CALCULATION OF REIDEMEISTER TORSION OF FLAG MANIFOLDS OF COMPACT SEMISIMPLE LIE GROUPS

by

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ABSTRACT

CALCULATION OF REIDEMEISTER TORSION OF FLAG MANIFOLDS OF COMPACT SEMISIMPLE LIE GROUPS

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In this thesis we calculate Reidemeister torsion of flag manifold K/T of compact semi-simple Lie group $K = SU_{n+1}$ using Reidemeister torsion formula and Schubert calculus. Here T is maximal torus of K. The last chapter of this thesis is completely original and it includes our calculations.

Keywords: Reidemeister torsion of manifolds, Flag Manifolds, Weyl Groups, Schubert Calculus.

ÖZET

TIKIZ YARI-BASİT LIE GRUPLARININ TEMEL KATMANLARININ REIDEMEISTER BURULMASININ HESAPLANMASI

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Bu tezde, katmanlar icin Reidemeister burulma formülü ve Schubert kalkulüs kullanılarak, tıkız yarı basit Lie grubu $K = SU_{n+1}$ nin temel katmanı SU_{n+1}/T'in Reidemeister burulmasını hesaplıyoruz. Burada T, K'nın maksimal abeliyen alt grubudur. Tezin son bölümü tamamen orijinal olup hesaplamalarımızı içermektedir.

Anahtar Kelimeler: Katmanların Reidemeister Burulması, Temel Katmanlar, Weyl Grupları, Schubert Kalkulüs.

To my family and my darling

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CHAPTER 1

INTRODUCTION

Reidemeister torsion is a topological invariant and was introduced by Reidemeister in 1935. Up to PL equivalence, he classified the lens spaces S^3/Γ, where Γ is a finite cyclic group of fixed point free orthogonal transformations [18]. In [8], Franz extended the Reidemeister torsion and classified the higher dimensional lens spaces S^{2n+1}/Γ, where Γ is a cyclic group acting freely and isometrically on the sphere S^{2n+1}.

In 1964, the results of Reidemeister and Franz were extended by de Rham to spaces of constant curvature $+1$ [7]. Kirby and Siebenmann proved the topological invariance of the Reidemeister torsion for manifolds in 1969 [12]. Chapman proved for arbitrary simplicial complexes [5, 6]. Hence, the classification of lens spaces of Reidemeister and Franz was actually topological (i.e. up to homeomorphism).

Using the Reidemeister torsion, Milnor disproved *Hauptvermutung* in 1961. He constructed two homeomorphic but combinatorially distinct finite simplicial complexes. He identified in 1962 the Reidemeister torsion with Alexander polynomial which plays an important role in knot theory and links [14, 16].

In paper [20], Y. Sozen explained the claim mentioned in [30, p. 187] about the relation between a symplectic chain complex with $\omega-$compatible bases and the Reidemeister torsion of it (Theorem 2.2). Moreover, he applied Theorem 2.2 to the chain-complex

$$0 \to C_2(\Sigma_g; \mathrm{Ad}_\varrho) \overset{\partial_2\otimes id}{\to} C_1(\Sigma_g; \mathrm{Ad}_\varrho) \overset{\partial_1\otimes id}{\to} C_0(\Sigma_g; \mathrm{Ad}_\varrho) \to 0,$$

where Σ_g is a compact Riemann surface of genus $g > 1$, where ∂ is the usual boundary operator, and where $\varrho : \pi_1(\Sigma_g) \to \mathrm{PSL}_2(\mathsf{R})$ is a discrete and faithful representation of the fundamental group $\pi_1(\Sigma_g)$ of Σ_g [20].

In the article [24] , oriented closed connected $2m-$manifolds ($m \geq 1$) are considered and he proved the following formula for computing the Reidemeister torsion of them. Namely,

Theorem 1.1. *Let M be an oriented closed connected $2m-$manifold ($m \geq 1$). For $p = 0, \ldots, 2m$, let $\mathbf{h}_p$ be a basis of $H_p(M)$. Then, the Reidemeister torsion of M satisfies the following formula:*

$$\left|\mathbb{T}(M, \{\mathbf{h}_p\}_0^{2m})\right| = \prod_{p=0}^{m-1} \left|\det H_{p,2m-p}(M)\right|^{(-1)^p} \sqrt{\left|\det H_{m,m}(M)\right|}^{(-1)^m},$$

where $\det H_{p,2m-p}(M)$ *is the determinant of the matrix of the intersection pairing* $(\cdot, \cdot)_{p,2m-p}$: $H_p(M) \times H_{2m-p}(M) \to \mathsf{R}$ *in bases* $\mathbf{h}_p, \mathbf{h}_{2m-p}$.

It is well known that Riemann surfaces and Grasmannians have many applications in a wide range of mathematics such as topology, differential geometry, algebraic geometry, symplectic geometry, and theoretical physics (see, e.g, [1]-[4], [9, 10], [20]-[30], and the references therein). He also applies Theorem 1.1 to Riemann surfaces and Grasmannians.

In this thesis we calculate Reidemeister torsion of compact flag manifold K/T for $G = A_l$, where K is compact simply connected semi-simple Lie group and T is maximal torus.

The content of the thesis is as follows. In Chapter 2 , we provide the basic definitions and facts about the Reidemeister torsion of a general chain complex. Moreover, we explain symplectic chain complex. In Chapter 3, we give all details of cup product formula in the cohomology ring of flag manifolds which is called Schubert calculus. In the last chapter, we calculate the Reidemesiter torison of flag manifold $SU(n)/T$ for $n \geq 4$.

CHAPTER 2

CALCULATION OF THE REIDEMEISTER TORSION OF COMPACT MANIFOLDS WITH APPLICATIONS

In this chapter, we will give the formula for calculation of Reidemeister torsion of compact manifolds. The general reference for this chapter is [24]. The proofs of the theorems in this chapter were given in [24].

2.1 Reidemeister torsion of a chain complex

In this section, the required definitions and the basic facts about the Reidemeister torsion are given. The detailed proofs and more information can be found in [17, 20, 30], and the references therein.

We shall reserve F to denote the field of real R or complex C numbers. Let $(C_*, \partial_*) = (C_n \stackrel{\partial_n}{\to} C_{n-1} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over F. Let $H_p(C_*) = Z_p(C_*)/B_p(C_*)$ denote the $p-$th homology of C_*, where $B_p(C_*) = \mathrm{Im}\{\partial_{p+1} : C_{p+1} \to C_p\}$, and $Z_p(C_*) = \ker\{\partial_p : C_p \to C_{p-1}\}$.

Clearly, we have the following short-exact sequences:

$$0 \to Z_p(C_*) \to C_p \to B_{p-1}(C_*) \to 0$$

and

$$0 \to B_p(C_*) \to Z_p(C_*) \to H_p(C_*) \to 0.$$

Assume that $\mathbf{b}_p$, $\mathbf{h}_p$ are bases of $B_p(C_*)$, $H_p(C_*)$, respectively. Assume also that ℓ_p :

$H_p(C_*) \to Z_p(C_*)$, $s_p : B_{p-1}(C_*) \to C_p$ are sections of $Z_p(C_*) \to H_p(C_*)$, $C_p \to B_{p-1}(C_*)$, respectively. Then, we obtain a new basis of C_p, namely $\mathbf{b}_p \oplus \ell_p(\mathbf{h}_p) \oplus s_p(\mathbf{b}_{p-1})$.

Definition 2.1. Let $C_* : C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$ be a chain complex of finite dimensional vector spaces over F. For $p = 0, \ldots, n$, let $\mathbf{c}_p$, $\mathbf{b}_p$, $\mathbf{h}_p$ be bases of C_p, $B_p(C_*)$, $H_p(C_*)$, respectively, and let $\ell_p : H_p(C_*) \to Z_p(C_*)$, $s_p : B_{p-1}(C_*) \to C_p$ be sections of $Z_p(C_*) \to H_p(C_*)$, $C_p \to B_{p-1}(C_*)$, respectively. The *Reidemeister torsion* of C_* with respect to bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is the alternating product

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n) = \prod_{p=0}^{n} \left[\mathbf{b}_p \oplus \ell_p(\mathbf{h}_p) \oplus s_p(\mathbf{b}_{p-1}), \mathbf{c}_p\right]^{(-1)^{(p+1)}},$$

where $\left[\mathbf{e}_p, \mathbf{f}_p\right]$ denotes the determinant of the change-base-matrix from basis $\mathbf{f}_p$ to $\mathbf{e}_p$ of C_p.

Remark 2.1. *Milnor proved that the Reidemeister torsion does not depend on bases* $\mathbf{b}_p$, *sections* s_p, ℓ_p *[15]. Let* $\mathbf{c}'_p, \mathbf{h}'_p$ *be other bases respectively for* C_p, $H_p(C_*)$. *Then, by an easy computation we have the following change-base-formula:*

$$\mathbb{T}(C_*, \{\mathbf{c}'_p\}_0^n, \{\mathbf{h}'_p\}_0^n) = \prod_{p=0}^{n} \left(\frac{[\mathbf{c}'_p, \mathbf{c}_p]}{[\mathbf{h}'_p, \mathbf{h}_p]}\right)^{(-1)^p} \mathbb{T}(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n). \tag{2.1}$$

By the independence of the Reidemeister torsion from $\mathbf{b}_p$ *and sections* s_p, ℓ_p, *formula (2.1) is easily obtained. Note that if, for example,* $[\mathbf{c}'_p, \mathbf{c}_p] = 1$, $[\mathbf{h}'_p, \mathbf{h}_p] = -1$, *then the torsions are the same for odd* n, *and torsions have opposite sign for even* n.

It follows from Snake Lemma that for short-exact sequence (2.2) of chain complexes

$$0 \to A_* \xrightarrow{\imath} B_* \xrightarrow{\pi} D_* \to 0, \tag{2.2}$$

there is also the long-exact sequence of vector spaces $\mathcal{C}_*$ of length $3n + 2$. Namely,

$$\mathcal{C}_* : \cdots \to H_p(A_*) \xrightarrow{\imath_p} H_p(B_*) \xrightarrow{\pi_p} H_p(D_*) \xrightarrow{\delta_p} H_{p-1}(A_*) \to \cdots, \tag{2.3}$$

where $C_{3p} = H_p(D_*)$, $C_{3p+1} = H_p(A_*)$, and $C_{3p+2} = H_p(B_*)$.

Clearly, the bases $\mathbf{h}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^B$ serve as bases for C_{3p}, C_{3p+1}, and C_{3p+2}, respectively.

The following result of Milnor states that the alternating product of the torsions of the chain complexes in (2.2) is equal to the torsion of (2.3). More precisely,

Theorem 2.1. *([15]) Let $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, and $\mathbf{c}_p^D$ be bases respectively for A_p, B_p, and D_p. Let $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ be bases of $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. If, moreover, $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$ are compatible in the sense that $[\mathbf{c}_p^B, \mathbf{c}_p^A \oplus \widetilde{\mathbf{c}_p^D}] = \pm 1$, where $\pi\left(\widetilde{\mathbf{c}_p^D}\right) = \mathbf{c}_p^D$, then*

$$\begin{aligned} \mathbb{T}(B_*, \{\mathbf{c}_p^B\}_0^n, \{\mathbf{h}_p^B\}_0^n) &= \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\mathbb{T}(D_*, \{\mathbf{c}_p^D\}_{p=0}^n, \{\mathbf{h}_p^D\}_0^n) \\ &\times \mathbb{T}(C_*, \{\mathbf{c}_{3p}\}_0^{3n+2}, \{0\}_0^{3n+2}). \end{aligned}$$

For future reference, let us give the following sum-lemma:

Lemma 2.1. *([24]) Let A_*, D_* be two chain complexes. Let $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^D$ be bases of A_p, D_p, $H_p(A_*)$, and $H_p(D_*)$, respectively. Then,*

$$\mathbb{T}(A_* \oplus D_*, \{\mathbf{c}_p^A \oplus \mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^A \oplus \mathbf{h}_p^D\}_0^n) = \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n).$$

Independently, it is explained in [1, 20] that a general chain complex can (unnaturally) be splitted as a direct sum of an acyclic and $\partial-$zero chain complexes. Moreover, it is proved independently in [1, Proposition 1.5] and [20, Theorem 2.0.4] that the Reidemeister torsion $\mathbb{T}(C_*)$ of a general complex C_* can be interpreted as an element of $\otimes_{p=0}^n(\det(H_p(C_*)))^{(-1)^{p+1}}$. For detailed proof and further information, we may refer the readers to [1, 20].

Definition 2.2. A *symplectic chain complex* of length q is $(C_*, \partial_*, \{\omega_{*,q-*}\})$, where $C_* : 0 \to C_q \stackrel{\partial_q}{\to} C_{q-1} \to \cdots \to C_{q/2} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0$ is a chain complex with $q \equiv 2(\text{mod } 4)$, and for $p = 0, \ldots, q/2$, $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathsf{R}$ is a $\partial-$compatible anti-symmetric non-degenerate bilinear form. To be more precise,

$$\omega_{p,q-p}(\partial_{p+1}a, b) = (-1)^{p+1}\omega_{p+1,q-(p+1)}(a, \partial_{q-p}b)$$

and

$$\omega_{p,q-p}(a, b) = (-1)^{p(q-p)}\omega_{q-p,p}(b, a).$$

Note that by $q \equiv 2(\mathrm{mod}\, 4)$, we easily have $\omega_{p,q-p}(a, b) = (-1)^p\omega_{q-p,p}(b, a)$. It follows from the $\partial-$compatibility of the non-degenerate anti-symmetric bilinear maps $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathsf{R}$ that one can easily extend these to homologies [20].

Definition 2.3. Let C_* be a symplectic chain complex. We say that bases $\mathbf{c}_p$ of C_p and $\mathbf{c}_{q-p}$ of C_{q-p} are $\omega-$*compatible* if the matrix of $\omega_{p,q-p}$ in bases $\mathbf{c}_p$, $\mathbf{c}_{q-p}$ equals to the $k \times k$ identity matrix $\mathrm{I}_{k\times k}$ when $p \neq q/2$ and $\begin{bmatrix} 0_{l\times l} & \mathrm{I}_{l\times l} \\ -\mathrm{I}_{l\times l} & 0_{l\times l} \end{bmatrix}$ when $p = q/2$, where $k = \dim C_p = \dim C_{q-p}$ and $2l = \dim C_{q/2}$.

Similarly, considering $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathsf{R}$, one can also define the $[\omega]$-compatibility of bases $\mathbf{h}_p$ of $H_p(C_*)$ and $\mathbf{h}_{q-p}$ of $H_{q-p}(C_*)$.

The existence of $\omega-$compatible bases enabled us to prove in [20] that a symplectic chain complex C_* can be splitted $\omega-$orthogonally as a direct sum of an exact and $\partial-$zero symplectic complexes. Moreover, Y. Sozen proved Theorem 2.2, which is one of the main results of [20]. Namely,

Theorem 2.2. *([20]) Let C_* be a symplectic chain complex. For $p = 0, \ldots, q$, let $\mathbf{c}_p$, $\mathbf{h}_p$ be any bases of C_p, $H_p(C_*)$, respectively. Then, for the Reidemeister torsion of C_* with respect to $\{\mathbf{c}_p\}_0^q$, $\{\mathbf{h}_p\}_0^q$, the following formula*

$$\mathbb{T}(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q) = \prod_{p=0}^{(q/2)-1} \left(\det[\omega_{p,q-p}]\right)^{(-1)^p} \sqrt{\det[\omega_{q/2,q/2}]}^{(-1)^{q/2}}$$

holds, where $\det[\omega_{p,q-p}]$ *is the determinant of the matrix of the non-degenerate pairing* $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathsf{R}$ *in bases* $\mathbf{h}_p$, $\mathbf{h}_{q-p}$.

The proof and unexplained subjects can be found in [20]. For further applications of Theorem 2.2, we refer the reader to [21, 22, 23, 24].

2.2 The Reidemeister torsion of a manifold

Let M be a smooth $m-$manifold with a cell decomposition K. If $\mathbf{c}_p = \{c_1^p, \ldots, c_{n_p}^p\}$ is the *geometric basis* for the $p-$cells $C_p(K; \mathsf{Z})$, $p = 0, \ldots, m$, then one can associate to M the following chain complex

$$0 \to C_m(K) \stackrel{\partial_m}{\to} C_{m-1}(K) \to \cdots \to C_1(K) \stackrel{\partial_1}{\to} C_0(K) \to 0,$$

where Z is the set of integers and ∂_p is the usual boundary operator.

Definition 2.4. Let M be an $m-$manifold with a cell decomposition K. For $p = 0, \ldots, m$, let $\mathbf{c}_p$ and $\mathbf{b}_p$ be bases of $C_p(K; \mathsf{Z})$ and $H_p(M; \mathsf{Z})$, respectively. $\mathbb{T}(C_*(K), \{\mathbf{c}_p\}_0^m, \{\mathbf{h}_p\}_0^m)$ is called the *Reidemeister torsion* of M.

Using similar arguments introduced in [20, Lemma 2.0.5], one can prove:

Lemma 2.2. *The Reidemeister torsion of M is independent of cell decomposition.*

Hence, the Reidemeister torsion $\mathbb{T}(C_*(K), \{\mathbf{c}_p\}_0^m, \{\mathbf{h}_p\}_0^m)$ of M is well-defined. Thus, we let $\mathbb{T}(M, \{\mathbf{h}_p\}_0^m)$ denote the Reidemeister torsion of M in the bases $\mathbf{h}_p$ of $H_p(M)$, $p = 0, \ldots, m$.

By [1, Proposition 1.5] and [20, Theorem 2.0.4], one concludes that the Reidemeister torsion of M is an element of the dual of 1−dimensional vector space $\otimes_{p=0}^n (\det(H_p(M))^{(-1)^p}$.

In this section, we give Theorem 1.1. To alleviate the notation, let us introduce the following which is used throughout the thesis. Let Y be an oriented closed connected smooth manifold of dimension d. For $p = 0, \ldots, d$, let $\mathbf{h}_p^Y$ and $\mathbf{h}_{d-p}^Y$ be bases of $H_p(Y)$ and $H_{d-p}(Y)$, respectively. We denote the matrix of the intersection pairing $(\cdot, \cdot)_{p,d-p}$: $H_p(Y) \times H_{d-p}(Y) \to \mathsf{R}$ in the bases $\mathbf{h}_p^Y$ and $\mathbf{h}_{d-p}^Y$ by $H_{p,d-p}(Y)$. As convention, we let $H_{p,d-p}(Y) = 1$ when $H_p(Y) = H_{d-p}(Y) = 0$.

Using sympletic chain complex, Y. Sözen proved in [24] the following formula for calculation of Reidemeister torsion on compact manifolds (Theorem 1.1 and Theorem 2.7 in [24]).

2.3 Application

In this section, we apply Theorem 1.1 to Riemann surfaces and Grasmannians.

2.3.1 Compact Riemann surfaces

Let Σ_g be a compact oriented Riemann surface of genus $g \geq 1$ without boundary. Let $\Gamma = \{\Gamma_1, \ldots, \Gamma_g, \Gamma_{1+g}, \ldots, \Gamma_{2g}\}$ be a canonical basis for $H_1(\Sigma_g)$, i.e. Γ_i intersects Γ_{i+g} once positively and does not intersect others. Then, we have

Theorem 2.3. *([24]) Let* $\mathbf{h}_0$, $\mathbf{h}_1 = \{\Omega_i\}_1^{2g}$, *and* $\mathbf{h}_2$ *be bases of* $H_0(\Sigma_g)$, $H_1(\Sigma_g)$, *and* $H_2(\Sigma_g)$, *respectively. Then,* $\left|\mathbb{T}(\Sigma_g, \{\mathbf{h}_p\}_0^2)\right| = \left|\frac{\det H_{0,2}(\Sigma_g)}{\det \wp(\mathfrak{h}^1, \Gamma)}\right|$, *where* $\mathbf{h}^1 = \{\omega_i\}_1^{2g}$ *is the Poincaré dual basis of* $H^1(\Sigma_g)$ *corresponding to the basis* $\mathbf{h}_1$ *of* $H_1(\Sigma_g)$, *where* $\wp(\mathbf{h}^1, \Gamma) = [\int_{\Gamma_i} \omega_j]$ *is the period matrix of* $\mathbf{h}^1$ *with respect to the canonical basis* $\Gamma = \{\Gamma_i\}_{i=1}^{2g}$ *of* $H_1(\Sigma_g)$.

Proof. From Theorem 1.1 it follows that $\left|\mathbb{T}(\Sigma_g, \{\mathbf{h}_p\}_0^2)\right| = \frac{|\det H_{0,2}(\Sigma_g)|}{\sqrt{\det H_{1,1}(\Sigma_g)}}$. For $\mathbf{h}_1 = \{\Omega_j\}_{j=1}^{2g}$, the non-degenerate skew-symmetric $2g \times 2g-$square matrix $H_{1,1}(\Sigma_g)$ is $[\Omega_{ij}]$, where $\Omega_{ij} = (\Omega_i, \Omega_j)_{1,1}$. By Poincaré duality, we also have $\Omega_{ij} = \int_{\Sigma_g} \omega_i \wedge \omega_j$. Change-base-formula results that $\sqrt{\det(H_{1,1}(\Sigma_g))} = \left|\det\left[(\Omega_j, \Gamma_i)_{1,1}\right]\right|$. If, moreover, we let $\gamma_i \in H^1(\Sigma_g)$ denote the Poincaré dual of $\Gamma_i \in H_1(\Sigma_g)$, then we have $(\Omega_i, \Gamma_j)_{1,1} = \int_{\Sigma_g} \omega_i \wedge \gamma_j = \int_{\Gamma_i} \omega_j$.

This completes the proof of Theorem 2.3. □

Before ending this section, we also would like to apply Theorem 1.1 to $M \times N$, where $M = \Sigma_g$, $N = \Sigma_{g'}$ are compact oriented Riemann surfaces of genus $g, g' \geq 1$ without boundary.

Let us start with the following well-known properties of tensor (or Kronecker) product of square matrices. Recall that if $A = [a_{ij}]$ is an $m \times m$ and $B = [b_{ij}]$ is an $n \times n$ matrix with real entries, then the tensor product of A and B is the $mn \times mn$ block matrix $A \otimes B = [a_{ij}B]$, where $a_{ij}B$ is the $n \times n$ matrix obtained by multiplying the matrix B with the scalar a_{ij}.

Recall that if A, B, C, D are square matrices such that the products AC and BD exist, then $(A \otimes B)(C \otimes D)$ exists and $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ (see, e.g, [19, p. 350]). Let A be an $n \times n$ and B be an $m \times m$ invertible matrix. Then, we clearly have $(A \otimes B)(A^{-1} \otimes B^{-1}) = I_{m\times m} \otimes I_{n\times n}$, where $I_{d\times d}$ is the $d \times d$ identity matrix. Note also that for the square matrices A and B, we have $(A \otimes B)^T = A^T \otimes B^T$, where A^T is the transpose of A. Finally, it is known that $\det(A \otimes B) = \det(A)^n \det(B)^m$; however, for the sake of completeness, we provide a proof for our case. More precisely,

Lemma 2.3. *([24])Let $A = [a_{ij}]$ be $2g \times 2g$ and $B = [b_{ij}]$ be a $2g' \times 2g'$ symmetric or skew-symmetric matrices with real entries. Then,* $\det(A \otimes B) = \det(A)^{2g'} \det(B)^{2g}$.

Proof. By the spectral theorem of normal matrices, symmetric and skew-symmetric matrices are orthogonally diagonalizable. Thus, there exist orthogonal $2g \times 2g$ real matrix P and $2g' \times 2g'$ real matrix Q so that $PAP^{-1} = D_1 = \operatorname{diag}\left(\lambda_1, \dots, \lambda_{2g}\right)$, $QBQ^{-1} = D_2 = \operatorname{diag}\left(\mu_1 \dots, \mu_{2g'}\right)$, respectively, where $\lambda_1, \dots, \lambda_{2g}$ and $\mu_1, \dots, \mu_{2g'}$ are real. Then, we have $A \otimes B = (PD_1P^{-1}) \otimes (QD_2Q^{-1}) = (P \otimes Q)(D_1 \otimes D_2)(P \otimes Q)^{-1}$.

Hence, $\det(A \otimes B) = \det(D_1 \otimes D_2) = \det(D_1)^{2g'} \det(D_2)^{2g} = \det(A)^{2g'} \det(B)^{2g}$.

This is the end of the proof of Lemma 2.3. □

Corollary 2.1. *([24]) Let $M = \Sigma_g$ and $N = \Sigma_{g'}$ be closed oriented Riemann surfaces of genus $g, g' \geq 1$, respectively. For $p = 0, 1, 2$, let $\mathbf{h}_p$ and $\mathbf{h}'_p$ be bases of $H_p(M)$ and $H_p(N)$, respectively. Then,*

$$\left|\mathbb{T}(M \times N, \{\oplus_{i+j=p}\mathbf{h}_i \otimes \mathbf{h}'_j\}_{p=0}^4)\right| = \left|\mathbb{T}(M, \{\mathbf{h}_p\}_0^2)\right|^{\chi(N)} \left|\mathbb{T}(N, \{\mathbf{h}'_p\}_0^2)\right|^{\chi(M)}. \tag{2.4}$$

Proof. From Künneth the formula it follows that $\mathbf{h}_0 \otimes \mathbf{h}'_0$, $\mathbf{h}_1 \otimes \mathbf{h}'_0 \oplus \mathbf{h}_0 \otimes \mathbf{h}'_1$, $\mathbf{h}_0 \otimes \mathbf{h}'_2 \oplus$

$\mathbf{h}_1 \otimes \mathbf{h}'_1 \oplus \mathbf{h}_2 \otimes \mathbf{h}'_0$, $\mathbf{h}_1 \otimes \mathbf{h}'_2 \oplus \mathbf{h}_2 \otimes \mathbf{h}'_1$, and $\mathbf{h}_2 \otimes \mathbf{h}'_2$ are bases of $H_0(M \times N)$, $H_1(M \times N)$, $H_2(M \times N)$, $H_3(M \times N)$, and $H_4(M \times N)$, respectively.

Using Theorem 1.1, we obtain

$$\begin{aligned} &\left|\mathbb{T}(M \times N, \{\oplus_{i+j=p}\mathbf{h}_i \otimes \mathbf{h}'_j\}_{p=0}^4)\right| \\ &= \left|\det H_{0,4}(M \times N)\right| \left|\det H_{1,3}(M \times N)\right|^{-1} \sqrt{\left|\det H_{2,2}(M \times N)\right|}. \end{aligned} \tag{2.5}$$

It follows from Lemma 2.3 that

$$\left|\det H_{0,4}(M \times N)\right| = \left|\det H_{0,2}(M)\right| \left|\det H_{0,2}(N)\right| \tag{2.6}$$

$$\begin{aligned} \left|\det H_{1,3}(M \times N)\right| &= \left|\det H_{0,2}(M)\right|^{\dim H_1(N)} \left|\det H_{1,1}(M)\right| \\ &\times \left|\det H_{0,2}(N)\right|^{\dim H_1(M)} \left|\det H_{1,1}(N)\right| \end{aligned} \tag{2.7}$$

$$\left|\det H_{2,2}(M \times N)\right| = \left|\det H_{0,2}(M)\right|^2 \left|\det H_{0,2}(N)\right|^2 \left|\det H_{1\otimes 1}(M \times N)\right|, \tag{2.8}$$

where $H_{1\otimes 1}(M \times N) = \left[\, (\cdot,\cdot) \text{ in } \mathbf{h}_1 \otimes \mathbf{h}'_1 \right]$.

By Lemma 2.3, we get

$$\det H_{1\otimes 1}(M \times N) = (\det H_{1,1}(M))^{\dim H_1(N)} (\det H_{1,1}(N))^{\dim H_1(M)}. \tag{2.9}$$

From (2.6)-(2.9) it follows that (2.5) is equal to

$$\begin{aligned} \left|\mathbb{T}(M \times N, \{\oplus_{i+j=p}\mathbf{h}_i \otimes \mathbf{h}'_j\}_{p=0}^4)\right| &= \left|\det H_{0,2}(M)\right|^{\chi(N)} \left|\det H_{1,1}(M)\right|^{-\chi(N)/2} \\ &\times \left|\det H_{0,2}(N)\right|^{\chi(M)} \left|\det H_{1,1}(N)\right|^{-\chi(M)/2} \\ &= \left|\mathbb{T}(M, \{\mathbf{h}_p\}_{p=0}^2)\right|^{\chi(N)} \left|\mathbb{T}(N, \{\mathbf{h}'_p\}_{p=0}^2)\right|^{\chi(M)} \end{aligned}$$

This concludes the proof of Corollary 2.3. □

Next, let us compute the Reidemeister torsion of the cartesian product $\times_{i=1}^n \Sigma_{g_i}$ of closed Riemann surfaces $\Sigma_{g_1}, \ldots, \Sigma_{g_n}$ of genus $g_1, \ldots, g_n \geq 1$, respectively. To do that, we shall first prove that formula (2.4) is valid for $M \times N$, where M is an oriented closed connected $2n-$manifold with $n \geq 1$ and $N = \Sigma_{g'}$ is a closed oriented Riemann surfaces of genus $g' \geq 1$. Namely,

Corollary 2.2. *([24]) Let M be an oriented closed connected $2n-$manifold with $n \geq 1$ and $N = \Sigma_{g'}$ be a closed oriented Riemann surface of genus $g' \geq 1$. For $i = 0, \ldots, 2n$, let $\mathbf{h}_i$ be a basis of $H_i(M)$. Let $\mathbf{h}'_j$ be a basis of $H_j(N)$, $j = 0, 1, 2$. Then,*

$$\left|\mathbb{T}(M \times N, \{\oplus_{i+j=p} \mathbf{h}_i \otimes \mathbf{h}'_j\}_{p=0}^{2n+2})\right| = \left|\mathbb{T}(M, \{\mathbf{h}_p\}_0^{2n})\right|^{\chi(N)} \left|\mathbb{T}(N, \{\mathbf{h}'_p\}_0^{2})\right|^{\chi(M)}. \tag{2.10}$$

Proof. Using the Künneth formula, we get $\mathbf{h}_0 \otimes \mathbf{h}'_0$, $\mathbf{h}_{2n} \otimes \mathbf{h}'_2$, $\mathbf{h}_1 \otimes \mathbf{h}'_0 \oplus \mathbf{h}_0 \otimes \mathbf{h}'_1$, $\mathbf{h}_{2n-1} \otimes \mathbf{h}'_2 \oplus \mathbf{h}_{2n} \otimes \mathbf{h}'_1$, and for $p = 2, \ldots, n+1$, $\mathbf{h}_p \otimes \mathbf{h}'_0 \oplus \mathbf{h}_{p-1} \otimes \mathbf{h}'_1 \oplus \mathbf{h}_{p-2} \otimes \mathbf{h}'_2$, $\mathbf{h}_{2n-p} \otimes \mathbf{h}'_2 \oplus \mathbf{h}_{2n-p+1} \otimes \mathbf{h}'_1 \oplus \mathbf{h}_{2n-p+2} \otimes \mathbf{h}'_0$ are bases of $H_0(M \times N)$, $H_{2n+2}(M \times N)$, $H_1(M \times N)$, $H_{2n+1}(M \times N)$, and for $p = 2, \ldots, n+1$, $H_p(M \times N)$, $H_{2n+2-p}(M \times N)$, respectively.

It follows from Theorem 1.1 that

$$\begin{aligned} &\left|\mathbb{T}(M \times N, \{\oplus_{i+j=p} \mathbf{h}_i \otimes \mathbf{h}'_j\}_{p=0}^{2n+2})\right| \\ &= \prod_{p=0}^{n} \left|\det H_{p,2n+2-p}(M \times N)\right|^{(-1)^p} \sqrt{\left|\det H_{p,2n+2-p}(M \times N)\right|^{(-1)^{n+1}}}. \end{aligned} \tag{2.11}$$

Using Lemma 2.3, we get

$$\left|\det H_{0,2n+2}(M \times N)\right| = \left|\det H_{0,2n}(M)\right|^{\dim H_0(N)} \left|\det H_{0,2}(N)\right|^{\dim H_0(M)}, \tag{2.12}$$

$$\begin{aligned} &\left|\det H_{1,2n+1}(M \times N)\right| = \left|\det H_{1,2n-1}(M)\right|^{\dim H_0(N)} \left|\det H_{0,2}(N)\right|^{\dim H_1(M)} \\ &\times \left|\det H_{0,2n}(M)\right|^{\dim H_1(N)} \left|\det H_{1,1}(N)\right|^{\dim H_0(M)} \end{aligned} \tag{2.13}$$

for $p = 2, \ldots, n$,

$$\left|\det H_{p,2n+2-p}(M \times N)\right| = \left|\det H_{p,2n-p}(M)\right|^{\dim H_0(N)} \tag{2.14}$$
$$\times \left|\det H_{p-1,2n-p+1}(M)\right|^{\dim H_1(N)} \left|\det H_{p-2,2n-p+2}(M)\right|^{\dim H_0(N)}$$
$$\times \left|\det H_{0,2}(N)\right|^{\dim H_p(M)+\dim H_{p-2}(M)} \left|\det H_{1,1}(N)\right|^{\dim H_{p-1}(M)},$$

$$\sqrt{\left|\det H_{n+1,n+1}(M \times N)\right|} = \left|\det H_{n-1,n+1}(M)\right|^{\dim H_0(N)} \tag{2.15}$$
$$\times \left|\det H_{0,2}(N)\right|^{\dim H_{n-1}(M)} \left|\det H_{n,n}(M)\right|^{\dim H_1(N)/2}$$
$$\times \left|\det H_{1,1}(N)\right|^{\dim H_n(M)/2}.$$

Using (2.12)-(2.15), (2.11) is equal to

$$\prod_{p=2}^{n} \left\{ \left|\det H_{p,2n-p}(M)\right|^{\dim H_0(N)} \left|\det H_{p-1,2n-p+1}(M)\right|^{\dim H_1(N)} \right. \tag{2.16}$$
$$\left. \times \left|\det H_{p-2,2n-p+2}(M)\right|^{\dim H_0(N)} \right\}^{(-1)^p} \prod_{p=2}^{n} \left\{ \left|\det H_{0,2}(N)\right|^{\dim H_p(M)} \right.$$
$$\left. \times \left|\det H_{1,1}(N)\right|^{\dim H_{p-1}(M)} \left|\det H_{0,2}(N)\right|^{\dim H_{p-2}(M)} \right\}^{(-1)^p}$$
$$\times \left|\det H_{0,2n}(M)\right|^{\dim H_0(N)} \left|\det H_{0,2}(N)\right|^{\dim H_0(M)}$$
$$\times \left|\det H_{0,2n}(M)\right|^{-\dim H_1(N)} \left|\det H_{1,1}(N)\right|^{-\dim H_0(M)}$$
$$\times \left|\det H_{1,2n-1}(M)\right|^{-\dim H_0(N)} \left|\det H_{0,2}(N)\right|^{-\dim H_1(M)}$$
$$\times \left\{ \left|\det H_{n-1,n+1}(M)\right|^{\dim H_0(N)} \left|\det H_{0,2}(N)\right|^{\dim H_{n-1}(M)} \right.$$
$$\left. \times \left|\det H_{n,n}(M)\right|^{\dim H_1(N)/2} \left|\det H_{1,1}(N)\right|^{\dim H_n(M)/2} \right\}^{(-1)^{n+1}}.$$

An easy computation gives us

$$\prod_{p=2}^{n}\Big\{\Big|\det H_{p,2n-p}(M)\Big|\,\Big|\det H_{p-1,2n-p+1}(M)\Big|^{\dim H_1(N)} \tag{2.17}$$
$$\times\Big|\det H_{p-2,2n-p+2}(M)\Big|\Big\}^{(-1)^p}\Big|\det H_{0,2n}(M)\Big|^{\dim H_0(N)-\dim H_1(N)}$$
$$\times\Big|\det H_{1,2n-1}(M)\Big|^{-\dim H_0(N)}\Big|\det H_{n-1,n+1}(M)\Big|^{(-1)^{n+1}\dim H_0(N)}$$
$$\times\Big|\det H_{n,n}(M)\Big|^{(-1)^{n+1}\dim H_1(N)/2}$$
$$=\left(\prod_{p=0}^{n-1}\Big|\det H_{p,2n-p}(M)\Big|^{(-1)^p}\right)^{\chi(N)}\left(\Big|\det H_{n,n}(M)\Big|^{((-1)^{n+1})/2}\right)^{\chi(N)}$$
$$=\Big|\mathbb{T}(M,\{\mathbf{h}_p\}_{p=0}^{2n})\Big|^{\chi(N)}.$$

Clearly, we have

$$\prod_{p=2}^{n}\Big\{\Big|\det H_{0,2}(N)\Big|^{\dim H_p(M)+\dim H_{p-2}(M)}\Big|\det H_{1,1}(N)\Big|^{\dim H_{p-1}(M)}\Big\}^{(-1)^p} \tag{2.18}$$
$$\times\Big|\det H_{0,2}(N)\Big|^{\dim H_0(M)-\dim H_1(M)}\Big|\det H_{1,1}(N)\Big|^{-\dim H_0(M)}$$
$$\times\Big|\det H_{0,2}(N)\Big|^{(-1)^{n+1}\dim H_{n+1}(M)/2}\Big|\det H_{0,2}(N)\Big|^{(-1)^{n+1}\dim H_{n-1}(M)/2}$$
$$\times\Big|\det H_{1,1}(N)\Big|^{(-1)^{n+1}\dim H_n(M)/2}$$
$$=\Big|\det H_{0,2}(N)\Big|^{\chi(M)}\Big|\det H_{1,1}(N)\Big|^{-\chi(M)/2}=\Big|\mathbb{T}(N,\{\mathbf{h}'_p\}_{p=0}^{2})\Big|^{\chi(M)}$$

Combining (2.17) and (2.18), we obtain (2.10).

This finishes the proof of Corollary 2.2. □

In particular, considering cartesian product of closed oriented Riemann surfaces of genus ≥ 1 and applying Corollary 2.2, we have

Corollary 2.3. *([24]) Let $\Sigma_{g_1},\ldots,\Sigma_{g_n}$ be closed oriented Riemann surfaces of genus $g_1,\ldots,g_n \geq 1$, respectively. For $p = 0,1,2$, and $i = 1,\ldots,n$, let $\mathbf{h}_{p,i}$ be a basis of*

$H_p(\Sigma_{g_i})$. *Then,*

$$\left|\mathbb{T}(\times_{i=1}^{n}\Sigma_{g_i}, \{\oplus_{|\alpha|=p}\mathbf{h}_{\alpha_1,1}\otimes\cdots\otimes\mathbf{h}_{\alpha_n,n}\}_{p=0}^{2n})\right|$$
$$= \prod_{i=1}^{n}\left|\mathbb{T}(\Sigma_{g_i}, \{\mathbf{h}_{p,i}\}_{p=0}^{2})\right|^{\chi(\Sigma_{g_1})\cdots\widehat{\chi(\Sigma_{g_i})}\cdots\chi(\Sigma_{g_n})},$$

where $\times_{i=1}^{n}\Sigma_{g_i}$ *is the cartesian product of* $\Sigma_{g_1},\ldots,\Sigma_{g_n}$, *where* $|\alpha| = \alpha_1 + \cdots + \alpha_n$ *is the length of the multi-index* $\alpha = (\alpha_1,\ldots,\alpha_n)$, *and where* $\widehat{\ }$ *in the product* $\chi(\Sigma_{g_1})\cdots\widehat{\chi(\Sigma_{g_i})}\cdots\chi(\Sigma_{g_n})$ *is deletion of* $\chi(\Sigma_{g_i})$.

2.3.2 Grassmannians and Schubert varieties

We provide the basic definitions and necessary facts about the Grassmannians, Lagrangian Grassmannians, Orthogonal Grassmannians, and Isotropic Grassmannians. For unexplained subject and further information, we refer the reader to [3, 4], [9, 10], [25]-[29], and the references therein.

Since the results corresponding to these manifolds are similar, we shall state for only one of them.

Grassmannian $G(d,N)$

Let E be $\mathbb{C}^N$ and let $G(d,E) = G(d,N)$ denote the Grassmannian of $d-$dimensional linear subspaces of E. This is a smooth algebraic variety of complex dimension dn, where $n = N-d$. It is well known that the Schubert cells stratify $G(d,N)$. The closures of these cells are called the *Schubert varieties*. More precisely, let $F_\bullet : 0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_N = E$ be a complete flag of subspaces of E with $\dim F_i = i$, $i = 0,\ldots,N$. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0)$ be a decreasing sequence of non-negative integers with $\lambda_1 \leq n$. Then, the Young diagram of the partition λ fits inside a $d\times n$ rectangle and this is denoted as $\lambda \subseteq (n^d)$.

The Schubert variety $X_\lambda(F_\bullet)$ associated to the complete flag $F_\bullet$ and the partition λ

is defined by

$$X_\lambda(F_\bullet) = \{\Lambda \in G(d,N) : \dim(\Lambda \cap F_{n+i-\lambda_i}) \geq i,\ i = 1,\dots,d\}.$$

This is a codimension $|\lambda|$ closed subvariety of $G(d,N)$, where $|\lambda| = \sum \lambda_i$ is the weight of λ. By Poincaré duality, $X_\lambda(F_\bullet)$ is associated to the Schubert class $\sigma_\lambda = [X_\lambda(F_\bullet)] \in H^{2|\lambda|}(G(d,N);\mathsf{Z})$. From the transitive action of $GL_N(\mathsf{C})$ on $G(d,N)$ and on the flags in E it follows that σ_λ is independent of the flag $F_\bullet$ used to define X_λ.

As an additive group $H^*(G(d,N);\mathsf{Z}) = \bigoplus_{\lambda\subseteq(n^d)} \mathsf{Z}\cdot\sigma_\lambda$ is a free abelian group generated by the Schubert classes. Odd dimensional cohomologies are all zero and the Euler characteristic $\chi(G(d,N)) = \begin{pmatrix} N \\ d \end{pmatrix}$. Recall also by Schubert Duality theorem that for any λ and μ with $|\lambda| + |\mu| = dn$, we have $\int_{G(d,N)} \sigma_\lambda\sigma_\mu = \delta_{\hat{\lambda},\mu}$, where $\hat{\lambda} = (\lambda_{N-d-\lambda_d},\dots,\lambda_{N-d-\lambda_1})$ is the dual partition of λ.

From Theorem 1.1 it follows that

Theorem 2.4. *([24]) Let $M = G(d,N)$ denote the Grassmannian of d-dimensional linear subspaces of C^N. For $p = 0,\dots,2m$, let $\mathbf{h}_p$ be a basis of $H_p(M)$, where $m = d(N-d)$. Then, the following formulas hold:*

(i) $\left|\mathbb{T}(M,\{\mathbf{h}_p\}_0^{2m})\right| = \prod_{p\in E_{m-1}} \left|\det H_{p,2m-p}(M)\right|$ *for m odd,*

(ii) $\left|\mathbb{T}(M,\{\mathbf{h}_p\}_0^{2m})\right| = \prod_{p\in E_{m-1}} \left|\det H_{p,2m-p}(M)\right| \sqrt{\left|\det H_{m,m}(M)\right|}$ *for m even,*

where E_{m-1} is the set of even numbers in $\{0,\dots,m-1\}$.

In particular, let us consider the complex projective space CP^m. For p even $H^p(\mathsf{CP}^m)$ is generated by ω_{FS}^p, where ω_{FS} is the Fubini-Study metric of CP^m and ω_{FS}^p denotes the p times wedge product of ω_{FS}. Using also the Poincare Duality, we have

Corollary 2.4. *([24]) If for $p = 0,\dots,2m$, $\mathbf{h}^p$ is a basis of $H^p(\mathsf{CP}^m)$, then*

(i) $\left|\mathbb{T}(\mathsf{CP}^m,\{\mathbf{h}_p\}_0^{2m})\right| = V_m \prod_{p\in E_{m-1}} \left|\lambda_p\right| \left|\lambda_{2m-p}\right|$ *for m odd,*

(ii) $\left|\mathbb{T}(\mathsf{CP}^m,\{\mathbf{h}_p\}_0^{2m})\right| = V_m \prod_{p\in E_{m-1}} \left|\lambda_p\right| \left|\lambda_{2m-p}\right| \left|\lambda_m\right|$ *for m even,*

where $\mathbf{h}_p \in H_p(\mathsf{CP}^m)$ *is the Poincare dual of* $\mathbf{h}^p \in H^p(\mathsf{CP}^m)$ *and* $\mathbf{h}^p = \lambda_p\ \omega_{\mathrm{FS}}^p$ *for some* $\lambda_p \in \mathsf{R}$, *where* ω_{FS} *is the Fubuni-Study form of* CP^m, *where* E_{m-1} *is the set of even numbers in* $\{0,\ldots,m-1\}$, *and where* $V_m = \left(\frac{1}{m!}\mathrm{Vol}(\mathsf{CP}^m)\right)^{\chi(\mathsf{CP}^m)/2}$.

We would like to conclude this section with the following example.

Example 2.1. Let $M = \mathsf{CP}^3$ and $N = \mathsf{CP}^6$. Using the Künneth formula, we get $H_0(M\times N) = H_0(M)\otimes H_0(N)$, $H_{18}(M\times N) = H_6(M)\otimes H_{12}(N)$, $H_2(M\times N) = H_0(M)\otimes H_2(N)\oplus H_2(M)\otimes H_0(N)$, $H_{16}(M\times N) = H_6(M)\otimes H_{10}(N)\oplus H_4(M)\otimes H_{12}(N)$, $H_4(M\times N) = H_0(M)\otimes H_4(N)\oplus H_2(M)\otimes H_2(N)\oplus H_4(M)\otimes H_0(N)$, $H_{14}(M\times N) = H_6(M)\otimes H_8(N)\oplus H_4(M)\otimes H_{10}(N)\oplus H_2(M)\otimes H_{12}(N)$, $H_6(M\times N) = H_0(M)\otimes H_6(N)\oplus H_2(M)\otimes H_4(N)\oplus H_4(M)\otimes H_2(N)\oplus H_6(M)\otimes H_0(N)$, $H_{12}(M\times N) = H_6(M)\otimes H_6(N)\oplus H_4(M)\otimes H_8(N)\oplus H_2(M)\otimes H_{10}(N)\oplus H_0(M)\otimes H_{12}(N)$, $H_8(M\times N) = H_0(M)\otimes H_8(N)\oplus H_2(M)\otimes H_6(N)\oplus H_4(M)\otimes H_4(N)\oplus H_6(M)\otimes H_2(N)$, and $H_{10}(M\times N) = H_6(M)\otimes H_4(N)\oplus H_4(M)\otimes H_6(N)\oplus H_2(M)\otimes H_8(N)\oplus H_0(M)\otimes H_{10}(N)$.

From these it follows that

$$|H_{0,18}(M\times N)| = |H_{0,6}(M)||H_{0,12}(N)|, \tag{2.19}$$

$$|H_{2,16}(M\times N)| = |H_{0,6}(M)||H_{2,4}(M)||H_{0,12}(N)||H_{2,10}(N)|, \tag{2.20}$$

$$|H_{4,14}(M\times N)| = |H_{0,6}(M)||H_{2,4}(M)|^2|H_{0,12}(N)||H_{2,10}(N)||H_{4,8}(N)|, \tag{2.21}$$

$$|H_{6,12}(M\times N)| = |H_{0,6}(M)|^2|H_{2,4}(M)|^2|H_{0,12}(N)||H_{2,10}(N)||H_{4,8}(N)||H_{6,6}(N)|, \tag{2.22}$$

$$|H_{8,10}(M\times N)| = |H_{0,6}(M)|^2|H_{2,4}(M)|^2|H_{2,10}(N)||H_{4,8}(N)|^2|H_{6,6}(N)|. \tag{2.23}$$

Combining (2.19)-(2.23), we obtain that $|\mathbb{T}(M\times N)| = |\mathbb{T}(M)|^{\chi(N)}\,|\mathbb{T}(N)|^{\chi(M)}$.

The Lagrangian Grassmannian $LG(n,2n)$

Let E be C^{2n} equipped with a symplectic form $<\cdot,\cdot>$. A subspace V of E is *isotropic* if the restriction of the symplectic form $<\cdot,\cdot>$ to V vanishes. Note that the maximal possible dimension of an isotropic subspace is n, and in this case V is called a *Lagrangian* subspace of E. The Lagrangian Grassmannian $LG(n, 2n)$ is a complex manifold of complex dimension $n(n+1)/2$ parametrizing the Lagrangian subspaces in E.

A complete isotropic flag $F_\bullet : 0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq E$ of subspaces of E is a flag of isotropic subspaces of E such that $\dim F_i = i$ for each i. Thus, a complete isotropic flag is a Lagrangian subspace F_n of E together with a complete flag of subspaces of F_n. In fact, any isotropic flag $F_\bullet : 0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq E$ can be completed to a complete flag by setting $F_{n+i} = F_{n-i}^\perp$, $i = 1, \ldots, n$.

Let $F_\bullet$ be a complete isotropic flag of E and $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with $\lambda_1 \leq n$ be a strictly decreasing partition. The codimension $|\lambda| = \sum \lambda_i$ Schubert variety $X_\lambda(F_\bullet) \subseteq LG(n, 2n)$ is defined by

$$X_\lambda(F_\bullet) = \{\Lambda \in LG(n, 2n) : \dim(\Lambda \cap F_{n+1-\lambda_i}) \geq i,\ i = 1, \ldots, \ell(\lambda)\ \},$$

where $\ell(\lambda)$ is the length of λ, i.e. the number of non-zero terms in λ.

Let $\sigma_\lambda = [X_\lambda(F_\bullet)] \in H^{2|\lambda|}(LG(n, 2n); \mathbb{Z})$ be the cohomology class of $X_\lambda(F_\bullet)$. $H^*(LG(n, 2n); \mathbb{Z})$ is a free abelian group generated by the Schubert classes σ_λ with strictly decreasing partition λ. Recall the Poincare duality $\int_{LG(n,2n)} \sigma_\lambda \sigma_\mu = \delta_{\check{\lambda},\mu}$, where $\check{\lambda} = \rho_n - \lambda$ is the dual partition of λ, and where $\rho_n = (n, n-1, \ldots, 1)$. Recall also that the Euler characteristic of $LG(n, 2n)$ is 2^n.

Moreover, for $LG(n, 2n)$, we have a result similar to Theorem 2.4, where $m = n(n+1)/2$.

The Orthogonal Grassmannian $OG(n+1, 2n+2)$

Let E be $\mathbb{C}^{2n+2}$ equipped with a non-degenerate symmetric form. The even orthogonal Grassmannian $OG(n+1, 2n+2)$ parametrizes one component of the locus of maximal isotropic subspaces of E. This is a complex manifold of complex dimension $n(n+1)/2$. There are two families of such subspaces. As convention, given a fixed isotropic flag $F_\bullet$ in E, only those isotropic Λ in E with $\Lambda \cap F_{n+1}$ even codimension in

F_{n+1} are considered. Recall that $OG(n+1, 2n+2)$ is isomorphic to the odd Orthogonal Grassmannian $OG(n, 2n+1)$.

As in $LG(n, 2n)$, the Schubert varieties $X_\lambda(F_\bullet)$ in $OG(n+1, 2n+2)$ are also parametrized by strictly decreasing partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with $\lambda_1 \leq n$ and defined by

$$X_\lambda(F_\bullet) = \{\Lambda \in OG(n+1, 2n+2) : \dim(\Lambda \cap F_{n+1-\lambda_i}) \geq i,\ i = 1, \ldots, \ell(\lambda)\ \}$$

with respect to a complete isotropic flag $F_\bullet$ in E. Let σ_λ be the cohomology class of $X_\lambda(F_\bullet)$. The abelian group $H^*(OG(n+1, 2n+2); \mathbb{Z})$ is generated by the Schubert classes σ_λ with strictly decreasing partition λ. Moreover, $\chi(OG(n+1, 2n+2)) = 2^n$.

Similar result of Theorem 2.4 also holds for $OG(n+1, 2n+2)$, where $m = n(n+1)/2$.

The Grassmannian $IG(n-k, 2n)$

Let us fix a vector space $E \cong \mathbb{C}^{2n}$ with a non-degenerate skew-symmetric bilinear form $< \cdot, \cdot >$, and let $d \leq n$ be a fixed non-negative integer. The Isotropic Grassmannian $IG(d, 2n)$ parametrizes $d-$dimensional isotropic subspaces of E. This is an algebraic variety of complex dimension $2d(n-d) + d(d+1)/2$.

Let k be a non-negative integer. The partition λ is said to be $k-$strict, if no part of λ greater than k is repeated, namely $\lambda_i > k \Rightarrow \lambda_{i+1} < \lambda_i$.

Now, let $k = n - d$. The Schubert varieties in $IG(d, 2n)$ are parametrized by the set $\mathcal{P}(k, n)$ of all $k-$strict partitions contained in a $d \times (n+k)$ rectangle.

Recall that an isotropic flag in E is a complete flag $F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = E$ of subspaces such that $F_{n+i} = F_{n-i}^\perp$, $i = 0, \ldots, n$. For each $\lambda \in \mathcal{P}(k, n)$, the Schubert variety relative to the isotropic flag $F_\bullet$ is

$$X_\lambda(F_\bullet) = \{\Lambda \in IG(d, 2n) : \dim(\Lambda \cap F_{p_j(\lambda)}) \geq j,\ j = 1, \ldots, \ell(\lambda)\ \},$$

where $p_j(\lambda) = n + k + 1 - \lambda_j + \sharp\{i < j : \lambda_i + \lambda_j \leq 2k + j - i\}$, and where $\ell(\lambda)$ is the length of λ.

This is a codimension $|\lambda|$ variety. Let σ_λ denote $[X_\lambda] \in H^{2|\lambda|}(IG(d,2n);\mathbb{Z})$. The cohomology ring $H^*(IG(d,2n);\mathbb{Z})$ is a free abelian group generated by these Schubert classes. Moreover, the $k-$strict partition λ has a unique dual partition $\check{\lambda} \in \mathcal{P}(k,n)$, for which $p_j(\check{\lambda}) = 2n+1-p_{d+1-j}(\lambda)$, $j = 1,\ldots,d$. We also have $\int_{IG(d,2n)} \sigma_\lambda\sigma_\mu = \delta_{\mu,\check{\lambda}}$. Finally, the Euler characteristic of $IG(d,2n)$ is the rank of $H^*(IG(d,2n);\mathbb{Z}) = \sharp\mathcal{P}(k,n) = 2^d \begin{pmatrix} n \\ k \end{pmatrix}$.

For $IG(d,2n)$, we also obtain a result similar to Theorem 2.4 where $m = 2d(n-d) + d(d+1)/2$.

The Grassmannian $OG(n-k, 2n+1)$

Let $E \cong \mathbb{C}^{2n+1}$ be a vector space with a non-degenerate symmetric bilinear form on E. For $d = n-k < n$, let $OG(d,2n+1)$ denote the Odd Orthogonal Grassmannian parametrizing the $d-$dimensional isotropic subspaces of E. Like $IG(d,2n)$, the algebraic variety $OG(n-k,2n+1)$ has also complex dimension $2d(n-d)+d(d+1)/2$. Furthermore, as in $IG(d,2n)$, the Schubert varieties are parametrized by the set of $k-$strict partitions $\mathcal{P}(k,n)$.

Recall that an isotropic flag $F_\bullet$ is a complete flag $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{2n+1} = E$ such that $F_{n+i} = F^\perp_{n+1-i}$, $i = 1,\ldots,n+1$. Let $F_\bullet$ be an isotropic flag and let $\lambda \in \mathcal{P}(k,n)$. The Schubert variety associated to $F_\bullet$ and λ is

$$X_\lambda(F_\bullet) = \{\Lambda \in OG(d,2n+1) : \dim(\Lambda \cap F_{\overline{p}_j(\lambda)}) \geq j,\ \ j = 1,\ldots,\ell(\lambda)\ \},$$

where $\overline{p}_j(\lambda) = p_j(\lambda) + \mathbf{1}_{\{0,\ldots,k\}}(\lambda_j)$, where $p_j(\lambda) = n+k+1-\lambda_j+\sharp\{i<j : \lambda_i+\lambda_j \leq 2k+j-i\}$, and where $\mathbf{1}_{\{0,\ldots,k\}}(\lambda_j) = \begin{cases} 1, & \lambda_j \leq k \\ 0, & \lambda_j > k \end{cases}$.

This variety has codimension $|\lambda|$. Let σ_λ denote the cohomology class of Poincare dual to the cycle given by $X_\lambda(F_\bullet)$. The abelian group $H^*(OG(d,2n+1);\mathbb{Z})$ is generated by these Schubert classes. We also have $\int_{OG(d,2n+1)} \sigma_\lambda\sigma_\mu = \delta_{\mu,\check{\lambda}}$, where $\overline{p}_j(\check{\lambda}) = 2n+2-$

$p_{d+1-j}(\lambda)$. The rank of $H^*(OG(d, 2n+1); \mathbb{Z})$ is equal to the rank of $H^*(IG(n-k, 2n); \mathbb{Z})$, i.e. $\sharp\mathcal{P}(k, n) = 2^d \begin{pmatrix} n \\ k \end{pmatrix}$.

Furthermore, for $OG(d, 2n + 1)$, there is a result similar to Theorem 2.4, where $m = 2d(n - d) + d(d + 1)/2$.

The Grassmannian $OG(n + 1 - k, 2n + 2)$

Let $E \cong \mathbb{C}^{2n+2}$ be a vector space with a non-degenerate symmetric bilinear form on E. For $d = n + 1 - k < n$, let $OG(d, 2n + 2)$ be the even Orthogonal Grassmannian parametrizing the $d-$dimensional isotropic subspaces of E. This is a variety of complex dimension $2d(n + 1 - d) + d(d - 1)/2$.

The subspaces U, V of E are *in the same family* if $\dim(U \cap V) \equiv (n + 1)(\text{mod } 2)$. Fix an isotropic subspace W of E with $\dim W = n + 1$. An isotropic flag is a complete flag $F_\bullet$ of subspaces of E such that $F_{n+1+i} = F^\perp_{n+1-i}$, $i = 0, \ldots, n$, and F_{n+1} and W are in the same family. Since the orthogonal space $F^\perp_n / F_n$ contains only two isotropic lines, to each such flag $F_\bullet$, there is an alternate isotropic flag $\tilde{F}_\bullet$ such that for $i \le n$ $\tilde{F}_i = F_i$ but with $\tilde{F}_{n+1}$ in the opposite family from F_{n+1}.

Let $k = n + 1 - d > 0$. The $k-$strict partition λ is of *type* 0 if it has no part equal to k. Otherwise, λ is of *type* 1 or 2. Type is a multi-valued function. Let $\tilde{\mathcal{P}}(k, n)$ be the set of all $k-$strict partitions contained in a $d \times (n + k)$ rectangle of all three possible types. For $\lambda \in \tilde{\mathcal{P}}(k, n)$, let us define an index set $P' = \{p'_1 < \cdots < p'_d\} \subseteq \{1, \ldots, 2n + 2\}$ with $p'_j(\lambda) = n + k - \lambda_j + \sharp\{i < j : \lambda_i + \lambda_j \le 2k - 1 + j - i\} + \begin{cases} 1, & \lambda_j > k \text{ or } \lambda_j = k < \lambda_{j-1} \text{ and } n + j + type(\lambda) \text{ is even} \\ 2, & \text{otherwise} \end{cases}$.

Let $F_\bullet$ be an isotropic flag. For each $\lambda \in \tilde{\mathcal{P}}(k, n)$, the codimension $|\lambda|$ Schubert variety is $X_\lambda(F_\bullet) = \{\Lambda \in OG(d, 2n + 2) : \dim(\Lambda \cap F_{p'_j(\lambda)}) \ge j, \text{ if } p'_j \ne n + 2, \dim(\Lambda \cap \tilde{F}_{n+1}) \ge j, \text{ if } p'_j = n + 2, \text{ for all } j = 1, \ldots, \ell(\lambda)\}$. Let σ_λ be the cohomology class in $H^{2|\lambda|}(OG(d, 2n + 2); \mathbb{Z})$ Poincare dual to the cycle determined by the Schubert variety associated to λ. The free abelian group $H^*(OG(d, 2n + 2); \mathbb{Z})$ is generated by the

Schubert classes and the rank of $H^*(OG(d, 2n+2); \mathsf{Z})$ is $2^{n+1-k}\begin{pmatrix} n+1 \\ k \end{pmatrix}$. For each $\lambda \in \tilde{\mathcal{P}}(k,n)$, define a dual partition $\check{\lambda} \in \tilde{\mathcal{P}}(k,n)$ by

$$p'_j(\check{\lambda}) = \begin{cases} 2n+3-p'_{d+1-j}(\lambda), & \text{if } n \text{ is odd or } p'_j(\lambda) \neq n+1, n+2 \\ p'_j(\lambda), & \text{if } n \text{ is even and } p'_j(\lambda) \in \{n+1, n+2\} \end{cases}.$$

Moreover, for $\lambda, \mu \in \tilde{\mathcal{P}}(k,n)$, we have $\int_{OG(d,2n+2)} \sigma_\lambda \sigma_\mu = \delta_{\mu,\check{\lambda}}$.

For $OG(d, 2n+2)$, similar result as Theorem 2.4 holds, where $m = 2d(n+1-d) + d(d-1)/2$.

CHAPTER 3

THE SCHUBERT CALCULUS AND COHOMOLOGY OF THE FLAG SPACE G/B FOR A KAČ-MOODY GROUP G

3.1 Schubert cells and integral cohomology of K/T for a compact Lie group K.

The general reference for this section is [31]. Let K be a compact semi-simple simply-connected Lie group. We fix a maximal torus $T \subseteq K$. The complexified Lie algebras of K and T will be denoted by $\mathbf{g}$ and $\mathbf{h}$ respectively. Let $\mathbf{b}$ be the Borel subalgebra of $\mathbf{g}$. The compact group K can be embedded into a complex Lie group G with Lie algebra $\mathbf{g}$. We choose a Borel subgroup B containing T. The analytic complexification $K/T \to G/B$ induces a complex structure on the flag space $K/T \cong G/B$. The flag space K/T will be denoted by X. In this section, the root system will be denoted by Δ, and the simple root system will be denoted by Σ. Δ_+ is the set of positive roots. From [31],

Theorem 3.1. *The finite dimensional flag space X is a non-singular complex projective variety.*

We give at this point two descriptions of the homology of X. The first of these makes use of the decomposition of X into cells, while the second involves the realization of two-dimensional cohomology classes as the Chern classes of one-dimensional holomorphic bundles.

Definition 3.1. Let W be the Weyl group of G. Then the *length* of an element $w \in W$

is the least number of factors in the decomposition relative to the set of the simple reflections r_α, is denoted by $\ell(w)$.

We know from [32] that $N_w = wN^- w^{-1} \cap N$ is a unipotent subgroup of G of (complex) dimension $\ell(w)$, where N is the unipotent radical of B and N^- is the opposite nilpotent subgroup of G. From [35], we have

Theorem 3.2. *Let G be a complex reductive Lie group. Then*

$$G = \bigsqcup_{w \in W} BwB.$$

In addition, there is an isomorphism of algebraic varieties

$$N_w \to BwB/B$$

given by $n \to nwB/B$.

Corollary 3.1.

$$X = \bigsqcup_{w \in W} BwB/B.$$

The cells $C_w = BwB/B$ are open and closed varieties in the Zariski topology. Let $\overline{C}_w$ be the closure of C_w in X respect to the usual topology, we have from [**?**],

Theorem 3.3. *Let Y be a projective variety and let Y° be the interior of Y with respect to the Zariski topology. Then the closure of Y° in the usual topology is Y.*

Since $C_w = BwB/B$ is a Zariski-open set, by Theorem 3.3, the closure of C_w coincides with the Zariski closure. $[\overline{C}_w] \in H_{2\ell(w)}(\overline{C}_w, \mathbb{Z})$ is the fundamental cycle of the complex algebraic variety $\overline{C}_w$. Let $s_w \in H_{2\ell(w)}(X, \mathbb{Z})$ be the image of $[\overline{C}_w]$ under the mapping induced by the embedding $\overline{C}_w \hookrightarrow X$.

Proposition 3.1. *The elements s_w form a basis of the free $\mathbb{Z}$-module $H_*(X, \mathbb{Z})$.*

Definition 3.2. A group W is a *Coxeter group* if there is a subset S of W such that W has the presentation

$$\langle s \in S : (ss')^{m_{ss'}} = 1 \rangle$$

where $m_{ss'} \in \{2, 3, \ldots, \infty\}$ is the order of ss', $s \neq s'$ and $m_{ss} = 1$. The pair (W, S) is called a *Coxeter system*.

Theorem 3.4. *[39] The Weyl group W is a Coxeter group.*

Definition 3.3. Let $w_1, w_2 \in W$, $\gamma \in \Delta_+$. Then we write $w_1 \xrightarrow{\gamma} w_2$ when $r_\gamma w_1 = w_2$ and $\ell(w_2) = \ell(w_1) + 1$. We put $w < w'$ if there is a chain

$$w = w_1 \to w_2 \to \cdots \to w_k = w'.$$

This order is called the *Bruhat order* on the Weyl group W.

Here are some properties of this ordering.

Lemma 3.1. *Let $w = r_{\alpha_1} \cdots r_{\alpha_l}$ be a reduced decomposition of an element $w \in W$. We put $\gamma_i = r_{\alpha_1} \cdots r_{\alpha_{i-1}}(\alpha_i)$. Then the roots $\gamma_1, \ldots, \gamma_l$ are distinct and the set $\{\gamma_1, \ldots, \gamma_l\}$ coincides with $\Delta_+ \cap w\Delta_-$.*

Lemma 3.2. *Let w, $w' \in W$ and let α be a simple root. Assume that $w < w'$. Then, either $r_\alpha w \leqslant w'$ or $r_\alpha w < r_\alpha w'$, either $w \leqslant r_\alpha w'$ or $r_\alpha w < r_\alpha w'$.*

The properties in Lemma 3.2 characterize the ordering $<$. From [31], we have

Proposition 3.2. *The Bruhat ordering $<$ on W is a partial order relation.*

Proposition 3.3. *Let $w \in W$ and let $w = r_{\alpha_1} \cdots r_{\alpha_l}$ be the reduced decomposition of w. If*

$$w' = r_{\alpha_{i_1}} \ldots r_{\alpha_{i_k}} \tag{3.1}$$

for $1 \leq i_1 < i_2 < \cdots < i_k \leq l$, then $w' \leqslant w$. If $w' < w$, then w' can be represented in the form (3.1) for some indexing set $\{i_j\}$. If $w' \to w$, then there is a unique index i satisfying $1 \leq i \leq l$ and such that

$$w' = r_{\alpha_1} \cdots r_{\alpha_{i-1}} r_{\alpha_{i+1}} \cdots r_{\alpha_l}.$$

Proposition 3.3 yields an alternative definition of the ordering on W in [49]. The geometrical interpretation of this partial ordering is very interesting and useful in what follows.

Theorem 3.5. *Let V be a finite-dimensional irreducible representation of the Lie algebra $\mathbf{g}$ with highest weight λ and let $\mathbf{n}$ be the nilpotent part of $\mathbf{g}$. Assume that all the weights $w\lambda$, $w \in W$, are distinct and select for each w a non-zero $f_w \in V$ of weight $w\lambda$. Then*

$$w' \leqslant w \quad \Longleftrightarrow \quad f_{w'} \in U(\mathbf{n}) f_w$$

where $U(\mathbf{n})$ is the enveloping algebra of Lie algebra $\mathbf{n}$.

We use Theorem 3.5 to describe the mutual disposition of the Schubert cells. From [49], we have

Theorem 3.6. *Let $w \in W$, $C_w \subseteq X$ be a Schubert cell, and $\overline{C}_w$ be its closure. Then*

$$C_{w'} \subseteq \overline{C}_w \quad \Longleftrightarrow \quad w' \leqslant w.$$

We turn to the other approach to the description of the cohomology of X. For this purpose, we introduce in $\mathbf{h}$ the coroot lattice

$$Q^{\vee} = \bigoplus_i \mathbb{Z} h_i$$

where h_i is the coroot to dual to $\alpha_i \in \Delta$. We have the weight lattice

$$P = \{\chi \in \mathbf{h}^* : \chi(h_i) \in \mathbb{Z} \text{ for all } \alpha_i \in \Delta\}$$

dual to $Q^{\vee}$. We set $P_{\mathbb{Q}} = P \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote by $\mathbf{h}_{\mathbb{Q}} \subseteq \mathbf{h}$ the vector space over $\mathbb{Q}$ spanned by the h_i. Let $R = S(P_{\mathbb{Q}})$ be the graded algebra of polynomial functions on $\mathbf{h}_{\mathbb{Q}}$ over rational coefficients where the graduation is given by the degree of polynomials. The

Weyl group W acts on $\mathbf{h}^*$ by the rule

$$r_{\alpha_i}(\chi) = \chi - \chi(h_i)\alpha_i \quad \text{for } \alpha \in \Sigma \text{ and } \chi \in \mathbf{h}^*.$$

We can extend the action of the Weyl group W on $\mathbf{h}^*$ to R by the rule $wf(h) = f(w^{-1}h)$ for $f \in R$. We denote by R^W the subring of W-invariant elements in R and set

$$R_+^W = \{f \in R^W : f(0) = 0\},$$
$$J = R_+^W R.$$

We want to construct a ring homomorphism $\psi_{\mathbb{Q}} : R \longrightarrow H^*(X, \mathbb{Q})$ in the following way. First let $\chi \in P$. Since G is simply-connected, there is a character $\theta \in \mathrm{Hom}(B, \mathbb{C}^*)$ such that $\theta(\exp b) = \exp\chi(b)$, for $b \in \mathbf{b}$, where $\exp : \mathbf{b} \to B$ is an exponential map which is a locally diffeomorphism. Since $G \to X$ is a principal bundle with structure group B, this θ defines a one-dimensional complex holomorphic line bundle

$$L_\chi = \{[g, \zeta] : [g\exp(b), \exp\chi(b)\zeta] = [g, \zeta] \text{ for } b \in \mathbf{b},\ g \in G \text{ and } \zeta \in \mathbb{C}\}$$

on X. We set $\psi(\chi) = c_\chi$, where $c_\chi \in H^2(X, \mathbb{Z})$ is the first Chern class of L_χ. Then ψ is a group homomorphism of P into $H^2(X, \mathbb{Z})$, which extends naturally to a homomorphism of graded rings

$$\psi_{\mathbb{Q}} : R \to H^*(X, \mathbb{Q}).$$

From [34, 33], we have

Proposition 3.4. *The homomorphism $\psi_{\mathbb{Q}}$ commutes with the action of W on R and $H^*(X, \mathbb{Q})$. $\ker\psi_{\mathbb{Q}} = J$ and the natural mapping $\overline{\psi_{\mathbb{Q}}} : R/J \to H^*(X, \mathbb{Q})$ is an isomorphism.*

We now study the rings R and $\overline{R} = R/J$. For each $w \in W$, we define an element $P_w \in \overline{R}$ and a functional D_w on R and investigate their properties. In the next section, we shall show that the D_w correspond to Schubert cell and that the P_w yield a basis, dual to the Schubert cell basis, for the rational cohomology of X. Let $\gamma \in \Delta$. We specify an operator A_γ on R by the rule

$$A_\gamma f = \frac{f - r_\gamma f}{\gamma}.$$

$A_\gamma f$ lies in R, since $f - r_\gamma f = 0$ on the hyperplane $\gamma = 0$ in $\mathbf{h}_{\mathbb{Q}}$. A_i will be called the *Bernstein-Gelfand-Gelfand operator* and it will be briefly indicated by *BGG-operator*. The properties of the A_γ are described in the following lemma.

Proposition 3.5. *For $\gamma \in \Delta$ and $w \in W$, we have*

$$\begin{aligned} A_{-\gamma} &= -A_\gamma, \\ A_\gamma^2 &= 0, \\ wA_\gamma w^{-1} &= A_{w\gamma}, \\ r_\gamma A_\gamma &= -A_\gamma r_\gamma = A_\gamma, \\ r_\gamma &= -\gamma A_\gamma + 1 = A_\gamma \gamma - 1, \\ A_\gamma f = 0 \quad &\Longleftrightarrow \quad r_\gamma f = f, \\ A_\gamma J &\subseteq J. \end{aligned}$$

Proposition 3.6. *Let $\chi \in \mathbf{h}_{\mathbb{Q}}^*$. Then the commutator of A_γ with the operator of multiplication by χ has the form $[A_\gamma, \chi] = \chi(h_\gamma) r_\gamma$.*

The following property of the BGG-operator A_γ is fundamental in what follows. From [31], we have

Theorem 3.7. *Let $\alpha_1, \ldots, \alpha_l \in \Sigma$. We put $w = r_{\alpha_1} \cdots r_{\alpha_l}$ and $A_{(\alpha_1, \cdots, \alpha_l)} = A_{\alpha_1} \cdots A_{\alpha_l}$. If $\ell(w) < l$, then $A_{(\alpha_1, \cdots, \alpha_l)} = 0$. If $\ell(w) = l$, then $A_{(\alpha_1, \cdots, \alpha_l)}$ depends only on w and not on the set $\alpha_1, \cdots, \alpha_l$; in this case we put $A_w = A_{(\alpha_1, \cdots, \alpha_l)}$.*

Proposition 3.7. *The operators A_w satisfy the following commutator relation:*

$$[w^{-1} A_w, \chi] = \sum_{w' \xrightarrow{\gamma} w} w' \chi(h_\gamma) w^{-1} A_{w'},$$

where h_γ is a coroot.

We put $S_i = R_i^*$, where $R_i \subseteq R$ is the space of homogeneous polynomials of degree i and R_i^* is the dual space of R_i and $S = \bigoplus_i S_i$. We denote by (,) the natural pairing $S \times R \to \mathbb{Q}$. Then W acts naturally on the graded ring S.

Definition 3.4. For any $\chi \in \mathbf{h}_{\mathbb{Q}}^*$, we let χ^* denote the transformation of S adjoint to the operator of multiplication by χ in R. We denote by $F_\gamma : S \to S$ the linear transformation adjoint to $A_\gamma : R \to R$.

The next lemma gives an explicit description of the F_γ.

Lemma 3.3. *Let $\gamma \in \Delta$. For any $D \in S$ there is a $\widetilde{D} \in S$ such that $\chi^*(\widetilde{D}) = D$. If $\widetilde{D}$ is any such operator, then $\widetilde{D} - r_\gamma \widetilde{D} = F_\gamma(D)$, in particular, the left-hand side of this equation does not depend on the choice of $\widetilde{D}$.*

It is often convenient to interpret S as a ring of differential operators on $\mathbf{h}$ with constant rational coefficients. Then the pairing (,) is given by the formula $(D, f) = (Df)(0)$, $D \in S$, $f \in R$. Also, it is easy to check that $\chi^*(D) = [D, \chi]$, where $\chi \in \mathbf{h}_{\mathbb{Q}}$ and $D \in S$ are regarded as operators on R .

Theorem 3.8. *Let $\alpha_1, \dots, \alpha_l \in \Sigma$. We put $w = r_{\alpha_1} \cdots r_{\alpha_l}$. If $\ell(w) < l$, then $F_{\alpha_l} \cdots F_{\alpha_1} = 0$. If $\ell(w) = l$, then $F_{\alpha_l} \cdots F_{\alpha_1}$ depends only on w and not on $\alpha_1 \dots \alpha_l$ and in this case we write $F_w = F_{\alpha_l} \cdots F_{\alpha_1}$. Also, $F_w = A_w^*$ and*

$$[\chi^*, F_w w] = \sum_{w' \xrightarrow{\gamma} w} w' \chi(h_\gamma) F_{w'} w,$$

where h_γ is a coroot.

We set $D_w = F_w(1)$. As we shall show in the next section, the functionals D_w correspond to the Schubert cells in $H_*(X, \mathbb{Q})$ in the sense that $(D_w, f) = \langle s_w, \psi_{\mathbb{Q}}(f) \rangle$ for all $f \in R$. The properties of the D_w are listed in the following theorem.

Theorem 3.9. *Let $w \in W$, let α be a simple root and $\chi, \chi_1, \ldots, \chi_l \in \mathbf{h}^*_{\mathbb{Q}}$ and $D_w \in S_{\ell(w)}$. Then*

$$F_\alpha D_w = \begin{cases} 0 & \text{if } \ell(wr_\alpha) < \ell(w), \\ D_{wr_\alpha} & \text{if } \ell(wr_\alpha) \geq \ell(w). \end{cases}$$

$$\chi^*(D_w) = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma) D_{w'}.$$

$$r_\alpha D_w = \begin{cases} -D_w & \text{if } \ell(wr_\alpha) < \ell(w), \\ -D_w + \sum_{w' \xrightarrow{\gamma} wr_\alpha} w'\alpha(h_\gamma) D_{w'} & \text{if } \ell(wr_\alpha) \geq \ell(w). \end{cases}$$

$$(D_w, \chi_1 \cdots \chi_l) = \sum \chi_1(h_{\gamma_1}) \cdots \chi_l(h_{\gamma_l}),$$

where the summation extends over all chains

$$e \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_1} w_2 \rightarrow \cdots \xrightarrow{\gamma_1} w_l = w^{-1}.$$

Let $J^\perp$ be the subspace of S orthogonal to the ideal $J \subseteq R$. It follows from Lemma 3.6 that $J^\perp$ is invariant with respect to all the F_γ. It is also clear that $1 \in J^\perp$. Thus, $D_w \in J^\perp$ for all $w \in W$. From [31], we have

Theorem 3.10. *For $w \in W$, the functionals D_w form a basis for $J^\perp$.*

The form (,) gives rise to a non-degenerate pairing between $\overline{R}$ and $J^\perp$. Let $\{P_w\}_{w \in W}$ be the basis of $\overline{R}$ dual to $\{D_w\}_{w \in W}$. The following properties of the P_w are immediate consequences of Theorem 3.9

Theorem 3.11. *Let $w \in W$, let α be a simple root and $\chi \in \mathbf{h}^*_{\mathbb{Q}}$. Then*

$$A_\alpha P_w = \begin{cases} 0 & \text{if } \ell(wr_\alpha) > \ell(w), \\ P_{wr_\alpha} & \text{if } \ell(wr_\alpha) \leqslant \ell(w). \end{cases}$$

$$\chi P_w = \sum_{w \xrightarrow{\gamma} w'} w\chi(h_\gamma) P_{w'}.$$

$$r_\alpha P_w = \begin{cases} P_w & \text{if } \ell(wr_\alpha) > \ell(w), \\ P_w - \sum_{wr_\alpha \xrightarrow{\gamma} w'} w\alpha(h_\gamma) P_{w'} & \text{if } \ell(wr_\alpha) \leqslant \ell(w). \end{cases}$$

From Theorem 3.11, it is clear that all the P_w can be expressed in terms of P_s, where $s \in W$ is the unique element of maximal length, $r = \ell(s)$. More precisely, let $w = r_{\alpha_1} \cdots r_{\alpha_l}$, $\ell(w) = l$. Then

$$P_w = A_{\alpha_l} \ldots A_{\alpha_1} P_s.$$

To find an explicit form for the P_w it therefore suffices to determine the $P_s \in \overline{R}$. From [31], we have

Theorem 3.12. $P_s = \frac{1}{|W|} \prod_{\gamma \in \Delta_+} \gamma \pmod{J}$.

We now give some results on products of the P_w in $\overline{R}$.

Theorem 3.13. *Let α be a simple root and let $w \in W$. Then*

$$P_{r_\alpha} P_w = \sum_{w \xrightarrow{\gamma} w'} \chi_\alpha(h_{w^{-1}\gamma}) P_{w'},$$

where $\chi_\alpha \in \mathbf{h}^*_{\mathbb{Z}}$ is the fundamental weight corresponding to the root α. Let $w_1, w_2 \in W$ and satisfying $\ell(w_1) + \ell(w_2) = r$. If $w_2 \neq w_1 s$, and $P_{w_1} P_{w_1 s} = P_s$, then $P_{w_1} P_{w_2} = 0$. Let $w \in W$ and $f \in \overline{R}$. Then

$$f P_w = \sum_{w' \geq w} c_{w'} P_{w'}$$

If $w_1 \nleq w_2 s$, then $P_{w_1} P_{w_2} = 0$.

We define the operator $\mathcal{P} : \overline{R} \to J^{\perp}$ of Poincaré duality by the formula

$$(\mathcal{P}f)(g) = D_s(fg), \ f, g \in \overline{R}, \ \mathcal{P}f \in J^{\perp}.$$

Corollary 3.2. $\mathcal{P}P_w = D_w$.

We will show that the functionals $D_w, w \in W$ introduced in last section correspond to Schubert cells $s_w, w \in W$ and give the cap-product formula in the cohomology of flag variety K/T. Let $s_w \in H_*(X, \mathbb{Q})$ be a Schubert cell. It gives rise to a linear functional on $H^*(X, \mathbb{Q})$, which, by means of the ring homomorphism $\psi_{\mathbb{Q}} : R \to H^*(X, \mathbb{Q})$, can be regarded as a linear functional on R. This functional takes the value 0 on all homogeneous components P_k with $k \neq \ell(w)$, and thus determines an element $\widehat{D}_w \in S_{\ell(w)}$. From [31], we have

Theorem 3.14. $\widehat{D}_w = D_w$

This theorem is a natural consequence of the next two propositions.

Proposition 3.8. $\widehat{D}_e = 1$*, and for any* $\chi \in \mathbf{h}^*_{\mathbb{Z}}$

$$\chi^*(\widehat{D}_w) = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma)\widehat{D}_w.$$

Proposition 3.9. *Suppose that for each* $w \in W$ *we are given an element* $\widehat{D}_w \in S_{\ell(w)}$*, with* $\widehat{D}_e = 1$*, for which Proposition 3.8 holds for any* $\chi \in \mathbf{h}^*_{\mathbb{Z}}$*. Then* $\widehat{D}_w = D_w$*.*

For any topological space Y there is a bilinear mapping called the cap-product

$$\cap : H^i(Y, \mathbb{Q}) \times H_j(Y, \mathbb{Q}) \to H_{j-i}(Y, \mathbb{Q}).$$

such that

$$\langle c \cap y, z \rangle = \langle y, c.z \rangle$$

for all $y \in H_j(Y, \mathbb{Q})$, $z \in H^{j-i}(Y, \mathbb{Q})$, $c \in H^i(Y, \mathbb{Q})$. If $f : Y_1 \to Y_2$ is a continuous mapping, then

$$f_*(f^*c \cap y) = c \cap f_*y$$

for all $y \in H_*(Y_1, \mathbb{Q})$, $c \in H^*(Y_2, \mathbb{Q})$. Then, we have for any $\chi \in \mathbf{h}^*_{\mathbb{Z}}$, $f \in R$

$$(\chi^*(\widehat{D}_w), f) = (\widehat{D}_w, \chi f) = \langle s_w, \psi(\chi)\psi_{\mathbb{Q}}(f)\rangle = \langle s_w \cap \psi(\chi), \psi_{\mathbb{Q}}(f)\rangle.$$

Therefore, Proposition 3.8 is equivalent to the following geometrical fact.

Proposition 3.10. *For all* $\chi \in \mathbf{h}^*_{\mathbb{Z}}$

$$s_w \cap \psi(\chi) = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma)s_{w'}.$$

We restrict the one complex dimensional holomorphic line bundle L_χ to $\overline{C}_w \subseteq X$ and let $c_\chi \in H^2(\overline{C}_w, \mathbb{Q})$ be the first Chern class of L_χ. Then, it is sufficient to prove that

$$s_w \cap c_\chi = \sum_{w' \xrightarrow{\gamma} w} w'\chi(h_\gamma)s_{w'} \tag{3.2}$$

in $H_{2\ell(w)-2}(\overline{C}_w, \mathbb{Q})$. To prove Equation 3.2, we use the following lemma, which can be verified by standard arguments involving relative Poincaré duality, see [38].

Lemma 3.4. *Let Y be a compact complex analytic space of dimension n, such that the codimension of the space of singularities of Y is greater than 1. Let E be an analytic vector bundle on Y, and $c \in H^2(Y, \mathbb{Q})$ the first Chern class of E. Let μ be a non-zero analytic section of E and $\sum_i m_iY_i = \mathrm{div}\mu$ the divisor of μ. Then $[Y] \cap c = \sum_i m_i[Y_i] \in H_{2n-2}(Y, \mathbb{Q})$ where $[Y]$ and $[Y_i]$ are the fundamental classes of Y and Y_i.*

Let $w \in W$, and let $C_w \subseteq X$ be the corresponding the Schubert cell. Then,

Proposition 3.11. *Let $w' \xrightarrow{\gamma} w$. Then $\overline{C}_w$ is non-singular at points $x \in C_{w'}$.*

We now give another proposition to prove Proposition 3.10

Proposition 3.12. *There is a section μ of the fibering E_χ over $\overline{C}_w$ such that*

$$\operatorname{div}\mu = \sum_{w' \xrightarrow{\gamma} w} w'\chi(H_\gamma)\overline{X}_{w'}.$$

3.2 Differential operators, Lie algebra cohomology and generalized Schubert cocycles.

First, we will give some facts about Kač-Moody Lie algebras and associated groups which will be used in this section. The general reference is [41] of V. Kač.

Definition 3.5. Let $A = \{a_{ij}\}_{n\times n}$ be a complex matrix of rank l. A *realization* of A is a triple $(\mathbf{h}, \pi, \pi^V)$, where $\mathbf{h}$ is a complex vector space of dimension $n + \operatorname{corank} A$, $\pi = \{\alpha_i\}_{1\leq i\leq n} \subseteq \mathbf{h}^*$ and $\pi^V = \{h_i\}_{1\leq i\leq n} \subseteq \mathbf{h}$ are free indexed sets satisfying $\alpha_j(h_i) = a_{ij}$.

Definition 3.6. Two realizations $(\mathbf{h}, \pi, \pi^V)$ and $(\mathbf{h}_1, \pi_1, \pi_1^V)$ are called *isomorphic* if there exists a vector space isomorphism $\phi : \mathbf{h} \to \mathbf{h}_1$ such that $\phi(\pi^V) = \pi_1^V$ and $\phi^*(\pi_1) = \pi$.

From [41], we have

Theorem 3.15. *There exists a unique up to isomorphism realization for every $n \times n$ matrix.*

Definition 3.7. A *generalized Cartan matrix* $A = \{a_{ij}\}_{n\times n}$ is a matrix of integers satisfying $a_{ii} = 2$ for all i and $a_{ij} \leqslant 0$ if $i \neq j$, $a_{ij} = 0$ implies $a_{ji} = 0$.

Definition 3.8. Given a realization $(\mathbf{h}, \pi, \pi^V)$ of a $n \times n$ generalized Cartan matrix A, the *Kač-Moody algebra* $\mathbf{g} = \mathbf{g}(A)$ is the Lie algebra over $\mathbb{C}$, generated by $\mathbf{h}$ and the elements e_i and f_i for $1 \leq i \leq l$ such that this basis elements satisfy the following relations:

$$[\mathbf{h}, \mathbf{h}] = 0,\ [h, e_i] = \alpha_i(h)e_i,\ [h, f_i] = -\alpha_i(h)f_i$$

for $h \in \mathbf{h}$ and all $1 \leq i \leq l$; $[e_i, f_i] = \delta_{ij}h_j$ for all $1 \leq i, j \leq n$;

$$(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0 = (\operatorname{ad} f_i)^{1-a_{ij}}(f_j)$$

for all $1 \le i \ne j \le n$. The elements h_i, e_i, f_i are called *Chevalley generators* and the subalgebra $\mathbf{h}$ of $\mathbf{g}(A)$ is called the *Cartan subalgebra*.

The Kač-Moody algebra $\mathbf{g} = \mathbf{g}(A)$ has a root space decomposition.

Theorem 3.16. *For $0 \ne \alpha$, the root space $\mathbf{g}_\alpha = \{x \in \mathbf{g} : [h, x] = \alpha(h)x, \forall\, h \in \mathbf{h}\}$ is finite dimensional and there is a root space decomposition*

$$\mathbf{g} = \mathbf{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbf{g}_\alpha = \mathbf{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathbf{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta^-} \mathbf{g}_\alpha$$

where Δ^+ (resp. Δ^-) is the positive (resp. negative) root system.

We define *fundamental reflections* $r_i \in \mathrm{Aut}_{\mathbb{C}}(\mathbf{h})$, $1 \le i \le n$, by $r_i(h) = h - \alpha_i(h)h_i$. They generate the *Weyl group* W, which is a Coxeter group on $\{r_i\}_{1 \le i \le n}$.

We define the following Lie algebras.

$$\mathbf{n} = \bigoplus_{\alpha \in \Delta^+} \mathbf{g}_\alpha, \quad \mathbf{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathbf{g}_\alpha.$$

Then, $\mathbf{g} = \mathbf{h} \oplus \mathbf{n} \oplus \mathbf{n}^-$, where $\mathbf{b} = \mathbf{h} \oplus \mathbf{n}$ is called the *Borel algebra*. We have a unique complex linear involution ω of $\mathbf{g}$ defined by $\omega(f_i) = e_i$ for all $1 \le i \le l$ and $\omega(h) = -h$ for $h \in \mathbf{h}$. The involution ω leaves the real points of $\mathbf{g}$ stable. ω is called the Chevalley involution. Also, we have a unique conjugate linear involution ω_0 which agrees with ω on the real points of $\mathbf{g}$. We can define a nondegenerate $\mathbf{g}$-invariant, symmetric complex bilinear form σ on $\mathbf{h}^*$ such that $\sigma(\alpha_i, \alpha_j) = \langle h_{\alpha_i}, h_{\alpha_j} \rangle$ where $\langle\, , \rangle$ is the standard complex inner product on $\mathbf{g}$. This form is called the *Killing form*. This gives a Hermitian form $\{\, , \}$ on $\mathbf{g}$ defined by $\{x, y\} = -\langle x, \omega_0(y) \rangle$ for $x, y \in \mathbf{g}$.

Now, we will mention the highest weight module category of a Kač-Moody algebra $\mathbf{g}$. The fundamental reference is [41] of V. Kač. Let V be a $\mathbf{g}$-module, $\lambda \in \mathbf{h}^*$. We define

$$V_\lambda = \{x \in V : h \cdot x = \lambda(h)x \text{ for } \forall h \in \mathbf{h}\}.$$

Then, V_λ is a subspace of V. If $V_\lambda \ne 0$, λ is called a *weight* of the $\mathbf{g}$-module V, V_λ the *weight space* corresponding to λ, and $\dim V_\lambda$ the *multiplicity* of λ. If λ is a weight of V, then any non-zero vector of V_λ is called a *weight vector* of λ. We denote by

$$P(V) = \{\lambda \in \mathbf{h}^* : V_\lambda \neq 0\},$$

the set of weights of the $\mathbf{g}$-module V.

Lemma 3.5. *For any $\alpha \in \Delta \cup \{0\}$ and $\lambda \in \mathbf{h}^*$, we have*

$$\mathbf{g}_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}.$$

We set

$$D(\lambda) = \{\lambda - \alpha : \alpha \in Q_+\},$$

where $Q_+ = \bigoplus_i \mathbb{Z}_+\alpha_i$. For any subset $F \subseteq \mathbf{h}^*$, we define

$$D(F) = \bigcup_{\lambda \in F} D(\lambda).$$

We can define a partial ordering $\geqslant$ on $\mathbf{h}^*$ by

$$\lambda \geqslant \mu \quad \Longleftrightarrow \quad \lambda - \mu \in Q_+ \quad \Longleftrightarrow \quad \mu \in D(\lambda).$$

We will give the definition of category O of $\mathbf{g}$-modules.

Definition 3.9. The objects of O are $\mathbf{g}$-modules V which satisfy the conditions

1. V is $\mathbf{h}$-diagonalizable, i.e.,
$$V = \bigoplus_{\lambda \in \mathbf{h}^*} V_\lambda,$$
2. $\dim V_\lambda < \infty$ for all $\lambda \in \mathbf{h}^*$,
3. there exists a finite set $F \subseteq \mathbf{h}^*$ such that $P(V) \subseteq D(F)$, and whose morphisms are $\mathbf{g}$-module homomorphisms.

By the property 3 of the definition of $\mathcal{O}$, we have

Proposition 3.13. *Every non-zero* $\mathbf{g}$*-module in* $\mathcal{O}$ *has at least one maximal weight.*

Definition 3.10. A $\mathbf{g}$-module V is called a *highest weight module*, if V has a unique maximal weight Λ and V is generated by some weight vector $v_\Lambda \in V_\Lambda$.

Theorem 3.17. *Let* V *be a highest weight module with maximal weight* Λ*. Then,*

$$V = U(\mathbf{g}) \cdot v_\Lambda = U(\mathbf{n}^-) \cdot v_\Lambda$$

for any weight vector v_Λ *of* Λ*;* $V \in \mathcal{O}$*,* $\dim V_\Lambda = 1$ *and* $P(V) \subseteq D(\Lambda)$*;* $\mathbf{g}_\alpha \cdot V_\Lambda = 0$ *for any* $\alpha \in \Delta_+$*;* V *has a unique maximal submodule, hence a unique quotient simple module; the homomorphic image of* V *is also a highest weight module with maximal weight* Λ*.*

Since $\mathbf{b} = \mathbf{h} \oplus \mathbf{n}^+$, we can regard V_Λ as a $\mathbf{b}$-module with $\mathbf{n}^+$ acting on it trivially. We define

$$M(\Lambda) = U(\mathbf{g}) \otimes_{U(\mathbf{b})} V_\Lambda$$

where V_Λ is 1-dimensional weight space corresponding to the weight Λ. The $\mathbf{g}$-module $M(\Lambda)$ is called *Verma module* corresponding to the weight Λ.

Theorem 3.18. *The Verma module* $M(\Lambda)$ *is a highest weight module with highest weight* Λ*. Any highest weight module with highest weight* Λ *is a homomorphic image of the Verma module* $M(\Lambda)$*.* $M(\Lambda)$ *has a unique maximal submodule with simple quotient* $L(\Lambda)$*. Any irreducible highest weight module with highest weight* Λ *is isomorphic to* $L(\Lambda)$*.*

Now, we will give the definition of lowest weight module. Let $L(\Lambda)$ be an irreducible highest weight module with the highest weight Λ. Let $L(\Lambda)^*$ be the $\mathbf{g}$-module contragredient to $L(\Lambda)$. Then

$$L(\Lambda)^* = \prod_{\lambda \in \mathbf{h}^*} L(\Lambda)^*_\lambda.$$

The subspace

$$L^*(\Lambda) = \bigoplus_{\lambda \in \mathbf{h}^*} L(\Lambda)^*_\lambda$$

is a submodule of the $\mathbf{g}$-module $L(\Lambda)^*$. The module $L^*(\Lambda)$ is irreducible and for $v \in L(\Lambda)^*_\lambda$, we have

$$\mathbf{n}_- \cdot v = 0 \quad \text{and} \quad h \cdot v = -\Lambda(h)v \text{ for } h \in \mathbf{h}.$$

The module $L^*(\Lambda)$ is called an *irreducible module with lowest weight* $-\Lambda$.

Theorem 3.19. *There is a bijection between* $\mathbf{h}^*$ *and irreducible lowest weight modules given by* $\Lambda \to L^*(-\Lambda)$.

We denote by π_Λ the action of $\mathbf{g}$ on $L(\Lambda)$. We give a new action π^*_Λ on the space $L(\Lambda)$ by

$$\pi^*_\Lambda(g)v = \pi_\Lambda(\omega(g))v,$$

where ω is the Chevalley involution of $\mathbf{g}$. $(L(\Lambda), \pi^*_\Lambda)$ is an irreducible $\mathbf{g}$-module with lowest weight $-\Lambda$. By the uniqueness of irreducible lowest weight modules with the lowest weight $-\Lambda$, this module can be identified with $L^*(\Lambda)$.

Definition 3.11. A $\mathbf{g}$-module L is called *quasi-simple* if it is a highest weight module with highest weight vector x_0 such that there exists $n \in \mathbb{Z}_+$ with $f_i{}^n(x_0) = 0$ for all $1 \leq i \leq i$.

From [37], we have

Proposition 3.14. *The quasi-simple* $\mathbf{g}$*-modules are indexed by the positive integral weights.*

We will denote by $L(\lambda)$ the quasi-simple module with highest weight λ. We will denote the derived algebra $[\mathbf{g}, \mathbf{g}]$ by $\mathbf{g}'$. From [50], we have

Theorem 3.20. $\mathbf{g}'$ *is the subalgebra of* $\mathbf{g}$ *generated by the Chevalley generators* e_i *anf* f_i *for* $1 \leq i \leq l$ *and we have the decomposition*

$$\mathbf{g}' = \mathbf{h}' \oplus \mathbf{n} \oplus \mathbf{n}^-,$$

where $\mathbf{h}' = \mathbf{g}' \cap \mathbf{h}$.

Definition 3.12. A $\mathbf{g}'$-module (V, π) is called *integrable* if $\pi(e)$ is locally nilpotent whenever $e \in \mathbf{g}_\alpha$ for any real root.

Let G^* be the free product of the additive groups $\{\mathbf{g}_\alpha\}_{\alpha\in\Delta_{\mathrm{re}}}$ with canonical inclusions $i_\alpha : \mathbf{g}_\alpha \to G^*$. For any integrable $\mathbf{g}'$-module (V, π), we define a homomorphism $\pi^* : G^* \to \mathrm{Aut}_{\mathbb{C}}(V)$ by $\pi^*(i_\alpha(e)) = \exp(\pi(e))$ for $e \in \mathbf{g}_\alpha$. Let N^* be the intersection of all $\ker \pi^*$ and let $q : G^* \to G^*/N^*$ be the canonical homomorphism. We put $G = G^*/N^*$. The next result comes from [44].

Proposition 3.15. *G is an algebraic group in the sense of Safarevič.*

We call G the *group associated to the Kač-Moody Lie algebra* $\mathbf{g}$. G may be of three different types: finite, affine and wild. The finite type Kač-Moody groups are simply-connected semi-simple finite dimensional algebraic groups. The affine type Kač-Moody groups are the circle group extension of the group of polynomial maps from $\mathbb{S}^1$ to a group of finite type, or a twisted analogue. There is no concrete realization of the wild type groups. Now, we will introduce some subgroups of the Kač-Moody group G. For $e \in \mathbf{g}_\alpha$, we put $\exp(e) = q(i_\alpha(e))$ so that $U_\alpha = \exp \mathbf{g}_\alpha$ is an additive one parameter subgroup of G. We denote by U (resp. U^-) the subgroup of G generated by the U_α (resp. $U_{-\alpha}$) for $\alpha \in \Delta_+$. For $1 \le i \le l$, there exists a unique homomorphism $\varphi_i : SL_2(\mathbb{C}) \to G$, satisfying $\varphi \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \exp(ze_i)$ and $\varphi \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \exp(zf_i)$ for all $z \in \mathbb{C}$. We define

$$H_i = \left\{ \varphi \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^* \right\};$$

$G_i = \varphi(SL_2(\mathbb{C}))$. Let N_i be the normalizer of H_i in G_i, H the subgroup of G generated by all H_i and N the subgroup of G generated by all N_i. There is an isomorphism $W \to N/H$. We put $B = HU$. B is called *standard Borel subgroup* of G. Also, we can define the negative Borel subgroup B^- as $B^- = HU^-$. G has Bruhat and Birkhoff decompositions. Details can be found in [43]. The conjugate linear involution ω_0 of $\mathbf{g}$

gives to an involution $\tilde{\omega}_0$ on G. Let K denote the set of fixed points of this involution. K is called the *standard real form* of G. Also, this involution preserves the subgroups G_i, H_i and H; we denote by K_i, T_i and T respectively the corresponding fixed point subgroups. Then, $K_i = \varphi(SU_2)$ and

$$T_i = \left\{ \varphi \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} : |u| = 1 \right\}$$

is a maximal torus of K_i and $T = \prod T_i$ is a maximal torus in K.

Now, we will give some facts about the topology of K. Let D (resp. D°) be the unit disk (resp. its interior) in $\mathbb{C}$ and let $\mathbb{S}^1$ be the unit circle. Given $u \in D$, let

$$z(u) = \begin{pmatrix} u & (1-|u|^2)^{1/2} \\ -(1-|u|^2)^{1/2} & \overline{u} \end{pmatrix} \in SU_2,$$

and $z_i(u) = \varphi_i(z(u))$. We also set

$$Y_i = \{z_i(u) : u \in D^\circ\} \subseteq K_i.$$

Let $w = r_{i_1} \cdots r_{i_n}$ be a reduced expression of $w \in W$. We put $Y_w = Y_{i_1} \cdots Y_{i_n}$. We have a fibration $\pi : K \to K/T$. The topological space K/T is called the *flag variety* of the K and G. Now, we will give the topological structure in the infinite dimensional case. We define $C_w = \pi(Y_w)$. From [42], we have

Proposition 3.16. *The decomposition*

$$K/T = \coprod_{w \in W} C_w$$

defines a CW structure on K/T.

The closure of C_w is given by

$$\overline{C}_w = \coprod_{w' \leqslant w} C_{w'}$$

The closures $\overline{C}_w$ are called Schubert varieties and they are finite dimensional complex spaces. The infinite type flag variety K/T is the inductive limit of these spaces and by Iwasawa decomposition in [43], we have a homeomorphism $K/T \to G/B$. From [42], we have

Proposition 3.17. *The flag variety K/T is an infinite dimensional complex projective variety.*

Proposition 3.18. *The elements $\overline{C}_w$ are a basis form of free $\mathbb{Z}$-module $H_*(K/T, \mathbb{Z})$.*

Now we will give the construction of the dual Schubert cocycles on the flag variety by using the relative Lie-algebra cohomology tools. This construction was done by B. Kostant in [45] for finite type and extended by S. Kumar in [47] for the Kač-Moody case.

First, we will introduce some notations for this section. Reference for the notations is [40]. By $\Lambda(V)$, we denote the exterior algebra on a $\mathbf{g}$-module V. For a Lie-algebra pair $(\mathbf{g}, \mathbf{h})$ and a left $\mathbf{g}$-module M, let $\Lambda(\mathbf{g}, \mathbf{h}, M^t)$ denote the standard chain complex with coefficients in the right module M^t, where M^t is the right $\mathbf{g}$-module, whose underlying space is M and on which $\mathbf{g}$ acts by the rule $m \cdot g = -g \cdot m$ for all $g \in \mathbf{g}$ and $m \in M$, and let $C(\mathbf{g}, \mathbf{h}, M)$ denote the standard cochain complex with coefficients in M.

Let L_λ be the quasi-simple $\mathbf{g}$-module with highest weight λ. Then, there is an invariant positive definite Hermitian form $\{,\}$ on $\Lambda(\mathbf{n}^-) \otimes L_\lambda$ due to V. Kač and D. Peterson [43]. Let $\partial : \Lambda(\mathbf{n}^-) \otimes L_\lambda \to \Lambda(\mathbf{n}^-) \otimes L_\lambda$ be the differential of degree -1 of the chain complex $\Lambda(\mathbf{n}^-, {L_\lambda}^t)$. We denote the adjoint of ∂ respect to $\{,\}$ by ∂^*. $\Lambda(\mathbf{n}^-) \otimes L_\lambda$ is a $\mathbf{h}$-module and ∂ is a $\mathbf{h}$-module map, as is ∂^*. We define the *Laplacian* by $\nabla = \partial\partial^* + \partial^*\partial$. We know from [37] that $\Lambda(\mathbf{n}^-) \otimes L_\lambda$ decomposes as a direct sum of finite dimensional irreducible $\mathbf{h}$-modules. From [47], we have

Theorem 3.21. *Let $\mathbf{g}$ be the Kač-Moody Lie algebra and let L_λ be the quasi-simple $\mathbf{g}$-module with the highest weight λ. Then the action of ∇ on $\Lambda(\mathbf{n}^-, {L_\lambda}^t)$ is as follows. Let S_β be an irreducible $\mathbf{h}$-submodule of $\Lambda(\mathbf{n}^-, {L_\lambda}^t)$ with highest weight β, then ∇ reduces* *to a scalar operator on S_β and the scalar is*

$$\frac{1}{2}[\sigma(\lambda+\rho,\lambda+\rho)-\sigma(\beta+\rho,\beta+\rho)]$$

where ρ is the sum of all simple roots.

From [37], we have

Theorem 3.22. *With the notations as in Theorem 3.21, the $2j$th homology space $H_{2j}(\mathbf{n}^-, {L_\lambda}^t)$ is finite dimensional and it is isomorphic as an* **h***-module to the direct sum*

$$\bigoplus_{\ell(w)=j} M_{(w(\lambda+\rho)-\rho)}$$

of non-isomorphic irreducible **h***-modules.*

$C(\mathbf{g},\mathbf{h})$ denotes the standard cochain complex with differential d associated to the Lie algebra pair $(\mathbf{g},\mathbf{h})$ with trivial coefficients. That is, $C(\mathbf{g},\mathbf{h})$ is defined to be $\sum_{s\geq 0}\mathrm{Hom}_{\mathbf{h}}(\Lambda^s(\mathbf{g}/\mathbf{h}),\mathbb{C})$ such that **h** acts trivially on $\mathbb{C}$. We define

$$\widetilde{C}=\sum_{s\geq 0}\widetilde{C^s}$$

where $\widetilde{C^s}=\mathrm{Hom}_{\mathbb{C}}(\Lambda^s(\mathbf{g}/\mathbf{h}),\mathbb{C})$. We put the topology of pointwise convergence on $\widetilde{C^s}$, i.e., $f_n\to f$ in $\widetilde{C^s}$ if and only if $f_n(x)\to f(x)$ in $\mathbb{C}$ with usual topology, for all $x\in\Lambda(\mathbf{g},\mathbf{h})$. From [36], we have

Theorem 3.23. *$\widetilde{C^s}$ is a complete, Hausdorff, topological vector space with respect to the pointwise topology.*

In [47], a continous map $\tilde{\partial}:\widetilde{C^s}\to\widetilde{C^{s-1}}$, and a cochain map of $\tilde{b}$ on $\widetilde{C}$ are defined. We define ∂, b to be the restrictions of $\tilde{\partial}$ and $\tilde{b}$ to the subspace $C(\mathbf{g},\mathbf{h})$. We define the following operators on $C(\mathbf{g},\mathbf{h})$: $S=d\partial+\partial d$ and $L=b\partial+\partial b$. From [47], we have

Proposition 3.19. $\ker S\oplus\operatorname{im}S=C$.

Theorem 3.24. *d and ∂ on $C(\mathbf{g}, \mathbf{h})$ are disjoint.*

Proposition 3.20 (Hodge type decomposition). *Let V be any vector space and $d, \partial : V \to V$ be two disjoint operators such that $d^2 = \partial^2 = 0$. Further, assume that* $\ker S \oplus \operatorname{im} S = V$ *where $S = d\partial + \partial d$. Then,* $\ker S \to \ker d / \operatorname{im} d$ *and* $\ker S \to \ker \partial / \operatorname{im} \partial$ *are both isomorphisms.*

By the Hodge type decomposition and Proposition 3.19, we have

Theorem 3.25. *The canonical maps $\psi_{d,S} : \ker S \to H(C, d)$ and $\psi_{\partial,S} : \ker S \to H(C, \partial)$ are both isomorphisms.*

Now, we describe a basis for $\ker L$. We fix $w \in W$ of length s. We define $\Phi_w = w\Delta_- \cap \Delta_+$. Φ_w consists of real roots $\{\gamma_1, \cdots, \gamma_s\}$. We pick $y_{\gamma_i} \in \mathbf{g}_{-\gamma_i}$ of unit norm with respect to the form $\{,\}$ and let $x_{\gamma_i} = -\omega_0(y_{\gamma_i})$. Let $M_{(w\rho-\rho)}$ be the irreducible $\mathbf{h}$-submodule with highest weight $(w\rho - \rho)$. By Proposition 2.5 in [37], the corresponding highest weight vector is $y_{\gamma_1} \wedge \cdots \wedge y_{\gamma_s}$. There exists a unique element $\overline{h^w} \in [M_{(w\rho-\rho)} \otimes \Lambda^s(\mathbf{n})]$ such that $\overline{h^w} = (2i)^s (y_{\gamma_1} \wedge \cdots \wedge y_{\gamma_s} \wedge x_{\gamma_1} \wedge \cdots \wedge x_{\gamma_s}) \mod P_w \otimes \Lambda^s(\mathbf{n})$, where P_w is the orthogonal complement of $y_{\gamma_1} \wedge \cdots \wedge y_{\gamma_s}$ in $M_{(w\rho-\rho)}$. Using the nondegenerate bilinear form $\langle\,\rangle$ on $\mathbf{g}$, we have embedding

$$e : \bigoplus_{k \geq 0} \Lambda^s(\mathbf{n} \oplus \mathbf{n}^-) \to \bigoplus_{k \geq 0} [\Lambda^s(\mathbf{n} \oplus \mathbf{n}^-)]^*.$$

Then $h_w = e(\overline{h_w}) \in \ker L$. These elements $\{h_w\}_{w \in W}$ is a $\mathbb{C}$-basis of $\ker L$. Then, we can define $s^w = {\psi_{\partial,S}}^{-1}([h^w]) \in H(C, \partial)$. From [47], [46], we have

Theorem 3.26. *Let $\mathbf{g}$ be the Kač-Moody Lie algebra and let G be the group associated to the Kač-Moody algebra $\mathbf{g}$ and B be standard Borel subgroup of G. Then*

$$\int_{C_{w'}} s^w = \begin{cases} 0 & \text{if } w \neq w', \\ (4\pi)^{2\ell(w)} \prod_{\nu \in w^{-1}\Delta \cap \Delta_+} \sigma(\rho, \nu)^{-1} & \text{if } w = w'. \end{cases}$$

This gives the expression for the d, ∂ harmonic forms $s_0^w = \dfrac{s^w}{d_w}$ which are dual to the Schubert cells where $d_w = \int_{C_w} s^w$.

Theorem 3.27. *(see [48])*

$$H(\int) : H^*(\mathbf{g}, \mathbf{h}) \to H^*(G/B, \mathbb{C})$$

is a graded algebra isomorphism.

Let ε^w denote the image of ${s_0}^w$ by the integral map in last theorem. These cohomology classes are dual to the closure of the Schubert cells, hence we have

Theorem 3.28. *The elements ε^w, $w \in W$, form a basis of the $\mathbb{Z}$-module $H^*(G/B, \mathbb{Z})$.*

For a finite type flag variety G/B, ε^w denotes $P_{w^{-1}}$ in the notation of Section 1. Now, we give a cup product formula in the cohomology of any type flag variety G/B. From [46],

Theorem 3.29. *Let χ_i be the fundamental weight of G for $1 \leq i \leq l$. For any simple reflection r_i and any element $w \in W$, and a coroot $\gamma^\vee$,*

$$\varepsilon^{r_i} \cdot \varepsilon^w = \sum_{w \xrightarrow{\gamma} w'} \chi_i(\gamma^\vee) \varepsilon^{w'}.$$

As an analogy of cohomology theory of the finite type flag space G/B, the cohomology of affine type flag space G/B and some operators will be introduced in this section. The fundamental reference is [42] of V. Kač.

Let $Q^\vee = \bigoplus_i \mathbb{Z} h_i$, where h_i is coroot, be the coroot lattice and let

$$P = \{\lambda \in \mathbf{h}'^* : \lambda(h_i) \in \mathbb{Z}\}$$

be the weight lattice dual to $Q^\vee$. Let $S(P) = \bigoplus_{j \geq 0} S^j(P)$ be the integral symmetric algebra over the lattice P, and $S(P)^+ = \bigoplus_{j > 0} S^j(P)$ the augmentation ideal. Given a commutative ring $\mathbb{F}$ with unit, we denote $S(P)_{\mathbb{F}} = S(P)_{\mathbb{F}} \otimes_{\mathbb{Z}} \mathbb{F}$. We define the *characteristic homomorphism* $\psi : S(P) \to H^*(G/B, \mathbb{Z})$ as follows: given $\lambda \in P$, we have the corresponding character of B and the associated line bundle L_λ on G/B. We put $\psi(\lambda) \in H^2(G/B, \mathbb{Z})$ equal to the Chern class of L_λ and we extend this multiplicativity to the whole $S(P)$. We denote by $\psi_{\mathbb{F}}$ the extension of ψ by linearity to $S(P)_{\mathbb{F}}$. In order to describe the properties of $\psi_{\mathbb{F}}$, we define BGG-operator Δ_i for $1 \leq i \leq l$ on $S(P)$ by

$$\Delta_i(f) = \frac{f - r_i(f)}{\alpha_i}$$

and we extend this by linearity to $S(P)_{\mathbb{F}}$. We define

$$I_{\mathbb{F}} = \{f \in S(P)^+_{\mathbb{F}} : \Delta_{i_1} \cdots \Delta_{i_n}(f) \in S(P)^+_{\mathbb{F}} \ \forall \text{ sequence } (i_1, \cdots .i_n)\}.$$

Theorem 3.30. *We have* $\ker \psi_{\mathbb{F}} = I_{\mathbb{F}}$ *and* $H^*(G/B, \mathbb{F})$ *is a free module over* $\operatorname{im} \psi_{\mathbb{F}}$.

We will introduce certain operators on cohomology of the flag space G/B which are basic tools in the study of this theory. These operators are extension of action of the BGG-operators Δ_i from the image of ψ to the whole cohomology operators. We know that the Weyl group W acts by right multiplication on K/T and this action induces an action of W on homology and cohomology of flag space. On the other hand, we have a fibration $p_i : K/T \to K/K_iT$ with fibre K_i/T_i. Since the odd degree cohomologies of K_i/T_i and K/K_iT are trivial, then the Leray-Serre spectral sequence of the fibration degenerates after the second term. So, $H^*(K/T, \mathbb{Z})$ is generated by $\operatorname{im} {p_i}^*$, which is r_i invariant and the element $\psi(\chi_i)$ where χ_i is fundamental weight. We define a $\mathbb{Z}$-linear operator A^i on $H^*(K/T, \mathbb{Z})$ lowering the degree by 2 such that r_i leaves the image of A^i invariant and

$$x - r_i(x) = A^i(x) \cup \psi(\alpha_i)$$

for $x \in H^*(K/T, \mathbb{Z})$. Similarly, we can define homology operators A_i on $H_*(K/T, \mathbb{Z})$

raising the degree by 2 such that $r_i(A_i(v)) = -A_i(v)$ and

$$v + r_i(v) = A_i(v) \cap \psi(\alpha_i)$$

for $v \in H_*(K/T, \mathbb{Z})$. The properties of the actions of the operators A^i (resp. A_i) on the cup product (resp. cap product) in the cohomology (resp. homology) can be found in [42]. Now, we will give the geometric interpretation of A_i. Given $w \in W$, we choose a reduced expression $w = r_{i_1} \cdots r_{i_s}$ and and define a map $\tau_w : D \to K/T$ given by $\tau_w(u_1, \cdots, u_s) = z_{i_1} \cdots z_{i_s} T$ where D is the unit disk in the complex space $\mathbb{C}^s$ and z_i has been defined in the previous section. By Proposition 3.16, the relative homology map τ_{w*} gives us an element $s_w \in H_{2\ell(w)}(K/T, \mathbb{Z})$. By Proposition 3.18, these elements are a basis of $H_*(K/T, \mathbb{Z})$; let ε^w be the dual basis of $H^*(K/T, \mathbb{Z})$.

Proposition 3.21.

$$r_i(\varepsilon^w) = \begin{cases} \varepsilon^w & \text{if } \ell(wr_i) > \ell(w), \\ \varepsilon^w - \sum\limits_{wr_i \xrightarrow{\gamma} w'} \langle \alpha_i, \gamma \rangle \varepsilon^{w'} & \text{otherwise.} \end{cases}$$

Similarly, we can give the reflection action on Schubert cycles s_w.

Proposition 3.22.

$$A^i(\varepsilon^w) = \begin{cases} \varepsilon^{wr_i} & \text{if } \ell(wr_i) < \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

$$A_i(s_w) = \begin{cases} s_{wr_i} & \text{if } \ell(wr_i) > \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.23.

$$c_1(L_\lambda) \cap s_w = \sum_{w' \xrightarrow{\gamma} w} \langle \lambda, \gamma \rangle s_{w'}$$

The set of all functions from the Weyl group W to $\mathbb{C}$ will be denoted by $\mathbb{C}\{W\}$. $\mathbb{C}\{W\}$ is an algebra under pointwise addition and multiplication. Now, we will give the relation between $\mathbb{C}\{W\}$ and $\operatorname{End}_{\mathbf{h}} H^*(\mathbf{n}^-, \mathbb{C})$. From [46], we have

Theorem 3.31. *Let $(\mathcal{A}, d)$ be a differential graded algebra over $\mathbb{C}$ and let δ be the derivation in $\operatorname{End}(\mathcal{A}, d)$ induced by d such that*

$$\delta\xi = d\xi - (-1)^i \xi d \, for \; \xi \in \operatorname{End}^i(\mathcal{A}).$$

Then $\iota : H(\operatorname{End}(\mathcal{A}, d), \delta) \to \operatorname{End} H(\mathcal{A}, d)$ is an isomorphism graded algebras.

Theorem 3.32. *The standard cochain complexes $C(\mathbf{g}, \mathbf{h})$ and $C(\mathbf{n}^-)$ with the topology of pointwise convergence are both differential graded algebras over $\mathbb{C}$.*

Also, we can put the topology of pointwise convergence on $\operatorname{End} C(\mathbf{g}, \mathbf{h})$ and $\operatorname{End} C(\mathbf{n}^-)$. Then, the derivation map $\delta : \operatorname{End} C(\mathbf{n}^-) \to \operatorname{End} C(\mathbf{n}^-)$ is continuous under this topology and it commutes with the action of $\mathbf{h}$ on $\operatorname{End} C(\mathbf{n}^-)$. We denote by δ_0, the restriction of δ to $\operatorname{End}_{\mathbf{h}} C(\mathbf{n}^-)$. From [46], we have

Proposition 3.24. *There exists a unique injective continuous map $\eta : C(\mathbf{g}, \mathbf{h}) \to \operatorname{End}_{\mathbf{h}} C(\mathbf{n}^-)$.*

Lemma 3.6. *We have $\eta(\ker S) \subseteq \ker \delta_0$.*

The map η induces a map $\tilde{\eta} : \ker S \to H(\operatorname{End}_{\mathbf{h}} C(\mathbf{n}^-), \delta_0)$ Also, ι induces a map $\iota_0 : H(\operatorname{End}_{\mathbf{h}} C(\mathbf{n}^-), \delta_0) \to \operatorname{End}_{\mathbf{h}} H^*(\mathbf{n}^-, \mathbb{C})$. By Theorem 3.22, as an $\mathbf{h}$-module, $H^{2j}(\mathbf{n}^-, \mathbb{C})$ is isomorphic to the direct sum

$$\bigoplus_{\ell(w)=j} M_{(w\rho-\rho)}$$

of non-isomorphic irreducible $\mathbf{h}$-submodules. By a property of the Hom functor, we have

$$\operatorname{End}_{\mathbf{h}} H^*(\mathbf{n}^-, \mathbb{C}) \cong \prod_{i \geq 0} \operatorname{End}_{\mathbf{h}} H^i(\mathbf{n}^-, \mathbb{C}) \cong \prod_{i \geq 0} \prod_{\ell(w)=i} \operatorname{End}_{\mathbf{h}} M_{(w\rho-\rho)}.$$

Since $M_{(w\rho-\rho)}$ is irreducible, $\operatorname{End}_{\mathbf{h}} M_{(w\rho-\rho)}$ is 1-dimensional with a canonical generator 1_w which is the identity map of $M_{(w\rho-\rho)}$. This identifies $\operatorname{End}_{\mathbf{h}} H^*(\mathbf{n}^-, \mathbb{C})$ with $\prod_{w \in W} \mathbb{C} 1_w$.

The space $\prod_{w\in W}\mathbb{C}1_w$ is the vector space $\mathbb{C}\{W\}$ of all functions from W to $\mathbb{C}$. Let $\overline{\eta}$ be the composite map

$$\ker S \xrightarrow{\tilde{\eta}} H(\mathrm{End}_{\mathbf{h}}\, C(\mathbf{n}^-), \delta_0) \xrightarrow{\iota_0} \mathrm{End}_{\mathbf{h}}\, H^*(\mathbf{n}^-, \mathbb{C}) \cong \mathbb{C}\{W\}.$$

Now, we will give filtrations of $C(\mathbf{g},\mathbf{h})$ and $\mathbb{C}\{W\}$. We define a decrasing filtration $\mathcal{G} = (\mathcal{G}_p)_{p\in\mathbb{Z}^-}$ by $\mathcal{G}_p = \sum_{0\geq k+q\geq p} C^{q,k}(\mathbf{g},\mathbf{h})$ where $C^{q,k}(\mathbf{g},\mathbf{h}) = \mathrm{Hom}_{\mathbf{h}}(\Lambda^q(\mathbf{n})\otimes\Lambda^k(\mathbf{n}^-))$. This gives rise to a filtration $\mathcal{F} = (\mathcal{F}_p)_{p\in\mathbb{Z}^-}$ of $\mathrm{End}_{\mathbf{h}}\, C(\mathbf{n})$ by defining $\mathcal{F}_p = \eta(\mathcal{G}_p)$. By ι_0, we have filtration $\mathcal{H} = (\mathcal{H}_p)_{p\in\mathbb{Z}^+}$ of $\mathbb{C}\{W\}$. $\mathrm{Gr}\,\mathbb{C}\{W\}$ will denote the associated graded algebra with respect to the filtration of $\mathbb{C}\{W\}$. That is, $\mathrm{Gr}\,\mathbb{C}\{W\} = \sum_{p\geq 0}\mathrm{Gr}^p$, where $\mathrm{Gr}^p = \mathcal{H}_p/\mathcal{H}_{p+1}$. From [46], we have

Theorem 3.33. *Let* $\mathbf{g}$ *be a symmetrizable Kač-Moody Lie algebra. Let* $\mathbf{h}$ *be the Cartan subalgebra. Then,* $H^*(\mathbf{g},\mathbf{h}) \to \mathrm{Gr}\,\mathbb{C}\{W\}$ *is a graded algebra isomorphism.*

By Theorem 3.27, we can give the following corollary.

Corollary 3.3. $H^*(G/B,\mathbb{C}) \to \mathrm{Gr}\,\mathbb{C}\{W\}$ *is a graded algebra isomorphism.*

Let $\mathbf{g}$ be an arbitrary Kač-Moody algebra associated to a generalized Cartan matrix A, with its Cartan subalgebra $\mathbf{h}$ and Weyl group W. Let $Q = Q(\mathbf{h}^*)$ be the field of the rational functions on $\mathbf{h}$. The Weyl group W acts as a group of automorphisms on the field Q. Let Q_W be the smash product of Q with the group algebra $\mathbb{C}[W]$, i.e., Q_W is a right Q-module with a basis $\{\delta_w\}_{w\in W}$ and the multiplication is given by

$$(\delta_v q_v)\cdot(\delta_w q_w) = \delta_{vw}(w^{-1}q_v)q_w$$

for $v, w \in W$ and $q_v, q_w \in Q$. The module Q_W admits an involutary anti-automorphism t, defined by $(\delta_w q)^t = \delta_{w^{-1}}(wq)$ for $w \in W$ and $q \in Q$. We define

$$x_i = -(\delta_{r_i} + \delta_e)\frac{1}{\alpha_i} = \frac{1}{\alpha_i}(\delta_{r_i} - \delta_e) \in Q_W$$

where $r_i \in W$ is a simple reflection and α_i is the simple root.

Proposition 3.25. *Let $w \in W$ and let $w = r_{i_1} \cdots r_{i_n}$ be a reduced expression. Then the element $x_{i_1} \cdots x_{i_n} \in Q_W$ does not depend upon the choice of reduced expression of w.*

The element $x_{i_1} \cdots x_{i_n} \in Q_W$ will be denoted by x_w and $(x_{w^{-1}})^t$ denoted by $\overline{x}_w$.

Proposition 3.26.

$$x_v \cdot x_w = \begin{cases} x_{vw} & \text{if } \ell(vw) = \ell(v) + \ell(w), \\ 0 & \text{otherwise}. \end{cases}$$

We know that Q_W is a right Q-module. Also, Q has a left Q_W-module structure defined by $(\delta_w q)q' = w(qq')$ for $w \in W$ and q, q'. We define subring $\mathcal{R} \subseteq Q_W$ given by

$$\mathcal{R} = \{x \in Q_W : x \cdot S \subseteq S\}$$

where $S = S(\mathbf{h}^*)$ is the polynomial algebra on $\mathbf{h}$. Let S_W be the smash product of S with the group algebra $\mathbb{C}[W]$. Obviously $S_W \subseteq \mathcal{R}$ since S has left S_W-module structure.

Theorem 3.34. *$\mathcal{R}$ is a free right S-module with basis $\{x_w\}_{w \in W}$ In particular, any $x \in \mathcal{R}$ can be uniquely written as $x = \sum_{w \in W} x_w p_w$ some $p_w \in S$.*

R will be referred as a *nil-Hecke ring*. Now, we will give the coproduct structure on Q_W. Let $Q_W \otimes_Q Q_W$ be the tensor product, considering both the copies of Q_W as right Q-modules. We define the diagonal map $\Delta : Q_W \to Q_W \otimes_Q Q_W$ by

$$\Delta(\delta_w q) = \delta_w q \otimes \delta_w = \delta_w \otimes \delta_w q$$

for $w \in W$ and $q \in Q$. Δ is right Q-linear.

Theorem 3.35. *For any $w \in W$, we have*

$$\Delta(\overline{x}_w) = \sum_{u,v \leqslant w} \overline{x}_u \otimes \overline{x}_v p^w_{u,v}$$

for some homogeneous polynomials $p^w_{u,v} \in S$ of degree $\ell(u) + \ell(v) - \ell(w)$. In particular, $p^w_{u,v} = 0$ unless $\ell(u) + \ell(v) \geq \ell(w)$.

Now, we will introduce some dual objects. Let $\Xi = \mathrm{Hom}_Q(Q_W, Q)$. Since any $\xi \in \Xi$ is determined by its restriction to the Q-basis $\{\delta_w\}_{w \in W}$, we can regard Ξ as the Q-module of all the functions $W \to Q$ with pointwise addition and scalar multiplication defined by the structure $(q\xi)w = q \cdot \xi(w)$ for $q \in Q$, $\xi \in \Xi$ and $w \in W$. Ξ has a commutative Q-algebra structure with the product as pointwise multiplication of functions on W. Also, Ξ has a left Q_W module structure defined by $(x \cdot \xi)y = \xi(x^t \cdot y)$ for $x, y \in Q_W$ and $\xi \in \Xi$. We have the Weyl group action as well as the *Hecke-type* operators A_w on Ξ defined by $w\xi = \delta_w \cdot \xi$ and $A_w\xi = x_w \cdot \xi$ for $w \in W$ and $\xi \in \Xi$. We define the important subring $\Lambda \subseteq \Xi$ as follows:

$$\Lambda = \{\xi \in \Xi : \xi(\mathcal{R}^t) \subseteq S \text{ and } \xi(\overline{x}_w) = 0 \text{ for all but a finite number of } w \in W\}$$

Proposition 3.27. *λ is a S-subalgebra of Ξ. $\{\xi^w\}_{w \in W}$ is a S-basis of Λ where ξ^w is dual to $\overline{x}_w$ for $w \in W$.*

Proposition 3.28. *$A_i\xi^w = \xi^{r_i w}$ if $r_i w < w$, 0 otherwise.*

Proposition 3.29. *$\xi^{r_i}(w) = \chi_i - w^{-1}\chi_i$ where χ_i is the fundamental weight dual to the coroot h_i corresponding to simple root α_i.*

Now, we will give the important formula equivalent to the cup product formula in the cohomology of G/B where G is a Kač-Moody group.

Proposition 3.30.

$$\xi^u \cdot \xi^v = \sum_{u,v \leqslant w} p^w_{u,v} \xi^w,$$

where $p^w_{u,v}$ is a homogeneous polynomial of degree $\ell(u) + \ell(v) - \ell(w)$.

Proposition 3.31.

$$r_i\xi^w = \begin{cases} \xi^w & \text{if } r_i w > w, \\ -(w^{-1}\alpha_i)\xi^{r_i w} + \xi^w - \sum\limits_{r_i w \xrightarrow{\gamma} w'} \alpha_i(\gamma^\vee)\xi^{w'} & \text{otherwise.} \end{cases}$$

Theorem 3.36. *Let $u, v \in W$. We write $w^{-1} = r_{i_1} \cdots r_{i_n}$ as a reduced expression.*

$$p^w_{u,v} = \sum_{\substack{j_1<\cdots<j_m \\ r_{j_1}\cdots r_{j_m}=v^{-1}}} A_{i_1} \circ \cdots \circ \hat{A}_{i_{j_1}} \circ \cdots \circ \hat{A}_{i_{j_m}} \circ \cdots \circ A_{i_n}(\xi^u)(e),$$

where $m = \ell(v)$ and the notation $\hat{A}_i$ means that the operator A_i is replaced by the Weyl group action r_i.

Let $\mathbb{C}_0 = S/S^+$ be the S-module where S^+ is the augmentation ideal of S. It is 1-dimensional as $\mathbb{C}$-vector space. Since Λ is a S-module, we can define $\mathbb{C}_0 \otimes_S \Lambda$. It is an algebra and the action of $\mathcal{R}$ on Λ gives an action of $\mathcal{R}$ on $\mathbb{C}_0 \otimes_S \Lambda$. The elements $\sigma^w = 1 \otimes \xi^w \in \mathbb{C}_0 \otimes_S \Lambda$ is a $\mathbb{C}$-basis form of $\mathbb{C}_0 \otimes_S \Lambda$.

Proposition 3.32. *$\mathbb{C}_0 \otimes_S \Lambda$ is a graded algebra associated with the filtration of length of the element of the Weyl group W.*

Proposition 3.33. *The complex linear map $f : \mathbb{C}_0 \otimes_S \Lambda \to \mathrm{Gr}\,\mathbb{C}\{W\}$ is a graded algebra homomorphism.*

Theorem 3.37. *Let K be the standard real form of the group G associated to a symmetrizable Kač-Moody Lie algebra* **g** *and let T denote the maximal torus of K. Then the map*

$$\theta : H^*(K/T, \mathbb{C}) \to \mathbb{C}_0 \otimes_S \Lambda$$

defined by $\theta(\varepsilon^w) = \sigma^w$ for any $w \in W$ is a graded algebra isomorphism. Moreover, the action of $w \in W$ and A^w on $H^*(K/T, \mathbb{C})$ corresponds respectively to that δ_w and $x_w \in \mathcal{R}$ on $\mathbb{C}_0 \otimes_S \Lambda$.

Corollary 3.4. *The operators A^i on $H^*(K/T, \mathbb{C})$ generate the nil-Hecke algebra.*

Corollary 3.5. *We can use Proposition 3.30 and Theorem 3.36 to determine the cup product $\varepsilon^u\varepsilon^v$ in terms of the Schubert basis $\{\varepsilon^w\}_{w\in W}$ of $H^*(K/T, \mathbb{Z})$.*

CHAPTER 4

THE REIDEMEISTER TORSION OF COMPACT FLAG MANIFOLDS K/T FOR $K = A_l$

This chapter is completely original and it includes our calculations about Reidemeister torsion of flag manifolds using Theorem 1.1 and Proposition 3.30.

We know that the Weyl group W of K acts on the Lie algebra of the maximal torus T. It is a finite group of isometries of the Lie algebra $\mathbf{t}$ of the maximal torus T. It preserves the coweight lattice T^{v}. For each simple root α, the Weyl group W contains an element r_α of order two represented by $e^{((\pi/2)(e_\alpha+e_{-\alpha}))}$ in $N(T)$. Since the roots α can be considered as the linear functionals on the Lie algebra $\mathbf{t}$ of the maximal torus T, the action of r_α on $\mathbf{t}$ is given by

$$r_\alpha(\xi) = \xi - \alpha(\xi)h_\alpha \quad \text{for} \quad \xi \in \mathbf{t},$$

where h_α is the coroot in $\mathbf{t}$ corresponding to simple root α. Also, we can give the action of r_α on the roots by

$$r_\alpha(\beta) = \beta - \alpha(h_\beta)\alpha \quad \text{for} \quad \alpha, \beta \in \mathbf{t}^*,$$

where $\mathbf{t}^*$ is the dual vector space of $\mathbf{t}$. The element r_α is the reflection in the hyperplane H_α of $\mathbf{t}$ whose equation is $\alpha(\xi) = 0$. These reflections r_α generate the Weyl group W.

Set $\alpha_1, \alpha_2, \ldots, \alpha_n$ be roots of Weyl Group of SU_{n+1}. Since the Cartan Matrix of Weyl Group of SU_{n+1} is

$$M_{ij} = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases},$$

$$r_{\alpha_i}(\alpha_j) = \begin{cases} -\alpha_i, & i = j \\ \alpha_i + \alpha_j, & |i-j| = 1 \\ \alpha_j, & \text{otherwise.} \end{cases}$$

Proposition 4.1. *The Weyl group W of SU_{n+1} is isomorphic to Coxeter Group A_n given by generators $s_1, s_2, \ldots, s_n$ and relations*

(i) $s_i^2 = 1 \qquad i = 1, 2, \ldots, n$

(ii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad i = 1, 2, \ldots, n-1$

(iii) $s_i s_j = s_j s_i \qquad 1 \leq i < j - 1 < n$

Proof. **(i)**

$$\begin{aligned} r_{\alpha_i} \circ r_{\alpha_i}(\beta) &= r_{\alpha_i}(\beta - <\alpha_i, \beta> \alpha_i) \\ &= \beta - <\alpha_i, \beta> \alpha_i - <\beta - <\alpha_i, \beta> \alpha_i, \alpha_i> \alpha_i \\ &= \beta - <\alpha_i, \beta> \alpha_i - <\beta, \alpha_i> \alpha_i + <\alpha_i, \beta><\alpha_i, \alpha_i> \alpha_i \\ &= \beta - <\alpha_i, \beta> \alpha_i - <\alpha_i, \beta> \alpha_i + 2 <\alpha_i, \beta> \alpha_i \\ &= \beta \end{aligned}$$

(ii)

$$
\begin{aligned}
r_{\alpha_i} \circ r_{\alpha_{i+1}} \circ r_{\alpha_i}(\beta) &= r_{\alpha_i} \circ r_{\alpha_{i+1}}(\beta - < \alpha_i, \beta > \alpha_i) \\
&= r_{\alpha_i}(\beta - < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta - < \alpha_i, \beta > \alpha_i > \alpha_{i+1}) \\
&= r_{\alpha_i}(\beta - < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_{i+1} \\
&\quad + < \alpha_{i+1}, < \alpha_i, \beta > \alpha_i > \alpha_{i+1}) \\
&= r_{\alpha_i}(\beta - < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_{i+1} \\
&\quad + < \alpha_i, \beta >< \alpha_{i+1}, \alpha_i > \alpha_{i+1}) \\
&= r_{\alpha_i}(\beta - < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_{i+1} - < \alpha_i, \beta > \alpha_{i+1}) \\
&= \beta - < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_{i+1} - < \alpha_i, \beta > \alpha_{i+1} \\
&\quad - < \alpha_i, \beta - < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_{i+1} \\
&\quad - < \alpha_i, \beta > \alpha_{i+1} > \alpha_i \\
&= \beta - < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_{i+1} - < \alpha_i, \beta > \alpha_{i+1} \\
&\quad - < \alpha_i, \beta > \alpha_i + < \alpha_i, \beta >< \alpha_i, \alpha_i > \alpha_i \\
&\quad + < \alpha_{i+1}, \beta >< \alpha_{i+1}, \alpha_i > \alpha_i + < \alpha_i, \beta >< \alpha_{i+1}, \alpha_i > \alpha_i \\
&= \beta - < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_{i+1} - < \alpha_i, \beta > \alpha_{i+1} \\
&\quad - < \alpha_i, \beta > \alpha_i + 2 < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_i \\
&\quad - < \alpha_i, \beta > \alpha_i \\
&= \beta - < \alpha_i, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_i - < \alpha_{i+1}, \beta > \alpha_{i+1} \\
&\quad - < \alpha_i, \beta > \alpha_{i+1} \\
&= \beta - (< \alpha_i, \beta > + < \alpha_{i+1}, \beta >)(\alpha_i + \alpha_{i+1}).
\end{aligned}
$$

$$
\begin{aligned}
r_{\alpha_{i+1}} \circ r_{\alpha_i} \circ r_{\alpha_{i+1}}(\beta) &= r_{\alpha_{i+1}} \circ r_{\alpha_i}(\beta - <\alpha_{i+1},\beta> \alpha_{i+1}) \\
&= r_{\alpha_{i+1}}(\beta - <\alpha_{i+1},\beta> \alpha_{i+1} - <\alpha_i, \beta - <\alpha_{i+1},\beta> \alpha_{i+1}> \alpha_i) \\
&= r_{\alpha_{i+1}}(\beta - <\alpha_{i+1},\beta> \alpha_{i+1} - <\alpha_i,\beta> \alpha_i \\
&\quad + <\alpha_{i+1},\beta><\alpha_i, \alpha_{i+1}> \alpha_i) \\
&= r_{\alpha_{i+1}}(\beta - <\alpha_{i+1},\beta> \alpha_{i+1} - <\alpha_i,\beta> \alpha_i - <\alpha_{i+1},\beta> \alpha_i) \\
&= \beta - <\alpha_{i+1},\beta> \alpha_{i+1} - <\alpha_i,\beta> \alpha_i - <\alpha_{i+1},\beta> \alpha_i \\
&\quad - <\alpha_{i+1}, \beta - <\alpha_{i+1},\beta> \alpha_{i+1} - <\alpha_i,\beta> \alpha_i \\
&\quad - <\alpha_{i+1},\beta> \alpha_i > \alpha_{i+1} \\
&= \beta - <\alpha_{i+1},\beta> \alpha_{i+1} - <\alpha_{i+1},\beta> \alpha_i - <\alpha_i,\beta> \alpha_i \\
&\quad - <\alpha_{i+1},\beta> \alpha_{i+1} + <\alpha_i,\beta><\alpha_{i+1}, \alpha_i> \alpha_{i+1} \\
&\quad + <\alpha_{i+1},\beta><\alpha_{i+1}, \alpha_i> \alpha_{i+1} \\
&\quad + <\alpha_{i+1},\beta><\alpha_{i+1}, \alpha_{i+1}> \alpha_{i+1} \\
&= \beta - <\alpha_{i+1},\beta> \alpha_{i+1} - <\alpha_i,\beta> \alpha_i - <\alpha_{i+1},\beta> \alpha_i \\
&\quad - <\alpha_{i+1},\beta> \alpha_{i+1} + 2 <\alpha_{i+1},\beta> \alpha_{i+1} - <\alpha_i,\beta> \alpha_{i+1} \\
&\quad - <\alpha_{i+1},\beta> \alpha_{i+1} \\
&= \beta - <\alpha_{i+1},\beta> \alpha_{i+1} - <\alpha_i,\beta> \alpha_i - <\alpha_{i+1},\beta> \alpha_i \\
&\quad - <\alpha_i,\beta> \alpha_{i+1} \\
&= \beta - (<\alpha_{i+1},\beta> + <\alpha_i,\beta>)(\alpha_{i+1} + \alpha_i).
\end{aligned}
$$

Hence $r_{\alpha_{i+1}} \circ r_{\alpha_i} \circ r_{\alpha_{i+1}}(\beta) = r_{\alpha_{i+1}} \circ r_{\alpha_i} \circ r_{\alpha_{i+1}}(\beta)$.

(iii)

$$\begin{aligned} r_{\alpha_i} \circ r_{\alpha_j}(\beta) &= r_{\alpha_i} \circ (\beta - < \alpha_j, \beta > \alpha_j) \\ &= \beta - < \alpha_j, \beta > \alpha_j - < \alpha_i, \beta - < \alpha_j, \beta > \alpha_j > \alpha_i \\ &= \beta - < \alpha_j, \beta > \alpha_j - < \alpha_i, \beta > \alpha_i + < \alpha_j, \beta >< \alpha_i, \alpha_j > \alpha_i \\ &= \beta - < \alpha_j, \beta > \alpha_j - < \alpha_i, \beta > \alpha_i. \end{aligned}$$

$$\begin{aligned} r_{\alpha_j} \circ r_{\alpha_i}(\beta) &= r_{\alpha_j} \circ (\beta - < \alpha_i, \beta > \alpha_i) \\ &= \beta - < \alpha_i, \beta > \alpha_i - < \alpha_j, \beta - < \alpha_i, \beta > \alpha_i > \alpha_j \\ &= \beta - < \alpha_i, \beta > \alpha_i - < \alpha_j, \beta > \alpha_j + < \alpha_i, \beta >< \alpha_j, \alpha_i > \alpha_j \\ &= \beta - < \alpha_i, \beta > \alpha_i - < \alpha_j, \beta > \alpha_j. \end{aligned}$$

Hence $r_{\alpha_i} \circ r_{\alpha_j}(\beta) = r_{\alpha_j} \circ r_{\alpha_i}(\beta)$.

□

After this point s_i will represent r_{α_i}.

Let us define the word

$$s_{ij} = \begin{cases} s_i s_{i+1} \cdots s_j & i < j \\ s_i & i = j \\ 1 & i > j \end{cases}$$

Theorem 4.1. *(Theorem 3.1 of [52])The reduced Gröbner-Shirshov basis of the Coxeter group A_n consist of relation*

$$s_{ij} s_i = s_{i+j} s_{ij} \quad 1 \leq i < j \leq n$$

together with defining relations of A_n.

The following lemma is equivalent of Lemma 3.2 in [52]. The only difference is that order of generators $s_1 > s_2 > \ldots > s_n$ in our setting.

Lemma 4.1. *Using elimination of leading words of relations, the reduced elements of A_n are in the form*

$s_{n+1,j_{n+1}} s_{n,j_n} s_{n-1,j_{n-1}} \cdots s_{i,j_i} \cdots s_{1,j_1} \qquad 1 \le i \le j_i + 1 \le n+1.$

Notice that $j_{n+1} + 1 = n + 1 \Longrightarrow j_{n+1} = n$ *and* $s_{n+1,n} = 1$.

Algorithm 4.1. (Finding Inverse) Let $w = s_{n,j_n} s_{n-1,j_{n-1}} \cdots s_{1,j_1}$. The inverse of w can be found using following algorithm.

$Invw = \{\}$;

$Conw = Reverse(w)$;

For $k = 1$ to $k = n$

Find maximum sequence in $Conw$;

$list = \{s_k, s_{k+1}, s_{k+2}, \ldots, s_{kj}\}$;

$Invw = list \cup Invw$;

End For.

Example 4.1. Let $s_{46}s_{35}s_{25}s_{13}$. The inverse of its $S_3 s_2 s_1 s_5 s_4 s_3 s_2 s_5 s_4 s_3 s_6 s_5 s_4$

$Invw = s_{14}$

$S_3 s_2 s_5 s_4 s_3 s_5 s_4 s_6 s_5$

$Invw = s_{25}s_{14}$

$S_3 s_5 s_4 s_5 s_6$

$Invw = s_{35}s_{25}s_{14}$

$s_5 s_6$

$Invw = s_{56}s_{35}s_{25}s_{14}$

Lemma 4.2. *Let* $w = (s_{n,j_n})(s_{n-1,j_{n-1}}) \cdots (s_{i+1,\, j_{i+1}})(s_{i,j_i}) \cdots (s_{1,J_1})$ *and*

$s_i w = (s_{n,\overline{j_n}})(s_{n-1,\, \overline{j_{n-1}}}) \cdots (s_{i+1,\, \overline{j_{i+1}}})(s_{i,\overline{j_i}}) \cdots (s_{1,\overline{j_1}})$ *where*

$$s_i w = \begin{cases} \overline{j_{i+1}} = j_i + 1,\ \overline{j_i} = j_{i+1} & if \quad j_i < j_{i+1} \\ \overline{j_{i+1}} = j_i,\ \overline{j_i} = j_{i+1} - 1 & if \quad j_i \geq j_{i+1} \\ \overline{j_k} = j_k & if \quad k \neq i, i+1 \end{cases}$$

Here if $i = n$, then we assume $j_{n+1} = n$.

Corollary 4.1. *Let $w = (s_{n,j_n})(s_{n-1,j_{n-1}}) \cdots (s_{i+1,j_{i+1}})(s_{i,j_i}) \cdots (s_{1,J_1})$ and*

$$s_{i-1}(s_i w) = (s_{n,\hat{j}_n})(s_{n-1,\widehat{j_{n-1}}}) \cdots (s_{i+1,\widehat{j_{i+1}}})(s_{i,\widehat{j_i}}) \cdots (s_{1,\widehat{J_1}})$$

where

$$s_{i-1}(s_i w) = \begin{cases} \widehat{j_{i+1}} = j_i + 1,\ \widehat{j_i} = j_{i-1} + 1,\ \widehat{j_{i-1}} = j_{i+1} & if \quad j_i < j_{i+1},\ j_{i-1} < j_{i+1} \\ \widehat{j_{i+1}} = j_i + 1,\ \widehat{j_i} = j_{i-1},\ \widehat{j_{i-1}} = j_{i+1} - 1 & if \quad j_i < j_{i+1},\ j_{i-1} \geq j_{i+1} \\ \widehat{j_{i+1}} = j_i,\ \widehat{j_i} = j_{i-1} + 1,\ \widehat{j_{i-1}} = j_{i+1} - 1 & if \quad j_i \geq j_{i+1},\ j_{i-1} < j_{i+1} - 1 \\ \widehat{j_{i+1}} = j_i,\ \widehat{j_i} = j_{i-1},\ \widehat{j_{i-1}} = j_{i+1} - 2 & if \quad j_i \geq j_{i+1},\ j_{i-1} \geq j_{i+1} - 1 \\ \widehat{j_k} = j_k & if \quad k \neq i-1,\ i,\ i+1 \end{cases}$$

Proof. Let $\overline{w} = s_i w = (s_{n,\overline{j_n}})(s_{n-1,\overline{j_{n-1}}}) \cdots (s_{i+1,\overline{j_{i+1}}})(s_{i,\overline{j_i}}) \cdots (s_{1,\overline{j_1}})$. Therefore

$$s_{i-1}(\overline{w}) = \begin{cases} \widehat{j_i} = \overline{j_{i-1}} + 1,\ \widehat{j_{i-1}} = \overline{j_i} & if \quad \overline{j_{i-1}} < \overline{j_i} \\ \widehat{j_i} = \overline{j_{i-1}},\ \widehat{j_{i-1}} = \overline{j_i} - 1 & if \quad \overline{j_{i-1}} \geq \overline{j_i} \\ \widehat{j_k} = \overline{j_k} & if \quad k \neq i-1,\ i \end{cases}$$

(i) $j_i < j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i + 1$, $\overline{j_i} = j_{i+1}$ So $\overline{j_{i-1}} < \overline{j_i} \Rightarrow j_{i-1} < j_{i+1}$, $\widehat{j_{i+1}} = \overline{j_{i+1}} = j_i + 1$, $\widehat{j_i} = \overline{j_{i-1}} + 1 = j_{i-1} + 1$, $\widehat{j_{i-1}} = \overline{j_i} = j_{i+1}$

(ii) $j_i < j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i + 1$, $\overline{j_i} = j_{i+1}$ So $\overline{j_{i-1}} \geq \overline{j_i} \Rightarrow j_{i-1} \geq j_{i+1}$, $\widehat{j_{i+1}} = \overline{j_{i+1}} = j_i + 1$, $\widehat{j_i} = \overline{j_{i-1}} = j_{i-1}$, $\widehat{j_{i-1}} = \overline{j_i} - 1 = j_{i+1} - 1$

(iii) $j_i \geq j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i$, $\overline{j_i} = j_{i+1} - 1$ So $\overline{j_{i-1}} < \overline{j_i} \Rightarrow j_{i-1} < j_{i+1}$, $\widehat{j_{i+1}} = \overline{j_{i+1}} = j_i + 1$, $\widehat{j_i} = \overline{j_{i-1}} = j_{i-1}$, $\widehat{j_{i-1}} = \overline{j_i} - 1 = j_{i+1} - 1$

(iv) $j_i \geq j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i$, $\overline{j_i} = j_{i+1} - 1$ So $\overline{j_{i-1}} \geq \overline{j_i} \Rightarrow j_{i-1} \geq j_{i+1} - 1$, $\widehat{j_{i+1}} = \overline{j_{i+1}} = j_i$, $\widehat{j_i} = \overline{j_{i-1}} = j_{i-1}$, $\widehat{j_{i-1}} = \overline{j_i} - 1 = j_{i+1} - 2$

□

Corollary 4.2. *Let* $w = (s_{n,j_n})(s_{n-1,j_{n-1}}) \cdots (s_{i+1,j_{i+1}})(s_{i,j_i}) \cdots (s_{1,J_1})$ *and*

$$s_{i+1}(s_i w) = (s_{n,\hat{j}_n})(s_{n-1,\widehat{j_{n-1}}}) \cdots (s_{i+1,\widehat{j_{i+1}}})(s_{i,\widehat{j_i}}) \cdots (s_{1,\widehat{j_1}}).$$

Then

$$s_{i+1}(s_i w) = \begin{cases} \widehat{j_{i+2}} = j_i + 2,\ \widehat{j_{i+1}} = j_{i+2},\ \widehat{j_i} = j_{i+1} & if \quad j_i < j_{i+1},\ j_{i+1} < j_{i+2} \\ \widehat{j_{i+2}} = j_i + 1,\ \widehat{j_{i+1}} = j_{i+2} - 1,\ \widehat{j_i} = j_{i+1} & if \quad j_i < j_{i+1},\ j_i + 1 \geq j_{i+2} \\ \widehat{j_{i+2}} = j_i + 1,\ \widehat{j_{i+1}} = j_{i+2},\ \widehat{j_i} = j_{i+1} - 1 & if \quad j_i \geq j_{i+1},\ j_i < j_{i+2} \\ \widehat{j_{i+2}} = j_i,\ \widehat{j_{i+1}} = j_{i+2} - 1,\ \widehat{j_i} = j_{i+1} - 1 & if \quad j_i \geq j_{i+1},\ j_i \geq j_{i+2} \\ \widehat{j_k} = j_k & if \quad k \neq i,\ i+1,\ i+2 \end{cases}$$

Proof. Let $\overline{w} = s_i w = (s_{n,\overline{j_n}})(s_{n-1,\overline{j_{n-1}}}) \cdots (s_{i+1,\overline{j_{i+1}}})(s_{i,\overline{j_i}}) \cdots (s_{1,\overline{j_1}})$. Therefore

$$s_{i+1}(\overline{w}) = \begin{cases} \widehat{j_{i+2}} = \overline{j_{i+1}} + 1,\ \widehat{j_{i+1}} = \overline{j_{i+2}} & if \quad \overline{j_{i+1}} < \overline{j_{i+2}} \\ \widehat{j_{i+2}} = \overline{j_{i+1}},\ \widehat{j_{i+1}} = \overline{j_{i+2}} - 1 & if \quad \overline{j_{i+1}} \geq \overline{j_{i+2}} \\ \widehat{j_k} = \overline{j_k} & if \quad k \neq i+1,\ i+2 \end{cases}$$

(i) $j_i < j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i + 1$, $\overline{j_i} = j_{i+1}$ So $\overline{j_{i+1}} < \overline{j_{i+2}} \Rightarrow j_i + 1 < j_{i+2}$, $\widehat{\overline{j_{i+2}}} = \overline{j_{i+1}} + 1 = j_i + 2$, $\widehat{\overline{j_{i+1}}} = \overline{j_{i+2}} = j_{i+2}$, $\widehat{\overline{j_i}} = \overline{j_i} = j_{i+1}$

(ii) $j_i < j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i + 1$, $\overline{j_i} = j_{i+1}$ So $\overline{j_{i+1}} \geq \overline{j_{i+2}} \Rightarrow j_i + 1 \geq j_{i+2}$, $\widehat{\overline{j_{i+2}}} = \overline{j_{i+1}} = j_i + 1$, $\widehat{\overline{j_{i+1}}} = \overline{j_{i+2}} - 1 = j_{i+2} - 1$, $\widehat{\overline{j_i}} = \overline{j_i} = j_{i+1}$

(iii) $j_i \geq j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i$, $\overline{j_i} = j_{i+1} - 1$ So $\overline{j_{i+1}} < \overline{j_{i+2}} \Rightarrow j_i < j_{i+2}$, $\widehat{\overline{j_{i+2}}} = \overline{j_{i+1}} + 1 = j_i + 1$, $\widehat{\overline{j_{i+1}}} = \overline{j_{i+2}} = j_{i+2}$, $\widehat{\overline{j_i}} = \overline{j_i} - 1 = j_{i+1} - 1$

(iv) $j_i \geq j_{i+1} \Rightarrow \overline{j_{i+1}} = j_i$, $\overline{j_i} = j_{i+1} - 1$ So $\overline{j_{i+1}} \geq \overline{j_{i+2}} \Rightarrow j_i \geq j_{i+2}$, $\widehat{\overline{j_{i+2}}} = \overline{j_{i+1}} = j_i$, $\widehat{\overline{j_{i+1}}} = \overline{j_{i+2}} - 1 = j_{i+2} - 1$, $\widehat{\overline{j_i}} = \overline{j_i} = j_{i+1} - 1$

□

Using Lemma 4.1 and definitions of A^i and r_i operators, we can obtain the followings.

Lemma 4.3. *Let* $w = (s_{n,j_n})(s_{n-1,j_{n-1}})\cdots(s_{i+1,\,j_{i+1}})(s_{i,j_i})\cdots(s_{1,J_1})$ *.Then*

$$A^i(\varepsilon^w) = \begin{cases} \varepsilon^{w_i} & if \quad j_i \geq j_{i+1} \\ 0 & if \quad j_i < j_{i+1} \end{cases}$$

where $w_i = (s_{n,\overline{j_n}})(s_{n-1,\overline{j_{n-1}}})\cdots(s_{i+1,\overline{j_{i+1}}})(s_{i,\overline{j_i}})\cdots(s_{1,\overline{j_1}})$ *with* $\overline{j_{i+1}} = j_i$, $\overline{j_i} = j_{i+1} - 1$ *and* $\overline{j_k} = j_k$ *if* $k \neq i, i+1$.

Lemma 4.4. $r_i(\varepsilon^{s_j}) = \begin{cases} \varepsilon^{s_{i-1}} - \varepsilon^{s_i} - \varepsilon^{s_{i+1}} & if \quad i = j \\ \varepsilon^{s_j} & if \quad i \neq j \end{cases}$

The integral cohomolgy of SU_{n+1}/T is generated by Schubert classes indexed

$$W = s_{nj_n}s_{n-1,j_{n-1}}\dots s_{1j_1}: \quad j_i = 0 \quad \text{or} \quad i \leq j_i \leq n.$$

Let $x_i = \varepsilon^{s_i} \in H^2(SU_{n+1}/T, R)$. We would like to define an order between generators of the integral cohomology of SU_{n+1}/T. Since each element $\varepsilon^{s_{nj_n}s_{n-1,j_{n-1}}\cdots s_{ij_i}\cdots s_{1j_1}}$ can be represented by an $n-$tuple

$$(j_n - n + 1, j_{n-1} - (n-1) + 1, \ldots, j_i - i + 1, \ldots, j_1 - 1 + 1),$$

we instead define an order between $n-$tuples.

Definition 4.1. (Graded Inverse Lexicographic Order)

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha > \beta$ if $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n > |\beta| = \beta_1 + \beta_2 + \ldots + \beta_n$ or $|\alpha| = |\beta|$ and in the vector difference $\alpha - \beta \in \mathbb{Z}^n$, the rightest nonzero entry is positive. We will write $\varepsilon^{s_{nj_n}s_{n-1,j_{n-1}}\cdots s_{ij_i}\cdots s_{1j_1}} > \varepsilon^{s_{nk_n}s_{n-1,j_{k-1}}\cdots s_{ik_i}\cdots s_{1j_1}}$ if $(j_n - n + 1, j_{n-1} - (n-1) + 1, \ldots, j_i - i - 1, \ldots, j_1 - 1 + 1) > (k_n - n + 1, k_{n-1} - (n-1) + 1, \ldots, k_i - i - 1, \ldots, k_1 - 1 + 1)$.

Example 4.2. $\varepsilon^{s_{35}s_{23}s_{14}} > \varepsilon^{s_{35}s_{24}s_{13}}$ since $(3, 2, 4) > (3, 3, 3)$ in graded inverse lex order.

We will try to find an quotient ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]/I$ which is isomorphic to $H^*(SU_{n+1}/T, R)$. We also define an order between monomials as follows:

Definition 4.2. We say $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} > x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$ if $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n > |\beta| = \beta_1 + \beta_2 + \ldots + \beta_n$ or $|\alpha| = |\beta|$ and in the vector difference $\alpha - \beta \in \mathbb{Z}^n$ the leftest non-zero entry is negative.

Example 4.3. $x_1^3 x_2^3 x_3^3 > x_1^4 x_2^2 x_3^3$,since $(3, 3, 3) - (4, 2, 3) = (-1, 1, 0)$.

Lemma 4.5. $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} = \varepsilon^{s_{n\alpha_n}s_{n-1,\alpha_{n-1}}\cdots s_{i\alpha_i}\cdots s_{1\alpha_1}}$ + lower terms

Proof. We use induction an degree of the monomial. By definition $x_i = \varepsilon^{s_i}$. Let us compute $x_i x_j = \varepsilon^{s_i}\varepsilon^{s_j}$. Here, we may assume that $i \leq j$. If $j - i > 1$, the inverse of $s_i s_j$ is $s_i s_j$. Hence,

$$P^{s_j s_i}_{s_i s_j} = r_j A^i(\varepsilon^{s_i}) = r_j(1) = 1$$

in cup product. If $j = i + 1$, the inverse of $s_{i+1}s_i$ is $s_i s_{i+1}$. In this case

$$P_{s_i, s_{i+1}} = A^i r_{i+1}(\varepsilon^{s_i}) = A^i(\varepsilon^{s_i}) = \varepsilon^{\{\}} = 1$$

.

If $i = j$, then we have to consider the word $s_{i,i+1}$. Its inverse $s_{i+1}s_i$

$$P^{s_{i,i+1}}_{s_i s_i} = r_{i+1}A^i(\varepsilon^{s_i}) = r_{i+1}(1) = 1$$

Now, we have to show that $P^{s_k s_l}_{s_i s_j} = 0$ if $\varepsilon^{s_k s_l} > \varepsilon^{s_j s_i}$. By definition of cup product the coefficient of $\varepsilon^{s_k s_l}$ is not zero only if $s_i \to s_k s_l$ and $s_j \to s_k s_l$. However , this is possible only if $s_k s_l = s_j s_i$ or $s_k s_l = s_{i,i+1}$ when $j = i+1$. Clearly $\varepsilon^{s_i s_{i+1}} < \varepsilon^{s_{i+1} s_i}$. Hence, $\varepsilon^{s_i}\varepsilon^{s_{i+1}} = \varepsilon^{s_{i+1} s_i}$+lower terms and $\varepsilon^{s_i}\varepsilon^{s_j} = \varepsilon^{s_j}\varepsilon^{s_i}$ if $j - i > 1$. In the case $i = j$, we have to look elements $s_i s_k$ and $s_k s_i$. The inverse of $s_k s_i = s_k s_i$, if $k - i > 1$, and thus

$$P^{s_k s_i}_{s_i s_j} = A^k r_i(\varepsilon^{s_i}) = A^k(\varepsilon^{s_{i-1}} - \varepsilon^{s_i} + \varepsilon^{s_{i+1}}) = 0$$

since $k - i > 1$. Clearly $\varepsilon^{s_i s_k} < \varepsilon^{s_i s_{i+1}}$ if $k < i$. Hence, $\varepsilon^{s_i}\varepsilon^{s_i} = \varepsilon^{s_i s_{i+1}}$+lower terms.

Assume

$x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} = \varepsilon^{s_{n\alpha_n} s_{n-1,\alpha_{n-1}} \cdots s_{i\alpha_i} \cdots s_{1\alpha_1}}$ + lower terms.

We have to show $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_i^{\alpha_i+1} \dots x_n^{\alpha_n} = \varepsilon^{s_{n\alpha_n} s_{n-1,\alpha_{n-1}} \cdots s_{i\alpha_i+1} \cdots s_{1\alpha_1}}$ + lower terms by Proposition 3.3.

$s_{n\alpha_n} s_{n-1,\alpha_{n-1}} \dots s_{i\alpha_i+1} \dots s_{1\alpha_1} \to w'$ only if $w' = s_{n\overline{\alpha_n}} s_{n-1,\overline{\alpha_{n-1}}} \dots s_{i\overline{\alpha_i}} \dots s_{1\overline{\alpha_1}}$, where there exists an index j for which $\overline{\alpha_j} = \alpha_j + 1$ and $\overline{\alpha_k} = \alpha_k$ if $k \neq j$.

By given ordering

$$w' = s_{n\overline{\alpha_n}} s_{n-1,\overline{\alpha_{n-1}}} \dots s_{i\overline{\alpha_i}} \dots s_{1\overline{\alpha_1}} > s_{n\alpha_n} s_{n-1,\alpha_{n-1}} \dots s_{i\alpha_i+1} \dots s_{1\alpha_1}$$

If $j > i$, then by Algorithm 4.1 in w'^{-1} we will not have a subsequence $s_{j-1}, s_{j-2} \dots s_i$ after the elements s_j. Therefore, in cup product before applying A^j we will not have the term ε^{s_j}.That means

$$P^{w'}_{s_i,w} = 0.$$

if $j = i$, then again by Algorithm 4.1 in w'^{-1} we will not have a subsequence $s_{j-1}, s_{j-2} \dots s_i$ after the elements s_j. Hence, in cup product before applying A^j we will not have the term ε^{s_j}.That means

$$P^{w'}_{s_i,w} = 1.$$

If and only if $j > i$. □

l denotes the length of w.

Example 4.4. Let $l = 3$;

$x_1x_2x_3 = \varepsilon^{s_3s_2s_1}$+ lower terms

$x_1^2x_2x_3 = \varepsilon^{s_3s_2s_{12}}$+ lower terms.

$\varepsilon^{s_3s_{23}s_1} > \varepsilon^{s_3s_2s_{12}} > \varepsilon^{s_{23}s_{12}} > \varepsilon^{s_3s_{13}} > \varepsilon^{s_2s_{13}}$

The inverse of $s_3s_{23}s_1$ is s_3s_{13} and the inverse of $s_3s_2s_1$ is s_{13}.

$A_3r_1r_2r_3(\varepsilon^{s_1}) = A_3r_1(\varepsilon^{s_1}) = A_3(-\varepsilon^{s_1} + \varepsilon^{s_2}) = 0.$

and

The inverse of $s_3s_2s_{12}$ is s_2s_{13}.

$A_2r_1r_2r_3(\varepsilon^{s_1}) = A_2r_1(\varepsilon^{s_1}) = A_2(-\varepsilon^{s_1} + \varepsilon^{s_2}) = 1.$

Before finding the quotient ring $\mathbb{Z}[x_1, \ldots, x_n]/I$, we give some information about ring $\Bbbk[x_1, \ldots, x_n]/I$, $\Bbbk$ is a field.

Fix a monomial ordering on $\Bbbk[x_1, \ldots, x_n]$. Let $f \in \Bbbk[x_1, \ldots, x_n]$. The leading monomial of f, denoted by $LM(f)$, is the highest degree monomial of f. The coefficient of $LM(f)$ is called leading coefficient of and denoted by $LC(f)$. The leading term of f, $LT(f) = LC(f)LM(f)$.

Let $I \subseteq \Bbbk[x_1, \ldots, x_n]$ be an ideal. Define

$LT(I) = \{LT(f) : f \in I\}$. Let $< LT(I) >$ be an ideal generated by $LT(I)$.

Proposition 4.2. *(Proposition 1 and 4 of 5.3 in [51])*

(i) *Every $f \in \Bbbk[x_1, \ldots, x_n]$ is congruent modulo I to a unique polynomial r which is a $\Bbbk$-linear combination of the monomials in the complement of $< LT(I) >$.*

(ii) *The elements of $\{x^\alpha : x^\alpha \notin < LT(I) >\}$ are linearly independent modulo I.*

(iii) *$\mathbb{k}[x_1, \ldots, x_n]/I$ is isomorphic as a $\mathbb{k}$ − vector space to*

$$S = Span\{x^\alpha : x^\alpha \notin < LT(I) >\}$$

.

Theorem 4.2. *(Theorem 6 of 5.3 in [51]) Let $I \subseteq \mathbb{k}[x_1, \ldots, x_n]$ be an ideal*

(i) *The $\mathbb{k}$-vector space $\mathbb{k}[x_1, \ldots, x_n]/I$ is finite dimensional.*

(ii) *For each i, $1 \leq i \leq n$, there is a polynomial $f_i \in I$ such that $LM(f_i) = x_i^{m_i}$ for some positive integer m_i.*

Theorem 4.3. *$H^*(SU_{n+1}/T, R)$ isomorphic to $\mathbb{Z}[x_1, x_2, \ldots, x_n]/ < f_1, f_2, \ldots, f_n >$ where $LT(f_i) = x_i^{n-i+2}$ with respect to monomial order given by Definition* 4.2.

Proof. Let I be the ideal such that $H^*(S_n/T, R) \cong \mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_n]/I$. Since we found one-to-one correspondence between length l elements of $H^*(SU_{n+1}/T, R)$ and monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha_1 + \alpha_2 + \cdots \alpha_n = l$ and for each i, $1 \leq i \leq n$, $\alpha_i \leq n - i + 1$, there should be a polynomial $f_i \in I$ such that $LT(f_i) = x_i^{n-i+2}$. □

Example 4.5. Let $n = 3$.

$\alpha_i \leq n - i + 1, \;\; i = 1, 2, 3$

$\alpha_1 \leq 3, \;\; \alpha_2 \leq 2, \;\; \alpha_3 \leq 1$

$l = 1; \;\; x_1, \;\; x_2, \;\; x_3$

$l = 2; \;\; x_1^2, \;\; x_1x_2, \;\; x_1x_3, \;\; x_2^2, \;\; x_2x_3$

We must have a polynomial f_3 with $LT(f_3) = x_3^2$.

$l = 3; \;\; x_1^3, \;\; x_1^2x_2, \;\; x_1^2x_3, \;\; x_1x_2x_3, \;\; x_1x_2^2, \;\; x_2^2x_3,$

We must have a polynomial f_2 with $LT(f_2) = x_2^3$.

$l = 4; \;\; x_1^3x_2, \;\; x_1^3x_3, \;\; x_1^2x_2x_3, \;\; x_1^2x_2^2, \;\; x_1x_2^2x_3,$

We must have a polynomial f_1 with $LT(f_1) = x_1^4$.

Since the unique highest element has length of $\frac{n(n+1)}{2}$, we now give the result about the multiplication of elements of length k and of length $\frac{n(n+1)}{2} - k$.

Theorem 4.4. *Let* $A = \varepsilon^{s_{nj_n} s_{n-1,j_{n-1}} \cdots s_{1j_1}}$ *be an element of length* k *and* $B = \varepsilon^{s_{np_n} s_{n-1,p_{n-1}} \cdots s_{1p_1}}$ *be an element of length* $\frac{n(n+1)}{2} - k$*. The corresponding polynomials in* $\mathbb{Z}[x_1, x_2, \ldots, x_n]/ < f_1, f_2, \ldots, f_n >$ *has leading monomials* $x_1^{j_1-1+1} x_2^{j_2-2+1} \cdots x_i^{j_i-i+1} \cdots x_1^{j_n-n+1}$ *and* $x_1^{p_1-1+1} x_2^{p_1-2+1} \cdots x_1^{p_n-n+1}$*, respectively.Then*

$$A \cdot B = \begin{cases} \varepsilon^{s_{n,n} s_{n-1,n}, \ldots, s_{in}, \ldots, s_{1n}}, & if \ j_i + p_i + 1 = n + i; \\ 0, & if \ j_i + p_i + 1 \neq n + i. \end{cases}$$

Proof. The unique highest degree monomial in $\mathbb{Z}[x_1, x_2, \ldots, x_n]/ < f_1, f_2, \ldots, f_n >$ is $x_1^n x_2^{n-1} \cdots x_i^{n-i+1} \cdots x_n$. The multiplication of leading monomials of corresponding monomials of A and B produce the monomial

$$x_1^{j_1+p_1} x_2^{j_2+p_2-2} \cdots x_i^{j_i+p_i-2i+2} \cdots x_n^{j_n+p_n-2n+2}.$$

If $j_i + p_i - 2i + 2 = n - i + 1 \rightarrow j_i + p_i + 1 = n + i$ for each i, $i \leq 1 \leq n$, then the multiplication gives the $x_1^n x_2^{n-1} \ldots x_n$. Since this monomial corresponds the element $\varepsilon^{s_{n,n} s_{n-1,n} \cdots s_{in} \cdots s_{1n}}$, $A \cdot B = \varepsilon^{s_{n,n} s_{n-1,n} \cdots s_{1n}}$. If $j_i + p_i + 1 \neq n + i$, then the leading monomial so the monomials of lower degree must reduce to zero modulo $< f_1, f_2, \ldots, f_n >$ in $\Bbbk[x_1, x_2, \ldots, x_n]$ when we apply the division algorithm. Hence, $A \cdot B = 0$.

□

Now, we give whole computation of the quotient ring $\mathbb{Z}[x_1, x_2, x_3]/ < f_1, f_2, f_3 >$.

Example 4.6. $x_1 = \varepsilon^{s_1}$, $x_2 = \varepsilon^{s_2}$, $x_3 = \varepsilon^{s_3}$

Let $l = 2$;

$$\begin{aligned}
x_2x_3 &= \varepsilon^{s_3s_2} + \varepsilon^{s_2s_3}\\
x_2^2 &= \varepsilon^{s_2s_3} + \varepsilon^{s_2s_1}\\
x_1x_3 &= \varepsilon^{s_3s_1}\\
x_1x_2 &= \varepsilon^{s_2s_1} + \varepsilon^{s_1s_2}\\
x_1^2 &= \varepsilon^{s_1s_2}
\end{aligned}$$

$$\begin{pmatrix} x_2x_3 \\ x_2^2 \\ x_1x_3 \\ x_1x_2 \\ x_1^2 \end{pmatrix} = M \begin{pmatrix} \varepsilon^{s_3s_2} \\ \varepsilon^{s_2s_3} \\ \varepsilon^{s_3s_1} \\ \varepsilon^{s_2s_1} \\ \varepsilon^{s_1s_2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon^{s_3s_2} \\ \varepsilon^{s_2s_3} \\ \varepsilon^{s_3s_1} \\ \varepsilon^{s_2s_1} \\ \varepsilon^{s_1s_2} \end{pmatrix} = M^{-1} \begin{pmatrix} x_2x_3 \\ x_2^2 \\ x_1x_3 \\ x_1x_2 \\ x_1^2 \end{pmatrix} \text{ where}$$

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
\varepsilon^{s_3s_2} &= x_2x_3 - x_2^2 + x_1x_2 - x_1^2\\
\varepsilon^{s_2s_3} &= x_2^2 - x_1x_2 + x_2^2\\
\varepsilon^{s_3s_1} &= x_1x_3\\
\varepsilon^{s_2s_1} &= x_1x_2 - x_1^2\\
\varepsilon^{s_1s_2} &= x_1^2
\end{aligned}$$

We must have a relation involving x_3^2.

$x_3^2 = \varepsilon^{s_3s_2} = x_2x_3 - x_2^2 + x_1x_2 - x_1^2$

Let $l = 3$;

$$\begin{aligned}
x_2^2 x_3 &= \varepsilon^{s_3 s_2 s_3} + \varepsilon^{s_3 s_2 s_1} + \varepsilon^{s_2 s_3 s_1} \\
x_1 x_2 x_3 &= \varepsilon^{s_3 s_2 s_1} + \varepsilon^{s_2 s_3 s_1} + \varepsilon^{s_3 s_1 s_2} + \varepsilon^{s_1 s_2 s_3} \\
x_1 x_2^2 &= \varepsilon^{s_2 s_3 s_1} + \varepsilon^{s_2 s_1 s_2} + \varepsilon^{s_1 s_2 s_3} \\
x_1^2 x_3 &= \varepsilon^{s_3 s_1 s_2} + \varepsilon^{s_1 s_2 s_3} \\
x_1^2 x_2 &= \varepsilon^{s_2 s_1 s_2} + \varepsilon^{s_1 s_2 s_3} \\
x_1^3 &= \varepsilon^{s_1 s_2 s_3}
\end{aligned}$$

$$\begin{pmatrix} x_2^2 x_3 \\ x_1 x_2 x_3 \\ x_1 x_2^2 \\ x_1^2 x_3 \\ x_1^2 x_2 \\ x_1^3 \end{pmatrix} = M \begin{pmatrix} \varepsilon^{s_3 s_2 s_3} \\ \varepsilon^{s_3 s_2 s_1} \\ \varepsilon^{s_2 s_3 s_1} \\ \varepsilon^{s_3 s_1 s_2} \\ \varepsilon^{s_2 s_1 s_2} \\ \varepsilon^{s_1 s_2 s_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon^{s_3 s_2 s_3} \\ \varepsilon^{s_3 s_2 s_1} \\ \varepsilon^{s_2 s_3 s_1} \\ \varepsilon^{s_3 s_1 s_2} \\ \varepsilon^{s_2 s_1 s_2} \\ \varepsilon^{s_1 s_2 s_3} \end{pmatrix} = M^{-1} \begin{pmatrix} x_2^2 x_3 \\ x_1 x_2 x_3 \\ x_1 x_2^2 \\ x_1^2 x_3 \\ x_1^2 x_2 \\ x_1^3 \end{pmatrix} \quad \text{where}$$

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
\varepsilon^{s_3s_2s_3} &= x_2^2x_3 - x_1x_2x_3 + x_1^2x_3 \\
\varepsilon^{s_3s_2s_1} &= x_1x_2x_3 - x_1x_2^2 - x_1^2x_3 + x_1^2x_2 \\
\varepsilon^{s_2s_3s_1} &= x_1x_2^2 - x_1^2x_2 \\
\varepsilon^{s_3s_1s_2} &= x_1^2x_3 - x_1^3 \\
\varepsilon^{s_2s_1s_2} &= x_1^2x_2 - x_1^3 \\
\varepsilon^{s_1s_2s_3} &= x_1^3
\end{aligned}$$

We must have a relation involving x_2^3

$x_2^3 = 2\varepsilon^{s_2s_3s_1} = 2(x_1x_2^2 - x_1^2x_2)$.

Let $l = 4$;

$$\begin{aligned}
x_1x_2^2x_3 &= \varepsilon^{s_3s_2s_3s_1} + \varepsilon^{s_3s_2s_1s_2} + 2\varepsilon^{s_2s_3s_1s_2} + 2\varepsilon^{s_3s_1s_2s_3} \\
x_1^2x_2x_3 &= \varepsilon^{s_3s_2s_1s_2} + \varepsilon^{s_2s_3s_1s_2} + \varepsilon^{s_3s_1s_2s_3} + \varepsilon^{s_2s_1s_2s_3} \\
x_1^2x_2^2 &= \varepsilon^{s_2s_3s_1s_2} + \varepsilon^{s_2s_1s_2s_3} \\
x_1^3x_3 &= \varepsilon^{s_3s_1s_2s_3} \\
x_1^3x_2 &= \varepsilon^{s_2s_1s_2s_3}
\end{aligned}$$

$$\begin{pmatrix} x_1x_2^2x_3 \\ x_1^2x_2x_3 \\ x_1^2x_2^2 \\ x_1^3x_3 \\ x_1^3x_2 \end{pmatrix} = M \begin{pmatrix} \varepsilon^{s_3s_2s_3s_1} \\ \varepsilon^{s_3s_2s_1s_2} \\ \varepsilon^{s_2s_3s_1s_2} \\ \varepsilon^{s_3s_1s_2s_3} \\ \varepsilon^{s_2s_1s_2s_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon^{s_3s_2s_3s_1} \\ \varepsilon^{s_3s_2s_1s_2} \\ \varepsilon^{s_2s_3s_1s_2} \\ \varepsilon^{s_3s_1s_2s_3} \\ \varepsilon^{s_2s_1s_2s_3} \end{pmatrix} = M^{-1} \begin{pmatrix} x_1x_2^2x_3 \\ x_1^2x_2x_3 \\ x_1^2x_2^2 \\ x_1^3x_3 \\ x_1^3x_2 \end{pmatrix} \quad \text{where}$$

$$M = \begin{pmatrix} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \varepsilon^{s_3s_2s_3s_1} &= x_1x_2^2x_3 - x_1^2x_2x_3 - x_1^2x_2^2 - x_1^3x_3 + 2x_1^3x_2 \\ \varepsilon^{s_3s_2s_1s_2} &= x_1^2x_2x_3 - x_1^2x_2^2 - x_1^3x_3 \\ \varepsilon^{s_2s_3s_1s_2} &= x_1^2x_2^2 - x_1^3x_2 \\ \varepsilon^{s_3s_1s_2s_3} &= x_1^3x_3 \\ \varepsilon^{s_2s_1s_2s_3} &= x_1^3x_2 \end{aligned}$$

We must have a relation involving x_1^4 which is

$x_1x_1^3 = \varepsilon^{s_1}.\varepsilon^{s_1s_2s_3} = 0$

Let $l = 5$;

$$\begin{aligned} x_1^2x_2^2x_3 &= \varepsilon^{s_3s_2s_3s_1s_2} + \varepsilon^{s_3s_2s_1s_2s_3} + \varepsilon^{s_2s_3s_1s_2s_3} \\ x_1^3x_2x_3 &= \varepsilon^{s_3s_2s_1s_2s_3} + \varepsilon^{s_2s_3s_1s_2s_3} \\ x_1^3x_2^2 &= \varepsilon^{s_2s_3s_1s_2s_3} \end{aligned}$$

$$\begin{pmatrix} x_1^2x_2^2x_3 \\ x_1^3x_2x_3 \\ x_1^3x_2^2 \end{pmatrix} = M \begin{pmatrix} \varepsilon^{s_3s_2s_3s_1s_2} \\ \varepsilon^{s_3s_2s_1s_2s_3} \\ \varepsilon^{s_2s_3s_1s_2s_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon^{s_3s_2s_3s_1s_2} \\ \varepsilon^{s_3s_2s_1s_2s_3} \\ \varepsilon^{s_2s_3s_1s_2s_3} \end{pmatrix} = M^{-1} \begin{pmatrix} x_1^2x_2^2x_3 \\ x_1^3x_2x_3 \\ x_1^3x_2^2 \end{pmatrix} \quad \text{where}$$

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \varepsilon^{s_3 s_2 s_3 s_1 s_2} &= x_1^2 x_2^2 x_3 - x_1^3 x_2 x_3 \\ \varepsilon^{s_3 s_2 s_1 s_2 s_3} &= x_1^3 x_2 x_3 - x_1^3 x_2^2 \\ \varepsilon^{s_2 s_3 s_1 s_2 s_3} &= x_1^3 x_2^2 \end{aligned}$$

We don't have any relation.

Let $l = 6$;

$x_1^3 x_2^2 x_3 = \varepsilon^{s_3 s_2 s_3 s_1 s_2 s_3}$

$\varepsilon^{s_3 s_2 s_3 s_1 s_2 s_3} = x_1^3 x_2^2 x_3$

Now let us multiply length of k and $6-k$ element,

$M_0 = 1$ and and $|det(M_0)| = 1$.

$\underline{Degree \quad 1 * Degree \quad 5}$

Elements	Leading Monomial in Polynomial Ring
ε^{s_1}	x_1
ε^{s_2}	x_2
ε^{s_3}	x_3
$\varepsilon^{s_3 s_{23} s_{12}}$	$x_1^2 x_2^2 x_3$
$\varepsilon^{s_3 s_2 s_{13}}$	$x_1^3 x_2 x_3$
$\varepsilon^{s_{23} s_{13}}$	$x_1^3 x_2^2$

$\varepsilon^{s_3} * \varepsilon^{s_3 s_{23} s_{12}}$	$x_1^2 x_2^2 x_3^2$	0
$\varepsilon^{s_3} * \varepsilon^{s_3 s_2 s_{13}}$	$x_1^3 x_2 x_3^2$	0
$\varepsilon^{s_3} * \varepsilon^{s_{23} s_{13}}$	$x_1^3 x_2^2 x_3$	1

$\varepsilon^{s_2} * \varepsilon^{s_3 s_{23} s_{12}}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_2} * \varepsilon^{s_3 s_2 s_{13}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_2} * \varepsilon^{s_{23} s_{13}}$	$x_1^3 x_2^3$	0

$\varepsilon^{s_1} * \varepsilon^{s_3 s_{23} s_{12}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_1} * \varepsilon^{s_3 s_2 s_{13}}$	$x_1^4 x_2 x_3$	0
$\varepsilon^{s_1} * \varepsilon^{s_{23} s_{13}}$	$x_1^4 x_2^2$	0

Now we will calculate Reidemeister torsion of SU_4/T using above multiplication. From multiplication of second cohomology then we have

$$M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad |det(M_2)| = 1.$$

$\underline{Degree \quad 2 * Degree \quad 4}$

Elements	Leading Monomial in Polynomial Ring
$\varepsilon^{s_3 s_2}$	$x_2 x_3$
$\varepsilon^{s_{23}}$	x_2^2
$\varepsilon^{s_3 s_1}$	$x_1 x_3$
$\varepsilon^{s_2 s_1}$	$x_1 x_2$
$\varepsilon^{s_{12}}$	x_1^2
$\varepsilon^{s_3 s_{23} s_1}$	$x_1 x_2^2 x_3$
$\varepsilon^{s_3 s_2 s_{12}}$	$x_1^2 x_2 x_3$
$\varepsilon^{s_{23} s_{12}}$	$x_1^2 x_2^2$
$\varepsilon^{s_3 s_{13}}$	$x_1^3 x_3$
$\varepsilon^{s_2 s_{13}}$	$x_1^3 x_2$

$\varepsilon^{s_3 s_2} * \varepsilon^{s_3 s_{23} s_1}$	$x_1 x_2^3 x_3^2$	0
$\varepsilon^{s_3 s_2} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^2 x_2^2 x_3^2$	0
$\varepsilon^{s_3 s_2} * \varepsilon^{s_{23} s_{12}}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_3 s_2} * \varepsilon^{s_3 s_{13}}$	$x_1^3 x_2 x_3^2$	0
$\varepsilon^{s_3 s_2} * \varepsilon^{s_2 s_{13}}$	$x_1^3 x_2^2 x_3$	1

$\varepsilon^{s_{23}} * \varepsilon^{s_3 s_{23} s_1}$	$x_1 x_2^4 x_3$	0
$\varepsilon^{s_{23}} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_{23}} * \varepsilon^{s_{23} s_{12}}$	$x_1^2 x_2^4$	0
$\varepsilon^{s_{23}} * \varepsilon^{s_3 s_{13}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_{23}} * \varepsilon^{s_2 s_{13}}$	$x_1^3 x_2^3$	0

$\varepsilon^{s_3 s_1} * \varepsilon^{s_3 s_{23} s_1}$	$x_1^2 x_2^2 x_3^2$	0
$\varepsilon^{s_3 s_1} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^3 x_2 x_3^2$	0
$\varepsilon^{s_3 s_1} * \varepsilon^{s_{23} s_{12}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_3 s_1} * \varepsilon^{s_3 s_{13}}$	$x_1^4 x_3^2$	0
$\varepsilon^{s_3 s_1} * \varepsilon^{s_2 s_{13}}$	$x_1^4 x_2 x_3$	0

$\varepsilon^{s_2 s_1} * \varepsilon^{s_3 s_{23} s_1}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_2 s_1} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_2 s_1} * \varepsilon^{s_{23} s_{12}}$	$x_1^3 x_2^3$	0
$\varepsilon^{s_2 s_1} * \varepsilon^{s_3 s_{13}}$	$x_1^4 x_2 x_3$	0
$\varepsilon^{s_2 s_1} * \varepsilon^{s_2 s_{13}}$	$x_1^4 x_2^2$	0

$\varepsilon^{s_{12}} * \varepsilon^{s_3 s_{23} s_1}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_{12}} * \varepsilon^{s_3 s_2 s_{12}}$	$x_1^4 x_2 x_3$	0
$\varepsilon^{s_{12}} * \varepsilon^{s_{23} s_{12}}$	$x_1^4 x_2^2$	0
$\varepsilon^{s_{12}} * \varepsilon^{s_3 s_{13}}$	$x_1^5 x_3$	0
$\varepsilon^{s_{12}} * \varepsilon^{s_2 s_{13}}$	$x_1^5 x_2$	0

Now we will calculate Reidemeister torsion of SU_4/T using above multiplication. From multiplication of fourth cohomology then we have

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad |det(M_4)| = 1.$$

$\underline{Degree \quad 3 * Degree \quad 3}$

Elements	Leading Monomial in Polynomial Ring
$\varepsilon^{s_3 s_{23}}$	$x_2^2 x_3$
$\varepsilon^{s_3 s_2 s_1}$	$x_2^2 x_3$
$\varepsilon^{s_{23} s_1}$	$x_1 x_2^2$
$\varepsilon^{s_3 s_{12}}$	$x_1^2 x_3$
$\varepsilon^{s_2 s_{12}}$	$x_1^2 x_2$
$\varepsilon^{s_{13}}$	x_1^3
$\varepsilon^{s_3 s_{23}}$	$x_2^2 x_3$
$\varepsilon^{s_3 s_2 s_1}$	$x_1 x_2 x_3$
$\varepsilon^{s_{23} s_1}$	$x_1 x_2^2$
$\varepsilon^{s_3 s_{12}}$	$x_1^2 x_3$
$\varepsilon^{s_2 s_{12}}$	$x_1^2 x_2$
$\varepsilon^{s_{13}}$	x_1^3

$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_3 s_{23}}$	$x_2^4 x_3^2$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_3 s_2 s_1}$	$x_1 x_2^3 x_3^2$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_{23} s_1}$	$x_1 x_2^4 x_3$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_3 s_{12}}$	$x_1^2 x_2^2 x_3^2$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_2 s_{12}}$	$x_1^2 x_2^3 x_3$	0
$\varepsilon^{s_3 s_{23}} * \varepsilon^{s_{13}}$	$x_1^3 x_2^2 x_3$	1

$\varepsilon^{s_3s_2s_1} * \varepsilon^{s_3s_{23}}$	$x_1x_2^3x_3^2$	0
$\varepsilon^{s_3s_2s_1} * \varepsilon^{s_3s_2s_1}$	$x_1^2x_2^2x_3^2$	0
$\varepsilon^{s_3s_2s_1} * \varepsilon^{s_{23}s_1}$	$x_1^2x_2^3x_3$	0
$\varepsilon^{s_3s_2s_1} * \varepsilon^{s_3s_{12}}$	$x_1^3x_2x_3^2$	0
$\varepsilon^{s_3s_2s_1} * \varepsilon^{s_2s_{12}}$	$x_1^3x_2^2x_3$	1
$\varepsilon^{s_3s_2s_1} * \varepsilon^{s_{13}}$	$x_1^4x_2x_3$	0

$\varepsilon^{s_{23}s_1} * \varepsilon^{s_3s_{23}}$	$x_1x_2^4x_3$	0
$\varepsilon^{s_{23}s_1} * \varepsilon^{s_3s_2s_1}$	$x_1^2x_2^3x_3$	0
$\varepsilon^{s_{23}s_1} * \varepsilon^{s_{23}s_1}$	$x_1^2x_2^4$	0
$\varepsilon^{s_{23}s_1} * \varepsilon^{s_3s_{12}}$	$x_1^3x_2^2x_3$	1
$\varepsilon^{s_{23}s_1} * \varepsilon^{s_2s_{12}}$	$x_1^3x_2^3$	0
$\varepsilon^{s_{23}s_1} * \varepsilon^{s_{13}}$	$x_1^4x_2^2$	0

$\varepsilon^{s_3s_{12}} * \varepsilon^{s_3s_{23}}$	$x_1^2x_2^2x_3^2$	0
$\varepsilon^{s_3s_{12}} * \varepsilon^{s_3s_2s_1}$	$x_1^3x_2x_3^2$	0
$\varepsilon^{s_3s_{12}} * \varepsilon^{s_{23}s_1}$	$x_1^3x_2^2x_3$	1
$\varepsilon^{s_3s_{12}} * \varepsilon^{s_3s_{12}}$	$x_1^4x_3^2$	0
$\varepsilon^{s_3s_{12}} * \varepsilon^{s_2s_{12}}$	$x_1^4x_2x_3$	0
$\varepsilon^{s_3s_{12}} * \varepsilon^{s_{13}}$	$x_1^5x_3$	0

$\varepsilon^{s_2s_{12}} * \varepsilon^{s_3s_{23}}$	$x_1^2x_2^3x_3$	0
$\varepsilon^{s_2s_{12}} * \varepsilon^{s_3s_2s_1}$	$x_1^3x_2^2x_3$	1
$\varepsilon^{s_2s_{12}} * \varepsilon^{s_{23}s_1}$	$x_1^3x_2^3$	0
$\varepsilon^{s_2s_{12}} * \varepsilon^{s_3s_{12}}$	$x_1^4x_2x_3$	0
$\varepsilon^{s_2s_{12}} * \varepsilon^{s_2s_{12}}$	$x_1^4x_2^2$	0
$\varepsilon^{s_2s_{12}} * \varepsilon^{s_{13}}$	$x_1^5x_2$	0

$\varepsilon^{s_{13}} * \varepsilon^{s_3 s_{23}}$	$x_1^3 x_2^2 x_3$	1
$\varepsilon^{s_{13}} * \varepsilon^{s_3 s_2 s_1}$	$x_1^4 x_2 x_3$	0
$\varepsilon^{s_{13}} * \varepsilon^{s_{23} s_1}$	$x_1^4 x_2^2$	0
$\varepsilon^{s_{13}} * \varepsilon^{s_3 s_{12}}$	$x_1^5 x_3$	0
$\varepsilon^{s_{13}} * \varepsilon^{s_2 s_{12}}$	$x_1^5 x_2$	0
$\varepsilon^{s_{13}} * \varepsilon^{s_{13}}$	x_1^6	0

Now we will calculate Reidemeister torsion of SU_4/T using above multiplication. From multiplication of sixth cohomology then we have

$$M_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad |det(M_6)| = 1.$$

Hence, the Reidemeister torsion of SU_n/T is 1 by the Reidemeister torsion formula.

Conjecture. The Reidemeister torsion of SU_{n+1}/T is always 1 for any $n \in \mathbb{Z}^+$.

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