# A NEW GEOMETRIC DUALITY AND APPROXIMATION ALGORITHMS FOR CONVEX VECTOR OPTIMIZATION PROBLEMS

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

MASTER OF SCIENCE

IN

INDUSTRIAL ENGINEERING

By

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July 2021

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

## A NEW GEOMETRIC DUALITY AND APPROXIMATION ALGORITHMS FOR CONVEX VECTOR OPTIMIZATION PROBLEMS

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In the literature, there are different algorithms for solving convex vector optimization problems, in the sense of approximating the set of all minimal points in the objective space. One of the main approaches is to provide outer approximations to this set and improve the approximation iteratively by solving scalarization models. In addition to the outer approximation algorithms, which are referred to as primal algorithms, there are also geometric dual algorithms which work on a dual space and approximate the set of all maximal elements of a geometric dual problem. In most of the primal and dual algorithms in the literature, the scalarization methods, the solution concepts and the design of the algorithms depend on a fixed direction vector from the ordering cone.

Recently, a primal algorithm that does not depend on a direction parameter is proposed in [1]. Using the primal algorithm in [1], we construct a new geometric dual algorithm based on a new geometric duality relation between the primal and dual images. This relation is shown by providing an inclusion reversing one-to-one correspondence between weakly minimal proper faces of the primal image and maximal proper faces of the dual image. For a primal problem with a $q$-dimensional objective space, we present a dual problem with a $q+1$-dimensional objective space. Consequently, the resulting dual image is a convex cone.

The primal algorithm in [1] is modified to give a finite $\epsilon$-solution to the dual problem as well as a finite weak $\epsilon$-solution to the primal problem. The constructed geometric dual algorithm gives a finite $\epsilon$-solution to the dual problem; moreover, it gives a finite weak $\tilde{\epsilon}$-solution to the primal problem, where $\tilde{\epsilon}$ is determined by

$\epsilon$ and the structure of the underlying ordering cone.

We implement primal and dual algorithms using MATLAB, and test the performance of the algorithms for randomly generated convex vector optimization problems. The tests are performed with different dimensions of the objective and decision spaces, different ordering cones, different $\ell_p$-norms and different stopping criteria. It is observed that the dual algorithm gives a fraction of the allowed approximation error, $\epsilon$, resulting in a longer runtime with $\epsilon$ stopping criterion. When runtime is used as stopping criterion, the dual algorithm returns a closer approximation for higher dimensions of the objective space.

*Keywords:* convex vector optimization, multiobjective optimization, approximation algorithm, scalarization.

# ÖZET

# DIŞBÜKEY VEKTÖR ENİYİLEME PROBLEMLERİ İÇİN YENİ BİR GEOMETRİK ÇİFTEŞLİK TEORİSİ VE DIŞ YAKLAŞIKLAMA ALGORİTMALARI

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Literatürde, dışbükey vektör eniyileme problemlerinin çözümü için farklı yöntemler geliştirilmiştir. Yaklaşıklama algoritmaları da bu yöntemlerden bir tanesidir. Yaklaşıklama algoritmaları, karar uzayındaki bütün verimli çözümlerin kümesini elde etmek yerine amaç uzayındaki Pareto kümeyi yaklaşıklamayı hedefler. Yaklaşıklama algoritmalarının temel yöntemlerinden bir tanesi, skalerizasyon modelleri çözerek, yaklaşık kümeyi her adımda iyileştirmektir. Temel algoritma olarak adlandırılan dış yaklaşıklama algoritmalarına ek olarak, geometrik çifteş algoritma olarak adlandırılan ve geometrik çifteş problemi yaklaşıklayan, çifteş uzayda tanımlı algoritmalar vardır. Literatürdeki pek çok temel ve çifteş algoritmanın skalerizasyon yöntemleri, çözüm tanımları ve yapıları, belirli bir yön vektörüne bağlı olarak tasarlanmıştır.

Yakın zamanda, yön vektörü parametresine bağlı olmayan bir temel algoritma geliştirildi [1]. Temel ve geometrik çifteş kümeler arasındaki geometrik çifteşlik ilişkisini kullanarak, geliştirilen bu temel algoritmaya yeni bir geometrik çifteş algoritma oluşturduk. Geometrik çifteşlik teorisi, çifteş problemin küme değerli eniyi değeri (alt görüntü kümesi) ile üst görüntü kümesi arasında, çokyüzlülerin arasındaki geometrik çifteşlik ilişkisine dayanmaktadır. Bu tezde sunulan temel ve geometrik çifteş algoritmalar bir yön vektörüne bağlı olmadıklarından, alt görüntü kümesi ve üst görüntü kümesi arasında geometrik çifteşlik ilişkisi kurabilmek için yeni bir yöntem geliştirdik. Bunun sonucunda, amaç uzayı boyutu $q$ olan bir temel problem için $q+1$ boyutlu amaç uzayına sahip bir geometrik çifteş problem geliştirdik. Geliştirilen bu geometrik çifteş problemin alt görüntü kümesini ise dışbükey bir koni olarak belirledik.

Temel algoritmayı bu yeni geometrik çifteş probleme de sonlu bir $\epsilon$-çözüm verecek şekilde değiştirdik. Geliştirdigimiz geometrik çifteş algoritmayı ise temel probleme sonlu zayıf $\tilde{\epsilon}$-çözüm ve geometrik çifteş probleme sonlu $\epsilon$-çözüm verecek şekilde geliştirdik.

Temel ve geometrik çifteş algoritmaları MATLAB programı kullanarak hayata geçirdik ve algoritmaların performanslarını kıyaslamak için rastgele dışbükey vektör eniyileme problemleri yarattık. Testler sırasında farklı amaç ve karar uzayı boyutları, farklı sıralama konileri, farklı $\ell_p$ norm değerleri ve farklı durma koşulları kullandık. Geometrik çifteş algoritmanın verilen hata payı durma koşulunun çok altında bir hata sonucu ile durduğunu gözlemledik. Bu durum algoritmanın temel algoritmaya kıyasla çok daha uzun bir çözüm süresine sahip olmasına neden oldu. Durma koşulu çözüm süresi olarak belirlendiğinde, geometrik çifteş algoritmanın özellikle büyük amaç uzayı boyutuna sahip problemlerde daha iyi bir yaklaşıklama sonucu verdiğini gözlemledik.

*Anahtar sözcükler*: dışbükey vektör eniyileme, çok amaçlı eniyileme, yaklaşıklama algoritmaları, skalerizasyon.

# Acknowledgement

First and foremost, I would like to express my everlasting gratitude to my advisors Firdevs Ulus and Çağın Ararat. I feel extremely fortunate that they always prioritized my development during this thesis. I cannot thank them enough for their guidance and support.

I would like to thank Özlem Karsu and Andreas Löhne for their valuable time to review this thesis. Their recommendations and remarks have been very helpful.

It was an honor to be a member of Bilkent University Department of Industrial Engineering. I feel privileged that I had the opportunity to learn from my professors. I am also thankful to my friends and particularly members of the office for creating a pleasant and cheerful environment.

Last but not least, I would like to thank my family for their endless love and support. I am deeply grateful to them.

I would like to acknowledge that this research is supported by grant 118M479 of Program 3501 of TÜBİTAK, The Scientific and Technological Research Council of Turkey.

# Contents

# List of Figures

# List of Tables

# Chapter 1

# Introduction

Multiobjective optimization (MO) is concerned with mathematical optimization problems involving more than one objective functions to be optimized simultaneously. It has been applied in many fields where optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives. These fields include medicine [3, 4], food processing [5], disaster management [6], economics and finance [7].

Vector optimization is a generalization of MO where the order relation over the objective vectors is determined by a more general ordering cone, instead of the componentwise order relation as in MO. Different from single-objective optimization problems, there is no unique optimal objective value of a vector optimization problem. Instead, one is interested in generating the set of all minimal/maximal elements (also called Pareto or nondominated points in MO terminology) in the objective space as well as the corresponding minimizers/maximizers (efficient solutions in MO) in the decision space.

Generally, in vector optimization, the aim is to generate either the whole set of minimal/maximal elements in the objective space (also called the Pareto/efficient frontier in MO) or an approximation of it; instead of generating the set of all efficient solutions in the decision space. This is mainly because the effort to generate all efficient solutions increases exponentially with the problem size in general and many points in the solution set might correspond to a single point in the objective space. By this motivation, Benson [8] introduced an outer approximation algorithm in 1998 that aims to generate the Pareto frontier of linear multiobjective optimization problems.

Benson's work [8] has led to vast literature in vector optimization. In the linear case, many variations of Benson's algorithm have been proposed [9, 10, 11, 12, 13, 14]. In [9, 10, 11, 12], two linear programming problems (LP) are solved at each iteration while in [13, 14], it is sufficient to solve only one. Solving one LP at each iteration is computationally more efficient as solving LPs takes most of the computational effort in Benson type algorihms [13].

Convex vector optimization problems (CVOPs) are generalizations of LVOPs and they are more difficult to handle. While various approximation techniques can be used to solve CVOPs [15], this thesis focuses on an outer approximation algorithm. In 2011, Ehrgott, Shao and Schöbel [16] proposed an extension of Benson's algorithm for convex MO problems. In addition to being restricted to the multiobjective case, the algorithm in [16] also assumes that the objective and constraint functions are differentiable. Later Löhne, Rudloff and Ulus [2] generalized this algorithm for CVOPs without assuming differentiability.

The primal algorithm in [2] solves a Pascoletti-Serafini [17] scalarization and a vertex enumeration problem in each iteration. This scalarization problem depends on two parameters: a direction vector and a reference point. Specifically, given a point $v$ in the objective space, it determines the closest (in the given fixed direction $c$) point to $v$ in the objective space as well as a corresponding feasible solution in the decision space.

In [2], the authors also proposed a geometric dual variant of their primal algorithm which is based on a geometric duality theory. The general theory of convex polytopes indicates that two polytopes are dual to each other if there exists an inclusion reversing one-to-one mapping between their faces [18]. Similar to the duality relation between polytopes, a duality relation between the polyhedral image of the primal problem and the polyhedral image of a dual problem was introduced in [19] for multiobjective linear programming problems. The duality relation was shown by providing an inclusion reversing one-to-one map between the set of weakly minimal proper faces of the primal image and the set of all maximal proper faces of the dual image. In [20], the geometric duality theory was generalized to CVOPs; in particular, the images of the primal and dual problems do not need to be polyhedral. Also, the generalized theory does not require polyhedral cones or cones with nonempty interior. The geometric dual algorithm that is proposed in [2] is based on the theory developed in [20]. Accordingly, the fixed direction $c$ that is used within the primal algorithm is also used in the design of the geometric dual problem and the algorithm.

Recently, for CVOPs, Ararat, Ulus and Umer [1] considered a norm-minimizing scalarization which requires only a reference point but no direction parameter, and proposed an outer approximation algorithm for CVOPs. As the proposed algorithm does not require a direction parameter, direction-biasedness is not a concern. However, the norm-minimizing scalarization has a nonlinear convex objection function, which is not the case for the Pascoletti-Serafini scalarization that is used in the primal algorithm in [2]. On the other hand, in the geometric dual algorithm in [2], only the well-known weighted sum scalarizations, which may be easier to deal with because of their simple structures, are solved.

In this thesis, after we present the algorithm and solution concept in [1], we propose a novel *direction-free* geometric dual problem and a solution concept for it. We modify the primal algorithm in [1] in a way that it returns a solution to the dual problem as well. Moreover, we propose a geometric dual algorithm which solves the geometric dual problem and the primal problem simultaneously. The proposed geometric dual algorithm solves only weighted sum scalarization problems.

Our geometric dual algorithm is based on a new duality relation between the images of the primal and dual problems, namely, the upper and lower images, as well as their approximations. Different from [20], the geometric dual problem does not depend on a fixed direction; but in order to handle this issue, the dimension of the objective space for the dual problem is increased by one. Accordingly, the image of the proposed dual problem is not only a convex set but a convex cone. Due to this conic structure of the lower image, the dimension increase compared to [20] is not an additional source of computational burden, as also confirmed by the numerical results.

The thesis is structured as follows. Chapter 2 presents the basic concepts and notation. Chapter 3 introduces the primal problem, dual problem, and solution concepts to these problems. The geometric duality between the primal and dual problems is studied in Chapter 4. Chapter 5 provides the primal algorithm (Algorithm 1) and its dual variant (Algorithm 2). In Chapter 6, we provide an illustrative example and some test results for the performance comparisons between Algorithm 1 and Algorithm 2. The final chapter, Chapter 7, includes the concluding remarks.

# Chapter 2

# Preliminaries

In this section, we provide the definitions and notations used throughout the paper. Let $q \in \mathbb{N} := \{1, 2, \ldots\}$ be a positive integer. We denote the $q$-dimensional Euclidean space by $\mathbb{R}^q$. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^q$. The associated dual norm is denoted by $\|\cdot\|_*$, that is, $\|z\|_* = \sup\{z^\mathsf{T} x \mid \|x\| \leq 1\}$ for each $z \in \mathbb{R}^q$. Throughout, $B(0, \epsilon) := \{z \in \mathbb{R}^q \mid \|z\| \leq \epsilon\}$ denotes the norm ball centered at $0 \in \mathbb{R}^q$ with radius $\epsilon > 0$.

For a set $A \subseteq \mathbb{R}^q$, we denote the convex hull, conic hull, interior, relative interior and closure of $A$ by $\operatorname{conv} A$, $\operatorname{cone} A$, $\operatorname{int} A$, $\operatorname{ri} A$ and $\operatorname{cl} A$, respectively. Recall that $\operatorname{cone} A := \{\lambda a \mid a \in A, \lambda \geq 0\}$. The *recession cone* of $A$ is defined by $\operatorname{recc} A = \{c \in \mathbb{R}^q \mid \forall \lambda \geq 0, a \in A \colon a + \lambda c \in A\}$. An element $d \in \operatorname{recc} A$ is a *(recession) direction* of $A$. Let $A \subseteq \mathbb{R}^q$ be convex and $F \subseteq A$ be a convex subset. If $\lambda y^1 + (1 - \lambda) y^2 \in F$ for some $0 < \lambda < 1$ holds only if both $y^1$ and $y^2$ are elements of $F$, then $F$ is called a *face* of $A$. A zero dimensional face is called an *extreme point*, a one dimensional face is called an *edge* and a $q - 1$ dimensional face is called a *facet* of $S$. A face of $A$ that is not the empty set and not $A$ itself is called a *proper face* of $A$.

Given nonempty sets $A, B \subseteq \mathbb{R}^q$, their Minkowski sum is defined by $A + B := \{a + b \mid a \in A, b \in B\}$. For $\lambda \in \mathbb{R}$, we also define $\lambda A := \{\lambda a \mid a \in A\}$. In particular, we have $A - B = A + (-1)B$.

Let $C \subseteq \mathbb{R}^q$ be a cone. The closed convex cone given by $C^+ := \{a \in \mathbb{R}^q \mid \forall c \in C \colon c^{\mathsf{T}} a \geq 0\}$ is called the *dual cone* of $C$. The cone $C$ is called *pointed* if it does not contain any lines, *solid* if it has nonempty interior, and *nontrivial* if $C \neq \emptyset$ and $C \neq \mathbb{R}^q$. If $C \subseteq \mathbb{R}^q$ is a convex pointed cone, then the relation $\leq_C := \{(x, y) \in \mathbb{R}^q \times \mathbb{R}^q \mid y - x \in C\}$ is an antisymmetric partial order on $\mathbb{R}^q$ [21, Theorem 1.18]; we write $x \leq_C y$ whenever $(x, y) \in \leq_C$.

Let $m \in \mathbb{N}$ and $\mathcal{X} \subseteq \mathbb{R}^m$ be a nonempty convex set. A function $f \colon \mathcal{X} \to \mathbb{R}^q$ is said to be *$C$-convex on $\mathcal{X}$* if

$$f(\lambda x^1 + (1-\lambda)x^2) \leq_C \lambda f(x^1) + (1-\lambda) f(x^2)$$

for every $x^1, x^2 \in \mathcal{X}$ and $\lambda \in [0, 1]$ [21, Definition 2.4].

Let $A \subseteq \mathbb{R}^q$ and let $C \subseteq \mathbb{R}^q$ be a closed convex pointed cone. The sets

$$\begin{aligned}
\operatorname{Min}_C A &:= \{y \in A \mid (\{y\} - C \setminus \{0\}) \cap A = \emptyset\}, \\
\operatorname{wMin}_C A &:= \{y \in A \mid (\{y\} - \operatorname{int} C) \cap A = \emptyset\}, \\
\operatorname{Max}_C A &:= \{y \in A \mid (\{y\} + C \setminus \{0\}) \cap A = \emptyset\}
\end{aligned}$$

are called the sets of *$C$-minimal*, *weakly $C$-minimal*, *$C$-maximal* elements of $A$, respectively [11, Definition 1.41]. A proper face of $A$ that only consists of (weakly) $C$-minimal elements is called a *(weakly) $C$-minimal proper face*. A proper face of $A$ that consists of only $C$-maximal elements is called a *$C$-maximal proper face* [11, Section 4.5].

**Remark 2.0.1.** *Let $A \subseteq \mathbb{R}^q$ be a nonempty convex set and let $C \subseteq \mathbb{R}^q$ be a convex cone such that $A = A + C$. For $w \in \mathbb{R}^q \setminus C^+$, we have $\inf_{c \in C} w^{\mathsf{T}} c = -\infty$. Hence, $\inf_{a \in A} w^{\mathsf{T}} a = \inf_{a \in A} w^{\mathsf{T}} a + \inf_{c \in C} w^{\mathsf{T}} c = -\infty$ holds. Therefore, given $w \in \mathbb{R}^q$, $\inf_{a \in A} w^{\mathsf{T}} a > -\infty$ implies that $w \in C^+$.*

**Remark 2.0.2.** *Let $A \subseteq \mathbb{R}^q$ be a nonempty convex set. Hence, $A = A + \operatorname{recc} A$ holds [22, Theorem 5.6]. Suppose that $A \subseteq H := \{z \in \mathbb{R}^q \mid w^{\mathsf{T}} z \geq r\}$ for some $w \in \mathbb{R}^q$ and $r \in \mathbb{R}$. This implies $\inf_{a \in A} w^{\mathsf{T}} a \geq r > -\infty$. Hence, by Remark 2.0.1, we have $w \in (\operatorname{recc} A)^+$.*

# Chapter 3

# Primal and Dual Problems

In this paper, we consider a convex vector optimization problem and its geometric dual. The primal problem is defined as

$$\text{minimize } f(x) \text{ with respect to } \leq_C \text{ subject to } x \in \mathcal{X}, \tag{P}$$

where the ordering cone $C \subseteq \mathbb{R}^q$ is nontrivial, pointed, solid, closed and convex; the vector-valued objective function $f\colon \mathcal{X} \to \mathbb{R}^q$ is $C$-convex and continuous; and the feasible set $\mathcal{X} \subseteq \mathbb{R}^m$ is compact and convex. The *upper image* of (P) is defined as

$$\mathcal{P} \coloneqq \mathrm{cl}(f(\mathcal{X}) + C),$$

where $f(\mathcal{X}) \coloneqq \{f(x) \mid x \in \mathcal{X}\}$ is the image of $\mathcal{X}$ under $f$.

**Proposition 3.0.1.** *The upper image $\mathcal{P}$ is a closed convex set, the image $f(\mathcal{X})$ is a compact set, and it holds*

$$\mathcal{P} = f(\mathcal{X}) + C.$$

*Proof.* By definition, $\mathcal{P}$ is closed. Next, we show that $f(\mathcal{X}) + C$ is convex. To that end, let $x^1, x^2 \in \mathcal{X}$, $c^1, c^2 \in C$, and let $\lambda \in [0, 1]$. Since $f$ is $C$-convex, there exists $c^3 \in C$ such that $\lambda f(x^1) + (1 - \lambda)f(x^2) = f(\lambda x^1 + (1 - \lambda)x^2) + c^3$, which can be written as

$$\lambda(f(x^1)+c^1)+(1-\lambda)(f(x^2)+c^2) = f(\lambda x^1+(1-\lambda)x^2)+c^3+\lambda c^1+(1-\lambda)c^2. \tag{3.0.1}$$

Since $\mathcal{X}$ is a convex set and $C$ is a convex cone, we have $\lambda x^1 + (1-\lambda)x^2 \in \mathcal{X}$ and $c^3 + \lambda c^1 + (1-\lambda)c^2 \in C$. Hence, by (3.0.1), we obtain

$$\lambda(f(x^1) + c^1) + (1-\lambda)(f(x^2) + c^2) \in f(\mathcal{X}) + C.$$

Therefore, $f(\mathcal{X}) + C$ is convex. Then, $\mathcal{P} = \mathrm{cl}(f(\mathcal{X}) + C)$ is convex as the closure of a convex set. Since $f$ is continuous and $\mathcal{X}$ is compact, $f(\mathcal{X})$ is also compact. Finally, since $f(\mathcal{X})$ is compact and $C$ is closed, the Minkowski sum $f(\mathcal{X}) + C$ is closed. Hence, $\mathcal{P} = f(\mathcal{X}) + C$. $\square$

**Proposition 3.0.2.** *The primal problem* (P) *is* bounded *in the sense that* $\{y\} + C \subseteq \mathcal{P}$ *for some* $y \in \mathbb{R}^q$.

*Proof.* By Proposition 3.0.1, $f(\mathcal{X})$ is a compact set. Hence, there exist $a \in \mathbb{R}^q$ and $r \geq 0$ such that $B(a,r) \supseteq f(\mathcal{X})$. Since $C$ is solid, there exists $y \in \mathbb{R}^q$ such that $\{y\} + C \supseteq B(a,r)$. Hence, we have $\{y\} + C \supseteq f(\mathcal{X})$, which implies that $\{y\} + C \supseteq f(\mathcal{X}) + C = \mathcal{P}$ by Proposition 3.0.1. $\square$

For a parameter vector $w \in C^+$, the convex program

$$\text{minimize } w^{\mathsf{T}} f(x) \text{ subject to } x \in \mathcal{X} \tag{WS($w$)}$$

is called the *weighted sum scalarization* of (P). Let $p^w$ be the optimal value of (WS($w$)), that is,

$$p^w := \inf_{x \in \mathcal{X}} w^{\mathsf{T}} f(x). \tag{3.0.2}$$

Since $\mathcal{X}$ is a compact set and $f$ is a continuous function, it follows that $p^w \in \mathbb{R}$. The next proposition is a well-known result that will be used in the design of the geometric dual problem.

**Proposition 3.0.3.** *[21, Corollary 5.29] Let* $w \in C^+ \setminus \{0\}$*. Then, an optimal solution* $x^w$ *of* (WS($w$)) *is a weak minimizer of* (P)*. The converse also holds: for each weak minimizer* $x \in \mathcal{X}$ *of* (P)*, there exists* $w \in C^+ \setminus \{0\}$ *such that* $x$ *is an optimal solution of* (WS($w$))*.*

Now, let us define the geometric dual problem of (P) as

$$\text{maximize } \xi(w) \text{ with respect to } \leq_K \text{ subject to } w \in \mathcal{W}. \tag{D}$$

In this problem, the objective function $\xi\colon \mathbb{R}^q \to \mathbb{R}^{q+1}$ is defined by

$$\xi(w) := (w_1, \ldots, w_q, p^w)^\mathsf{T}, \quad w \in \mathcal{W}; \tag{3.0.3}$$

the ordering cone $K$ is defined by

$$K := \operatorname{cone}\{e^{q+1}\} = \{\lambda e^{q+1} \mid \lambda \geq 0\},$$

where $e^{q+1} = (0, \ldots, 0, 1)^\mathsf{T} \in \mathbb{R}^{q+1}$; and the feasible set is $\mathcal{W} := C^+$. The *lower image* of (D) is defined as

$$\mathcal{D} := \xi(\mathcal{W}) - K = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid w \in \mathcal{W},\ \alpha \leq p^w\}.$$

**Proposition 3.0.4.** *The lower image $\mathcal{D}$ is a closed convex cone.*

*Proof.* Let $((w^i)^\mathsf{T}, \alpha_i)^\mathsf{T} \in \mathcal{D}$ and $\lambda_i \geq 0$ for each $i \in \{1, \ldots, n\}$, where $n \in \mathbb{N}$. Since $\mathcal{W} = C^+$ is a convex cone and we have $w^i \in C^+$ for each $i \in \{1, \ldots, n\}$, we have $\sum_{i=1}^n \lambda_i w^i \in \mathcal{W}$. Moreover, we have $\alpha_i \leq \inf_{x \in \mathcal{X}} (w^i)^\mathsf{T} f(x)$ for each $i \in \{1, \ldots, n\}$, which implies that

$$\sum_{i=1}^n \lambda_i \alpha_i \leq \sum_{i=1}^n \lambda_i \inf_{x \in \mathcal{X}} (w^i)^\mathsf{T} f(x) \leq \inf_{x \in \mathcal{X}} \left( \sum_{i=1}^n \lambda_i w^i \right)^\mathsf{T} f(x).$$

Hence, $\sum_{i=1}^n \lambda_i ((w^i)^\mathsf{T}, \alpha_i)^\mathsf{T} \in \mathcal{D}$. It follows that $\mathcal{D}$ is a convex cone.

Note that the function $w \mapsto p^w$ is continuous as a concave function on $\mathcal{W}$ with finite values. Moreover, $\mathcal{D} \subseteq \mathcal{W} \times \mathbb{R}$ coincides with the hypograph of this function. Hence, $\mathcal{D}$ is a closed set. □

We define exact and approximate solution concepts for the primal problem (P). Let us start with the following definition.

**Definition 3.0.5.** *[11, Definition 2.20, Proposition 4.7] A point $\bar{x} \in \mathcal{X}$ is said to be a* (weak) minimizer *for* (P) *if $f(\bar{x})$ is a (weakly) $C$-minimal element of $f(\mathcal{X})$. A nonempty set $\bar{\mathcal{X}} \subseteq \mathcal{X}$ is called an* infimizer *of* (P) *if*

$$\operatorname{cl}\operatorname{conv}(f(\bar{\mathcal{X}}) + C) = \mathcal{P}.$$

*An infimizer $\bar{\mathcal{X}}$ of* (P) *is called* (weak) solution to (P) *if it consists of only (weak) minimizers.*

When the upper image of a convex vector optimization problem is considered, a finite set $\bar{\mathcal{X}}$ does not satisfy the exact solution concept in Definition 3.0.5, in general. Hence, we give an approximate solution concept below. In what follows, let us fix $\epsilon > 0$.

**Definition 3.0.6.** *[1, Definition 3.5] A nonempty finite set $\bar{\mathcal{X}} \subseteq \mathcal{X}$ is called a* finite $\epsilon$-infimizer *of* (P) *if*

$$\operatorname{conv} f(\bar{\mathcal{X}}) + C + B(0, \epsilon) \supseteq \mathcal{P}.$$

*A finite $\epsilon$-infimizer $\bar{\mathcal{X}}$ of* (P) *is called a* finite (weak) $\epsilon$-solution *to* (P) *if it consists of only (weak) minimizers.*

Note that if $\bar{\mathcal{X}}$ is a finite (weak) $\epsilon$-solution, then we have the following inner and outer approximations of the upper image:

$$\operatorname{conv} f(\bar{\mathcal{X}}) + C + B(0, \epsilon) \supseteq \mathcal{P} \supseteq \operatorname{conv} f(\bar{\mathcal{X}}) + C.$$

Now, similar to Definition 3.0.5, we provide an exact solution concept for the dual problem (D).

**Definition 3.0.7.** *[11, Definition 2.53, Corollary 2.54] A point $\bar{w} \in \mathcal{W}$ is called a* maximizer *for* (D) *if $\xi(\bar{w})$ is a $K$-maximal element of $\xi(\mathcal{W})$. A nonempty set $\bar{\mathcal{W}} \subseteq \mathcal{W}$ is called a* supremizer *of* (D) *if*

$$\operatorname{cone} \operatorname{conv} \xi(\bar{\mathcal{W}}) - K = \mathcal{D}.$$

*A supremizer $\bar{\mathcal{W}}$ of* (D) *is called a* solution *to* (D) *if it consists of only maximizers.*

As for the upper image, in general, the lower image cannot be represented by a finite set $\bar{\mathcal{W}}$ using the exact solution concept in Definition 3.0.7. In the next definition, we propose a novel approximate solution that is tailor-made for the lower image $\mathcal{D}$, which is a convex cone; see Remark 3.0.9 below for the technical motivation.

**Definition 3.0.8.** *A nonempty finite set* $\bar{\mathcal{W}} \subseteq \mathcal{W} \cap \mathbb{S}^{q-1}$ *is called a* finite $\epsilon$-supremizer *of* (D) *if*

$$\operatorname{cone}(\operatorname{conv}\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D}, \tag{3.0.4}$$

*where* $\mathbb{S}^{q-1} := \{z \in \mathbb{R}^q \mid \|z\|_* = 1\}$ *denotes the unit sphere in* $\mathbb{R}^q$ *with respect to the dual norm. A finite* $\epsilon$*-supremizer* $\bar{\mathcal{W}}$ *of* (D) *is called a* finite $\epsilon$-solution *to* (D) *if it consists of only maximizers.*

If $\bar{\mathcal{W}}$ is a finite $\epsilon$-solution of (D), then one obtains the following inner and outer approximations of the lower image:

$$\operatorname{cone}(\operatorname{conv}\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D} \supseteq \operatorname{cone}\operatorname{conv}\xi(\bar{\mathcal{W}}) - K.$$

**Remark 3.0.9.** *Let us comment on the particular structure of* (3.0.4) *in Definition 3.0.8. First, we add the error term* $\epsilon(e^{q+1})$ *to the set* $\xi(\bar{\mathcal{W}})$ *of objective values produced by the candidate solution* $\bar{\mathcal{W}}$*. Since the lower image* $\mathcal{D}$ *is a convex cone by Proposition 3.0.4, we evaluate the conic hull of the Minkowski sum* $\operatorname{conv}\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}$ *so that the error values are scaled properly. With this operation, we ensure that the resulting conic hull is comparable with* $\mathcal{D}$ *(up to the subtraction of the ordering cone* $K$*).*

The next proposition will be used in Chapter 4 to prove some geometric duality results.

**Proposition 3.0.10.** *Let* $w \in \mathcal{W}$*. Then,* $\xi(w)$ *is a* $K$*-maximal element of* $\mathcal{D}$*.*

*Proof.* Let $\epsilon > 0$. We prove that $\xi(w) + \epsilon e^{q+1} \notin \mathcal{D} = \xi(\mathcal{W}) - K$. To get a contradiction, assume that there exist $\bar{w} \in \mathcal{W}$ and $\bar{\epsilon} \geq 0$ such that $\xi(w) + \epsilon e^{q+1} = \xi(\bar{w}) - \bar{\epsilon}e^{q+1}$. Recalling (3.0.2) and (3.0.3), we have

$$(w_1, \dots, w_q, p^w)^\mathsf{T} + \epsilon e^{q+1} = (\bar{w}_1, \dots, \bar{w}_q, p^{\bar{w}})^\mathsf{T} - \bar{\epsilon}e^{q+1},$$

that is,

$$(w_1, \dots, w_q, p^w + \epsilon)^\mathsf{T} = (\bar{w}_1, \dots, \bar{w}_q, p^{\bar{w}} - \bar{\epsilon})^\mathsf{T}.$$

Hence, $w = \bar{w}$, which implies that $\inf_{x \in \mathcal{X}} w^\mathsf{T} f(x) = \inf_{x \in \mathcal{X}} \bar{w}^\mathsf{T} f(x)$ and $\epsilon = -\bar{\epsilon}$. This contradicts $\epsilon > 0$ and $\bar{\epsilon} \geq 0$. Hence, $\xi(w) + \epsilon e^{q+1} \notin \mathcal{D}$. Since $\epsilon > 0$ is arbitrary, $\xi(w)$ is a $K$-maximal element of $\mathcal{D}$. ∎

# Chapter 4

# Geometric Duality

In this section, we will investigate the duality relation between the primal and dual problems, (P) and (D). First, we will provide the main duality theorem which relates the weakly $C$-minimal proper faces of $\mathcal{P}$ and the $K$-maximal proper faces of $\mathcal{D}$. Then, we will establish some duality properties regarding $\mathcal{P}$ and $\mathcal{D}$, as well as their polyhedral approximations.

## 4.1 Geometric duality between $\mathcal{P}$ and $\mathcal{D}$

We will prove that there is a one-to-one correspondence between the set of all weakly $C$-minimal proper faces of the upper image $\mathcal{P}$ and the set of all $K$-maximal proper faces of the lower image $\mathcal{D}$. Moreover, we will show that the upper and lower images can be recovered from each other.

Let us start by defining

$$\varphi\colon \mathbb{R}^q \times \mathbb{R}^{q+1} \to \mathbb{R}, \quad \varphi(y, w, \alpha) := w^{\mathsf{T}} y - \alpha, \tag{4.1.1}$$

and the following set-valued maps:

$$\begin{aligned}
\mathcal{H}\colon \mathbb{R}^{q+1} \rightrightarrows \mathbb{R}^q, &\qquad \mathcal{H}(w,\alpha) := \{y \in \mathbb{R}^q \mid \varphi(y,w,\alpha) \geq 0\},\\
H\colon \mathbb{R}^{q+1} \rightrightarrows \mathbb{R}^q, &\qquad H(w,\alpha) := \operatorname{bd}\mathcal{H}(w,\alpha) = \{y \in \mathbb{R}^q \mid \varphi(y,w,\alpha) = 0\},\\
\mathcal{H}^*\colon \mathbb{R}^q \rightrightarrows \mathbb{R}^{q+1}, &\qquad \mathcal{H}^*(y) := \{(w^\mathsf{T},\alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \varphi(y,w,\alpha) \geq 0\},\\
H^*\colon \mathbb{R}^q \rightrightarrows \mathbb{R}^{q+1}, &\qquad H^*(y) := \operatorname{bd}\mathcal{H}^*(y) = \{(w^\mathsf{T},\alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \varphi(y,w,\alpha) = 0\}.
\end{aligned}$$

These functions are essential to define a duality map between $\mathcal{P}$ and $\mathcal{D}$. Moreover, for arbitrary $y \in \mathbb{R}^q$ and $(w^\mathsf{T},\alpha)^\mathsf{T} \in \mathbb{R}^{q+1}$, we have the following statement:

$$(w^\mathsf{T},\alpha)^\mathsf{T} \in H^*(y) \iff y \in H(w,\alpha). \tag{4.1.2}$$

With the next proposition we show that, by solving a weighted sum scalarization, one finds supporting hyperplanes to the upper and lower images.

**Proposition 4.1.1.** *Let $w \in \mathcal{W}$ and $x^w$ be an optimal solution to* (WS($w$)). *Then, $H(\xi(w))$ is a supporting hyperplane of $\mathcal{P}$ at $f(x^w)$ such that $\mathcal{P} \subseteq \mathcal{H}(\xi(w))$, and $H^*(f(x^w))$ is a supporting hyperplane of $\mathcal{D}$ at $\xi(w)$ such that $\mathcal{D} \subseteq \mathcal{H}^*(f(x^w))$.*

*Proof.* To prove the first statement, we show that $f(x^w) \in H(\xi(w))$ and $\mathcal{P} \subseteq \mathcal{H}(\xi(w))$. As $\xi(w) = (w^\mathsf{T}, w^\mathsf{T} f(x^w))^\mathsf{T}$, we have $\varphi(f(x^w),\xi(w)) = w^\mathsf{T} f(x^w) - w^\mathsf{T} f(x^w) = 0$, which implies that $f(x^w) \in H(\xi(w))$. To prove $\mathcal{P} \subseteq \mathcal{H}(\xi(w))$, let $x \in \mathcal{X}$, $c \in C$ be arbitrary. Then, $\varphi(f(x)+c,\xi(w)) = w^\mathsf{T} f(x) + w^\mathsf{T} c - w^\mathsf{T} f(x^w) \geq 0$ as $w \in C^+$ and $x^w$ is an optimal solution of (WS($w$)). Therefore, $\mathcal{H}(\xi(w)) \supseteq f(\mathcal{X}) + C = \mathcal{P}$, where the equality follows by Proposition 3.0.1.

On the other hand, having $\varphi(f(x^w),\xi(w)) = w^\mathsf{T} f(x^w) - w^\mathsf{T} f(x^w) = 0$ also implies that $\xi(w) \in H^*(f(x^w))$. To complete the proof, it is enough to show that $\mathcal{D} \subseteq \mathcal{H}^*(f(x^w))$. Let $(w^\mathsf{T},\alpha)^\mathsf{T} \in \mathcal{D}$ be arbitrary. It follows from the definition of $\mathcal{D}$ that $\alpha \leq \inf_{x\in\mathcal{X}} w^\mathsf{T} f(x) = w^\mathsf{T} f(x^w)$. Hence, $\varphi(f(x^w),w,\alpha) = w^\mathsf{T} f(x^w) - \alpha \geq 0$, that is, $(w^\mathsf{T},\alpha)^\mathsf{T} \in \mathcal{H}^*(f(x^w))$. ☐

The following lemmas show the relationship between the weakly $C$-minimal elements of $\mathcal{P}$ and the $K$-maximal proper faces of $\mathcal{D}$, as well as that between the $K$-maximal elements of $\mathcal{D}$ and the weakly $C$-minimal proper faces of $\mathcal{P}$.

**Lemma 4.1.2.** *The following statements hold true:*

*(a) Let $y \in \mathbb{R}^q$. Then, $y$ is a weakly $C$-minimal element of $\mathcal{P}$ if and only if $H^*(y) \cap \mathcal{D}$ is a $K$-maximal proper face of $\mathcal{D}$.*

*(b) For every $K$-maximal proper face $F^*$ of $\mathcal{D}$, there exists some $y \in \mathcal{P}$ such that $F^* = H^*(y) \cap \mathcal{D}$.*

*Proof.* (a) Suppose that $y$ is a weakly $C$-minimal element of $\mathcal{P}$. By Proposition 3.0.1, $y = f(\bar{x}) + \bar{c}$ for some $\bar{x} \in \mathcal{X}$ and $\bar{c} \in C$. Moreover, as $y$ is a weakly $C$-minimal element of $\mathcal{P}$, we have $y \in \operatorname{bd} \mathcal{P}$ [11, Corollary 1.48]. Then, there exist $\bar{w} \in \mathbb{R}^q$ and $\bar{\alpha} \in \mathbb{R}$ such that $H := \{\tilde{y} \in \mathbb{R}^q \mid \bar{w}^\mathsf{T}\tilde{y} = \bar{\alpha}\}$ supports $\mathcal{P}$ at $y$. In particular, we have

$$\inf_{\tilde{y}\in\mathcal{P}} \bar{w}^\mathsf{T}\tilde{y} = \bar{w}^\mathsf{T}y = \bar{\alpha}. \tag{4.1.3}$$

By Remark 2.0.1, (4.1.3) implies that $\bar{w} \in C^+$. Moreover, using Proposition 3.0.1 and the fact that $y = f(\bar{x}) + \bar{c}$, (4.1.3) can be rewritten as

$$\inf_{x\in\mathcal{X}} \bar{w}^\mathsf{T} f(x) + \inf_{c\in C} \bar{w}^\mathsf{T} c = \bar{w}^\mathsf{T} f(\bar{x}) + \bar{w}^\mathsf{T}\bar{c}.$$

Noting that we have $\inf_{c\in C} \bar{w}^\mathsf{T} c = 0$, $\inf_{x\in\mathcal{X}} \bar{w}^\mathsf{T} f(x) \leq \bar{w}^\mathsf{T} f(\bar{x})$ and $\bar{w}^\mathsf{T}\bar{c} \geq 0$, we obtain $\bar{w}^\mathsf{T}\bar{c} = 0$ and $\bar{w}^\mathsf{T} f(\bar{x}) = \inf_{x\in\mathcal{X}} \bar{w}^\mathsf{T} f(x)$. It follows that $\mathcal{H}^*(f(\bar{x})) = \mathcal{H}^*(y)$ since $y = f(\bar{x}) + \bar{c}$ and $\bar{w}^\mathsf{T}\bar{c} = 0$. Moreover, $\bar{x}$ is an optimal solution to the problem (WS($\bar{w}$)). Hence, $H^*(f(\bar{x})) = H^*(y)$ is a supporting hyperplane of $\mathcal{D}$ at $\xi(\bar{w})$ such that $\mathcal{D} \subseteq \mathcal{H}^*(f(\bar{x}))$ by Proposition 4.1.1. Therefore, $H^*(f(\bar{x})) \cap \mathcal{D}$ is a proper face of $\mathcal{D}$ [23, Section 18, page 162].

To show that $H^*(f(\bar{x})) \cap \mathcal{D}$ is a $K$-maximal proper face of $\mathcal{D}$, let $(\tilde{w}^\mathsf{T}, \tilde{\alpha})^\mathsf{T} \in H^*(f(\bar{x})) \cap \mathcal{D}$ be arbitrary. Note that $\varphi(f(\bar{x}), \tilde{w}, \tilde{\alpha}) = \tilde{w}^\mathsf{T} f(\bar{x}) - \tilde{\alpha} = 0$. On the other hand, the fact $\mathcal{D} \subseteq \mathcal{H}^*(f(\bar{x}))$ implies that $\varphi(f(\bar{x}), w, \alpha) = w^\mathsf{T} f(\bar{x}) - \alpha \geq 0$ for each $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}$. Together, these imply that $(\tilde{w}^\mathsf{T}, \hat{\alpha})^\mathsf{T} \notin \mathcal{D}$ for every $\hat{\alpha} > \tilde{\alpha}$. Hence, $(\tilde{w}^\mathsf{T}, \tilde{\alpha})^\mathsf{T}$ is a $K$-maximal element of $\mathcal{D}$.

Conversely, suppose that $H^*(y) \cap \mathcal{D}$ is a $K$-maximal proper face of $\mathcal{D}$. Hence, $H^*(y)$ is a supporting hyperplane of $\mathcal{D}$ [23, Section 18, page 162], and we have either $\mathcal{D} \subseteq \mathcal{H}^*(y)$ or $\mathcal{D} \subseteq \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \varphi(y, w, \alpha) \leq 0\}$. We claim that the former relation holds. Indeed, letting $(\tilde{w}^\mathsf{T}, \tilde{\alpha})^\mathsf{T} \in H^*(y) \cap \mathcal{D}$, we have $\tilde{\alpha} \leq \inf_{x \in \mathcal{X}} \tilde{w}^\mathsf{T} f(x)$ and $\varphi(y, \tilde{w}, \tilde{\alpha}) = \tilde{w}^\mathsf{T} y - \tilde{\alpha} = 0$. Then, $(\tilde{w}^\mathsf{T}, \tilde{\alpha} - 1)^\mathsf{T} \in \mathcal{D}$ and $\varphi(y, \tilde{w}, \tilde{\alpha} - 1) = \tilde{w}^\mathsf{T} y - \tilde{\alpha} + 1 > 0$. Hence, $(\tilde{w}^\mathsf{T}, \tilde{\alpha} - 1)^\mathsf{T} \notin \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \varphi(y, w, \alpha) \leq 0\}$. Therefore, the claim holds and we have $\mathcal{D} \subseteq \mathcal{H}^*(y)$.

Next, we show that $y \in \mathcal{P}$. To get a contradiction, suppose that $y \notin \mathcal{P}$. By separation theorem, there exists $\bar{w} \in \mathbb{R}^q \setminus \{0\}$ such that

$$\bar{w}^\mathsf{T} y < \inf_{\bar{y} \in \mathcal{P}} \bar{w}^\mathsf{T} \bar{y} =: \bar{\alpha}.$$

Note that $\bar{w} \in C^+$ by Remark 2.0.1. Therefore, we have $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \mathcal{D} \subseteq \mathcal{H}^*(y)$, which contradicts $\bar{w}^\mathsf{T} y - \bar{\alpha} < 0$. Hence, we have $y \in \mathcal{P}$.

Finally, we show that $y$ is a weakly $C$-minimal element of $\mathcal{P}$. To get a contradiction, suppose that there exists $c \in \operatorname{int} C$ with $y - c \in \mathcal{P}$. Without loss of generality, assume that $y - c \in \operatorname{wMin}_C \mathcal{P}$. From the proof of the previous implication, $\mathcal{D} \subseteq \mathcal{H}^*(y - c)$. Let $(w^\mathsf{T}, \alpha)^\mathsf{T} \in H^*(y) \cap \mathcal{D} \subseteq \mathcal{H}^*(y - c)$. Then, we have $w^\mathsf{T}(y - c) - \alpha = w^\mathsf{T} y - w^\mathsf{T} c - \alpha \geq 0$. Note that $w^\mathsf{T} y - \alpha = 0$ since $(w^\mathsf{T}, \alpha)^\mathsf{T} \in H^*(y) \cap \mathcal{D}$. Hence, we have $w^\mathsf{T} y - w^\mathsf{T} c - \alpha = -w^\mathsf{T} c \geq 0$, which contradicts $w \in \mathcal{W}$ and $c \in \operatorname{int} C$. Therefore, $y$ is a weakly $C$-minimal element of $\mathcal{P}$.

(b) Let $F^*$ be a $K$-maximal proper face of $\mathcal{D}$. Then, there exists a supporting hyperplane $H^* \subseteq \mathbb{R}^{q+1}$ of $\mathcal{D}$ such that $F^* = H^* \cap \mathcal{D}$ [23, Section 18, page 162]. Then, we may write

$$H^* = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid a^\mathsf{T}(w^\mathsf{T}, \alpha) = b\}$$

for some $a \in \mathbb{R}^{q+1}, b \in \mathbb{R}$. Without loss of generality, we may assume that

$$\mathcal{H}^* := \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid a^\mathsf{T}(w^\mathsf{T}, \alpha)^\mathsf{T} \geq b\} \supseteq \mathcal{D}.$$

Since $\mathcal{D}$ is a convex cone, $F^*$ is also a convex cone [24, Lemma 10.2]. Moreover, $F^*$ is closed as $\mathcal{D}$ is closed by Proposition 3.0.4. Therefore, $0 \in F^* \subseteq H^*$. Hence, $b = 0$.

Next, we show that $a_{q+1} < 0$. For every $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in F^*$, the point $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T}$ is a $K$-maximal element of $\mathcal{D}$ and it holds $a^\mathsf{T}(\bar{w}^\mathsf{T}, \bar{\alpha}) = 0$. Moreover, $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in$

$\mathcal{H}^*$ implies $a^\mathsf{T}(w^\mathsf{T},\alpha) \geq 0$. For every $\gamma > 0$, since $(\bar{w}^\mathsf{T}, \bar{\alpha}-\gamma)^\mathsf{T} \in \mathcal{D} \subseteq \mathcal{H}^*$, we have $a^\mathsf{T}(\bar{w}^\mathsf{T}, \bar{\alpha}-\gamma) = a^\mathsf{T}(w^\mathsf{T},\alpha) - \gamma a_{q+1} \geq 0$. Therefore, $a_{q+1} \leq 0$ holds. If $a_{q+1} = 0$, then $a^\mathsf{T}(\bar{w}^\mathsf{T}, \bar{\alpha}-1)^\mathsf{T} = 0$ implies that $(\bar{w}^\mathsf{T}, \bar{\alpha}-1)^\mathsf{T} \in F^*$ contradicting the $K$-maximality of $F^*$. Therefore, $a_{q+1} < 0$.

By setting

$$y := \left(\frac{-a_1}{a_{q+1}}, \frac{-a_2}{a_{q+1}}, \cdots, \frac{-a_q}{a_{q+1}}\right)^\mathsf{T},$$

we obtain $H^* = H^*(y)$ and $\mathcal{D} \subseteq \mathcal{H}^*(y)$.

Finally, we show that $y \in \mathcal{P}$. Assuming otherwise, there exists $\tilde{w} \in C^+$ such that $\tilde{w}^\mathsf{T} y < \inf_{x\in\mathcal{X}} \tilde{w}^\mathsf{T} f(x) =: \tilde{\alpha}$ by separation arguments. Then, we obtain $(\tilde{w}^\mathsf{T}, \tilde{\alpha})^\mathsf{T} \in \mathcal{D} \setminus \mathcal{H}^*(y)$, which contradicts with $\mathcal{D} \subseteq \mathcal{H}^*(y)$.

$\square$

**Lemma 4.1.3.** *The following statements hold true:*

*(a) Let $(w^\mathsf{T},\alpha)^\mathsf{T} \in \mathbb{R}^{q+1}$. Then, $(w^\mathsf{T},\alpha)^\mathsf{T}$ is a $K$-maximal element of $\mathcal{D}$ if and only if $H(w,\alpha)\cap\mathcal{P}$ is a weakly $C$-minimal proper face of $\mathcal{P}$ satisfying $\mathcal{H}(w,\alpha) \supseteq \mathcal{P}$.*

*(b) For every $C$-minimal proper face $F$ of $\mathcal{P}$, there exists some $(w^\mathsf{T},\alpha)^\mathsf{T} \in \mathcal{D}$ such that $F = H(w,\alpha) \cap \mathcal{P}$.*

*Proof.* (a) Suppose that $(w^\mathsf{T},\alpha)^\mathsf{T}$ is a $K$-maximal point of $\mathcal{D}$. Clearly, $w \in C^+$, $\alpha = \inf_{x\in\mathcal{X}} w^\mathsf{T} f(x)$ and $(w^\mathsf{T},\alpha)^\mathsf{T} = \xi(w)$. Since $\mathcal{X}$ is a compact set, there exists an optimal solution $x^w \in \mathcal{X}$ to (WS($w$)). By Proposition 4.1.1, $H(\xi(w)) = H(w,\alpha)$ is a supporting hyperplane of $\mathcal{P}$ at $f(x^w)$ satisfying $\mathcal{H}(w,\alpha) \supseteq \mathcal{P}$. Then, $H(w,\alpha)\cap\mathcal{P}$ is a proper face of $\mathcal{P}$ [23, Section 18, page 162]. To show that $H(w,\alpha)\cap\mathcal{P}$ is weakly $C$-minimal, let $\bar{y} \in H(w,\alpha)\cap\mathcal{P}$ be arbitrary. Since $\bar{y} \in H(w,\alpha)$ and $w \in C^+$, we have $\varphi(\bar{y}-c,w,\alpha) = w^\mathsf{T}\bar{y} - w^\mathsf{T}c - \alpha < 0$ for every $c \in \operatorname{int} C$. Note that each $y \in \mathcal{P} \subseteq \mathcal{H}(w,\alpha)$ satisfies $\varphi(y,w,\alpha) = w^\mathsf{T}y - \alpha \geq 0$. Then, $(\{\bar{y}\} - \operatorname{int} C)\cap\mathcal{P} = \emptyset$, hence $\bar{y}$ is weakly $C$-minimal.

Conversely, suppose that $H(w,\alpha)\cap\mathcal{P}$ is a weakly $C$-minimal proper face of $\mathcal{P}$ such that $\mathcal{H}(w,\alpha) \supseteq \mathcal{P}$. Then, $H(w,\alpha)$ is a supporting hyperplane of

$\mathcal{P}$ [23, Section 18, page 162]. By Remark 2.0.2, we have $w \in (\operatorname{recc}\mathcal{P})^+ \subseteq C^+$. Moreover, for each $y \in \mathcal{P}$, $\varphi(y, w, \alpha) = w^\mathsf{T} y - \alpha \geq 0$. This implies $\alpha \leq \inf_{y \in \mathcal{P}} w^\mathsf{T} y$, hence $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}$. On the other hand, let $y = f(x) + c \in H(w, \alpha) \cap \mathcal{P}$ for some $x \in \mathcal{X}$ and $c \in C$. Since $\varphi(y, w, \alpha) = w^\mathsf{T} f(x) + w^\mathsf{T} c - \alpha = 0$, we have $\varphi(y, w, \alpha + \epsilon) = w^\mathsf{T} f(x) + w^\mathsf{T} c - \alpha - \epsilon < 0$ for every $\epsilon > 0$. This implies

$$\alpha + \epsilon > \inf_{x \in \mathcal{X}} w^\mathsf{T} f(x) + \inf_{c \in C} w^\mathsf{T} c = \inf_{x \in \mathcal{X}} w^\mathsf{T} f(x).$$

Since $\epsilon > 0$ is arbitrary, we have $\alpha \geq \inf_{x \in \mathcal{X}} w^\mathsf{T} f(x)$. Together, we obtain $\alpha = \inf_{x \in \mathcal{X}} w^\mathsf{T} f(x)$, which implies that $(w^\mathsf{T}, \alpha)^\mathsf{T}$ is $K$-maximal.

(b) Let $F$ be a $C$-minimal proper face of $\mathcal{P}$. Then, there exists a supporting hyperplane $H$ of $\mathcal{P}$ such that $F = H \cap \mathcal{P}$ [23, Section 18, page 162]. We may write $H = \{y \in \mathbb{R}^q \mid w^\mathsf{T} y = \alpha\}$ for some $w \in \mathbb{R}^q$ and $\alpha \in \mathbb{R}$, and assume that $\mathcal{P} \subseteq \mathcal{H} := \{y \in \mathbb{R}^q \mid w^\mathsf{T} y \geq \alpha\}$ without loss of generality. By Remark 2.0.2, we have $w \in (\operatorname{recc}\mathcal{P})^+ \subseteq C^+$. Moreover, as $\mathcal{P} \subseteq \mathcal{H}$, it holds true that $\alpha \leq \inf_{x \in \mathcal{X}} w^\mathsf{T} f(x)$. Hence, $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}$.

$\square$

We now proceed with the main geometric duality result. Let $\mathcal{F}_\mathcal{P}$ be the set of all weakly $C$-minimal proper faces of $\mathcal{P}$ and $\mathcal{F}^*_\mathcal{D}$ be the set of all $K$-maximal proper faces of $\mathcal{D}$. Consider the set-valued function

$$\Psi \colon \mathcal{F}^*_\mathcal{D} \rightrightarrows \mathbb{R}^q, \quad \Psi(F^*) := \bigcap_{(w^\mathsf{T}, \alpha)^\mathsf{T} \in F^*} H(w, \alpha) \cap \mathcal{P}. \tag{4.1.4}$$

**Theorem 4.1.4.** *$\Psi$ is an inclusion-reversing one-to-one correspondence between $\mathcal{F}^*_\mathcal{D}$ and $\mathcal{F}_\mathcal{P}$. The inverse map is given by*

$$\Psi^{-1}(F) := \bigcap_{y \in F} H^*(y) \cap \mathcal{D}.$$

*Proof.* First, for a $K$-maximal proper face $F^*$ of $\mathcal{D}$, we show that $\Psi(F^*)$ is a weakly $C$-minimal proper face of $\mathcal{P}$. By Lemma 4.1.3 (a), $H(w, \alpha) \cap \mathcal{P}$ is a weakly $C$-minimal proper face of $\mathcal{P}$ for each $(w^\mathsf{T}, \alpha)^\mathsf{T} \in F^*$. From the definition given by (4.1.4), $\Psi(F^*)$ is a weakly $C$-minimal proper face of $\mathcal{P}$ if it is nonempty. By

Lemma 4.1.2 (b), there exists some $y \in \mathcal{P}$ such that $F^* = H^*(y) \cap \mathcal{D}$. Therefore, for each $(w^\mathsf{T}, \alpha)^\mathsf{T} \in F^*$, we have $(w^\mathsf{T}\alpha)^\mathsf{T} \in H^*(y)$, equivalently, $y \in H(w, \alpha)$, see (4.1.2). Then, $\Psi(F^*)$ is nonempty as $y \in \Psi(F^*)$.

For a weakly $C$-minimal proper face $F$ of $\mathcal{P}$, define $\Psi^*(F) := \bigcap_{y \in F}(H^*(y) \cap \mathcal{D})$. To show that $\Psi^*(F)$ is a $K$-maximal proper face of $\mathcal{D}$, let $y \in F$. By Lemma 4.1.2 (a), $H^*(y) \cap \mathcal{D}$ is a $K$-maximal proper face of $\mathcal{D}$. Therefore, $\Psi(F^*)$ is a $K$-maximal proper face of $\mathcal{D}$, if it is nonempty. From Lemma 4.1.3 (b), there exists some $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}$ such that $F = H(w, \alpha) \cap \mathcal{P}$. Therefore, for each $y \in F$, we have $y \in H(w, \alpha)$, equivalently, $(w^\mathsf{T}, \alpha)^\mathsf{T} \in H^*(y)$, see (4.1.2). Then, $\Psi^*(F)$ is nonempty as $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \Psi^*(F)$.

In order to show that $\Psi$ is a bijection and $\Psi^{-1} = \Psi^*$, we will show the following two statements:

(a) $\Psi^*(\Psi(F^*)) = F^*$ for every $K$-maximal proper face $F^*$ of $\mathcal{D}$,

(b) $\Psi(\Psi^*(F)) = F$ for every weakly $C$-minimal proper face $F$ of $\mathcal{P}$.

(a) Let $F^*$ be a $K$-maximal proper face of $\mathcal{D}$. Assume for a contradiction that $F^* \nsubseteq \Psi^*(\Psi(F^*))$. Let $(w^\mathsf{T}, \alpha)^\mathsf{T} \in F^* \setminus \Psi^*(\Psi(F^*))$. Since $\Psi(F^*)$ is nonempty, $(w^\mathsf{T}, \alpha)^\mathsf{T} \notin \Psi^*(\Psi(F^*))$ means that there exists $\bar{y} \in \Psi(F^*)$ such that $(w^\mathsf{T}, \alpha)^\mathsf{T} \notin H^*(\bar{y}) \cap \mathcal{D}$. This implies $(w^\mathsf{T}, \alpha)^\mathsf{T} \notin H^*(\bar{y})$ since $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}$. Using (4.1.2), we have $\bar{y} \notin H(w, \alpha)$. Therefore $\bar{y} \notin \Psi(F^*)$, a contradiction. Hence, $F^* \subseteq \Psi^*(\Psi(F^*))$. For the reverse inclusion, first note that, from Lemma 4.1.2 (b), there exists $\bar{y} \in \mathcal{P}$ such that $F^* = H^*(\bar{y}) \cap \mathcal{D}$. Therefore, for each $(w^\mathsf{T}, \alpha)^\mathsf{T} \in F^*$, we have $(w^\mathsf{T}, \alpha)^\mathsf{T} \in H^*(\bar{y})$, equivalently, $\bar{y} \in H(w, \alpha)$, see (4.1.2). Hence,

$$\bar{y} \in \bigcap_{(w^\mathsf{T}, \alpha)^\mathsf{T} \in F^*} \big(H(w, \alpha) \cap \mathcal{P}\big) = \Psi(F^*).$$

Therefore,

$$\Psi^*(\Psi(F^*)) = \bigcap_{y \in \Psi(F^*)} \big(H^*(y) \cap \mathcal{D}\big) \subseteq H^*(\bar{y}) \cap \mathcal{D} = F^*.$$

Hence, the equality $\Psi^*(\Psi(F^*)) = F^*$ holds.

(b) Let $F$ be a weakly $C$-minimal proper face of $\mathcal{P}$. Assume for a contradiction that $F \not\subseteq \Psi(\Psi^*(F))$. Let $y \in F \setminus \Psi(\Psi^*(F))$. Then, there exists $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \Psi^*(F)$ such that $y \notin H(\bar{w}, \bar{\alpha}) \cap \mathcal{P}$. This implies $y \notin H(\bar{w}, \bar{\alpha})$ since $y \in \mathcal{P}$. By (4.1.2), $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \notin H^*(y)$, which implies $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \notin \Psi^*(F)$, a contradiction. Hence, $\Psi(\Psi^*(F)) \supseteq F$. For the reverse inclusion, first note that, by Lemma 4.1.3 (b), there exists $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \mathcal{D}$ such that $F = H(\bar{w}, \bar{\alpha}) \cap \mathcal{P}$. Then, for each $y \in F$, we have $y \in H(\bar{w}, \bar{\alpha})$, equivalently, $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in H^*(y)$, see (4.1.2). Hence,

$$(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \bigcap_{y \in F} \big(H^*(y) \cap \mathcal{D}\big) = \Psi^*(F).$$

Therefore,

$$\Psi(\Psi^*(F)) = \bigcap_{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \Psi^*(F)} \big(H(w, \alpha) \cap \mathcal{P}\big) \subseteq H(\bar{w}, \bar{\alpha}) \cap \mathcal{P} = F.$$

Hence, the equality $\Psi(\Psi^*(F)) = F$ holds. □

Now, we will see that the upper and dual images $\mathcal{P}$ and $\mathcal{D}$ can be recovered from each other using the function $\varphi$ introduced in (4.1.1). The following definition will be used to simplify the notation in later steps.

**Definition 4.1.5.** *For closed and convex sets $\bar{\mathcal{P}}$ and $\bar{\mathcal{D}}$, we define*

$$\mathcal{D}_{\bar{\mathcal{P}}} := \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in \bar{\mathcal{P}} \colon \varphi(y, w, \alpha) \geq 0\},$$
$$\mathcal{P}_{\bar{\mathcal{D}}} := \{y \in \mathbb{R}^q \mid \forall (w^\mathsf{T}, \alpha)^\mathsf{T} \in \bar{\mathcal{D}} \colon \varphi(y, w, \alpha) \geq 0\}.$$

**Proposition 4.1.6.** *For the upper image $\mathcal{P}$ and the lower image $\mathcal{D}$, it holds (a) $\mathcal{D}_{\mathcal{P}} = \mathcal{D}$, (b) $\mathcal{P}_{\mathcal{D}} = \mathcal{P}$.*

*Proof.* (a) Let $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_{\mathcal{P}}$ be arbitrary. Since $\varphi(y, w, \alpha) = w^\mathsf{T} y - \alpha \geq 0$ holds for every $y \in \mathcal{P}$, we have $\alpha \leq \inf_{y \in \mathcal{P}} w^\mathsf{T} y$. By Remark 2.0.1, $w \in C^+$. Therefore, $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}$, which implies $\mathcal{D}_{\mathcal{P}} \subseteq \mathcal{D}$. Now let $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}$. Since $\alpha \leq \inf_{x \in \mathcal{X}} w^\mathsf{T} f(x)$, $w \in C^+$ and $\mathcal{P} = f(\mathcal{X}) + C$ by Proposition 3.0.1, we have $\alpha \leq w^\mathsf{T} y$ for every $y \in \mathcal{P}$. Therefore, $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_{\mathcal{P}}$.

(b) Let $y \in \mathcal{P}$ be arbitrary. By Proposition 3.0.1, we may write $y = f(x) + c \in \mathcal{P}$ for some $x \in \mathcal{X}, c \in C$. For every $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}$, we have $\varphi(f(x) + c, w, \alpha) =$

$w^\mathsf{T}(f(x)+c)-\alpha \geq \inf_{x\in\mathcal{X}} w^\mathsf{T} f(x) - \alpha + w^\mathsf{T} c \geq 0$, where the last inequality follows from $w \in C^+$ and $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}$. This implies $f(x)+c \in \mathcal{P}_\mathcal{D}$, hence, $\mathcal{P}_\mathcal{D} \supseteq \mathcal{P}$. For the reverse inclusion, let $y \in \mathcal{P}_\mathcal{D}$ be arbitrary. Assume for contradiction that $y \notin \mathcal{P}$. By separation theorem, there exists $\bar{w} \in \mathbb{R}^q \setminus \{0\}$ with $\bar{w}^\mathsf{T} \bar{y} < \inf_{\bar{y}\in\mathcal{P}} \bar{w}^\mathsf{T} \bar{y} =: \bar{\alpha}$. By the definition of $\bar{\alpha}$, $\bar{w}^\mathsf{T} \bar{y} \geq \bar{\alpha}$ holds for every $\bar{y} \in \mathcal{P}$. Therefore, $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \mathcal{D}_\mathcal{P}$ and using the result from part (a), $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \mathcal{D}$ as well. Now, $\varphi(y, \bar{w}, \bar{\alpha}) = \bar{w}^\mathsf{T} y - \bar{\alpha} < 0$, which contradicts with $y \in \mathcal{P}_\mathcal{D}$. Hence, $\mathcal{P}_\mathcal{D} \subseteq \mathcal{P}$.

□

## 4.2 Geometric duality between the approximations of $\mathcal{P}$ and $\mathcal{D}$

In Theorem 4.1.4, we have seen that the upper and lower images can be recovered from each other. In this section, we show that similar relations hold for polyhedral approximations of these sets. Throughout, we call a set $A \subseteq \mathbb{R}^q$ ($A \subseteq \mathbb{R}^{q+1}$) an *upper set* (a *lower set*) if $A = A + C$ ($A = A - K$).

We start by showing that a closed convex upper set can be recovered using the transformations introduced in Definition 4.1.5.

**Proposition 4.2.1.** *Let $\emptyset \neq \bar{\mathcal{P}} \subsetneq \mathbb{R}^q$ be a closed convex upper set and let $\mathcal{D}_{\bar{\mathcal{P}}}$ be defined by Definition 4.1.5. Then, $\bar{\mathcal{P}} = \mathcal{P}_{\mathcal{D}_{\bar{\mathcal{P}}}}$, that is,*

$$\bar{\mathcal{P}} = \{y \in \mathbb{R}^q \mid \forall (w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_{\bar{\mathcal{P}}} \colon \varphi(y, w, \alpha) \geq 0\}.$$

*Proof.* To show that $\bar{\mathcal{P}} \subseteq \mathcal{P}_{\mathcal{D}_{\bar{\mathcal{P}}}}$, let $y \in \bar{\mathcal{P}}$. For every $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_{\bar{\mathcal{P}}}$, we have $\varphi(y, w, \alpha) \geq 0$ as $y \in \bar{\mathcal{P}}$. Hence, $y \in \mathcal{P}_{\mathcal{D}_{\bar{\mathcal{P}}}}$. Next, we show that $\bar{\mathcal{P}} \supseteq \mathcal{P}_{\mathcal{D}_{\bar{\mathcal{P}}}}$. Assume that the inclusion does not hold. Then, there exists $\tilde{y} \in \mathcal{P}_{\mathcal{D}_{\bar{\mathcal{P}}}} \setminus \bar{\mathcal{P}}$. By the definition of $\mathcal{P}_{\mathcal{D}_{\bar{\mathcal{P}}}}$, $\varphi(\tilde{y}, w, \alpha) = w^\mathsf{T} \tilde{y} - \alpha \geq 0$ holds for every $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_{\bar{\mathcal{P}}}$. Using separation theorem, we may find $\bar{w} \in \mathbb{R}^q \setminus \{0\}$ such that $\bar{w}^\mathsf{T} \tilde{y} < \inf_{\bar{y}\in\bar{\mathcal{P}}} \bar{w}^\mathsf{T} \bar{y} =: \bar{\alpha}$. Clearly, for every $y \in \bar{\mathcal{P}}$, $\bar{w}^\mathsf{T} y \geq \bar{\alpha}$, that is, $\varphi(y, \bar{w}, \bar{\alpha}) \geq 0$ holds. Hence, $(\bar{w}^\mathsf{T}, \bar{\alpha}) \in \mathcal{D}_{\bar{\mathcal{P}}}$. But $\varphi(\tilde{y}, \bar{w}, \bar{\alpha}) = \bar{w}^\mathsf{T} \tilde{y} - \bar{\alpha} < 0$, contradicting the definition of $\mathcal{P}_{\mathcal{D}_{\bar{\mathcal{P}}}}$. Therefore, $\bar{\mathcal{P}} \supseteq \mathcal{P}_{\mathcal{D}_{\bar{\mathcal{P}}}}$. □

Next, for a closed convex lower set $\bar{\mathcal{D}}$, we want to investigate the relationship between $\bar{\mathcal{D}}$ and $\mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$. While the equality of these sets may not hold in general, we will see by Propositions 4.2.4 and 4.2.5 that it holds for some special cases. Before these results, we provide two lemmas. The first one shows that, when computing $\mathcal{P}_{\bar{\mathcal{D}}}$, considering the extreme directions of $\bar{\mathcal{D}}$ is sufficient under a special structure for $\bar{\mathcal{D}}$.

**Lemma 4.2.2.** *Let $\bar{\mathcal{D}} = \operatorname{cone}\operatorname{conv} \xi(\bar{\mathcal{W}}) - K$ for some $\bar{\mathcal{W}} \subseteq \mathcal{W}$. Then,*

$$\mathcal{P}_{\bar{\mathcal{D}}} = \{y \in \mathbb{R}^q \mid \forall (w^\mathsf{T}, \alpha)^\mathsf{T} \in \xi(\bar{\mathcal{W}}) \colon \varphi(y, w, \alpha) \geq 0\}.$$

*Proof.* Let $\tilde{\mathcal{P}} := \{y \in \mathbb{R}^q \mid \forall (w^\mathsf{T}, \alpha)^\mathsf{T} \in \xi(\bar{\mathcal{W}}) \colon \varphi(y, w, \alpha) \geq 0\}$. Since $\bar{\mathcal{D}} \supseteq \xi(\bar{\mathcal{W}})$, we obtain $\tilde{\mathcal{P}} \supseteq \mathcal{P}_{\bar{\mathcal{D}}}$. To show the reverse inclusion, let us fix $y \in \tilde{\mathcal{P}}$. Let $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \bar{\mathcal{D}}$, that is, there exist $n \in \mathbb{N}$, $\lambda_i \geq 0$, $(\tilde{w}_i^\mathsf{T}, \tilde{\alpha}_i)^\mathsf{T} \in \xi(\bar{\mathcal{W}})$ for $i \in \{1, \dots, n\}$, $\beta \geq 0$ such that $(w^\mathsf{T}, \alpha + \beta)^\mathsf{T} = \sum_{i=1}^n \lambda_i (\tilde{w}_i^\mathsf{T}, \tilde{\alpha}_i)^\mathsf{T}$. In particular, $\tilde{w}_i^\mathsf{T} y - \tilde{\alpha}_i \geq 0$ for each $i \in \{1, \dots, n\}$. Then,

$$\varphi(y, w, \alpha) = (w^\mathsf{T} y - \alpha - \beta) + \beta = \sum_{i=1}^n \lambda_i \tilde{w}_i^\mathsf{T} y - \sum_{i=1}^n \lambda_i \tilde{\alpha}_i + \beta = \sum_{i=1}^n \lambda_i (\tilde{w}_i^\mathsf{T} y - \tilde{\alpha}_i) + \beta \geq 0.$$

Since $\varphi(y, w, \alpha) \geq 0$ for each $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \bar{\mathcal{D}}$, we conclude that $y \in \mathcal{P}_{\bar{\mathcal{D}}}$. □

A similar approach as in Lemma 4.2.2 can be developed using the vertices and facets of an upper set. Recall that a closed convex set containing no lines can be represented by the set of its extreme points and extreme directions, see [23, Theorem 18.5]. Hence, the following holds.

**Lemma 4.2.3.** *Let $\emptyset \neq \bar{\mathcal{P}} \subsetneq \mathbb{R}^q$ be a closed convex set such that $\bar{\mathcal{P}} = \bar{\mathcal{P}} + C$. Let $V$ be the set of its extreme points and $B$ be the set of its extreme directions so that*

$$\bar{\mathcal{P}} = \operatorname{conv} V + \operatorname{cone}\operatorname{conv} B.$$

*Then,*

$$\mathcal{D}_{\bar{\mathcal{P}}} = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall v \in V \colon \varphi(v, w, \alpha) \geq 0\} \cap ((\operatorname{cone} B)^+ \times \mathbb{R}). \quad (4.2.1)$$

*Proof.* Let us denote by $\tilde{\mathcal{D}}$ the set on the right of (4.2.1). Since $\bar{\mathcal{P}} \supseteq V$, the inclusion $\mathcal{D}_{\bar{\mathcal{P}}} \subseteq \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall v \in V \colon \varphi(v, w, \alpha) \geq 0\}$ holds. To show that $\mathcal{D}_{\bar{\mathcal{P}}} \subseteq (\operatorname{cone} B)^+ \times \mathbb{R}$, let $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_{\bar{\mathcal{P}}}$, that is, for every $y \in \bar{\mathcal{P}}$, the inequality $\varphi(y, w, \alpha) \geq 0$ holds. Let us fix $n, n' \in \mathbb{N}$, $v_i \in V$ for each $i \in \{1, \ldots, n\}$, $b_j \in B$ for each $j \in \{1, \ldots, n'\}$; and consider $y \in \bar{\mathcal{P}}$ of the form

$$y = \sum_{i=1}^{n} \lambda_i v_i + \sum_{j=1}^{n'} \mu_j b_j, \quad (4.2.2)$$

where $\lambda_i \geq 0$ for each $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^n \lambda_i = 1$, $\mu_j \geq 0$ for each $j \in \{1, \ldots, n'\}$. We have

$$\varphi(y, w, \alpha) = w^\mathsf{T} y - \alpha = \sum_{i=1}^{n} \lambda_i w^\mathsf{T} v_i + \sum_{j=1}^{n'} \mu_j w^\mathsf{T} b_j - \alpha \geq 0.$$

We claim that $w^\mathsf{T} b_j \geq 0$ for each $j \in \{1, \ldots, n'\}$. To get a contradiction, assume that there exists $j' \in \{1, \ldots, n'\}$ such that $w^\mathsf{T} b_{j'} < 0$. Then, we may choose $\mu_{j'} > 0$ such that $\mu_{m'} w^\mathsf{T} b_{j'} < \alpha - \sum_{i=1}^n \lambda_i w^\mathsf{T} v_i - \sum_{j \in \{1, \ldots, n'\} \setminus j'} \mu_j w^\mathsf{T} b_j$ holds. Therefore, we have

$$\varphi(y, w, \alpha) = \sum_{i=1}^{n} \lambda_i w^\mathsf{T} v_i + \sum_{j \in \{1, \ldots, n'\} \setminus j'} \mu_j w^\mathsf{T} b_j + \mu_{j'} w^\mathsf{T} b_{j'} - \alpha < 0,$$

a contradiction. Hence, the claim holds, which implies that $w \in (\operatorname{cone} B)^+$ so that $\mathcal{D}_{\bar{\mathcal{P}}} \subseteq (\operatorname{cone} B)^+ \times \mathbb{R}$. This completes the proof of $\tilde{\mathcal{D}} \supseteq \mathcal{D}_{\bar{\mathcal{P}}}$. For the reverse inclusion, let $(\tilde{w}^\mathsf{T}, \tilde{\alpha})^\mathsf{T} \in \tilde{\mathcal{D}}$ and $y \in \bar{\mathcal{P}}$ be arbitrary. Then, $y$ can be written in the form (4.2.2) and we have

$$\varphi(y, \tilde{w}, \tilde{\alpha}) = \tilde{w}^\mathsf{T} y - \tilde{\alpha} = \sum_{i=1}^{n} \lambda_i (\tilde{w}^\mathsf{T} v_i - \tilde{\alpha}) + \sum_{j=1}^{n'} \mu_j \tilde{w}^\mathsf{T} b_j \geq 0$$

as $\tilde{w}^\mathsf{T} v - \tilde{\alpha} \geq 0$ for every $v \in V$ and $\tilde{w}^\mathsf{T} b \geq 0$ for every $b \in B$. Therefore, $\varphi(y, \tilde{w}, \tilde{\alpha}) \geq 0$ for every $y \in \bar{\mathcal{P}}$, meaning that $(\tilde{w}^\mathsf{T}, \tilde{\alpha})^\mathsf{T} \in \mathcal{D}_{\bar{\mathcal{P}}}$. Hence, $\tilde{\mathcal{D}} \subseteq \mathcal{D}_{\bar{\mathcal{P}}}$ holds. $\square$

Next, under a special structure, we show that a lower set $\bar{\mathcal{D}}$ can be recovered from the transformations introduced in Definition 4.1.5.

**Proposition 4.2.4.** *Let* $\bar{\mathcal{D}} = \operatorname{cone}\operatorname{conv}\xi(\bar{\mathcal{W}}) - K$ *for some* $\bar{\mathcal{W}} \subseteq \mathcal{W}$ *and let* $\mathcal{P}_{\bar{\mathcal{D}}}$ *be defined by Definition 4.1.5. Then,* $\bar{\mathcal{D}} = \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$*, that is,*

$$\bar{\mathcal{D}} = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in \mathcal{P}_{\bar{\mathcal{D}}} \colon \varphi(y, w, \alpha) \geq 0\}.$$

*Proof.* To prove that $\bar{\mathcal{D}} \subseteq \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$, let $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \bar{\mathcal{D}}$ and $y \in \mathcal{P}_{\bar{\mathcal{D}}}$. We have $\varphi(y, \bar{w}, \bar{\alpha}) \geq 0$. Therefore, $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$.

As a preparation for the proof of $\bar{\mathcal{D}} \supseteq \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$, let us define the following projections:

$$\bar{\bar{\mathcal{W}}} := \{w \in \mathbb{R}^q \mid \exists \alpha \in \mathbb{R} \colon (w^\mathsf{T}, \alpha)^\mathsf{T} \in \bar{\mathcal{D}}\},$$
$$\tilde{\mathcal{W}} := \{w \in \mathbb{R}^q \mid \exists \alpha \in \mathbb{R} \colon (w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}\}.$$

First, note that $\bar{\bar{\mathcal{W}}}$ is a cone satisfying $\bar{\bar{\mathcal{W}}} \subseteq \mathcal{W} = C^+$ as $\bar{\mathcal{D}}$ is a cone and $\bar{\mathcal{W}} \subseteq \mathcal{W}$. We claim that $\bar{\bar{\mathcal{W}}} = \tilde{\mathcal{W}}$. Indeed, since the inclusion $\bar{\mathcal{D}} \subseteq \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$ is shown above, we immediately have $\bar{\bar{\mathcal{W}}} \subseteq \tilde{\mathcal{W}}$. Let us prove that $\bar{\bar{\mathcal{W}}} \supseteq \tilde{\mathcal{W}}$. To get a contradiction, assume that there exists $\tilde{w} \in \tilde{\mathcal{W}} \setminus \bar{\bar{\mathcal{W}}}$. Using separation theorem, we may find $\beta \in \mathbb{R}^q \setminus \{0\}$ such that

$$-\infty < \tilde{w}^\mathsf{T}\beta < \inf_{w \in \bar{\bar{\mathcal{W}}}} w^\mathsf{T}\beta. \tag{4.2.3}$$

Note that $\inf_{w \in \bar{\bar{\mathcal{W}}}} w^\mathsf{T}\beta = 0$. Indeed, if there exists $w \in \bar{\bar{\mathcal{W}}}$ with $w^\mathsf{T}\beta < 0$, then we have $\inf_{w \in \bar{\bar{\mathcal{W}}}} w^\mathsf{T}\beta \leq \inf_{\lambda > 0} \lambda w^\mathsf{T}\beta = -\infty$ since $\bar{\bar{\mathcal{W}}}$ is a cone and $\lambda w \in \bar{\bar{\mathcal{W}}}$ for every $\lambda \geq 0$. Hence, $\inf_{w \in \bar{\bar{\mathcal{W}}}} w^\mathsf{T}\beta \geq 0$ holds. Moreover, equality holds again as $\bar{\bar{\mathcal{W}}}$ is a cone. Then, by (4.2.3),

$$\tilde{w}^\mathsf{T}\beta < 0. \tag{4.2.4}$$

Next, we prove that $\beta \in \operatorname{recc}\mathcal{P}_{\bar{\mathcal{D}}}$. Let $y \in \mathcal{P}_{\bar{\mathcal{D}}}$ and $\mu > 0$. For every $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \bar{\mathcal{D}}$, we have $\varphi(y, w, \alpha) = w^\mathsf{T}y - \alpha \geq 0$. On the other hand, it is already shown that $w^\mathsf{T}\beta \geq 0$ for each $w \in \bar{\bar{\mathcal{W}}}$. Hence,

$$\varphi(y + \mu\beta, w, \alpha) = w^\mathsf{T}y + \mu w^\mathsf{T}\beta - \alpha \geq 0$$

for every $(w^\mathsf{T},\alpha)^\mathsf{T} \in \bar{\mathcal{D}}$, which implies that $y+\mu\beta \in \mathcal{P}_{\bar{\mathcal{D}}}$. So $\beta \in \operatorname{recc}\mathcal{P}_{\bar{\mathcal{D}}}$.

Note that $\mathcal{P}_{\bar{\mathcal{D}}}$ is an intersection of closed halfspaces, hence it is a closed convex set. It also satisfies $\mathcal{P}_{\bar{\mathcal{D}}} = \mathcal{P}_{\bar{\mathcal{D}}} + C$. To see this, let $y \in \mathcal{P}_{\bar{\mathcal{D}}}$, $c \in C$, and $(w^\mathsf{T},\alpha)^\mathsf{T} \in \bar{\mathcal{D}}$. We have $\varphi(y,w,\alpha) = w^\mathsf{T}y - \alpha \geq 0$. Moreover, $w^\mathsf{T}c \geq 0$ since $\bar{\bar{\mathcal{W}}} \subseteq C^+$. Then, $\varphi(y+c,w,\alpha) = w^\mathsf{T}y + w^\mathsf{T}c - \alpha \geq 0$. Hence, $y+c \in \mathcal{P}_{\bar{\mathcal{D}}}$ so that we have $\mathcal{P}_{\bar{\mathcal{D}}} = \mathcal{P}_{\bar{\mathcal{D}}} + C$. Using the setting of Lemma 4.2.3, let $V$ be the set of all extreme points, and $B$ the set of all extreme directions of $\mathcal{P}_{\bar{D}}$. In particular, $\operatorname{recc}\mathcal{P}_{\bar{\mathcal{D}}} = \operatorname{cone} B$ and $\beta \in \operatorname{cone} B$ by the arguments above. By Lemma 4.2.3, we have

$$\mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}} = \{(w^\mathsf{T},\alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall v \in V \colon \varphi(v,w,\alpha) \geq 0\} \cap ((\operatorname{cone} B)^+ \times \mathbb{R}).$$

Therefore, $(\operatorname{cone} B)^+ \times \mathbb{R} \supseteq \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$, which implies that $(\operatorname{cone} B)^+ \supseteq \tilde{\mathcal{W}}$ and hence $\operatorname{cone} B \subseteq \tilde{\mathcal{W}}^+$. Thus, we have $\beta \in \tilde{\mathcal{W}}^+$, meaning that $\tilde{w}^\mathsf{T}\beta \geq 0$ for every $\tilde{w} \in \tilde{\mathcal{W}}$. Clearly, this contradicts (4.2.4), which completes the proof of $\bar{\bar{\mathcal{W}}} \supseteq \tilde{\mathcal{W}}$. Hence, $\bar{\bar{\mathcal{W}}} = \tilde{\mathcal{W}}$.

Finally, we prove that $\bar{\mathcal{D}} \supseteq \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$. To get a contradiction, we assume that there exists $(\tilde{w}^\mathsf{T},\tilde{\alpha})^\mathsf{T} \in \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}} \setminus \bar{\mathcal{D}}$. By separation theorem, there exists $a \in \mathbb{R}^{q+1} \setminus \{0\}$ such that

$$a^\mathsf{T}(\tilde{w}^\mathsf{T},\tilde{\alpha})^\mathsf{T} < \inf_{(w^\mathsf{T},\alpha)^\mathsf{T} \in \bar{\mathcal{D}}} a^\mathsf{T}(w^\mathsf{T},\alpha)^\mathsf{T} =: b. \tag{4.2.5}$$

Note that

$$H^* := \{(w^\mathsf{T},\alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid a^\mathsf{T}(w^\mathsf{T},\alpha)^\mathsf{T} = b\} \tag{4.2.6}$$

is a supporting hyperplane of $\bar{\mathcal{D}}$. Then, $F^* := H^* \cap \bar{\mathcal{D}}$ is a proper face of $\bar{\mathcal{D}}$ [23, Section 18]. Since $\bar{\mathcal{D}}$ is a closed convex cone, $F^*$ is also a closed convex cone [24, Lemma 10.2]. Therefore, $0 \in F^*$ and $0 \in H^*$. In particular, we have $b = 0$, and $a^\mathsf{T}(w^\mathsf{T},\alpha)^\mathsf{T} \geq 0$ holds for each $(w^\mathsf{T},\alpha)^\mathsf{T} \in \bar{\mathcal{D}}$.

We claim that $a_{q+1} < 0$. By definition, $\bar{\mathcal{D}}$ is a lower set so that the inequality

$$a^\mathsf{T}(w^\mathsf{T},\alpha-\gamma)^\mathsf{T} = a^\mathsf{T}w + a_{q+1}\alpha - a_{q+1}\gamma \geq 0$$

holds for each $(w^\mathsf{T},\alpha)^\mathsf{T} \in \bar{\mathcal{D}}$ and $\gamma \geq 0$. Therefore, we must have $a_{q+1} \leq 0$. If $a_{q+1} = 0$ holds, then we have $a^\mathsf{T}w \geq 0$ for every $w \in \bar{\bar{\mathcal{W}}}$. In particular, $a^\mathsf{T}\tilde{w} \geq 0$ since $\tilde{w} \in \tilde{\mathcal{W}} = \bar{\bar{\mathcal{W}}}$. Clearly, this contradicts (4.2.5). Therefore, we have $a_{q+1} < 0$.

We proceed by setting

$$y := \left(\frac{-a_1}{a_{q+1}}, \frac{-a_2}{a_{q+1}}, \dots, \frac{-a_q}{a_{q+1}}\right)^{\mathsf{T}}$$

and obtain $H^* = H^*(y)$ by following the definition in (4.2.6). Since $\varphi(y, w, \alpha) = w^{\mathsf{T}}y - \alpha \geq 0$ for each $(w^{\mathsf{T}}, \alpha)^{\mathsf{T}} \in \bar{\mathcal{D}}$, we have $y \in \mathcal{P}_{\bar{\mathcal{D}}}$. Then, as $(\tilde{w}^{\mathsf{T}}, \tilde{\alpha})^{\mathsf{T}} \in \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$, we must have $\varphi(y, \tilde{w}, \tilde{\alpha}) = \tilde{w}^{\mathsf{T}}y - \tilde{\alpha} \geq 0$ by the definition of $\mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$. However, by (4.2.5), we have $\varphi(y, \tilde{w}, \tilde{\alpha}) = \tilde{w}^{\mathsf{T}}y - \tilde{\alpha} < 0$, a contradiction. This completes the proof of $\bar{\mathcal{D}} \supseteq \mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}}$. □

The property shown in Proposition 4.2.4 also holds when one considers a different lower set. The construction in the next proposition will be useful in designing the dual algorithm in Chapter 5. Recall that $\mathbb{S}^{q-1}$ denotes the unit sphere in $\mathbb{R}^q$ with respect to the dual norm $\|\cdot\|_*$.

**Proposition 4.2.5.** *Let*

$$\mathcal{D}_\epsilon = \operatorname{cone}\Big(\big(\operatorname{cone}\operatorname{conv}\xi(\bar{\mathcal{W}}) \cap (\mathbb{S}^{q-1} \times \mathbb{R})\big) + \epsilon\{e^{q+1}\}\Big) - K$$

*for some $\bar{\mathcal{W}} \subseteq \mathcal{W}$ and let $\mathcal{P}_{\mathcal{D}_\epsilon}$ be as in Definition 4.1.5. Then, $\mathcal{D}_\epsilon = \mathcal{D}_{\mathcal{P}_{\mathcal{D}_\epsilon}}$, that is,*

$$\mathcal{D}_\epsilon = \{(w^{\mathsf{T}}, \alpha)^{\mathsf{T}} \in \mathbb{R}^{q+1} \mid \forall y \in \mathcal{P}_{\mathcal{D}_\epsilon} \colon \varphi(y, w, \alpha) \geq 0\}.$$

*Proof.* To prove that $\mathcal{D}_\epsilon \subseteq \mathcal{D}_{\mathcal{P}_{\mathcal{D}_\epsilon}}$, let $(\bar{w}^{\mathsf{T}}, \bar{\alpha})^{\mathsf{T}} \in \mathcal{D}_\epsilon$ and $y \in \mathcal{P}_{\mathcal{D}_\epsilon}$. We have $\varphi(y, \bar{w}, \bar{\alpha}) \geq 0$. Therefore, $(\bar{w}^{\mathsf{T}}, \bar{\alpha})^{\mathsf{T}} \in \mathcal{D}_{\mathcal{P}_{\mathcal{D}_\epsilon}}$ as well. Hence, the desired inclusion follows.

Let us define the projections

$$\begin{aligned}\bar{\bar{\mathcal{W}}} &:= \{w \in \mathbb{R}^q \mid \exists \alpha \in \mathbb{R} \colon (w^{\mathsf{T}}, \alpha)^{\mathsf{T}} \in \mathcal{D}_\epsilon\},\\ \tilde{\mathcal{W}} &:= \{w \in \mathbb{R}^q \mid \exists \alpha \in \mathbb{R} \colon (w^{\mathsf{T}}, \alpha)^{\mathsf{T}} \in \mathcal{D}_{\mathcal{P}_{\mathcal{D}_\epsilon}}\}.\end{aligned}$$

First, we show that $\bar{\bar{\mathcal{W}}}$ is a cone satisfying $\bar{\bar{\mathcal{W}}} \subseteq \mathcal{W} = C^+$. To that end, let $w \in \bar{\bar{\mathcal{W}}}$, which implies the existence of $b_1 \in \mathbb{R}$ such that $(w^{\mathsf{T}}, b_1)^{\mathsf{T}} \in \mathcal{D}_\epsilon$. By the definition of $\mathcal{D}_\epsilon$, there exist $\alpha \in \mathbb{R}$ and $b_2 \in \mathbb{R}_+$ such that $b_1 = \alpha - b_2$ and

$$(w^{\mathsf{T}}, \alpha)^{\mathsf{T}} \in \operatorname{cone}\Big(\big(\operatorname{cone}\operatorname{conv}\xi(\bar{\mathcal{W}}) \cap (\mathbb{S}^{q-1} \times \mathbb{R})\big) + \epsilon\{e^{q+1}\}\Big). \tag{4.2.7}$$

Let $\lambda \geq 0$. One can see that $(\lambda w^\mathsf{T}, \lambda\alpha)^\mathsf{T}$ belongs to the set on the right of (4.2.7). Therefore, $(\lambda w^\mathsf{T}, \lambda\alpha - \lambda b_2)^\mathsf{T} = (\lambda w^\mathsf{T}, \lambda b_1)^\mathsf{T} \in \mathcal{D}_\epsilon$ so that $\lambda w \in \bar{\bar{\mathcal{W}}}$. Hence, $\bar{\bar{\mathcal{W}}}$ is a cone.

On the other hand, by (4.2.7), there exist $\delta \geq 0$; $n \in \mathbb{N}$; $\lambda_i \geq 0$ and $(w_i^\mathsf{T}, \alpha_i)^\mathsf{T} \in \xi(\bar{\mathcal{W}})$ for each $i \in \{1, \ldots, n\}$ such that

$$(w^\mathsf{T}, \alpha)^\mathsf{T} = \delta\left(\frac{\sum_{i=1}^n \lambda_i (w_i^\mathsf{T}, \alpha_i)^\mathsf{T}}{\|\sum_{i=1}^n \lambda_i w_i\|_*} + \epsilon e^{q+1}\right).$$

In particular,

$$w = \frac{\delta}{\|\sum_{i=1}^n \lambda_i w_i\|_*} \sum_{i=1}^n \lambda_i w_i.$$

Since $w_i \in \bar{\mathcal{W}} \subseteq \mathcal{W}$ and $\mathcal{W} = C^+$ is a convex cone, $\sum_{i=1}^n \lambda_i w_i \in \mathcal{W}$. Hence, $w \in \mathcal{W}$ as well and $\bar{\bar{\mathcal{W}}} \subseteq \mathcal{W} = C^+$.

Following the same lines of reasoning as in the proof of Proposition 4.2.4, it can be shown that $\bar{\bar{\mathcal{W}}} = \tilde{\mathcal{W}}$ as well as $\mathcal{D}_\epsilon \supseteq \mathcal{D}_{\mathcal{P}_{\mathcal{D}_\epsilon}}$. To avoid repetitions, the details are omitted. $\square$

# Chapter 5

# Algorithms

In this section, we will present two approximation algorithms, namely the primal and dual algorithms, for solving the primal and dual problems, (P) and (D), simultaneously. First, we will explain the primal algorithm, which is proposed in [1] for solving (P) only, and show that by simple modifications, this algorithm also yields a solution to the dual problem (D). Next, we will describe the dual algorithm which uses the geometric duality results from Chapter 4.

Recall that $C$ is assumed to be a nontrivial, pointed, solid, closed and convex cone; the vector-valued objective function $f\colon \mathcal{X} \to \mathbb{R}^q$ is $C$-convex and continuous; and the feasible set $\mathcal{X} \subseteq \mathbb{R}^m$ is compact and convex. We further assume the following from now on.

**Assumption 5.0.1.** *(a) The feasible region of* (P) *has nonempty interior, that is,* $\operatorname{int} \mathcal{X} \neq \emptyset$*; and (b) the ordering cone $C$ is polyhedral.*

Under Assumption 5.0.1 (b), it is known that $C^+$ is also polyhedral. We denote the generating vectors of $C^+$ by $w^1, \dots, w^J$ and assume without loss of generality that $\|w^j\|_* = 1$ for each $j \in \{1, \dots, J\}$.

## 5.1 Primal algorithm

The primal algorithm in [1] is an outer approximation algorithm, that is, it works with polyhedral outer approximations of the upper image $\mathcal{P}$. In particular, it starts by finding a polyhedral outer approximation $\mathcal{P}_0$ of the upper image and iterates by updating the outer approximation with the help of supporting halfspaces of $\mathcal{P}$ until the approximation is sufficiently fine.

In each iteration of the primal algorithm (Algorithm 1), the following norm-minimizing scalarization problem is solved:

$$\text{minimize } \|z\| \text{ subject to } f(x) - z - v \in -C,\ z \in \mathbb{R}^q,\ x \in \mathcal{X}, \tag{P($v$)}$$

where $v \in \mathbb{R}^q$ is some parameter to be set by the algorithm. The Lagrangian dual of this problem is given by

$$\text{maximize } \inf_{x \in \mathcal{X}} w^\mathsf{T} f(x) - w^\mathsf{T} v \text{ subject to } \|w\|_* \leq 1,\ w \in C^+. \tag{D($v$)}$$

Before explaining the details of the algorithm, we present some results regarding (P($v$)) and (D($v$)); see [1] for details and further results.

**Proposition 5.1.1.** *Let $v \in \mathbb{R}^q$. The following statements hold true under Assumption 5.0.1:*

*(a) [1, Proposition 4.2] There exist optimal solutions $(x^v, z^v)$ and $w^v$ to* (P($v$)) *and* (D($v$))*, respectively, and the optimal values coincide.*

*(b) [1, Proposition 4.6] If $v \notin \operatorname{int} \mathcal{P}$, then $x^v$ is a weak minimizer for* (P).

*(c) [1, Remark 4.4] $x^v$ is an optimal solution to* (WS($w^v$))*, that is, $\inf_{x \in \mathcal{X}} (w^v)^\mathsf{T} f(x) = (w^v)^\mathsf{T} f(x^v)$.*

The primal algorithm is initialized by solving the weighted sum scalarization problem (WS($w$)) for each generating vector of the dual ordering cone $C^+$. Let $x^j \in \mathcal{X}$ denote the optimal solution of (WS($w^j$)) for each $j \in \{1, \ldots, J\}$. By

Proposition 3.0.3, each $x^j$ is a weak minimizer for (P). Moreover, from Proposition 3.0.10 and Definition 3.0.8, it is known that each $w^j$ is a maximizer for (D). Hence, we initialize the set to be returned as a weak $\epsilon$-solution for (P) as $\bar{\mathcal{X}} = \{x^1, \dots, x^J\}$; and the set to be returned as an $\epsilon$-solution to (D) as $\bar{\mathcal{W}} = \{w^1, \dots, w^J\}$. Note that by Proposition 4.1.1, for each $j \in \{1, \dots, J\}$, the set $\mathcal{H}(\xi(w^j))$ is a supporting halfspace of $\mathcal{P}$ such that $\mathcal{H}(\xi(w^j)) \supseteq \mathcal{P}$. Then, the initial outer approximation of $\mathcal{P}$ is defined as

$$\mathcal{P}_0 := \bigcap_{j=1}^{J} \mathcal{H}(\xi(w^j)).$$

As part of the initialization, we introduce a set $\mathcal{V}_{\text{known}}$, which stores the set of vertices that have already been considered by the algorithm and initialize it as $\mathcal{V}_{\text{known}} = \emptyset$, see lines 1-3 of Algorithm 1. Note that we later introduce a set $\mathcal{V}_{\text{unknown}}$, which stores the set of vertices of the current approximation that are not yet considered, see line 7.

In each iteration $k$, the first step is to compute the vertices $\mathcal{V}_k$ of the current outer approximation $\mathcal{P}_k$ by solving a vertex enumeration problem (line 6). Then, for each vertex in $\mathcal{V}_k$ which have not been considered before, optimal solutions for (P($v$)) and (D($v$)) are found; the respective solutions are added to sets $\bar{\mathcal{X}}$ and $\bar{\mathcal{W}}$ (see Proposition 5.1.1(b) and Proposition 3.0.10); and $\mathcal{V}_{\text{known}}$ is updated (lines 7-10). Note that by Proposition 5.1.1(c) and Proposition 4.1.1, $\mathcal{H}(\xi(w^v))$ is a supporting halfspace of $\mathcal{P}$. If the distance of a vertex $v$ to the upper image, namely $\|z^v\|$, is not sufficiently small, then $\mathcal{H}(\xi(w^v)) = \{y \in \mathbb{R}^q \mid \varphi(y, w^v, (w^v)^\mathsf{T} f(x^v)) \geq 0\}$ is stored in order to be used in updating the current outer approximation. After each vertex in $\mathcal{V}_k$ is considered, then the current approximation is updated by intersecting it with those halfspaces (lines 11-16). The algorithm terminates when all the vertices of $\mathcal{P}_k$ are in $\epsilon$ distance to the upper image (lines 5 and 18).

**Remark 5.1.2.** *A 'break' command can be placed between lines 12 and 13 in the algorithm. In the current version, the algorithm goes over all the vertices of the current outer approximation without updating it. If such command is added, then the algorithm updates the outer approximation as soon as it detects a vertex $v$ with $\|z^v\| > \epsilon$.*

**Algorithm 1** Primal algorithm

Compute an optimal solution $x^j$ of (WS($w^j$)) for each $j \in \{1, \dots, J\}$;
Let $\mathcal{P}_0 = \bigcap_{j=1}^{J} \mathcal{H}(\xi(w^j))$;
$k \leftarrow 0, \bar{\mathcal{X}} \leftarrow \{x^1, \dots, x^J\}, \bar{\mathcal{W}} \leftarrow \{w^1, \dots, w^J\}, \mathcal{V}_{\text{known}} = \emptyset$;
**repeat**
  $M \leftarrow \mathbb{R}^q$;
  Compute the set $\mathcal{V}_k$ of vertices of $\mathcal{P}_k$;
  $\mathcal{V}_{\text{unknown}} \leftarrow \mathcal{V}_k \setminus \mathcal{V}_{\text{known}}$;
  **for** $v \in \mathcal{V}_{\text{unknown}}$ **do**
    Compute optimal solutions $(x^v, z^v)$ and $w^v$ to (P($v$)) and (D($v$));
    $\bar{\mathcal{X}} \leftarrow \bar{\mathcal{X}} \cup \{x^v\}, \bar{\mathcal{W}} \leftarrow \bar{\mathcal{W}} \cup \{\frac{w^v}{\|w^v\|_*}\}, \mathcal{V}_{\text{known}} \leftarrow \mathcal{V}_{\text{known}} \cup \{v\}$;
    **if** $\|z^v\| > \epsilon$ **then**
      $M \leftarrow M \cap \mathcal{H}(\xi(w^v))$;
    **end if**
  **end for**
  **if** $M \neq \mathbb{R}^q$ **then**
    $\mathcal{P}_{k+1} = \mathcal{P}_k \cap M,\ k \leftarrow k+1$;
  **end if**
**until** $M = \mathbb{R}^q$
**return** $\begin{cases} \bar{\mathcal{X}} & : \text{A finite weak } \epsilon\text{-solution to (P);} \\ \bar{\mathcal{W}} & : \text{A finite } \epsilon\text{-solution to (D);} \end{cases}$

The following proposition states that the primal algorithm gives a finite weak $\epsilon$-solution $\bar{\mathcal{X}}$ to problem (P).

**Proposition 5.1.3.** *[1, Theorem 5.4] If the primal algorithm terminates, then it returns a finite weak $\epsilon$-solution $\bar{\mathcal{X}}$ to* (P)*.*

Next, we show that the primal algorithm yields also a finite $\epsilon$-solution, $\bar{\mathcal{W}}$, for the dual problem (D). To that end, we provide the following lemma which shows that inner and outer approximations of the lower image $\mathcal{D}$ can be obtained by using a finite $\epsilon$-solution $\bar{\mathcal{X}}$ of (P). Then, this lemma will be used in order to prove the main result of this section.

**Lemma 5.1.4.** *For $\epsilon > 0$, let $\bar{\mathcal{X}}$ be a finite weak $\epsilon$-solution of* (P)*, and $\mathcal{P}_\epsilon := \operatorname{conv} f(\bar{\mathcal{X}}) + C + B(0, \epsilon)$. Then, $\mathcal{D}_\epsilon := \mathcal{D}_{\mathcal{P}_\epsilon} = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in \mathcal{P}_\epsilon \colon \varphi(y, w, \alpha) \geq 0\}$ is an inner approximation of $\mathcal{D}$ and satisfies*

$$\operatorname{cone}\left(\left(\mathcal{D}_\epsilon \cap (\mathbb{S}^{q-1} \times \mathbb{R})\right) + \epsilon\{e^{q+1}\}\right) - K \supseteq \mathcal{D} \supseteq \mathcal{D}_\epsilon. \tag{5.1.1}$$

*Proof.* Since $\bar{\mathcal{X}}$ is a finite weak $\epsilon$-solution of (P), $\mathcal{P}_\epsilon$ is an outer approximation of the upper image $\mathcal{P}$ by Definition 3.0.6, that is, $\mathcal{P}_\epsilon \supseteq \mathcal{P}$. By Proposition 4.1.6(a), we have $\mathcal{D} \supseteq \mathcal{D}_\epsilon$.

In order to show the first inclusion in (5.1.1), we first note that $\mathcal{P}' := \operatorname{conv} f(\bar{\mathcal{X}}) + C \subseteq \mathcal{P}$ and, by Proposition 4.1.6(a), we have

$$\mathcal{D}' := \mathcal{D}_{\mathcal{P}'} = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in \mathcal{P}' \colon \varphi(y, w, \alpha) \geq 0\} \supseteq \mathcal{D}. \tag{5.1.2}$$

We claim that

$$(\mathcal{D}_\epsilon \cap (\mathbb{S}^{q-1} \times \mathbb{R})) + \epsilon\{e^{q+1}\} \supseteq \mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R}) \tag{5.1.3}$$

holds. Indeed, observe that

$$\begin{aligned}
&(\mathcal{D}_\epsilon \cap (\mathbb{S}^{q-1} \times \mathbb{R})) + \epsilon\{e^{q+1}\} \\
&= \{(w^\mathsf{T}, \alpha + \epsilon)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \|w\|_* = 1,\ \forall y \in \mathcal{P}_\epsilon \colon \varphi(y, w, \alpha) \geq 0\} \\
&= \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \|w\|_* = 1,\ \forall y \in \mathcal{P}_\epsilon \colon \varphi(y, w, \alpha - \epsilon) \geq 0\} \\
&= \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \|w\|_* = 1,\ \forall y \in \mathcal{P}',\ \forall \gamma \in B(0,1) \colon \varphi(y + \gamma\epsilon, w, \alpha - \epsilon) \geq 0\}.
\end{aligned}$$

On the other hand, we have

$$\mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R}) = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \|w\|_* = 1,\ \forall y \in \mathcal{P}' \colon \varphi(y, w, \alpha) \geq 0\}.$$

Let $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R})$ be arbitrary. Note that $\varphi(y, w, \alpha) = w^\mathsf{T} y - \alpha \geq 0$ for every $y \in \mathcal{P}'$ and $\|w\|_* = 1$. Moreover, for every $\gamma \in B(0,1)$ we have $|w^\mathsf{T}\gamma| \leq \|\gamma\| \, \|w\|_* \leq 1$, hence $w^\mathsf{T}\gamma \geq -1$. Then, $\varphi(y+\gamma\epsilon, w, \alpha-\epsilon) = w^\mathsf{T} y + \epsilon w^\mathsf{T}\gamma - \alpha + \epsilon \geq 0$ holds, that is, $(w^\mathsf{T}, \alpha)^\mathsf{T} \in (\mathcal{D}_\epsilon \cap (\mathbb{S}^{q-1} \times \mathbb{R})) + \epsilon\{e^{q+1}\}$, which implies (5.1.3).

The next step is to show that

$$\operatorname{cone}(\mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R})) - K = \mathcal{D}'. \tag{5.1.4}$$

Note that an arbitrary element of $\operatorname{cone}(\mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R})) - K$ can be written as $(\lambda w^\mathsf{T}, \lambda\alpha - \gamma)^\mathsf{T}$ for some $\lambda, \gamma \geq 0$ and $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1}$ satisfying $\|w\|_* = 1$ and $\varphi(y, w, \alpha) = w^\mathsf{T} y - \alpha \geq 0$ for all $y \in \mathcal{P}'$. Note that we have $\varphi(y, \lambda w, \lambda\alpha - \gamma) = \lambda w^\mathsf{T} y - \lambda\alpha + \gamma \geq 0$. Hence, $(\lambda w^\mathsf{T}, \lambda\alpha - \gamma)^\mathsf{T} \in \mathcal{D}'$, which shows that

$\operatorname{cone}(\mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R})) - K \subseteq \mathcal{D}'$. For the reverse inclusion, let $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}'$. As a first case, suppose that $w \neq 0$. Let $\lambda := \frac{1}{\|w\|_*} > 0$. Clearly, $\|\lambda w\|_* = 1$ and $\varphi(y, \lambda w, \lambda\alpha) = \lambda w^\mathsf{T} y - \lambda\alpha \geq 0$ holds for each $y \in \mathcal{P}'$. Hence, $(\lambda w^\mathsf{T}, \lambda\alpha)^\mathsf{T} \in \mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R})$ and $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \operatorname{cone}(\mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R})) - K$. Now, suppose that $w = 0$. By the definition of $\mathcal{D}'$, we have $\alpha \leq 0$. Since $\operatorname{cone}(\mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R}))$ is a closed cone, $0 \in \operatorname{cone}(\mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R}))$ and $(w, \alpha) = (0, \alpha) \in \operatorname{cone}(\mathcal{D}' \cap (\mathbb{S}^{q-1} \times \mathbb{R})) - K$. Therefore, (5.1.4) holds.

Finally, (5.1.2), (5.1.3) and (5.1.4) imply that $\operatorname{cone}((\mathcal{D}_\epsilon \cap (\mathbb{S}^{q-1} \times \mathbb{R})) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D}$. □

**Proposition 5.1.5.** *If the primal algorithm terminates, then it returns a finite $\epsilon$-solution $\bar{\mathcal{W}}$ to* (D)*.*

*Proof.* By the structure of the algorithm, $\bar{\mathcal{W}}$ is nonempty and it consists of maximizers by Proposition 3.0.10. Also the inclusion $\bar{\mathcal{W}} \subseteq \mathcal{W} \cap \mathbb{S}^{q-1}$ holds, see line 10 of Algorithm 1. To prove the statement, it is sufficient to show that $\mathcal{D} \subseteq \operatorname{cone}(\operatorname{conv}\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K$.

Let $\bar{\mathcal{D}} := \operatorname{cone}\operatorname{conv}\xi(\bar{\mathcal{W}}) - K$ and $\bar{\mathcal{P}} := \mathcal{P}_{\bar{\mathcal{D}}}$. By Lemma 4.2.2, we have

$$\bar{\mathcal{P}} = \{y \in \mathbb{R}^q \mid \forall (w^\mathsf{T}, \alpha)^\mathsf{T} \in \xi(\bar{\mathcal{W}})\colon \varphi(y, w, \alpha) \geq 0\}.$$

Though not part of the original algorithm, we introduce an alternative for $\bar{\mathcal{W}}$ that is updated only when the current vertex $v$ is sufficiently far from the upper image, that is, when $\|z^v\| > \epsilon$. More precisely, let us introduce a set $\bar{\bar{\mathcal{W}}}$ that is initialized as $\bar{\bar{\mathcal{W}}} = \{w^1, \ldots, w^J\}$ in line 3 of Algorithm 1 and updated as $\bar{\bar{\mathcal{W}}} \leftarrow \bar{\bar{\mathcal{W}}} \cup \{\frac{w^v}{\|w^v\|_*}\}$ in line 12 throughout the algorithm. Observe that $\bar{\bar{\mathcal{W}}} \subseteq \bar{\mathcal{W}}$.

Suppose that the algorithm terminates at the $\bar{k}^{\text{th}}$ iteration and let $\mathcal{P}_{\bar{k}}$ denote the resulting outer approximation of the upper image. Due to the construction of $\bar{\bar{\mathcal{W}}}$, we have

$$\mathcal{P}_{\bar{k}} = \{y \in \mathbb{R}^q \mid \forall (w^\mathsf{T}, \alpha)^\mathsf{T} \in \xi(\bar{\bar{\mathcal{W}}})\colon \varphi(y, w, \alpha) \geq 0\}.$$

Since $\bar{\bar{\mathcal{W}}} \subseteq \bar{\mathcal{W}}$, the inclusion $\bar{\mathcal{P}} \subseteq \mathcal{P}_{\bar{k}}$ holds.

Let us define

$$\mathcal{P}_\epsilon := \operatorname{conv} f(\bar{\mathcal{X}}) + C + B(0,\epsilon).$$

By the structure and the termination criterion of the algorithm, for every vertex $v \in \mathcal{V}_{\bar{k}}$ of $\mathcal{P}_{\bar{k}}$, the scalarization problem (P($v$)) is solved; $x^v$ is added to $\bar{\mathcal{X}}$; and we have $\|z^v\| \leq \epsilon$. Moreover, $v + z^v \in \{f(x^v)\} + C \subseteq \operatorname{conv} f(\bar{\mathcal{X}}) + C$ holds for every $v \in \mathcal{V}_{\bar{k}}$. Thus, $\mathcal{V}_{\bar{k}} \subseteq \mathcal{P}_\epsilon$. From [1, Lemma 5.2], the recession cone of $\mathcal{P}_{\bar{k}}$ is the ordering cone; hence, $\mathcal{P}_{\bar{k}} = \operatorname{conv} \mathcal{V}_{\bar{k}} + C$. Moreover, $\mathcal{P}_\epsilon$ is a convex upper set. As a result, we have $\mathcal{P}_{\bar{k}} \subseteq \mathcal{P}_\epsilon$. Together with $\bar{\mathcal{P}} \subseteq \mathcal{P}_{\bar{k}}$, the last inclusion implies that $\bar{\mathcal{P}} \subseteq \mathcal{P}_\epsilon$.

Define $\mathcal{D}_\epsilon := \mathcal{D}_{\mathcal{P}_\epsilon} = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in \mathcal{P}_\epsilon \colon \varphi(y, w, \alpha) \geq 0\}$. By Proposition 4.2.4 and since $\bar{\mathcal{P}} \subseteq \mathcal{P}_\epsilon$, $\bar{\mathcal{D}} \supseteq \mathcal{D}_\epsilon$ holds. On the other hand, using Proposition 5.1.3 and Lemma 5.1.4, we know $\operatorname{cone}((\mathcal{D}_\epsilon \cap (\mathbb{S}^{q-1} \times \mathbb{R})) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D}$. Together with $\bar{\mathcal{D}} \supseteq \mathcal{D}_\epsilon$, this implies

$$\operatorname{cone}\left(\left(\bar{\mathcal{D}} \cap (\mathbb{S}^{q-1} \times \mathbb{R})\right) + \epsilon\{e^{q+1}\}\right) - K \supseteq \mathcal{D}. \tag{5.1.5}$$

We claim that

$$(\bar{\mathcal{D}} \cap (\mathbb{S}^{q-1} \times \mathbb{R})) + \epsilon\{e^{q+1}\} \subseteq \operatorname{cone}(\operatorname{conv} \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K. \tag{5.1.6}$$

Let $(\bar{w}, \bar{\alpha}) \in (\bar{\mathcal{D}} \cap (\mathbb{S}^{q-1} \times \mathbb{R})) + \epsilon\{e^{q+1}\}$, which can be written as

$$(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} = \sum_{i=1}^{n} \lambda_i((w^i)^\mathsf{T}, p^i)^\mathsf{T} - ae^{q+1} + \epsilon e^{q+1}$$

for some $a \geq 0$, $n \in \mathbb{N}$, $\lambda_i \geq 0$, $w^i \in \bar{\mathcal{W}}$ for $i \in \{1, \ldots, n\}$, satisfying $\|\sum_{i=1}^n \lambda_i w^i\|_* = 1$, where $p^i := \inf_{x \in \mathcal{X}} (w^i)^\mathsf{T} f(x)$ for $i \in \{1, \ldots, n\}$. Then, we can write

$$(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} = \bar{\lambda}\left(\sum_{i=1}^{n} \frac{\lambda_i}{\bar{\lambda}}((w^i)^\mathsf{T}, p^i)^\mathsf{T} + \epsilon e^{q+1}\right) - ae^{q+1} - (\bar{\lambda} - 1)\epsilon e^{q+1},$$

where $\bar{\lambda} := \sum_{i=1}^n \lambda_i$. Noting that

$$1 = \left\|\sum_{i=1}^{n} \lambda_i w^i\right\|_* \leq \sum_{i=1}^{n} \lambda_i \left\|w^i\right\|_* = \bar{\lambda},$$

we have $(\bar{w}^{\mathsf{T}}, \bar{\alpha})^{\mathsf{T}} \in \operatorname{cone}(\operatorname{conv} \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K$. Hence, (5.1.6) holds.

Finally, (5.1.6) implies that

$$\operatorname{cone}\big((\bar{\mathcal{D}} \cap (\mathbb{S}^{q-1} \times \mathbb{R})\big) + \epsilon\{e^{q+1}\}\big) - K \subseteq \operatorname{cone}(\operatorname{conv} \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K.$$

Together with (5.1.5), we obtain $\operatorname{cone}(\operatorname{conv} \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D}$; hence, $\bar{\mathcal{W}}$ is a finite $\epsilon$-solution to (D). □

## 5.2 Dual algorithm

In this section, we describe a geometric dual algorithm for solving problems (P) and (D). The main idea is to construct outer approximations of the lower image $\mathcal{D}$, iteratively. Recall that, by Proposition 3.0.4, $\mathcal{D}$ is a closed convex cone; similarly, we will see that the outer approximations found through the iterations of the dual algorithm are polyhedral convex cones.

The dual algorithm (Algorithm 2) is initialized by solving a weighted sum scalarization for some weight vector from $\operatorname{int} C^+$. In particular, we take

$$w^0 := \frac{\sum_{j=1}^J w^j}{\left\|\sum_{j=1}^J w^j\right\|_*}$$

and solve (WS($w^0$)). By Propositions 3.0.3 and 3.0.10, an optimal solution $x^0$ of (WS($w^0$)) is a weak minimizer for (P) and $w^0$ is a maximizer for (D). Hence, we set $\bar{\mathcal{X}} = \{x^0\}, \bar{\mathcal{W}} = \{w^0\}$. Moreover, using Proposition 4.1.1 and the definition of the lower image $\mathcal{D}$, we define the initial outer approximation of $\mathcal{D}$ as

$$\mathcal{D}_0 := \mathcal{H}^*(f(x^0)) \cap (C^+ \times \mathbb{R}) \supseteq \mathcal{D};$$

see lines 1-3 of Algorithm 2. Note that $\mathcal{D}_0$ satisfies $\mathcal{D}_0 = \mathcal{D}_0 - K$ and, under Assumption 5.0.1, it is a polyhedral convex cone.

Throughout the algorithm, weighted sum scalarizations will be solved for some weight vectors from $\mathcal{W}$. In order to keep track of the already used $w \in \mathcal{W}$, we keep a list $\mathcal{W}_{\text{known}}$ and initialize it as the empty set.

In each iteration $k$, first, the set $\mathcal{D}_k^{\text{dir}}$ of extreme directions of the current outer approximation $\mathcal{D}_k$ is computed (line 6). The extreme directions in $\mathcal{D}_k^{\text{dir}} \setminus (\{0\} \times \mathbb{R})$ are normalized such that $\|w\|_* = 1$, and $\mathcal{D}_k^{\text{dir}}$ is updated such that it is a subset of $\mathbb{S}^{q-1} \times \mathbb{R}$ (line 7). It will be seen that $\mathcal{D}_k$ is constructed by intersecting $\mathcal{D}_0$ with a set of halfspaces of the form $\mathcal{H}^*(f(x))$, where $x$ is a weak minimizer. Then, by the definitions of $\mathcal{D}_0$ and $\mathcal{H}^*(\cdot)$, it is clear that $\mathcal{D}_k = \mathcal{D}_k - K$; moreover, $e^{q+1}$ is not a recession direction for $\mathcal{D}_k$. Hence, we have

$$\mathcal{D}_k = \operatorname{cone}\operatorname{conv}(\mathcal{D}_k^{\text{dir}} \cup \{-e^{q+1}\}) = \operatorname{cone}\operatorname{conv}\mathcal{D}_k^{\text{dir}} - K. \tag{5.2.1}$$

For each unknown extreme direction $(w^\mathsf{T}, \alpha)^\mathsf{T}$, an optimal solution $x^w$ and the optimal value $p^w$ of (WS($w$)) are computed (lines 8-10). Recall that $x^w$ is a weak minimizer by Proposition 3.0.3 and $\xi(w) = (w, p^w)$ is a $K$-maximal element of $\mathcal{D}$ by Proposition 3.0.10; we update $\bar{\mathcal{X}}$ and $\bar{\mathcal{W}}$ accordingly, and add $(w^\mathsf{T}, \alpha)^\mathsf{T}$ to the set of known extreme directions (line 11). Since $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_k$ and $\mathcal{D}_k \supseteq \mathcal{D}$, it is true that $\alpha \geq p^w$. If the difference between $\alpha$ and $p^w$ is greater than the allowed error value $\epsilon$, then the supporting halfspace $\mathcal{H}^*(f(x^w))$ of $\mathcal{D}$ is computed and stored. Once all unknown extreme directions are explored, the current outer approximation is updated by using the stored supporting halfspaces (lines 12-14 and 17). The algorithm terminates when every unknown extreme direction $(w^\mathsf{T}, \alpha)^\mathsf{T}$ is sufficiently close to the lower image in terms of the "vertical distance" $\alpha - p^w$ (line 19).

**Remark 5.2.1.** *A 'break' command can be placed between lines 13 and 14 in the algorithm. With the current version, the algorithm goes through all the extreme directions of the current outer approximation without updating it. If such command was added, the algorithm would update the outer approximation as soon as it detects an extreme direction $w$ with $\alpha - p^w > \epsilon$.*

First, we will prove that Algorithm 2 returns a finite $\epsilon$-solution to problem (D).

**Proposition 5.2.2.** *If the dual algorithm terminates, then it returns a finite $\epsilon$-solution $\bar{\mathcal{W}}$ to* (D).

**Algorithm 2** Dual algorithm

Compute an optimal solution $x^0$ to (WS($w^0$)) for $w^0 = \frac{\sum_{j=1}^{J} w^j}{\left\|\sum_{j=1}^{J} w^j\right\|_*}$;
Let $\mathcal{D}_0 = \mathcal{H}^*(f(x^0)) \cap (C^+ \times \mathbb{R})$;
$k \leftarrow 0, \bar{\mathcal{X}} \leftarrow \{x^0\}, \bar{\mathcal{W}} \leftarrow \{w^0\}, \mathcal{D}_{\text{known}} = \emptyset$;
**repeat**
  $M \leftarrow \mathbb{R}^{q+1}$;
  Compute the set $\mathcal{D}_k^{\text{dir}}$ of extreme directions of $\mathcal{D}_k$;
  $\mathcal{D}_k^{\text{dir}} \leftarrow \{\frac{(w^\mathsf{T}, \alpha)^\mathsf{T}}{\|w\|_*} \in \mathbb{R}^{q+1} \mid (w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_k^{\text{dir}} \setminus (\{0\} \times \mathbb{R})\}$;
  $\mathcal{D}_{\text{unknown}} \leftarrow \mathcal{D}_k^{\text{dir}} \setminus \mathcal{D}_{\text{known}}$;
  **for** $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_{\text{unknown}}$ **do**
    Compute an optimal solution $x^w$ to (WS($w$)) and let $p^w := w^\mathsf{T} f(x^w)$;
    $\bar{\mathcal{X}} \leftarrow \bar{\mathcal{X}} \cup \{x^w\}$, $\bar{\mathcal{W}} \leftarrow \bar{\mathcal{W}} \cup \{w\}$, $\mathcal{D}_{\text{known}} \leftarrow \mathcal{D}_{\text{known}} \cup \{(w^\mathsf{T}, \alpha)^\mathsf{T}\}$;
    **if** $\alpha - p^w > \epsilon$ **then**
      $M \leftarrow M \cap \mathcal{H}^*(f(x^w))$;
    **end if**
  **end for**
  **if** $M \neq \mathbb{R}^{q+1}$ **then**
    $\mathcal{D}_{k+1} = \mathcal{D}_k \cap M, k \leftarrow k + 1$;
  **end if**
**until** $M = \mathbb{R}^{q+1}$
**return** $\begin{cases} \bar{\mathcal{X}} & : \text{A finite weak } \tilde{\epsilon}\text{-solution to (P)}; \\ \bar{\mathcal{W}} & : \text{A finite } \epsilon\text{-solution to (D)}; \end{cases}$

*Proof.* By the structure of the algorithm and Proposition 3.0.10, $\bar{\mathcal{W}}$ is nonempty and consists of maximizers. Also the inclusion $\bar{\mathcal{W}} \subseteq \mathcal{W} \cap \mathbb{S}^{q-1}$ holds. To prove the statement, one needs to show that

$$\mathcal{D} \subseteq \operatorname{cone}(\operatorname{conv} \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K.$$

Suppose that the algorithm terminates at the $\bar{k}^{\text{th}}$ iteration and let $\mathcal{D}_{\bar{k}}$ denote the resulting outer approximation. By Proposition 4.1.1, $\mathcal{D}_{\bar{k}} \supseteq \mathcal{D}$. The algorithm terminates only if every extreme direction $(w^\mathsf{T}, \alpha)^\mathsf{T}$ of $\mathcal{D}_{\bar{k}}$ satisfies $\alpha - p^w \leq \epsilon$, where $p^w = w^\mathsf{T} f(x^w)$. As every $w$ considered through the algorithm is added to $\bar{\mathcal{W}}$, clearly, we have $w \in \bar{\mathcal{W}}$ for every $(w^\mathsf{T}, \alpha)^\mathsf{T}$ of $\mathcal{D}_{\bar{k}}^{\text{dir}}$, where $\mathcal{D}_{\bar{k}}^{\text{dir}}$ denotes the set of all extreme directions of $\mathcal{D}_{\bar{k}}$ such that $\mathcal{D}_{\bar{k}}^{\text{dir}} \subseteq \mathbb{S}^{q-1} \times \mathbb{R}$. Without loss of generality, we write

$$\mathcal{D}_{\bar{k}}^{\text{dir}} := \{((\bar{w}^1)^\mathsf{T}, \alpha_1)^\mathsf{T}, \ldots, ((\bar{w}^m)^\mathsf{T}, \alpha_m)^\mathsf{T}\},$$

and

$$\bar{\mathcal{W}} = \{\bar{w}^1, \dots, \bar{w}^s\},$$

where $s \geq m$. Note that we have

$$\xi(\bar{\mathcal{W}}) = \{((\bar{w}^1)^\mathsf{T}, p^{\bar{w}^1})^\mathsf{T}, \dots, ((\bar{w}^s)^\mathsf{T}, p^{\bar{w}^s})^\mathsf{T}\}.$$

Moreover, from (5.2.1), $\mathcal{D}_{\bar{k}} = \operatorname{cone}\operatorname{conv}\{((\bar{w}^1)^\mathsf{T}, \alpha_1)^\mathsf{T}, \dots, ((\bar{w}^m)^\mathsf{T}, \alpha_m)^\mathsf{T}\} - K$. Using $s \geq m$, the definition of $K$ and $\alpha_i \leq p^{\bar{w}^i} + \epsilon$ for all $i \in \{1, \dots, m\}$, we have

$$\begin{aligned}
\operatorname{cone}(\operatorname{conv}\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K &= \operatorname{cone}(\operatorname{conv}\{((\bar{w}^1)^\mathsf{T}, p^{\bar{w}^1} + \epsilon)^\mathsf{T}, \dots, ((\bar{w}^s)^\mathsf{T}, p^{\bar{w}^s} + \epsilon)^\mathsf{T}\}) - K \\
&\supseteq \operatorname{cone}(\operatorname{conv}\{((\bar{w}^1)^\mathsf{T}, p^{\bar{w}^1} + \epsilon)^\mathsf{T}, \dots, ((\bar{w}^m)^\mathsf{T}, p^{\bar{w}^m} + \epsilon)^\mathsf{T}\}) - K \\
&\supseteq \operatorname{cone}(\operatorname{conv}\{((\bar{w}^1)^\mathsf{T}, \alpha_1)^\mathsf{T}, \dots, ((\bar{w}^m)^\mathsf{T}, \alpha_m)^\mathsf{T}\}) - K \\
&= \mathcal{D}_{\bar{k}} \supseteq \mathcal{D}.
\end{aligned}$$

Therefore, $\bar{\mathcal{W}}$ is a finite $\epsilon$-solution to (D). □

The next step is to prove that the algorithm returns also a solution to the primal problem (P). In order to prove this result, the following lemma and propositions will be useful. To that end, for each $n \in \{2, 3, \dots\}$, let us define

$$\Delta^{n-1} := \left\{\lambda \in \mathbb{R}^n_+ \mid \sum_{j=1}^n \lambda_j = 1\right\}, \qquad \Delta^{n-1}_+ := \left\{\lambda \in \mathbb{R}^n_+ \mid \sum_{j=1}^n \lambda_j \geq 1\right\}.$$

**Lemma 5.2.3.** *Let $d^1, \dots, d^n$ be the generating vectors of a pointed convex cone $D \subseteq \mathbb{R}^q$, where $n \geq 2$.*

*(a) For every $\lambda \in \Delta^{n-1}$, $\sum_{j=1}^n \lambda_j d^j \neq 0$ holds.*

*(b) It holds $\min_{\lambda \in \Delta^{n-1}} \|\sum_{j=1}^n \lambda_j d^j\|_* > 0$.*

*(c) It holds*

$$\min_{\lambda \in \Delta^{n-1}} \left\|\sum_{j=1}^n \lambda_j d^j\right\|_* = \min_{\lambda \in \Delta^{n-1}_+} \left\|\sum_{j=1}^n \lambda_j d^j\right\|_*.$$

*Proof.* (a) Let $\lambda \in \Delta^{n-1}$. Assume to the contrary that $\sum_{j=1}^n \lambda_j d^j = 0$. Since $\sum_{j=1}^n \lambda_j = 1$, we have $\lambda_j > 0$ for at least one $j \in \{1, \dots, n\}$. Without loss of generality, we may assume that $j = 1$. Hence, $\sum_{j=2}^{n} \lambda_j d^j = -\lambda_1 d^1$, that is,

$$\frac{1}{\lambda_1} \sum_{j=2}^{n} \lambda_j d^j = -d^1.$$

Since $D$ is a convex cone, $\frac{1}{\lambda_1} \sum_{j=2}^{n} \lambda_j d^j \in D$. Therefore, $d^1, -d^1 \in D$, contradicting the pointedness of $D$ [25, Section 2.4.1]. Therefore, $\sum_{j=1}^{n} \lambda_j d^j \neq 0$.

(b) By (a), $\|\sum_{j=1}^{n} \lambda_j d^j\|_* > 0$ for every $\lambda \in \Delta^{n-1}$. Since the feasible set $\Delta^{n-1}$ is compact, the minimum of the continuous function $\lambda \mapsto \|\sum_{j=1}^{n} \lambda_j d^j\|_*$ is attained at some $\bar{\lambda} \in \Delta^{n-1}$. Hence, the minimum $\|\sum_{j=1}^{n} \bar{\lambda}_j d^j\|_*$ is also strictly positive.

(c) Since $\Delta^{n-1} \subseteq \Delta_+^{n-1}$, we have

$$\min_{\lambda \in \Delta^{n-1}} \left\| \sum_{j=1}^{n} \lambda_j d^j \right\|_* \geq \inf_{\lambda \in \Delta_+^{n-1}} \left\| \sum_{j=1}^{n} \lambda_j d^j \right\|_* .$$

To prove the reverse inequality, assume to the contrary that there exists $\bar{\lambda} \in \Delta_+^{n-1}$ such that

$$\left\| \sum_{j=1}^{n} \bar{\lambda}_j d^j \right\|_* < \min_{\lambda \in \Delta^{n-1}} \left\| \sum_{j=1}^{n} \lambda_j d^j \right\|_* . \tag{5.2.2}$$

Then, using $\bar{\lambda} \in \Delta_+^{n-1}$, we have

$$\left\| \sum_{j=1}^{n} \bar{\lambda}_j d^j \right\|_* \geq \frac{1}{\sum_{i=1}^{n} \bar{\lambda}_i} \left\| \sum_{j=1}^{n} \bar{\lambda}_j d^j \right\|_* = \left\| \sum_{j=1}^{n} \frac{\bar{\lambda}_j}{\sum_{i=1}^{n} \bar{\lambda}_i} d^j \right\|_* \geq \min_{\lambda \in \Delta^{n-1}} \left\| \sum_{j=1}^{n} \lambda_j d^j \right\|_* ,$$

where the last inequality holds since $\sum_{j=1}^{n} \frac{\bar{\lambda}_j}{\sum_{i=1}^{n} \bar{\lambda}_i} = 1$. Hence, we get a contradiction to (5.2.2). Therefore,

$$\min_{\lambda \in \Delta^{n-1}} \left\| \sum_{j=1}^{n} \lambda_j d^j \right\|_* = \inf_{\lambda \in \Delta_+^{n-1}} \left\| \sum_{j=1}^{n} \lambda_j d^j \right\|_* .$$

Since the minimum is attained on the left and $\Delta^{n-1} \subseteq \Delta_+^{n-1}$, the infimum on the right is also a minimum.

□

In the following proposition, it is shown that an inner $\tilde{\epsilon}$-approximation of the upper image $\mathcal{P}$ can be obtained by using a finite $\epsilon$-solution $\bar{\mathcal{W}}$ of (D).

**Proposition 5.2.4.** *For $\epsilon > 0$, let $\bar{\mathcal{W}}$ be a finite $\epsilon$-solution of* (D)*, and define*

$$\mathcal{D}_\epsilon := \operatorname{cone}(\operatorname{conv} \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K.$$

*Then, the set*

$$\mathcal{P}_\epsilon := \mathcal{P}_{\mathcal{D}_\epsilon} = \{y \in \mathbb{R}^q \mid \forall (w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_\epsilon \colon \varphi(y, w, \alpha) \geq 0\}$$

*is an inner approximation of $\mathcal{P}$ and it satisfies $\mathcal{P}_\epsilon + B(0, \tilde{\epsilon}) \supseteq \mathcal{P} \supseteq \mathcal{P}_\epsilon$, where*

$$\tilde{\epsilon} = \frac{\epsilon}{\min\limits_{\lambda \in \Delta^{J-1}} \left\| \sum_{j=1}^J \lambda_j w^j \right\|_*}.$$

*Proof.* Since $\bar{\mathcal{W}}$ is a finite $\epsilon$-solution of (D), $\mathcal{D}_\epsilon$ is an outer approximation of the lower image $\mathcal{D}$ by Definition 3.0.8, that is, $\mathcal{D}_\epsilon \supseteq \mathcal{D}$. Therefore, by Proposition 4.1.6(b), the inclusion $\mathcal{P} \supseteq \mathcal{P}_\epsilon$ holds.

Next, we show that $\mathcal{P}_\epsilon + B(0, \tilde{\epsilon}) \supseteq \mathcal{P}$. To get a contradiction, assume that there exists $\bar{y} \in \mathcal{P} \setminus (\mathcal{P}_\epsilon + B(0, \tilde{\epsilon}))$. Hence, there exists $\bar{w} \in \mathbb{R}^q \setminus \{0\}$ such that

$$\bar{w}^\mathsf{T} \bar{y} < \inf_{y \in \mathcal{P}_\epsilon} \bar{w}^\mathsf{T} y + \inf_{\gamma \in B(0,1)} \tilde{\epsilon} \bar{w}^\mathsf{T} \gamma. \tag{5.2.3}$$

Without loss of generality, we may assume that $\|\bar{w}\|_* = 1$ so that

$$\inf_{\gamma \in B(0,1)} \tilde{\epsilon} \bar{w}^\mathsf{T} \gamma = -\tilde{\epsilon} \|\bar{w}\|_* = -\tilde{\epsilon}.$$

Hence, (5.2.3) becomes

$$\bar{w}^\mathsf{T} \bar{y} + \tilde{\epsilon} < \inf_{y \in \mathcal{P}_\epsilon} \bar{w}^\mathsf{T} y =: \bar{\alpha}. \tag{5.2.4}$$

The definition of $\bar{\alpha}$ ensures that $\varphi(y, \bar{w}, \bar{\alpha}) = \bar{w}^\mathsf{T} y - \bar{\alpha} \geq 0$ for each $y \in \mathcal{P}_\epsilon$, that is, $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \mathcal{D}_{\mathcal{P}_\epsilon}$. By Proposition 4.2.5, we have $\mathcal{D}_{\mathcal{P}_\epsilon} = \mathcal{D}_{\mathcal{P}_{\mathcal{D}_\epsilon}} = \mathcal{D}_\epsilon$. Therefore, $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \mathcal{D}_\epsilon$. Hence, there exist $\delta \geq 0$, $k \geq 0$, $n \in \mathbb{N}$ and $\mu \in \Delta^{n-1}$, $((\bar{w}^i)^\mathsf{T}, \alpha_i)^\mathsf{T} \in \xi(\bar{\mathcal{W}})$ for each $i \in \{1, \dots, n\}$ such that

$$(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} = \delta \left( \sum_{i=1}^n \mu_i ((\bar{w}^i)^\mathsf{T}, \alpha_i)^\mathsf{T} + \epsilon e^{q+1} \right) - k e^{q+1}. \tag{5.2.5}$$

Using (5.2.5), we have $\bar{w} = \delta \sum_{i=1}^n \mu_i \bar{w}^i$ and $\bar{\alpha} = \delta(\sum_{i=1}^n \mu_i \alpha_i + \epsilon) - k$. In particular, having $\|\bar{w}\|_* = 1$ implies that

$$\delta = \frac{1}{\left\|\sum_{i=1}^n \mu_i \bar{w}^i\right\|_*}.$$

Next, we claim that $(\bar{w}^\mathsf{T}, \bar{\alpha} - \tilde{\epsilon}) \in \mathcal{D}$. Since $\mathcal{W} = C^+$ is a convex cone, we have $\bar{w} \in \mathcal{W}$. Therefore, using the definition of $\mathcal{D}$, proving the inequality

$$\bar{\alpha} - \tilde{\epsilon} \leq \inf_{x \in \mathcal{X}} \bar{w}^\mathsf{T} f(x) \tag{5.2.6}$$

is enough to conclude that $(\bar{w}^\mathsf{T}, \bar{\alpha} - \tilde{\epsilon}) \in \mathcal{D}$. Since $((\bar{w}^i)^\mathsf{T}, \alpha_i) \in \xi(\bar{\mathcal{W}})$, we have $\alpha_i = \inf_{x \in \mathcal{X}} (\bar{w}^i)^\mathsf{T} f(x)$ for each $i \in \{1, \dots, n\}$. Hence, using (5.2.5) and $k \geq 0$, we may write

$$\begin{aligned}
\bar{\alpha} &= \delta \left( \sum_{i=1}^n \mu_i \alpha_i + \epsilon \right) - k \\
&\leq \delta \left( \sum_{i=1}^n \mu_i \inf_{x \in \mathcal{X}} (\bar{w}^i)^\mathsf{T} f(x) + \epsilon \right) \\
&\leq \delta \left( \inf_{x \in \mathcal{X}} \left( \sum_{i=1}^n \mu_i \bar{w}^i \right)^\mathsf{T} f(x) \right) + \delta\epsilon = \inf_{x \in \mathcal{X}} \bar{w}^\mathsf{T} f(x) + \delta\epsilon.
\end{aligned}$$

Hence, we have $\bar{\alpha} - \delta\epsilon \leq \inf_{x \in \mathcal{X}} \bar{w}^\mathsf{T} f(x)$.

Let us show that $\delta\epsilon \leq \tilde{\epsilon}$ so that (5.2.6) follows. For each $i \in \{1, \dots, n\}$, since $\bar{w}^i \in \bar{\mathcal{W}} \subseteq \mathcal{W} \cap (\mathbb{S}^{q-1} \times \mathbb{R})$, there exists $(\gamma_{i1}, \dots, \gamma_{iJ})^\mathsf{T} \in \Delta^{J-1}$ such that

$$\bar{w}^i = \sum_{j=1}^J \frac{\gamma_{ij}}{\left\|\sum_{j'=1}^J \gamma_{ij'} w^{j'}\right\|_*} w^j,$$

where $w^1, \dots, w^J$ are the generating vectors of $C^+$. Then,

$$\sum_{i=1}^n \mu_i \bar{w}^i = \sum_{i=1}^n \mu_i \sum_{j=1}^J \frac{\gamma_{ij}}{\left\|\sum_{j'=1}^J \gamma_{ij'} w^{j'}\right\|_*} w^j = \sum_{j=1}^J \lambda_j w^j,$$

where, for each $j \in \{1, \dots, J\}$,

$$\lambda_j := \sum_{i=1}^n \mu_i \frac{\gamma_{ij}}{\left\|\sum_{j'=1}^J \gamma_{ij'} w^{j'}\right\|_*} \geq 0.$$

Note that $\sum_{j=1}^J \lambda_j \geq 1$ since we have $\| \sum_{j=1}^J \gamma_{ij} w^j \|_* \leq \sum_{j=1}^J \|\gamma_{ij} w^j\|_* = \sum_{j=1}^J \gamma_{ij} \|w^j\|_* = 1$, $\sum_{j=1}^J \gamma_{ij} = 1$ for each $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n \mu_i = 1$. Therefore, $\lambda = (\lambda_1, \dots, \lambda_J)^\mathsf{T} \in \Delta_+^{J-1}$. Since the dual cone $C^+$ is convex and pointed, we may use Lemma 5.2.3 to write

$$\begin{aligned}
\delta\epsilon &= \frac{\epsilon}{\|\sum_{i=1}^n \mu_i \bar{w}^i\|_*} = \frac{\epsilon}{\left\|\sum_{j=1}^J \lambda_j w^j\right\|_*} \\
&\leq \frac{\epsilon}{\min\limits_{\lambda' \in \Delta_+^{J-1}} \left\|\sum_{j=1}^J \lambda'_j w^j\right\|_*} = \frac{\epsilon}{\min\limits_{\lambda' \in \Delta^{J-1}} \left\|\sum_{j=1}^J \lambda'_j w^j\right\|_*} \\
&= \tilde{\epsilon}.
\end{aligned}$$

Therefore, (5.2.6) follows and we have $(\bar{w}^\mathsf{T}, \bar{\alpha} - \tilde{\epsilon})^\mathsf{T} \in \mathcal{D}$. However, by (5.2.4), we have $\varphi(\bar{y}, \bar{w}, \bar{\alpha} - \tilde{\epsilon}) = \bar{w}^\mathsf{T}\bar{y} - \bar{\alpha} + \tilde{\epsilon} < 0$ for $\bar{y} \in \mathcal{P}$ and $(\bar{w}^\mathsf{T}, \bar{\alpha} - \tilde{\epsilon})^\mathsf{T} \in \mathcal{D}$, a contradiction to Proposition 4.1.6. Hence, $\mathcal{P}_\epsilon + B(0, \tilde{\epsilon}) \supseteq \mathcal{P}$. □

**Proposition 5.2.5.** *For $\epsilon > 0$, let $\bar{\mathcal{W}}$ be a finite $\epsilon$-solution of* (D). *Define*

$$\mathcal{D}_\epsilon \coloneqq \operatorname{cone}(\operatorname{conv} \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K.$$

*Let $\mathcal{F} = \{F_1, \dots, F_T\}$ be the set of facets of $\mathcal{D}_\epsilon$. For each $i \in \{1, \dots, T\}$, let*

$$\left\{((w^{i1})^\mathsf{T}, \alpha_{i1})^\mathsf{T}, \dots, ((w^{iJ_i})^\mathsf{T}, \alpha_{iJ_i})^\mathsf{T}\right\}$$

*be the set of extreme directions of $F_i$ and define*

$$f_{\min}^i \coloneqq \min_{\lambda \in \Delta^{J_i - 1}} \left\| \sum_{j=1}^{J_i} \lambda_j w^{ij} \right\|_* .$$

*Then,*

$$\mathcal{P}_\epsilon \coloneqq \mathcal{P}_{\mathcal{D}_\epsilon} = \{y \in \mathbb{R}^q \mid \forall (w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}_\epsilon \colon \varphi(y, w, \alpha) \geq 0\}$$

*is an inner approximation of $\mathcal{P}$ and it satisfies $\mathcal{P}_\epsilon + B(0, \tilde{\epsilon}) \supseteq \mathcal{P} \supseteq \mathcal{P}_\epsilon$, where*

$$\tilde{\epsilon} = \frac{\epsilon}{\min\{f_{\min}^1, \dots, f_{\min}^T\}}.$$

*Proof.* To start with the proof, we have the following observation:

$$\mathcal{D}_\epsilon = \operatorname{cone}\operatorname{conv}\left((\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) \cup \{-e^{q+1}\}\right).$$

This implies that the set of extreme directions of $\mathcal{D}_\epsilon$ is a subset of $(\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) \cup \{-e^{q+1}\}$. Let $i \in \{1, \dots, T\}$. By [23, Section 18, page 162], we also have

$$\{((w^{i1})^\mathsf{T}, \alpha_{i1})^\mathsf{T}, \dots, ((w^{iJ_i})^\mathsf{T}, \alpha_{iJ_i})^\mathsf{T}\} \subseteq (\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) \cup \{-e^{q+1}\}.$$

In particular, for $j \in \{1, \dots, J_i\}$, we have $w^{ij} \in \bar{\mathcal{W}} \cup \{0\}$ and $\alpha_{ij} = \inf_{x\in\mathcal{X}}(w^{ij})^\mathsf{T} f(x) + \epsilon = p^{w^{ij}} + \epsilon$ if $w^{ij} \neq 0$.

By Proposition 5.2.4, we have $\mathcal{P} \supseteq \mathcal{P}_\epsilon$. Following similar steps as in the proof of Proposition 5.2.4, in order to show that $\mathcal{P}_\epsilon + B(0, \tilde{\epsilon}) \supseteq \mathcal{P}$, we assume the contrary. Then, we obtain $\bar{y} \in \mathcal{P}$, $\bar{w} \in \mathbb{R}^q$ with $\|\bar{w}\|_* = 1$ such that

$$\bar{w}^\mathsf{T}\bar{y} + \tilde{\epsilon} < \inf_{y\in\mathcal{P}_\epsilon} \bar{w}^\mathsf{T} y =: \bar{\alpha},$$

and we can check that $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} \in \mathcal{D}_\epsilon$.

By the construction of $\mathcal{D}_\epsilon$, there exists $k \geq 0$ such that $(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} + ke^{q+1} \in \operatorname{bd} \mathcal{D}_\epsilon$ is a $K$-maximal element of $\mathcal{D}_\epsilon$. Hence, there exists $I \in \{1, \dots, T\}$ such that $F_I$ is a $K$-maximal facet of $\mathcal{D}_\epsilon$ and

$$(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} + ke^{q+1} \in F_I = \operatorname{cone}\operatorname{conv}\{((w^{I1})^\mathsf{T}, \alpha_{I1})^\mathsf{T}, \dots, ((w^{IJ_I})^\mathsf{T}, \alpha_{IJ_I})^\mathsf{T}\}. \quad (5.2.7)$$

Using the $K$-maximality of $F_I$, we may assume that $w^{Ij} \neq 0$, hence $\alpha_{Ij} = p^{w^{Ij}} + \epsilon$ for each $j \in \{1, \dots, J_I\}$. Then, we can rewrite (5.2.7) as

$$(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} + ke^{q+1} \in F_I = \operatorname{cone}\big(\operatorname{conv}\{((w^{I1})^\mathsf{T}, p^{w^{I1}})^\mathsf{T}, \dots, ((w^{IJ_I})^\mathsf{T}, p^{w^{IJ_I}})^\mathsf{T}\} + \epsilon\{e^{q+1}\}\big).$$

Hence, there exist $\delta \geq 0$ and $\mu \in \Delta^{J_I - 1}$ such that

$$(\bar{w}^\mathsf{T}, \bar{\alpha})^\mathsf{T} = \delta\left(\sum_{j=1}^{J_I} \mu_j((w^{Ij})^\mathsf{T}, p^{w^{Ij}})^\mathsf{T} + \epsilon e^{q+1}\right) - ke^{q+1}. \quad (5.2.8)$$

As exactly in the proof of Proposition 5.2.4, the aim is to show that $(\bar{w}^\mathsf{T}, \bar{\alpha} - \tilde{\epsilon})^\mathsf{T} \in \mathcal{D}$ in order to get a contradiction. For this purpose, it is sufficient to show that

$$\bar{\alpha} - \tilde{\epsilon} \leq \inf_{x\in\mathcal{X}} \bar{w}^\mathsf{T} f(x). \quad (5.2.9)$$

Following the same steps as in the proof of Proposition 5.2.4, one can use (5.2.8) in order to obtain $\bar{\alpha} - \delta\epsilon \leq \inf_{x\in\mathcal{X}} \bar{w}^\mathsf{T} f(x)$ as well as

$$\delta\epsilon \leq \frac{\epsilon}{\min\limits_{\lambda\in\Delta^{J_I-1}} \left\| \sum_{j=1}^{J_I} \lambda_j w_{Ij} \right\|_*} = \frac{\epsilon}{f^I_{\min}} \leq \frac{\epsilon}{\min\{f^1_{\min}, \dots, f^T_{\min}\}} = \tilde{\epsilon}.$$

Then, (5.2.9) follows from $\bar{\alpha} - \delta\epsilon \leq \inf_{x\in\mathcal{X}} \bar{w}^\mathsf{T} f(x)$ and $\delta\epsilon \leq \tilde{\epsilon}$. □

**Proposition 5.2.6.** *If the algorithm terminates, then it returns a finite weak $\tilde{\epsilon}$-solution $\bar{\mathcal{X}}$ to* (P)*, where $\tilde{\epsilon}$ is either as in Proposition 5.2.4 or as in Proposition 5.2.5.*

*Proof.* Note that every element of $\bar{\mathcal{X}}$ is of the form $x^w$ which is an optimal solution to (WS($w$)) for some $w \in C^+ \setminus \{0\}$; by Proposition 3.0.3, $x^w$ is a weak minimizer of (P). To prove the statement, we need to show that $\operatorname{conv} f(\bar{\mathcal{X}}) + C + B(0, \tilde{\epsilon}) \supseteq \mathcal{P}$ holds. Let us define

$$\bar{\mathcal{P}} := \operatorname{conv} f(\bar{\mathcal{X}}) + C, \qquad \bar{\mathcal{D}} := \mathcal{D}_{\bar{\mathcal{P}}} = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in \bar{\mathcal{P}} \colon \varphi(y, w, \alpha) \geq 0\}.$$

Note that $\bar{\mathcal{D}} \subseteq \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in f(\bar{\mathcal{X}}) \colon \varphi(y, w, \alpha) \geq 0\}$ holds since $\bar{\mathcal{P}} \supseteq f(\bar{\mathcal{X}})$.

Suppose that the algorithm terminates at the $\bar{k}^{\text{th}}$ iteration. In particular, $\mathcal{D}_{\bar{k}}$ is the outer approximation of $\mathcal{D}$ returned by the algorithm. Consider an artificial set $\bar{\bar{\mathcal{X}}}$ that is initialized in line 3 of Algorithm 2 as the empty set, and updated in line 13 as $\bar{\bar{\mathcal{X}}} \leftarrow \bar{\bar{\mathcal{X}}} \cup \{x^w\}$. By the definition of $\bar{\bar{\mathcal{X}}}$, we have

$$\mathcal{D}_{\bar{k}} = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in f(\bar{\bar{\mathcal{X}}}) \colon \varphi(y, w, \alpha) \geq 0\}$$

and $\bar{\bar{\mathcal{X}}} \subseteq \bar{\mathcal{X}}$. Therefore, the inclusion $\mathcal{D}_{\bar{k}} \supseteq \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in f(\bar{\mathcal{X}}) \colon \varphi(y, w, \alpha) \geq 0\}$ holds. We will show that $\mathcal{D}_{\bar{k}} \subseteq \mathcal{D}_\epsilon$, where $\mathcal{D}_\epsilon$ is defined by $\mathcal{D}_\epsilon := \operatorname{cone}(\operatorname{conv} \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K$. Note that for every $(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathcal{D}^{\text{dir}}_{\bar{k}} \subseteq \mathbb{S}^{q-1} \times \mathbb{R}$, the scalarization problem (WS($w$)) is solved; $w$ is added to $\bar{\mathcal{W}}$ and we have $\alpha - p^w \leq \epsilon$. Then, we have $\mathcal{D}^{\text{dir}}_{\bar{k}} \subseteq \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\} - K$. On the other hand, $\mathcal{D}_{\bar{k}} = \operatorname{cone}\operatorname{conv} \mathcal{D}^{\text{dir}}_{\bar{k}} - K$, see (5.2.1). These imply

$$\mathcal{D}_{\bar{k}} \subseteq \operatorname{cone}\operatorname{conv}(\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K = \mathcal{D}_\epsilon$$

Therefore, we have

$$\bar{\mathcal{D}} \subseteq \{(w^{\mathsf{T}}, \alpha)^{\mathsf{T}} \in \mathbb{R}^{q+1} \mid \forall y \in f(\bar{\mathcal{X}})\colon \varphi(y, w, \alpha) \geq 0\} \subseteq \mathcal{D}_{\bar{k}} \subseteq \mathcal{D}_{\epsilon}.$$

By Proposition 4.2.1 and since $\bar{\mathcal{D}} \subseteq \mathcal{D}_{\epsilon}$, we have $\bar{\mathcal{P}} = \mathcal{P}_{\bar{\mathcal{D}}} \supseteq \mathcal{P}_{\mathcal{D}_{\epsilon}}$. Using Proposition 5.2.2 and Proposition 5.2.4 (or Proposition 5.2.5), we have $\mathcal{P}_{\mathcal{D}_{\epsilon}} + B(0, \tilde{\epsilon}) \supseteq \mathcal{P}$. Since we have $\bar{\mathcal{P}} \supseteq \mathcal{P}_{\mathcal{D}_{\epsilon}}$, we conclude $\bar{\mathcal{P}} + B(0, \tilde{\epsilon}) = \operatorname{conv} f(\bar{\mathcal{X}}) + C + B(0, \tilde{\epsilon}) \supseteq \mathcal{P}$, hence $\bar{\mathcal{X}}$ is a finite $\tilde{\epsilon}$-solution to (P). $\square$

# Chapter 6

# Numerical examples

We implemented the primal and dual algorithms given in Chapter 5 using MATLAB software. For the problems (P($v$)) and (WS($w$)) we employ CVX, a package for specifying and solving convex programs [26, 27]. For solving vertex enumeration problems within the algorithms, we use *bensolve tools* [28, 29, 30]. We run all the examples on a computer with i5-8265U CPU and 8 GB RAM.

## 6.1 An illustrative example

To illustrate the approximations obtained by the algorithms, we consider the following example.

**Example 6.1.1.** *Consider the problem*

$$\begin{aligned} &\text{minimize } f(x) = x \text{ with respect to } \leq_{\mathbb{R}^q_+} \\ &\text{subject to } \|x - e\|_2 \leq 1, \quad x \in \mathbb{R}^q_+, \end{aligned}$$

*where $e \in \mathbb{R}^q$ is the vector of ones.*

The motivation of this thesis was to eliminate direction-biasedness of the algorithm in [2]. Indeed, the algorithm's performance changes based on the chosen direction parameter. Table 6.1 shows the results obtained by solving Example 6.1.1 for $q = 2$, $q = 3$ and $q = 4$ with the dual algorithm in [2]. From these results, one can see the impact of the direction parameter $c$ on the number of optimization problems solved (Opt), number of vertex enumeration problems solved (En) and the realized error.

| $q$ | $\epsilon$ | $c$ | Opt | En | Error |
|---|---|---|---|---|---|
| 2 | 0.01 | $(0.05, 1)^\mathsf{T}$ | 25 | 8 | 0.0077 |
| | | $(0.5, 1)^\mathsf{T}$ | 21 | 7 | 0.0028 |
| | | $(1, 1)^\mathsf{T}$ | 19 | 6 | 0.0062 |
| | 0.05 | $(0.05, 1)^\mathsf{T}$ | 13 | 7 | 0.0233 |
| | | $(0.5, 1)^\mathsf{T}$ | 9 | 5 | 0.0408 |
| | | $(1, 1)^\mathsf{T}$ | 11 | 5 | 0.0231 |
| 3 | 0.1 | $(0.05, 0.05, 1)^\mathsf{T}$ | 77 | 6 | 0.0904 |
| | | $(0.5, 0.5, 1)^\mathsf{T}$ | 53 | 6 | 0.0730 |
| | | $(1, 1, 1)^\mathsf{T}$ | 55 | 6 | 0.0624 |
| | 0.5 | $(0.05, 0.05, 1)^\mathsf{T}$ | 21 | 5 | 0.4050 |
| | | $(0.5, 0.5, 1)^\mathsf{T}$ | 13 | 4 | 0.3382 |
| | | $(1, 1, 1)^\mathsf{T}$ | 13 | 4 | 0.2393 |
| 4 | 0.3 | $(0.05, 0.05, 0.05, 1)^\mathsf{T}$ | 566 | 7 | 0.1959 |
| | | $(0.5, 0.5, 0.5, 1)^\mathsf{T}$ | 106 | 5 | 0.1815 |
| | | $(1, 1, 1, 1)^\mathsf{T}$ | 87 | 5 | 0.1739 |
| | 0.8 | $(0.05, 0.05, 0.05, 1)^\mathsf{T}$ | 108 | 6 | 0.7247 |
| | | $(0.5, 0.5, 0.5, 1)^\mathsf{T}$ | 24 | 4 | 0.4144 |
| | | $(1, 1, 1, 1)^\mathsf{T}$ | 21 | 4 | 0.4142 |

Table 6.1: Results for Example 6.1.1 with $q = 2$, $q = 3$ and $q = 4$ solved by the dual algorithm in [2].

Example 6.1.1 is solved with Algorithm 2 for $q = 2$ and $q = 3$ with $\epsilon = 0.05$ and $\epsilon = 0.005$. The outer approximations of the upper and a lower images obtained from these runs for $q = 2$ can be found in Figures 6.1 and 6.2. (More precisely, for visualization purposes, we indeed provide the set $\mathrm{conv}(\mathcal{D}^{\mathrm{dir}}_{\bar{k}} \cup \{0\}) - K$ as a portion of the approximation of the lower image.) For $q = 3$, since the lower image $\mathcal{D}$ is in $\mathbb{R}^4$, we only provide the outer approximations of the upper image, see Figure 6.3.

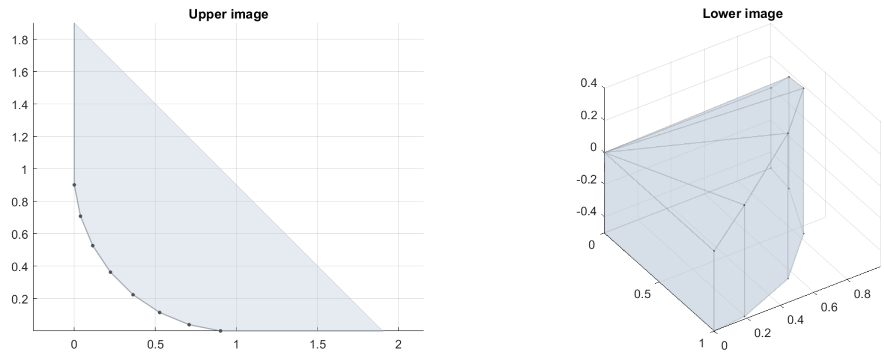


Figure 6.1: Approximations of upper (left) and lower (right) images for Example 6.1.1 for $q = 2$ where $\epsilon = 0.05$.

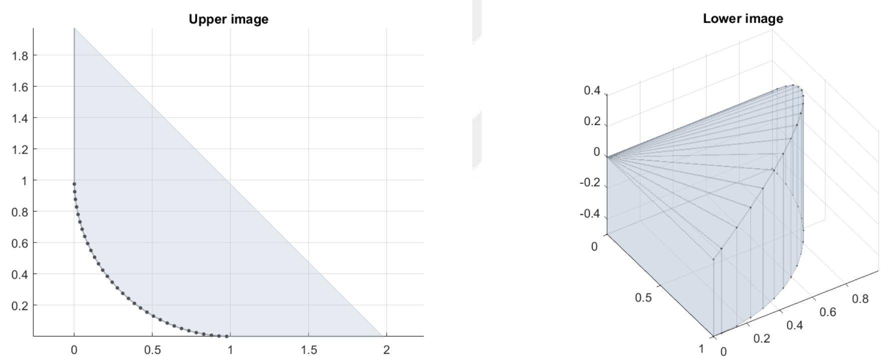


Figure 6.2: Approximations of upper (left) and lower (right) images for Example 6.1.1 for $q = 2$ where $\epsilon = 0.005$.

## 6.2 Proximity measures

We measure the performance of the primal and dual algorithms using two indicators as we explain next. The distance between $\mathcal{P}$ and the furthest point from the outer approximation $\mathcal{P}_o$ returned by the algorithm to $\mathcal{P}$ is referred to as the *primal error indicator*. If the recession cone of $\mathcal{P}_o$ is $C$, then the primal error indicator is nothing but the Hausdorff distance between $\mathcal{P}_o$ and $\mathcal{P}$, see [1, Lemma 5.3]. The primal error indicator of an algorithm is calculated by solving (P($v$)) for the set $\mathcal{V}_o$ of the vertices of $\mathcal{P}_o$. For each $v \in \mathcal{V}_o$, we find an optimal solution $z^v$ and the corresponding optimal value $\|z^v\|$ by solving (P($v$)). Then, the primal

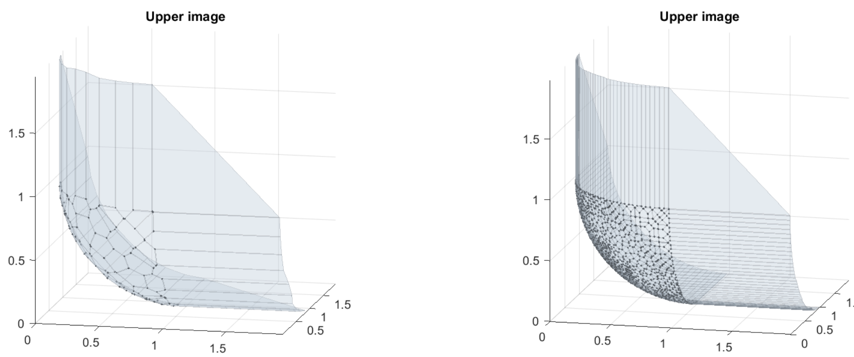


Figure 6.3: Approximations of upper images for Example 6.1.1 for $q = 3$ with $\epsilon = 0.05$ (left) and $\epsilon = 0.005$ (right).

error indicator is calculated by

$$\mathrm{PE} := \max\{\|z^v\| \mid v \in \mathcal{V}_o\}.$$

Note that the norm used in the definition of PE is the same (primal) norm that is used in the algorithm. Hence, this indicator depends on the choice of the norm. Motivated by this, we consider another indicator which is free of norm-biasedness. We define a *hypervolume indicator for CVOPs* which is in the same spirit as the hypervolume indicator defined for multiobjective optimization problems, see for instance [31, 32, 33]. Recall that for a multiobjective optimization problem, the hypervolume of a set $S \subseteq \mathbb{R}^q$ of points with respect to a reference point $r \in \mathbb{R}^q$ is computed as

$$\Lambda\Big(\bigcup_{s \in S, s \leq r} \{y \in \mathbb{R}^q \mid s \leq_{\mathbb{R}^q_+} y \leq_{\mathbb{R}^q_+} r\}\Big),$$

where $\Lambda$ is the Lebesgue measure. Different from this hypervolume measure, which can be used for convex as well as nonconvex multiobjective optimization problems, we use the fact that (P) is a convex problem and the underlying order relation is induced by cone $C$.

In this thesis, we define the hypervolume of a set $S \subseteq \mathbb{R}^q$ with respect to a bounding polytope $\mathcal{Q} \in \mathbb{R}^q$ as

$$\mathrm{HV}(S, \mathcal{Q}) := \Lambda\big((\operatorname{conv} S + C) \cap \mathcal{Q}\big).$$

Let $\mathcal{P}_o, \mathcal{P}_i$ be, respectively, the outer and inner approximations of $\mathcal{P}$ returned by an algorithm and $\mathcal{V}_o, \mathcal{V}_i$ be the set of vertices of them. Then, we compute the *hypervolume indicator* by

$$\text{HV} := \left( \frac{\text{HV}(\mathcal{V}_o, \mathcal{Q}) - \text{HV}(\mathcal{V}_i, \mathcal{Q})}{\text{HV}(\mathcal{V}_o, \mathcal{Q})} \right) \times 100,$$

where $\mathcal{Q} \in \mathbb{R}^q$ is a polytope satisfying $\mathcal{V}_o \cup \mathcal{V}_i \subseteq \mathcal{Q}$. Suppose that the problem is solved by finitely many algorithms and let $A \subseteq \mathbb{R}^q$ be the set of all vertices of the outer and inner approximations returned by *all* algorithms. For our computational tests, in order to have a fair comparison, we fix the polytope $\mathcal{Q}$ such that $A \subseteq \mathcal{Q}$. For this, we find the polytope $\mathcal{Q}$ as follows:

$$\mathcal{Q} = \bigcap_{j \in \{1,\dots,J\}} \left\{ y \in \mathbb{R}^q \mid (w^j)^\mathsf{T} y \leq \max_{a \in A} \{(w^j)^\mathsf{T} a\} \right\},$$

where $w^1, w^2, \dots, w^J$ are the generating vectors of $C^+$. Note that a smaller hypervolume indicator is more desirable in terms of an algorithm's performance since it means less of a difference between the inner and outer sets.

## 6.3 Computational results

We assess the performance of the primal and dual algorithms by solving randomly generated problem instances. A problem structure that is simple and versatile is required for scaling purposes, in terms of both the decision space and the image space. To this end, we work with a linear objective function and a quadratic constraint as described in Example 6.3.1.

**Example 6.3.1.** *Consider the problem*

$$\begin{aligned} &\text{minimize } f(x) = A^\mathsf{T} x \text{ with respect to } \leq_{\mathbb{R}^q_+} \\ &\text{subject to } x^\mathsf{T} P x - 1 \leq 0, \end{aligned}$$

*where* $P \in \mathbb{R}^{n \times n}$ *is a symmetric positive definite matrix, and* $A \in \mathbb{R}^{n \times q}_+$. *In the computational experiments, we generate* $A$ *as the instance of a random matrix with independent entries having the uniform distribution over* $[0, 50]$. *To construct*

*$P$, we first create a matrix $U \in \mathbb{R}^{n\times n}$ following the same procedure as for $A$. We assume that $U$ has at least one nonzero entry, which occurs with probability one. Then, $S := (U+U^\mathsf{T})/2$ is a symmetric matrix and we calculate its diagonalization as $S = Q\bar{D}Q^\mathsf{T}$, where $\bar{D} \in \mathbb{R}^{n\times n}$ is the diagonal matrix of the eigenvalues of $S$, and $Q$ is the orthogonal matrix of the corresponding eigenvectors. Denoting by $D \in \mathbb{R}^{n\times n}$ the diagonal matrix whose entries are the absolute values of the entries of $\bar{D}$, we let $P := QDQ^\mathsf{T}$, which is guaranteed to be symmetric and positive definite.*

### 6.3.1 Results for multiobjective problem instances

This subsection provides the results obtained by solving randomly generated instances of Example 6.3.1. The computational results are presented in Tables 6.2 and 6.3, which show the stopping criteria (Stop), number of optimization problems solved (Opt), time taken to solve optimization problems ($\mathrm{T_{opt}}$), number of vertex enumeration problems solved (En), time taken to solve vertex enumeration problems ($\mathrm{T_{en}}$), and total runtime of the algorithm (T) in terms of seconds. As a measure of efficiency for the algorithms, we also provide the average runtime spent per optimization problem ($\mathrm{T_{opt}}$/Opt) and total runtime per weak minimizer in $\bar{\mathcal{X}}$ (T/$|\bar{\mathcal{X}}|$).

The performance indicators for the tests are primal error (PE) and hypervolume (HV). For the computations of these, we take the following: For Algorithm 1, $\mathcal{P}_o = \mathcal{P}_{\bar{k}}$, where $\bar{k}$ is the iteration number at which the algorithm terminates; for Algorithm 2, $\mathcal{P}_o = \bigcap_{w\in\bar{\mathcal{W}}} \mathcal{H}(w,p^w)$ where $\bar{\mathcal{W}}$ is the solution to (D) that Algorithm 2 returns. For both algorithms, $\mathcal{P}_i = \operatorname{conv} f(\bar{\mathcal{X}}) + C$, where $\bar{\mathcal{X}}$ is the solution to (P) that the corresponding algorithm returns.

The results for $q = 2$ and various choices of the dimension of the decision space ($n$) can be found in Table 6.2. For each value of $n$, 50 random problem instances are generated using the structure of Example 6.3.1 and solved by both Algorithms 1 and 2 twice with different stopping criteria. The averages of the results and performance indicators of the 50 instances are presented in the table in corresponding cells.

In Table 6.2, the first two rows for each value of $n$ show the results of the primal and dual algorithms when the given $\epsilon_i$ value is fed to the algorithms as stopping criteria. It can be seen that the $\epsilon_i$ values that are used in the algorithms are different. We run Algorithm 1 and obtain a weak $\epsilon_1$-solution to problem (P). When working with Algorithm 2, to obtain also a weak $\epsilon_1$-solution to problem (P), we calculate $\epsilon_2$ such that

$$\epsilon_1 = \frac{\epsilon_2}{\min\limits_{\lambda \in \Delta^{J-1}} \left\| \sum_{j=1}^{J} \lambda_j w^j \right\|_*}$$

based on Proposition 5.2.4. As a result, we take $\epsilon_1 = 0.05$ for Algorithm 1 and $\epsilon_2 = 0.0354$ for Algorithm 2 as stopping criteria. We observe from the first two rows for each $n$ in Table 6.2 that Algorithm 1 stops in shorter runtime; however, Algorithm 2 returns smaller primal error and hypervolume indicators.

For further comparison of the algorithms, we run each algorithm once more with different stopping criteria. In the third row, we aim to observe the runtime it takes for Algorithm 1 to reach similar primal error that Algorithm 2 returns. Therefore, the PE value in the second row is fed to Algorithm 1 as the stopping criteria. Finally, in the fourth row, we aim to observe the primal error and hypervolume indicators of Algorithm 2 when it is terminated after a similar runtime as of Algorithm 1 from the first row. Therefore, T from the first row is fed to Algorithm 2 as stopping criteria. (Note that the actual termination time is slightly higher than the predetermined time limit as we check the time only at the beginning of the loop in the implementation and it takes a couple of more seconds to terminate the algorithm afterwards.)

From Table 6.2, one can observe that Algorithms 1 and 2 in the first two rows are not comparable. Indeed, while Algorithm 1 in the first row has shorter runtime for each $n$ value, it also gives larger hypervolume results, which indicates worse performance in comparison with Algorithm 2 in the second row. The main reason is that, when Algorithm 2 is run with a stopping criterion that guarantees obtaining a weak $\epsilon_1$-solution to (P), the solution that it returns has much higher proximity, for instance, compare $\epsilon_2 = 0.0354$ and PE$= 0.0066$ in the second row

| $n$ | Alg | Stop | Opt | $T_{opt}$ | $T_{opt}$/Opt | En | $T_{en}$ | T | T/$\lvert\bar{\mathcal{X}}\rvert$ | PE | HV |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 1 | $\epsilon_1 = 0.0500$ | 16.20 | 6.15 | 0.3790 | 3.98 | 0.03 | 14.90 | 0.92 | 0.0352 | 0.9282 |
| | 2 | $\epsilon_2 = 0.0354$ | 18.54 | 6.28 | 0.3388 | 4.24 | 0.05 | 15.97 | 0.87 | 0.0066 | 0.3372 |
| | 1 | $\epsilon_3 = 0.0066$ | 39.16 | 14.89 | 0.3802 | 5.34 | 0.04 | 36.06 | 0.92 | 0.0057 | 0.1500 |
| | 2 | T = 14.90 | 19.70 | 6.66 | 0.3380 | 4.74 | 0.07 | 16.88 | 0.86 | 0.0105 | 0.3603 |
| 10 | 1 | $\epsilon_1 = 0.0500$ | 21.56 | 8.36 | 0.3923 | 4.44 | 0.04 | 19.47 | 0.90 | 0.0377 | 0.4481 |
| | 2 | $\epsilon_2 = 0.0354$ | 25.96 | 9.12 | 0.3500 | 4.82 | 0.07 | 23.15 | 0.89 | 0.0068 | 0.1516 |
| | 1 | $\epsilon_3 = 0.0068$ | 54.28 | 20.49 | 0.3764 | 5.80 | 0.05 | 48.21 | 0.89 | 0.0053 | 0.0683 |
| | 2 | T = 19.47 | 24.54 | 8.61 | 0.3499 | 5.12 | 0.08 | 21.57 | 0.88 | 0.0129 | 0.2119 |
| 15 | 1 | $\epsilon_1 = 0.0500$ | 22.84 | 8.57 | 0.3774 | 4.56 | 0.04 | 20.20 | 0.89 | 0.0361 | 0.3560 |
| | 2 | $\epsilon_2 = 0.0354$ | 26.42 | 8.90 | 0.3378 | 4.88 | 0.07 | 22.31 | 0.85 | 0.0070 | 0.1361 |
| | 1 | $\epsilon_3 = 0.0070$ | 54.08 | 20.06 | 0.3714 | 5.86 | 0.05 | 47.62 | 0.88 | 0.0060 | 0.0648 |
| | 2 | T = 20.20 | 26.62 | 8.91 | 0.3350 | 5.24 | 0.08 | 22.16 | 0.84 | 0.0116 | 0.1631 |
| 20 | 1 | $\epsilon_1 = 0.0500$ | 28.84 | 10.84 | 0.3792 | 4.88 | 0.04 | 25.13 | 0.87 | 0.0334 | 0.2208 |
| | 2 | $\epsilon_2 = 0.0354$ | 31.84 | 10.68 | 0.3353 | 5.10 | 0.07 | 26.56 | 0.84 | 0.0063 | 0.0791 |
| | 1 | $\epsilon_3 = 0.0063$ | 64.32 | 23.81 | 0.3702 | 6.08 | 0.05 | 56.33 | 0.88 | 0.0053 | 0.0366 |
| | 2 | T =25.13 | 32.66 | 10.91 | 0.3341 | 5.60 | 0.11 | 27.11 | 0.83 | 0.0106 | 0.1015 |
| 25 | 1 | $\epsilon_1 = 0.0500$ | 28.24 | 10.89 | 0.3869 | 4.94 | 0.04 | 27.52 | 0.98 | 0.0352 | 0.2260 |
| | 2 | $\epsilon_2 =0.0354$ | 32.96 | 11.60 | 0.3520 | 5.22 | 0.08 | 31.54 | 0.96 | 0.0066 | 0.0744 |
| | 1 | $\epsilon_3 =0.0066$ | 69.24 | 26.60 | 0.3841 | 6.20 | 0.06 | 68.01 | 0.98 | 0.0055 | 0.0334 |
| | 2 | T = 27.52 | 31.12 | 10.95 | 0.3520 | 5.56 | 0.11 | 29.78 | 0.96 | 0.0126 | 0.1171 |
| 30 | 1 | $\epsilon_1 = 0.0500$ | 31.48 | 12.55 | 0.3983 | 5.00 | 0.04 | 33.45 | 1.06 | 0.0332 | 0.2012 |
| | 2 | $\epsilon_2 = 0.0354$ | 36.32 | 13.19 | 0.3635 | 5.36 | 0.09 | 37.79 | 1.04 | 0.0064 | 0.0771 |
| | 1 | $\epsilon_3 = 0.0064$ | 76.60 | 30.35 | 0.3963 | 6.26 | 0.06 | 81.42 | 1.06 | 0.0056 | 0.0364 |
| | 2 | T = 33.45 | 34.62 | 12.55 | 0.3626 | 5.74 | 0.12 | 35.95 | 1.04 | 0.0124 | 0.1021 |
| 35 | 1 | $\epsilon_1 = 0.0500$ | 31.60 | 12.70 | 0.4017 | 4.98 | 0.04 | 34.27 | 1.08 | 0.0312 | 0.1609 |
| | 2 | $\epsilon_2 = 0.0354$ | 35.58 | 12.90 | 0.3625 | 5.28 | 0.09 | 36.89 | 1.04 | 0.0069 | 0.0640 |
| | 1 | $\epsilon_3 = 0.0069$ | 71.36 | 28.49 | 0.3991 | 6.22 | 0.06 | 76.88 | 1.08 | 0.0061 | 0.0308 |
| | 2 | T = 34.27 | 35.48 | 12.85 | 0.3620 | 5.84 | 0.13 | 36.74 | 1.04 | 0.0095 | 0.0767 |
| 40 | 1 | $\epsilon_1 = 0.0500$ | 33.02 | 13.16 | 0.3985 | 5.12 | 0.04 | 34.86 | 1.06 | 0.0351 | 0.1431 |
| | 2 | $\epsilon_2 = 0.0354$ | 39.46 | 14.25 | 0.3610 | 5.58 | 0.10 | 40.55 | 1.03 | 0.0072 | 0.0535 |
| | 1 | $\epsilon_3 = 0.0072$ | 81.94 | 32.48 | 0.3963 | 6.44 | 0.06 | 86.70 | 1.06 | 0.0064 | 0.0244 |
| | 2 | T = 34.86 | 36.06 | 13.06 | 0.3622 | 5.82 | 0.13 | 37.20 | 1.03 | 0.0156 | 0.0856 |
| 45 | 1 | $\epsilon_1 = 0.0500$ | 32.20 | 12.74 | 0.3954 | 5.04 | 0.04 | 34.08 | 1.06 | 0.0339 | 0.1570 |
| | 2 | $\epsilon_2 =0.03540$ | 37.20 | 13.39 | 0.3598 | 5.32 | 0.09 | 38.22 | 1.03 | 0.0068 | 0.0597 |
| | 1 | $\epsilon_3 = 0.0068$ | 77.14 | 30.42 | 0.3940 | 6.32 | 0.06 | 81.58 | 1.06 | 0.0064 | 0.0282 |
| | 2 | T = 34.08 | 35.50 | 12.76 | 0.3594 | 5.80 | 0.12 | 36.48 | 1.03 | 0.0102 | 0.0764 |
| 50 | 1 | $\epsilon_1 = 0.0500$ | 35.20 | 14.19 | 0.4028 | 5.16 | 0.05 | 38.02 | 1.08 | 0.0330 | 0.1306 |
| | 2 | $\epsilon_2 = 0.0354$ | 40.88 | 14.67 | 0.3593 | 5.46 | 0.10 | 41.63 | 1.02 | 0.0065 | 0.0479 |
| | 1 | $\epsilon_3 = 0.0065$ | 88.64 | 34.98 | 0.3945 | 6.56 | 0.06 | 94.06 | 1.06 | 0.0057 | 0.0221 |
| | 2 | T = 38.02 | 39.54 | 14.21 | 0.3594 | 5.98 | 0.14 | 40.42 | 1.02 | 0.0100 | 0.0623 |

Table 6.2: Results of randomly generated problems with $q = 2$.

of Table 6.2.[1] On the other hand, when we set $\epsilon_3$ in a way that Algorithms 1 and 2 return similar PE values, Algorithm 1 returns better HV results but it requires higher runtime. In order to have further insights, we analyze the results of Algorithm 1 from the first row and Algorithm 2 from the fourth row in more detail, since they have similar runtimes, see Figure 6.4. Similarly, as they yield similar PE values, we analyze the results of Algorithm 1 from the third row and Algorithm 2 from the second row in Figure 6.5.

In Figure 6.4, we plot the PE and HV values from rows one and four of Table 6.2 for each $n$, together with the underlying CPU times. We observe that, when the two algorithms work for similar runtimes, Algorithm 2 returns smaller primal error. Moreover, the algorithms perform similarly in terms of the hypervolume indicator, especially for higher values of $n$. When we consider the total runtimes, we see approximately 2 seconds of difference between the algorithms. Since the runtime is used as a stopping criterion for Algorithm 2 in the fourth row, it takes approximately 2 more seconds to terminate the algorithm, as explained before.

In Figure 6.5, we plot the runtime and HV values from rows two and three of Table 6.2 as well as the PE values on the right vertical axes. As we have anticipated, when the primal error value of Algorithm 2 in row two is fed to Algorithm 1 in row three as $\epsilon_3$, the primal error of Algorithm 1 is lower than that of Algorithm 2. Besides, the average hypervolume indicator of Algorithm 1 is smaller than that of Algorithm 2. However, Algorithm 1 requires higher runtime for termination.

The results for the case $q = 3$ are presented in Table 6.3. For each value of $n$, we generate 20 random instances of the problem in Example 6.3.1 and solve them by both algorithms twice. The four rows for each $n$ are constructed as in Table 6.2.

In Figure 6.6, the plots of the PE and HV values for rows one and four of

[1]After running Algorithm 2 with the predetermined $\epsilon_2$ value (to get a weak $\epsilon_1$-solution for (P)), in order to compute the realized primal error for (P), we solve (P($v$)) problems as explained in Section 6.2. Instead, in order to compute an upper bound for the PE value (which would be a better bound than $\epsilon_1$), one can also use Proposition 5.2.5.

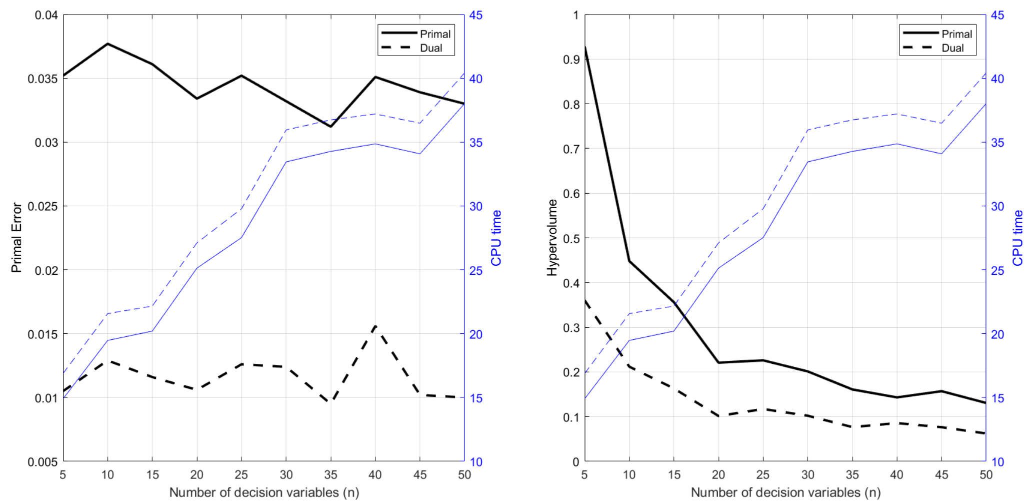


Figure 6.4: Average PE (left) and HV (right) values of 50 random instances of Example 6.3.1 for $q = 2$ under nearly equal runtime (rows one and four of Table 6.2).

Table 6.3 are shown. One can see that Algorithm 2 has consistently better performance for each value of $n$ in terms of both primal error and hypervolume indicator.

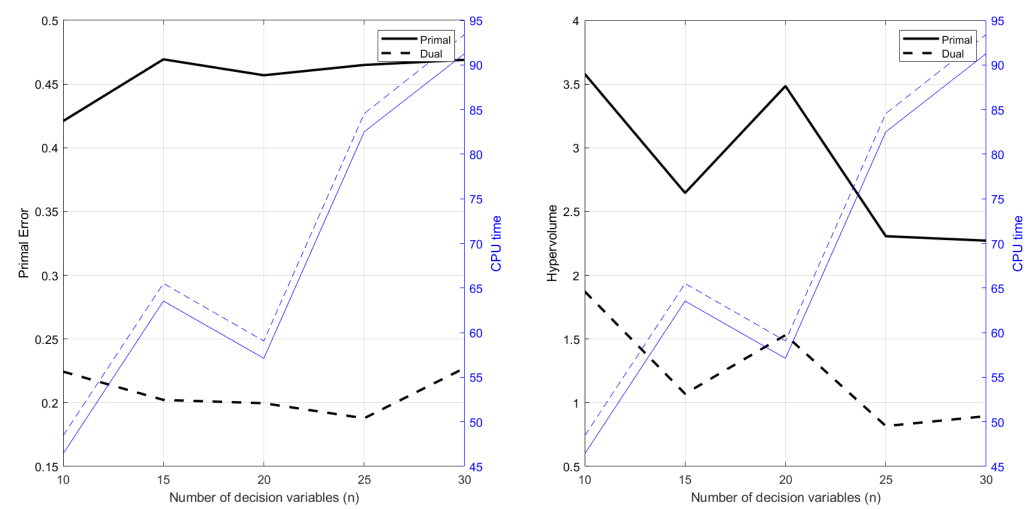


Figure 6.6: Average PE (left) and HV (right) values of 20 random instances of Example 6.3.1 for $q = 3$ under nearly equal runtime (rows one and four of Table 6.3).

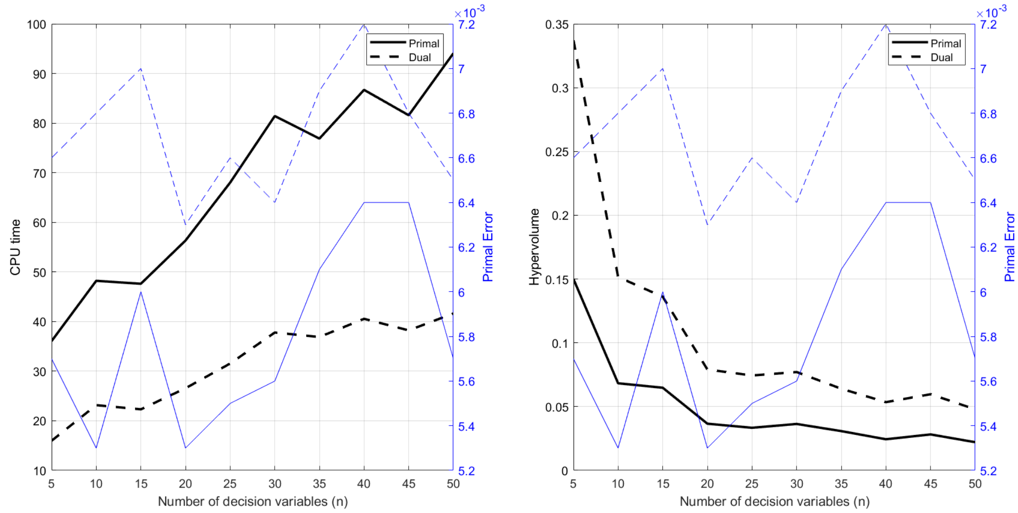


Figure 6.5: Average CPU time (left) and HV (right) values of 50 random instances of Example 6.3.1 for $q = 2$ when the dual algorithm uses the PE of the primal algorithm as approximation error (rows two and three of Table 6.2).

In Figure 6.7, the plots of total runtimes and HV values corresponding to rows two and three of Table 6.3 can be seen together with the PE values on the right vertical axis. We observe that the difference between the primal error indicators of both algorithms are very similar to each other, although, Algorithm 1 has smaller primal error as expected. With very similar primal error indicators, we observe that Algorithm 2 has around half of the runtime of Algorithm 1. In line with the primal error results, Algorithm 1 has a better HV value than Algorithm 2.

| $n$ | Alg | Stop | Opt | $\mathrm{T_{opt}}$ | $\mathrm{T_{opt}}$/Opt | En | $\mathrm{T_{en}}$ | T | T/$\lvert\bar{\mathcal{X}}\rvert$ | PE | HV |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 1 | $\epsilon_1 = 0.5000$ | 52.35 | 19.92 | 0.3801 | 5.05 | 0.09 | 46.41 | 0.91 | 0.4209 | 3.5805 |
| | 2 | $\epsilon_2 = 0.2887$ | 86.25 | 29.03 | 0.3363 | 4.40 | 0.20 | 71.36 | 70.84 | 0.1106 | 1.1973 |
| | 1 | $\epsilon_3 = 0.1106$ | 232.80 | 85.32 | 0.3655 | 6.90 | 0.16 | 200.13 | 0.92 | 0.1052 | 1.2846 |
| | 2 | T = 46.41 | 60.70 | 20.32 | 0.3336 | 4.05 | 0.24 | 48.43 | 0.80 | 0.2244 | 1.8735 |
| 15 | 1 | $\epsilon_1 = 0.5000$ | 70.85 | 26.98 | 0.3777 | 5.60 | 0.09 | 63.54 | 0.93 | 0.4694 | 2.6458 |
| | 2 | $\epsilon_2 = 0.2887$ | 105.15 | 34.87 | 0.3311 | 4.70 | 0.26 | 84.58 | 0.80 | 0.1033 | 0.7146 |
| | 1 | $\epsilon_3 = 0.1033$ | 295.30 | 108.74 | 0.3677 | 7.45 | 0.21 | 251.56 | 0.91 | 0.0991 | 0.5690 |
| | 2 | T = 63.54 | 81.75 | 27.01 | 0.3307 | 4.40 | 0.36 | 65.51 | 0.81 | 0.2022 | 1.0703 |
| 20 | 1 | $\epsilon_1 = 0.5000$ | 65.45 | 24.62 | 0.3797 | 5.50 | 0.09 | 57.12 | 0.92 | 0.4569 | 3.4845 |
| | 2 | $\epsilon_2 = 0.2887$ | 101.95 | 33.66 | 0.3302 | 4.65 | 0.24 | 81.31 | 0.80 | 0.1052 | 1.0783 |
| | 1 | $\epsilon_3 = 0.1052$ | 284.25 | 104.18 | 0.3672 | 7.35 | 0.19 | 238.73 | 0.91 | 0.1020 | 0.7881 |
| | 2 | T = 57.12 | 74.40 | 24.63 | 0.3324 | 4.30 | 0.29 | 59.06 | 0.80 | 0.1996 | 1.5307 |
| 25 | 1 | $\epsilon_1 = 0.5000$ | 85.95 | 32.63 | 0.3786 | 5.95 | 0.12 | 82.51 | 0.98 | 0.4650 | 2.3063 |
| | 2 | $\epsilon_2 = 0.2887$ | 139.25 | 46.13 | 0.3306 | 4.95 | 0.34 | 112.65 | 0.81 | 0.1071 | 0.6154 |
| | 1 | $\epsilon_3 = 0.1071$ | 368.60 | 137.91 | 0.3736 | 7.70 | 0.26 | 340.28 | 1.01 | 0.1034 | 0.5063 |
| | 2 | T = 82.51 | 106.10 | 35.02 | 0.3297 | 4.80 | 0.56 | 84.55 | 0.80 | 0.1878 | 0.8167 |
| 30 | 1 | $\epsilon_1 = 0.5000$ | 95.70 | 36.59 | 0.3818 | 6.00 | 0.12 | 91.27 | 0.99 | 0.4690 | 2.2715 |
| | 2 | $\epsilon_2 = 0.2887$ | 150.70 | 51.13 | 0.3386 | 5.15 | 0.43 | 131.15 | 0.87 | 0.1066 | 0.6110 |
| | 1 | $\epsilon_3 = 0.1066$ | 452.70 | 170.22 | 0.3751 | 8.05 | 0.32 | 419.02 | 1.02 | 0.1046 | 0.5420 |
| | 2 | T = 91.27 | 109.00 | 36.89 | 0.3381 | 4.60 | 0.48 | 93.41 | 0.86 | 0.2272 | 0.8947 |

Table 6.3: Results of randomly generated problems with $q = 3$.

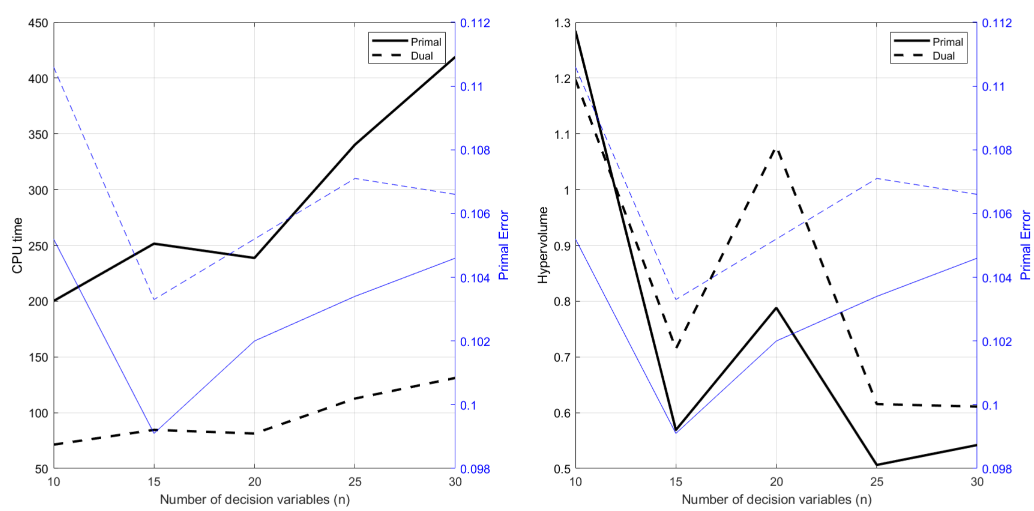


Figure 6.7: Average CPU time (left) and HV (right) values of 20 random instances of Example 6.3.1 for $q = 3$ when the dual algorithm uses the PE of the primal algorithm as approximation error (rows two and three of Table 6.3).

### 6.3.2 Results for different ordering cones and norms

In order to test the performance of the algorithms on problems with different ordering cones and with different norms employed in (P($v$)) and (D($v$)), we design some additional experiments. Although the original stopping criteria in Algorithms 1 and 2 are based on $\epsilon$-closeness, we implement an alternative stopping criterion for the rest of our analysis. In particular, motivated by our analysis in Section 6.3.1, we set a predefined time limit for the algorithms. Note that since $\epsilon$-closeness depends on the choice of the norm used in the scalarization models, using runtime as the stopping criterion results in a fair comparison of the algorithms when considering different norms.

First, we consider Example 6.1.1 from Section 6.1 for $q \in \{2, 3, 4\}$. We denote the nonnegative cone by $C_1 = \mathbb{R}^q_+$ for $q \in \{2, 3, 4\}$. The non-standard ordering cones $C_2, C_3$ that will be used throughout are given below[2].

For $q = 2$,

$$C_2 = \operatorname{conv}\operatorname{cone}\{(1,2)^\mathsf{T}, (2,1)^\mathsf{T}\},$$

$$C_3 = \operatorname{conv}\operatorname{cone}\{(2,-1)^\mathsf{T}, (-1,2)^\mathsf{T}\}.$$

For $q = 3$,

$$C_2 = \operatorname{conv}\operatorname{cone}\{(4,2,2)^\mathsf{T}, (2,4,2)^\mathsf{T}, (4,0,2)^\mathsf{T}, (1,0,2)^\mathsf{T}, (0,1,2)^\mathsf{T}, (0,4,2)^\mathsf{T}\},$$

$$C_3 = \operatorname{conv}\operatorname{cone}\{(-1,-1,3)^\mathsf{T}, (2,2,-1)^\mathsf{T}, (1,0,0)^\mathsf{T}, (0,-1,2)^\mathsf{T}, (-1,0,2)^\mathsf{T}, (0,1,0)^\mathsf{T}\}.$$

For $q = 4$,

$$C_2 = \operatorname{conv}\operatorname{cone}\left\{\begin{array}{l} (3,4,-4,4)^\mathsf{T}, (1,-4,-2,0)^\mathsf{T}, (5,5,-3,5)^\mathsf{T}, (5,0,3,-4)^\mathsf{T}, \\ (-1,4,3,5)^\mathsf{T}, (2,-5,3,4)^\mathsf{T}, (2,3,1,-1)^\mathsf{T}, (2,-3,2,-5)^\mathsf{T} \end{array}\right\},$$

[2]For $q \in \{2, 3\}$, these cones are taken from [1].

$$C_3 = \operatorname{conv\,cone}\left\{\begin{array}{l}(1,-1,0,0)^\mathsf{T},(0,-1,1,-1)^\mathsf{T},(1,0,0,0)^\mathsf{T},(1,0,1,-1)^\mathsf{T}\\(1,-1,1,0)^\mathsf{T},(1,0,1,1)^\mathsf{T},(1,-1,1,1)^\mathsf{T}\end{array}\right\}.$$

Table 6.4 shows the results and performance indicators obtained by solving Example 6.1.1 for $q=2$, $q=3$ and $q=4$. The problems are solved with respect to ordering cones $C_1$, $C_2$ and $C_3$, where the norm in (P($v$)) is taken as the $\ell_2$ norm; moreover, they are solved with respect to the standard ordering cone $C_1$, where the norm in (P($v$)) is set as $\ell_1$ and $\ell_\infty$. For different $q$ values, different runtime limits are implemented. For $q=2$, $q=3$, $q=4$, the problems are terminated after $25, 50, 75$ seconds, respectively.

In Table 6.4 for $q=2$, the primal error values that Algorithms 1 and 2 return are very similar to each other for each case. Although primal error values are similar, we see that Algorithm 2 returns a better HV value. For $q=3$, we see that Algorithm 1 performs better than Algorithm 2 and the difference is more prominent in HV values.

For $q=4$, most of the HV calculations could not be performed. The hypervolume calculator that we have implemented, which employs *bensolve tools*, returned some errors for these instances. Also, Algorithm 2 did not return a solution due to a vertex enumeration error for one of the instances. According to the results that we could obtain, Algorithm 2 has better performance compared to Algorithm 1 in terms of both PE and HV values. Hence, we observe that Algorithm 2 may have an advantage over Algorithm 1 when the dimension of the objective space is increased.

Next, we consider the 20 instances of Example 6.3.1 with $q=3$ that were generated randomly for our analysis in Section 6.3.1. This time, we solve all instances under the runtime limit of 50 seconds. Similar to the previous analysis with Example 6.1.1, we consider ordering cones $C_1, C_2, C_3$ and $\ell_p$ norms for $p \in \{1, 2, \infty\}$. While solving some of the instances with different ordering cones or different norms, we encountered some errors in the solvers that we have employed in the algorithms. Hence, we consider a subset of instances which can be solved in all settings and we indicate the size of this subset in a separate column in

| $q$ | T | Cone | $p$ | Alg | Opt | $\mathrm{T_{opt}}$ | $\mathrm{T_{opt}}$/Opt | En | $\mathrm{T_{en}}$ | T/$\lvert\bar{\mathcal{X}}\rvert$ | PE | HV |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 25 | $C_1$ | 1 | 1 | 26 | 13.25 | 0.5096 | 5 | 0.18 | 1.09 | 0.0015 | 0.1652 |
| | | | | 2 | 26 | 10.88 | 0.4185 | 5 | 0.09 | 1.07 | 0.0016 | 0.1392 |
| | | | 2 | 1 | 25 | 12.60 | 0.5040 | 5 | 0.17 | 1.13 | 0.0012 | 0.1806 |
| | | | | 2 | 24 | 10.23 | 0.4264 | 5 | 0.09 | 1.13 | 0.0012 | 0.1618 |
| | | | $\infty$ | 1 | 24 | 12.90 | 0.5374 | 5 | 0.16 | 1.19 | 0.0011 | 0.1962 |
| | | | | 2 | 26 | 10.10 | 0.3885 | 5 | 0.08 | 1.05 | 0.0011 | 0.1392 |
| | | $C_2$ | 2 | 1 | 26 | 13.17 | 0.5065 | 5 | 0.15 | 1.09 | 0.0031 | 0.1499 |
| | | | | 2 | 25 | 10.49 | 0.4195 | 5 | 0.08 | 1.07 | 0.0031 | 0.1303 |
| | | $C_3$ | 2 | 1 | 26 | 13.22 | 0.5086 | 5 | 0.15 | 1.08 | 0.0002 | 0.1696 |
| | | | | 2 | 26 | 10.88 | 0.4186 | 5 | 0.09 | 1.04 | 0.0002 | 0.1365 |
| 3 | 50 | $C_1$ | 1 | 1 | 50 | 23.47 | 0.4693 | 5 | 0.17 | 1.08 | 0.0339 | 4.6710 |
| | | | | 2 | 52 | 21.50 | 0.4134 | 4 | 0.16 | 1.02 | 0.0354 | 4.6897 |
| | | | 2 | 1 | 54 | 23.61 | 0.4372 | 5 | 0.16 | 0.99 | 0.0267 | 3.7475 |
| | | | | 2 | 51 | 20.10 | 0.3942 | 4 | 0.13 | 1.02 | 0.0259 | 4.4461 |
| | | | $\infty$ | 1 | 53 | 23.78 | 0.4488 | 5 | 0.16 | 1.02 | 0.0183 | 3.8963 |
| | | | | 2 | 53 | 20.89 | 0.3942 | 4 | 0.13 | 1.00 | 0.0166 | 4.0549 |
| | | $C_2$ | 2 | 1 | 56 | 24.66 | 0.4404 | 4 | 0.16 | 1.02 | 0.0378 | 0.3730 |
| | | | | 2 | 50 | 20.55 | 0.4110 | 3 | 0.12 | 1.04 | 0.0856 | 0.5077 |
| | | $C_3$ | 2 | 1 | 55 | 24.54 | 0.4463 | 4 | 0.16 | 1.05 | 0.0225 | 2.2475 |
| | | | | 2 | 51 | 20.29 | 0.3978 | 3 | 0.10 | 1.04 | 0.0289 | 1.8664 |
| 4 | 75 | $C_1$ | 1 | 1 | 71 | 34.09 | 0.4801 | 4 | 0.21 | 1.15 | 0.1471 | - |
| | | | | 2 | 72 | 31.57 | 0.4385 | 4 | 0.89 | 1.08 | 0.1270 | - |
| | | | 2 | 1 | 70 | 33.72 | 0.4817 | 4 | 0.19 | 1.18 | 0.1281 | - |
| | | | | 2 | 72 | 30.53 | 0.4241 | 4 | 1.04 | 1.08 | 0.0824 | - |
| | | | $\infty$ | 1 | 68 | 36.40 | 0.5353 | 4 | 0.23 | 1.22 | 0.0849 | - |
| | | | | 2 | - | - | - | - | - | - | - | - |
| | | $C_2$ | 2 | 1 | 70 | 34.53 | 0.4933 | 3 | 0.21 | 1.27 | 0.0842 | 6.8793 |
| | | | | 2 | 70 | 30.73 | 0.4390 | 3 | 1.04 | 1.10 | 0.0488 | 3.1927 |
| | | $C_3$ | 2 | 1 | 74 | 34.84 | 0.4708 | 3 | 0.15 | 1.11 | 0.8384 | - |
| | | | | 2 | 66 | 29.48 | 0.4466 | 3 | 1.13 | 1.18 | 0.2077 | - |

Table 6.4: Results for Example 6.1.1 under limited runtime with different ordering cones and norms.

Tables 6.5 and 6.6 (Size).

First, we fix the $\ell_2$ norm and solve the problem instances with respect to the ordering cones $C_1, C_2$ and $C_3$, see Table 6.5. Second, we fix the ordering cone as $C_1$ and solve the problem instances where we take the norms in (P($v$)) as $\ell_1, \ell_2$ and $\ell_\infty$, see Table 6.6. In order to summarize the results in these tables, we plot the average PE and HV values obtained by the algorithms, see Figures 6.8-6.11. From these figures, we observe that for all considered ordering cones and norms, Algorithm 2 has better performance in terms of both primal error and hypervolume indicators under time limit.

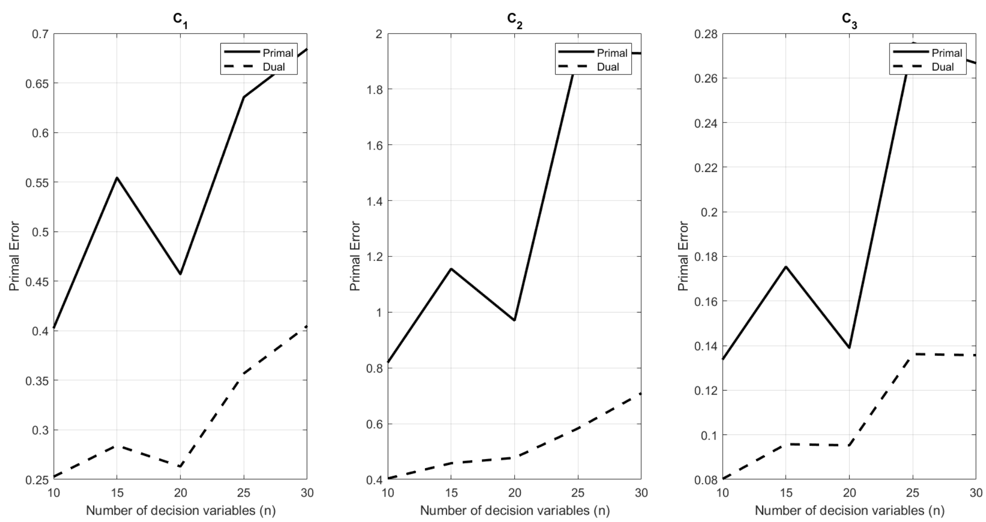


Figure 6.8: Average primal error values of random instances of Example 6.3.1 for $q = 3$ for ordering cones $C_1$ (left), $C_2$ (middle) and $C_3$ (right) when the algorithms are run under time limit of 50 seconds.

From the tests we have performed in this section, Chapter 6, we found that Algorithm 2 works correctly, i.e., returns a finite weak $\epsilon$-solution to primal problem and a finite $\epsilon$-solution to dual problem. However, the primal error indicator that Algorithm 2 returns is a very small fraction of the input value $\epsilon$. To achieve such a close approximation, Algorithm 2 solves higher number of scalarizations, causing the algorithm to terminate with a longer runtime. We observe from the tests performed with a runtime stopping criterion with respect to different ordering cones and $\ell_p$-norms that, the Algorithm 2 returns a better approximation for objective spaces with higher dimensions.

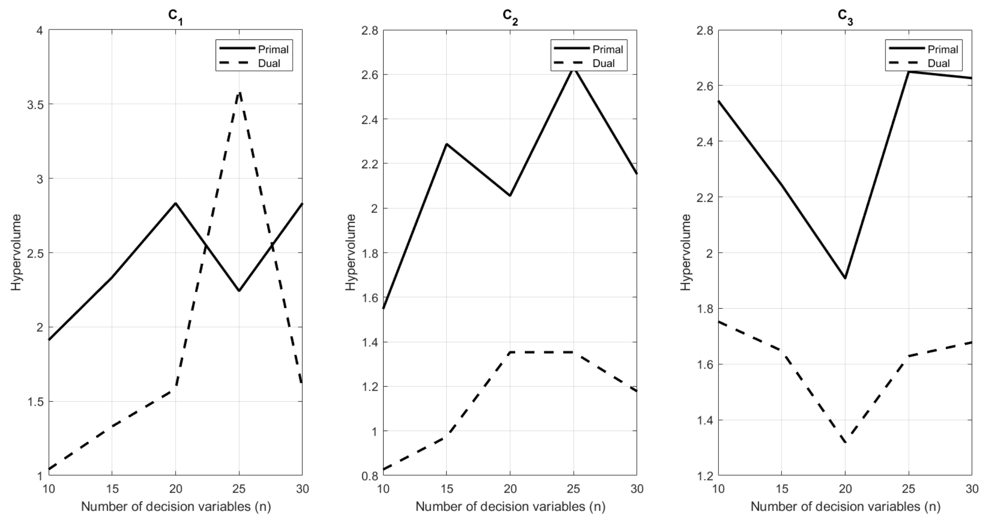


Figure 6.9: Average HV values of random instances of Example 6.3.1 for $q = 3$ for ordering cones $C_1$ (left), $C_2$ (middle) and $C_3$ (right) when the algorithms are run under time limit of 50 seconds.

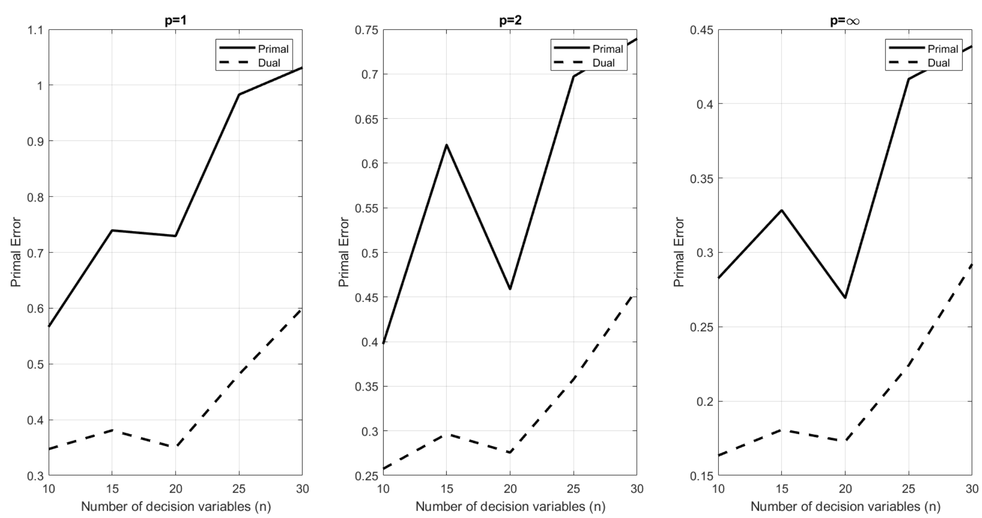


Figure 6.10: Average primal error values of random instances of Example 6.3.1 for $q = 3$ with $\ell_p$ norms for $p = 1$ (left), $p = 2$ (middle) and $p = \infty$ (right), when the algorithms are run under time limit of 50 seconds.

| $n$ | Size | Cone | Alg | Opt | $T_{opt}$ | $T_{opt}$/Opt | En | $T_{en}$ | $T/\lvert\bar{\mathcal{X}}\rvert$ | PE | HV |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 15 | $C_1$ | 1 | 57.47 | 21.60 | 0.3763 | 5.00 | 0.09 | 0.93 | 0.4024 | 1.9112 |
| | | | 2 | 60.73 | 20.42 | 0.3363 | 4.00 | 0.16 | 0.86 | 0.2527 | 1.0413 |
| | | $C_2$ | 1 | 60.60 | 22.65 | 0.3743 | 3.80 | 0.07 | 0.92 | 0.8190 | 1.5472 |
| | | | 2 | 60.73 | 20.49 | 0.3376 | 3.80 | 0.35 | 0.86 | 0.4035 | 0.8266 |
| | | $C_3$ | 1 | 61.20 | 22.73 | 0.3719 | 3.93 | 0.07 | 0.91 | 0.1337 | 2.5455 |
| | | | 2 | 60.20 | 20.40 | 0.3392 | 3.93 | 0.39 | 0.87 | 0.0802 | 1.7527 |
| 15 | 12 | $C_1$ | 1 | 54.92 | 21.38 | 0.3908 | 5.00 | 0.10 | 0.97 | 0.5544 | 2.3338 |
| | | | 2 | 60.17 | 20.44 | 0.3401 | 4.00 | 0.17 | 0.87 | 0.2843 | 1.3300 |
| | | $C_2$ | 1 | 58.58 | 22.44 | 0.3839 | 3.67 | 0.07 | 0.95 | 1.1556 | 2.2875 |
| | | | 2 | 60.25 | 20.49 | 0.3405 | 3.92 | 0.41 | 0.87 | 0.4582 | 0.9734 |
| | | $C_3$ | 1 | 60.33 | 22.81 | 0.3785 | 4.00 | 0.07 | 0.92 | 0.1754 | 2.2426 |
| | | | 2 | 61.33 | 20.72 | 0.3380 | 4.00 | 0.42 | 0.85 | 0.0958 | 1.6479 |
| 20 | 15 | $C_1$ | 1 | 58.40 | 22.36 | 0.3837 | 5.00 | 0.09 | 0.91 | 0.4571 | 2.8328 |
| | | | 2 | 62.67 | 21.24 | 0.3390 | 4.00 | 0.16 | 0.83 | 0.2631 | 1.5824 |
| | | $C_2$ | 1 | 60.73 | 23.10 | 0.3809 | 3.80 | 0.07 | 0.91 | 0.9700 | 2.0552 |
| | | | 2 | 61.80 | 20.88 | 0.3382 | 4.00 | 0.41 | 0.85 | 0.4777 | 1.3532 |
| | | $C_3$ | 1 | 62.33 | 23.39 | 0.3756 | 4.00 | 0.07 | 0.89 | 0.1390 | 1.9080 |
| | | | 2 | 62.27 | 21.06 | 0.3384 | 4.00 | 0.41 | 0.84 | 0.0953 | 1.3192 |
| 25 | 13 | $C_1$ | 1 | 54.31 | 21.10 | 0.3897 | 5.00 | 0.08 | 0.98 | 0.6356 | 2.2417 |
| | | | 2 | 61.08 | 20.56 | 0.3370 | 4.00 | 0.16 | 0.86 | 0.3569 | 3.6038 |
| | | $C_2$ | 1 | 55.77 | 21.52 | 0.3866 | 3.54 | 0.06 | 1.00 | 1.9313 | 2.6329 |
| | | | 2 | 61.23 | 20.69 | 0.3381 | 3.92 | 0.39 | 0.85 | 0.5833 | 1.3532 |
| | | $C_3$ | 1 | 57.08 | 21.79 | 0.3822 | 3.77 | 0.07 | 0.97 | 0.2757 | 2.6498 |
| | | | 2 | 62.00 | 20.85 | 0.3364 | 4.00 | 0.41 | 0.84 | 0.1362 | 1.6287 |
| 30 | 11 | $C_1$ | 1 | 55.45 | 21.11 | 0.3829 | 5.00 | 0.08 | 0.98 | 0.6844 | 2.8334 |
| | | | 2 | 59.55 | 20.08 | 0.3375 | 4.00 | 0.16 | 0.88 | 0.4049 | 1.5810 |
| | | $C_2$ | 1 | 58.91 | 22.26 | 0.3783 | 3.73 | 0.07 | 0.95 | 1.9284 | 2.1519 |
| | | | 2 | 59.36 | 20.03 | 0.3378 | 3.91 | 0.43 | 0.88 | 0.7094 | 1.1776 |
| | | $C_3$ | 1 | 60.09 | 22.50 | 0.3747 | 3.82 | 0.07 | 0.93 | 0.2666 | 2.6265 |
| | | | 2 | 59.73 | 20.18 | 0.3382 | 3.91 | 0.39 | 0.87 | 0.1357 | 1.6780 |

Table 6.5: Results for randomly generated instances of Example 6.3.1 for $q = 3$ with different ordering cones, when the algorithms are run for T=50 seconds.

| $n$ | Size | $p$ | Alg | Opt | $T_{opt}$ | $T_{opt}$/Opt | En | $T_{en}$ | $T/\lvert\bar{\mathcal{X}}\rvert$ | PE | HV |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 17 | 1 | 1 | 53.71 | 21.09 | 0.3931 | 5.00 | 0.08 | 1.00 | 0.5665 | 2.1873 |
| | | | 2 | 57.35 | 19.79 | 0.3450 | 4.00 | 0.17 | 0.91 | 0.3473 | 1.2154 |
| | | 2 | 1 | 56.00 | 21.14 | 0.3777 | 5.00 | 0.07 | 0.95 | 0.3973 | 2.0040 |
| | | | 2 | 57.88 | 19.98 | 0.3453 | 4.00 | 0.16 | 0.90 | 0.2574 | 1.1113 |
| | | $\infty$ | 1 | 53.41 | 21.23 | 0.3976 | 5.06 | 0.07 | 0.99 | 0.2826 | 1.7489 |
| | | | 2 | 58.18 | 20.07 | 0.3450 | 4.00 | 0.17 | 0.90 | 0.1634 | 1.0354 |
| 15 | 17 | 1 | 1 | 57.65 | 22.04 | 0.3831 | 5.00 | 0.08 | 0.92 | 0.7394 | 2.1807 |
| | | | 2 | 63.24 | 20.72 | 0.3278 | 4.00 | 0.16 | 0.82 | 0.3808 | 0.8593 |
| | | 2 | 1 | 61.82 | 22.47 | 0.3637 | 5.00 | 0.07 | 0.86 | 0.6205 | 2.2184 |
| | | | 2 | 63.88 | 20.83 | 0.3261 | 4.00 | 0.16 | 0.82 | 0.2966 | 1.0616 |
| | | $\infty$ | 1 | 60.94 | 22.54 | 0.3700 | 5.18 | 0.07 | 0.87 | 0.3284 | 2.0562 |
| | | | 2 | 64.06 | 20.90 | 0.3264 | 4.00 | 0.16 | 0.81 | 0.1807 | 1.2320 |
| 20 | 18 | 1 | 1 | 61.17 | 22.51 | 0.3686 | 5.00 | 0.08 | 0.87 | 0.7294 | 3.0080 |
| | | | 2 | 63.83 | 20.90 | 0.3274 | 4.00 | 0.15 | 0.82 | 0.3499 | 1.2083 |
| | | 2 | 1 | 60.11 | 22.12 | 0.3684 | 5.00 | 0.07 | 0.88 | 0.4588 | 2.6591 |
| | | | 2 | 64.00 | 20.98 | 0.3279 | 4.00 | 0.16 | 0.81 | 0.2757 | 1.4626 |
| | | $\infty$ | 1 | 59.28 | 22.21 | 0.3753 | 5.00 | 0.07 | 0.90 | 0.2694 | 2.6331 |
| | | | 2 | 64.00 | 21.10 | 0.3300 | 4.00 | 0.15 | 0.81 | 0.1729 | 1.5567 |
| 25 | 14 | 1 | 1 | 58.79 | 21.74 | 0.3703 | 5.00 | 0.07 | 0.90 | 0.9830 | 2.3330 |
| | | | 2 | 64.64 | 21.23 | 0.3286 | 4.00 | 0.16 | 0.81 | 0.4813 | 0.8697 |
| | | 2 | 1 | 57.93 | 21.49 | 0.3714 | 5.00 | 0.07 | 0.91 | 0.6971 | 2.3461 |
| | | | 2 | 64.79 | 21.25 | 0.3282 | 4.00 | 0.15 | 0.81 | 0.3578 | 1.2700 |
| | | $\infty$ | 1 | 57.36 | 21.60 | 0.3767 | 5.00 | 0.06 | 0.92 | 0.4166 | 2.6032 |
| | | | 2 | 64.71 | 21.26 | 0.3286 | 4.00 | 0.16 | 0.81 | 0.2241 | 1.4588 |
| 30 | 16 | 1 | 1 | 57.44 | 21.51 | 0.3751 | 5.00 | 0.08 | 0.93 | 1.0316 | 2.1772 |
| | | | 2 | 59.75 | 20.08 | 0.3363 | 4.00 | 0.16 | 0.88 | 0.5999 | 0.8015 |
| | | 2 | 1 | 55.63 | 20.98 | 0.3777 | 5.00 | 0.07 | 0.96 | 0.7395 | 2.5195 |
| | | | 2 | 60.00 | 20.11 | 0.3355 | 4.00 | 0.16 | 0.87 | 0.4591 | 1.3388 |
| | | $\infty$ | 1 | 56.00 | 21.35 | 0.3815 | 5.00 | 0.07 | 0.95 | 0.4387 | 2.4787 |
| | | | 2 | 60.69 | 20.29 | 0.3345 | 4.00 | 0.16 | 0.86 | 0.2922 | 1.4639 |

Table 6.6: Results for randomly generated instances of Example 6.3.1 for $q = 3$ with different norms used in (P($v$)), when the algorithms are run for T=50 seconds.

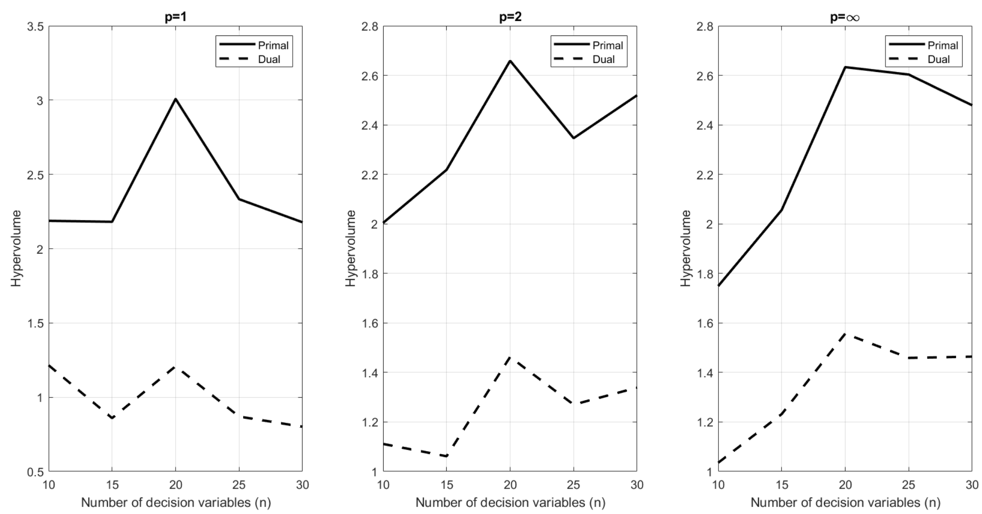


Figure 6.11: Average HV values of random instances of Example 6.3.1 for $q = 3$ with $\ell_p$ norms for $p = 1$ (left), $p = 2$ (middle) and $p = \infty$ (right), when the algorithms are run under time limit of 50 seconds.

# Chapter 7

# Conclusion

In this paper, a geometric dual algorithm to the primal algorithm in [1] is developed. The proposed algorithm is based on a new geometric dual problem and a new duality relation between the upper and lower images of the primal and dual problems, respectively. The new geometric dual problem is different from the one in [20] in the sense that it does not require a fixed direction vector. Before presenting the algorithm, we provide geometric duality relations between the weakly minimal proper faces of the upper image and the maximal proper faces of the lower image. Moreover, we provide geometric duality relations between polyhedral approximations of the upper and lower images.

The primal algorithm presented as Algorithm 1 is essentially the algorithm in [1] with some changes. It starts by obtaining an initial outer approximation and solves a norm-minimizing scalarization problems at each iteration to update the outer approximation of the upper image. The algorithm stops when the approximation is sufficiently fine. Different from [1], we show that, with our modifications, Algorithm 1 returns also an $\epsilon$-solution to the dual problem.

In addition to the primal algorithm, we have developed a geometric dual algorithm, Algorithm 2, which starts by obtaining an initial outer approximation to the lower image of the geometric dual problem. It solves weighted sum scalarization problems at each iteration to update the outer approximation. When Algorithm 2 terminates, it returns a finite $\epsilon$-solution to dual problem and a finite weak $\tilde{\epsilon}$-solution to the primal problem, where $\tilde{\epsilon} \geq \epsilon$ is an escalated approximation error that can be computed depending on the ordering cone.

We compare the performance of Algorithms 1 and 2 through computational experiments for which we solve randomly generated test problems. We employ different stopping criteria for the algorithms and measure the proximity of the final outer approximations returned by the algorithms via primal error and hypervolume indicators. We conclude that Algorithm 2 returns a small fraction of the allowed approximation error, $\epsilon$, resulting in a longer runtime with $\epsilon$ stopping criterion. When runtime is used as stopping criterion, the Algorithm 2 returns a closer approximation for higher dimensions of the objective space, $q$.

# Bibliography

[1] Ç. Ararat, F. Ulus, and M. Umer, "A norm minimization-based convex vector optimization algorithm," *arXiv preprint arXiv:2104.10282*, 2021.

[2] F. Ulus, B. Rudloff, and A. Löhne, "Primal and dual approximation algorithms for convex vector optimization problems," *Journal of Global Optimization*, vol. 60, no. 4, pp. 713–736, 2014.

[3] P. Agarwalla and S. Mukhopadhyay, *Feature Selection Using Multi-Objective Optimization Technique for Supervised Cancer Classification*, pp. 195–213. Singapore: Springer Singapore, 2018.

[4] S. Chakraborty and K. Mali, *Application of Multiobjective Optimization Techniques in Biomedical Image Segmentation*, pp. 181–194. Singapore: Springer Singapore, 2018.

[5] J. O. H. Sendín, A. A. Alonso, and J. R. Banga, "Efficient and robust multi-objective optimization of food processing: A novel approach with application to thermal sterilization," *Journal of Food Engineering*, vol. 98, no. 3, pp. 317–324, 2010.

[6] W. J. Gutjahr and P. C. Nolz, "Multicriteria optimization in humanitarian aid," *European Journal of Operational Research*, vol. 252, no. 2, pp. 351–366, 2016.

[7] M. G. C. Tapia and C. A. C. Coello, "Applications of multi-objective evolutionary algorithms in economics and finance: A survey," in *2007 IEEE Congress on Evolutionary Computation*, pp. 532–539, 2007.

[8] H. P. Benson, "An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem," *Journal of Global Optimization*, vol. 13, pp. 1–24, 1998.

[9] L. Shao and L. Ehrgott, "Approximately solving multiobjective linear programmes in objective space and an application in radiotherapy treatment planning," *Mathematical Methods of Operations Research*, vol. 68, no. 2, pp. 257–276, 2008.

[10] L. Shao and L. Ehrgott, "Approximating the nondominated set of an molp by approximately solving its dual problem," *Mathematical Methods of Operations Research*, vol. 68, no. 3, pp. 469–492, 2008.

[11] A. Löhne, *Vector Optimization with Infimum and Supremum*. Springer, 2011.

[12] M. Ehrgott, A. Löhne, and L. Shao, "A dual variant of bensons outer approximation algorithm for multiple objective linear programming," *Journal of Global Optimization*, vol. 52, no. 4, pp. 757–778, 2012.

[13] A. H. Hamel, A. Löhne, and B. Rudloff, "Benson type algorithms for linear vector optimization and applications," *Journal of Global Optimization*, vol. 59, no. 4, pp. 811–836, 2014.

[14] L. Csirmaz, "Using multiobjective optimization to map the entropy region," *Computational Optimization and Applications*, vol. 63, no. 1, pp. 45–67, 2016.

[15] S. Ruzika and M. M. Wiecek, "Approximation methods in multiobjective programming," *Journal of optimization theory and applications*, vol. 126, no. 3, pp. 473–501, 2005.

[16] M. Ehrgott, L. Shao, and A. Schöbel, "An approximation algorithm for convex multi-objective programming problems," *Journal of Global Optimization*, vol. 50, no. 3, pp. 397–416, 2011.

[17] A. Pascoletti and S. P., "Scalarizing vector optimization problems," *Journal of Optimization Theory and Applications*, vol. 42, no. 4, pp. 499–524, 1984.

[18] B. Grünbaum, *Convex polytopes*, vol. 221. Springer Science & Business Media, 2013.

[19] F. Heyde and A. Löhne, "Geometric duality in multiple objective linear programming," *SIAM Journal of Optimization*, vol. 19, no. 2, pp. 836–845, 2008.

[20] F. Heyde, "Geometric duality for convex vector optimization problems," *Journal of Convex Analysis*, vol. 20, no. 3, pp. 813–832, 2013.

[21] J. Jahn, *Vector Optimization: Theory, Applications, and Extensions.* Springer, 2004.

[22] V. Soltan, *Lectures on Convex sets.* World Scientific, 2015.

[23] R. T. Rockafellar, *Convex Analysis.* Princeton University Press, 1970.

[24] H. Glöckner, "Positive definite functions on infinite-dimensional convex cones," *Memoirs of the American Mathematical Society*, vol. 166, no. 789, 2003.

[25] S. Boyd and L. Vandenberghe, *Convex Optimization.* Cambridge University Press, 2004.

[26] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1." `http://cvxr.com/cvx`, 2014.

[27] M. Grant and S. Boyd, "Graph implementations for nonsmooth convex programs," in *Recent Advances in Learning and Control* (V. Blondel, S. Boyd, and H. Kimura, eds.), Lecture Notes in Control and Information Sciences, pp. 95–110, Springer-Verlag Limited, 2008. `http://stanford.edu/~boyd/graph_dcp.html`.

[28] D. Ciripoi, A. Löhne, and B. Weißing, "A vector linear programming approach for certain global optimization problems," *Journal of Global Optimization*, vol. 72, no. 2, pp. 347–372, 2018.

[29] A. Löhne and B. Weißing, “The vector linear program solver *Bensolve* – notes on theoretical background,” *European Journal of Operational Research*, vol. 260, pp. 807–813, 2017.

[30] A. Löhne and B. Weißing, “Equivalence between polyhedral projection, multiple objective linear programming and vector linear programming,” *Mathematical Methods of Operations Research*, vol. 84, no. 2, pp. 411–426, 2016.

[31] E. Zitzler and L. Thiele, “Multiobjective evolutionary algorithms: a comparative case study and the strength pareto approach,” *IEEE Transactions on Evolutionary Computation*, vol. 3, no. 4, pp. 257–271, 1999.

[32] E. Zitzler, D. Brockhoff, and L. Thiele, “The hypervolume indicator revisited: On the design of pareto-compliant indicators via weighted integration,” in *International Conference on Evolutionary Multi-Criterion Optimization*, pp. 862–876, Springer, 2007.

[33] A. Auger, J. Bader, D. Brockhoff, and E. Zitzler, “Hypervolume-based multiobjective optimization: Theoretical foundations and practical implications,” *Theoretical Computer Science*, vol. 425, pp. 75–103, 2012.