Optimal Posted Price Mechanism vs Auction Under Common Value Environment

by

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Graduate School of Social Sciences and Humanities

This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
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[To my family]

ABSTRACT

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This paper investigates the revenue maximizing mechanisms in the simple common value environment. We analyzed optimal posted price mechanism under the discrete common value environment in which bidders get some noisy signals about the value of the object and seller sets a fixed posted price but she determines the selling probabilities of the good conditional on the signals that bidders state. In this mechanism, whenever seller decides to sell the good she picks a bidder randomly and sells to him. By doing so we aimed to eliminate so-called winner's curse and to increase revenue for the seller under this common value environment. Moreover, under the same environment we also calculated the expected revenue of the second-price auction mechanism. After finding optimal posted price mechanism, we compared the expected revenues of these two mechanisms. Finally, we offered an alternative posted-price mechanism for easy comparison of auction and posted price mechanism when number of bidders tends to infinity.

ÖZETÇE

Ortak Değer Ortamında Müzayede ve Optimal Etiket Fiyat Mukayesesi
İrfan Tekdir
Ekonomi, Yüksek Lisans
15 Ağustos 2021

Bu makale basit ortak değer ortamında geliri maksimum yapan mekanizmaları araştırmaktadır. Teklif verenlerin satılan objenin değeri hakkında gürültülü sinyaller aldığı, satıcının da satma kararını gelen sinyallere göre verip sabit etiket fiyat uygulaması yaptıgı ortak değer ortamında optimal etiket fiyat mekanizması analizini yaptık. Bu mekanizmada satıcı ürünü satmaya karar verdiğinde rastgele bir teklif vereni seçip ürünü ona satıyor. Böyle yaparak ortak değer ortamında 'Kazananın Laneti' olarak bilinen etkiyi ortadan kaldırarak satıcı için geliri artırmayı amaçladık. Ayrıca aynı ortamda ikincil fiyat müzayede mekanizmasının beklenen gelirini hesapladık. Optimal etiket fiyat mekanizmasını bulduktan sonra bu iki mekanizmanın gelirlerini mukayese ettik. Son olarak teklif verenlerin sayısının sonsuza gittiği durumda etiket fiyat mekanizması ve müzayede mekanizmasının kolay bir mukayesesi için alternatif bir etiket fiyat mekanizması önerdik.

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NOMENCLATURE

MLRP	Monotone Likelihood Ratio Property

Chapter 1

INTRODUCTION

When we buy something from an auction, the valuation of the item that is sold in the auction may change from environment to environment. If we were in the art auction, most probably this value would be depending on our taste of art. The other bidders' valuation would not effect our valuation. Of course, our bidding strategy would change with the auction setup. However, the value that we give for the item would remain same. We call this environment as private value environment. In the independent private value environment, *The Revenue Equivalence Theorem* [Myerson, 1981] states that, with some regularity conditions, revenue of the seller remains same regardless of the auction setup if the object is sold to the one who gives highest valuation for the object. [Milgrom, 1985] also analyzed the optimal auction design when there is an affiliation between the bidders' valuations. Most of the theory is known for the private valuation environment. However, if we were in the auction of mineral track, valuation of the item would not vary with the personal tastes. It would contain some common valuation. Whoever wins the item would benefit from it almost equally. To see this more easily, suppose there is a jar full of coins and it is auctioned off. All bidders looks the jar and estimates the overall value of the jar. However, whoever wins the jar will get the same amount of coins. We call this environment as common value environment. According to the common value auction theory, the object that is sold has ex-post common value for each bidders and bidders get some noisy signals about the value of the object. They construct their expectation about the value of the object conditional on the signals that they take. In this more general environment, 'The Revenue Equivalance Theorem' fails. [Milgrom and Weber, 1982] gives the condition for ranking of revenues for the auction mechanisms under the common value environment which is so called "Linkage Principle". For the optimal auction setup under the common value environment, [Bulow and Klemperer, 1996] generalized the Revenue Equivalance theorem. The most problematic issue under the common value environment is the Winner's Curse which firstly mentioned [Capen et al., 1971] and can be summarized as follows: In the auction mechanisms under the common value environment bidders set some interim expectation about the object's value after getting their signal. However, when they are announced that they won the auction, since the object is sold to the highest bidder, they would conclude that their signals were higher than the other bidders' signals. At that moment their expectation about the value of the object would become lower than their interim expectation. This means that 'win announcement' brought bad news to the winner. In order not to face so-called winner's curse in the equilibrium bidders lower their bids and this situation may decrease expected revenue of the seller severely. By following this argument the direct question comes to mind is as follows: Can we set different selling mechanism in which the object is sold to the bidders randomly, not to the highest one, so that when winners are announced this announcement brings no new information about the value of the object and so there is zero winner's curse. This selling mechanism is called as posted price mechanism and it is introduced by [Bulow and Klemperer, 2002]. [Bergemann et al., 2020] characterized the revenue maximizing mechanisms under the common value environment. They showed that under the maximum signal model in which valuation of the item that is auctioned off is equal to the maximum signal of the bidders', since winner's curse is too strong in this environment, simple posted price mechanism provides higher expected revenue than the auction mechanism. Their result was surprising and pretty amazing. By following their approach we also planned to have posted price mechanism that beats the auction mechanism in different common valuation setup. Our setup is briefly as follows: Suppose that there is an oil track which is commonly known that it is either useless, the value of the oil track is V=0, or it has value V=1(You can think of this as a 1 Billion dollar). Seller and bidders have prior common belief about the value of this

oil track which is shown by $\mu_0 = Pr(V = 0)$ and we are assuming that this prior belief is equal to 1/2. That is, $\mu_0 = Pr(V = 1) = 1/2$. There are N bidders and they get state contingent independent signals as either Low or High. In each state we are assuming that getting an informative signal is more probable. That is, if we show the signals with S_i, then getting a Low signal when the true state is V=0 and getting a High signal when the true state is V=1 is probabilistically higher than 1/2.
To show it formally, $Pr(S_i = L|V = 0) = Pr(S_i = H|V = 1) = \alpha > 1/2$. According to these signals bidders will construct their posterior beliefs and then they will bid strategically by using these posteriors. We will show these posterior beliefs via μ_L and μ_H which are respectively the posterior probabilities about the value of the common value object after getting Low or High signal. That is $\mu_L = Pr(V = 1|S_i = L) = 1 - \alpha$ and $\mu_H = Pr(V = 1|Si = H) = \alpha$. These probabilities can easily be calculated by using Bayes' rule. So, if someone gets Low signal his posterior will be lower than his prior and if he gets High signal his posterior will be higher than his prior.($\mu_H > \mu_0 = 1/2 > \mu_L$). In this environment we will compare the expected revenues from second price auction and the expected revenues of the our new posted price mechanism. The reason of why we only consider second-price auction to compare with our posted -price mechanism is following so called 'Linkage Principle'. According to the Linkage Principle, as the linkage between a bidder's own signal and price of the auction increases, bidders will pay higher expected price for the object when they won. Since in the second price auction it is equilibrium that all bidders bid their valuation, the linkage between a bidder's own valuation and price of the auction is higher compared to the first price auction. So expected revenue in the second price auction is higher than the first price auction for the symmetric common value environment. To give the idea of our posted price mechanism, suppose that there is only one bidder and this bidder gets either Low or High signal. Seller sets a fixed price p and asks the signal of the bidder. She announces that she will keep the object if bidder says Low and she will sell the object if bidder says High at fixed posted price p. There may be another posted price mechanism in which seller could say that whatever

signal bidder says she will sell the object at another fixed posted price. To find the optimal posted price in the former, we will consider two important constraints which are individual rationality and incentive compatibility for the bidder. Individual rationality implies that if bidder's expected utility is less than zero, he could choose not to buy this object and just get zero. We also want from bidder to state his signal truthfully. To give an example, in the former posted price mechanism if $p > \alpha$ then the bidder who gets a High signal may prefer to state his signal as if it is Low and conversely if $p < 1 - \alpha$ then the bidder who gets Low signal may prefer to state his signal as if it is High. With the individual rationality and incentive compatibility constraints, the optimal posted price is $p = \alpha$ in the former mechanism and the expected revenue is $\alpha/2$. Under the second mechanism with the individual rationality and incentive compatibility constraints the optimal posted price is $p = 1 - \alpha$ and the expected revenue for the seller is $1 - \alpha$. According to the value of α optimal mechanism varies. However, there are actually infinitely many posted price mechanism if we set selling probabilities conditional on the signals that bidders state as free. So, our aim is to find optimal posted price mechanism with selling probabilities amoung all posted price mechanisms not only finding optimal posted price after selling conditions are given. After finding optimal posted price mechanism for N=2 and N=3 case, we will compare the expected revenue with the second price auction. Moreover, we will offer another relatively simple posted price mechanism to make the comparison of auction and posted price mechanism when number of bidder tends to infinity.

The organization of the rest of our paper as follows: Chapter 2, will provide simple model environment. Section 3, will analyze bidding function in the second price auction and provides way of expected revenue calculation. Section 4, will be analyzing the optimal posted price mechanism when N=1,2,3 and will be comparing auction and posted price mechanism. Section 5, will give another posted price mechanism in which we will be able to compare auction and posted price mechanism asymptotically. Section 6 will be the conclusion part.

Chapter 2

MODEL ENVIRONMENT

Basic Model

- There is an oil track that has the value V which is either 0 or 1 with equal probability.
  $\mu_0 = Pr(V = 0) = Pr(V = 1) = \dfrac{1}{2}$

- There are N risk-neutral bidders that are trying to maximize their expected utility.

- Bidders get state contingent independent signals $\mathcal{S}_i \in \{L, H\}$ where $i = 1, 2, 3, ..., N$.

- $Pr(\mathcal{S}_i = L|V = 0) = Pr(\mathcal{S}_i = H|V = 1) = \alpha > \frac{1}{2}$

- $Pr(\mathcal{S}_i = L) = Pr(\mathcal{S}_i = H) = \frac{1}{2}$

- $\mathcal{S} =_{i=1}^{n} \mathcal{S}_i$ such that $Pr(\mathcal{S} = s) = Pr(\mathcal{S}_1 = s_1, \mathcal{S}_2 = s_2, ..., \mathcal{S}_N = s_N)$ has the following symmetric probability distributions:
  $Pr(H, H, ..., H) = \dfrac{\alpha^N + (1-\alpha)^N}{2}$
  $Pr(\underbrace{H, H, .., H}_{\text{j-many}}, \underbrace{L, L, .., L}_{\text{N-j-many}}) = \dbinom{N}{j} \dfrac{\alpha^j (1-\alpha)^{N-j} + (1-\alpha)^j \alpha^{N-j}}{2}$

- The value of the object for the seller is zero.

Chapter 3

LINKAGE PRINCIPLE AND SECOND-PRICE AUCTION

The Revenue Equivalance Theorem states that under the private value auction setup with the independent valuations, revenue for the seller remains same if object is sold to the highest bidder with first price and second price auction. However, when we analyze revenues of the auction mechanisms under the common value environment, so-called Linkage Principle comes to the scene. "Linkage Principle" basically states that in the general symmetric common value environment, the more information on which the winner's payment is based, the higher will be the expected revenue. So, we won't deal with the first auction because we know that under this environment Linkage Principle works and so second price auction provides more expected revenue than the first price auction. Moreover, it is known that in the common value environment bidders face the winner's curse and bid accordingly under the auction mechanisms. On the specificity of the second price auction bidders do not bid the expected value of the object conditional on their signal but they bid according to the expected value of the object conditional on winning. In order to calculate equilibrium bid functions for each type we will exploit [Pesendorfer, 1997]approach from the opposite side. We will slightly convert our discrete signal structure to the continuous case such that each agent gets signal $S \in [0, 1]$. If $S \leq \frac{1}{2}$, it is labelled as Low in the previous model and if $S > \frac{1}{2}$, it is labelled as High.

Now, by using this idea we will construct our signal distributions conditional on the value of the object.

- $P(S = L|V = 0) = \alpha \implies f(s|0) = \begin{cases} 2\alpha, \text{ if } s \leq \frac{1}{2} \\ 2(1\text{-}\alpha), \text{ if } s > \frac{1}{2} \end{cases}$

- $P(S = H|V = 1) = \alpha \implies f(s|1) = \begin{cases} 2(1\text{-}\alpha), \text{ if } s \leq \frac{1}{2} \\ 2\alpha, \text{ if } s > \frac{1}{2} \end{cases}$

- $F(s|0) = \begin{cases} 2\alpha s, \text{ if } s \leq \frac{1}{2} \\ 2\alpha - 1 + 2(1-\alpha)s, \text{ if } s > \frac{1}{2} \end{cases}$

- $F(s|1) = \begin{cases} 2(1\text{-}\alpha)s, \text{ if } s \leq \frac{1}{2} \\ 1\text{-}2\alpha + 2\alpha s, \text{ if } s > \frac{1}{2} \end{cases}$

- This setup satisfies *Monotone Likelihood Ratio Property*(**MLRP**) which means that all signals between [0,1/2] and all signals between (1/2,1] provides the same information to the bidders.

  ∀s and s' $\in [0, 1/2], \dfrac{f(s|1)}{f(s|0)} = \dfrac{f(s'|1)}{f(s'|0)}$

  ∀s and s' $\in (1/2, 1], \dfrac{f(s|1)}{f(s|0)} = \dfrac{f(s'|1)}{f(s'|0)}$

- By following [Pesendorfer, 1997] there is unique symmetric increasing equilibrium.

  Proposition1: Unique symmetric increasing equilibrium in the second-price auction, if Y_1 represents the maximum of the other $n-1$ signals, as follows:

  $$\beta(s) = \mathbf{E}[V|Y_1 = s, S_1 = s] = \frac{F(s|1)^{n-2} f(s|1)^2}{F(s|1)^{n-2} f(s|1)^2 + F(s|0)^{n-2} f(s|0)^2}$$

- For the Low type ($s \leq \frac{1}{2}$) bid function takes the following form:

  $$\beta(s) = \mathbf{E}[V|Y_1 = s, S_1 = s] = \frac{(1-\alpha)^n}{\alpha^n + (1-\alpha)^n}$$

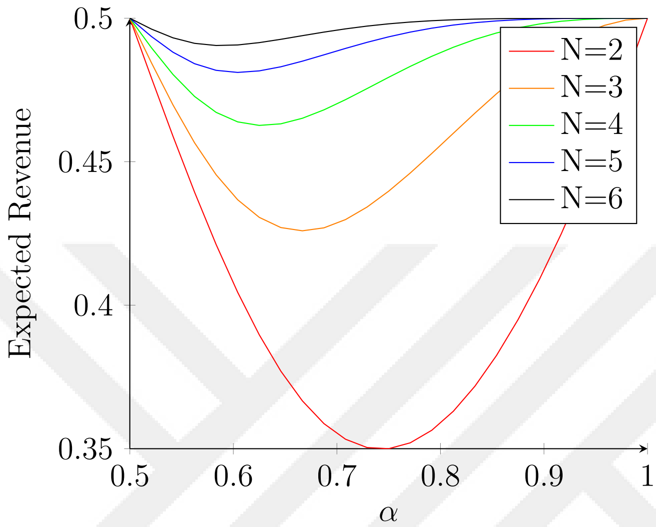
- For the High type ($s > \frac{1}{2}$) bid function takes the following form:

  $$\beta(s) = \mathbf{E}[V|Y_1 = s, S_1 = s] = \frac{(1 - 2\alpha + 2\alpha s)^{n-2}(2\alpha)^2}{((1 - 2\alpha + 2\alpha s)^{n-2}(2\alpha)^2 + (2\alpha - 1 + 2(1-\alpha)s)^{n-2}(2(1-\alpha)^2}$$

Proposition2: In the second price auction with N players, Low type uses pure strategy and bids $\beta(L) = \dfrac{(1-\alpha)^n}{\alpha^n + (1-\alpha)^n}$, and High type uses mixed strategy over the $[\dfrac{(1-\alpha)^{N-4}}{\alpha^{N-4} + (1-\alpha)^{N-4}}, \dfrac{\alpha^2}{\alpha^2 + (1-\alpha)^2}]$ interval.
Interpretation of Proposition 2 as follows: Low type uses pure strategy because when he bids he bids by considering that all other bidders also took low signal. However, for the High type this situation changes. He gets High signal and he only considers that maximum of others also got High signal. This could happen with different cases. On the one boundary, he thinks that he got High signal and only one of the rest got High signal too but other $N-2$ people got Low signal. This case represents the left boundary of High type's bid interval. On the other boundary, he thinks that he got highest signal, in the continuous case this coincides with the signal of 1, and bids by considering that one of the other $N-1$ bidders also got that highest signal. However, this case actually gives zero information about other $N-2$ signals because he had already known that signals are between 0 and 1.

Expected Revenue of Second Price Auction To calculate the expected revenue, let z be second-order statistic of signals amoung n bidders given $V = v$ and $g(.|v)$ be a distribution of z given $V = v$.
$g(z|v) = n(n-1)(1-F(z|v))(F(z|v))^{n-2}f(z|v)$.
Expected Revenue of the seller can be calculated with the following formula:
$\sum_{v\in\mathcal{V}} P(V=v)(\int_0^1 \beta(z)g(z|v)dz) = \dfrac{\int_0^1 \beta(z)g(z|0)dz + \int_0^1 \beta(z)g(z|1)dz}{2}$

Proposition3: Expected revenue of second-price auction with N bidders as follows: $\frac{\frac{N\alpha^{N-1}(1-\alpha)^{N-1}(1-2\alpha)}{\alpha^N+(1-\alpha)^N}+1}{2}$. As the number of bidders increase, expected revenue of the seller increases in second-price auction. When N tends to ∞, seller extracts full expected surplus in the second-price auction.



Chapter 4

OPTIMAL POSTED-PRICE MECHANISM

One alternative selling mechanism for the auction mechanism is posted-price mechanism. In this mechanism, for given model environment, seller as a mechanism designer, sets a fixed price p and asks bidders if their signal is Low or High. Seller also sets probabilities to sell the object conditional on the signals that bidders reveal. In the two bidders case suppose seller assigns q_1, q_2, q_3 to show the probabilities for selling the object respectively if both bidders say Low, if only one bidder says High, and if both bidders say High. Two important concepts should be considered when we design such mechanism which are individual rationality and incentive compatibility. For the former one, a bidder should not end up with negative expected payoff, because he could prefer not to participate and just get zero. For the latter one, seller as a mechanism designer wants to set an optimal posted price p such that no bidder could have profitable deviation by deviating from his true signal. Moreover, when seller decides to sell the object she chooses a random bidder and sell the object to this bidder at the fixed posted price p. When a winner is announced, since winner is randomly chosen not the highest bidder, this announcement will bring no new information to the winner.By doing so, we can eliminate so-called Winner's Curse. One more thing is that bidders and seller both fully commit to the mechanism. To give an example again in two bidders case, suppose seller said that he sells the object if both bidders say High and won't sell the object in other cases. In this case if the seller set the price too low, then the bidder who gets a Low signal might want to say that his signal is High because by doing so he could increase his probability of getting the object and the expected utility. Conversely, sup-

pose that the seller set the price too high, then the bidder who gets a High signal might want to say that his signal is Low because by doing so he could get rid off paying too high price and could increase his utility. We will look at the optimal posted price mechanism that maximizes expected revenue of the seller by considering the individual rationality and the incentive compatibility conditions. After finding the optimal posted price mechanism we will compare the revenue from optimal posted price mechanism with the revenue from second-price auction. After comparing the posted-price mechanism and the second price auction for $N = 2$ and $N = 3$ bidders, we will try to investigate the asymptotic behaviour of these two mechanisms. That is, when the number of bidders goes to infinity, how would the expected revenue change in these two mechanisms? We will also investigate which mechanism provides more expected revenue for the seller. At chapter 7, we will give another simple posted mechanism in which we will be aiming to compare posted-price mechanism and auction mechanism asymptotically.
Now, for a given model environment we can give the formal representation of posted price mechanism.

- Seller sets $q : \mathcal{S} \to \{0,1\}$ such that $q(s) = q(\underbrace{H, H, .., H}_{\text{j-many}}, \underbrace{L, L, .., L}_{(N-j)\text{-many}}) = q_{j+1} \in \{0,1\}$ as a selling probability of the object conditional on the signals that bidders state. , seller **fix the price** as p and when he sells the object, he picks a **random bidder** and sells to him.
- By **Revelation Principle** we can work over the direct mechanism which can be stated as follows: A **direct posted price mechanism** $(\mathcal{Q}, \mathcal{M})$ consists of a pair of functions.
  $\mathcal{Q} : \mathcal{S} \to \Delta$ and $\mathcal{M} : \mathcal{S} \to \mathcal{R}^N$ where
- $\mathcal{Q}_i(s)$ is the probability that bidder i will get the object and $\mathcal{Q}_i(s) = \dfrac{q(s)}{N}$
- $\mathcal{M}_i(s)$ is the expected payment that bidder i will pay and $\mathcal{M}_i(s) = p\mathcal{Q}_i(s)$

- $m_i^{s_i}(z_i) = \sum_{s_{-i} \in \mathcal{S}_{-i}} \mathcal{M}_i(z_i, s_{-i}) Pr(\mathcal{S}_{-i} = s_{-i} | \mathcal{S}_i = s_i)$ is the expected payment of bidder i when he gets signal s_i and reports z_i and all other bidders report truthfully.
- Since the signals are not independent $m_i^{s_i}(z_i)$ depends on the signal that bidder gets. So we can define $m_i^L(z_i)$ and $m_i^H(z_i)$ as the expected payment of bidder i when he gets Low signal and reports z_i and when he gets High and reports z_i, respectively. So expected payment of bidder i when he reports truthfully is equal to the following
$Pr(\mathcal{S}_i = L) m_i^L(L) + Pr(\mathcal{S}_i = H) m_i^H(H) = \dfrac{m_i^L(L) + m_i^H(H)}{2}$
- **Expected Revenue** of the seller N times above equation which is equal to
$$\frac{N(m_i^L(L) + m_i^H(H))}{2}$$
- The expected payoff of bidder i when he gets signal s_i and he reports z_i, assuming that all other bidders report truthfully, is as follows:
$U_i^{s_i}(z_i) = \sum_{s_{-i} \in \mathcal{S}_{-i}} \mathcal{Q}_i(z_i, s_{-i})(E[V|\mathcal{S} = s] - p) Pr(\mathcal{S}_{-i} = s_{-i} | \mathcal{S}_i = s_i)$
- The direct posted-price mechanism $(\mathcal{Q}, \mathcal{M})$ is said to be Incentive Compatible(IC) if $\forall$ i, $\forall s_i$ and $\forall z_i$, $U_i^{s_i}(s_i) \geq U_i^{s_i}(z_i)$.
- The direct posted-price mechanism $(\mathcal{Q}, \mathcal{M})$ is said to be Individually Rational(IR) if $\forall$i and $\forall s_i$ the equilibrium expected payoff $U_i^{s_i}(s_i) \geq 0$.
  Now we can state the seller's problem.

Seller's Problem

$$\underset{\{p,q_1,q_2,..,q_{N+1}\}}{\text{Max}} \frac{N(m_i^L(L)+m_i^H(H))}{2} \quad \text{subject to}$$

$$\text{U}_i^H(H) \geq U_i^H(L) \text{ (IC-H)}$$

$$\text{U}_i^H(H) \geq 0 \text{ (IR-H)}$$

$$\text{U}_i^L(L) \geq U_i^L(H) \text{ (IC-L)}$$

$$\text{U}_i^L(L) \geq 0 \text{ (IR-L)}$$

4.1 Optimal Posted Price Mechanism with $N = 1$ bidder

Suppose there is only one bidder, and the seller assigns q_1 probability to sell the object if the bidder say Low and q_2 probability to sell the object if the bidders say High and sets the price to p that is constant. What is the optimal posted price mechanism in this environment in which the seller also takes the individual rationality and the incentive compatibility into the consideration. We have two Individual Rationality Conditions and two Incentive Compatibility Conditions which are as follows:

Incentive Compatibility Condition-L:

$(q_1 - q_2)(1 - \alpha - p) \geq 0$

Individual Rationality Condition-L:

$q_1(1 - \alpha - p) \geq 0$

Incentive Compatibility Condition-H:

$(q_2 - q_1)(\alpha - p) \geq 0$

Individual Rationality Condition-H:

$q_2(\alpha - p) \geq 0$

Now we can state the seller's problem as the following:

$$\underset{\{p,q_1,q_2\}}{\text{Max}} \frac{(q_1+q_2)p}{2} \quad \text{subject to}$$

$$(\text{q}_1 - q_2)(1-\alpha-p) \geq 0 \text{ (IC-L)}$$

$$\text{q}_1(1-\alpha-p) \geq 0 \text{ (IR-L)}$$

$$(\text{q}_2 - q_1)(\alpha-p) \geq 0 \text{ (IC-H)}$$

$$\text{q}_2(\alpha-p) \geq 0 \text{ (IR-H)}$$

We have four cases:

Case1:$q_1 = q_2 = 0$

Expected Revenue=0

Case2:$q_1 = 0, q_2 = 1$

From (IC-L) $p \geq 1-\alpha$ and from (IC-H) $p \leq \alpha$. $p = \alpha$ and $q_2 = 1$ is the optimal mechanism.

Expected Revenue=$\frac{\alpha}{2}$

Case3:$q_1 = 1, q_2 = 0$

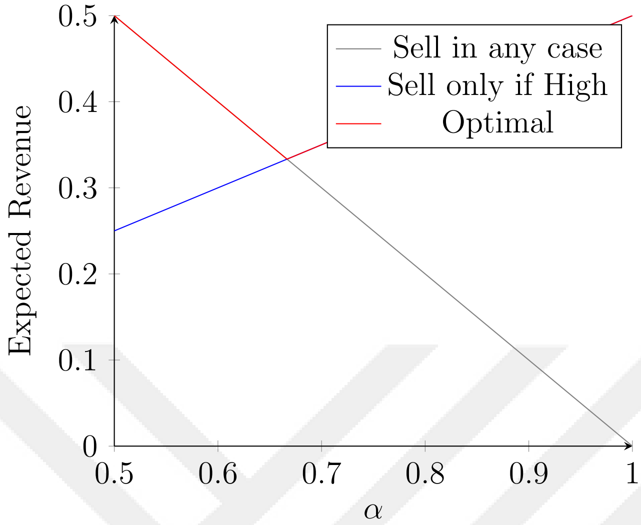
From (IC-L) $p \leq 1-\alpha$ and from (IC-H) $p \geq \alpha$, which is not possible.

Case4:$q_1 = q_2 = 1$

From (IR-L) $p \leq 1-\alpha$. $q_1 = q_2 = 1$ and $p = 1-\alpha$ is the optimal mechanism.

Expected Revenue=$(1-\alpha)$

Proposition4: When $\alpha \geq \frac{2}{3}$, the optimal mechanism is to sell the object at price $p = \alpha$ only if bidder says High, when $\alpha < \frac{2}{3}$ the optimal mechanism is to sell the good at price $p = 1 - \alpha$ regardless of the situation.



4.2 Optimal Posted Price Mechanism with $N = 2$ bidders

Now suppose there are two bidders and seller wants from bidders to reveal their signals. According to these signals that bidders said suppose the seller sets q_1, q_2, and $q_3$probability to sell the object respectively if both says Low, if one of them says Low and other says High, and if both says High. What are the optimal selling probabilities and price in this environment in which the seller also takes the individual rationality and the incentive compatibility into the consideration again.

Now we can state the seller's problem as the following:

$$\underset{\{p,q_1,q_2,q_3\}}{\text{Max}} p[(\alpha^2 + (1-\alpha)^2)\frac{(q_1+q_3)}{2} + 2\alpha(1-\alpha)q_2] \quad \text{subject to}$$

$$(q_2 - q_1)[\alpha(1-\alpha)(1-2p)] + (q_3 - q_2)[\alpha^2 - p(\alpha^2 + (1-\alpha)^2)] \geq 0 \text{ (IC-H)}$$

$$q_2[\alpha(1-\alpha)(1-2p)] + q_3[\alpha^2 - p(\alpha^2 + (1-\alpha)^2)] \geq 0 \text{ (IR-H)}$$

$$(q_1 - q_2)[(1-\alpha)^2 - p(\alpha^2 + (1-\alpha)^2)] + (q_2 - q_3)[\alpha(1-\alpha)(1-2p)] \geq 0 \text{ (IC-L)}$$

$$q_1[(1-\alpha)^2 - p(\alpha^2 + (1-\alpha)^2)] + q_2[\alpha(1-\alpha)(1-2p)] \geq 0 \text{ (IR-L)}$$

Case1:p$\geq \dfrac{\alpha^2}{\alpha^2+(1-\alpha)^2}$

$From(IR-1)$,q$_1 = q_2 = 0$.So, $q_3 = 1$ and $p = \dfrac{\alpha^2}{(\alpha^2+(1-\alpha)^2)}$ is the optimal mechanism.

Expected Revenue:$\dfrac{\alpha^2}{2}$

Case2:$\dfrac{1}{2} < p < \dfrac{\alpha^2}{\alpha^2+(1-\alpha)^2}$

$From(IR-1)$,q$_1 = q_2 = 0$. So, $q_3 = 1$ is the optimal mechanism.

$ExpectedRevenue : Lessthan\dfrac{\alpha^2}{2}$

Case3:$\dfrac{(1-\alpha)^2}{\alpha^2+(1-\alpha)^2}< p \leq\dfrac{1}{2}$

$From(IR-1)$,q$_1 = 0$. So $q_2 = q_3 = 1$ and $p = 1/2$ is the optimal mechanism.

Expected Revenue:$\dfrac{\alpha^2+(1-\alpha)^2}{4} + \alpha(1-\alpha)$

Case4:p$\leq \dfrac{(1-\alpha)^2}{\alpha^2+(1-\alpha)^2}$

q$_1 = q_2 = q_3 = 1$ is the optimal mechanism that satisfies all conditions.

Expected Revenue:$\dfrac{(1-\alpha)^2}{\alpha^2+(1-\alpha)^2}$

Proposition 5: When there are two bidders, if α is close to 1, then optimal posted price mechanism is to sell the object at price $p = \dfrac{\alpha^2}{(\alpha^2+(1-\alpha)^2)}$whenever both bidders say High and if α is close to $\dfrac{1}{2}$ optimal is selling the object at price $p = \dfrac{(1-\alpha)^2}{(\alpha^2+(1-\alpha)^2)}$ regardless of the signals .

4.3 Optimal Posted Price Mechanism with $N = 3$ bidders

Now suppose there are three bidders and seller wants from bidders to reveal their signals. According to these signals that bidders stated suppose the seller sets q_1, q_2, q_3 and q_4 probabilities to sell the object respectively if no one says High, if only one of them says High, if two of them say High and if all of them says High. What are the optimal selling probabilities and price in this environment in which the seller also takes the individual rationality and the incentive compatibility into the consideration again.

Now we can state the seller's problem as the following:

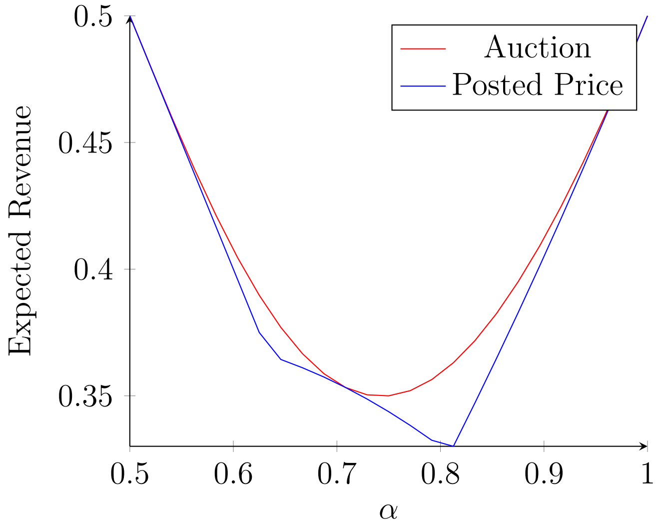
$$\underset{\{p,q_1,q_2,q_3,q_4\}}{\text{Max}} \frac{p}{2}[(\alpha^3+(1-\alpha)^3)(q_1+q_4)+3\alpha(1-\alpha)(q_2+q_3)]$$

subject to

$$(\mathrm{q}_4-q_3)[\alpha^3-p(\alpha^3+(1-\alpha)^3)]+(q_3-q_2)[2\alpha(1-\alpha)(\alpha-p)$$
$$+(q_2-q_1)[\alpha(1-\alpha)(1-\alpha-p)+\geq 0 \text{ (IC-H)}$$
$$\mathrm{q}_4[\alpha^3-p(\alpha^3+(1-\alpha)^3)]+q_3[2\alpha(1-\alpha)(\alpha-p)$$
$$+q_2[\alpha(1-\alpha)(1-\alpha-p)+\geq 0 \text{ (IR-H)}$$
$$(\mathrm{q}_1-q_2)[(1-\alpha)^3-p(\alpha^3+(1-\alpha)^3)]+(q_2-q_3)[2\alpha(1-\alpha)(1-\alpha-p)$$
$$+(q_3-q_4)[\alpha(1-\alpha)(\alpha-p)+\geq 0 \text{ (IC-L)}$$
$$\mathrm{q}_1[(1-\alpha)^3-p(\alpha^3+(1-\alpha)^3)]+q_2[2\alpha(1-\alpha)(1-\alpha-p)+$$
$$q_3[\alpha(1-\alpha)(\alpha-p)+\geq 0 \text{ (IR-L)}$$

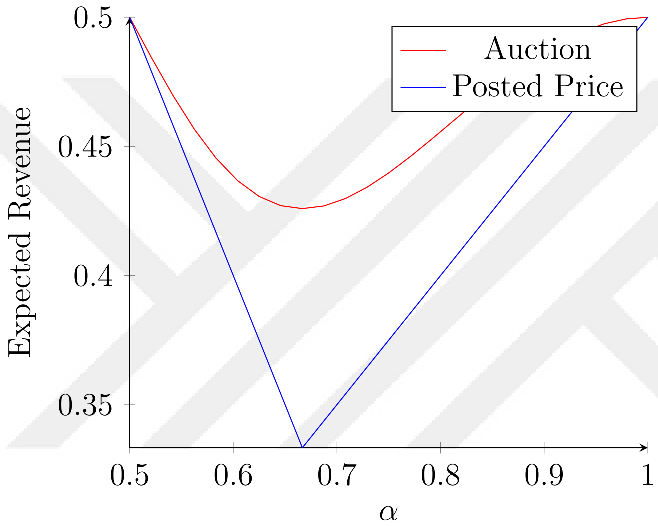
4.4 Revenue Comparison of Auction and Optimal Posted Price Mechanism

Observation 1: When there are only 2 bidders, second-price auction always provides higher expected revenue for the seller than posted-price mechanism.



Above figure shows that although expected revenues of the auction and posted price mechanism go hand in hand, auction mechanism gives weakly higher payoff for any value of α. When we increase the number of bidders as in the below Observation 2 auction mechanism keeps giving higher expected revenue than posted price mechanism.

Observation 2: When there are 3 bidders, second-price auction always provides higher expected revenue for the seller than posted-price mechanism.



Chapter 5

ALTERNATIVE POSTED-PRICE MECHANISM AND ASYMPTOTIC ANALYSIS

One alternative posted price mechanism could be as follows: Seller announces that she will sell the object if and only if share of the High signal bidders amoung all bidders is greater or equal to α, which is known by both bidders and seller. That is, if $q(s)$ represents selling probability of the object conditional on the signals bidders stated, $q(s)$ is 1 if share of the High signal bidders amoung all bidders is greater or equal to α, otherwise $q(s)$ is 0.

$$q(s) = \begin{cases} 1, \frac{\#H}{N} \geq \alpha \\ 0, \frac{\#H}{N} < \alpha \end{cases}$$

What would be the optimal posted price in that selling environment?

Claim: The following (p, q) where $p = Pr(V = 1 | \lceil N\alpha \rceil$ [1]$High)$ and

$$q(s) = \begin{cases} 1, \frac{\#H}{N} \geq \alpha \\ 0, \frac{\#H}{N} < \alpha \end{cases}$$

produces **individually rational** and **incentive compatible** mechanism.

Proof of Claim:

i) Let us first show that this mechanism is incentive compatible.

For the low type, he knows that(by considering all other bidders report truthfully) if he says High, the object will be sold with some positive probability $r > 0$ and otherwise seller will keep the object. That is we are just look-

[1]This is the ceiling function

ing at the point that this type is **pivotal**. First of all $r > 0$ because for any (α, N) pair there exists a threshold such that selling probability change with the pivotal guy with some positive probability. To give an example, suppose $N = 10$ and $\alpha = 0.6$, seller says that she will sell the object if and only if there are 6 High. In that environment, if one thinks that amoung 9 people ,except himself, with some positive probability 5 of them reports High and he is the pivotal with that probability. That is selling decision will change with his statement. If he says High, seller will sell otherwise seller will keep the object. In all other cases this guy's decision will not change the selling probabilities. That is saying Low or High will be the same for him. When he says High, the object will be sold with $r > 0$ probability and with $\frac{r}{N}$ probability he will pay for that object. At that point his expectation is $\mathbb{E}[V|\lceil N\alpha\rceil - 1High] = Pr(V = 1|\lceil N\alpha\rceil - 1High)$ but the price he is going to pay is $p = Pr(V = 1|\lceil N\alpha\rceil\, High)$.

Since $p = Pr(V = 1|\lceil N\alpha\rceil\, High) > Pr(V = 1|\lceil N\alpha\rceil - 1High)$, he will not report High when he gets Low signal.

Similarly, for High type, he will be the pivotal with some $s > 0$ probability. For all other cases reporting Low or High will be the same for him. So when he gets High signal, if he reports High he will get the object $\frac{s}{N} > 0$ probability. At that point his expectation is $\mathbb{E}[V|\lceil N\alpha\rceil\, High] = Pr(V = 1|\lceil N\alpha\rceil\, High)$ and the price he is going to pay $p = Pr(V = 1|\lceil N\alpha\rceil\, High)$.

Assume we resolve indiferences to increase the selling probability. Under this assumption, High type will also report truthfully.

ii) Now let us show that this mechanism is individually rational.

For the low type, when he reports low, in the case of the object is sold his expectation about the object is at least $Pr(V = 1|\lceil N\alpha\rceil\, High)$.So he will never get negative expected payoff. For the high type, again conditional on the object is sold he never has an expectation less than $p = Pr(V = 1|\lceil N\alpha\rceil\, High)$.

Expected Revenue for the seller can be calculated as follows:
Pr(Selling the object)xPrice=$Pr(\frac{\#H}{N} \geq \alpha)xPr(V = 1|\lceil N\alpha\rceil\ High)$

Observation 3: Seller gets the full surplus when number of bidders tends to infinity conditional on that object is sold. That is price, p, converges to 1 when N goes to infinity.

Proof of Observation 3: $p = Pr(V = 1|\lceil N\alpha\rceil\ High) = \frac{\alpha^{2\lceil N\alpha\rceil - N}}{\alpha^{2\lceil N\alpha\rceil - N} + (1-\alpha)^{2\lceil N\alpha\rceil - N}}$

Since $\alpha > 1/2$, $\frac{1-\alpha}{\alpha} < 1$ and for a given α, when N tends to infinity $2\lceil N\alpha\rceil - N$ goes to infinity as well and p converges to 1.

Proposition 6: Expected revenue from that mechanism converges to the 1/4 when number of bidders tends to infinity.

Proof of Proposition 6:$Pr(\frac{\#H}{N} \geq \alpha) = \frac{Pr(\frac{\#H}{N} \geq \alpha|V=1) + Pr(\frac{\#H}{N} \geq \alpha|V=0)}{2}$. By Law of Large Numbers, $Pr(\frac{\#H}{N} \geq \alpha|V = 1)$ converges to $\frac{1}{2}$ and $Pr(\frac{\#H}{N} \geq \alpha|V = 0)$ converges to 0. So, $Pr(\frac{\#H}{N} \geq \alpha)$ converges to $\frac{1}{4}$. By using Observation 3 we can conclude that expected revenue from that mechanism converges to the $\frac{1}{4}$.

Chapter 6

CONCLUSION

If we are working under the common value environment, we need to take Winner's Curse into the consideration. In the auction mechanism, this may decrease the expected revenue of the seller remarkably. In the main paper[Bergemann et al., 2020] which gave us insight about the posted price mechanism, the authors provided that in the maximum signal model, simple posted price mechanism can beat the auction mechanism. By following their insight we also aimed to find an optimal posted price mechanism that also beats auction mechanism in the different discrete common value setup. However, our results show that in our environment second price auction provides higher expected revenue for the seller when the number of bidders is relatively small. When number of bidders tends to infinity, in order to be able to compare auction and posted price mechanism we offered an alternative posted price mechanism in which we guaranteed to be able to sell the object with the almost highest price in the true state. However, when we calculate the expected revenues, auction mechanism still provided more revenue for the seller. One further analysis would be to compare convergence rates of the prices of our alternative mechanism and auction mechanism.

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