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HESITANT T-SPHERICAL FUZZY SETS AND THEIR APPLICATIONS IN DECISION-MAKING

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ABDULRASOOL HASAN SULTAN AL-HUSSEINAWI
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By Abdulrasool Hasan Sultan AL-HUSSEINAWI

August 2021

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

HESITANT T-SPHERICAL FUZZY SETS AND THEIR APPLICATIONS IN DECISION-MAKING

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Master of Science in Mathematics

Advisor: Assoc. Prof. Dr. Faruk KARAASLAN

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This thesis consists of eight sections. In Section 1, detailed literarature review and motivation of the paper are presented. In Section 2, generalizations of fuzzy sets. In Section 3, the concept of the hesitant T-spherical fuzzy (HTSF) set, the score and accuracy functions of the hesitant T-Spherical fuzzy elements, and the set operations between the hesitant T-Spherical fuzzy sets are defined and examples of them are given. In Section 4, Dombi operators among HTSF elements are introduced and based on Dombi operators, hesitant T-spherical Dombi fuzzy (HTSDF) weighted arithmetic averaging operator, HTSDF weighted geometric averaging operator, HTSDF ordered weighted arithmetic averaging operator and HTSDF ordered weighted geometric averaging operator are defined and some properties of them are investigated. In Section 5, a multi-criteria group decision making method has been developed in the HTSF environment and an algorithm for the proposed method is presented. In addition, an example of personal selection is given to demonstrate the operation of the proposed method. In Section 6, an analysis by the parameter in the aggregation operators presented is presented. In Section 7, a table and commentary showing comparisons between the generalizations of fuzzy sets are given. In Section 8, some explanations about the obtained results and works to be presented in the future.

2021, 46 pages

Keywords: Hesitant fuzzy sets, Spherical fuzzy sets, T-spherical fuzzy Sets, Hesitant T-spherical fuzzy sets, Dombi operators, Decision-making.

ÖZET

TEREDDÜTLÜ T-KÜRESEL BULANIK KÜMELER VE KARAR VERMEDEKİ UYGULAMALARI

Abdulrasool Hasan Sultan AL-HUSSEINAWI
Matematik, Yüksek Lisans
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Ağustos 2021

Bu tez sekiz bölümden oluşmaktadır. Bölüm 1'de, detaylı literatur incelemesi ve tezin motivasyonu sunulmuştur. Bölüm 2'de bulanık kümelerin genelleştirmeleri hakkında ön bilgiler verilmiştir. Bölüm 3'te tereddütlü T-Küresel bulanık (TTKB) küme kavramı, tereddütlü T-Küresel bulanık elemanların skor ve kesinlik fonksiyonları ve tereddütlü T-Küresel bulanık kümeler arasındaki küme işlemleri tanımlanarak örnekler verilmiştir. Bölüm 4'te, TTKB elemanlar arasındaki Dombi operatörleri tanıtılmakta ve Dombi operatörlerine dayanarak tereddütlü T-küresel Dombi bulanık (TTKDB) ağırlıklı aritmetik ortalama operatörü, TTKDB ağırlıklı geometrik ortalama operatörü, TTKDB sıralı ağırlıklı aritmetik ortalama operatörü ve TTKDBsıralı ağırlıklı geometrik ortalama operatörü tanımlanmış ve baz özellikleri incelenmiştir. Bölüm 5'te, TTKB ortamda bir çok kriterli grup karar verme yöntemi geliştirilmiş ve önerilen yöntem için bir algoritma sunulmuştur. Ayrıca önerilen yöntemin işleyişini göstermek için personel seçimini konu alan bir örnek verilmiştir. Bölüm 6'da sunulan birleştirme operatörlerindeki parametreye göre bir analiz sunulmuştur. Bölüm 7'de, bulanık kümelerin genelleştirmeleri arasındaki karşılaştırmaları gösteren bir tablo ve yorum verilmiştir. Bölüm 8'de, elde edilen sonuçlar ve gelecekte yapılacak çalışmalar hakkında bazı açıklamalar sunulmuştur.

2021, 46 sayfa

Anahtar Kelimeler: Tereddütlü bulanık kümeler, Küresel bulanık kümeler, T-küresel bulanık kümeler, Tereddütlü T-küresel bulanık kümeler, Dombi operatörleri, Karar verme.

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Abdulrasool Hasan Sultan AL-HUSSEINAWI
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CONTENTS

LIST OF SYMBOLS

$\mathscr{X}$	Universal set
$\mathscr{A}$	Many types of fuzzy sets
$\mathfrak{x}$	An element in the universal set
$s_{\mathscr{A}}(\mathfrak{x})$	Membership function of the element $\mathfrak{x}$
$i_{\mathscr{A}}(\mathfrak{x})$	The degree of neutral membership of $\mathfrak{x}$
$d_{\mathscr{A}}(\mathfrak{x})$	Non-membership function of the element $\mathfrak{x}$
$R_{\mathscr{A}}(\mathfrak{x})$	The degree of hesitancy of $\mathfrak{x}$
$\mathfrak{X}$	A non-empty set
$\ell_{\mathfrak{h}}$	Length of $HT-SFE$

LIST OF ABBREVIATIONS

FS	Fuzzy Set
IFS	Intutionistic Fuzzy Set
IFN	Intutionistic Fuzzy Number
PyFS	Pythagorean Fuzzy Set
PyFN	Pythagorean fuzzy number
q-ROFS	q-rung orthopair fuzzy set
q-ROFN	q-rung orthopair fuzzy number
SFS	Spherical Fuzzy Set
SFN	Spherical Fuzzy number
T-SFS	T-Spherical Fuzzy Set
T-SFN	T-Spherical Fuzzy Number
HFS	Hesitant Fuzzy Set
HFE	Hesitant Fuzzy Element
IHFS	Intuitionistic Hesitant Fuzzy Set
IHFE	Intuitionistic Hesitant Fuzzy Element
PyHFS	Pythagorean Hesitant Fuzzy Set
PyHFE	Pythagorean Hesitant Fuzzy Element
Hq-ROFS	Hesitant q-rung orthopair fuzzy set
Hq-ROFE	Hesitant q-rung orthopair fuzzy Element
HT-SFS	Hesitant T-spherical Fuzzy Set
HT-SFE	Hesitant T-spherical Fuzzy Element

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1. INTRODUCTION

The notion of fuzzy sets (FS) was defined by Zadeh in 1965 in order to model some problems involving uncertainty. The FS has found application in several different fields for instance, computer science, medical science, robotics and data mining. Zadeh (1965) characterized an FS by the membership function of which codomain is $[0,1]$. In an FS, the membership degree (MD) of an element is μ and the non-membership degree (NMD) is $1-\mu$. Namely, in an FS, hesitation degree of an element is accepted as "0". However, this perspective has some constraints. To overcome with these constraints, Atanassov (1986) introduced the concept of intuitionistic FS (IFS) as a generalization of FSs. An IFS is defined by assigning two values from the range $[0,1]$, named MD μ and NMD ν, under the condition $\mu+\nu \leq 1$ for all elements of the working universe. However, this set is not useful when $\mu+\nu > 1$. Therefore, Yager (2013a, 2013b) defined the Pythagorean FS (PyFS) as an extension of IFS with condition $\mu^2+\nu^2 \leq 1$. Another extension of IFS is Picture FS (PFS) defined by Cuong (2013a, 2013b). PFS is a beneficial tool for representing human opinion because a PFS can model judgments about an object or idea using degrees of yes, neutral, no, and refusal. A PFS is identified three degrees of an element, called MD (μ), abstinence degree (AD) or neutral degree (γ) and, NMD (ν) with the condition $0 \leq \mu+\gamma+\nu \leq 1$. Although PFS has wide applications in some field such as decision making (DM) (Cao 2020, Garg 2017, Joshi 2020, Peng *et al.* 2017, Tian *et al.* 2020, Wei 2017a, Wei 2018), similarity measure (Rafiq *et al.* 2019, Thao 2020, Wei 2017b, Wei and Gao 2018b, Wei 2018c), correlation coefficient (Ganie *et al.* 2020, Singh 2015), and clustering (Hao *et al.* 2016, Son 2016), it is not enough to modelling some problems when $\mu+\gamma+\nu > 1$. For this reason, Gündoğdu and Kahraman (2019a, 2019b) inaugurated the design of spherical FS (SFS) which is an expantion of PFS satisfying the condition $0 \leq \mu^2+\gamma^2+\nu^2 \leq 1$. They also studied on SFS operations and applications of this set in DM problems. Gündoğdu and Kahraman (2019a) proposed a DM method by integrating the SFS and TOPSIS method and gave an application of the proposed method in the choosing of hospital location. Mahmood *et al.* (2019a) defined the T-spherical FS (T-SFS) as an extension of the SFS with condition $0 \leq \mu^q+\gamma^q+\nu^q \leq 1$ and gave some applications in medical diagnosis and decision making problems of T-SFS and SFS. Ullah *et al.* (2018a) introduced the similarity measures for T-SFSs and presented an application in pattern recognition. Garg *et al.*

(2018) presented improved interactive aggregation operators for T-SFSs and studied on operational laws of these operators. Ullah *et al.* (2018b) described some ordered weighted geometric (OWG) and hybrid geometric (HG) operators and gave a numerical example involving multi-attribute decision making (MADM) problem. Ullah *et al.* (2019) presented the concept of interval valued T-SFS (IVT-SFS) and essential operations of them. In addition, two aggregation operators were defined including weighted averaging and weighted geometric operators for IVT-SFS and presented an MCDM method. Liu *et al.* (2019a) pointed out some limitations in operational laws of SFS and T-SFS and suggested some novel operational laws for SFS and T-SFS. They also introduced Power Muirhead Mean Operator for T-SFS by combining power average operator with Muirhead Mean operator and presented an MAGDM method constructive proposed operators. Recently, T-SFS has gained attention of researchers working on MCDM methods, MCGDM methods, and aggregation operators. For example, divergence measure of T-SFSs Wu *et al.* (2019), immediate probabilistic Interactive averaging aggregation operators of T-SFSs Zeng *et al.* (2019), T-SF soft sets and their aggregation operators Guleria and Bajaj (2021), generalized T-SF weighted aggregation operators on neutrosophic sets Quek *et al.* (2019), T-SF Einstein Hybrid Aggregation operators Munir *et al.* (2020), correlation coefficients for T-SFSs Ullah *et al.* (2020a), T-SF Hamacher aggregation operators Ullah *et al.* (2020b), and complex T-SF aggregation operators Ali *et al.* (2020) are some of them. The hesitant FS (HFS) is another extension of the FS for modelling the problems in which decision-makers have different opinions about an alternative or element in considered universe. The HFS was defined by Torra and Narukawa (2009), Torra (2010). To explain the basic idea behind of the concept of the hesitant fuzzy set, we give an example: two decision-makers discuss the membership grade of an element to a set, and while one of them assigns membership grade 0.7 for the element, the other may assign 0.3. In such cases, making a common decision is difficult. In a such case, the HFS is a useful tool. Because of advantages of HFS, many researchers have developed multiple decision making methods and they have presented their applications under HF environment (Hu *et al.* 2018, Li *et al.* 2015, Xu and Xia 2011a, Xu and Xia 2011b, Zeng and Xiao 2018, Zeng *et al.* 2016). Xia *et al.* (2013) described some HF aggregation operators and developed a group decision making method. Chen *et al.* (2013) interpreted the idea of interval-valued hesitant fuzzy sets (IvHFSs) which is a generalization of HFS. Peng *et al.* (2014)

investigated the continuous HF aggregation operators with the aid of continuous OWA operator and they defined the C-HFOWA operator and C-HFOWG operator with their essential properties. They also extended these operators interval-valued HFS. Mu *et al.* (2015) introduced a novel aggregation principle for HF elements (HFE). Amin *et al.* (2018) defined some aggregation operators for triangular cubic linguistic hesitant fuzzy sets. Fahmi *et al.* (2019) defined some new operation laws for trapezoidal cubic hesitant fuzzy (TrCHF) numbers and introduced some new aggregation operators. Jiang *et al.* (2019) introduced the interval-valued HFS concept and described aggregation operators under interval-valued Dual HF enviroment based on Hamacher t-norm and t-conorm. Liu *et al.* (2020a) introduced the Dombi aggregation operators of interval-valued hesitant fuzzy set based on Dombi t-norm and t-conorm. Some studies related to aggregation operator of HFS, extension of HFS and decision making can be found (Bai *et al.* 2020, Ding *et al.* 2020, Li and Huang 2020, Liao *et al.* 2020, Liang *et al.* 2019, Liu *et al.* 2019b, Liu *et al.* 2020b, Mahmood *et al.* 2019b, Mahmood *et al.* 2020, Mo *et al.* 2020, Qiao 2019, Wang and Li 2018, Wang *et al.* 2020, Wu *et al.* 2020, Zeng *et al.* 2018). This thesis presents a novel concept called the hesitant T-spherical fuzzy set(HT-SFS) by combining concepts of HFs and T-SFS. In HT-SFS, more than one T-spherical fuzzy value can be assigned to the elements of the set containing the elements to be evaluated. T-SFS theory deals only one T-spherical fuzzy value for an element. Therefore, it doesn't suffice to model problems including disagreements of the opinion of decision-makers about an element or object. On the other hand, an HT-SFS can handle such situation. Thus, keeping the advantages of HFS and T-SFS, we define the concept of HT-SFS. We also introduce some aggregation operators based on Dombi operators, including hesitant T-spherical Dombi fuzzy weighted arithmetic averaging (HTSDFWAA) operator, hesitant T-spherical Dombi fuzzy weighted geometric averaging (HTSDFWGA) operator, hesitant T-spherical Dombi fuzzy ordered weighted arithmetic averaging (HTSDFOWAA) operator and hesitant T-spherical Dombi fuzzy ordered weighted geometric averaging (HTSDFOWGA) operator and obtain some properties of them. Furthermore, we give a multi criteria group decision-making (MCGDM) method and algorithm of the introduced method under the HT-SF environment. To demonstrate the presented method's process, We offer an example of how to choose the best candidate for an assistant professorship job at a university.

2. PRELIMINARIES

2.1 Fuzzy Set and Its Generalizations

Zadeh (1965) defined the concept of the fuzzy set as follows:

2.1.1 Fuzzy sets

Definition 2.1. (Zadeh 1965) Let $\mathscr{X}$ be a non-empty universal set. A fuzzy set $\mathscr{A}$ on $\mathscr{X}$ is described by a membership function:

$$s_{\mathscr{A}} : \mathscr{X} \to [0,1].$$

A fuzzy set can be written as a set of pairs as follow:

$$\mathscr{A} = \{(\mathfrak{x}, s_{\mathscr{A}}(\mathfrak{x})) : \mathfrak{x} \in \mathscr{X}\}$$

where $s_{\mathscr{A}}(\mathfrak{x})$ is the degree of membership of the element $\mathfrak{x} \in \mathscr{X}$.

Example 2.2. Let $\mathscr{X} = \{1,2,3,4\}$ be a universal set. A fuzzy set $\mathscr{A}$ (two or so) over $\mathscr{X}$ can be written as follows:

$$\mathscr{A} = \{(1,0.5),(2,1),(3,0.5),(4,0)\}.$$

FS $\mathscr{A}$ can be illustrated by Figure 2.1.

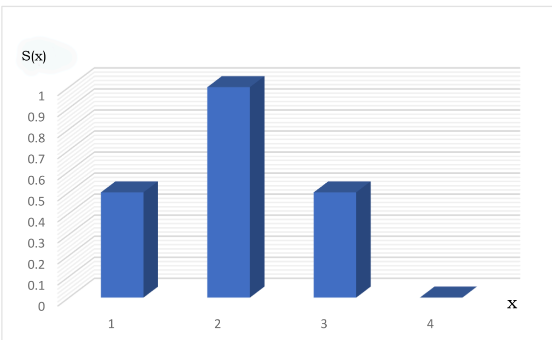

Figure 2.1. Fuzzy Set

2.1.2 Intuitioinistic Fuzzy Sets

Atanassov (1986) defined the concept of intuitionistic fuzzy set which is a generalization of fuzzy sets as follows:

Definition 2.3. (Atanassov 1986) Let $\mathscr{X}$ be a non-empty universal set. An intuitionistic fuzzy set (IFS) on $\mathscr{X}$ is defined as:

$\mathscr{A} = \{(\mathfrak{x}, s_{\mathscr{A}}(\mathfrak{x}), d_{\mathscr{A}}(\mathfrak{x}))|\forall \mathfrak{x} \in \mathscr{X}\}$. Here $s : \mathscr{X} \to [0,1]$ and $d : \mathscr{X} \to [0,1]$ are called membership function and non-membership function of intuitionistic fuzzy set, respectively. $0 \leq s_{\mathscr{A}}(\mathfrak{x}) + d_{\mathscr{A}}(\mathfrak{x}) \leq 1$. $R_{\mathscr{A}}(\mathfrak{x}) = 1 - s_{\mathscr{A}}(\mathfrak{x}) - d_{\mathscr{A}}(\mathfrak{x})$ denotes the degree of hesitancy of $\mathfrak{x}$ in $\mathscr{A}$. The pair (s,d) is called intuitionistic fuzzy number (IFN).

Example 2.4. Let a person applies to take a position in a company in his cv.,

$\mathscr{A} = \{(\text{experience}, 0.5, 0.4), (\text{computer Skills}, 0.3, 0.6), (\text{ profiency}, 0.2, 0.3),$
$(\text{work under pressure}, 0.6, 0.2)\}$. Now when we check the condition of IFS, we find it satisfies the condition as shown:

For experience, $0 \leq 0.5 + 0.4 \leq 1$ and it is the same for others.

2.1.3 Pythagorean fuzzy sets

In an IFS, the summation of the membership degree and non-membership degree of an element in the initial universe can not be greater than 1. This is an inherent limitation of the IFSs. To cope with this limitation, (Yager Yager (2013a,1)) put forward the notion of Pythagorean fuzzy sets (PyFSs) as a generalization of IFSs.

Definition 2.5. (Yager 2013a, 2013b) Let $\mathscr{X}$ be a non-empty universal set. A Pythagorean fuzzy set (PyFS) on $\mathscr{X}$ is defined as:

$\mathscr{A} = \{(\mathfrak{x}, s_{\mathscr{A}}(\mathfrak{x}), d_{\mathscr{A}}(\mathfrak{x}))|\forall \mathfrak{x} \in \mathscr{X}\}$. Here $s : \mathscr{X} \to [0,1]$ and $d : \mathscr{X} \to [0,1]$ are called membership function and non-membership function of Pythagorean fuzzy set, respectively. $0 \leq s^2_{\mathscr{A}}(\mathfrak{x}) + d^2_{\mathscr{A}}(\mathfrak{x}) \leq 1$. And $R_{\mathscr{A}}(\mathfrak{x}) = \sqrt{1 - s^2_{\mathscr{A}}(\mathfrak{x}) - d^2_{\mathscr{A}}(\mathfrak{x})}$ denotes the degree of hesitancy of $\mathfrak{x}$ in $\mathscr{A}$. The pair (s,d) is called Pythagorean fuzzy number (PyFN).

Example 2.6. We consider the same set in Example 2.4 with other values $\mathscr{A} = \{(\text{experience}, 0.5, 0.6), (\text{computer skills}, 0.6, 0.6), (\text{English languge}, 0.8, 0.3),$
$(\text{work under pressure}, 0.2, 0.9)\}$. Here, $0.5 + 0.6 = 1.1 > 1$, so the condition of IFS does

not satisfy, but $0.5^2 + 0.6^2 = 0.25 + 0.36 = 0.61 \leq 1$, the condition satisfies. The values (pairs) corresponding to other elements in the set $\mathscr{X}$ satisfy the same condition. Hence $\mathscr{A}$ is an PyFS.

2.1.4 q-rung orthopair fuzzy sets

In PyFSs, the summation of the square of membership degree and square of non-membership degree of an element in the initial universe can not be greater than 1. This is an inherent limitation of the PyFSs. Yager (2017) put forward the q-rung orthopair fuzzy sets (q-ROFSs) as a generalization of PyFS.

Definition 2.7. (Yager 2017) Let $\mathscr{X}$ be a non-empty universal set. A q-rung orthopair fuzzy set $\mathscr{A}$ is defined as:

$$\mathscr{A} = \{(\mathfrak{x}, s_{\mathscr{A}}(\mathfrak{x}), d_{\mathscr{A}}(\mathfrak{x}))|\mathfrak{x} \in \mathscr{X}\}$$

where $s_{\mathscr{A}} : \mathscr{X} \to [0,1]$,$d_{\mathscr{A}} : \mathscr{X} \to [0,1]$ satisfying the condition: $0 \leq s^q_{\mathscr{A}}(\mathfrak{x}) + d^q_{\mathscr{A}}(\mathfrak{x}) \leq 1, q > 1$. $R_{\mathscr{A}}(\mathfrak{x}) = (1 - (s^q_{\mathscr{A}}(\mathfrak{x}) + d^q_{\mathscr{A}}(\mathfrak{x})))^{\frac{1}{q}}$ is considered as hesitancy degree.$(s_{\mathscr{A}}, d_{\mathscr{A}})$ is called a q-rung orthopair fuzzy number (q-ROFN).

2.1.5 Picture fuzzy sets

The fuzzy sets do not take into account the neutral situation., Cuong (2013a;2013b) proposed this definition.

Definition 2.8. (Cuong 2013a, 2013b) A picture fuzzy set $\mathscr{A}$ on a non-empty universal set $\mathscr{X}$ is an object in the form of

$$\mathscr{A} = \{(\mathfrak{x}, s_{\mathscr{A}}(\mathfrak{x}), i_{\mathscr{A}}(\mathfrak{x}), d_{\mathscr{A}}(\mathfrak{x}))|\mathfrak{x} \in \mathscr{X}\}$$

where $s_{\mathscr{A}}(\mathfrak{x}) \in [0,1]$ is called the degree of positive membership of $\mathfrak{x}$ in $\mathscr{A}$, $i_{\mathscr{A}}(\mathfrak{x}) \in [0,1]$ is called the degree of neutral membership of $\mathfrak{x}$ in $\mathscr{A}$ and $d_{\mathscr{A}}(\mathfrak{x}) \in [0,1]$ is called the degree of negative membership of $\mathfrak{x}$ in $\mathscr{A}$, where $s_{\mathscr{A}}(\mathfrak{x}), i_{\mathscr{A}}(\mathfrak{x})$ and $d_{\mathscr{A}}(\mathfrak{x})$ satisfy the following condition:

$$s_{\mathscr{A}}(\mathfrak{x}) + i_{\mathscr{A}}(\mathfrak{x}) + d_{\mathscr{A}}(\mathfrak{x}) \leq 1, \quad \text{(for all} \quad \mathfrak{x} \in \mathscr{X}).$$

Here $R_{\mathscr{A}}(\mathfrak{x}) = (1 - (s_{\mathscr{A}}(\mathfrak{x}) + i_{\mathscr{A}}(\mathfrak{x}) + d_{\mathscr{A}}(\mathfrak{x})))$ can be called the degree of refusal membership of $\mathfrak{x}$ in $\mathscr{A}$. From now on $PFS(\mathscr{X})$ denotes the set of all the picture fuzzy sets on a non-empty universal set $\mathscr{X}$.

Example 2.9. A student in a high school took an exam in many subjects. He got degrees which shown in the set:
$SU = \{$english $,0.3,0.2,0.1),($mathematics$,0.4,0.1,0.2),($biology$,0.2,0.2,0.2),$
$($physics$,0.3,0.1,0.1),($chemistry$,0.4,0.2,0.2)\}$.
It is clear that SU is PFS.

2.1.6 Spherical fuzzy sets

In a PFS, the summation of the membership degree and the degree of neutral and the non-membership degree of an element in the initial universe can not be greater than 1.This is an inherent limitation of the PFSs. To cope with this limitation, the concept of the Spherical fuzzy sets (SFSs) was defined as a generalization of PFSs.

Definition 2.10. (Gündoğdu and Kahraman 2019a) An SFS $\mathscr{A}$ on a non-empty universal set $\mathscr{X}$ is defined as follows:

$$\mathscr{A} = \{(\mathfrak{x}, s_{\mathscr{A}}(\mathfrak{x}), i_{\mathscr{A}}(\mathfrak{x}), d_{\mathscr{A}}(\mathfrak{x})) | \mathfrak{x} \in \mathscr{X}\}$$

where $s_{\mathscr{A}}(\mathfrak{x}), i_{\mathscr{A}}(\mathfrak{x}), d_{\mathscr{A}}(\mathfrak{x}) \in [0,1]$, $0 \leq s^2_{\mathscr{A}}(\mathfrak{x}) + i^2_{\mathscr{A}}(\mathfrak{x}) + d^2_{\mathscr{A}}(\mathfrak{x}) \leq 1$ for all $\mathfrak{x} \in \mathscr{X}$. We consider the triplet (s,i,d) as SF number (SFN). Here s, i and d are the membership degree (MD), abstinence degree (AD) and non-membership degree (NMD) of $\mathfrak{x} \in \mathscr{A}$, respectively. Further $R_{\mathscr{A}}(\mathfrak{x}) = \sqrt{1 - (s^2_{\mathscr{A}}(\mathfrak{x}) + i^2_{\mathscr{A}}(\mathfrak{x}) + d^2_{\mathscr{A}}(\mathfrak{x}))}$ is the hesitancy degree of $\mathfrak{x}$ in $\mathscr{A}$.

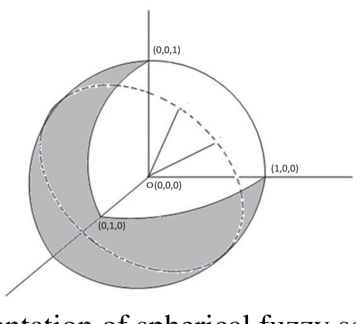

Figure 2.2. Geometrical representation of spherical fuzzy sets. (Gündoğdu and Kahraman 2019a)

Example 2.11. We consider the same universal set in Example 2.9 and SU will be
$SU = \{$english$, 0.3, 0.7, 0.3),($mathematics$, 0.4, 0.6, 0.5),($biology$, 0.5, 0.5, 0.5),$
$($physics$, 0.3, 0.3, 0.5),($chemistry$, 0.4, 0.3, 0.5)\}$
We see that the set does not satisfy the condition of PFS. When we consider the english class,
$0.3+0.7+0.3 = 1.3 > 1$; However $0.3^2+0.7^2+0.3^2 = 0.67 \leq 1$.
For mathematics class, $0.4+0.6+0.5 = 1.5 > 1$; However $0.4^2+0.6^2+0.5^2 = 0.77 \leq 1$.
For the biology class, $0.5+0.5+0.5 = 1.5 > 1$; However $0.5^2+0.5^2+0.5^2 = 0.75 \leq 1$.
For physics class, $0.3+0.3+0.5 = 1.1 > 1$; However $0.3^2+0.3^2+0.5^2 = 0.43 \leq 1$.
For chemistry class, $0.4+0.3+0.5 = 1.2 > 1$; However $0.4^2+0.3^2+0.5^2 = 0.50 \leq 1$.
This condition is satisfied for all triple values and SU is SFS.

2.1.7 T-Spherical Fuzzy Sets

In SFSs, the summation of the square of membership degree and square of neutral membership degree and square of non-membership degree of an element in the initial universe can not be greater than 1. This is an inherent limitation of the (SFSs). Mahmood *et al.* (2019a) put forward the T-Spherical Fuzzy Sets (T-SFSs) as a generalization of (SFSs).

Definition 2.12. (Mahmood *et al.* 2019a) A T-SF Set (T-SFS) on $\mathscr{X}$ is defined as:

$$\mathscr{A} = \{(\mathfrak{x}, s_{\mathscr{A}}(\mathfrak{x}), i_{\mathscr{A}}(\mathfrak{x}), d_{\mathscr{A}}(\mathfrak{x}))|\mathfrak{x} \in \mathscr{X}\}$$

for some natural number n, where $s_{\mathscr{A}}, i_{\mathscr{A}}, d_{\mathscr{A}} : \mathscr{X} \longrightarrow [0,1]$ are membership, abstinence, non-membership functions with a condition that:

$$0 \leq s^n_{\mathscr{A}}(\mathfrak{x}) + i^n_{\mathscr{A}}(\mathfrak{x}) + d^n_{\mathscr{A}}(\mathfrak{x}) \leq 1.$$

The refusal degree of $T-SFS$ is defined as:

$$R_{\mathscr{A}}(\mathfrak{x}) = \sqrt[n]{1-(s^n_{\mathscr{A}}(\mathfrak{x}) + i^n_{\mathscr{A}}(\mathfrak{x}) + d^n_{\mathscr{A}}(\mathfrak{x}))}$$

and the triplet (s,i,d) is called the T-SF number $(TSFN)$.

Example 2.13. We consider the same universal set in Example 2.9. A T-SFS can be written as follows:
$SU = \{\text{english}, 0.8, 0.7, 0.6), (\text{mathematics}, 0.7, 0.7, 0.7), (\text{biology}, 0.9, 0.5, 0.4),$
$(\text{physics}, 0.8, 0.7, 0.7), (\text{chemistry}, 0.4, 0.8, 0.5)\}$
When we check the condition of SFS, we see that it does not satisfy. For the case of english,
$0.8^2 + 0.7^2 + 0.6^2 = 0.64 + 0.49 + 0.36 = 1.49 > 1$,
Now $0.8^3 + 0.7^3 + 0.6^3 = 0.512 + 0.343 + 0.216 = 1.071 > 1$,
but $0.8^4 + 0.7^4 + 0.6^4 = 0.4096 + 0.2401 + 0.1296 = 0.3697 \leq 1$, For the case of mathematics, $0.7^4 + 0.7^4 + 0.7^4 = 0.7203 \leq 1$,
For the case of biology, $0.9^4 + 0.5^4 + 0.4^4 = 0.7442 \leq 1$,
For the case of physics, $0.8^4 + 0.7^4 + 0.7^4 = 0.8898 \leq 1$,
For the case of chemistry, $0.4^4 + 0.9^4 + 0.5^4 = 0.4977 \leq 1$,
i.e. the condition of T-SFS satisfies with $n = 4$ and SU is T-SFS.

2.2 Hesitant Fuzzy Sets and Its Generalizations

2.2.1 Hesitant fuzzy sets

Definition 2.14. (Torra and Narukawa 2009, Torra 2010) Let $\mathscr{X}$ be a fixed set, a hesitant fuzzy set (HFS) on $\mathscr{X}$ is in terms of a function that applied to $\mathscr{X}$ returns of $[0,1]$. The mathematical symbol of HFS:

$$\mathscr{A} = \{< x, h_{\mathscr{A}}(\mathfrak{x}) > | \mathfrak{x} \in \mathscr{X}\}$$

where $h_{\mathscr{A}}(\mathfrak{x})$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $\mathfrak{x} \in \mathscr{X}$ to the set $\mathscr{A}$. $h = h_{\mathscr{A}}(\mathfrak{x})$ is called a HF element (HFE).

2.2.2 Intuitionistic hesitant fuzzy sets

An extension of IFSs was introduced to manage those situations in which several values are possible for the definition of a membership function and non-membership function. The concept of Intuitionistic Hesitant Fuzzy Sets were proposed by keeping in view the importance of IFSs and HFSs. IHFSs are defined in terms of a function that returns a set of membership values and a set of non-membership values for each element in the domain.

Definition 2.15. (Beg and Rashid 2014) An IHFS on $\mathscr{X}$ are functions h and $\acute{h}$ that when applied to $\mathscr{X}$ return subsets of $[0,1]$ which can be represented as the following mathematical symbol:

$$\mathscr{A} = \{(\mathfrak{x}, h(\mathfrak{x}), \acute{h}(\mathfrak{x}))|\mathfrak{x} \in \mathscr{X}\}$$

where $h(\mathfrak{x})$ and $\acute{h}(\mathfrak{x}))$ are sets of some values in $[0,1]$ denoting the possible membership degrees and non-membership degrees of the element $\mathfrak{x} \in \mathfrak{X}$ to the set $\mathscr{A}$ with the conditions that $max(h(\mathfrak{x})) + min(\acute{h}(\mathfrak{x})) \leq 1$ and $min(h(\mathfrak{x})) + max(\acute{h}(\mathfrak{x})) \leq 1$. For convenience, $(h(\mathfrak{x}), \acute{h}(\mathfrak{x}))$ is an intuitionistic hesitant fuzzy element(IHFE).

2.2.3 Pythagorean hesitant fuzzy sets

On the basis of the PyFSs and HFSs, in the following, the concept of Pythagorean Hesitant Fuzzy Sets (PyHFSs) were proposed, which permit the membership of an element to be a set of several possible PyFNs.

Definition 2.16. (Wei *et al.* 2018d) Given a fixed set $\mathscr{X}$, then a PyHFS on $\mathscr{X}$ is given in terms of a function that applied to $\mathscr{X}$ returns a subset of $(0,1)$. The PyHFS can be expressed by the following mathematical symbol:

$$\mathscr{A} = \{< \mathfrak{x}, h_{\mathscr{A}}(\mathfrak{x}) >\},$$

where $h_{\mathscr{A}}(\mathfrak{x})$ is a set of some PyFNs in $(0,1)$, denoting the possible membership degree and non-membership degree of the element $\mathfrak{x} \in \mathscr{X}$ to the set $\mathscr{A}$. For convenience, we call $h = h_{\mathscr{A}}(\mathfrak{x})$ is an PyHFN and H is the set of all PyHFNs. If $\alpha \in P$, then α is a PyHFN, and it can be denoted by $\alpha = (s_{\mathscr{A}}, d_{\mathscr{A}})$ and $s^2_{\mathscr{A}} + d^2_{\mathscr{A}} \leq 1$.

2.2.4 Hesitant q-rung orthopair fuzzy sets (Hq-ROFSs)

The concept of Hesitant q-rung orthopair fuzzy sets (Hq-ROFSs) are more effective and flexible for the decision makers because this concept provides more space for the decision makers in assigning values as compared to IHFSs and PyHFSs.

Definition 2.17. (Hussain *et al.* 2020) Let $\mathscr{X}$ be a non-empty universal set. Then, an Hq-ROFS $\mathscr{A}$ defined on $\mathscr{X}$ is an object given by the following:

$$\mathscr{A} = \{< \mathfrak{x}, h_{\mathscr{A}}(\mathfrak{x}), g_{\mathscr{A}}(\mathfrak{x}) > | \mathfrak{x} \in \mathscr{X}, q > 1\},$$

where $h_{\mathscr{A}}(\mathfrak{x})$ and $g_{\mathscr{A}}(\mathfrak{x})$ are two subsets of $[0,1]$ which represents the hesitant q-rung orthopair membership and hesitant q-rung orthopair non-membership grades of an object $\mathfrak{x} \in \mathscr{X}$ to the set $\mathscr{A}$. Moreover for each element $\mathfrak{x} \in \mathscr{X}$, for all $s_{\mathscr{A}}(\mathfrak{x}) \in h_{\mathscr{A}}(\mathfrak{x})$ there exist $d_{\mathscr{A}}(\mathfrak{x}) \in g_{\mathscr{A}}(\mathfrak{x})$, which holds the condition that $0 \leq (s_{\mathscr{A}}(\mathfrak{x}))^q + (d_{\mathscr{A}}(\mathfrak{x}))^q \leq 1$ and similarly, for all $d_{\mathscr{A}}(\mathfrak{x}) \in g_{\mathscr{A}}(\mathfrak{x})$ there exist $s_{\mathscr{A}}(\mathfrak{x}) \in h_{\mathscr{A}}(\mathfrak{x})$, which holds the condition that $0 \leq (s_{\mathscr{A}}(\mathfrak{x}))^q + (d_{\mathscr{A}}(\mathfrak{x}))^q \leq 1$.

The comparative study of Hq-ROFSs with HFSs, IFSs, IHFSs, PyFSs, HPyFSs, q-ROFSs, Hq-ROFSs are mentioned in Table 2.1.

Table 2.1 The comparison of the extension of HFSs

The evaluation format	The No. of elements in membership grade		The No. of elements in non-membership grade	
	One	Many	One	Many
HFSs	✓	✓		
IFSs	✓		✓	
IHFSs	✓	✓	✓	✓
PyFSs	✓		✓	
HPYFSs	✓	✓	✓	✓
q-ROFSs	✓		✓	
Hq-ROFSs	✓	✓	✓	✓

3. HESITANT T-SPHERICAL FUZZY SETS

In this part, we define the concept of hesitant T-Spherical fuzzy sets and introduce their set theoretical operations.

Definition 3.18. Let $\mathfrak{X}$ be a non-empty set. A hesitant T-SFS (HT-SFS) over $\mathfrak{X}$ denoted by $\mathbb{T}_H$, is defined as follows:

$$\mathbb{T}_H = \left\{(x, \mathfrak{h}(x)) : x \in \mathfrak{X}\right\}$$

Here $\mathfrak{h}(x) = \mathfrak{h}$ is collection of T-SFNs and $\mathfrak{h}$ is called HT-SF element ($HT-SFE$). The number of elements of an $HT-SFE$ is the length of $HT-SFE$, $\mathfrak{h}$ and denoted by $\ell_{\mathfrak{h}}$. In other words, an $HT-SFS$ is collection of $HT-SFEs$.

Example 3.19. Let us consider a set $\mathfrak{X} = \{x_1, x_2, x_3, x_4\}$. Then, for $n = 3$ (n represents the value, where $s^n + i^n + d^n \leq 1$), we can write an $HT-SFS$ $\mathbb{T}$ as follows:

$$\begin{aligned}\mathbb{T} = \; & \Big\{\big(x_1, \{(0.6,0.8,0.4),(0.4,0.5,0.9),(0.6,0.2,0.7)\}\big), \big(x_2, \{(0.3,0.9,0.5),\\ & (0.2,0.4,0.7)\}\big), \big(x_3, \{(0.9,0.3,0.2),(0.6,0.6,0.6),(0.4,0.3,0.8),\\ & (0.5,0.7,0.6)\}\big), \big(x_4, \{(0.7,0.2,0.2)\}\big)\Big\}.\end{aligned}$$

3.1 The Score Function of HT-SFEs

Definition 3.20. Let $\mathfrak{h}$ be an $HT-SFE$. Then, score value of $HT-SFE$ $\mathfrak{h}$ denoted by $SV(\mathfrak{h})$ is defined as:

$$SV(\mathfrak{h}) = \frac{1}{\ell_{\mathfrak{h}}} \sum_{k=1}^{\ell_{\mathfrak{h}}} (s_k^n - d_k^n) \tag{3.1}$$

for some positive integer n. Here $SV(\mathfrak{h}) \in [-1, 1]$.

3.2 The Accuracy Function of HT-SFEs

Definition 3.21. Let $\mathfrak{h}$ be an $HT-SFE$. Then, accuracy value of $HT-SFE$ $\mathfrak{h}$ denoted by $AV(\mathfrak{h})$ is defined as follows:

$$AV(\mathfrak{h}) = \frac{1}{\ell_{\mathfrak{h}}} \sum_{k=1}^{\ell_{\mathfrak{h}}} (s_k^n + i_k^n + d_k^n) \tag{3.2}$$

for the positive integer n. Here $AV(\mathfrak{h}) \in [0,1]$.

3.3 The Comparison Between Two HT-SFNs

Definition 3.22. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two $HT-SFEs$, $SV(\mathfrak{h}_1)$ and $SV(\mathfrak{h}_2)$ are the score values of $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively, and $AV(\mathfrak{h}_1)$ and $AV(\mathfrak{h}_2)$ are the accuracy values of $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. Then,

1. If $SV(\mathfrak{h}_1) < SV(\mathfrak{h}_2)$, then $\mathfrak{h}_1 < \mathfrak{h}_2$
2. If $SV(\mathfrak{h}_1) > SV(\mathfrak{h}_2)$, then $\mathfrak{h}_1 > \mathfrak{h}_2$
3. If $SV(\mathfrak{h}_1) = SV(\mathfrak{h}_2)$, then there are three cases
   (a) If $AV(\mathfrak{h}_1) < AV(\mathfrak{h}_2)$, then $\mathfrak{h}_1 < \mathfrak{h}_2$
   (b) If $AV(\mathfrak{h}_1) > AV(\mathfrak{h}_2)$, then $\mathfrak{h}_1 > \mathfrak{h}_2$
   (c) If $AV(\mathfrak{h}_1) = AV(\mathfrak{h}_2)$, then $\mathfrak{h}_1 = \mathfrak{h}_2$

Example 3.23. Let
$\mathfrak{h}_1 = \{(0.6,0.8,0.4),(0.4,0.5,0.9),(0.6,0.2,0.7)\}$ and
$\mathfrak{h}_2 = \{(0.3,0.9,0.5),(0.2,0.4,0.7)\}$ are two elements of HT-SFS $\mathbb{T}$ given in Example 3.19, $n = 3$, Then,

$$\begin{aligned} SV(\mathfrak{h}_1) &= \frac{1}{\ell_{\mathfrak{h}_1}}[(0.6^3-0.4^3)+(0.4^3-0.9^3)+(0.6^3-0.7^3)] \\ &= -0.2133, \\ SV(\mathfrak{h}_2) &= \frac{1}{\ell_{\mathfrak{h}_2}}[(0.3^3-0.5^3)+(0.2^3-0.7^3)] \\ &= -0.2165 \end{aligned}$$

Since $SV(\mathfrak{h}_2) < SV(\mathfrak{h}_1)$, then $\mathfrak{h}_2 < \mathfrak{h}_1$

Definition 3.24. Let $\mathbb{T}_1 = \{(x,\mathfrak{h}_1(x)) : x \in \mathfrak{X}\}$ and $\mathbb{T}_2 = \{(x,\mathfrak{h}_2(x)) : x \in \mathfrak{X}\}$ be two HT-SFSs over a common universe $\mathfrak{X}$. If for all $x \in \mathfrak{X}$, $SV(\mathfrak{h}_1(x)) \leq SV(\mathfrak{h}_2(x))$, then $\mathbb{T}_1$ is said to be HT-SF subset of $\mathbb{T}_2$, and denoted by $\mathbb{T}_1 \lhd \mathbb{T}_2$.

Example 3.25. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two HT-SFSs over $\mathfrak{X} = \{x_1,x_2,x_3\}$ for $n = 3$ given as follows:

$$
\begin{aligned}
\mathbb{T}_1 =\Big\{&(x_1,\{(0.5,0.4,0.3),(0.6,0.5,0.4),(0.7,0.2,0.6)\}),\\
&(x_2,\{(0.6,0.2,0.4),(0.7,0.1,0.6),(0.5,0.5,0.2)\}),\\
&(x_3,\{(0.9,0.6,0.1),(0.4,0.4,0.4),(0.6,0.2,0.5)\})\Big\}\\
\mathbb{T}_2 =\Big\{&(x_1,\{(0.7,0.6,0.5),(0.7,0.5,0.5),(0.8,0.3,0.6)\}),\\
&(x_2,\{(0.8,0.3,0.3),(0.6,0.2,0.2),(0.5,0.7,0.1)\}),\\
&(x_3,\{(0.7,0.3,0.3),(0.9,0.2,0.2),(0.5,0.1,0.3)\})\Big\}.
\end{aligned}
$$

Then, by using Eq. (3.1), for $x_i \in \mathfrak{X}$ $(i = 1,2,3)$ SVs of HT-SFEs are obtained as follow:

	x_1	x_2	x_3
$SV(\mathfrak{h}_1)$	0.126	0.132	0.273
$SV(\mathfrak{h}_2)$	0.244	0.270	0.378

From the table, it is clear that $\mathbb{T}_1 \lhd \mathbb{T}_2$.

3.4 Set Theoretical Operations of HT-SFSs

In this subsection, union, intersection and complement of an HT-SFS are defined with their examples.

Definition 3.26. Let $\mathfrak{h} = \{(s_t,i_t,d_t) : 1 \le t \le \ell_{\mathfrak{h}}\}$ be a T-SFE over $\mathfrak{X}$. Then, lower and upper bound of $\mathfrak{h}$ are defined as follows:

$$\mathfrak{h}^- = \min_t (s_t^n - d_t^n)$$

$$\mathfrak{h}^+ = \max_t (s_t^n - d_t^n),$$

respectively.

The following example is to explain lower and upper bound of $\mathfrak{h}$.

Let $\mathfrak{h} = \{(0.5,0.4,0.3),(0.6,0.5,0.4),(0.7,0.2,0.5)\}$ be an HT-FS element, $n = 3, l_{\mathfrak{h}} = 3$

$$\begin{aligned}
\mathfrak{h}^- &= min\{(0.5^3-0.3^3),(0.6^3-0.4^3),(0.7^3-0.5^3)\}\\
&= min\{0.116,0.152,0.218\}\\
&= 0.116\\
\mathfrak{h}^+ &= max\{0.116,0.152,0.218\}\\
&= 0.218
\end{aligned}$$

Definition 3.27. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two HT-SFSs over $\mathfrak{X}$ and let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be HT-SFEs of $\mathbb{T}_1$ and $\mathbb{T}_2$ for all $x \in \mathfrak{X}$. Then, based on HT-SFEs, set theoretical operations between $\mathbb{T}_1$ and $\mathbb{T}_2$ are defined as follows:

1. Union:

$$\mathbb{T}_1 \Cup \mathbb{T}_2 = \bigcup_{x\in\mathfrak{X}} \Big\{ \Big(x, \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2}} \{(s_k,i_k,d_k) : s_k^n - d_k^n = max\{s_1^n-d_1^n, s_2^n-d_2^n\}, k=1,2\} \Big) \Big\}.$$

2. Intersection:

$$\mathbb{T}_1 \Cap \mathbb{T}_2 = \bigcup_{x\in\mathfrak{X}} \Big\{ \Big(x, \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2}} \{(s_k,i_k,d_k) : s_k^n - d_k^n = min\{s_1^n-d_1^n, s_2^n-d_2^n\}, k=1,2\} \Big) \Big\}.$$

3. Complement:

$$\mathbb{T}_1^c = \bigcup_{x\in\mathfrak{X}} \Big\{ \Big(x, \bigcup_{(s_1,i_1,d_1)\in\mathfrak{h}_1} \{(d_1,i_1,s_1)\} \Big) \Big\}.$$

Example 3.28. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two HT-SFSs over $\mathfrak{X} = \{x_1,x_2,x_3\}$ for $n=4$ given as follows:

$$\begin{aligned}
\mathbb{T}_1 =& \Big\{ (x_1,\{(0.5,0.7,0.9),(0.3,0.2,0.7),(0.6,0.5,0.4)\}),(x_2,\{(0.4,0.6,0.3),\\
&(0.2,0.5,0.9)\}),(x_3,\{(0.8,0.4,0.3)\}) \Big\}\\
\mathbb{T}_2 =& \Big\{ (x_1,\{(0.4,0.8,0.5),(0.5,0.7,0.6)\}),(x_2,\{(0.2,0.7,0.6),(0.4,0.3,0.8)\}),\\
&(x_3,\{(0.5,0.4,0.8),(0.6,0.7,0.2),(0.9,0.2,0.4)\}) \Big\}.
\end{aligned}$$

Then, by using Definition 3.27, union, intersection and complement of HT-SFSs

$\mathbb{T}_1$ and $\mathbb{T}_2$ are obtained as follow:

$$\begin{aligned}
\mathbb{T}_1 \Cup \mathbb{T}_2 &= \Big\{(x_1,\{(0.4,0.8,0.5),(0.5,0.7,0.6),(0.6,0.5,0.4)\}),(x_2,\{(0.4,0.6,\\
&\quad 0.3),(0.2,0.7,0.6),(0.4,0.3,0.8)\}),(x_3,\{(0.8,0.4,0.3),(0.9,0.2,\\
&\quad 0.4)\})\Big\}\\
\mathbb{T}_1 \Cap \mathbb{T}_2 &= \Big\{(x_1,\{(0.5,0.7,0.9),(0.3,0.2,0.7),(0.4,0.8,0.5),(0.5,0.7,0.6)\}),\\
&\quad (x_2,\{(0.2,0.7,0.6),(0.4,0.3,0.8),(0.2,0.5,0.9)\}),(x_3,\{(0.5,0.4,\\
&\quad 0.8),(0.6,0.7,0.2),(0.8,0.4,0.3)\})\Big\},\\
\mathbb{T}_1^c &= \Big\{(x_1,\{(0.9,0.7,0.5),(0.7,0.2,0.3),(0.4,0.5,0.6)\}),(x_2,\{(0.3,0.6,\\
&\quad 0.4),(0.9,0.5,0.2)\}),(x_3,\{(0.3,0.4,0.8)\})\Big\}.
\end{aligned}$$

respectively. From now on set of all T-spherical fuzzy numbers is denoted by Υ.

4. HESITANT T-SPHERICAL FUZZY AGGREGATION OPERATORS

In this section, we define some aggregation operators based on Dombi T-norm and T-conorm for Hesitant T-spherical elements. Dombi product and Dombi sum which are specific types of triangular norms and conorms given in Dombi (1982) as follows:

Definition 4.29. (Dombi 1982) Let f and g be two real numbers in the interval $[0,1]$. Then, Dombi t-norm is given by

$$f \otimes g = \frac{1}{1+\left((\frac{f}{1-f})^{-\eta} + (\frac{g}{1-g})^{-\eta} \right)^{\frac{1}{\eta}}}, \quad \eta > 0.$$

The Dombi t-conorm is defined as:

$$f \oplus g = 1 - \frac{1}{1+\left((\frac{1-f}{f})^{-\eta} + (\frac{1-g}{g})^{-\eta} \right)^{\frac{1}{\eta}}}, \quad \eta > 0,$$

respectively.

4.1 Dombi operations of HT-SFEs

In this subsection, we define some Dombi operations between HT-SFEs.

Definition 4.30. Let $\mathfrak{h}_1 = \{(s_{1t}, i_{1t}, d_{1t}) : 1 \le t \le \ell_{\mathfrak{h}_1}\}$ and $\mathfrak{h}_2 = \{(s_{2r}, i_{2r}, d_{2r}) : 1 \le r \le \ell_{\mathfrak{h}_2}\}$ be two $HT-SFEs$ and $\gamma > 0$, the Dombi operations for HT-SF elements are defined as follows:

1. $\mathfrak{h}_1 \oplus \mathfrak{h}_2 = \bigcup_{\substack{(s_{1t}, i_{1t}, d_{1t}) \in \mathfrak{h}_1 \\ (s_{2r}, i_{2r}, d_{2r}) \in \mathfrak{h}_2}}$

$$\left\{ \left(\sqrt[n]{1 - \frac{1}{1+\left\{ \left(\frac{s_{1t}^n}{1-s_{1t}^n} \right)^{\gamma} + \left(\frac{s_{2r}^n}{1-s_{2r}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \sqrt[n]{\frac{1}{1+\left\{ \left(\frac{1-i_{1t}^n}{i_{1t}^n} \right)^{\gamma} + \left(\frac{1-i_{2r}^n}{i_{2r}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \sqrt[n]{\frac{1}{1+\left\{ \left(\frac{1-d_{1t}^n}{d_{1t}^n} \right)^{\gamma} + \left(\frac{1-d_{2r}^n}{d_{2r}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}} \right) \right\}$$

2. $\mathfrak{h}_1 \otimes \mathfrak{h}_2 = \bigcup_{\substack{(s_{1t}, i_{1t}, d_{1t}) \in \mathfrak{h}_1 \\ (s_{2r}, i_{2r}, d_{2r}) \in \mathfrak{h}_2}}$

$$\left\{ \left(\sqrt[n]{\frac{1}{1+\left\{ \left(\frac{1-s_{1t}^n}{s_{1t}^n} \right)^{\gamma} + \left(\frac{1-s_{2r}^n}{s_{2r}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \sqrt[n]{\frac{1}{1+\left\{ \left(\frac{1-i_{1t}^n}{i_{1t}^n} \right)^{\gamma} + \left(\frac{1-i_{2r}^n}{i_{2r}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \sqrt[n]{1 - \frac{1}{1+\left\{ \left(\frac{d_{1t}^n}{1-d_{1t}^n} \right)^{\gamma} + \left(\frac{d_{2r}^n}{1-d_{2r}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}} \right) \right\}$$

3. $\lambda \mathfrak{h}_1 = \bigcup_{(s_{1t}, i_{1t}, d_{1t}) \in \mathfrak{h}_1} \left\{ \left(\sqrt[n]{1 - \frac{1}{1+\left\{ \lambda \left(\frac{s_{1t}^n}{1-s_{1t}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \sqrt[n]{\frac{1}{1+\left\{ \lambda \left(\frac{1-i_{1t}^n}{i_{1t}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \sqrt[n]{\frac{1}{1+\left\{ \lambda \left(\frac{1-d_{1t}^n}{d_{1t}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}} \right) \right\}$

4. $\mathfrak{h}_1^{\lambda} = \bigcup_{(s_{1t}, i_{1t}, d_{1t}) \in \mathfrak{h}_1} \left\{ \left(\sqrt[n]{\frac{1}{1+\left\{ \lambda \left(\frac{1-s_{1t}^n}{s_{1t}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \sqrt[n]{\frac{1}{1+\left\{ \lambda \left(\frac{1-i_{1t}^n}{i_{1t}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}}, \sqrt[n]{1 - \frac{1}{1+\left\{ \lambda \left(\frac{d_{1t}^n}{1-d_{1t}^n} \right)^{\gamma} \right\}^{\frac{1}{\gamma}}}} \right) \right\}$

Example 4.31. Let $\mathfrak{h}(x_1) = \mathfrak{h}_1 = \{(0.6, 0.8, 0.4), (0.4, 0.5, 0.9), (0.6, 0.2, 0.7)\}$ and $\mathfrak{h}(x_2) = \mathfrak{h}_2 = \{(0.3, 0.9, 0.5), (0.2, 0.4, 0.7)\}$ be two HT-SF elements in Example 3.19. For $n = 3$, $\gamma = 1$ and $\lambda = 2$:

$$\begin{aligned}
\mathfrak{h}_1 \oplus \mathfrak{h}_2 &= \{(0.6151, 0.7549, 0.3536), (0.6045, 0.3922, 0.3849), \\
&\quad (0.4443, 0.4925, 0.4925), (0.4141, 0.3536, 0.6726), \\
&\quad (0.6151, 0.1998, 0.4655), (0.6045, 0.1928, 0.5915)\}. \\
\mathfrak{h}_1 \otimes \mathfrak{h}_2 &= \{(0.2908, 0.7549, 0.5587), (0.1981, 0.3922, 0.7187), \\
&\quad (0.2685, 0.4925, 0.9041), (0.1928, 0.3536, 0.9136), \\
&\quad (0.2908, 0.1998, 0.7364), (0.1981, 0.1928, 0.7994)\}.
\end{aligned}$$

$$\begin{aligned}
2\mathfrak{h}_1 &= \{(0.7082, 0.7007, 0.3209), (0.4937, 0.4055, 0.8309), \\
&\quad (0.7082, 0.1590, 0.5915)\}. \\
\mathfrak{h}_1^2 &= \{(0.4947, 0.7007, 0.4937), (0.3209, 0.4055, 0.9448), \\
&\quad (0.4947, 0.1590, 0.7994)\}.
\end{aligned}$$

4.2 Hesitant T-Spherical Fuzzy Aggregation Operators

In this subsection, we define hesitant T-spherical Dombi fuzzy weighted arithmetic averaging operator, Hesitant T-spherical Dombi fuzzy weighted geometric averaging operator, hesitant T-spherical Dombi fuzzy ordered weighted arithmetic averaging operator, and hesitant T-spherical Dombi fuzzy ordered weighted geometric averaging operator.

4.2.1 Hesitant T-spherical Dombi fuzzy weighted arithmetic averaging (HTSDFWAA) operator

Definition 4.32. Let $\mathscr{H}^m = \{\mathfrak{h}_k = \{(s_{kj}, i_{kj}, d_{kj}) : 1 \leq j \leq \ell_{\mathfrak{h}_k}, k = 1,2,...,m\}$ be an m dimensional collection of HT-SFEs. A hesitant T-spherical Dombi fuzzy weighted arithmetic averaging (HTSDFWAA) operator is defined by a function $HTSDFWAA : \mathscr{H}^m \to \mathscr{H}$ as follows:

$$\begin{aligned} HTSDFWAA(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, ..., \mathfrak{h}_m) &= \bigoplus_{z=1}^{m} (w_z \mathfrak{h}_z) \\ &= (w_1 \mathfrak{h}_1) \oplus (w_2 \mathfrak{h}_2) \oplus ... \oplus (w_m \mathfrak{h}_m), \end{aligned}$$

where w_z is weight vector of $\mathfrak{h}_z (z = 1,2,...,m)$, $0 \leq w_z \leq 1$ and $\sum_{z=1}^{m} w_z = 1$.

We get the following theorem that follows the Dombi operations on HT-SFEs.

Theorem 4.33. Let $\mathfrak{h}_k \in \mathscr{H}^m$. Then,

$$\begin{aligned} HTSDFWAA(\mathfrak{h}_1, \mathfrak{h}_2, ..., \mathfrak{h}_m) &= \bigoplus_{z=1}^{m} (w_z \mathfrak{h}_z) \\ &= \bigcup_{\substack{(s_1,i_1,d_1) \in \mathfrak{h}_1 \\ (s_2,i_2,d_2) \in \mathfrak{h}_2 \\ \cdots \\ (s_m,i_m,d_m) \in \mathfrak{h}_m}} \left(\begin{array}{c} \sqrt[n]{1 - \frac{1}{1+(\sum_{z=1}^{m} w_z (\frac{s_z^n}{1-s_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{m} w_z (\frac{1-i_z^n}{i_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{m} w_z (\frac{1-d_z^n}{d_z^n})^{\gamma})^{\frac{1}{\gamma}}}} \end{array} \right) \end{aligned}$$

where $W = (w_1, w_2, ..., w_m)^{\tau}$ be the m weight vector of $\mathfrak{h}_k$ $(k = 1,2,...,m)$ such that $w_k > 0$ and $\sum_{k=1}^{m} w_k = 1$.

Proof. The theorem can be proved by the method of mathematical induction as follows:
(i) When m=2, based on Dombi operations on HT-SFEs we obtain the following results:

$$w_1\mathfrak{h}_1 = \bigcup_{(s_1,i_1,d_1)\in\mathfrak{h}_1} \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+(w_1(\frac{s_1^n}{1-s_1^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_1(\frac{1-i_1^n}{i_1^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_1(\frac{1-d_1^n}{d_1^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right),$$

$$w_2\mathfrak{h}_2 = \bigcup_{(s_2,i_2,d_2)\in\mathfrak{h}_2} \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+(w_2(\frac{s_2^n}{1-s_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_2(\frac{1-i_2^n}{i_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_2(\frac{1-d_2^n}{d_2^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right),$$

$$w_1\mathfrak{h}_1 \oplus w_2\mathfrak{h}_2 = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1 \\ (s_2,i_2,d_2)\in\mathfrak{h}_2}} \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+(w_1(\frac{s_1^n}{1-s_1^n})^\gamma+w_2(\frac{s_2^n}{1-s_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_1(\frac{1-i_1^n}{i_1^n})^\gamma+w_2(\frac{1-i_2^n}{i_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_1(\frac{1-d_1^n}{d_1^n})^\gamma+w_2(\frac{1-d_2^n}{d_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \end{array} \right),$$

$$w_1\mathfrak{h}_1 \oplus w_2\mathfrak{h}_2 = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1 \\ (s_2,i_2,d_2)\in\mathfrak{h}_2}} \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{2} w_z(\frac{s_z^n}{1-s_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{2} w_z(\frac{1-i_z^n}{i_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{2} w_z(\frac{1-d_z^n}{d_z^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right).$$

Then, the theorem holds for $m = 2$

(ii) Suppose the theorem holds when $z = k$

i.e.

$$\oplus_{z=1}^{k}(w_z\mathfrak{h}_z) = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\\cdots\\(s_k,i_k,d_k)\in\mathfrak{h}_k}} \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{s_z^n}{1-s_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-i_z^n}{i_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-d_z^n}{d_z^n})^{\gamma})^{\frac{1}{\gamma}}}} \end{array} \right)$$

When $z = k+1$

$$\oplus_{z=1}^{k}(w_z\mathfrak{h}_z)\oplus w_{k+1}\mathfrak{h}_{k+1} = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\\cdots\\(s_k,i_k,d_k)\in\mathfrak{h}_k}} \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{s_z^n}{1-s_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-i_z^n}{i_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-d_z^n}{d_z^n})^{\gamma})^{\frac{1}{\gamma}}}} \end{array} \right) \oplus w_{k+1}\mathfrak{h}_{k+1}$$

$$\oplus_{z=1}^{k}(w_z\mathfrak{h}_z)\oplus w_{k+1}\mathfrak{h}_{k+1} = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\\cdots\\(s_k,i_k,d_k)\in\mathfrak{h}_k}} \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{s_z^n}{1-s_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-i_z^n}{i_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-d_z^n}{d_z^n})^{\gamma})^{\frac{1}{\gamma}}}} \end{array} \right) \oplus$$

$$\bigcup_{(s_{k+1},i_{k+1},d_{k+1})\in\mathfrak{h}_{k+1}} \begin{pmatrix} \sqrt[n]{1-\frac{1}{1+(w_{k+1}(\frac{s^n_{k+1}}{1-s^n_{k+1}})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_{k+1}(\frac{1-i^n_{k+1}}{i^n_{k+1}})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_{k+1}(\frac{1-d^n_{k+1}}{d^n_{k+1}})^\gamma)^{\frac{1}{\gamma}}}} \end{pmatrix},$$

$$\oplus_{z=1}^{k+1}(w_z\mathfrak{h}_z) = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\ \dots \\(s_{k+1},i_{k+1},d_{k+1})\in\mathfrak{h}_{k+1}}} \begin{pmatrix} \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{k+1} w_z(\frac{s^n_z}{1-s^n_z})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k+1} w_z(\frac{1-i^n_z}{i^n_z})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k+1} w_z(\frac{1-d^n_z}{d^n_z})^\gamma)^{\frac{1}{\gamma}}}} \end{pmatrix}.$$

Then, the theorem holds for $z = k+1$. Hence, the proof is completed.

□

Example 4.34. Let us consider $\mathfrak{h}_1 = \{(0.6,0.8,0.4),(0.4,0.5,0.9),(0.6,0.2,0.7)\}$, $\mathfrak{h}_2 = \{(0.3,0.9,0.5),(0.2,0.4,0.7)\}$ and $\mathfrak{h}_3 = \{(0.5,0.4,0.3)\}$ for $n = 3$.
When $\gamma = 1$, with weighted vector $w = (0.5,0.3,0.2)$ we get:
$HTSDFWAA(\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3) =$

$$\oplus_{z=1}^{3}(w_z\mathfrak{h}_z) = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\(s_3,i_3,d_3)\in\mathfrak{h}_3}} \begin{pmatrix} \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{3} w_z(\frac{s^n_z}{1-s^n_z})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{3} w_z(\frac{1-i^n_z}{i^n_z})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{3} w_z(\frac{1-d^n_z}{d^n_z})^\gamma)^{\frac{1}{\gamma}}}} \end{pmatrix}$$

$$\begin{aligned} HTSDFWAA(\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3) = \{&(0.5298,0.6051,0.3843),(0.5246,0.4846,0.3961), \\ &(0.4049,0.5100,0.4568),(0.3941,0.4391,0.4813), \\ &(0.5298,0.2474,0.4461),(0.5246,0.2423,0.4683)\}. \end{aligned}$$

Theorem 4.35. (Idempotency) Let $\mathfrak{h}_k$ $(k = 1,2,..,m)$ be a number of HT-SFNs. Then, $\mathfrak{h}_k = (s_k, i_k, d_k)(k = 1,2,...,m)$ be a number of $HT-SFEs$ are all equal, i.e., $\mathfrak{h}_k = \mathfrak{h}$ for all k , then $HTSDFWAA(\mathfrak{h}_1, \mathfrak{h}_2, ..., \mathfrak{h}_m) = \mathfrak{h}$

Proof. Since $\mathfrak{h}_k = (s,i,d)(k = 1,2,...,m)$ then,
$HTSDFWAA(\mathfrak{h}_1, \mathfrak{h}_2, ..., \mathfrak{h}_m) =$

$$\oplus_{z=1}^{m}(w_z\mathfrak{h}_z) = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\...\\(s_k,i_k,d_m)\in\mathfrak{h}_m}} \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{m} w_z(\frac{s_z^n}{1-s_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{m} w_z(\frac{1-i_z^n}{i_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{m} w_z(\frac{1-d_z^n}{d_z^n})^{\gamma})^{\frac{1}{\gamma}}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+((\frac{s_z^n}{1-s_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+((\frac{1-i_z^n}{i_z^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+((\frac{1-d_z^n}{d_z^n})^{\gamma})^{\frac{1}{\gamma}}}} \end{array} \right) = \left(\begin{array}{c} \sqrt[n]{1-\frac{1}{1+(\frac{s_z^n}{1-s_z^n})}}, \\ \sqrt[n]{\frac{1}{1+(\frac{1-i_z^n}{i_z^n})}}, \\ \sqrt[n]{\frac{1}{1+(\frac{1-d_z^n}{d_z^n})}} \end{array} \right) = (s,i,d) = \mathfrak{h}.$$

Thus, the proof is completed.

□

4.2.2 Hesitant T-spherical Dombi fuzzy weighted geometric averaging (HTSDFWGA) operator

Definition 4.36. Let $\mathscr{H}^m = \{\mathfrak{h}_k = \{(s_{kj}, i_{kj}, d_{kj}) : 1 \leq j \leq \ell_{\mathfrak{h}_k}, k = 1,2,...,m\}$ be an m dimensional collection of HT-SFEs. An HTSDFWGA operator is defined by a function

$HTSDFWGA : \mathscr{H}^m \rightarrow \mathscr{H}$ as follows:

$$HTSDFWGA(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, ..., \mathfrak{h}_m) = \bigotimes_{z=1}^{m} (\mathfrak{h}_z^{w_z}) = (\mathfrak{h}_1^{w_1}) \otimes (\mathfrak{h}_2^{w_2}) \otimes ... \otimes (\mathfrak{h}_m^{w_m}),$$

where w_z is weight vector of $\mathfrak{h}_z (z = 1, 2, ..., m)$, $0 \leq w_z \leq 1$ and $\sum_{z=1}^{m} w_z = 1$.

Theorem 4.37. Let $\mathfrak{h}_k \in \mathscr{H}^m$. Then,

$$HTSDFWGA(\mathfrak{h}_1, \mathfrak{h}_2, ..., \mathfrak{h}_m) = \bigotimes_{z=1}^{m} (\mathfrak{h}_z^{\omega_z}) = \bigcup_{\substack{(s_1,i_1,d_1) \in \mathfrak{h}_1 \\ (s_2,i_2,d_2) \in \mathfrak{h}_2 \\ \cdots \\ (s_m,i_m,d_m) \in \mathfrak{h}_m}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{m} w_z(\frac{1-s_z^n}{s_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{m} w_z(\frac{1-i_z^n}{i_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1 - \frac{1}{1+(\sum_{z=1}^{m} w_z(\frac{d_z^n}{1-d_z^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right)$$

where $w = (w_1, w_2, ..., w_m)^\tau$ be the m weighted vector of $\mathfrak{h}_k (k = 1, 2, ..., m)$ such that $w_k > 0$ and $\sum_{k=1}^{m} w_k = 1$.

Proof. The theorem can be proved by mathematical induction as follows:

(i) When $m = 2$, we have

$$\mathfrak{h}_1^{w_1} = \bigcup_{(s_1,i_1,d_1) \in \mathfrak{h}_1} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(w_1(\frac{1-s_1^n}{s_1^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_1(\frac{1-i_1^n}{i_1^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1 - \frac{1}{1+(w_1(\frac{d_1^n}{1-d_1^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right)$$

$$\mathfrak{h}_2^{w_2} = \bigcup_{(s_2,i_2,d_2) \in \mathfrak{h}_2} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(w_2(\frac{1-s_2^n}{s_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_2(\frac{1-i_2^n}{i_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1 - \frac{1}{1+(w_2(\frac{d_2^n}{1-d_2^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right)$$

$$\mathfrak{h}_1^{w_1} \otimes \mathfrak{h}_2^{w_2} = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1 \\ (s_2,i_2,d_2)\in\mathfrak{h}_2}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(w_1(\frac{1-s_1^n}{s_1^n})^\gamma + w_2(\frac{1-s_2^n}{s_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_1(\frac{1-i_1^n}{i_1^n})^\gamma + w_2(\frac{1-i_2^n}{i_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(w_1(\frac{d_1^n}{1-d_1^n})^\gamma + w_2(\frac{d_2^n}{1-d_2^n})^\gamma)^{\frac{1}{\gamma}}}}, \end{array} \right)$$

$$\mathfrak{h}_1^{w_1} \otimes \mathfrak{h}_2^{w_2} = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1 \\ (s_2,i_2,d_2)\in\mathfrak{h}_2}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{2} w_z(\frac{1-s_z^n}{s_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{2} w_z(\frac{1-i_z^n}{i_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{2} w_z(\frac{d_z^n}{1-d_z^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right).$$

Then, the theorem holds for $m = 2$.

(ii) Suppose that the theorem holds for $z = k$ i.e.

$$\otimes_{z=1}^{k}(\mathfrak{h}_z^{w_z}) = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1 \\ (s_2,i_2,d_2)\in\mathfrak{h}_2 \\ \cdots \\ (s_k,i_k,d_k)\in\mathfrak{h}_k}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-s_z^n}{s_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-i_z^n}{i_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{d_z^n}{1-d_z^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right).$$

We prove that equation is true for $z = k+1$

$$\otimes_{z=1}^{k}(\mathfrak{h}_z^{w_z}) \otimes \mathfrak{h}_{k+1}^{w_{k+1}} = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\\cdots\\(s_k,i_k,d_k)\in\mathfrak{h}_k}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-s_z^n}{s_z^n})\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-i_z^n}{i_z^n})\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{d_z^n}{1-d_z^n})\gamma)^{\frac{1}{\gamma}}}} \end{array} \right) \otimes \mathfrak{h}_{k+1}^{w_{k+1}}$$

$$\otimes_{z=1}^{k}(\mathfrak{h}_z^{w_z}) \otimes \mathfrak{h}_{k+1}^{w_{k+1}} = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\\cdots\\(s_k,i_k,d_k)\in\mathfrak{h}_k}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-s_z^n}{s_z^n})\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{1-i_z^n}{i_z^n})\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{k} w_z(\frac{d_z^n}{1-d_z^n})\gamma)^{\frac{1}{\gamma}}}} \end{array} \right) \otimes$$

$$\bigcup_{(s_{k+1},i_{k+1},d_{k+1})\in\mathfrak{h}_{k+1}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(w_{k+1}(\frac{1-s_{k+1}^n}{s_{k+1}^n})\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(w_{k+1}(\frac{1-i_{k+1}^n}{i_{k+1}^n})\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(w_{k+1}(\frac{d_{k+1}^n}{1-d_{k+1}^n})\gamma)^{\frac{1}{\gamma}}}} \end{array} \right)$$

$$\otimes_{z=1}^{k+1}(\mathfrak{h}_z^{w_z}) = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\ \cdots \\(s_{k+1},i_{k+1},d_{k+1})\in\mathfrak{h}_{k+1}}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k+1} w_z(\frac{1-s_z^n}{s_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{k+1} w_z(\frac{1-i_z^n}{i_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{k+1} w_z(\frac{d_z^n}{1-d_z^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right).$$

Then, the theorem holds for $z = k+1$. Hence, the proof is completed.

□

Example 4.38. Let us consider HT-SFEs given in Example 3.19. Then, we get $HTSDFWGA(\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3) =$

$$\otimes_{z=1}^{3}(\mathfrak{h}_z^{w_z}) = \bigcup_{\substack{(s_1,i_1,d_1)\in\mathfrak{h}_1\\(s_2,i_2,d_2)\in\mathfrak{h}_2\\(s_3,i_3,d_3)\in\mathfrak{h}_3}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{3} w_z(\frac{1-s_z^n}{s_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{3} w_z(\frac{1-i_z^n}{i_z^n})^\gamma)^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{3} w_z(\frac{d_z^n}{1-d_z^n})^\gamma)^{\frac{1}{\gamma}}}} \end{array} \right)$$

$$\begin{aligned} HTSDFWGA(\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3) = \{&(0.4053,0.6051,0.4241),(0.2890,0.4846,0.5475),\\ &(0.3652,0.5100,0.8350),(0.2773,0.4391,0.8440),\\ &(0.4052,0.2474,0.6183),(0.2890,0.2423,0.6675)\} \end{aligned}$$

Theorem 4.39. (Idempotency) Let $\mathfrak{h}_k \in \mathscr{H}^m$ $(k = 1,2,...,m)$. If $\mathfrak{h}_k = \mathfrak{h}$ for $k = 1,2,...,m$, then $HTSDFWGA(\mathfrak{h}_1,\mathfrak{h}_2,...,\mathfrak{h}_m) = \mathfrak{h}$.

Proof. Straightforward in Theorem 4.35. So, the proof is omitted. □

4.2.3 Hesitant T-spherical Dombi fuzzy ordered weighted arithmetic averaging (HTSDFOWAA) operator

Definition 4.40. Let $\mathscr{H}^m = \{\mathfrak{h}_k = \{(s_{kj},i_{kj},d_{kj}) : 1 \le j \le \ell_{\mathfrak{h}_k}, k = 1,2,...,m\}$ be an m dimensional collection of HT-SFEs. An HTSDFOWAA operator is defined by a function

HTSDFOWAA : $\mathscr{H}^m \to \mathscr{H}$ as follows:

$$\begin{aligned} HTSDFOWAA(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, ..., \mathfrak{h}_m) &= \bigoplus_{z=1}^{m} (w_z \mathfrak{h}_{\sigma(z)}) \\ &= (w_1 \mathfrak{h}_{\sigma(1)}) \oplus (w_2 \mathfrak{h}_{\sigma(2)}) \oplus ... \oplus (w_m \mathfrak{h}_{\sigma(m)}), \end{aligned}$$

where $\mathfrak{h}_{\sigma(z)}$ is the z-th largest of $\mathfrak{h}_z$ and w_z is weight vector of $\mathfrak{h}_z (z = 1, 2, ..., m)$, $0 \leq w_z \leq 1$ and $\sum_{z=1}^{m} w_z = 1$.

Theorem 4.41. Let $\mathfrak{h}_k \in \mathscr{H}^m$ $(k = 1, 2, ..., m)$. Then,

$$HTSDFOWAA(\mathfrak{h}_1, \mathfrak{h}_2, ..., \mathfrak{h}_m) = \bigoplus_{z=1}^{m} (w_z \mathfrak{h}_{\sigma(z)})$$

$$= \bigcup_{\substack{(s_{\sigma(1)}, i_{\sigma(1)}, d_{\sigma(1)}) \in \mathfrak{h}_{\sigma(1)} \\ (s_{\sigma(2)}, i_{\sigma(2)}, d_{\sigma(2)}) \in \mathfrak{h}_{\sigma(2)} \\ \dots \\ (s_{\sigma(m)}, i_{\sigma(m)}, d_{\sigma(m)}) \in \mathfrak{h}_{\sigma(m)}}} \left(\begin{array}{c} \sqrt[n]{1 - \frac{1}{1 + (\sum_{z=1}^{m} w_z (\frac{s^n_{\sigma(z)}}{1 - s^n_{\sigma(z)}})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1 + (\sum_{z=1}^{m} w_z (\frac{1 - i^n_{\sigma(z)}}{i^n_{\sigma(z)}})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1 + (\sum_{z=1}^{m} w_z (\frac{1 - d^n_{\sigma(z)}}{d^n_{\sigma(z)}})^{\gamma})^{\frac{1}{\gamma}}}} \end{array} \right)$$

where $\mathfrak{h}_{\sigma(z)}$ is the z-th largest of $\mathfrak{h}_z$ and $W = (w_1, w_2, ..., w_m)^{\tau}$ be the m weight vector of $\mathfrak{h}_k (k = 1, 2, ..., m)$ such that $0 < w_k < 1$ and $\sum_{k=1}^{m} w_k = 1$

Proof. The proof is carried out in the same manner as the proof of the obvious Theorem. 4.33. □

4.2.4 Hesitant T-spherical Dombi fuzzy ordered weighted geometric averaging (HTSDFOWGA) operator

Definition 4.42. Let $\mathscr{H}^m = \{\mathfrak{h}_k = \{(s_{kj}, i_{kj}, d_{kj}) : 1 \leq j \leq \ell_{\mathfrak{h}_k}\}, k = 1, 2, ..., m\}$ be an m dimensional collection of HT-SFEs. An HTSDFOWGA operator is defined by a function $HTSDFOWGA : \mathscr{H}^m \to \mathscr{H}$ as follows:

$$\begin{aligned} HTSDFOWGA(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, ..., \mathfrak{h}_m) &= \bigotimes_{z=1}^{m} (\mathfrak{h}^{w_z}_{\sigma(z)}) \\ &= (\mathfrak{h}^{w_1}_1) \otimes (\mathfrak{h}^{w_2}_2) \otimes ... \otimes (\mathfrak{h}^{w_m}_{\sigma(m)}), \end{aligned}$$

where w_z is weight vector of $\mathfrak{h}_z (z = 1, 2, ..., m)$, $0 \leq w_z \leq 1$ and $\sum_{z=1}^{m} w_z = 1$.

Theorem 4.43. Let $\mathfrak{h}_k \in \mathscr{H}^m$. Then,

$$HTSDFOWGA(\mathfrak{h}_1,\mathfrak{h}_2,...,\mathfrak{h}_m) = \bigotimes_{z=1}^{m}(\mathfrak{h}_{\sigma(z)}^{w_z})$$

$$= \bigcup_{\substack{(s_{\sigma(1)},i_{\sigma(1)},d_{\sigma(1)}) \in \mathfrak{h}_{\sigma(1)} \\ (s_{\sigma(2)},i_{\sigma(2)},d_{\sigma(2)}) \in \mathfrak{h}_{\sigma(2)} \\ \cdots \\ (s_{\sigma(m)},i_{\sigma(m)},d_{\sigma(m)}) \in \mathfrak{h}_{\sigma(m)}}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{m} w_z(\frac{1-s_{\sigma(z)}^n}{s_{\sigma(z)}^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{m} w_z(\frac{1-i_{\sigma(z)}^n}{i_{\sigma(z)}^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{m} w_z(\frac{d_{\sigma(z)}^n}{1-d_{\sigma(z)}^n})^{\gamma})^{\frac{1}{\gamma}}}} \end{array} \right)$$

where $\mathfrak{h}_{\sigma(z)}$ is the z-th largest of $\mathfrak{h}_z$ and $W = (w_1, w_2, ..., w_m)$ be the m weighted vector of $\mathfrak{h}_k(k = 1, 2, ..., m)$ such that $0 < w_k < 1$ and $\sum_{k=1}^{m} w_k = 1$.

Example 4.44. Let us consider $\mathfrak{h}_1 = \{(0.6, 0.8, 0.4), (0.4, 0.5, 0.9), (0.6, 0.2, 0.7)\}$, $\mathfrak{h}_2 = \{(0.3, 0.9, 0.5), (0.2, 0.4, 0.7)\}$ and $\mathfrak{h}_3 = \{(0.5, 0.4, 0.3)\}$ for $n = 3$. When $\gamma = 1$, with weight vector $w = (0.5, 0.3, 0.2)$ we get

$$HTSDFOWGA(\mathfrak{h}_1,\ \ \mathfrak{h}_2,\ \ \mathfrak{h}_3) = \bigotimes_{z=1}^{3}(\mathfrak{h}_{\sigma(z)}^{w_z})$$

$$= \bigcup_{\substack{(s_{\sigma(1)},i_{\sigma(1)},d_{\sigma(1)}) \in \mathfrak{h}_{\sigma(1)} \\ (s_{\sigma(2)},i_{\sigma(2)},d_{\sigma(2)}) \in \mathfrak{h}_{\sigma(2)} \\ (s_{\sigma(3)},i_{\sigma(3)},d_{\sigma(3)}) \in \mathfrak{h}_{\sigma(3)}}} \left(\begin{array}{c} \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{3} w_z(\frac{1-s_{\sigma(z)}^n}{s_{\sigma(z)}^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{\frac{1}{1+(\sum_{z=1}^{3} w_z(\frac{1-i_{\sigma(z)}^n}{i_{\sigma(z)}^n})^{\gamma})^{\frac{1}{\gamma}}}}, \\ \sqrt[n]{1-\frac{1}{1+(\sum_{z=1}^{3} w_z(\frac{d_{\sigma(z)}^n}{1-d_{\sigma(z)}^n})^{\gamma})^{\frac{1}{\gamma}}}} \end{array} \right).$$

By using Eq. 3.1, SVs of HT-SFEs are obtained as follows:

$$SV(\mathfrak{h}_1) = -0.2133,$$
$$SV(\mathfrak{h}_2) = -0.2165,$$
$$SV(\mathfrak{h}_3) = 0.089.$$

Here, $SV(\mathfrak{h}_3) > SV(\mathfrak{h}_1) > SV(\mathfrak{h}_2)$ and

$$\begin{aligned}
\mathfrak{h}_{\sigma(1)} &= \mathfrak{h}_3 = \{(0.5, 0.4, 0.3)\} \\
\mathfrak{h}_{\sigma(2)} &= \mathfrak{h}_1 = \{(0.6, 0.8, 0.4), (0.4, 0.5, 0.9), (0.6, 0.2, 0.7)\} \\
\mathfrak{h}_{\sigma(3)} &= \mathfrak{h}_2 = \{(0.3, 0.9, 0.5), (0.2, 0.4, 0.7)\}.
\end{aligned}$$

Then,

$$\begin{aligned}
&HTSDFOWGA(\mathfrak{h}_1, \quad \mathfrak{h}_2, \quad \mathfrak{h}_3) \\
&\qquad = \{(0.4275, 0.4867, 0.3898), (0.3961, 0.4569, 0.7716), \\
&\qquad\quad (0.4275, 0.2799, 0.5496), (0.3205, 0.3105, 0.4958), \\
&\qquad\quad (0.3096, 0.3051, 0.7833), (0.3205, 0.2423, 0.5997)\}
\end{aligned}$$

5. MULTIPLE CRITERIA GROUP DECISION MAKING (MCGDM)

Let $\kappa = \{\kappa_1, \kappa_2, ..., \kappa_l\}$ be set of alternatives, $\varepsilon = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_s\}$ be a set of criteria and the set of decision makers is $\partial = \{\partial_1, \partial_2, ..., \partial_t\}$. $W = (w_1, w_2, ...w_s)$ such that $w_j \in (0,1]$ and $\sum_{j=1}^{s} w_j = 1$ is the weight vector of the criteria which is determined by decision makers. The steps of the MCGDM method are given as follows:

Step 1: The evaluation of the alternative κ_i according to criteria ε_j performed by decision makers ∂_y $(y = 1,2,...,t)$ can be written as: $\zeta_{yj}(i = 1,2,...,l; j = 1,2,...,s; y = 1,2,...,t)$. Hence, HT-SF-decision matrix $DM_{\kappa_i} = [\zeta_{yj}]_{t\times s}$ can be put together as follows:

$$DM_{\kappa_i} = [\zeta_{yj}]_{t\times s} = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1s} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2s} \\ \vdots & \vdots & \cdots & \vdots \\ \zeta_{t1} & \zeta_{t2} & \cdots & \zeta_{ts} \end{pmatrix}.$$

Step 2: For all $i = 1,2,...,l$, HT-SFS denoted by $HTSF_i$ is obtained as follows:

$$HTSF_i = \Big\{(\varepsilon_j, \mathfrak{h}_{\varepsilon_j}) : j = 1,2,...,s\Big\}.$$

Here, $\mathfrak{h}_{\varepsilon_j} = \cup_{y=1}^{t}\{\zeta_{yj}\}$.

Step 3: For $\kappa_i, i = 1,2,3,...,l$ HT-SF element related to κ_i denoted by $\mathfrak{A}_i$, is defined as follows:

$$\mathfrak{A}_i = \bigoplus_{j=1}^{s} w_j \mathfrak{h}_{\varepsilon_j}.$$

Step 4: Find score values of $\mathfrak{A}_i$ $(i = 1,2,3,...,l)$.

Step 5: Order score values of $\mathfrak{A}_i$ $(i = 1,2,3,...,l)$.

Step 6:Choose the one with the highest score value. Flowchart of the algorithm is given in Figure 5.3.

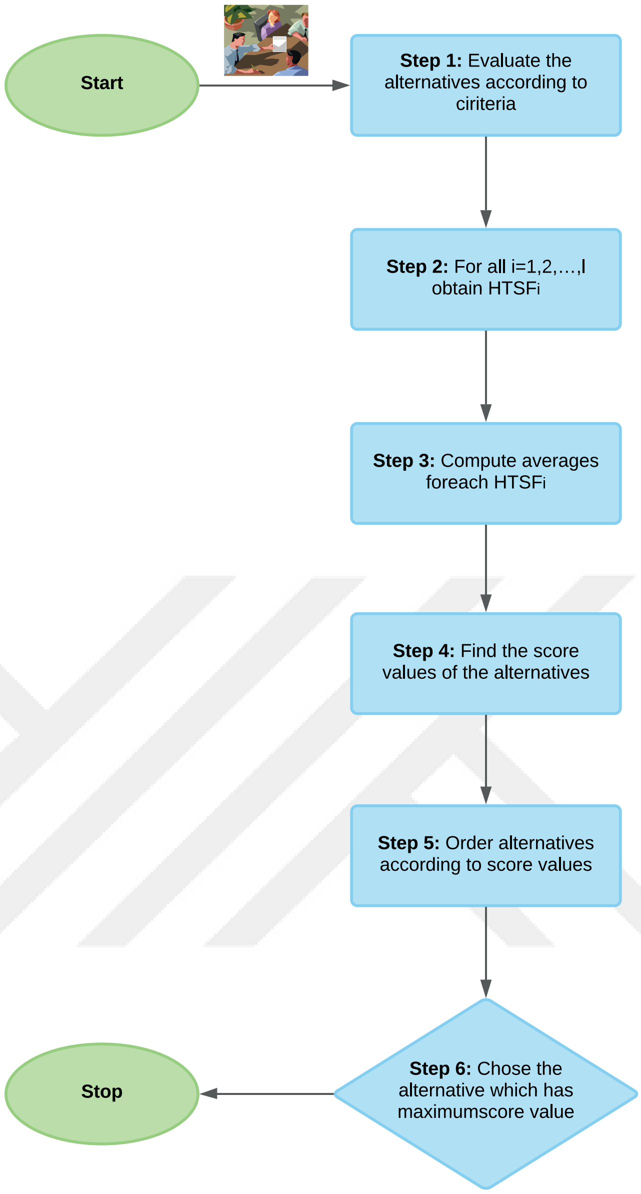

Figure 5.3. Flowchart of the proposed method

5.1 Illustrative Example

We consider that university wants to fulfill the position of one assistant professorship in a department. After the announcement for this vacant position, seven candidates $\kappa_1, \kappa_2, \ldots, \kappa_7$ apply for the position. University rector assigns three experts ∂_1, ∂_2, and ∂_3 to evaluate alternatives according to criterion ε_1 =experience, ε_2 = scientific works and ε_3 =quality of the researches. After interview, experts determine the weight vector of the criteria as $(0.35, 0.25, 0.40)^T$.

Step 1: Experts evaluate the alternatives by HT-SFNs corresponding to linguistic variables given in Table 5.2 for each criteria and $n = 4$.

Table 5.2 Linguistic variables table for evaluation of the candidates

Grades	HT-SFNs
Very Poor (VP)	(0.100,0.700,0.900)
Poor (P)	(0.233,0.634,0.767)
Medium Poor (MP)	(0.367,0.567,0.634)
Fairly (F)	(0.500,0.500,0.500)
Medium Good (MG)	(0.633,0.436,0.367)
Good (G)	(0.764,0.370,0.234)
Very Good (VG)	(0.900,0.300,0.100)

		ε_1	ε_2	ε_3
	∂_1	(0.633,0.436,0.367)	(0.233,0.634,0.767)	*
D_{κ_1}	∂_2	(0.100,0.700,0.900)	(0.900,0.300,0.100)	(0.500,0.500,0.500)
	∂_3	*	(0.764,0.370,0.234)	*

		ε_1	ε_2	ε_3
	∂_1	*	(0.233,0.634,0.767)	*
D_{κ_2}	∂_2	(0.233,0.634,0.767)	(0.367,0.567,0.634)	(0.500,0.500,0.500)
	∂_3	(0.764,0.370,0.234)	(0.100,0.700,0.900)	(0.633,0.436,0.367)

		ε_1	ε_2	ε_3
	∂_1	(0.367,0.567,0.634)	(0.633,0.436,0.367)	(0.100,0.700,0.900)
D_{κ_3}	∂_2	(0.900,0.300,0.100)	(0.764,0.370,0.234)	*
	∂_3	(0.633,0.436,0.367)	(0.100,0.700,0.900)	(0.500,0.500,0.500)

		ε_1	ε_2	ε_3
	∂_1	(0.900,0.300,0.100)	(0.500,0.500,0.500)	(0.100,0.700,0.900)
D_{κ_4}	∂_2	(0.900,0.300,0.100)	*	(0.500,0.500,0.500)
	∂_3	*	(0.100,0.700,0.900)	(0.500,0.500,0.500)

		ε_1	ε_2	ε_3
	∂_1	(0.367,0.567,0.634)	(0.500,0.500,0.500)	(0.100,0.700,0.900)
D_{κ_5}	∂_2	(0.100,0.700,0.900)	(0.367,0.567,0.634)	(0.633,0.436,0.367)
	∂_3	(0.900,0.300,0.100)	(0.100,0.700,0.900)	(0.500,0.500,0.500)

		ε_1	ε_2	ε_3
	∂_1	(0.500,0.500,0.500)	(0.633,0.436,0.367)	(0.764,0.370,0.234)
D_{κ_6}	∂_2	*	(0.367,0.567,0.634)	(0.633,0.436,0.367)
	∂_3	*	(0.233,0.634,0.767)	(0.500,0.500,0.500)

		ε_1	ε_2	ε_3
	∂_1	(0.500,0.500,0.500)	(0.633,0.436,0.367)	(0.500,0.500,0.500)
D_{κ_7}	∂_2	(0.367,0.567,0.634)	(0.100,0.700,0.900)	(0.233,0.634,0.767)
	∂_3	(0.233,0.634,0.767)	(0.100,0.700,0.900)	*

Step 2: By using HT-SF decision matrices given in Step 1, $HTSF_i\,(i=1,2,..,7)$ are obtained as follows:

$$\begin{aligned}
HTSF_1 &= \Big\{\Big(\varepsilon_1,\{(0.633,0.436,0.367),(0.100,0.700,0.900)\}\Big),\Big(\varepsilon_2,\{(0.233,0.634,0.767),\\
&\quad (0.900,0.300,0.100),(0.764,0.370,0.234)\}\Big),\Big(\varepsilon_3,\{(0.500,0.500,0.500)\}\Big)\Big\},\\
HTSF_2 &= \Big\{\Big(\varepsilon_1,\{(0.764,0.370,0.234),(0.233,0.634,0.767\}\Big),\Big(\varepsilon_2,\{(0.100,0.700,0.900),\\
&\quad (0.233,0.634,0.767),(0.367,0.567,0.634)\}\Big),\Big(\varepsilon_3,\{(0.500,0.500,0.500),\\
&\quad (0.633,0.436,0.367)\}\Big)\Big\},\\
HTSF_3 &= \Big\{\Big(\varepsilon_1,\{(0.367,0.567,0.634),(0.900,0.300,0.100),(0.633,0.436,0.367)\}\Big),\\
&\quad \Big(\varepsilon_2,\{(0.633,0.436,0.367),(0.764,0.370,0.234),(0.100,0.700,0.900)\}\Big),\\
&\quad \Big(\varepsilon_3,\{(0.100,0.700,0.900),(0.500,0.500,0.500)\}\Big)\Big\},\\
HTSF_4 &= \Big\{\Big(\varepsilon_1,\{(0.900,0.300,0.100),(0.900,0.300,0.100)\}\Big),\Big(\varepsilon_2,\{(0.100,0.700,0.900),\\
&\quad (0.500,0.500,0.500)\}\Big),\Big(\varepsilon_3,\{(0.100,0.700,0.900),(0.500,0.500,0.500),\\
&\quad (0.500,0.500,0.500)\}\Big)\Big\},\\
HTSF_5 &= \Big\{\Big(\varepsilon_1,\{(0.367,0.567,0.634),(0.100,0.700,0.900),(0.900,0.300,0.100)\}\Big),\\
&\quad \Big(\varepsilon_2,\{(0.500,0.500,0.500),(0.367,0.567,0.634),(0.100,0.700,0.900)\}\Big),\\
&\quad \Big(\varepsilon_3,\{(0.100,0.700,0.900),(0.633,0.436,0.367),(0.500,0.500,0.500)\}\Big)\Big\},\\
HTSF_6 &= \Big\{\Big(\varepsilon_1,\{(0.500,0.500,0.500)\}\Big),\Big(\varepsilon_2,\{(0.633,0.436,0.367),(0.367,0.567,0.634),\\
&\quad (0.233,0.634,0.767)\}\Big),\Big(\varepsilon_3,\{(0.764,0.370,0.234),(0.633,0.436,0.367),\\
&\quad (0.500,0.500,0.500)\}\Big)\Big\},\\
HTSF_7 &= \Big\{\Big(\varepsilon_1,\{(0.500,0.500,0.500),(0.367,0.567,0.634),(0.233,0.634,0.767)\}\Big),\\
&\quad \Big(\varepsilon_2,\{(0.633,0.436,0.367),(0.100,0.700,0.900),(0.100,0.700,0.900)\}\Big),\\
&\quad \Big(\varepsilon_3,\{(0.500,0.500,0.500),(0.233,0.634,0.767)\}\Big)\Big\}
\end{aligned}$$

Step 3: For $n = 4$ and $\gamma = 1$, HTSDFWAA and HTSDFWGA values of $HTSF_i$, $(i = 1, 2, ..., 7)$ are obtained as in Table 5.3.

Table 5.3 HTSDFWAA and HTSDFWGA values of $HTSF_i$ $(i = 1, 2, ..., 7)$

	HTSDFWAA	HTSDFWGA
$\mathfrak{A}_1$	{(0.542, 0.488, 0.441),(0.776, 0.382, 0.141),(0.653, 0.429, 0.309) (0.404, 0.571, 0.601),(0.761, 0.401, 0.141),(0.606, 0.466, 0.324)}	{(0.322, 0.488, 0.614), (0.578, 0.382, 0.423), (0.572, 0.429, 0.425), (0.129, 0.571, 0.820), (0.130, 0.401, 0.800), (0.130, 0.466, 0.800)}
$\mathfrak{A}_2$	{(0.408, 0.561, 0.592), (0.519, 0.510, 0.453), (0.421, 0.549, 0.578), (0.525, 0.503, 0.449), (0.405, 0.568, 0.599), (0.518, 0.514, 0.454), (0.644, 0.440, 0.300), (0.673, 0.423, 0.291), (0.647, 0.437, 0.299), (0.675, 0.420, 0.291), (0.644, 0.442, 0.300) (0.673, 0.425, 0.291)}	{(0.263, 0.561, 0.711), (0.264, 0.510, 0.704), (0.291, 0.549, 0.674), (0.293, 0.503, 0.664), (0.140, 0.568, 0.799), (0.140, 0.514, 0.796), (0.323, 0.440, 0.609), (0.326, 0.423, 0.593), (0.466, 0.437, 0.515), (0.490, 0.420, 0.481), (0.141, 0.442, 0.761), (0.141, 0.425, 0.756)}
$\mathfrak{A}_3$	{(0.476, 0.538, 0.496), (0.523, 0.495, 0.460), (0.588, 0.483, 0.328), (0.611, 0.456, 0.323), (0.284, 0.636, 0.750), (0.423, 0.551, 0.578), (0.804, 0.373, 0.130), (0.808, 0.365, 0.130), (0.816, 0.362, 0.129), (0.820, 0.355, 0.129), (0.795, 0.384, 0.130), (0.800, 0.375, 0.130), (0.566, 0.484, 0.415), (0.593, 0.457, 0.399), (0.636, 0.449, 0.314), (0.653, 0.429, 0.309) (0.501, 0.533, 0.471), (0.541, 0.492, 0.443)}	{(0.126, 0.538, 0.821), (0.438, 0.495, 0.548), (0.126, 0.483, 0.821), (0.441, 0.456, 0.543), (0.111, 0.636, 0.868), (0.141, 0.551, 0.776), (0.126, 0.373, 0.812), (0.586, 0.365, 0.417), (0.126, 0.362, 0.811), (0.601, 0.355, 0.404), (0.111, 0.384, 0.863), (0.141, 0.375, 0.761), (0.126, 0.484, 0.813), (0.560, 0.457, 0.437), (0.126, 0.449, 0.812), (0.572, 0.429, 0.425), (0.111, 0.533, 0.863), (0.141, 0.492, 0.762)}
$\mathfrak{A}_4$	{(0.798, 0.378, 0.130), (0.798, 0.378, 0.130), (0.798, 0.378, 0.130), (0.798, 0.378, 0.130), (0.803, 0.370, 0.130), (0.803, 0.370, 0.130), (0.800, 0.375, 0.130), (0.800, 0.375, 0.130), (0.800, 0.375, 0.130), (0.800, 0.375, 0.130), (0.800, 0.375, 0.130), (0.800, 0.375, 0.130)}	{(0.126, 0.378, 0.814), (0.126, 0.378, 0.814), (0.126, 0.378, 0.814), (0.126, 0.378, 0.814), (0.550, 0.370, 0.452), (0.550, 0.370, 0.452), (0.141, 0.375, 0.761), (0.141, 0.375, 0.761), (0.141, 0.375, 0.761), (0.141, 0.375, 0.761), (0.141, 0.375, 0.761), (0.141, 0.375, 0.761)}
$\mathfrak{A}_5$	{(0.388, 0.577 0.620), (0.549, 0.482, 0.434), (0.467, 0.519, 0.531), (0.324, 0.605, 0.694), (0.533, 0.493, 0.444), (0.437, 0.535, 0.561), (0.284, 0.636, 0.750), (0.526, 0.504, 0.449), (0.423, 0.551, 0.578), (0.358, 0.612, 0.664), (0.540, 0.496, 0.440), (0.452, 0.539, 0.550), (0.261, 0.652, 0.781), (0.523, 0.509, 0.451), (0.417, 0.559, 0.586), (0.100, 0.700, 0.900), (0.516, 0.521, 0.457), (0.402, 0.579, 0.608), (0.798, 0.378, 0.130), (0.811, 0.362, 0.130), (0.803, 0.370, 0.130), (0.796, 0.381, 0.130), (0.809, 0.364, 0.130), (0.801, 0.372, 0.130), (0.795, 0.384, 0.130), (0.808, 0.367, 0.130), (0.800, 0.375, 0.130)}	{(0.126, 0.577, 0.823), (0.444, 0.482, 0.538), (0.428, 0.519, 0.562), (0.125, 0.605, 0.827), (0.409, 0.493, 0.575), (0.399, 0.535, 0.594), (0.111, 0.636, 0.868), (0.141, 0.504, 0.772), (0.141, 0.551, 0.776), (0.107, 0.612, 0.877), (0.130, 0.496, 0.800), (0.130, 0.539, 0.803), (0.107, 0.652, 0.879), (0.130, 0.509, 0.805), (0.130, 0.559, 0.808), (0.100, 0.700, 0.900), (0.114, 0.521, 0.855), (0.114, 0.579, 0.857), (0.126, 0.378, 0.814), (0.614, 0.362, 0.392), (0.550, 0.370, 0.452), (0.126, 0.381, 0.818), (0.494, 0.364, 0.479), (0.469, 0.372, 0.514), (0.111, 0.384, 0.863), (0.141, 0.367, 0.756), (0.141, 0.375, 0.761)}
$\mathfrak{A}_6$	{(0.683, 0.415, 0.284), (0.599, 0.454, 0.394), (0.546, 0.479, 0.444), (0.660, 0.430, 0.290), (0.555, 0.478, 0.430), (0.477, 0.513, 0.521), (0.658, 0.433, 0.291), (0.550, 0.484, 0.433), (0.469, 0.521, 0.529)}	{(0.589, 0.415, 0.410), (0.567, 0.454, 0.430), (0.521, 0.479, 0.477), (0.470, 0.430, 0.510), (0.462, 0.478, 0.520), (0.444, 0.513, 0.547), (0.323, 0.433, 0.607), (0.322, 0.484, 0.612), (0.319, 0.521, 0.627)}
$\mathfrak{A}_7$	{(0.546, 0.479, 0.444), (0.510, 0.508, 0.470), (0.467, 0.526, 0.533), (0.393, 0.575, 0.612), (0.467, 0.526, 0.533), (0.393, 0.575, 0.612), (0.523, 0.495, 0.460), (0.479, 0.529, 0.492), (0.423, 0.551, 0.578), (0.295, 0.617, 0.721), (0.423, 0.551, 0.578), (0.295, 0.617, 0.721), (0.515, 0.504, 0.466), (0.467, 0.542, 0.500), (0.405, 0.568, 0.599), (0.217, 0.648, 0.791), (0.405, 0.568, 0.599), (0.217, 0.648, 0.791)}	{(0.521, 0.479, 0.477), (0.289, 0.508, 0.663), (0.141, 0.526, 0.766), (0.139, 0.575, 0.803), (0.141, 0.526, 0.766), (0.139, 0.575, 0.803), (0.438, 0.495, 0.548), (0.283, 0.529, 0.686), (0.141, 0.551, 0.776), (0.139, 0.617, 0.810), (0.141, 0.551, 0.776), (0.139, 0.617, 0.810), (0.298, 0.504, 0.650), (0.250, 0.542, 0.732), (0.140, 0.568, 0.799), (0.138, 0.648, 0.826), (0.140, 0.568, 0.799), (0.138, 0.648, 0.826)}

Step 4: Score values of $\mathfrak{A}_i, (i = 1, 2, ..., 7)$ under score function are obtained as in Table 5.4:

Table 5.4 Score values of $\mathfrak{A}_i$ according to HTSDFWAA and HTSDFWGA values

	HTSDFWAA	HTSDFWGA
$S(\mathfrak{A}_1)$	0.156	-0.208
$S(\mathfrak{A}_2)$	0.076	-0.217
$S(\mathfrak{A}_3)$	0.165	-0.285
$S(\mathfrak{A}_4)$	0.409	-0.305
$S(\mathfrak{A}_5)$	0.063	-0.335
$S(\mathfrak{A}_6)$	0.088	-0.035
$S(\mathfrak{A}_7)$	-0.104	-0.319

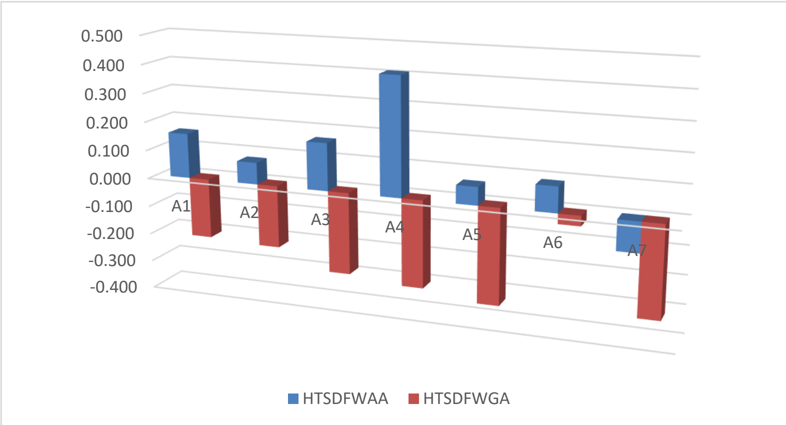

Figure 5.4. Score values of $\mathfrak{A}_i$ according to HTSDFWAA and HTSDFWGA values

Step 5: By using Eq. 3.1 ordering of the candidates are obtained as in Table 5.5.

Table 5.5 Score values of $\mathfrak{A}_i$ obtained by using the proposed aggregation operators

	Ordering
HTSDFWAA	$S(\mathfrak{A}_4) > S(\mathfrak{A}_3) > S(\mathfrak{A}_1) > S(\mathfrak{A}_6) > S(\mathfrak{A}_2) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7)$
HTSDFWGA	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_4) > S(\mathfrak{A}_7) > S(\mathfrak{A}_5)$

Step 6: From the above illustration, Despite the fact that the overall ranking values of the options fluctuate due to the employment of two operators, optimum alternatives are κ_4 and κ_6 for the two operators, respectively.

6. ANALYSIS OF THE EFFECT OF PARAMETER γ ON THE RESULTS

To show the effect of the γ variable in the formula of $HTSDFWAA$ and $HTSDFWGA$ on MCGDM results, we assign different values to γ from 1 to 10 and order candidates according to score values based on $HTSDFWAA$ and $HTSDFWGA$. Ranking orders of the candidates according to score values and their ranking orders based on $HTSDFWAA$ and $HTSDFWGA$ operators are shown in Table 5.6. It is clear when γ value is changed in the formula $HTSDFWAA$, the optimum candidate is always the same person and orderings of candidates
$(S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7))$ are same except in condition $\gamma = 1$. By Table 5.7, it is clear that when the value of γ is changed for $HTSDFWGA$ operator, the ranking orders of candidates
$(S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7) > S(\mathfrak{A}_4))$ are same except from in case $\gamma = 1$. Also, best suitable candidate is identical for $1 \leq \gamma \leq 10$.

Table 5.6 Ranking order for different γ values in the HTSDFWAA operator

γ	$S(\mathfrak{A}_1)$	$S(\mathfrak{A}_2)$	$S(\mathfrak{A}_3)$	$S(\mathfrak{A}_4)$	$S(\mathfrak{A}_5)$	$S(\mathfrak{A}_6)$	$S(\mathfrak{A}_7)$	Ranking order
1	0.273	0.148	0.265	0.573	0.156	0.134	-0.049	$S(\mathfrak{A}_4) > S(\mathfrak{A}_3) > S(\mathfrak{A}_1) > S(\mathfrak{A}_6) > S(\mathfrak{A}_2) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7)$
2	0.237	0.125	0.235	0.530	0.130	0.118	-0.067	$S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7)$
3	0.273	0.148	0.265	0.573	0.156	0.134	-0.049	$S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7)$
4	0.292	0.161	0.281	0.595	0.170	0.143	-0.040	$S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7)$
5	0.305	0.169	0.291	0.607	0.178	0.149	-0.033	$S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7)$
6	0.313	0.175	0.298	0.615	0.183	0.153	-0.029	$S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7)$
7	0.319	0.179	0.303	0.621	0.187	0.157	-0.026	$S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7)$
8	0.323	0.182	0.307	0.626	0.190	0.159	-0.023	$S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7)$
9	0.327	0.185	0.310	0.629	0.192	0.161	-0.022	$S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7)$
10	0.330	0.187	0.312	0.632	0.194	0.162	-0.020	$S(\mathfrak{A}_4) > S(\mathfrak{A}_1) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_2) > S(\mathfrak{A}_6) > S(\mathfrak{A}_7)$

Graphical representation of the Table 5.6 is given in Figure 5.5.

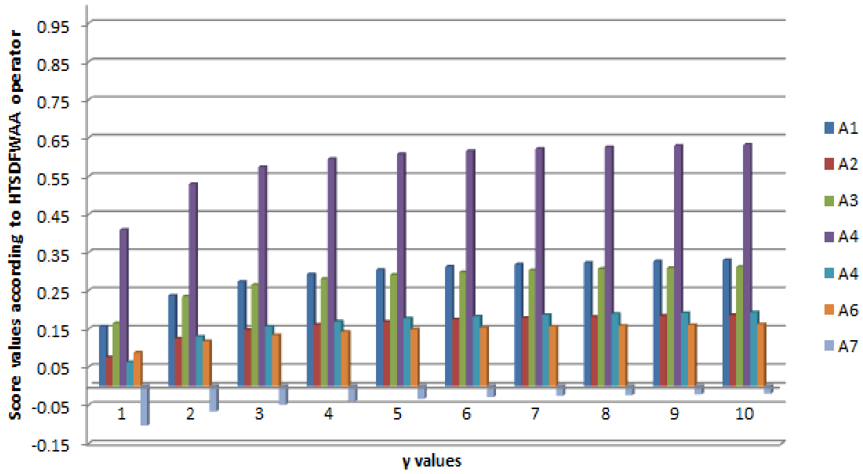

Figure 5.5. Ranking order for different γ values in the HTSDFWAA operator

Table 5.7 Ranking order for different γ values in the HTSDFWGA operator

γ	$S(\mathfrak{A}_1)$	$S(\mathfrak{A}_2)$	$S(\mathfrak{A}_3)$	$S(\mathfrak{A}_4)$	$S(\mathfrak{A}_5)$	$S(\mathfrak{A}_6)$	$S(\mathfrak{A}_7)$	Ranking order
1	-0.316	-0.333	-0.392	-0.465	-0.435	-0.099	-0.434	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_4) > S(\mathfrak{A}_7) > S(\mathfrak{A}_5)$
2	-0.284	-0.297	-0.363	-0.422	-0.406	-0.076	-0.395	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_7) > S(\mathfrak{A}_5) > S(\mathfrak{A}_4)$
3	-0.316	-0.333	-0.392	-0.465	-0.435	-0.099	-0.434	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_7) > S(\mathfrak{A}_5) > S(\mathfrak{A}_4)$
4	-0.333	-0.352	-0.408	-0.486	-0.449	-0.113	-0.455	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7) > S(\mathfrak{A}_4)$
5	-0.344	-0.364	-0.417	-0.498	-0.458	-0.123	-0.468	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7) > S(\mathfrak{A}_4)$
6	-0.350	-0.372	-0.423	-0.507	-0.464	-0.129	-0.477	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7) > S(\mathfrak{A}_4)$
7	-0.355	-0.378	-0.427	-0.512	-0.468	-0.133	-0.484	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7) > S(\mathfrak{A}_4)$
8	-0.359	-0.383	-0.431	-0.517	-0.472	-0.137	-0.489	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7) > S(\mathfrak{A}_4)$
9	-0.362	-0.386	-0.433	-0.520	-0.474	-0.140	-0.492	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7) > S(\mathfrak{A}_4)$
10	-0.364	-0.389	-0.435	-0.523	-0.476	-0.142	-0.495	$S(\mathfrak{A}_6) > S(\mathfrak{A}_1) > S(\mathfrak{A}_2) > S(\mathfrak{A}_3) > S(\mathfrak{A}_5) > S(\mathfrak{A}_7) > S(\mathfrak{A}_4)$

Graphical representation of the Table 5.7 is given in Figure 5.6.

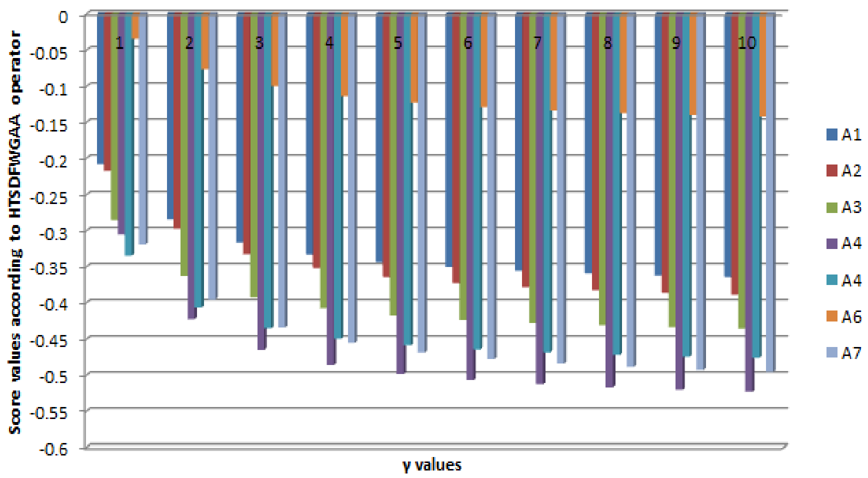

Figure 5.6. Ranking order for different γ values in the HTSDFWGA operator

7. A COMPARISON AMONG THE EXTENSIONS OF FUZZY SETS

In this section, we compare HT-SFS with other extensions of the fuzz sets. Let $\mathbb{T}_H = \left\{(x, \{(s_k, i_k, d_k) : 1 \leq k \leq l_x\}) : x \in \mathfrak{X}\right\}$. Then, comparison table can be given as in Table 6.8.

Table 6.8 Comparison table of the HT-SFS with some extensions of fuzzy set

	n (degrees of component)			**k** (lenght)		**Condition**
	1	2	>2	1	>1	
Fuzzy Set (1965)	✓	×	×	✓	×	$0 \leq s_k + d_k = 1$, $i_k = 0$
Intuitionistic Fuzzy Set (1986)	✓	×	×	✓	×	$s_k + i_k + d_k = 1$
Pythagorean Fuzzy Set (2013a)	✓	✓	×	✓	×	$0 \leq s_k^2 + d_k^2 \leq 1$
Picture Fuzzy Set (2013a)	✓	×	×	✓	×	$0 \leq s_k + i_k + d_k \leq 1$
Spherical Fuzzy Set (2019a)	✓	✓	×	✓	×	$0 \leq s_k^2 + i_k^2 + d_k^2 \leq 1$
q-rung Orthopair Fuzzy Set (2013b)	✓	✓	✓	✓	×	$0 \leq s_k^n + d_k^n \leq 1$
T-Spherical Fuzzy Set (2019a)	✓	✓	✓	✓	×	$0 \leq s_k^n + i_k^n + d_k^n \leq 1$
Hesitant Fuzzy Set (2009, 2010)	✓	×	×	✓	✓	$0 \leq s_k + d_k = 1$, $i_k = 0$
Picture Hesitant Fuzzy Set (2018)	✓	✓	×	✓	✓	$0 \leq s_k + i_k + d_k \leq 1$
Hesitant T-Spherical Fuzzy Set	✓	✓	✓	✓	✓	$0 \leq s_k^n + i_k^n + d_k^n \leq 1$

Here, we see that HT-SFS is an extension of sets specified in the Table 6.8. Therefore, the set structure defined in this thesis has advantages of the other extensions of fuzzy sets specified in Table 6.8 in the modelling. It also model some problems which can no be modelled with existing set theories.

8. CONCLUSION

In this thesis, the concept of HT-SFSs and its set theoretical operations such as union, intersection and complement have been defined. To be more understandable, some examples are given related to defined operations. Based on Dombi t-norm and Domb t-conorm operations, arithmetic operations between two HT-SFEs and some aggregation operators such as HTSDFWAA, HTSDFWGA, HTSDFOWAA and HTSDFOWGA operators have been introduced. Furthermore, an MCGDM method has been developed and and we presented an application to MCGDM problem involving selecting a person for a vacant position in any department of the university. Obtained results have been compared for different parameters. Also, the proposed sets have been compared by other extensions of the fuzzy sets and mentioned its advantages. Our long-term goals include researching additional aggregation operators, distance measures, similarity measures, and TOPSIS, VIKOR, AHP, and other decision-making methods are used.

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SCIENTIFIC ACTIVITIES

a) Publications

1. F. Karaaslan, A.H.S Al-Husseinawi, Hesitant T-spherical Dombi Fuzzy Aggregation Operators and Their Applications in Multiple Criteria Group Decision-Making, Under Review.