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INVESTIGATION OF IMPULSIVE DIRAC EQUATIONS

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ABDULLAH MHMOOD JASIM JASIM

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INVESTIGATION OF IMPULSIVE DIRAC EQUATIONS

By Abdullah Mhmood Jasim JASIM

September 2021

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Advisor : Asst. Prof. Dr. Şerifenur CEBESOY ERDAL

Examining Committee Members:

Chairman : Assoc. Prof. Dr. Emre TAŞ
Mathematics
Kırşehir Ahi Evran University

Member : Asst. Prof. Dr. Şerifenur CEBESOY ERDAL
Mathematics
Çankırı Karatekin University

Member : Asst. Prof. Dr. Celalettin KAYA
Mathematics
Çankırı Karatekin University

Approved for the Graduate School of Natural and Applied Sciences

Prof. Dr. İbrahim ÇİFTÇİ
Director of Graduate School

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Abdullah Mhmood Jasim JASIM

ABSTRACT

INVESTIGATION OF IMPULSIVE DIRAC EQUATIONS

Abdullah Mhmood Jasim JASIM
Master of Science in Mathematics
Advisor: Asst. Prof. Dr. Şerifenur CEBESOY ERDAL
September 2021

This thesis consists of five chapters.
The first chapter is devoted to the introduction.
In the second chapter, some well known concepts of functional analysis and spectral theory are mentioned and an example of an eigenvalue problem is presented.
In the third chapter, the matrix equation

$$K\frac{d\psi}{dx}+P(x)\psi=\mu\psi,\quad \psi(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix},\quad x\in[0,\pi]$$

is introduced, where μ is a spectral parameter, $K=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $P=\begin{pmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix}$ and P_{ij} are real valued functions which are continuous on the interval $[0,\pi]$ for $i,j=1,2$.
The fourth chapter consists of two sections. In the first section, the impulsive Dirac operator is defined, theory on the eigenvalues and the spectral singularities of this operator is given. In the second section, some special cases are examined, where the impulsive condition at the origin has certain symmetries.
The fifth and last chapter is devoted to the conclusion and recommendations.

2021, 42 pages

Keywords: Dirac equation, Impulsive Dirac operator, Jost solution, Eigenvalue, Spectral singularity, Symmetry.

ÖZET

İMPALSİF DİRAK DENKLEMLERİNİN İNCELENMESİ

Abdullah Mhmood Jasim JASIM

Matematik, Yüksek Lisans Tezi

Danışman: Dr. Öğr. Üyesi Şerifenur CEBESOY ERDAL

Eylül 2021

Bu tez beş bölümden oluşmaktadır.
Birinci bölüm giriş kısmına ayrılmıştır.
İkinci bölümde, fonksiyonel analiz ve spektral teoride bilinen bazı temel kavramlardan bahsedilmiş ve bir özdeğer problemi örneği sunulmuştur.
Üçüncü bölümde, μ bir spektral parametre, $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ ve $i, j = 1, 2$ için P_{ij}, $[0, \pi]$ aralığında reel değerli sürekli fonksiyonlar olmak üzere,

$$K\frac{d\psi}{dx} + P(x)\psi = \mu\psi, \quad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad x \in [0, \pi]$$

matris denklemi tanıtılmış, bu denklem yardımıyla bir boyutlu kanonik Dirac sistemi elde edilmiş ve bu sistemin bazı spektral özellikleri verilmiştir.
Dördüncü bölüm, iki kısımdan oluşmaktadır. İlk kısımda, impalsif Dirak operatörü tanımlanmış, bu operatörün özdeğerleri ve spektral tekillikleri üzerindeki teori verilmiştir. İkinci kısımda ise, orjindeki impalsif koşulun sahip olduğu bazı simetrilere ait özel durumlar incelenmiştir.
Beşinci ve son bölüm ise sonuç ve öneriler için ayrılmıştır.

2021, 42 sayfa

Anahtar Kelimeler: Dirak denklemi, İmpalsif Dirak operatörü, Jost çözümü, Özdeğer, Spektral tekillik, Simetri.

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Abdullah Mhmood Jasim JASIM
Çankırı-2021

CONTENTS

LIST OF SYMBOLS

$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{Z}$	Set of integers
$\mathbb{N}$	Set of natural numbers
$\mathrm{Re}z$	Real part of the complex number z
$\mathrm{Im}z$	Imaginary part of the complex number z
$\mathbb{C}_{\pm}$	$\{z \in \mathbb{C} : \pm\mathrm{Im}z > 0\}$
$\overline{\mathbb{C}}_{\pm}$	$\{z \in \mathbb{C} : \pm\mathrm{Im}z \geq 0\}$
$\sigma(L)$	Spectrum of the operator L
$\sigma_c(L)$	Continuous spectrum of the operator L
$\sigma_d(L)$	Set of eigenvalues of the operator L
$\sigma_{ss}(L)$	Set of spectral singularities of the operator L
$L_p(-\infty,\infty)$	$\left\{f : \int_{-\infty}^{\infty} \lvert f(x)\rvert^p dx < \infty\right\}$
$L_2(\mathbb{R},\mathbb{C}^2)$	$\left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{C}^2 : \int_{-\infty}^{\infty} \left(\lvert y_1\rvert^2 + \lvert y_2\rvert^2\right) dx < \infty \right\}$
$\parallel f \parallel$	Norm of the function f
$< f,g >$	Inner product of f and g
$D(L)$	Domain of the operator L
$W[y,z]$	Wronskian of the solutions y and z

1. INTRODUCTION

Mathematical models are the most effective tools used in solving real life problems. A mathematical model imitates a real world problem of daily life encountered in the fields of engineering, economics, biology, physics and chemistry with the mathematical language by using the closest features of the problem. Mostly, the modelling techniques are very useful and successful since general results are achieved from simple models. In applied mathematics, differential equations theory is used for the solution of many problems in these mathematical models which are commonly created by the help of initial value problems *(IVP)* or boundary value problems *(BVP)*. The investigations of such problems first started by Birkhoff (Birkhoff 1908). In the application of both first and higher order differential equations, the problems involve both a differential equation and one or more supplementary conditions which the solution of the given differential equation must satisfy. If all the associated supplementary conditions relate to one independent variable value, the problem is called an *IVP*, if the conditions relate more than one independent variable values, the problem is called *BVP* (Ross 1989).

Spectral analysis, a sub–branch of applied mathematics and functional analysis, examines the solutions of these problems using "operator theory". Operator theory is no doubt a diverse area that has grown out of linear algebra and complex analysis, and is often described as the branch of functional analysis that deals with bounded linear operators and their spectral properties. In mathematical models, an operator is determined compatible with the differential equation used and the properties of the operator are investigated. Differential operators generated by the differential equations are the first operator type handled in the literature. In particular, Sturm–Liouville operator, known as one dimensional Schrödinger operator is the most widely studied operator having a tremendous potential for applications in quantum theory.

As is well known, quantum mechanics forms the basis of spectral theory and wave equations form the basis of quantum theory. In the scientific studies carried out in 19th century, how the state of atomic particles will change over time is estimated by the help of wave functions. It was Erwin Schrödinger who first found out the equation pointing up the variation of wave function with respect to space and time, that is why the equation is called as "Schrödinger equation". The time independent Schrödinger equation explains a physical system and one dimensional motion of a particle of mass m in an external field V is represented by the equation

$$\frac{-\hbar^2}{2m}\Psi''(x)+V(x)\Psi(x)=E\Psi(x), \tag{1.1}$$

in which Ψ is the wave function, E is the total energy of the particle and $\hbar$ is the Plank constant divided by 2π. (1.1) is said to be *Schrödinger equation* which is a special form of a Sturm–Liouville equation. As is known in literature, the second order linear ordinary differential equation

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=\lambda w(x) \tag{1.2}$$

is called the *Sturm–Liouville equation*, the operator generated by (1.2) or the operator generated by some boundary conditions together with (1.2) is called the *Sturm–Liouville operator*, the spectral problems of this operator are called the *Sturm–Liouville problems*. For instance, the one-dimensional time independent Schrödinger equation (1.1) is a Sturm–Liouville problem. Due to their physical importance, Sturm–Liouville problems have been studied for a long time. The foundations of Sturm–Liouville theory were laid by two French mathematicians Charles Sturm and Joseph Liouville who gave such theory their names. In 1836, in their own article, they studied the boundary value problem

$$-y''+q(x)y=\lambda y, \quad a\leqslant x\leqslant b \tag{1.3}$$

$$\begin{cases} y(a)cos\alpha+y'(a)sin\alpha=0 \\ y(b)cos\beta+y'(b)sin\beta=0 \end{cases} \tag{1.4}$$

in a finite interval (Sturm 1836, Liouville 1836).

Since the beginning of 18th century, various investigations have been done by various authors, but with the pioneering of Naimark. In his first study, Naimark considered the Sturm–Liouville operator generated by the Sturm–Liouville equation

$$-y''+q(x)y=\lambda y, \quad x\in\mathbb{R}_+:=[0,\infty) \tag{1.5}$$

and the initial condition

$$y'(0) - hy(0) = 0, \tag{1.6}$$

in the Hilbert space

$$L_2(0,\infty) := \left\{ f : \int_0^\infty |f(x)|^2 dx < \infty \right\},$$

where q is a complex potential and λ is a spectral parameter (Naimark 1960).

In spectral theory, operators are classified according to their domains or the potential function q. If the domain region of the operator is finite and the coefficients are continuous, operator is said to be *regular*, if the domain region is infinite or the coefficients are discontinuous functions, operator is said to be *singular*. Furthermore, if the potential function of the *BVP* and the parameters of the boundary conditions are real valued, then the operator corresponding to such *BVP* is equal to its adjoint operator. That kind of operators are called *selfadjoint operators*. On the other hand, if the potential function of *BVP* or the parameters of the boundary conditions are complex valued, then the operator corresponding to such *BVP* is called a *non-selfadjoint operator*. Non-selfadjoint operators are not equal to their adjoints. Clearly, the operator in Naimark's study is singular and non-selfadjoint. Naimark proved that the spectrum of non-selfadjoint operator consists of the continuous spectrum, the eigenvalues and the spectral singularities. Keep in mind that, all eigenvalues of selfadjoint operators have to be real, but the eigenvalues of the non-selfadjoint operators may not be real in general. In non-selfadjoint case, operators have spectral singularities.

The spectral singularities are the poles of the kernel of the resolvent and are also embedded in the continuous spectrum, but they are not eigenvalues. The spectral analysis of differential operators with spectral singularities was investigated in detail by several mathematicians. Schwartz studied the spectral singularities of a certain class of abstract linear operators in a Hilbert space and provided new definitions to the literature (Schwartz 1960). The sets of the spectral singularities for closed linear operators on a Banach space was given by Nagy (Nagy 1986). Nagy showed that the set of spectral singularities defined according to his general definition coincides in the case of differential operators as defined by Naimark and Lyance (Naimark 1960, Lyance 1967).

Lyance considered the effects of spectral singularities in the spectral expansion of Sturm–Liouville operators in terms of the principal functions. Pavlov established the dependence of the structure of the spectral singularities of Sturm–Liouville operators on the behaviour of the potential function of infinity (Pavlov 1967). For some of the other techniques for differential and abstract operators, we may consult with the studies of Kemp, Naimark, Gasymov, Maksudov and Allahverdiev (Kemp 1958, Naimark 1968, Gasymov 1968, Gasymov and Maksudov 1972, Maksudov and Allahverdiev 1984). Another approach for the discussion of the spectral analysis of the non-selfadjoint differential operators has been given by Marchenko (Marchenko 1963). Such results of differential operators have been extended to the operators on the whole axis over the years (Blashak 1966, Marchenko 1986).

One of the greatest contribution to the spectral theory of differential equations is that Sturm–Liouville equations have Jost solutions which are defined by Marchenko, so that the set of eigenvalues and the set of spectral singularities can be characterized in terms of the zeros of Jost functions, many crucial properties such as the finiteness and multiplicities of eigenvalues and spectral singularities can be obtained. Since the Jost solutions are of great importance on the spectral and scattering analysis of quantum calculus, various authors have been dealt with Jost solutions in their studies (Krall 1965, Gasymov and Levitan 1966, Jaulent and Jean 1972, Levitan and Sargsjan 1975, Tunca and Bairamov 1999, etc.).

As a result of developing technology, differential equations have become very insufficient to overcome the requirements of new modellings. So, the theory of difference equations has grown at an accelerated pace in the past decade. For the studies on the spectral and scattering theory of difference equations, we refer to various references (Atkinson 1964, Guseinov 1976, Agarwal and Wong 1997, Agarwal 2000, Krall *et al.* 2001, Bairamov *et al.* 2001, Adıvar and Bairamov 2001, Adıvar and Bairamov 2003). Also, the modellings of certain problems in engineering, physics, control theory and quantum mechanics and other areas have led to a rapid development of the theory of quadratic pencil of Schrödinger, Klein-Gordon and Dirac equations (Gasymov and Levitan 1966, Levitan and Sargsjan 1991, Krall *et al.* 1999, Bairamov and Karaman 2002).

The main objective of this master thesis is to investigate the spectral properties of an impulsive Dirac equation. As is known, Dirac systems are of the form

$$\begin{cases} y_2' + p(x)y_1 + r(x)y_2 = \lambda y_1 \\ -y_1' + r(x)y_1 + q(x)y_2 = \lambda y_2, \end{cases} \tag{1.7}$$

where λ is a complex parameter, p, q and r are real-valued and integrable functions on some interval $(a,b) \subseteq \mathbb{R}$. The system (1.7) plays a central role in relativistic quantum theory. In fact, the system (1.7) corresponds to Dirac's radial relativistic wave equation for a particle in a central field (Roos and Sangren 1962, Levitan and Sargsjan 1991). One of the important problems of the system (1.7) is to describe solutions belonging to squarely integrable space on some singular intervals, that is, intervals in which at least one of the potentials p, q and r increase boundlessly. In order to understand the nature of basic particles, the Dirac equation has been studied for many years. The study of Dirac equations has recently attracted interest on the solutions of physical systems.

Before introducing the so-called impulsive Dirac system, let us shortly give an overview on the existing literature of impulsive equations.

In all the problems of differential, difference and Dirac equations mentioned above, the processes in which mathematical modelling is done do not always continue uninterruptedly and continuously. While a physical or chemical phenomena takes place, the instantaneous and rapid situation changes may occur at some stages of these processes. Although, when compared with the total process, period of this sharp change is neglectable, the functioning of the system changes. These short-term effects are called "impulse (jump)" effects, the problems that occur as a result of impulse effects are called "impulsive problems". Naturally, in an impulsive problem, additional conditions arise in discontinuity points called "impulsive conditions", "transmission conditions" or "point interactions". Impulsive problems can be seen in various scientific facts. To be more clear, many biological phenomena involving thresholds, bursting rhythm models in medicine, optimal control models in economics, pharmacokinetics, and frequency modulated systems do exhibit impulsive effects (Lakshmikantham *et al.* 1989). As a natural response to developing technology, the interest in impulsive equations has grown and impulsive equations have been a subject of both theoretical and experimental researchs. The theory is improved very well by the in differential equations (Lakshmikantham *et al.* 1989, Samoilenko and Perestyuk 1995, Bainov and Simeonov 1998), and in difference equations, as well (Zhang 2002, He and Zhang 2004, Wang and Wang 2006, Bairamov *et al.* 2019). Notably, spectral theory of singular and regular Sturm–Liouville operators with impulsive effects has been studied extensively in previous papers (Mukhtarov *et al.* 2004, Mostafazadeh and Dehnavi 2009, Mostafazadeh 2011, Ugurlu and Bairamov 2013).

This master thesis is concerned with the investigation of an impulsive Dirac operator. In the next section, some well known concepts of spectral analysis are mentioned and required preliminary informations are given. Then, the Dirac system is obtained on the whole axis and related operator is introduced in the Hilbert space $L_2(\mathbb{R},\mathbb{C}^2)$. The main results and theorems of this thesis are given in Chapter 4, in which the situations of spectral singularities and eigenvalues of an impulsive operator corresponding to an impulsive Dirac system are investigated. Moreover, some particular occasions, where the point interaction at a single $x = 0$ point possess certain symmetries are examined. Throughout this master thesis, the paper (Bairamov *et al.* 2020) is used as a main reference and investigated in detail in Chapter 4.

2. BASIC CONCEPTS AND DEFINITIONS

In this section, we establish some basic properties of spectral theory and recall some definitions from functional analysis. The results of following sections are mostly based on the concepts of this section.

Definition 2.1. (Vector space) A vector space (or linear space) over a field K is a nonempty set X of elements $x, y, \ldots$ (called vectors) together with two algebraic operations. First of these operations is vector addition having the following properties for all $x, y, z \in X$:

(i) $x + y \in X$ (Closure property)

(ii) $x + y = y + x$ (Commutative property)

(iii) $x + (y + z) = (x + y) + z$ (Associative property)

(iv) There exists a vector θ, called the zero vector such that $x + \theta = x$ (Identity element property)

(v) There exists a vector $-x$ for every vector x such that $x + (-x) = \theta$ (Inverse element property).

The other operation is multiplication of vectors by scalars having the following properties for all $x, y \in X$ and for all α, β scalars:

(i) $\alpha x \in X$ (Closure property)

(ii) $\alpha(\beta x) = (\alpha\beta)x$ (Associative property)

(iii) $\alpha(x + y) = \alpha x + \alpha y$ (Distributive property)

(iv) $(\alpha + \beta)x = \alpha x + \beta x$ (Distributive property)

(v) There exists a vector $1 \in X$ such that $1x = x$ (Identity element property)

(vi) $0x = \theta$, where "0" is the zero scalar (Absorbing element property).

From the definition, we conclude that vector addition is a mapping $X \times X \to X$, whereas multiplication by scalars is a mapping $K \times X \to X$. The vector space X is denoted by $(X, +, .)$ and K is called the scalar field of X. Moreover, X is called a real vector space if $K = \mathbb{R}$ (the field of real numbers), and a complex vector space if $K = \mathbb{C}$ (the field of complex numbers).

Throughout this master thesis, we consider an impulsive Dirac operator. As known, operators are certain transformations between two vector spaces. But a vector space may not supply the needs of calculations, that's why we need to mention the definitions of linear operator, normed space, Hilbert space and inner product of two functions in the given Hilbert space.

Definition 2.2. (Linear operator) A linear operator T is an operator such that

(i) the domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field,

(ii) for all $x, y \in D(T)$ and scalars α, β

$$T(\alpha x+\beta y) = \alpha Tx+\beta Ty$$

satisfies.

Note that, the notation Tx is used instead of $T(x)$ which is a standard simplification in functional analysis (Kreyszig 1978).

Definition 2.3. (Normed space) A norm on a (real or complex) vector space X is a real valued function represented as

$$\|.\| : X \to \mathbb{R},$$

which has the properties

(i) $\| x \| \geqslant 0$

(ii) $\| x \| = 0 \Leftrightarrow x = 0$

(iii) $\| \alpha x \| = | \alpha | \| x \|$

(iv) $\| x+y \| \leqslant \| x \| + \| y \|$ (Triangle inequality),

here x and y are arbitrary vectors in X and α is any scalar.

A normed space X is a vector space with a norm function defined on it and denoted by $(X, \| \,.\, \|)$ (Kreyszig 1978).

Definition 2.4. (Inner product space) An inner product on a (real or complex) vector space X is a mapping represented as

$$<,>: X \times X \to K,$$

where K is a scalar field of X; that is, with every pair of vectors x and y, there is associated a scalar which is written as

$$< x, y >$$

and is called the inner product of x and y such that for all vectors x, y, z and scalars α, we have

(i) $< x+y, z > = < x, z > + < y, z >$

(ii) $<\alpha x,y>=\alpha<x,y>$

(iii) $<x,y>=\overline{<y,x>}$

(iv) $<x,x>\geqslant 0 \quad (<x,x>=0 \Leftrightarrow x=0)$.

An inner product space is a vector space X with an inner product on it and denoted by $(X,<,>)$. An inner product on X defines a norm on X given by

$$\| x \|= \sqrt{<x,x>}$$

(Kreyszig 1978).

Definition 2.5. The metric space X is said to be *complete* if every Cauchy sequence in X converges. A *Banach space* is a complete normed space (complete in the metric defined by the norm), a *Hilbert space* is a complete inner product space (complete in the metric defined by $\| x \|= \sqrt{<x,x>}$ norm) (Kreyszig 1978).

The theory of Hilbert spaces is richer than that of general Banach spaces.

Definition 2.6. ($L_p(-\infty,\infty)$ space) The vector space of measurable functions for which the p-th power of the absolute value is Lebesgue integrable on the interval $(-\infty,\infty)$ is said to be L_p space and represented as follows:

$$L_p(-\infty,\infty) := \left\{ f : \int_{-\infty}^{\infty} |f(x)|^p dx < \infty \right\}.$$

Note that, a L_p space forms a Banach space for $p \geqslant 1$. In particular, for $p=2$, L_2 space forms a Hilbert space. Obviously,

$$L_1(-\infty,\infty) := \left\{ f : \int_{-\infty}^{\infty} |f(x)| dx < \infty \right\}$$

is a Banach space with the norm

$$\| f \|_1 := \int_{-\infty}^{\infty} |f(x)| dx, \quad f \in L_1(-\infty,\infty).$$

On the other hand,

$$L_2(-\infty,\infty) := \left\{ f : \int\limits_{-\infty}^{\infty} |f(x)|^2 dx < \infty \right\}$$

is a Hilbert space with the norm and inner product

$$\| f \|_2 := \left(\int\limits_{-\infty}^{\infty} |f(x)|^2 dx \right)^{\frac{1}{2}}, \quad f \in L_2(-\infty,\infty), \tag{2.1}$$

$$< f,g > := \int\limits_{-\infty}^{\infty} f(x)\overline{g(x)} dx, \quad f,g \in L_2(-\infty,\infty), \tag{2.2}$$

respectively.

Example 2.7. Let $f,g : (-\infty,\infty) \to \mathbb{R}$ are real valued functions such that

$$f(x) = e^{-|x|} \quad \text{and} \quad g(x) = \frac{1}{1+x}.$$

The function f belongs to both $L_1(-\infty,\infty)$ and $L_2(-\infty,\infty)$ spaces, whereas the function g only belongs to $L_2(-\infty,\infty)$, that is, $g \notin L_1(-\infty,\infty)$.

The remainder of this section is devoted to some basic definitions of spectral theory which have certainly contributed significantly to the rapid development of the theory of Sturm–Liouville operators, even Dirac operators that we investigate in this thesis.

Definition 2.8. Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ is a linear operator with domain $D(T) \subset X$. If the operator

$$T_\lambda := T - \lambda I,$$

where λ is a complex number and I is the identity operator on $D(T)$, has an inverse, we denote it by $R_\lambda(T)$, that is,

$$R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}$$

and call it the resolvent operator of T or simply, the resolvent of T

(Lusternik and Sobolev 1974).

Many properties of T_λ and R_λ depend on λ and spectral theory is concerned with those properties such as existence, boundedness, density, etc.

Definition 2.9. Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ is a linear operator with domain $D(T) \subset X$. A regular value λ of T is a complex number such that $R_\lambda(T)$ exists, bounded and is defined on a set which is dense in X. The resolvent set $\rho(T)$ of T is the set of all regular values λ of T and represented as follows:

$$\rho(T) := \left\{ \lambda : \lambda \in \mathbb{C}, \quad \begin{array}{l} R_\lambda(T) \text{ exists} \\ R_\lambda(T) \text{ is bounded} \\ \overline{D(R_\lambda(T))} = X \end{array} \right\}$$

(Lusternik and Sobolev 1974).

Definition 2.10. The complement of the resolvent set in the complex plane, that is, the set

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

is called the spectrum of T and every $\lambda \in \sigma(T)$ number is called a spectral value of T (Lusternik and Sobolev 1974).

Definition 2.11. The set of λ complex numbers of the spectrum such that $R_\lambda(T)$ does not exist is said to be point spectrum or discrete spectrum and represented as follows:

$$\sigma_d(T) := \{\lambda : \lambda \in \mathbb{C},\ R_\lambda(T) \text{ does not exist}\}$$

(Lusternik and Sobolev 1974).

It is time to mention in passing that the set $\sigma_d(T)$ is called the set of eigenvalues (bound states) of the operator T and all $\lambda \in \sigma_d(T)$ is called an eigenvalue or a bound state of T.

Definition 2.12. The set of λ complex numbers of the spectrum such that $R_\lambda(T)$ exists, is defined on a dense set in X, but unbounded is said to be the continuous spectrum and represented as follows:

$$\sigma_c(T) := \left\{ \lambda : \lambda \in \mathbb{C}, \quad \begin{array}{l} R_\lambda(T) \text{ exists} \\ R_\lambda(T) \text{ is unbounded} \\ \overline{D(R_\lambda(T))} = X \end{array} \right\}$$

(Lusternik and Sobolev 1974).

Definition 2.13. The set of λ complex numbers of the spectrum such that $R_\lambda(T)$ exists (and may be bounded or not) but does not satisfy the last condition, that is, the domain of $R_\lambda(T)$ is not dense in X is said to be the residual spectrum and represented as follows:

$$\sigma_r(T) := \left\{ \lambda : \lambda \in \mathbb{C}, \quad \begin{array}{l} R_\lambda(T) \text{ exists} \\ R_\lambda(T) \text{ is bounded} \\ \overline{D(R_\lambda(T))} \neq X \end{array} \right\}$$

(Lusternik and Sobolev 1974).

The sets given in Definition 2.11, Definition 2.12 and Definition 2.13 are pairwise disjoint and union of them generates the spectrum, then we write

$$\sigma(T) = \sigma_d(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Now, let us give the definitions of eigenvalues (bound states) and spectral singularities that we frequently use in the main section of this thesis.

Definition 2.14. Let X be a vector space and $T : X \to X$ be a linear operator. If the equation

$$Ty = \lambda y, \quad y \in D(T)$$

has a non-zero $y(x,\lambda_0)$ solution for $\lambda = \lambda_0$ complex number, then λ_0 is said to be an *eigenvalue* or a *bound state* of T, the function $y(x,\lambda_0)$ is said to be the *eigenfunction* or the *eigenvector* of T corresponding to λ_0 value (Naimark 1968).

Recall that, the set of eigenvalues are written in Definition 2.11.

Example 2.15. In $L_2(0,\pi)$, consider the differential operator L generated by the differential expression

$$\ell(y) := -y'', \qquad 0 \leqslant x \leqslant \pi$$

and the boundary conditions

$$y'(0) = y(\pi) = 0.$$

Obviously,

$$D(L) = \left\{ y : y \in L_2(0,\pi), \quad \begin{array}{l} (i)\, y'' \text{ exists} \\ (ii)\, y'' \in L_2(0,\pi) \\ (iii)\, y'(0) = y(\pi) = 0 \end{array} \right\}$$

denotes the domain of L. Let λ be an eigenvalue of L and y be an eigenfunction of L corresponding to λ. Then, we write

$$Ly = \lambda y, \quad y \neq 0, \; y \in D(L),$$

which implies the boundary value problem

$$-y'' = \lambda y \tag{2.3}$$

$$y'(0) = y(\pi) = 0. \tag{2.4}$$

Determining the eigenvalues of the operator L means determining the eigenvalues of the *BVP* (2.3)–(2.4). Hence, we need to find the λ values satisfying (2.3)–(2.4). Since the roots of the characteristic equation $r^2 + \lambda = 0$ of (2.3) are $r_{1,2} = \mp i\sqrt{\lambda}$, $y_1(x) = \cos(\sqrt{\lambda}x)$ and $y_2(x) = \sin(\sqrt{\lambda}x)$ are two linearly independent solutions of the homogeneous differential equation (2.3) for $\lambda > 0$. Then, the general solution of (2.3) can be written as

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \quad |c_1| + |c_2| \neq 0. \tag{2.5}$$

Moreover, since

$$y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x),$$

using the boundary condition $y'(0) = 0$, we have

$$y'(0) = \sqrt{\lambda}c_2 = 0.$$

Since $\lambda > 0$, we get

$$c_2 = 0.$$

Regulating (2.5), we rewrite the general solution as

$$y(x) = c_1 \cos(\sqrt{\lambda} x).$$

Similarly, substituting the other boundary condition $y(\pi) = 0$ into the last equation, we get

$$y(\pi) = c_1 \cos(\sqrt{\lambda}\pi) = 0.$$

Since $c_1 \neq 0$, we conclude that

$$\cos(\sqrt{\lambda}\pi) = 0,$$

which yields

$$\lambda_k = \left(k + \frac{1}{2}\right)^2, \quad k = 0, 1, 2, \ldots$$

On the other hand, in the case of $\lambda = 0$, Equation (2.3) turns into

$$y'' = 0. \tag{2.6}$$

Since the roots of the characteristic equation are repeated real zeros, we can write the general solution of (2.6) as

$$y(x) = c_1 x + c_2, \qquad |c_1| + |c_2| \neq 0.$$

Since

$$y'(x) = c_1,$$

we find

$$y'(0) = c_2 = 0$$

from the boundary condition. Moreover, using the next boundary condition, we see

$$y(\pi) = c_1 = 0,$$

this means that $y = 0$ is the only solution satisfying the boundary condition. It contradicts the definition of eigenfunction given in Definition 2.14. Therefore, $\lambda = 0$ can not be an eigenvalue, $y=0$ can not be an eigenfunction corresponding to $\lambda=0$. Consequently, the eigenvalues and the eigenfunctions of the operator L can be listed respectively as follows:

$$\lambda_k=\left(k+\frac{1}{2}\right)^2, \quad k=0,1,2,...$$
$$y_k(x)=\cos\left[\left(k+\frac{1}{2}\right)x\right], \quad k=0,1,2,...$$

Definition 2.16. A *spectral singularity* is said to be the λ complex number belonging continuous spectrum but not an eigenvalue, which is the pole of resolvent operator. The set of spectral singularities of T is denoted by $\sigma_{ss}(T)$.

3. THEORY OF DIRAC SYSTEMS

Let us consider the matrix equation

$$K\frac{d\psi}{dx}+P(x)\psi=\mu\psi, \quad \psi(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}, \quad x\in[0,\pi], \tag{3.1}$$

where

$$K=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ P(x)=\begin{pmatrix}P_{11}(x) & P_{12}(x)\\ P_{21}(x) & P_{22}(x)\end{pmatrix},\ P_{12}(x)=P_{21}(x),$$

P_{ij} are real valued functions which are continuous on the interval $[0,\pi]$ for $i,j=1,2$ and μ is a spectral parameter. It follows from (3.1) that

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\psi_1'(x)\\ \psi_2'(x)\end{pmatrix}+\begin{pmatrix}P_{11}(x) & P_{12}(x)\\ P_{21}(x) & P_{22}(x)\end{pmatrix}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}=\mu\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}.$$

Using the last equality, we have

$$\begin{pmatrix}\psi_2'(x)\\ -\psi_1'(x)\end{pmatrix}+\begin{pmatrix}P_{11}(x)\psi_1(x)+P_{12}(x)\psi_2(x)\\ P_{21}(x)\psi_1(x)+P_{22}(x)\psi_2(x)\end{pmatrix}=\begin{pmatrix}\mu\psi_1(x)\\ \mu\psi_2(x)\end{pmatrix}.$$

Then, we arrive at that the Equation (3.1) is equivalent to the system of two simultaneous first order ordinary differential equations

$$\begin{cases}\psi_2'+P_{11}(x)\psi_1+P_{12}(x)\psi_2=\mu\psi_1,\\ -\psi_1'+P_{21}(x)\psi_1+P_{22}(x)\psi_2=\mu\psi_2.\end{cases} \tag{3.2}$$

In the case that $P_{12}(x)=P_{21}(x)=0$, $P_{11}(x)=s(x)+m$, $P_{22}(x)=s(x)-m$, where s is a potential function and m is the mass of a particle, the system (3.2) is called a *one dimensional stationary Dirac system* in relativistic quantum theory (Levian and Sargsjan 1991). Let T be an orthogonal transformation in two dimensional space. As is well known, every orthogonal transformation in two dimensional space has a matrix of the form

$$T(x)=\begin{pmatrix}\cos\varphi(x) & -\sin\varphi(x)\\ \sin\varphi(x) & \cos\varphi(x)\end{pmatrix}$$

with respect to a fixed orthonormal basis. It can be noticed that the matrices T and K are commutable, that is, $TK = KT$. Substituting the value $\psi := T(x)z$ into (3.1) and multiplying both sides of the resulting equation by T^{-1} from the left, we obtain

$$T^{-1}K\frac{d}{dx}(Tz) + T^{-1}P(x)Tz = T^{-1}\mu Tz, \tag{3.3}$$

where

$$T^{-1} = \begin{pmatrix} cos\varphi(x) & sin\varphi(x) \\ -sin\varphi(x) & cos\varphi(x) \end{pmatrix}.$$

By the help of commutative property of the matrices T and K, (3.3) turns into

$$K\frac{dz}{dx} + \left(T^{-1}K\frac{d}{dx}T + T^{-1}P(x)T\right)z = \mu z. \tag{3.4}$$

By choosing $T^{-1}K\dfrac{d}{dx}T := U$, we have

$$\begin{aligned} U &= \begin{pmatrix} \cos\varphi(x) & \sin\varphi(x) \\ -\sin\varphi(x) & \cos\varphi(x) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\varphi'(x)\sin\varphi(x) & -\varphi'(x)\cos\varphi(x) \\ \varphi'(x)\cos\varphi(x) & -\varphi'(x)\sin\varphi(x) \end{pmatrix} \\ &= \begin{pmatrix} \varphi'(x) & 0 \\ 0 & \varphi'(x) \end{pmatrix} \end{aligned}$$

and similarly by choosing $T^{-1}P(x)T := V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$, we have

$$\begin{aligned} v_{11} &= P_{11}(x)\cos^2\varphi(x) + P_{12}(x)\sin 2\varphi(x) + P_{22}(x)\sin^2\varphi(x), \\ v_{12} &= \frac{1}{2}\left(P_{22}(x) - P_{11}(x)\right)\sin 2\varphi(x) + P_{12}(x)\cos 2\varphi(x), \\ v_{21} &= \frac{1}{2}(P_{22}(x) - P_{11}(x))\sin 2\varphi(x) + P_{12}(x)\cos 2\varphi(x), \\ v_{22} &= P_{11}(x)\sin^2\varphi(x) - P_{12}(x)\sin 2\varphi(x) + P_{22}(x)\cos^2\varphi(x). \end{aligned}$$

Thus, the matrix $U + V := Q$ gets the form

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix},$$

where

$$\begin{aligned}
q_{11} &= \varphi'(x) + P_{11}(x)\cos^2\varphi(x) + P_{12}(x)\sin 2\varphi(x) + P_{22}(x)\sin^2\varphi(x), \\
q_{12} &= \frac{1}{2}(P_{22}(x) - P_{11}(x))\sin 2\varphi(x) + P_{12}(x)\cos 2\varphi(x), \\
q_{21} &= \frac{1}{2}(P_{22}(x) - P_{11}(x))\sin 2\varphi(x) + P_{12}(x)\cos 2\varphi(x), \\
q_{22} &= \varphi'(x) + P_{11}(x)\sin^2\varphi(x) - P_{12}(x)\sin 2\varphi(x) + P_{22}(x)\cos^2\varphi(x).
\end{aligned}$$

Now, we shall select the function Q so that $q_{12}(x) = 0$. Thus, we write

$$\frac{1}{2}(P_{22}(x) - P_{11}(x))\sin 2\varphi(x) + P_{12}(x)\cos 2\varphi(x) = 0.$$

If $P_{11}(x) \neq P_{22}(x)$, then

$$\varphi(x) = \frac{1}{2}\arctan\left(\frac{2P_{12}(x)}{P_{11}(x) - P_{22}(x)}\right)$$

and the matrix Q takes the form

$$Q = \begin{pmatrix} q_{11}(x) & 0 \\ 0 & q_{22}(x) \end{pmatrix}.$$

Thus, the Equation (3.4) can be rewritten as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{dz}{dx} + \begin{pmatrix} q_{11}(x) & 0 \\ 0 & q_{22}(x) \end{pmatrix} z = \mu z. \tag{3.5}$$

Now, by choosing the function φ so that $tr(Q) = 0$, where $tr(Q)$ denotes the trace of the matrix Q, we arrive at

$$q_{11}(x) + q_{22}(x) = 2\varphi'(x) + P_{11}(x) + P_{22}(x) = 0,$$

which yields

$$\varphi(x) = -\frac{1}{2}\int_0^x (P_{11}(t) + P_{22}(t))\,dt.$$

Then, the Equation (3.4) turns into

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{dz}{dx} + \begin{pmatrix} q_{11}(x) & q_{12}(x) \\ q_{12}(x) & -q_{11}(x) \end{pmatrix} z = \mu z. \tag{3.6}$$

The Equations (3.5) and (3.6) are said to be the *canonical forms* of the Equation (3.1) or the system (3.2). In order to seek answers to some questions in spectral and scattering theory of (3.1), the canonical form (3.5), either the canonical form (3.6) is preferred in accordance with the convenience. For instance, in studying the asymptotic behaviour of bound states and eigenfunctions of (3.1) (with some homogeneous boundary conditions at 0 and π), it is useful to apply the canonical Equation (3.5). But, in studying such questions as the asymptotic distribution of the bound states of (3.1), given on an infinite interval and also inverse problem, it is useful to handle the canonical Equation (3.6).

By reducing the Equation (3.1) to the canonical form (3.5), we get the boundary value problem as follows:

$$\psi_2' - \{\mu + q_{11}(x)\}\,\psi_1 = 0, \quad \psi_1' + \{\mu + q_{22}(x)\}\,\psi_2 = 0, \tag{3.7}$$

$$\psi_1(0)\sin\alpha + \psi_2(0)\cos\alpha = 0, \tag{3.8}$$

$$\psi_1(\pi)\sin\beta + \psi_2(\pi)\cos\beta = 0, \tag{3.9}$$

where the functions q_{11} and q_{22} are continuous on the interval $[0,\pi]$. Suppose that the boundary value problem (3.7)–(3.9) has a nontrivial solution

$$\psi(x,\mu_0) = \begin{pmatrix} \psi_1(x,\mu_0) \\ \psi_2(x,\mu_0) \end{pmatrix}$$

for certain μ_0. Recall from Definition 2.14 that, we call the number μ_0 an eigenvalue of the *BVP* (3.7)–(3.9) and we call the solution $\psi(x,\mu_0)$ a vector valued eigenfunction corresponding to that eigenvalue.

Before giving two notable lemmas related with the eigenvalues of the *BVP* (3.7)–(3.9), it should be better to define the Hilbert space that we have been working on.

Let $L_2(I,\mathbb{C}^2)$ denote the Hilbert space consisting of all vector-valued $\psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

functions in $\mathbb{C}^2$ satisfying

$$\int_I \left(|\psi_1|^2+|\psi_2|^2\right)dx<\infty$$

with the usual inner product

$$<\psi,z>=\int_I \psi^T z dx,$$

where $I:=[0,\pi]$ and $\psi^T=(\psi_1,\psi_2)$ is the transpose of the vector ψ. Note that

$$\|\ \psi\ \|_I:=\int_I \left(|\psi_1|^2+|\psi_2|^2\right)dx \tag{3.10}$$

denotes the norm of the vector-valued function ψ in $L_2(I,\mathbb{C}^2)$.

Lemma 3.1. The vector valued eigenfunctions corresponding to different eigenvalues are orthogonal.

Proof. Suppose that μ_1,μ_2 are eigenvalues of the *BVP* (3.7)–(3.9) such that $\mu_1\neq\mu_2$ and ψ,z are eigenfunctions corresponding to μ_1 and μ_2, respectively. We need to prove that

$$<\psi,z>=\int_0^\pi \psi^T z dx=0.$$

Due to the fact that the functions $\psi(x,\mu_1)$ and $z(x,\mu_2)$ are solutions of the system (3.7), we obtain

$$\psi_2'(x,\mu_1)-\{\mu_1+q_{11}(x)\}\,\psi_1(x,\mu_1)=0,$$
$$\psi_1'(x,\mu_1)+\{\mu_1+q_{22}(x)\}\,\psi_2(x,\mu_1)=0,$$

and

$$z_2'(x,\mu_2)-\{\mu_2+q_{11}(x)\}\,z_1(x,\mu_2)=0,$$
$$z_1'(x,\mu_2)+\{\mu_2+q_{22}(x)\}\,z_2(x,\mu_2)=0.$$

Multiplying these equations by $-z_1(x,\mu_2),z_2(x,\mu_2),\psi_1(x,\mu_1)$ and $-\psi_2(x,\mu_1)$, respectively and adding together, we get

$$-\frac{d}{dx}\left\{\psi_1(x,\mu_1)z_2(x,\mu_2)-\psi_2(x,\mu_1)z_1(x,\mu_2)\right\}$$
$$=(\mu_1-\mu_2)\quad\left\{\psi_1(x,\mu_1)z_1(x,\mu_2)+\psi_2(x,\mu_1)z_2(x,\mu_2)\right\}.$$

Now, integrating last equation from 0 to π, we see that

$$\left\{-\psi_1(x,\mu_1)z_2(x,\mu_2)+\psi_2(x,\mu_1)z_1(x,\mu_2)\right\}|_0^\pi$$
$$=(\mu_1-\mu_2)\int_0^\pi\quad\left\{\psi_1(x,\mu_1)z_1(x,\mu_2)+\psi_2(x,\mu_1)z_2(x,\mu_2)\right\}dx,$$

where the left side vanishes by the help of the boundary conditions (3.8) and (3.9). Thus, we have

$$(\mu_1-\mu_2)\int_0^\pi\psi^T(x,\mu_1)z(x,\mu_2)dx$$
$$=(\mu_1-\mu_2)\int_0^\pi\quad\left\{\psi_1(x,\mu_1)z_1(x,\mu_2)+\psi_2(x,\mu_1)z_2(x,\mu_2)\right\}dx=0,$$

which proves the lemma because of the hypothesis, $\mu_1\neq\mu_2$. □

Lemma 3.2. All eigenvalues of the boundary value problem (3.7)–(3.9) are real.

Proof. Let μ_1 be an eigenvalue of the *BVP* (3.7)–(3.9) and $\psi(x,\mu_1)$ be an eigenfunction corresponding to μ_1. It is easy to see that $\mu_2=\overline{\mu_1}$ is also an eigenvalue of the *BVP* (3.7)–(3.9) and $\overline{\psi(x,\mu_1)}$ be an eigenfunction corresponding to $\overline{\mu_1}$. Then, by the previous lemma, we have

$$(\mu_1-\overline{\mu_1})\int_0^\pi\left\{|\psi_1(x,\mu_1)|^2+|\psi_2(x,\mu_1)|^2\right\}dx=0 \tag{3.11}$$

Since $\psi(x,\mu_1)\neq 0$, in view of (3.10) and Definition 2.3., we get

$$\int_0^\pi\left\{|\psi_1(x,\mu_1)|^2+|\psi_2(x,\mu_1)|^2\right\}dx>0.$$

Substituting the last inequality in (3.11), we obtain that $\mu_1-\overline{\mu_1}=0$, that is, $\mu_1\in\mathbb{R}$. □

4. IMPULSIVE DIRAC SYSTEMS

Let us introduce the Dirac operator L in $L_2(\mathbb{R},\mathbb{C}^2)$ generated by the equation

$$i\sigma_2\frac{d\psi}{dx}+m\sigma_3\psi=\lambda\psi,\quad \psi(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix},\quad x\in\mathbb{R}\backslash\{0\} \tag{4.1}$$

with the impulsive condition

$$\begin{pmatrix}\psi_1(0^+)\\ \psi_2(0^+)\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\psi_1(0^-)\\ \psi_2(0^-)\end{pmatrix},\ a,b,c,d\in\mathbb{C}, \tag{4.2}$$

where $\sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, m is the mass of a particle and λ is a spectral parameter.

Note that, Equation (4.1) is a special case of the Equation (3.1) for $P_{12}(x)=P_{21}(x)=0$. Furthermore, $x=0$ is the impulsive point of the problem (4.1)–(4.2) and the matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is used to continue the solution of (4.1) from $\mathbb{R}^-$ to $\mathbb{R}^+$.

Let $\psi(x^+)$ and $\psi(x^-)$ be the restrictions of ψ to the sets of positive and negative real numbers respectively, that is

$$\begin{cases}\psi(x^+):=\psi(x), & x\in\mathbb{R}^+\\ \psi(x^-):=\psi(x), & x\in\mathbb{R}^-\end{cases}$$

and

$$\begin{cases}\psi(0^+):=\lim\limits_{\varepsilon\to 0^+}\psi(\varepsilon)\\ \psi(0^-):=\lim\limits_{\varepsilon\to 0^-}\psi(\varepsilon)\end{cases}.$$

Introducing the two component wave function

$$\begin{cases} \Psi(x^+) := \begin{pmatrix} \psi_1(x^+) \\ \psi_2(x^+) \end{pmatrix}, \quad x \geqslant 0 \\ \\ \Psi(x^-) := \begin{pmatrix} \psi_1(x^-) \\ \psi_2(x^-) \end{pmatrix}, \quad x \leqslant 0 \end{cases} \tag{4.3}$$

we can restate the impulsive condition (4.2) as

$$\Psi(0^+) = B\Psi(0^-), \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{C} \tag{4.4}$$

by imposing the matching condition.

4.1 Spectral Singularities and Bound States of Impulsive Dirac Operator

One can easily deduce that, the functions

$$\eta_1(x) = \begin{pmatrix} \dfrac{\lambda+m}{k} ie^{ikx} \\ e^{ikx} \end{pmatrix}, \quad x \in \mathbb{R} \setminus \{0\} \tag{4.5}$$

and

$$\eta_2(x) = \begin{pmatrix} -\dfrac{\lambda+m}{k} ie^{-ikx} \\ e^{-ikx} \end{pmatrix}, \quad x \in \mathbb{R} \setminus \{0\} \tag{4.6}$$

are the linearly independent solutions of Equation (4.1) for $\lambda \in \mathbb{C} \setminus \{\pm m\}$, where $k := \sqrt{\lambda^2 - m^2}$. Thus, the general solution of (4.1) can be expressed by

$$\psi(x^-) = A_- \eta_1(x) + B_- \eta_2(x), \quad x < 0,$$
$$\psi(x^+) = A_+ \eta_1(x) + B_+ \eta_2(x), \quad x > 0.$$

Taking into account impulsive condition (4.4), we have

$$A_+ \eta_1(0^+) + B_+ \eta_2(0^+) = B\left[A_- \eta_1(0^-) + B_- \eta_2(0^-)\right].$$

Regulating the last equation, we obtain

$$A_+\begin{pmatrix}\dfrac{\lambda+m}{k}i\\1\end{pmatrix}+B_+\begin{pmatrix}-\dfrac{\lambda+m}{k}i\\1\end{pmatrix}=B\left[A_-\begin{pmatrix}\dfrac{\lambda+m}{k}i\\1\end{pmatrix}+B_-\begin{pmatrix}-\dfrac{\lambda+m}{k}i\\1\end{pmatrix}\right].$$

By the previous equation, the equality

$$\begin{pmatrix}\dfrac{\lambda+m}{k}i & -\dfrac{\lambda+m}{k}i\\1 & 1\end{pmatrix}\begin{pmatrix}A_+\\B_+\end{pmatrix}=B\begin{pmatrix}\dfrac{\lambda+m}{k}i & -\dfrac{\lambda+m}{k}i\\1 & 1\end{pmatrix}\begin{pmatrix}A_-\\B_-\end{pmatrix}$$

can be obtained easily. Choosing

$$N:=\begin{pmatrix}\dfrac{\lambda+m}{k}i & -\dfrac{\lambda+m}{k}i\\1 & 1\end{pmatrix}, \tag{4.7}$$

we obtain a transfer matrix $\mathscr{M}$ satisfying

$$\begin{pmatrix}A_+\\B_+\end{pmatrix}=\mathscr{M}\begin{pmatrix}A_-\\B_-\end{pmatrix}, \tag{4.8}$$

such that

$$\mathscr{M}:=N^{-1}BN, \tag{4.9}$$

where $\mathscr{M}=(\mathscr{M}_{ij});\ i,j=1,2.$

On the other side, (4.1) admits a pair of solutions $\psi_{\lambda\pm}(x)=\begin{pmatrix}\psi^{(1)}_{\lambda\pm}(x)\\\psi^{(2)}_{\lambda\pm}(x)\end{pmatrix}$ fulfilling the asymptotic boundary conditions

$$\lim_{x\to-\infty}\frac{\psi_{\lambda_-}(x)}{\eta_2(x)}=1,\quad \lambda_-\in\overline{\mathbb{C}}_+$$

and

$$\lim_{x\to\infty}\frac{\psi_{\lambda_+}(x)}{\eta_1(x)}=1,\quad \lambda_+\in\overline{\mathbb{C}}_+,$$

where $\overline{\mathbb{C}}_+:=\{\lambda:\lambda\in\mathbb{C},\ \mathrm{Im}\lambda\geqslant 0\}$. These are called Jost solutions of Equation (4.1). By the way, the Wronskian of the Jost solutions $\psi_{\lambda\pm}$ of (4.1) for the operator acting in $L_2(\mathbb{R},\mathbb{C}^2)$ is defined as

$$W[\psi_{\lambda_+},\psi_{\lambda_-}] := \begin{vmatrix} \psi^{(1)}_{\lambda_+}(x) & \psi^{(1)}_{\lambda_-}(x) \\ \psi^{(2)}_{\lambda_+}(x) & \psi^{(2)}_{\lambda_-}(x) \end{vmatrix} = \psi^{(1)}_{\lambda_+}(0)\psi^{(2)}_{\lambda_-}(0) - \psi^{(2)}_{\lambda_+}(0)\psi^{(1)}_{\lambda_-}(0).$$

Keep in mind that, a spectral singularity of an operator is a point λ of the continuous spectrum of that operator such that ψ_{λ_-} and ψ_{λ_+} are linearly dependent, that is, the solutions have a vanishing Wronskian.

Now, consider the right- and left-going scattering solutions of $L\psi = \lambda\psi$ that we denote by ψ^r_λ and ψ^l_λ, respectively, which are expressed as

$$\psi^r_\lambda(x) = \begin{cases} A^-_+\eta_1(x) + B^-_+\eta_2(x), & x \to +\infty \\ A^-_-\eta_1(x) + B^-_-\eta_2(x), & x \to -\infty \end{cases} \tag{4.10}$$

and

$$\psi^l_\lambda(x) = \begin{cases} A^+_+\eta_1(x) + B^+_+\eta_2(x), & x \to +\infty \\ A^+_-\eta_1(x) + B^+_-\eta_2(x), & x \to -\infty, \end{cases} \tag{4.11}$$

where $A^\pm_\pm$ and $B^\pm_\pm$ are λ-dependent complex coefficients. Note that, denoting the coefficients $A_\pm$ and $B_\pm$ for the Jost solutions $\psi_{\lambda\pm}$ by $A^\pm_\pm$ and $B^\pm_\pm$, we get ψ^r_λ and ψ^l_λ. Hence, comparing right- and left-going scatterings with Jost solutions, we find that ψ^r_λ and ψ^l_λ are proportional to the Jost solutions $\psi_{\lambda-}$ and $\psi_{\lambda+}$, respectively. Also, at a spectral singularity λ, the scattering solutions ψ^r_λ and ψ^l_λ become linearly dependent.

Theorem 4.1. The following asymptotic equations hold.

$$W[\psi^l_\lambda,\psi^r_\lambda] = \frac{2i(\lambda+m)\mathscr{M}_{22}}{k}, \quad x \to +\infty, \tag{4.12}$$

$$W[\psi^l_\lambda,\psi^r_\lambda] = \frac{2i(\lambda+m)\mathscr{M}_{22}}{k\det\mathscr{M}}, \quad x \to -\infty. \tag{4.13}$$

Proof. The right- and left-going scattering solutions are defined in terms of their asymptotic behaviours

$$\psi^r_\lambda(x) \to \eta_2(x), \quad x \to -\infty$$

and

$$\psi^l_\lambda(x) \to \eta_1(x), \quad x \to +\infty,$$

respectively. Thus, using (4.10) and (4.11), we uniquely get

$$A^+_+ = B^-_- = 1, \quad A^-_- = B^+_+ = 0. \tag{4.14}$$

Next, for the left-going scattering solution, using (4.8) and (4.11), we write

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathscr{M}_{11} & \mathscr{M}_{12} \\ \mathscr{M}_{21} & \mathscr{M}_{22} \end{pmatrix} \begin{pmatrix} A^+_- \\ B^+_- \end{pmatrix}.$$

This implies that

$$A^+_- = \frac{\mathscr{M}_{22}}{\det \mathscr{M}}, \quad B^+_- = -\frac{\mathscr{M}_{21}}{\det \mathscr{M}}. \tag{4.15}$$

Similarly, for the right-going scattering solution, we get

$$\begin{pmatrix} A^-_+ \\ B^-_+ \end{pmatrix} = \begin{pmatrix} \mathscr{M}_{11} & \mathscr{M}_{12} \\ \mathscr{M}_{21} & \mathscr{M}_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

using (4.8) and (4.10). This implies that

$$A^-_+ = \mathscr{M}_{12}, \quad B^-_+ = \mathscr{M}_{22}. \tag{4.16}$$

Now, substituting these eight coefficients into (4.10) and (4.11), we can express the right- and left-going scattering solutions as

$$\psi^r_\lambda(x) = \begin{cases} \mathscr{M}_{12}\eta_1(x) + \mathscr{M}_{22}\eta_2(x), & x \to +\infty \\ \eta_2(x), & x \to -\infty \end{cases} \tag{4.17}$$

and

$$\psi_\lambda^l(x) = \begin{cases} \eta_1(x), & x \to +\infty \\ \frac{\mathscr{M}_{22}}{\det \mathscr{M}}\eta_1(x) - \frac{\mathscr{M}_{21}}{\det \mathscr{M}}\eta_2(x), & x \to -\infty. \end{cases} \tag{4.18}$$

It is easily concluded from the independency of Wronskian from x that, (4.17) and (4.18) can be used in order to calculate the Wronskian of the Jost solutions for $x \to -\infty$ and for $x \to +\infty$.

For $x \to -\infty$, we obtain

$$\begin{aligned} W[\psi_\lambda^l, \psi_\lambda^r] &= \left\{ \left(\frac{\mathscr{M}_{22}}{\det \mathscr{M}}\eta_1^{(1)}(x) - \frac{\mathscr{M}_{21}}{\det \mathscr{M}}\eta_2^{(1)}(x) \right) \eta_2^{(2)}(x) \right. \\ &\qquad \left. - \left(\frac{\mathscr{M}_{22}}{\det \mathscr{M}}\eta_1^{(2)}(x) - \frac{\mathscr{M}_{21}}{\det \mathscr{M}}\eta_2^{(2)}(x) \right) \eta_2^{(1)}(x) \right\} \\ &= \left\{ \frac{\lambda + m}{k \det \mathscr{M}} i \left(\mathscr{M}_{22} e^{ikx} + \mathscr{M}_{21} e^{-ikx} \right) e^{-ikx} \right. \\ &\qquad \left. + \frac{\lambda + m}{k \det \mathscr{M}} i \left(\mathscr{M}_{22} e^{ikx} - \mathscr{M}_{21} e^{-ikx} \right) e^{-ikx} \right\} \\ &= \frac{\lambda + m}{k \det \mathscr{M}} i(\mathscr{M}_{22} + \mathscr{M}_{21}) + \frac{\lambda + m}{k \det \mathscr{M}} i(\mathscr{M}_{22} - \mathscr{M}_{21}) \\ &= \frac{2i(\lambda + m)\mathscr{M}_{22}}{k \det \mathscr{M}}. \end{aligned}$$

For $x \to +\infty$, we see

$$\begin{aligned} W[\psi_\lambda^l, \psi_\lambda^r] &= \left\{ \eta_1^{(1)}(x) \left[\mathscr{M}_{12}\eta_1^{(2)}(x) + \mathscr{M}_{22}\eta_2^{(2)}(x) \right] \right. \\ &\qquad \left. - \eta_1^{(2)}(x) \left[\mathscr{M}_{12}\eta_1^{(1)}(x) + \mathscr{M}_{22}\eta_2^{(1)}(x) \right] \right\} \\ &= \left\{ \frac{\lambda + m}{k} i e^{ikx} \left(\mathscr{M}_{12} e^{ikx} + \mathscr{M}_{22} e^{-ikx} \right) \right. \\ &\qquad \left. - \frac{\lambda + m}{k} i e^{ikx} \left(\mathscr{M}_{12} e^{ikx} - \mathscr{M}_{22} e^{-ikx} \right) \right\} \\ &= \frac{\lambda + m}{k} i(\mathscr{M}_{12} + \mathscr{M}_{22}) - \frac{\lambda + m}{k} i(\mathscr{M}_{12} - \mathscr{M}_{22}) \\ &= \frac{2i(\lambda + m)\mathscr{M}_{22}}{k}. \end{aligned}$$

A direct consequence of (4.12) and (4.13) is

$$\det \mathscr{M} = \det B = ad - bc = 1. \tag{4.19}$$

The impulsive conditions violating this condition is called anomalous impulsive conditions. □

By the help of Theorem 4.1 and (4.19), we can give the following corollary.

Corollary 4.2. A necessary and sufficient condition to investigate the bound states and spectral singularities of the impulsive Dirac operator L is to investigate the zeros of the function $\mathscr{M}_{22}$.

Then, we need to get the $\mathscr{M}_{22}$ component of the matrix $\mathscr{M}$ by using (4.9). Since

$$N^{-1} = \frac{k}{2i(\lambda+m)} \begin{pmatrix} 1 & \frac{\lambda+m}{k}i \\ -1 & \frac{\lambda+m}{k}i \end{pmatrix},$$

we write

$$\begin{aligned} \mathscr{M} &= \frac{k}{2i(\lambda+m)} \begin{pmatrix} 1 & \frac{\lambda+m}{k}i \\ -1 & \frac{\lambda+m}{k}i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{\lambda+m}{k}i & -\frac{\lambda+m}{k}i \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2\xi} \begin{pmatrix} a+\xi c & b+\xi d \\ -a+\xi c & -b+\xi d \end{pmatrix} \begin{pmatrix} \xi & -\xi \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2\xi} \begin{pmatrix} c\xi^2+(a+d)\xi+b & -c\xi^2+(d-a)\xi+b \\ c\xi^2+(d-a)\xi-b & -c\xi^2+(a+d)\xi-b \end{pmatrix}, \end{aligned} \tag{4.20}$$

where

$$\xi := \frac{\lambda+m}{k}i.$$

Since the spectral singularities and bound states of the impulsive Dirac operator L are found by way of the zeros of $\mathscr{M}_{22}$, according to (4.20), they correspond to λ values, i.e. $\mathscr{M}_{22}(\xi)=0$, that is,

$$c\xi^2-(a+d)\xi+b=0. \tag{4.21}$$

For this reason, with regard to te definitions of bound states and spectral singularities of an operator in $L_2(\mathbb{R},\mathbb{C}^2)$, we introduce the sets of bound states and spectral singularities of impulsive Dirac operator L as

$$\sigma_d(L) = \left\{\lambda : \lambda \in \mathbb{C}\setminus\mathbb{R}^*,\ \lambda = \left(1 - \frac{2}{\xi^2+1}\right)m,\ \mathscr{M}_{22}(\xi) = 0\right\} \tag{4.22}$$

and

$$\sigma_{ss}(L) = \left\{\lambda : \lambda \in \mathbb{R}^*,\ \lambda = \left(1 - \frac{2}{\xi^2+1}\right)m,\ \mathscr{M}_{22}(\xi) = 0\right\}, \tag{4.23}$$

where $\mathbb{R}^* := (-\infty, -m) \cup (m, +\infty)$.

In order to study numerical properties of the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$, we investigate the zeros of the Equation (4.21). For this purpose, the following cases should be investigated.

Case (I) Let $c \neq 0$.

In this case, it follows from (4.21) that

$$\xi_{1,2} = \frac{(a+d) \mp \sqrt{(a+d)^2 - 4bc}}{2c},$$

that is,

$$\xi_{1,2} = \frac{a+d}{2c} \mp \sqrt{\left(\frac{a+d}{2c}\right)^2 - \frac{b}{c}}\ .$$

Then, it can be written

$$\xi_{1,2} = u \mp \sqrt{u^2 - v}\ ,$$

where

$$u := \frac{a+d}{2c}, \quad v := \frac{b}{c}.$$

Finally, we obtain

$$\lambda = \left(1 - \frac{2}{\left(u \mp \sqrt{u^2-v}\right)^2 + 1}\right)m.$$

Thus, bound states appear whenever

$$\operatorname{Re}\left(u \mp \sqrt{u^2 - v}\right) \neq 0 \tag{4.24}$$

and spectral singularities exist whenever

$$\operatorname{Re}\left(u \mp \sqrt{u^2 - v}\right) = 0, \quad \operatorname{Im}\left(u \mp \sqrt{u^2 - v}\right) \neq \mp 1. \tag{4.25}$$

Case (II) Let $c = 0$ and $a + d = tr(B) \neq 0$.
In this case, (4.21) gives

$$\xi = \frac{b}{a+d}$$

and therefore, we have

$$\lambda = \left(1 - \frac{2}{\left(\frac{b}{a+d}\right)^2 + 1}\right) m.$$

Thus, a bound state appears whenever

$$\operatorname{Re}\left(\frac{b}{a+d}\right) \neq 0$$

and a spectral singularity exists if

$$\operatorname{Re}\left(\frac{b}{a+d}\right) = 0, \quad \operatorname{Im}\left(\frac{b}{a+d}\right) \neq \mp 1.$$

Case (III) Let $c = 0$ and $a + d = tr(B) = 0$.
In that case, condition of the existence of bound states or spectral singularities, namely $\mathscr{M}_{22} = 0$, implies that $b = 0$. Therefore, (4.20) gives $B = a\sigma_3$ and $\mathscr{M} = -a\sigma_1$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, since $\mathscr{M}_{22}$ vanishes identically, $\mathscr{M}$ is independent of λ and the impulsive condition is anomalous for $a \neq \mp i$.

Now, in consideration of three cases above, the following theorem can be given.

Theorem 4.3. Assume that $c \neq 0$. For the impulsive condition (4.4), we can summarize the conditions for the existence of bound states and spectral singularities as follows:

(i) There is a bound state located at $\lambda = \left(1 - \frac{2}{\xi^2+1}\right)m$, where $\mathrm{Re}\xi \neq 0$.

(ii) There is a spectral singularity located at $\lambda = \left(1 - \frac{2}{\xi^2+1}\right)m$, for pure imaginary $\xi \neq \mp i$.

4.2 $\mathscr{P}$, $\mathscr{T}$ and $\mathscr{P}\mathscr{T}$–Symmetries

In this subsection, the definitions of $\mathscr{P}$,$\mathscr{T}$ and $\mathscr{P}\mathscr{T}$–symmetries on the impulsive condition (4.4) for the Dirac system will be given. Moreover, their spectral singularities and bound states will be examined.

4.2.1 $\mathscr{P}$–Symmetry

Definition 4.4. Suppose that $\mathscr{P}$ is the parity (reflection) operator acting in the space of all differentiable complex vector valued functions $\psi(x) = \begin{pmatrix} \psi^{(1)}(x) \\ \psi^{(2)}(x) \end{pmatrix}$, $\psi : \mathbb{R} \to \mathbb{C}^2$. Then for all $x \in \mathbb{R}$, we have

$$(\mathscr{P}\psi)(x) := \psi(-x).$$

Definition 4.5. The impulsive condition (4.4) has $\mathscr{P}$–symmetry (or (4.4) is $\mathscr{P}$–invariant) if

$$(\mathscr{P}\Psi)(0^+) = B(\mathscr{P}\Psi)(0^-), \tag{4.26}$$

where the action of $\mathscr{P}$, on a two component wave function Ψ, is defined componentwise.

Theorem 4.6. The impulsive condition (4.4) has $\mathscr{P}$–symmetry if and only if

$$B = B^{-1}. \tag{4.27}$$

Proof. Let the matrix B be equal to its inverse. By the help of (4.4) and definition of parity operator $\mathscr{P}$, we can write directly

$$
\begin{aligned}
B^{-1} = B \quad &\Rightarrow \quad B^{-1}\Psi(0^-) = B\Psi(0^-) \\
&\Rightarrow \quad B^{-1}\Psi(0^-) = \Psi(0^+) \\
&\Rightarrow \quad \Psi(0^-) = B\Psi(0^+) \\
&\Rightarrow \quad (\mathscr{P}\Psi)(0^+) = B(\mathscr{P}\Psi)(0^-),
\end{aligned}
$$

which yields that the impulsive condition (4.4) has $\mathscr{P}$–symmetry.
The contrary part of the proof proceeds similarly. □

Theorem 4.7. If the impulsive condition (4.4) has $\mathscr{P}$–symmetry, then the operator does not have any bound state and spectral singularity.

Proof. Assume that (4.4) has $\mathscr{P}$–symmetry. (4.27) gives

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
$$

and since $\det B = 1$, we have

$$
a = d = \mp 1, \quad b = c = 0.
$$

Thus, we obtain

$$
\xi = \frac{\lambda + m}{k} i = 0
$$

and this means that there is no bound state and spectral singularity. This completes the proof. □

4.2.2 $\mathscr{T}$–Symmetry

Definition 4.8. Let $\mathscr{T}$ be the time-reversal operator acting on complex vector valued functions $\psi : \mathbb{R} \to \mathbb{C}^2$ according to

$$
(\mathscr{T}\psi)(x) := \psi^*(x),
$$

where ψ^* is the complex conjugate of ψ.

Definition 4.9. The impulsive condition (4.4) has $\mathscr{T}$–symmetry (or (4.4) is the time-reversal invariant) if

$$(\mathscr{T}\Psi)(0^+) = B(\mathscr{T}\Psi)(0^-), \tag{4.28}$$

where the action of operator $\mathscr{T}$, on a two component wave function Ψ, is defined componentwise.

Theorem 4.10. The impulsive condition (4.4) has $\mathscr{T}$–symmetry if and only if B is real matrix.

Proof. Assume that the matrix B is real. Using (4.4) and definition of time-reversal operator $\mathscr{T}$, we have

$$\begin{aligned} B^* = B \quad &\Rightarrow \quad B^*\Psi^*(0^-) = B\Psi^*(0^-) \\ &\Rightarrow \quad \left[B\Psi(0^-)\right]^* = B\Psi^*(0^-) \\ &\Rightarrow \quad \Psi^*(0^+) = B\Psi^*(0^-) \\ &\Rightarrow \quad (\mathscr{T}\Psi)(0^+) = B(\mathscr{T}\Psi)(0^-), \end{aligned}$$

which yields that the impulsive condition (4.4) has $\mathscr{T}$–symmetry.
The contrary part of the proof proceeds similarly.

□

Theorem 4.11. If the impulsive condition (4.4) has $\mathscr{T}$–symmetry, then the situationss for the existence of bound states and spectral singularities can be summarized as below:

(i) If $c \neq 0$, then there exist spectral singularities whenever $a+d=0$, $b \neq c$ and $bc > 0$, also there exist bound states whenever $a+d \neq 0$.

(ii) If $c = 0$ and $a+d = tr(B) \neq 0$, then there exist only bound states whenever $b \neq 0$.

(iii) If $c = 0$ and $a+d = tr(B) = 0$, then the impulsive condition is anomalous.

Proof. (i) Assume that $c \neq 0$. Since (4.4) has $\mathscr{T}$–symmetry, from the previous theorem, we have $B = B^*$ such that $u, v \in \mathbb{R}$. Using (4.24) and (4.25), two special subcases can be examined.

Let $u^2 > v$. In this case, since

$$\mathrm{Re}\left(u \mp \sqrt{u^2-v}\right) = u \mp \sqrt{u^2-v}, \quad \mathrm{Im}\left(u \mp \sqrt{u^2-v}\right) = 0,$$

there exist spectral singularities whenever $v = 0$, that is, $b = 0$. Also there exist bound states if $b \neq 0$.
Let $u^2 < v$. In this case, since

$$\mathrm{Re}\left(u \mp \sqrt{u^2-v}\right) = u, \quad \mathrm{Im}\left(u \mp \sqrt{u^2-v}\right) = \sqrt{v-u^2},$$

there exist bound states if $u \neq 0$, that is, $a+d \neq 0$. Also, there exist spectral singularities if $u = 0$, $v > 0$ and $v \neq 1$, namely, $a+d = 0$, $b \neq c$ and $bc > 0$.

(ii) Assume that $c = 0$ and $a+d = tr(B) \neq 0$. Since $B = B^*$, using (4.21), we have

$$\xi = \frac{b}{a+d} \in \mathbb{R}.$$

There exist spectral singularities if and only if $b = 0$, that is, $\lambda = -m$. But it contradicts that λ can not be $\mp m$. Thus, the operator does not have any spectral singularity, it only has bound states whenever $b \neq 0$.

(iii) Assume that $c = 0$ and $a+d = tr(B) = 0$. In this case, b must be zero so that the Equation (4.21) is supplied. Since

$$b = c = 0, \quad d = -a,$$

we have $\det \mathscr{M} = -a^2$. Since $\det \mathscr{M} \neq 1$, the impulsive condition is anomalous.

□

4.2.3 $\mathscr{P}\mathscr{T}$–Symmetry

Definition 4.12. : The impulsive condition (4.4) has $\mathscr{P}\mathscr{T}$–symmetry (or (4.4) is $\mathscr{P}\mathscr{T}$–invariant) if

$$(\mathscr{P}\mathscr{T}\Psi)(0^+) = B(\mathscr{P}\mathscr{T}\Psi)(0^-), \tag{4.29}$$

where the action of $\mathscr{P}\mathscr{T}$, on a two component wave function Ψ, is defined componentwise.

Theorem 4.13. The impulsive condition (4.4) has $\mathscr{P}\mathscr{T}$–symmetry if and only if

$$B^* = B^{-1} \tag{4.30}$$

holds.

Proof. By the help of (4.4), definition of time-reversal operator $\mathscr{T}$, definition of parity operator $\mathscr{P}$ and (4.29) successively, we have

$$\begin{aligned}
\Psi(0^+) = B\Psi(0^-) \;&\Leftrightarrow\; \Psi^*(0^+) = B^*\Psi^*(0^-) \\
&\Leftrightarrow\; (\mathscr{T}\Psi)(0^+) = B^*(\mathscr{T}\Psi)(0^-) \\
&\Leftrightarrow\; [\mathscr{P}(\mathscr{T}\Psi)]\,(0^-) = B^*\,[\mathscr{P}(\mathscr{T}\Psi)]\,(0^+) \\
&\Leftrightarrow\; (B^*)^{-1}(\mathscr{P}\mathscr{T}\Psi)(0^-) = (\mathscr{P}\mathscr{T}\Psi)(0^+) \\
&\Leftrightarrow\; (B^*)^{-1}(\mathscr{P}\mathscr{T}\Psi)(0^-) = B(\mathscr{P}\mathscr{T}\Psi)(0^-) \\
&\Leftrightarrow\; (B^*)^{-1} = B \\
&\Leftrightarrow\; (B^*) = B^{-1}.
\end{aligned}$$

This completes the proof. □

Remark 4.14. If the impulsive condition (4.4) has $\mathscr{P}\mathscr{T}$–symmetry, then (4.30) gives

$$\begin{aligned}
B^* = B^{-1} \;&\Leftrightarrow\; \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&\Leftrightarrow\; ad-bc = 1, \quad a^* = d, \quad b^* = -b, \quad c^* = -c \\
&\Leftrightarrow\; ad-bc = 1, \quad a^* = d, \quad \mathrm{Re}b = 0, \quad \mathrm{Re}c = 0,
\end{aligned}$$

and so, in terms of the entries of matrix B, (4.30) is equivalent to

$$a+d = 2\mathrm{Re}a, \quad \mathrm{Re}b = 0, \quad \mathrm{Re}c = 0.$$

Theorem 4.15. Let the impulsive condition (4.4) have $\mathscr{P}\mathscr{T}$–symmetry. Then the conditions for the existence of bound states and spectral singularities can be summarized as follows:

(i) If $c \neq 0$, then there exist bound states whenever $\left(\frac{\mathrm{Re}a}{\mathrm{Im}c}\right)^2 + \frac{b}{c} < 0$, also there exist spectral singularities whenever $\left(\frac{\mathrm{Re}a}{\mathrm{Im}c}\right)^2 + \frac{b}{c} \geqslant 0$.

(ii) If $c = 0$ and $a + d = tr(B) \neq 0$, then there exist only spectral singularities if $\frac{\mathrm{Im}b}{2\mathrm{Re}a} \neq \mp 1$.

(iii) If $c = 0$ and $a + d = tr(B) = 0$, then the impulsive condition is anomalous if $a \neq \mp i$.

Proof. (i) Assume that $c \neq 0$. (4.21) implies that

$$\xi = -i\left\{\frac{\mathrm{Re}a}{\mathrm{Im}c} \mp \sqrt{\left(\frac{\mathrm{Re}a}{\mathrm{Im}c}\right)^2 + \frac{b}{c}}\right\}.$$

Thus, by (4.24) and (4.25), there exist bound states when $\left(\frac{\mathrm{Re}a}{\mathrm{Im}c}\right)^2 + \frac{b}{c} < 0$. Also there exist spectral singularities when $\left(\frac{\mathrm{Re}a}{\mathrm{Im}c}\right)^2 + \frac{b}{c} \geqslant 0$.

(ii) Assume that $c = 0$ and $a + d = tr(B) \neq 0$. In this case, (4.21) implies that

$$\xi = \frac{b}{a+d} = \frac{\mathrm{Im}b}{2\mathrm{Re}a} i.$$

Hence, by (4.25) there exist only spectral singularities when $\frac{\mathrm{Im}b}{2\mathrm{Re}a} \neq \mp 1$.

(iii) Assume that $c = 0$ and $a + d = tr(B) = 0$. In this case, b must be zero so that the Equation (4.21) is supplied. Then, we have

$$b = c = 0, \quad d = -a$$

which yields $\det \mathscr{M} = -a^2$. Therefore, the impulsive condition is anomalous for $a \neq \mp i$.

□

5. CONCLUSIONS AND RECOMMENDATIONS

In the last decade, the equations under impulse effect attract the attention of scientists due to the expanding application area in various disciplines and necessity in modeling real processes, this leads to a requirement of developing the theory. As in other fields of applied mathematics, the theory of impulsive equations has been studied by various mathematicians so far and it has been well developed for differential and difference equations. Also, the spectral and scattering theory for the one-dimensional Dirac equations has attracted a great amount of interest for years. However, there is no enough study concerning impulsive Dirac systems. Considering this inadequacy, studies on this subject have become substantial for us.

Throughout this master thesis, we study some spectral properties of impulsive Dirac system on the entire axis, two papers in this area were used as main references and investigated in detail (Mostafazadeh 2011 and Bairamov *et al.* 2020). In this thesis, we examine the eigenvalues (so-called bound states) and spectral singularities of a Dirac operator generated by the Dirac system with a single impulsive point at the origin. We emphasize that if a discontinuity appears in Dirac system, this may create some structural changes on the solutions of the system. Thus, the locations of eigenvalues and spectral singularities (if exist), depend on the choice of the constants given in impulsive condition. Unlike traditional methods, a transfer matrix is determined first, then some special cases of eigenvalues and spectral singularites are investigated by the help of the zeros of a quadratic polynomial obtained from this matrix. At last, we deduce some consequences about $\mathscr{P},\mathscr{T}$ and $\mathscr{P}\mathscr{T}$–symmetries which are directly related with mathematical physics. Further, as a continuation of this thesis, it is possible to investigate the similar impulsive problem on Dirac systems with multiple interior impulsive points.

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CURRICULUM VITAE

PERSONAL INFORMATION

Name Surname	:	ABDULLAH MHMOOD JASIM JASIM

EDUCATION STATUS

High school	:	Al-Furat High School for Boys, Salahuddin Governorate - Al- Tuz, 2014
Bachelor	:	University of Tikrit, College of Computer Sciences and Mathematics, Department of Mathematics, 2018
Master	:	Çankırı Karatekin University, Graduate School of Natural and Applied Sciences, Department of Mathematics, Mathematics Program, 2021