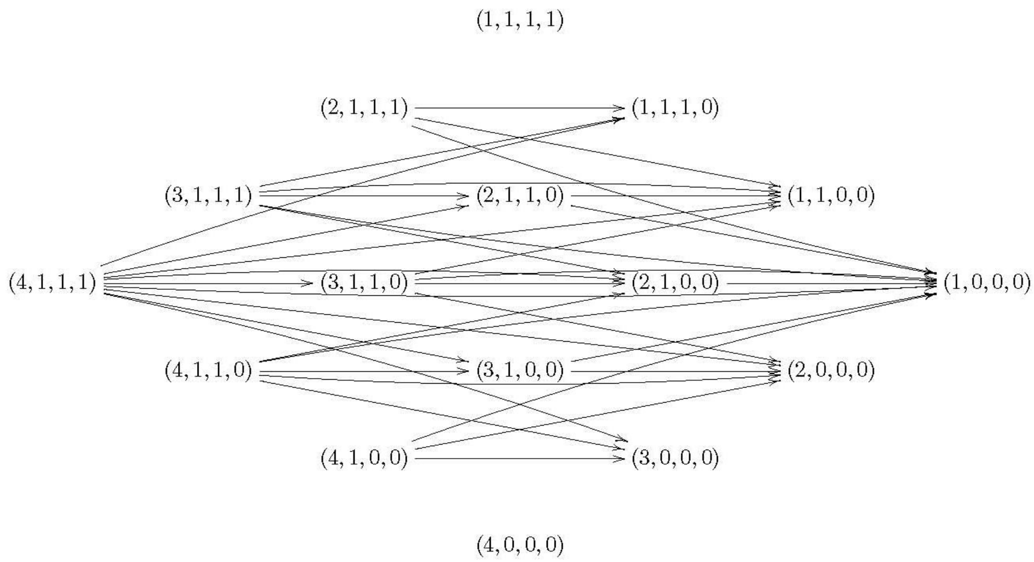


Figure 3.2: Partitions of $\widetilde{A_4}$

is connected to a_2. Continuing the process until reaching the $(0, \ldots, 0)$, one can obtain a sequence of connected basic partitions. □

Example 3.13. If we apply the partition for $n = 4$, then the graph is at figure 3.2,

$$\begin{aligned}(4,1,1,1) \oplus (3,1,1,0) \oplus (1,1,0,0) &= \sigma^0((4,1,1,1)) + \sigma^1((3,1,1,0)) + \sigma^2((1,1,0,0)) \\ &= (4,1,1,1) + (0,3,1,1) + (0,0,1,1) \\ &= (4,4,3,3).\end{aligned}$$

$$\begin{aligned}(3,1,1,1) \oplus (2,1,0,0) \oplus (1,0,0,0) &= \sigma^0((3,1,1,1)) + \sigma^1((2,1,0,0)) + \sigma^2((1,0,0,0)) \\ &= (3,1,1,1) + (0,2,1,0) + (0,0,1,0) \\ &= (3,3,3,1).\end{aligned}$$

$$\begin{aligned}(4,1,1,1) \oplus (1,0,0,0) &= \sigma^0((4,1,1,1)) + \sigma^1((1,0,0,0)) \\ &= (4,1,1,1) + (0,1,0,0) \\ &= (4,2,1,1).\end{aligned}$$

3.6 Main Result

Definition 3.20. (q-Binomials) Let m and r be positive integers. The q-binomial is defined by

$$\begin{pmatrix} m \\ r \end{pmatrix}_q = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}.$$

Although the formula in the first clause appears to involve a rational function, it actually designates a polynomial, because the division is exact in $\mathbb{Z}[q]$. A standard combinatorial interpretation for q-binomial is that it counts the number of partitions that will fit into a box of size $r \times (m-r)$, weighted by the size of the partition. In particular the q-binomial

$$p(x) = \begin{pmatrix} 2n \\ n \end{pmatrix}_x = \frac{(1-x^{2n})(1-x^{2n-1})\cdots(1-x^{n+1})}{(1-x)(1-x^2)\cdots(1-x^n)}$$

counts the number of partitions in which there are at most n parts and in which no parts is larger than n. Hence the coefficient of the term x^α in $p(x)$ is equal to number of words $\widetilde{w}$ whose length is α.

Theorem 3.6. *Let*

$$u = (b_{2n})^{m_{2n}}(b_{2n-1})^{m_{2n-1}}\cdots(b_{n+1})^{m_{n+1}}.$$

Define

$$\overline{u} = (b_{2n})^{\alpha_{2n}}(b_{2n-1})^{\alpha_{2n-1}}\cdots(b_{n+1})^{\alpha_{n+1}}$$

where $\alpha_i \geq 1$ if b_i is a marked component of u and otherwise $\alpha_i \geq 0$. Let $w = uv$ for a fixed word v. Define $\overline{w} = \overline{u}v$. Then the words $\overline{u}$ and $\overline{w}$ are reduced forms.

The proof is similar to the proof of Proposition 3.1. Now, we can prove the main result of this paper.

Theorem 3.7. *The reduced Gröbner-Shirshov basis of the affine Weyl group $\widetilde{A}_n$ is the set R'. Moreover all the reduced forms are the form $s\overline{w}$ or $s\overline{u}$ where s is a reduced form not including r_0 and $\overline{u}, \overline{w}$ are words given by above theorem.*

Proof. Let $\widetilde{u}$ be the marked components of u and $\widetilde{w} = \widetilde{u}v$. The reduced forms not including r_0 is given in Lemma 3.3. It is easy to see that the number of such words given by the generating function $(1 + x)(1 + x + x^2)\ldots(1 + x + \ldots + x^n)$. Let

$$F(x) = \frac{x^{\alpha}}{(1 - x^{2n})(1 - x^{2n-1}\ldots(1 - x^{n+1})}$$

where α is the length of the word $\widetilde{w}$ (or $\widetilde{u}$ in case $w = u$). Then the coefficient of x^k in $F(x)$ gives the number of words in $\overline{w}$ (or $\overline{u}$ in case $w = u$)with length k. Considering every possible $\widetilde{w}$ and $\widetilde{u}$, the number of words of the form $\overline{w}$ or $\overline{u}$ given by the following generating function.

$$\begin{aligned} \frac{\begin{pmatrix} 2n \\ n \end{pmatrix}_x}{(1 - x^{2n})(1 - x^{2n-1})\cdots(1 - x^{n+1})} &= \frac{\frac{(1-x^{2n})(1-x^{2n-1})\cdots(1-x^{n+1})}{(1-x)(1-x^2)\cdots(1-x^n)}}{(1 - x^{2n})(1 - x^{2n-1})\cdots(1 - x^{n+1})} \\ &= \frac{1}{(1 - x)(1 - x^2)\cdots(1 - x^n)}. \end{aligned}$$

Hence the number of reduced forms of the form $s\overline{w}$ or $s\overline{u}$ is given by the generating function

$$\frac{(1 + x)(1 + x + x^2)\cdots(1 + x + \cdots + x^n)}{(1 - x)(1 - x^2)\cdots(1 - x^n)}$$

which is well known Poincaré polynomial of the affine Weyl group $\widetilde{A}_n$. (see [19]). Therefore these are all reduced forms of $\widetilde{A}_n$. Hence by Composition-Diamond Lemma, R' is a Gröbner-Shirshov basis of $\widetilde{A}_n$. In fact, R' is a reduced Gröbner-Shirshov basis. □

REFERENCES

[1] L. A. Bokut and L. S. Shiao, *Gröbner-Shirshov Bases for Coxeter Groups*, Comm. Algebra (29), pp. 178-218, (2001).

[2] L. A. Bokut, *Imbeddings into Simple Associative Algebras*, Algebra Logika (15), pp. 117-142, (1976).

[3] L. A. Bokut, *Unsolvability of the Word Problem, and Subalgebras of Finitely Presented Lie Algebras*, Izv. Akad. Nauk. SSSR Ser. Mat. (36), pp. 1173-1219, (1972).

[4] L. A. Bokut and G. Kukin, *Algorithmic and Combinatorial Algebra*, Kluwer, Dordrecht, (1994).

[5] L. A. Bokut and P. Kolesnikov, *Gröbner-Shirshov Bases : from incipient to nowadays*, Proceedings of the POMI (272), pp. 26-67, (2000).

[6] L. A. Bokut and P. Kolesnikov, *Gröbner-Shirshov Bases : from their incipiency to the present*, Journal of Mathematical Sciences (116(1)), pp. 2894-2916, (2003).

[7] L. A. Bokut and Y. Chen, *Gröbner-Shirshov Bases : some new results*, Proceedings of the Second, International Congress in Algebra and Combinatorics, World Scientific, pp. 35-56, (2008).

[8] G. M. Bergman, *The Diamond Lemma for Ring Theory*, Adv. Math. (29), pp. 178-218, (1978).

[9] Y.Chen and C. Liu, *Gröbner-Shirshov Bases for Coxeter groups I*, arXiv:0910.0096v1, (2009).

[10] D. Lee, *Gröbner-Shirshov Bases and Normal Forms for the Coxeter Groups E_6 and E_7*, Advances in Algebra and Combinatorics, World Scientific, pp. 243-255, (2008).

[11] M. H. A. Newman, *On Theories with a Combinatorial Definition of Equivalence*, Ann. Math. (43), pp. 223-243, (1942).

[12] H. Hironaka, *Resolution of Singulatities of an Algebtaic Variety over a Field of Characteristic Zero, I, II*, Ann. Math. (79), pp. 109-203 and pp. 205-326, (1964).

[13] B. Buchberger, *An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-dimensional Polynomial Ideal*, Ph.D. Thesis, University of Innsbruck, Austria, (1965).

[14] B. Buchberger, *An Algorithmical Criteria for the Solvability of Algebraic Systems of Equations*, Aequationes Math. (4), pp. 374-383, (1970).

[15] A. I. Shirshov, *Some Algorithmic Problem for Lie Algebras*, Sib. Mat. Zh. (3), pp. 292-296, (1962). [English translation: SIGSAM Bull. (33 (2)), pp. 3-6, (1999)].

[16] O. Svechkarenko, *Gröbner-Shirshov Bases for the Coxeter Group E_8*, Master Thesis, Novosibirsk State University, (2007).

[17] N. Bourbaki, *Groupes et Algébres de Lie*, Chapitres IV, V et VI , Hermann, Paris, (1968).

[18] R. Rouquier, *Weyl Groups, Affine Weyl Groups and Reflection Groups*, Isaac Newton Institute for Mathematical Sciences, University of Cambridge.

[19] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, 29, Cambridge University Press, Cambridge, (1990).

[20] G. H. Hardy and S. Ramanujan, *Asymptotic Formula in Combinatory Analysis*, Proc. London Math. Soc. 17, pp. 175-115, (1918).

[21] Hans Rademacher, *On the Partition Function* $p(n)$, Proc. London Math. Soc. 43, pp. 241-254, (1937).

[22] Herbert S. Wilf, *Lectures on Integer Partitions*, University of Pennsylvania, (2000).