

A NEW APPROACH TO THE JORDAN BRANS DICKE COSMOLOGY

by

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ABSTRACT

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Based on the idea that the components of a cosmological metric may be determined by the total gravitational potential of the universe, the scalar field $\phi = 1/G$ in the Jordan-Brans-Dicke (JBD) theory is introduced as evolving with the inverse square of the scale factor. Since the gravitational potential is related to the field ϕ resulting from Mach's principle and depends on time due to the expansion of space, the temporal evolution of the field should be in accord with the evolution of time and space intervals in the metric tensor. For the same reason, the time dependence of the field makes these comoving intervals relative for different points on the time axis. Thus, it is shown that introduction of the cosmic gravitational potential as a time dependent scalar field proportional to $1/a^2$ may resolve the flatness, the horizon and the late-time accelerating expansion problems of the standard model of cosmology. The luminosity distance vs redshift data of Type Ia supernovae is in agreement with this approach. In addition to these, after finding the first order field equation and the Boltzmann equations for the metric, which is coordinate transformed, we first focus on cosmological applications such as the neutron abundance, the helium fraction and the redshift of the recombination in our model. Then, the equations that show how radiation and matter perturbations evolve over time in different epochs of the universe are obtained.

ÖZET

JORDAN BRANS DICKE KOZMOLOJİSİNE YENİ BİR YAKLAŞIM

Kozmolojik bir metrik tensörün bileşenlerinin evrenin toplam kütleçekim potansiyeli tarafından belirlenebileceği fikrine dayanarak, Jordan-Brans-Dicke (JBD) teorisindeki $\phi = 1/G$ skaler alanının zamanla gelişimini ölçek faktörün ters karesi ile orantılı olarak sunuyoruz. Kütleçekim potansiyeli Mach ilkesinden dolayı ϕ alanıyla ilişkili olduğundan ve uzayın genişlemesi nedeniyle zamana bağımlı olduğundan, alanın zamansal evrimi, metrik tensördeki zaman ve uzay aralıklarının evrimiyle uyumlu olmalıdır. Aynı sebepten dolayı alanın zamana bağımlılığı, bu eş-hareketli çerçevedeki zaman ve uzay aralıklarını zaman eksenindeki farklı noktalar için göreceli hale getirir. Böylece, kozmik kütleçekim potansiyelinin $1/a^2$ ile orantılı olarak zamana bağlı bir skaler alan olarak dahil edilmesinin, standart kozmolojinin düzlük, ufuk ve geç zamandaki hızlanan genişleme sorunlarını çözebileceği gösterilmiştir. Tip Ia süpernovalarının parlaklık mesafesi ve kırmızıya kayma verileri bu yaklaşımla uyumludur. Ek olarak birinci dereceden alan denklemi ve koordinat dönüşümüyle elde edilen yeni metrik tensör için Boltzmann denklemleri bulunduğundan sonra öncelikle modelimizde nötron bolluğu, helyum fraksiyonu ve rekombinasyon döneminin kırmızıya kayma değeri gibi kozmolojik uygulamalara odaklanıyoruz. Daha sonra evrenin farklı dönemlerinde radyasyon ve madde tedirginliklerinin zaman içinde nasıl evrildiğini gösteren denklemler elde ediliyor.

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LIST OF SYMBOLS

a	Scale factor
a_{eq}	Scale factor of the radiation matter equality
a_*	Scale factor of the recombination
A	Comoving surface area
c	Speed of light
c_s	Speed of sound
$C[f]$	Collision term in the Boltzmann equations
E	Energy
f	Distribution function
F	Flux
$\vec{F}$	Force vector
g_s	Degeneracy number
$g(\eta)$	Visibility function
$g_{\mu\nu}$	Metric tensor
G	Gravitational constant
$G_{\mu\nu}$	Einstein tensor
h	Dimensionless equivalent of the Hubble parameter
H	Hubble parameter in the standard cosmology
$\bar{H}$	Hubble parameter in our model
I_n	Modified Bessel function of the first kind in order n
j_l	Spherical Bessel function
k	Wavenumber
k_D	Wavenumber of the damping scale
K_n	Modified Bessel function of the second kind in order n
L	Luminosity
$\mathcal{L}$	Lagrangian density
m	Apparent magnitude of a supernova
m_s	Mass of a type of particle

M	Observable universe mass <i>or</i> supernova absolute magnitude
$\mathcal{M}$	Scattering amplitude
n_s	Number density of a type of particle
p^i	Momentum components
P	Pressure
P_l	Legendre polynomials
P^μ	Four Momentum
r	Radial coordinate
r_s	Comoving sound horizon
R	Ricci scalar
$R_{\mu\nu}$	Ricci tensor
S	Action <i>or</i> the source function
$S_\kappa(\chi)$	Metric function depending on the curvature of space
t	Time coordinate
T	Temperature
$T_{\mu\nu}$	Energy momentum tensor
U_μ	Four velocity
v	Velocity
w	The factor of P/ρ in the equation of state
x^i	Cartesian coordinates
X_e	Free electron fraction
X_n	Neutron fraction
y	Variable defined as a/a_{eq}
Y_P	Helium mass fraction
z	Redshift
α	Coupling parameter for the zeroth order solutions
α_{QED}	Fine structure constant
β	Coupling parameter for the first order solutions
$\Gamma_{\mu\nu}^\lambda$	Christoffel symbol
δ	Matter perturbations

δ_D	Dirac delta function
ϵ	Ratio α/β
η	Conformal time
η_b	Ratio of baryon number to photon number
Θ	Temperature fluctuations
Θ_l	Moments of temperature fluctuations
κ	Spatial curvature parameter in the metric tensor
λ	Wavelength
Λ	Cosmological constant
μ	Distance modulus <i>or</i> the angle between two unit vectors
μ_s	Chemical potential of a type of particle
ρ	Energy density
σ	Cross-section
σ_T	Thomson cross-section
τ	Optical depth
ϕ	Jordan Brans Dicke scalar field
Φ	Perturbation to the spatial curvature in the metric tensor
ψ_i	Matter fields in the action
Ψ	Perturbation to the temporal component of the metric tensor
ω	Jordan Brans Dicke coupling constant
Ω	Energy density parameter

LIST OF ACRONYMS/ABBREVIATIONS

BBN	Big Bang Nucleosynthesis
CMB	Cosmic Microwave Background
FLRW	Friedmann Lemaitre Robertson Walker
ISW	Integrated Sachs Wolfe
JBD	Jordan Brans Dicke
LSS	Large Scale Structures
Λ CDM	Lambda Cold Dark Matter

1. INTRODUCTION

The standard model of cosmology is based upon the Einstein field equation that can be derived from the Einstein-Hilbert action as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (1.1)$$

While the left hand side of the equation is determined by the geometry of space-time or simply by the Einstein tensor $G_{\mu\nu}$, the right hand side is designated by the energy-momentum tensor $T_{\mu\nu}$ that is given by

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + Pg_{\mu\nu}. \quad (1.2)$$

Since the content of the universe is assumed to correspond to a perfect fluid. In Equation (1.1), $R_{\mu\nu}$, R and G stand for the Ricci tensor, the Ricci scalar and Newton's gravitational constant, respectively. Furthermore, in Equation (1.2), ρ , P , U_μ and $g_{\mu\nu}$ denote the energy density, pressure, four-velocity and metric tensor. The metric tensor is a crucial component for the standard cosmology, which is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, in order to calculate the Ricci tensor, Ricci scalar and energy momentum tensor for the comoving frame. The metric tensor should normally be found as the solution of the field equation but in cosmology one uses the assumptions of homogeneity and spatial isotropy of the universe to write the metric tensor phenomenologically. The FLRW metric represents a space-time in which the space expands with the scale factor and the time is as measured for comoving observers. The line element for its general form without any perturbation, which is used to draw a basic picture for the expansion history of the universe in the standard cosmology, is given by

$$ds^2 = -dt^2 + a^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right). \quad (1.3)$$

Here, κ corresponds to the curvature parameter of space, which has the dimension of $length^{-2}$ ($c = 1$ and subscript 0 denotes today's value of any cosmological parameter unless it is stated explicitly throughout the manuscript). Furthermore, κ can be negative, zero and positive for an open, flat and closed universe, respectively. Using the

FLRW metric for the field equation, one can obtain the Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}. \quad (1.4)$$

Here, H is the Hubble parameter and its current value is $H_0 = (67.4 \pm 0.5)$ km/s/Mpc according to the latest observation [1] on the basis of the Λ CDM (Lambda Cold Dark Matter) cosmology. Another fundamental equation, the continuity equation, is derived from the conservation of energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ as

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \quad (1.5)$$

with the solution $\rho(t) = \rho_0 a^{-3(1+w)}$ for an equation of state $P = w\rho$. Based on this solution, while the matter density of the universe is related to the scale factor by $\rho_m \sim \frac{1}{a^3}$, the radiation density is related by $\rho_r \sim \frac{1}{a^4}$, in which the extra factor of $1/a$ comes from the red-shift of photons due to the expansion of space. Photons arriving us today from distant stars and galaxies, are red-shifted by

$$z = \frac{\lambda_{ob} - \lambda_{em}}{\lambda_{em}}, \quad (1.6)$$

and the relation between the scale factor and the red-shift is

$$a(t) = \frac{1}{1+z}. \quad (1.7)$$

Measuring the red-shift of photons and the luminosity distance of standard candles like Type Ia supernovae, it is possible to determine how fast the expansion of the space is, and to make inferences about essential quantities such as the age and the energy content of the universe. In the standard model, the luminosity distance [2] is given by

$$d_L(z) = (1+z) \frac{H_0^{-1}}{\sqrt{|\Omega_{\kappa 0}|}} S_\kappa \left(\sqrt{|\Omega_{\kappa 0}|} \int \frac{dz'}{E(z')} \right), \quad (1.8)$$

where $\Omega_{\kappa 0}$ is today's value of the curvature density parameter, S_κ is a function for different spatial curvatures ($S_\kappa(\chi)$ is $\sin(\chi)$, χ and $\sinh(\chi)$ for a closed, flat and open universe, respectively) and $E(z') = [\Omega_{m0}(1+z')^3 + \Omega_{r0}(1+z')^4 + \Omega_{\kappa 0}(1+z')^2]^{1/2}$ that stands for all types of energy contribution.

In the last two decades, it has been unveiled that the expansion rate predicted by the standard model does not fit the data of Type Ia supernovae [3,4] when only the

contributions of matter and radiation are taken into account. According to the data the expansion is accelerating and the problem can be solved by adding some exotic constituents like dark energy. The cosmological constant Λ , one of the candidates, has been introduced into the action. However, this type of solution brought its own problems such as the cosmic coincidence [5] and the cosmological constant [6, 7] problems. In the former one, with "coincidence", it is meant that the era in which the cosmological constant dominates the evolution of the universe matches up with the period of time we started to make observations. Second, the cosmological constant is many orders of magnitude smaller than the value estimated in particle physics. Even though some approaches [6], [8] have been suggested, they were found inconclusive. So, the problems still remain unsolved.

To explain the late-time accelerating expansion, alternative gravity theories like scalar-tensor theories in which a scalar field is non-minimally coupled to the geometry of space-time or to matter fields in the action [9] (depending on the frame chosen, the Jordan frame or the Einstein frame, respectively) have been proposed. In the Jordan-Brans-Dicke (JBD) theory [10, 11], which is one of the favored scalar-tensor theories, a scalar field has been introduced with a motivation based on Mach's principle stating that inertial forces on a body are originating from the gravitational effects of matter distribution in the universe.

The following relation, where the only time dependent parameter in expanding space is the radius of the observable universe R (this is a finite distance regardless of the spatial curvature) contains a connection between the gravitational potential of the universe and natural constants G and c like

$$\frac{GM}{Rc^2} \sim 1. \quad (1.9)$$

Here, M is the finite mass of the observable universe. According to Brans and Dicke, this relation is not a coincidence and they argued that there should be at least one other time dependent parameter so that Equation (1.9) has an explanation. In their opinion, the most suitable one is the gravitational constant G , and the relation (by

taking $c = 1$ again) is

$$\frac{1}{G} = \phi \sim \frac{M}{R}. \quad (1.10)$$

Since the JBD theory was first introduced in 1961, it has been studied for many different cases in numerous ways. Considering that it tried to fully integrate Mach's principle into general relativity in the years it was first put forward, it can be seen as a fundamental correction to the existing theory. Over the years, it has become one of the modified gravity theories applied to explain the cosmic puzzles, in particular, the accelerating expansion in the late times of the universe. In general, the solution requires a potential term for the self-interacting JBD scalar field to be added in the action. Thus, different extensions of the theory have emerged [12–17]. One of the common points of such approaches is that the JBD coupling parameter should be negative in order to fit the data. Although it has been deduced from solar system observations that this parameter must be a positive large number, its value on cosmic scales does not have to be necessarily the same. In other words, it is a viable scenario for the JBD coupling parameter to be scale dependent. Since the expansion of space is ignored in local observations, the solar system is a system that has reached equilibrium in this sense. However, this is not the case on cosmic scales.

The late-time accelerating expansion of the universe is not the only problem with the standard model of cosmology. The flatness and horizon problems are almost equally important and the theory of inflation [18, 19] is referred to as a solution to these two. Although the standard inflation theory has its own problems [20–24], it is also necessary to explain the formation of the Large Scale Structures (LSS) and the Cosmic Microwave Background (CMB) anisotropy spectrum in the Λ CDM cosmology which is the current concordance model. Therefore, it is also possible to see the applications of the JBD theory as extensions of the standard theory of inflationary expansion [25–32] in the early universe. Although such approaches solve the problems of inflation theory up to a point, the search for a model that solves all three problems and includes a specific JBD scalar field that evolves consistently between the two periods of time

continues (because the horizon and flatness problems are related to the early universe, while the accelerated expansion concerns the late times). The constraints, which have been put on the evolution of the JBD scalar field and the JBD coupling parameter by fitting the standard JBD theory and its extensions with various potential terms to the Type Ia supernova, the CMB, the Baryon Acoustic Oscillation (BAO) and other cosmological data sets (or the combinations of these), can be found in the literature [33–39]. However, since the model we come up with is a distinctive modification of the JBD theory rather than just adding a potential term and the observational tests mentioned above are model dependent, one should not question the validity of our model on the basis of these constraints.

The aim of this thesis is to solve the three basic problems at once, by modifying the JBD theory without using any potential term and including the inflationary expansion. Our approach starts with the same relation in Equation (1.9) but what we’re trying to do is quite different from the JBD theory extensions aforementioned. In addition, since there will be no need for the dark energy in the late times and the inflationary expansion in the early universe, our model is free from their problems. This is a more fundamental modification with a relation assumed to exist between the JBD scalar field and the metric tensor. Although we use the standard JBD action in this new model, the field equation will be different due to the assumption made based on some motivations. In the next chapter, the FLRW metric will be coordinate transformed for a reason that will be explained in detail. In Chapter 3, the field equation of the modified JBD theory will be obtained. Then, in Chapter 4, our main focus will be presenting possible solutions to these basic problems of the standard model of cosmology thanks to this modification. In Chapter 5, we will obtain the first order field equation, and in Chapter 6, we will obtain the Boltzmann equations for the coordinate transformed metric tensor. Finally, in Chapter 7, we will mention some applications of our model in cosmology and how matter, radiation and potential perturbations evolve over time.

2. TRANSFORMED FLRW METRIC

As a starting point, we consider the perturbed metric due to the presence of a gravitational field in a non-expanding space. Its most general form is given by

$$ds^2 = -(1 + 2\Psi)dt^2 + (1 + 2\Phi)(dr^2 + r^2d\Omega^2). \quad (2.1)$$

Ψ and Φ correspond to the Newtonian gravitational potential and the perturbation to the spatial curvature, respectively. Furthermore, due to the weak field limit, perturbations on time and space can be set as $\Phi = -\Psi$. The same relation is also valid for a cosmological scenario. Since radiation strongly coupled to matter (coupling of the relativistic particles to the baryons) in the radiation dominated era and the energy density of radiation can be ignored in the matter dominated era, radiation and matter behave like a fluid. So, one can safely disregard the anisotropic stress in both epochs. Thus, the perturbed FLRW metric takes the same form in the Newtonian gauge, except for the scale factor in front of its spatial part.

On the metric tensor in Equation (2.1), the effect of gravitational potential term is quite apparent as perturbation. It alters time and space intervals for a space-time point by reshaping the metric components. These intervals are relative and dependent on the magnitude of the potential. In the same manner, the total gravitational potential of the universe may affect the cosmological metric. In the FLRW universe, space is expanding physically with time and there is only one type of time passing with the same pace since the very beginning of the Big Bang. However, we live in a universe full of matter. So, in a way, the cosmological space-time intervals in the metric should be formed by a zeroth order potential term originating from the matter distribution. Since this potential term changes with time due to the expansion of space, our relative space and time intervals should also change according to any other observer standing on a different point on the time axis. So, there should be two kinds of time intervals. One is comoving and the other one is physical time just as for spatial intervals. One can perform a coordinate transformation to get back into the FLRW space-time in which

a physical time coordinate is used, but a few arguments in the following section will be presented about why one should keep using the comoving one. In addition, if the matter distribution is assumed to be homogeneous at zeroth order, the gravitational potential and the related JBD scalar field should be independent of spatial coordinates.

To explain the relation in Equation (1.9), the JBD theory proposes that the gravitational constant G is a time dependent parameter as mentioned in the introduction. There exist research papers [36], [40–54] providing constraints on the possible variation of G with time based on different observations. It is also known from the Shapiro delay [55] that the speed of light seems to slow down due to the gravitational potential near an astrophysical object in the coordinate system of an observer that is infinitely far away from the object. For a cosmological scenario in which the gravitational potential evolves with time, hypothesizing the speed of light to be time dependent would lead to inconsistencies. However, it can safely be chosen as relatively changing among coordinate systems. In this way, while the speed of light is a universal constant for all observers in their comoving coordinates, its value relatively changes (similar to the case of Shapiro delay) for observers separated by time. The future and the past light cones are different than those in the standard FLRW metric. This change in light cones results from relative cosmological space-time intervals. By taking the arguments and the observation mentioned above into consideration, we select $1/G = \phi \sim M/R$ as in the JBD theory, i.e. Equation (1.10). Thus, the perturbed metric in Equation (2.1) can be put in the following form by ignoring higher order terms

$$ds^2 = - \left(1 - \frac{2\delta\phi}{\phi_0}\right) dt^2 + \left(1 - \frac{2\delta\phi}{\phi_0}\right)^{-1} (dr^2 + r^2 d\Omega^2), \quad (2.2)$$

where $\delta\phi$ represents contributions coming from inhomogeneities and ϕ_0 is the present value of ϕ whose reciprocal stands for G . The metric form in Equation (2.2) is valid for any region and can define a space-time point inside a non-expanding universe. The point of view that must be emphasized, however, is that any such $\delta\phi$ can be regarded as a perturbation normalized with respect to ϕ_0 (i.e., $|\delta\phi| \ll \phi_0$) and it inversely affects time and space intervals.

When we look at any point in space, we are looking back in time which means we are getting photons from an era in which the value of ϕ is bigger than the value today due to the expansion of space. The difference in the field ϕ or the expanding space causes photons arriving us to be red-shifted. For this reason, the time dependent scalar field must be in accord with the red-shift phenomenon i.e., the expansion of space. In addition, in Newtonian mechanics, the gravitational force is given by

$$\vec{F} = -m\vec{\nabla}\phi, \quad (2.3)$$

so the field ϕ may take part in the components of a cosmological metric tensor to be able to define gravity in a purely geometrical way as done by Einstein himself for the general relativity. Therefore, in order to construct a metric for changing time and space intervals, which are determined by the normalization of the scalar field with respect to its value today as in the case of Equation (2.2), it is better to choose $\phi = \phi_0/a^2$ to embed the time dependence into the metric as the scale factor. Since the scale factor increases with time, ϕ will be decreasing correspondingly. Incidentally, once the equation of ϕ and the solutions to the scale factor for different epochs are obtained in the next section, it will be shown that our choice of $\phi = \phi_0/a^2$ is a solution. A zeroth order cosmological metric covering the properties above, after normalization with respect to ϕ_0 , should have the following form that can be called as the coordinate-transformed or shortly transformed FLRW metric

$$ds^2 = -\frac{dt^2}{a^2} + a^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right). \quad (2.4)$$

At this point, it may look like that 00 component of the metric is the only change and a simple coordinate transformation compared to the FLRW metric. It is, however, more than that because the meaning behind the metric components has also been changed. There is no absolute time and space intervals for observers separated by time. They can be measured relatively because of the time dependence of the field ϕ . Length and time intervals do not vary in a way that one is used to think for the FLRW metric. The contraction and the expansion of space-time intervals are what we are experiencing in our comoving coordinate system for the past and the future.

Furthermore, whenever an observation takes place, it should be considered as being performed from a new coordinate system because of change in ϕ . This is a more appropriate and meaningful interpretation for a cosmological metric constructed based on Mach's principle.

By selecting a single coordinate time for all observers at different times in an expanding space, the form of the FLRW metric in Equation (1.3) is written phenomenologically, and the field equation is found by calculating the Ricci tensor elements. Although the FLRW metric has generally been used in this form in cosmology, some criticisms [56–58] about this use and some counter arguments [59] to these criticisms have been expressed recently. The focus of these criticisms is the reduction of g_{00} to -1 using a gauge transformation. In general, while this element of the metric can be any function of coordinate time for a comoving metric which is non-inertial because of non-inertial Hubble flow (if the universe has acceleration and deceleration for its spatial expansion), its elimination by coordinate transformation is against some principles underlying general relativity, even though it seems mathematically feasible. Because with a coordinate transformation, one passes from the comoving frame to the inertial reference frame in which the time dilation is absent. One should not assume to be allowed to use the same form of the energy momentum tensor for the comoving and the inertial frames simultaneously. So, the time dilation effect in the comoving reference frame must be considered for the universe that has non-linear expansion phases.

In the case of inhomogeneous space (Equation (2.2)), light cones shrink a bit depending on the value of $\delta\phi$ in regions, where $\delta\phi$ is bigger by comparison. In the cosmological case, a light cone at any time in the past, where ϕ is bigger, seems much more open in our comoving coordinate system. These two seem to contradict each other since we are considering a space-time point on which the fields are relatively bigger in both cases. However, while $\delta\phi$ changes with the spatial coordinate r in the first case, it is the time coordinate assigning the coordinate system in the latter one. Their opposite effects on the metric components can be explained by the scale-dependent gravity as in Mannheim's conformal gravity [60, 61]. Thus, gravity can be attractive on small

scales but repulsive on cosmic scales and this can be the reason behind this apparent contradiction.

In addition, the scale-dependent gravitational coupling in scalar tensor theories is one of the subjects that have been studied [62–66]. In our model, it is the JBD parameter that will determine how gravity acts on different scales. We will discuss this issue in more detail in the following chapters and we will see that the coupling parameter behaves differently to meet certain conditions in zeroth order and first order calculations. While the condition in the zeroth order cosmology is that the Hubble parameter must be positive, the condition in the latter is to obtain the Poisson equation and thus gravity is attractive. This behavior change of the coupling parameter is one of the frequently encountered results in the literature. While the JBD parameter is negative on cosmic scales, it takes positive and large values on solar system scales, thus reducing the JBD theory to the theory of general relativity. A similar inevitable change in the behavior of the coupling parameter is present in our model too.

3. MODIFIED JBD FIELD EQUATION

Since the FLRW metric has been coordinate-transformed, it is necessary to make all basic calculations for the new metric tensor. At this point, it must be emphasized again that one can always make a coordinate transformation between comoving and physical times with the relation $dt/a(t) = dt'$ to get back to the FLRW metric. Although solutions for both metric tensors will be mathematically equivalent, we continue with the transformed one (one written in comoving time) not to hide the effect of the field ϕ on time intervals. It is better to start with the action that is the same as that of the JBD theory. However, the metric tensor is now a function of the scalar field, which is an important modification. In light of this, the JBD action is written as

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R - \omega g^{\mu\nu}(\phi) \frac{\partial_\mu \phi \partial_\nu \phi}{\phi} + 16\pi \mathcal{L}_M(g_{\mu\nu}(\phi), \psi_i) \right). \quad (3.1)$$

Here, the first term represents the non-minimal coupling of the scalar field to the Ricci scalar (we study in the Jordan frame) while the second one is the kinetic term of the field. The third one is there because of the energy content of the universe coming from different matter fields ψ_i as in any general relativistic Lagrangian density. Since the matter Lagrangian is a function of the Jordan frame metric, test particles follow geodesics set by this metric tensor.

In the standard JBD theory, there are two field equations obtained by varying the action with respect to $g^{\mu\nu}$ and ϕ (involves $\partial_\mu \phi$ variation). In our case, however, there will be a single equation resulting from the variation with respect to ϕ since the metric tensor is also a function of ϕ . Hence, varying the action with respect to ϕ and $\partial_\mu \phi$ yields

$$\delta S = \frac{1}{16\pi} \int d^4x \left(\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \frac{dg^{\mu\nu}}{d\phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta(\partial_\mu \phi) \right). \quad (3.2)$$

For the first term in Equation (3.2), one needs to vary the Lagrangian density with respect to $g^{\mu\nu}$ that is a laborious process. We do not give its details here, but can be found in Appendix A. The rest of the variation process, which is the part involving

with respect to ϕ and $\partial_\mu\phi$, is straightforward. These calculations can be carried out to obtain the field equation as

$$\begin{aligned} \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) \frac{dg^{\mu\nu}}{d\phi} = & \left[\frac{8\pi}{\phi} T_{\mu\nu} + \frac{1}{\phi} (\nabla_\mu \partial_\nu \phi - g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \partial_\beta \phi) \right. \\ & + \frac{\omega}{\phi^2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right) \left. \right] \frac{dg^{\mu\nu}}{d\phi} \\ & - \left(\frac{R}{\phi} + 2\omega g^{\alpha\beta} \frac{\nabla_\alpha \partial_\beta \phi}{\phi^2} - \omega g^{\alpha\beta} \frac{\partial_\alpha \phi \partial_\beta \phi}{\phi^3} \right), \end{aligned} \quad (3.3)$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \quad (3.4)$$

In order to see how the matter and the radiation densities of the universe evolve with time, one should look at the continuity equation. Since test particles move on geodesics determined by the Jordan frame metric, it is obtained by using the transformed FLRW metric for the relation $\nabla_\mu T^{\mu\nu} = 0$ or simply by making a transformation for the time coordinate in Equation (1.5). This equation does not change under coordinate transformation and the continuity equation is still $\dot{\rho} + 3\dot{a}/a(\rho + P) = 0$. So, the matter and the radiation densities evolve with the scale factor as in the case of the FLRW metric. Using $P_m = 0$ for the pressure of ordinary matter (or dust) and $P_r = \rho_r/3$ for the pressure of radiation gives the evolution of the energy densities as

$$\rho_m = \frac{\rho_{m0}}{a^3}, \quad (3.5)$$

$$\rho_r = \frac{\rho_{r0}}{a^4}. \quad (3.6)$$

From the extra $1/a$ factor in the evolution of the radiation density, it is easily seen that the wavelength of a photon is again proportional to the scale factor in the transformed FLRW metric. So, the relation in Equation (1.7) is still valid.

As the next step, the Ricci tensor components and the Ricci scalar for the transformed FLRW metric are calculated as

$$R_{00} = -3 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right), \quad (3.7)$$

$$R_{11} = \frac{\ddot{a}a^3 + 3\dot{a}^2a^2 + 2\kappa}{1 - \kappa r^2}, \quad (3.8)$$

$$R_{22} = r^2(\ddot{a}a^3 + 3\dot{a}^2a^2 + 2\kappa), \quad (3.9)$$

$$R_{33} = r^2 \sin^2 \theta (\ddot{a}a^3 + 3\dot{a}^2a^2 + 2\kappa), \quad (3.10)$$

and

$$R = 6 \left(\ddot{a}a + 2\dot{a}^2 + \frac{\kappa}{a^2} \right). \quad (3.11)$$

By using Equation (3.7)-Equation (3.11), $\phi = \phi_0/a^2$ and the relations

$$\frac{dg^{00}}{d\phi} = \frac{\phi_0}{\phi^2}, \quad (3.12)$$

$$\frac{dg^{11}}{d\phi} = \frac{1 - \kappa r^2}{\phi_0}, \quad (3.13)$$

$$\frac{dg^{22}}{d\phi} = \frac{1}{\phi_0 r^2}, \quad (3.14)$$

$$\frac{dg^{33}}{d\phi} = \frac{1}{\phi_0 r^2 \sin^2 \theta}, \quad (3.15)$$

for the summation in Equation (3.3), the field equation is obtained as

$$\ddot{a}a = \frac{3}{3 + 2\omega} \left(\frac{4\pi}{3} \frac{(\rho + 3P)}{\phi} - \frac{\kappa}{a^2} \right). \quad (3.16)$$

By the definition

$$\omega \equiv -\frac{3}{2}(1 + \alpha), \quad (3.17)$$

Equation (3.16) reduces to

$$\ddot{a}a = -\frac{4\pi}{3\alpha\phi}(\rho + 3P) + \frac{\kappa}{\alpha a^2}. \quad (3.18)$$

As a field equation, we have just gained the acceleration equation, which is similar with the one in the standard model. If one ignores the changes resulting from using the transformed FLRW metric, the differences are the curvature term and the factor of α in the denominators coming from the JBD parameter. Thus, based on our model, the acceleration of the universe depends not only on $(\rho + 3P)$ but also on the spatial

curvature, which is not the case in the standard model of cosmology.

In order to have the equation of $\dot{a}^2$, which is the other Friedmann-like equation, one can firstly rewrite Equation (3.18) by adding and subtracting $8\pi\rho/(3\alpha\phi)$ as

$$\ddot{a}a = -\frac{4\pi}{\alpha\phi}(\rho + P) + \frac{8\pi}{3\alpha\phi}\rho + \frac{\kappa}{\alpha a^2} \quad (3.19)$$

and multiply Equation (3.19) with $\dot{a}/a$ to get

$$\dot{a}\ddot{a} = -\frac{4\pi}{\alpha\phi}\frac{\dot{a}}{a}(\rho + P) + \frac{8\pi}{3\alpha\phi}\frac{\dot{a}}{a}\rho + \frac{\kappa\dot{a}}{\alpha a^3}. \quad (3.20)$$

Then, considering $\phi = \phi_0/a^2$ and using the continuity equation to get rid of pressure in Equation (3.20) yields

$$\dot{a}\ddot{a} = \frac{4\pi}{3\alpha\phi_0}a^2\dot{\rho} + \frac{8\pi}{3\alpha\phi_0}a\dot{a}\rho + \frac{\kappa\dot{a}}{\alpha a^3}. \quad (3.21)$$

Finally, both sides of this equation can be written as time derivatives like

$$\frac{1}{2}\frac{d}{dt}(\dot{a}^2) = \frac{d}{dt}\left(\frac{4\pi}{3\alpha\phi_0}a^2\rho - \frac{\kappa}{2\alpha a^2}\right) \quad (3.22)$$

to obtain

$$\dot{a}^2 = \bar{H}^2 = \frac{8\pi}{3\alpha\phi}\rho - \frac{\kappa}{\alpha a^2}. \quad (3.23)$$

Here, as a new and more suitable parameter, the Hubble parameter is defined as $\bar{H} = \dot{a}$ since $v_{phy} = \dot{a}r_{phy}$ in our framework.

At this point, one can also look for the pressure equation by expanding Equation (3.18) like

$$\ddot{a}a = -\frac{4\pi}{\alpha\phi}P - \left(\frac{4\pi}{3\alpha\phi}\rho - \frac{\kappa}{2\alpha a^2}\right) + \frac{\kappa}{2\alpha a^2}, \quad (3.24)$$

where the terms in the parenthesis is equal to $\dot{a}^2/2$. Taking this term to the left hand side and multiplying whole equation with -2 , one ends up with the pressure equation

$$-2\ddot{a}a - \dot{a}^2 = \frac{8\pi}{\alpha\phi}P - \frac{\kappa}{\alpha a^2}. \quad (3.25)$$

From Equation (3.23), one may conclude that the matter density dominates the evolution of the universe after the spatial curvature and the radiation density. In addition,

the scale factors for the matter, radiation and curvature dominated universe are found as $a \sim t^{2/3}$, $a \sim t^{1/2}$ and $a \sim t^{1/2}$ respectively. These power-law solutions may not seem interesting because they look like the same as in the standard model of cosmology, except the one for the curvature dominated era. However, it should be remembered that these solutions are obtained in our comoving time coordinate. In other words, while spatial intervals are expanding, time intervals will be getting smaller or contracted with the scale factor in the current scenario. Their combined effect can not be distinguished from the accelerating expansion of space when calculating the luminosity distance of a type Ia supernova. Instead, as it is stated before, one may get back to the FLRW metric via $dt' = dt/a(t)$, and obtain solutions such as $a \sim t'^2$, $a \sim t'^1$ and $a \sim t'^1$ in physical time for the matter, radiation and curvature dominated universe in that order. Now, it is more obvious that the space undergoes a linear expansion in the era dominated by the curvature and the radiation, and an accelerating expansion in the matter dominated one. However, we will stick to using comoving time or the former interpretation to avoid the impression that the ordinary matter or any other form of matter results in an accelerating expansion of space. In addition to this, it would be a bit strange to use physical time while using comoving space coordinate. Equation (3.23), which is a Friedmann-like equation, differs from the original one with the definition of the Hubble parameter and the time dependence of $1/G$, which will be very helpful when solving the flatness and the horizon problems.

When constructing the transformed FLRW metric on the arguments presented in Chapter 2, the evolution of the field has been chosen $1/a^2$. At this point, the time dependence of ϕ can be derived by taking an integral over all contributions coming from the observable universe. The quantity to be integrated is

$$d\phi = \frac{dM}{r_{ltr}}, \quad (3.26)$$

where $dM = 4\pi r_{ltr}^2 \frac{\rho_0}{a^3(1+w)} dr_{ltr}$ and r_{ltr} is the light travelled distance. In comoving time, the distance taken by the gravitational effect of a shell of dust (it travels with the speed of light) for a general scale factor $a \sim t^s$ (since the solutions found in the

previous section is power law) is

$$r_{ltr} = \int_{t'}^t \frac{dt''}{a(t'')}. \quad (3.27)$$

Hence, the field at a point in space is calculated with the integral

$$\phi = 4\pi\rho_0 \int_0^t \frac{dt'}{a^{3(1+w)+1}(t')} \int_{t'}^t \frac{dt''}{a(t'')}. \quad (3.28)$$

One can use $a \sim t^s$ and w values for the matter and the radiation dominated epochs accordingly in Equation (3.28) to find

$$\phi = \frac{4\pi\rho_0 t_0^2}{2s(1+s)} \left(\frac{t}{t_0} \right)^{-2s}. \quad (3.29)$$

The result of the integration, Equation (3.29), is valid for both eras, the type of energy content does not matter. Thus, it confirms that the field is proportional to $1/a^2 = (t/t_0)^{-2s}$, which agrees with our motivation.

4. SOLUTIONS TO THE PROBLEMS OF STANDARD COSMOLOGY

4.1. The Flatness Problem

On the basis of the original Friedmann equation, the absolute value of the curvature density parameter increases with time in the matter and radiation dominated eras. Since its value is nearly zero today or negligible [1], [67] compared to other density parameters, it must have been extremely close to zero at very early times. In other words, the initial condition had to be fine tuned to $\kappa = 0$, which is an unstable point. This is called the flatness problem in the standard model of cosmology. By using the solutions to the scale factor, it can be checked that the following curvature density parameter defined based on Equation (3.23) is constant in the radiation dominated era and decreases with time in the matter dominated era ($a^2\dot{a}^2$ is constant for radiation and $a^2\dot{a}^2 \propto t^{2/3}$ for matter)

$$|\Omega_\kappa| = \left| -\frac{\kappa}{\alpha a^2 \bar{H}^2} \right| = \left| -\frac{\kappa}{\alpha a^2 \dot{a}^2} \right|. \quad (4.1)$$

According to Equation (4.1), it does not matter which curvature parameter the universe had at the beginning. Even if it started to evolve as open or closed, the universe we see today is very likely to exist because the curvature density parameter is dragged to zero with time. So, there is no fine-tuning or naturalness problem in this approach. This substantial change in the evolution of $|\Omega_\kappa|$ is actually a solution to the flatness problem.

4.2. The Horizon Problem

Now, we continue with the solution to the horizon problem, which is simply to understand how it is possible for the CMB to be nearly homogeneous (by ignoring small temperature fluctuations) for all parts of the sky although the comoving distance from the decoupling era (the period of time in which photons decoupled from combining

protons and electrons and started to travel freely in space) to the present is much larger than the comoving particle horizon at the time of decoupling in the standard cosmology. As a solution to this, one can compare two comoving distances, which have been just mentioned, in our approach. In the transformed FLRW metric, a null path taken by a photon, which is the fastest way of interaction, is

$$\begin{aligned} R_0 S_\kappa \left(R_0^{-1} \int \frac{dt}{a^2} \right) &= R_0 S_\kappa \left(R_0^{-1} \int \frac{da}{a^2 \bar{H}} \right) \\ &= R_0 S_\kappa \left((R_0 \bar{H}_0)^{-1} \int \frac{dz}{E(z)} \right), \end{aligned} \quad (4.2)$$

where $E(z) = [\Omega_{m0}(1+z) + \Omega_{r0}(1+z)^2 + \Omega_{\kappa0}(1+z)^2]^{1/2}$. In order to calculate the comoving particle horizon at the period of decoupling, one needs to figure out the redshift at which it occurred and so solve the Boltzmann equations for the change in free electron density over time. Since our model does not make a notable change in the radiation and matter densities (based on the fit with the Type Ia supernovae data which will be shown next), the baryon to photon ratio is similar to the one in the conventional cosmology. Furthermore, as it will be seen in Chapter 7, the differential equation that needs to be numerically solved for the free electron density evolution predicts the redshift of decoupling around $z_* \approx 1000$, depending on the coupling parameter of our approach. Considering the redshift of the Big Bang as infinite makes the integration in Equation (4.2) to diverge. Since our classical model should be valid at least up to the Planck scale, the redshift at which the universe was as small as the Planck scale, can be taken as the upper limit of the integration. By dividing the Planck temperature (1.42×10^{32} K) by the temperature today (2.73 K), the redshift of the Planck scale is found to be $z_{Pl} \simeq 5 \times 10^{31}$. In addition, for the energy content of the universe one can take $\Omega_{m0} \simeq 1$ and $\Omega_{r0} \simeq 10^{-4}$ as suggested by our model. Thus, it is easy to show for a flat universe (for simplicity a flat universe scenario is taken into consideration but it can be shown for a closed and open universe as well) the comoving particle horizon at the decoupling ($c = 1$) is

$$\bar{H}_0^{-1} \int_{z_*}^{z_{Pl}} \frac{dz}{E(z)} \simeq 6454 \bar{H}_0^{-1} \quad (4.3)$$

and the comoving distance from the decoupling to the present is

$$\bar{H}_0^{-1} \int_0^{z_*} \frac{dz}{E(z)} \simeq 60 \bar{H}_0^{-1}. \quad (4.4)$$

Equation (4.3) and Equation (4.4) show the observed comoving scale that is uniform in temperature is much smaller than the comoving particle horizon of the decoupling era. Hence, the horizon problem is eliminated.

4.3. The Late Time Accelerating Expansion Problem

In cosmology, providing the luminosity distance of a supernova in terms of the red-shift of photons in a way compatible with observations is an important theoretical competence that any successful theory should have. By using the ratio of flux to luminosity

$$\frac{F}{L} = \frac{1}{(1+z)^2 A}, \quad (4.5)$$

where $A = 4\pi R_0^2 S_\kappa^2(\chi)$, F is flux and L is the actual luminosity of a standard candle, the relation for the luminosity distance is found as

$$d_L = (1+z) R_0 S_\kappa(\chi). \quad (4.6)$$

In an expanding space, flux is reduced by a factor $(1+z)^2$ due to the red-shift of photons and the dilution of their number. This is why the factor $(1+z)^2$ lies in front of the comoving surface area A . Although the matter, radiation and curvature density parameters evolve as $\Omega_{m0}(1+z)^3$, $\Omega_{r0}(1+z)^4$ and $\Omega_{\kappa0}(1+z)^2$ in the right hand side of Equation (3.23) respectively, their effective changes with the red-shift are $\Omega_{m0}(1+z)$, $\Omega_{r0}(1+z)^2$ and $\Omega_{\kappa0}(1+z)^2$ since $\phi = \phi_0/a^2 = \phi_0(1+z)^2$. Thus, in our model, the luminosity distance formula that may look similar to the one in Equation (1.8) at the first sight is actually very different due to a totally different energy content and the factor of α coupled to the curvature density parameter

$$d_L(z) = (1+z) \frac{\bar{H}_0^{-1}}{\sqrt{\alpha|\Omega_{\kappa0}|}} S_\kappa \left(\sqrt{\alpha|\Omega_{\kappa0}|} \int \frac{dz'}{E(z')} \right), \quad (4.7)$$

where $E(z') = [\Omega_{m0}(1+z') + \Omega_{r0}(1+z')^2 + \Omega_{\kappa0}(1+z')^2]^{1/2}$. When the curvature and the radiation densities are neglected, Equation (4.7) is simplified to

$$d_L(z) = (1+z)\bar{H}_0^{-1} \int (1+z')^{-1/2} dz'. \quad (4.8)$$

The data published by the Supernova Cosmology Project [68] can be used to test the conformity between the observation and the theory for the late time accelerating expansion. Based upon our model, the expansion of space is not accelerating because the power of comoving time coordinate in the scale factor is smaller than one for all eras. However, time intervals are contracted over time as well. Hence, the combined effect corresponds to an acceleration in the FLRW universe. Since, in general, the redshift vs. distance modulus graph is more common among cosmologists, we prefer to use the following form of the distance modulus μ

$$\mu = m - M = 25 + 5 \log_{10} \left(\frac{d_L}{\text{Mpc}} \right), \quad (4.9)$$

where m and M are apparent and absolute magnitudes of an astronomical light source.

In Figure 4.1, as dots represent the Type Ia supernovae data, the fitted solid curve is the relation between the redshift and the distance modulus for a flat universe predicted by our model (as a figure we put only the one for a flat case since the figures for closed and open cases look exactly the same). The fit for a flat Λ CDM cosmology is shown with the dashed line. Both are obtained with the weighted least square method. Considering the best fit parameters, the flat universe is slightly preferred over closed and open universe in our model. In addition, the best fit values of the curvature density parameters in closed and open states are very close to zero. Although we compare our model with the flat Λ CDM, which is currently accepted as a concordance model, there is an ongoing debate in the literature about the curvature of space [69–73]. For a flat universe in our model, the parameter α does not go into the fit and the Hubble parameter, which is the only free parameter, is found $\bar{H}_0 = 70.37 \pm 0.23$ km/s/Mpc. The best-fit is achieved for a closed universe with $\Omega_{\kappa0} = -1.21 \times 10^{-7}$, $\alpha = 0.55$ and $\bar{H}_0 = 70.37$ km/s/Mpc. For an open universe, they are $\Omega_{\kappa0} = 9.04 \times 10^{-7}$,

$\alpha = 0.18$ and $\bar{H}_0 = 70.37$ km/s/Mpc. Note that the parameters in these fits are the ones that naturally arise in the JBD theory. There is no need to add any energy source other than the matter (baryonic and dark) and the radiation. Since, ϕ_0 does not go into this fit and the relation $\phi = 1/G$ is our motivation from the beginning, $\phi_0 = 1/G_0 = 1/(6.67 \times 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2}) \simeq 1.50 \times 10^{10} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2}$, where G_0 is the value of Newton's constant. By using fitted values and ϕ_0 , one can find today's values of the matter and the radiation densities as $\rho_{m0} \simeq 5.12 \times 10^{-27} \text{ kg m}^{-3}$ and $\rho_{r0} \simeq 4.64 \times 10^{-31} \text{ kg m}^{-3}$. Although the value of α makes ω negative and the solar system observations predict it to be a large positive number, i.e. $\omega > 10^4$, the JBD theory with a negative coupling parameter [12, 14, 15, 74] is common in the literature and a viable scenario.

In the flat Λ CDM scenario, on the other hand, the best fit parameters are $H_0 = 70.00 \pm 0.34$ km/s/Mpc and $\Omega_{A0} = 0.72 \pm 0.02$. To compare the goodness of the fits of the two models, we use χ^2 statistics, Bayesian Information Criterion (BIC) and Akaike Information Criterion (AIC). In our approach χ^2/dof values (*dof* stands for the degree of freedom) are found as $\chi^2/dof = 585.741/579 = 1.012$, $\chi^2/dof = 585.741/577 = 1.015$ and $\chi^2/dof = 585.741/577 = 1.015$ for a flat, closed and open universe respectively while in a flat Λ CDM it is $\chi^2/dof = 562.227/578 = 0.973$. Using the χ^2 values above and ignoring the constant term coming from the natural logarithm of the maximized value of the likelihood function in the BIC and AIC definitions, they are, for all cases, calculated as follows. For the flat case BIC=592 and AIC=588, for the closed and open case BIC=605 and AIC=592 whereas for the flat Λ CDM cosmology BIC=575 and AIC=566. Taking these values into consideration, we can conclude that the flat Λ CDM cosmology is favored over our model. However, the JBD scalar field decreasing as $1/a^2$ is able to elucidate not only the late time accelerating expansion but also the flatness and horizon problems. From this point of view, since our model does not need the dark energy as well as the inflationary era in the early universe, it stands in a very different place from the Λ CDM cosmology and the extensions of the JBD theory. Indeed, incorporating the cosmological constant term Λ in the JBD theory [75, 76] can provide a better fit and solve the Hubble tension [77] with a more slowly decreasing

scalar field compared to the one evolving as $1/a^2$. At this point, however, it is not necessary because we have not implemented our approach to the CMB and the LSS calculations yet. In other words, for now, there is no tension for the Hubble constant in our model and we prefer to avoid adding any dark energy term.

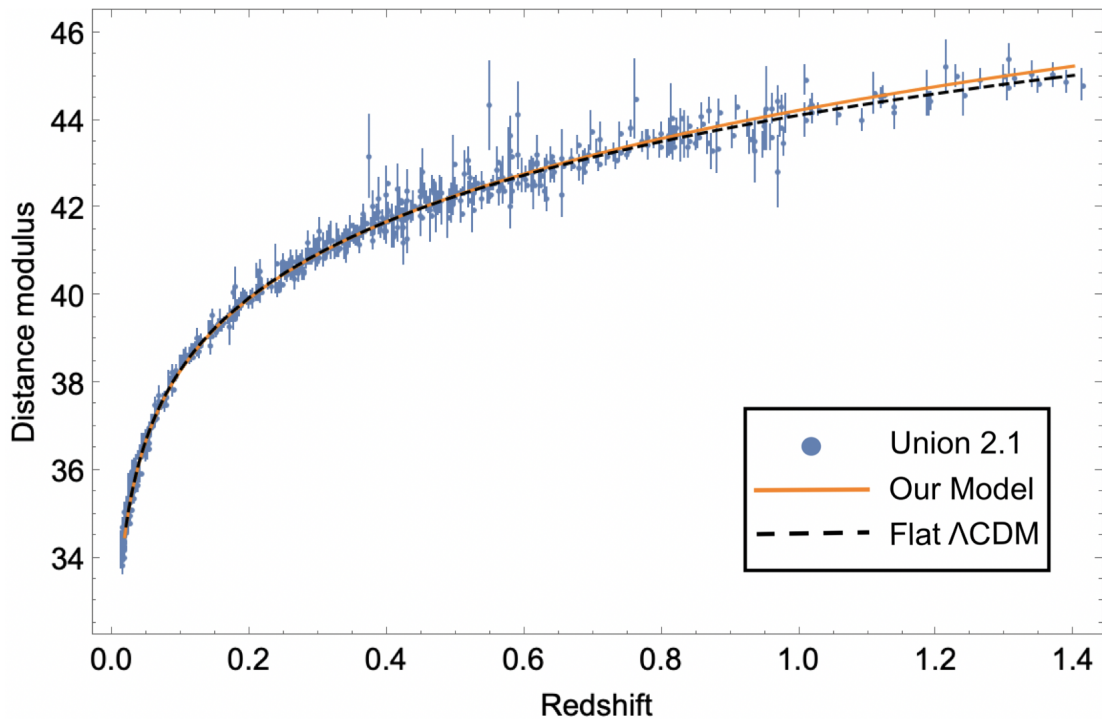


Figure 4.1. Redshift vs. distance modulus diagram for the Union 2.1 Type Ia supernovae compilation. The solid orange line represents the best-fit for our model for a flat universe scenario and the dashed line is the best-fit for a flat Λ CDM cosmology.

On the temporal evolution of G , there is a wide range of upper limits in the literature obtained from different observations. Some of them are based on terrestrial measurements [41,42], Lunar Laser Ranging [43,44], helioseismology [45,46], the period change of pulsating white dwarfs [47], the precision timing of a millisecond pulsar [48] and the gravitational wave observation of a binary neutron star system [49]. There are some others deduced from cosmological observations [36], [50–54], which are extremely dependent on the model chosen, such as the CMB, the Big Bang Nucleosynthesis (BBN) and the Baryon Acoustic Oscillations (BAO). For a very detailed compilation,

one may check [40] by Uzan. While the upper bounds on $|\dot{G}/G|_{t_0}$ from non-cosmological approaches range from $\approx 5 \times 10^{-8} \text{ yr}^{-1}$ to $\approx 7 \times 10^{-14} \text{ yr}^{-1}$, the interval of cosmological ones is narrower and they vary between $\approx 10^{-12} \text{ yr}^{-1}$ and $\approx 10^{-13} \text{ yr}^{-1}$. The upper bounds in [50, 51] were given in terms of G_{rec}/G_0 , where G_{rec} is the value of G at the recombination, and they correspond to a similar $|\dot{G}/G|_{t_0}$ bounds in the interval given above.

In our case, the variation of G today can be found as

$$\left. \frac{\dot{G}}{G} \right|_{t_0} = 2 \left. \frac{\dot{a}}{a} \right|_{t_0} = 2\bar{H}_0 = 1.44 \times 10^{-10} \text{ yr}^{-1}, \quad (4.10)$$

which is not a stringent bound, especially when the cosmological observations are taken into consideration. However, the upper limits obtained from local and cosmological observations depend on the theory of gravity. It should also be added that while obtaining such upper limits, in addition to choosing a gravity theory, some assumptions (related to the theory of gravity itself and/or other theoretical models in observation) are made within the theory (for details the review [40] by Uzan can be checked). For instance, the solar system constraints are sensitive to the models for asteroids. The same data but different asteroid models result in a difference of about one order of magnitude in $\dot{G}/G$. Furthermore, they differ according to pulsar models in pulsar timing tests and nuclear reaction models for the interior of a star in surface temperature tests. For the same pulsar, the difference of about two orders of magnitude in $\dot{G}/G$ is obtained from different nuclear reaction models. In cosmological observations, it is necessary to consider how the temporal variation of G affects the expansion of the universe from beginning to end. So, using cosmological constraints derived from theories having different expansion histories to test a specific model of varying G is not a healthy approach. Even if the standard JBD theory was used to get these cosmological constraints, one should not rely on them when making an inference about our model because we modified the JBD theory. This modification brought many changes for the background equations regarding the spatial expansions in different epochs and the age of the universe, both of which affect the BBN. In addition to this, the perturbation calculations for our approach are needed to check the consistency with the angular power spectrum of the

CMB. All of these will alter upper bounds on the variation of G attained from cosmological observations. At this point, it is very important to note that we do not question whether the mentioned limits or the way they are obtained are correct. We simply argue that these limits are highly dependent on the theoretical models (not only the theory of gravity but also other physical models in the observations). Therefore, theories with varying Newton's constant not obeying the strictest upper bounds derived based on another model may still be valid.

The age of the universe is another parameter that gives some constraints for a theoretical model and is calculated by measuring the Hubble parameter. One can make an estimation about its value just by regarding the evolution of matter, radiation and curvature. So, the age of the universe is given by

$$t_0 = \int_0^{t_0} dt = \int_0^1 \frac{da}{\dot{a}} = \int_0^1 \frac{da}{\bar{H}_0 \sqrt{(1 + a_{eq}a^{-1})\Omega_{m0}a^{-1} + \Omega_{\kappa0}a^{-2}}}, \quad (4.11)$$

where $dt = da/\dot{a}$ is used to change the variable, and for $\dot{a}$ we have made use of Equation (3.23). As mentioned before, although the matter, radiation and curvature density parameters evolve as Ω_{m0}/a^3 , Ω_{r0}/a^4 and $\Omega_{\kappa0}/a^2$ in Equation (3.23) respectively, their effective changes with the scale factor are Ω_{m0}/a , Ω_{r0}/a^2 and $\Omega_{\kappa0}/a^2$ due to $\phi = \phi_0/a^2$. In addition, a_{eq} is the value of the scale factor at which the matter and radiation densities are equal (so $\Omega_{r0} = a_{eq}\Omega_{m0}$). The value calculated from Equation (4.11) for the fitted parameters is smaller than the age calculated in the Λ CDM cosmology. So, our model seems to contradict some astrophysical constraints [78] related to the age of oldest stars. However, Equation (4.11) gives the amount of time elapsed in comoving time. The physical age, which is the actual age, is calculated as

$$\int_0^{t_0} \frac{dt}{a} = \int_0^1 \frac{da}{a\dot{a}} = \int_0^1 \frac{da}{\bar{H}_0 \sqrt{(1 + a_{eq}a^{-1})\Omega_{m0}a + \Omega_{\kappa0}}}. \quad (4.12)$$

Now, for the best-fit parameters, one can find 27.6×10^9 years, which is a bigger value in comparison with the age in the comoving time. So, even the oldest stars had enough time to be in their present phase in our model.

5. FIRST ORDER FIELD EQUATION

In the previous chapters, we have obtained the field equation for the unperturbed cosmological metric and solved the problems of the standard cosmology. Now, one can make use of the following line element for the perturbed metric form

$$ds^2 = -(1 + 2\Psi)\frac{dt^2}{a^2} + \frac{a^2(1 + 2\Phi)}{(1 + \frac{\kappa r^2}{4})^2}(dr^2 + r^2 d\Omega^2) \quad (5.1)$$

in order to find the first order field equation. The line element in Equation (5.1) is written in comoving space and time coordinates based on our motivations in the previous chapters. In addition, it is stated in the Newtonian gauge. Ψ and Φ are scalar perturbations corresponding the Newtonian potential and the curvature perturbation, respectively. In order to test our model whether or not it is able to explain the CMB anisotropy spectrum and the LSS formation, we need the first order field equation of the modified JBD theory gained for the metric in Equation (5.1).

The first order field equations in the JBD theory for a flat or non-flat space can be found in the literature. However, since our approach is a modified version of the JBD theory due to the connection between the JBD scalar field and the metric tensor components, the equation will be a bit different in comparison with that in the standard JBD theory. So, finding this equation is the main aim of this chapter and all calculations can be found in detail (the long but straightforward intermediate steps will not be given since they are unnecessary at this point). Since we are interested only in the zeroth and the first order terms, the terms higher than the first order will be ignored in calculations. As a starting point, we need connection coefficients and they can be obtained for the perturbed metric in Equation (5.1) as

$$\Gamma_{00}^0 = -\frac{\dot{a}}{a} + \dot{\Psi}, \quad (5.2)$$

$$\Gamma_{0i}^0 = \partial_i \Psi, \quad (5.3)$$

$$\Gamma_{00}^i = \frac{(1 + \frac{\kappa r^2}{4})^2}{a^4} \partial_i \Psi, \quad (5.4)$$

$$\Gamma_{0i}^i = \frac{\dot{a}}{a} + \dot{\Phi}, \quad (5.5)$$

$$\Gamma_{ij}^0 = \begin{cases} \frac{\dot{a}a^3(1+2\Phi-2\Psi)}{(1+\frac{\kappa r^2}{4})^2} + \frac{a^4\dot{\Phi}}{(1+\frac{\kappa r^2}{4})^2} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad (5.6)$$

$$\Gamma_{jk}^i = \begin{cases} 0 & \text{if } i \neq j \neq k, \\ \partial_i\Phi - \frac{1}{2}\frac{\kappa x_i}{(1+\frac{\kappa r^2}{4})} & \text{if } i = j = k, \\ -\partial_i\Phi + \frac{1}{2}\frac{\kappa x_i}{(1+\frac{\kappa r^2}{4})} & \text{if } i \neq j = k, \\ \partial_k\Phi - \frac{1}{2}\frac{\kappa x_k}{(1+\frac{\kappa r^2}{4})} & \text{if } i = j \neq k. \end{cases} \quad (5.7)$$

As the next step, starting with the Riemann tensor whose form is given by

$$R_{\sigma\mu\nu}^\rho = \partial_\mu\Gamma_{\nu\sigma}^\rho - \partial_\nu\Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho\Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho\Gamma_{\mu\sigma}^\lambda, \quad (5.8)$$

one can obtain the general form of the Ricci tensor once the first and the third indices of the Riemann tensor are contracted like

$$R_{\sigma\nu} = R_{\sigma\mu\nu}^\mu = \partial_\mu\Gamma_{\nu\sigma}^\mu - \partial_\nu\Gamma_{\mu\sigma}^\mu + \Gamma_{\mu\lambda}^\mu\Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\mu\Gamma_{\mu\sigma}^\lambda. \quad (5.9)$$

Then, using the connection coefficients (Equation (5.2)-Equation (5.7)) in Equation (5.9) gives the Ricci tensor components as

$$R_{00} = -3\frac{\ddot{a}}{a} - 3\frac{\dot{a}^2}{a^2} - 3\ddot{\Phi} + \frac{\dot{a}}{a}(3\dot{\Psi} - 9\dot{\Phi}) + \frac{(1+\frac{\kappa r^2}{4})^2}{a^4}\nabla^2\Psi \quad (5.10)$$

and

$$\begin{aligned} R_{ij} = \delta_{ij} \frac{a^2}{(1+\frac{\kappa r^2}{4})^2} & \left[(1+2\Phi-2\Psi)(\ddot{a}a+3\dot{a}^2) + \frac{2\kappa}{a^2} + \dot{a}(7\dot{\Phi}-\dot{\Psi}) \right. \\ & \left. + a^2\ddot{\Phi} - \frac{(1+\frac{\kappa r^2}{4})^2}{a^2}\nabla^2\Phi \right] - \partial_i\partial_j(\Phi+\Psi). \end{aligned} \quad (5.11)$$

Contracting Equation (5.10) and Equation (5.11) with the metric tensor yields the Ricci scalar

$$\begin{aligned} R = & (6-12\Psi)(\ddot{a}a+2\dot{a}^2) + (6-12\Phi)\frac{\kappa}{a^2} + 6a^2\ddot{\Phi} \\ & + 6\dot{a}a(5\dot{\Phi}-\dot{\Psi}) - \frac{(1+\frac{\kappa r^2}{4})^2}{a^2}\nabla^2(4\Phi+2\Psi). \end{aligned} \quad (5.12)$$

In the field equation (Equation (3.3)), for the summations on the left and right hand sides, one needs the derivatives of the metric components with respect to the scalar field. Since, the metric components are

$$g^{00} = -\frac{a^2}{(1+2\Psi)} = -\frac{\phi_0}{\phi}, \quad (5.13)$$

and

$$g^{ii} = \frac{(1 + \frac{\kappa r^2}{4})^2}{a^2(1+2\Phi)} = \left(1 + \frac{\kappa r^2}{4}\right)^2 \frac{(1-2\Phi)}{a^2} = \left(1 + \frac{\kappa r^2}{4}\right)^2 \frac{\phi}{\phi_0}, \quad (5.14)$$

their derivatives are found as

$$\frac{dg^{00}}{d\phi} = \frac{\phi_0}{\phi^2} = \frac{a^4(1-4\Psi)}{\phi_0}, \quad (5.15)$$

and

$$\frac{dg^{ii}}{d\phi} = \frac{(1 + \frac{\kappa r^2}{4})^2}{\phi_0}. \quad (5.16)$$

The contribution coming from the ${}^0_0$ component on the left hand side of Equation (3.3) can be calculated by using $R_{00} = \bar{R}_{00} + \delta R_{00}$ and $R = \bar{R} + \delta R$ (summation of the zeroth order and the first order terms) like

$$\begin{aligned} \left(R_{00} - \frac{1}{2}Rg_{00}\right) \frac{dg^{00}}{d\phi} &= (\bar{R}_{00} + \delta R_{00}) \frac{a^4(1-4\Psi)}{\phi_0} \\ &\quad + \frac{1}{2}(\bar{R} + \delta R) \frac{(1+2\Psi)}{a^2} \frac{a^4(1-4\Psi)}{\phi_0} \\ &= \frac{a^4}{\phi_0}(\bar{R}_{00} + \delta R_{00}) - \frac{4a^4\Psi}{\phi_0}\bar{R}_{00} \\ &\quad + \frac{a^2}{2\phi_0}(\bar{R} + \delta R) - \frac{a^2\Psi}{\phi_0}\bar{R}. \end{aligned} \quad (5.17)$$

Using Equation (5.10) and Equation (5.12) for the Ricci tensor component and the Ricci scalar respectively gives the first order terms as

$$\begin{aligned} G_{00} \frac{dg^{00}}{d\phi} \Big|_{1\text{st order}} &= \frac{1}{\phi_0} \left[6a^3 \dot{a} \dot{\Phi} - 12a^2 \dot{a}^2 \Psi \right. \\ &\quad \left. - 2 \left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2 \Psi - 6\kappa(\Phi + \Psi) \right]. \end{aligned} \quad (5.18)$$

The very same calculations can be done for the $^i{}_i$ component

$$\begin{aligned}
\left(R_{ii} - \frac{1}{2}Rg_{ii}\right) \frac{dg^{ii}}{d\phi} &= (\bar{R}_{ii} + \delta R_{ii}) \frac{(1 + \frac{\kappa r^2}{4})^2}{\phi_0} \\
&\quad - \frac{1}{2}(\bar{R} + \delta R) \frac{a^2(1 + 2\Phi)(1 + \frac{\kappa r^2}{4})^2}{(1 + \frac{\kappa r^2}{4})^2 \phi_0} \\
&= (\bar{R}_{ii} + \delta R_{ii}) \frac{(1 + \frac{\kappa r^2}{4})^2}{\phi_0} - \frac{a^2}{2\phi_0}(\bar{R} + \delta R) \\
&\quad - \frac{a^2\phi}{\phi_0}\bar{R},
\end{aligned} \tag{5.19}$$

and the first order terms are obtained as

$$\begin{aligned}
G_{ii} \frac{dg^{ii}}{d\phi} \Big|_{\text{1st order}} &= \frac{1}{\phi_0} \left[6a^3\dot{a}(\dot{\Psi} - 4\dot{\Phi}) + 18a^2\dot{a}^2(\Psi - \Phi) + 12a^3\ddot{a}(\Psi - \Phi) \right. \\
&\quad \left. - 6a^4\ddot{\Phi} + 2\left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2(\Psi + \Phi) \right].
\end{aligned} \tag{5.20}$$

Hence, after using Equation (5.18) and Equation (5.20) for the left hand side of Equation (3.3), we are left with

$$\begin{aligned}
G_{\mu\nu} \frac{dg^{\mu\nu}}{d\phi} \Big|_{\text{1st order}} &= \frac{1}{\phi_0} \left[6a^3\dot{a}(\dot{\Psi} - 3\dot{\Phi}) + 6a^2\dot{a}^2(\Psi - 3\Phi) + 12a^3\ddot{a}(\Psi - \Phi) \right. \\
&\quad \left. - 6a^4\ddot{\Phi} - 6\kappa(\Phi + \Psi) + 2\left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2\Psi \right].
\end{aligned} \tag{5.21}$$

After very long but straightforward calculations for $^0{}_0$ component, the terms in the square bracket on the right hand side of Equation (3.3) (the terms which go into summation) give

$$\begin{aligned}
&\left[\frac{8\pi}{\phi} T_{00} + \frac{1}{\phi} (\nabla_0 \partial_0 \phi - g_{00} g^{\alpha\beta} \nabla_\alpha \partial_\beta \phi) + \frac{\omega}{\phi^2} \left(\partial_0 \phi \partial_0 \phi - \frac{1}{2} g_{00} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right) \right] \frac{dg^{00}}{d\phi} \\
&= \frac{1}{\phi_0} \left[\frac{8\pi}{\phi_0} a^6 \delta T_{00} + (4\omega - 6) a^2 \dot{a}^2 \Psi - (4\omega + 6) a^3 \dot{a} \dot{\Psi} + 6a^3 \dot{a} \dot{\Phi} - 18\kappa \Psi \right. \\
&\quad \left. + 2\left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2 \Psi \right],
\end{aligned} \tag{5.22}$$

and for ^i_i component, they yield

$$\begin{aligned}
& \left[\frac{8\pi}{\phi} T_{ii} + \frac{1}{\phi} (\nabla_i \partial_i \phi - g_{ii} g^{\alpha\beta} \nabla_\alpha \partial_\beta \phi) + \frac{\omega}{\phi^2} \left(\partial_i \phi \partial_i \phi - \frac{1}{2} g_{ii} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right) \right] \frac{dg^{ii}}{d\phi} \\
&= \frac{1}{\phi_0} \left[\frac{8\pi}{\phi_0} a^2 \left(1 + \frac{\kappa r^2}{4} \right)^2 3\delta T_{ii} - 4a^3 \ddot{a} (2\omega\Psi + 3\Phi) - (16\omega + 6)a^2 \dot{a}^2 \Psi \right. \\
&\quad \left. + 12\omega a^2 \dot{a}^2 \Phi - 12a^3 \dot{a} (\omega\dot{\Psi} + \dot{\Phi}) + 6a^4 \ddot{\Psi} - 6\kappa\Psi - 4 \left(1 + \frac{\kappa r^2}{4} \right)^2 \nabla^2 \Psi \right]. \tag{5.23}
\end{aligned}$$

The remaining terms on the right hand side of Equation (3.3) (the ones which do not go into summation, in other words the last three terms in the curly parentheses) in the same order are

$$\begin{aligned}
-\frac{R}{\phi} &= -\frac{a^2}{\phi_0} (1 - 2\Psi) R = \frac{1}{\phi_0} \left[-6(a^3 \ddot{a} + 2a^2 \dot{a}^2) (1 - 4\Psi) + 6a^3 \dot{a} (\dot{\Psi} - 5\dot{\Phi}) \right. \\
&\quad \left. - 6a^4 \ddot{\Phi} - 6\kappa(1 - 2\Psi - 2\Phi) \right. \\
&\quad \left. + \left(1 + \frac{\kappa r^2}{4} \right)^2 \nabla^2 (2\Psi + 4\Phi) \right], \tag{5.24}
\end{aligned}$$

$$\begin{aligned}
-2\omega g^{\alpha\beta} \frac{\nabla_\alpha \partial_\beta \phi}{\phi^2} &= -\frac{a^4}{\phi_0^2} 2\omega (1 - 4\Psi) g^{\alpha\beta} \nabla_\alpha \partial_\beta \phi \\
&= \frac{2\omega}{\phi_0} \left[-2a^3 \ddot{a} - 2a^2 \dot{a}^2 + 8a^3 \ddot{a} \Psi + 8a^2 \dot{a}^2 \Psi + 2a^3 \dot{a} (\dot{\Psi} - 3\dot{\Phi}) \right. \\
&\quad \left. + 2a^4 \ddot{\Psi} - 2 \left(1 + \frac{\kappa r^2}{4} \right)^2 \nabla^2 \Psi \right], \tag{5.25}
\end{aligned}$$

and

$$\begin{aligned}
\omega g^{\alpha\beta} \frac{\partial_\alpha \phi \partial_\beta \phi}{\phi^3} &= \frac{a^6}{\phi_0^3} \omega (1 - 6\Psi) g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \\
&= \frac{\omega}{\phi_0} \left[-4a^2 \dot{a}^2 (1 - 4\Psi) + 8a^3 \dot{a} \dot{\Psi} \right]. \tag{5.26}
\end{aligned}$$

Now, since we have been equipped with all the necessary terms, using Equation (5.21) for the left hand side and Equation (5.22)-Equation (5.26) for the right hand

side of Equation (3.3) results in

$$\begin{aligned}
& (6 + 4\omega)a^3\dot{a}\dot{\Psi} + (18 + 12\omega)a^3\dot{a}\dot{\Phi} - (30 + 20\omega)a^2\dot{a}^2\Psi - (18 + 12\omega)a^2\dot{a}^2\Phi \\
& - (12 + 8\omega)a^3\ddot{a}\Psi - (6 + 4\omega)a^4\ddot{\Psi} + 6\kappa(\Psi - 3\Phi) + (2 + 4\omega)\left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2\Psi \\
& - 4\left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2\Phi = \frac{8\pi}{\phi_0}\left(a^6\delta T_{00} + 3a^2\left(1 + \frac{\kappa r^2}{4}\right)^2 \delta T_{ii}\right).
\end{aligned} \tag{5.27}$$

At this point, one can take advantage of the fact that $\Phi = -\Psi$ to simplify Equation (5.27). This comes from our motivation on which the modified JBD theory has been constructed. The same relation, as an assumption, is used in the perturbation calculations of the standard model of cosmology for a different reason. Thus, Equation (5.27) turns into

$$\begin{aligned}
& - (12 + 8\omega)a^3\dot{a}\dot{\Psi} - (12 + 8\omega)a^2\dot{a}^2\Psi - (12 + 8\omega)a^3\ddot{a}\Psi - (6 + 4\omega)a^4\ddot{\Psi} \\
& + 24\kappa\Psi + (6 + 4\omega)\left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2\Psi = \frac{8\pi}{\phi_0}\left[a^6\delta T_{00} + 3a^2\left(1 + \frac{\kappa r^2}{4}\right)^2 \delta T_{ii}\right].
\end{aligned} \tag{5.28}$$

Here, one needs to have δT_{00} and δT_{ii} terms for dust and radiation. They can be obtained from the definition of energy momentum tensor in Equation (1.2). For δT_{00} , one can make use of

$$T_{00} = (\rho + P)U_0U_0 + Pg_{00} \tag{5.29}$$

to get, for matter,

$$\bar{T}_{00} + \delta T_{00} = (\bar{\rho}_m + \delta\rho_m)\frac{(1 + 2\Psi)}{a^2} = \frac{1}{a^2}(\bar{\rho}_m + 2\bar{\rho}_m\Psi + \delta\rho_m). \tag{5.30}$$

For radiation, one should use the energy density formula obtained thanks to the distribution function. From the next chapter, Equation (6.41) and Equation (6.42) can be borrowed to write the first line of the following equation

$$\begin{aligned}
\bar{T}_{00} + \delta T_{00} &= \left[2 \int \frac{d^3p}{(2\pi)^3} p \left(f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta\right)\right] \frac{(1 + 2\Psi)}{a^2} \\
&= \bar{\rho}_r(1 + 4\Theta_0)\frac{(1 + 2\Psi)}{a^2} = \frac{1}{a^2}(\bar{\rho}_r + 2\bar{\rho}_r\Psi + 4\bar{\rho}_r\Theta_0),
\end{aligned} \tag{5.31}$$

where Θ_0 is the monopole term for temperature fluctuations $\Theta = \delta T/T$. There is no pressure for matter particles but for radiation (photons and neutrinos) it is again

gained by the definition of energy momentum tensor

$$T_{ii} = (\rho + P)U_i U_i + P g_{ii} \quad (5.32)$$

as

$$\begin{aligned} \bar{T}_{ii} + \delta T_{ii} &= \left[2 \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{3E} \left(f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta \right) \right] \frac{a^2(1 + 2\Phi)}{(1 + \frac{\kappa r^2}{4})^2} \\ &= \frac{\bar{\rho}_r}{3} (1 + 4\Theta_0) \frac{a^2(1 + 2\Phi)}{(1 + \frac{\kappa r^2}{4})^2} \\ &= \frac{a^2}{3(1 + \frac{\kappa r^2}{4})^2} (\bar{\rho}_r + 2\bar{\rho}_r \Phi + 4\bar{\rho}_r \Theta_0). \end{aligned} \quad (5.33)$$

Before continuing further, one point needs to be emphasized again. We mentioned in Chapter 1 that the JBD coupling parameter may be scale dependent. By this we mean that the JBD coupling parameter may take different values on cosmic and solar system scales. This is what we deduced from the perturbation calculations in our model. When the weak field limit of the perturbed field equation is taken, where one can ignore the change of potential over time, it is inevitable to define a different coupling parameter than the one defined in the zeroth order calculations in order for the equation to be reduced to a Poisson-like equation and therefore for gravity to be an attractive force. Thus, incorporating the energy density and the pressure terms and defining ω (as we have done before in Chapter 3 but using β instead of α) like

$$\omega \equiv -\frac{3}{2}(1 + \beta) \quad (5.34)$$

or

$$\beta \equiv -1 - \frac{2\omega}{3}, \quad (5.35)$$

Equation (5.28) reduces to

$$\begin{aligned} 2a^3 \dot{a} \dot{\Psi} + 2a^2 \dot{a}^2 \Psi + 2a^3 \ddot{a} \Psi + a^4 \ddot{\Psi} + \frac{4\kappa}{\beta} \Psi - \left(1 + \frac{\kappa r^2}{4} \right)^2 \nabla^2 \Psi \\ = \frac{4\pi}{3\beta\phi_0} a^4 (\delta\rho_m + 2\bar{\rho}_m \Psi + 8\bar{\rho}_r \Theta_0 + 2\bar{\rho}_r \Psi + 2\bar{\rho}_r \Phi) \end{aligned} \quad (5.36)$$

for baryonic, dark matter and relativistic particles. One can make use of the Friedmann-

like equation (Equation 3.23) to get rid of the second and the fourth terms on the right to get

$$\begin{aligned} & 2a^3\dot{a}\dot{\Psi} + a^2\dot{a}^2\ddot{\Psi} + 2a^3\ddot{a}\ddot{\Psi} + a^4\ddot{\Psi} + \frac{3\kappa}{\beta}\Psi - \left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2\Psi \\ &= \frac{4\pi}{3\beta\phi_0}a^4(\delta\rho_m + 8\bar{\rho}_r\Theta_0 + 2\bar{\rho}_r\Phi). \end{aligned} \quad (5.37)$$

Then, implementing the pressure equation (Equation 3.25) for the third term on the right yields

$$2a^3\dot{a}\dot{\Psi} + a^4\ddot{\Psi} + \frac{4\kappa}{\beta}\Psi - \left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2\Psi = \frac{4\pi}{3\beta\phi_0}a^4(\delta\rho_m + 8\bar{\rho}_r\Theta_0). \quad (5.38)$$

Writing this equation in conformal time (derivatives w.r.t. conformal time are denoted by $'$) simplifies it to

$$\Psi'' - \left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2\Psi + \frac{4\kappa}{\beta}\Psi = \frac{4\pi}{3\beta\phi_0}a^4(\delta\rho_m + 8\bar{\rho}_r\Theta_0). \quad (5.39)$$

After redefining $\delta\rho_m = \bar{\rho}_m\delta_m$ where $\delta_m = \delta\rho_m/\bar{\rho}_m$, we are left with

$$\Psi'' - \left(1 + \frac{\kappa r^2}{4}\right)^2 \nabla^2\Psi + \frac{4\kappa}{\beta}\Psi = \frac{4\pi}{3\beta\phi_0}a^4(\bar{\rho}_m\delta_m + 8\bar{\rho}_r\Theta_0). \quad (5.40)$$

This is our first order field equation and when the change of the potential over time and the spatial curvature are ignored, it reduces to the Poisson-like equation at the cost of a negative value for β that means $\omega > -3/2$, which is a different behaviour compared to $\omega < -3/2$ on cosmic scales.

6. BOLTZMANN EQUATIONS

The model developed in the previous chapters is free of the horizon and flatness problems, and it is able to explain the late time accelerating expansion of the universe. So, as the next step it should be taken into consideration to explain the nucleosynthesis in the early universe, the CMB and the LSS formation. To be able to do that we need the Boltzmann equations for radiation (photons and neutrinos) and massive particles such as baryonic and dark matter. The Boltzmann equations for a flat space can be found in the literature. However, we derive them for all spatial scenarios because our model may favor a spatially non-flat universe. So, finding these equations will be the main goal of this chapter and all calculations can be found in detail.

To obtain the Boltzmann equations, we start with the general form of the metric tensor in terms of spatial curvature. In order to study nucleosynthesis process we will consider a homogeneous universe, so the metric tensor in Equation (2.4) is used. For the CMB anisotropy spectrum and the structure formation we should use a perturbed metric as in Equation (5.1). When deriving Boltzmann equations, one needs the energy-momentum four vector in terms of particle's coordinate independent energy E and momentum p^i

$$P^\mu = \left[(1 - \Psi)aE, (1 - \Phi)\frac{(1 + \frac{\kappa r^2}{4})}{a}p^i \right]. \quad (6.1)$$

For a homogeneous universe scenario, one can disregard the metric perturbations Ψ and Φ in Equation (6.1). The Boltzmann equation, which relates the change in the number of particles over time to the collisions among these particles, is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \cdot \frac{dx^i}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i} \cdot \frac{d\hat{p}^i}{dt} = C[f], \quad (6.2)$$

where f is the distribution function and $C[f]$ is the collision term. For now, we will only focus on how to calculate the derivatives on the left hand side of the equation in homogeneous and inhomogeneous space. We will mention what the distribution function and the collision term are in the next sections when appropriate.

6.1. The Boltzmann Equation for Homogeneous Space

In homogeneous expanding space, $\frac{dx^i}{dt}$ can be found in terms of E and p by applying the following chain rule

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} = \frac{P^i}{P^0} = \frac{(1 + \frac{\kappa r^2}{4})p^i}{a} \frac{1}{aE} = \frac{(1 + \frac{\kappa r^2}{4})p\hat{p}^i}{a^2 E}. \quad (6.3)$$

In Equation (6.3), λ parameterizes the trajectory $x^\mu(\lambda)$ and, P^0 and P^i are taken from Equation (6.1). For $\frac{dp}{dt}$, as a first step, one should look at the time component of the geodesic equation

$$\frac{dP^0}{d\lambda} = -\Gamma_{\alpha\beta}^0 P^\alpha P^\beta. \quad (6.4)$$

Since $\frac{dt}{d\lambda} = P^0$, one can write the Equation (6.4) like

$$P^0 \frac{dP^0}{dt} = -\Gamma_{\alpha\beta}^0 P^\alpha P^\beta. \quad (6.5)$$

The summation in the right hand side of the Equation (6.5) can be expanded and the connection coefficients can be used to obtain

$$\begin{aligned} P^0 \frac{dP^0}{dt} &= -(\Gamma_{00}^0 P^0 P^0 + \Gamma_{11}^0 P^1 P^1 + \Gamma_{22}^0 P^2 P^2 + \Gamma_{33}^0 P^3 P^3) \\ &= \frac{\dot{a}}{a} P^0 P^0 - \frac{\dot{a}a^3}{(1 + \frac{\kappa r^2}{4})^2} (P^1 P^1 + r^2 P^2 P^2 + r^2 \sin^2 \theta P^3 P^3). \end{aligned} \quad (6.6)$$

Since, the square of momentum is

$$p^2 = \frac{a^2}{(1 + \frac{\kappa r^2}{4})^2} (P^1 P^1 + r^2 P^2 P^2 + r^2 \sin^2 \theta P^3 P^3), \quad (6.7)$$

and $P^0 = aE$, Equation (6.6) simplifies to

$$P^0 \frac{dP^0}{dt} = \dot{a}aE^2 - \dot{a}ap^2. \quad (6.8)$$

In order to obtain the left hand side of Equation (6.8) in terms of the energy of a particle, one can make use of the following two relations

$$P^0 \frac{dP^0}{dt} = \frac{1}{2} \frac{d(a^2 E^2)}{dt}, \quad (6.9)$$

$$E^2 = p^2 + m^2, \quad (6.10)$$

to write

$$P^0 \frac{dP^0}{dt} = \frac{1}{2} \frac{d(a^2 p^2 + a^2 m^2)}{dt}. \quad (6.11)$$

Finally, one gets the left hand side

$$P^0 \frac{dP^0}{dt} = a^2 p \frac{dp}{dt} + \dot{a} a E^2, \quad (6.12)$$

and Equation (6.8) is found as

$$\frac{dp}{dt} = -\frac{\dot{a}}{a} p = -\frac{\bar{H}}{a} p. \quad (6.13)$$

In a homogeneous universe, the distribution function does not depend on the position and the direction. In other words, $\frac{df}{dx^i}$ and $\frac{\partial f}{\partial \hat{p}^i}$ are first order terms and at this point they are ignored. Thus, Equation (6.13) is inserted in Equation (6.2) to get the Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} - \frac{\bar{H}}{a} p \frac{\partial f}{\partial p} = C[f]. \quad (6.14)$$

Equation (6.14) can be integrated over momentum space (after taking $\hbar = 1$) as

$$\int \frac{d^3 p}{(2\pi)^3} \frac{\partial f}{\partial t} - \frac{\bar{H}}{a} \int \frac{d^3 p}{(2\pi)^3} p \frac{\partial f}{\partial p} = \int \frac{d^3 p}{(2\pi)^3} C[f], \quad (6.15)$$

where for the first term on the left, we can use the fact that

$$\int \frac{d^3 p}{(2\pi)^3} \frac{\partial f}{\partial t} = \frac{d}{dt} \int \frac{d^3 p}{(2\pi)^3} f = \frac{dn(t)}{dt}, \quad (6.16)$$

and for the second term that

$$\begin{aligned} -\frac{\bar{H}}{a} \int \frac{d^3 p}{(2\pi)^3} p \frac{\partial f}{\partial p} &= -\frac{\bar{H}}{a} \int \frac{\sin \theta dp d\theta d\phi}{(2\pi)^3} p^3 \frac{\partial f}{\partial p} \\ &= -\frac{\bar{H}}{a} \int \frac{\sin \theta dp d\theta d\phi}{(2\pi)^3} \left[\frac{\partial}{\partial p} (p^3 f) - 3p^2 f \right] \\ &= 3 \frac{\bar{H}}{a} \int \frac{p^2 \sin \theta dp d\theta d\phi}{(2\pi)^3} f \\ &= 3 \frac{\bar{H}}{a} n(t). \end{aligned} \quad (6.17)$$

In the first line of Equation (6.17), $d^3 p$ has been expanded. In the second line, the method of integration by parts has been implemented. Finally, in the third line, in-

tegration over the first term in the square parentheses has been set to zero since $p^3 f$ term is zero at the boundaries, i.e., at $p = 0$ and at $p = \infty$. Hence, the change in the number density of a particle (Equation (6.15)) is found

$$\frac{dn(t)}{dt} + 3\frac{\bar{H}}{a}n(t) = \int \frac{d^3p}{(2\pi)^3} C[f]. \quad (6.18)$$

6.2. The Boltzmann Equation for Inhomogeneous Space

To derive the Boltzmann equation for inhomogeneous and expanding space, a similar procedure to the one in homogeneous space can be followed. Determining $\frac{dx^i}{dt}$ is an easy step and can be done by using the energy momentum four vector like

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} = \frac{P^i}{P^0} = (1 - \Phi + \Psi) \frac{(1 + \frac{\kappa r^2}{4})^2 p \hat{p}^i}{a^2 E}. \quad (6.19)$$

For $\frac{dp^i}{dt}$, we start with the following relation by applying the chain rule

$$\frac{dp^i}{dt} = \frac{d\lambda}{dt} \frac{dp^i}{d\lambda}, \quad (6.20)$$

where

$$\frac{d\lambda}{dt} = \frac{1}{P^0} = \frac{(1 + \Psi)}{aE}. \quad (6.21)$$

Differentiation of particle momentum with respect to the parameter λ is straightforward and it is

$$\frac{dp^i}{d\lambda} = \frac{d}{d\lambda} \left[\frac{a(1 + \Phi)}{(1 + \frac{\kappa r^2}{4})} P^i \right] = P^i \frac{d}{d\lambda} \left[\frac{a(1 + \Phi)}{(1 + \frac{\kappa r^2}{4})} \right] + \left[\frac{a(1 + \Phi)}{(1 + \frac{\kappa r^2}{4})} \right] \frac{dP^i}{d\lambda}. \quad (6.22)$$

Here, the following chain rule

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \frac{d}{dx^\mu} = P^\mu \frac{d}{dx^\mu} \quad (6.23)$$

can be applied to write the derivative in the first term on the right hand side of Equation (6.22) as

$$\begin{aligned} \frac{d}{d\lambda} \left[\frac{a(1 + \Phi)}{(1 + \frac{\kappa r^2}{4})} \right] &= P^\mu \frac{d}{dx^\mu} \left[\frac{a(1 + \Phi)}{(1 + \frac{\kappa r^2}{4})} \right] \\ &= P^0 \left[\frac{\dot{a}(1 + \Phi) + a\dot{\Phi}}{(1 + \frac{\kappa r^2}{4})} \right] + P^k \left[\frac{a\partial_k \Phi}{(1 + \frac{\kappa r^2}{4})} \right] - \frac{P^k}{2} \left[\frac{a\kappa x_k}{(1 + \frac{\kappa r^2}{4})^2} \right]. \end{aligned} \quad (6.24)$$

For the derivative in the second term on the right hand side of Equation (6.22), we use the geodesic equation given by

$$\frac{dP^i}{d\lambda} = -\Gamma_{\alpha\beta}^i P^\alpha P^\beta = -(\Gamma_{00}^i P^0 P^0 + 2\Gamma_{0j}^i P^0 P^j + \Gamma_{jk}^i P^j P^k). \quad (6.25)$$

Once the relevant connection coefficients in Equation (5.3)-Equation (5.7) are inserted in Equation (6.25), it turns out to be

$$\begin{aligned} \frac{dP^i}{d\lambda} = & -\frac{(1 + \frac{\kappa r^2}{4})^2}{a^4} \partial_i \Psi P^0 P^0 - 2 \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) P^0 P^i \\ & - 2 \left(\partial_k \Phi - \frac{1}{2} \frac{\kappa x_k}{(1 + \frac{\kappa r^2}{4})} \right) P^i P^k \\ & + \left(\partial_i \Phi - \frac{1}{2} \frac{\kappa x_k}{(1 + \frac{\kappa r^2}{4})} \right) P^i P^i. \end{aligned} \quad (6.26)$$

Here, one can take advantage of the energy-momentum four vector to state Equation (6.26) in terms of a particle's energy and momentum by ignoring the higher order terms as

$$\begin{aligned} \frac{dP^i}{d\lambda} = & -\frac{(1 + \frac{\kappa r^2}{4})^2}{a^2} \partial_i \Psi E^2 - 2(1 - \Psi - \Phi) \left(1 + \frac{\kappa r^2}{4} \right) \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) E p^i \\ & - 2 \frac{(1 + \frac{\kappa r^2}{4})^2}{a^2} \left(\partial_k \Phi - \frac{1}{2} \frac{\kappa x_k}{(1 + \frac{\kappa r^2}{4})} \right) p^i p^k \\ & - \frac{(1 + \frac{\kappa r^2}{4})^2}{a^2} \left(\partial_i \Phi - \frac{1}{2} \frac{\kappa x_k}{(1 + \frac{\kappa r^2}{4})} \right) p^2. \end{aligned} \quad (6.27)$$

Now, in order to determine $\frac{dp^i}{dt}$, one should insert Equation (6.24) and Equation (6.27) into Equation (6.22) at first. Then, Equation (6.21) and Equation (6.22) can be plugged into Equation (6.20) to get

$$\begin{aligned} \frac{dp^i}{dt} = \frac{d\lambda}{dt} \frac{dp^i}{d\lambda} = & \frac{\dot{a}(1 + \Phi) + a\dot{\Phi}}{(1 + \frac{\kappa r^2}{4})} P^i + \frac{a\partial_k \Phi}{(1 + \frac{\kappa r^2}{4})} \frac{P^i P^k}{P^0} - \frac{1}{2} \frac{a\kappa x_k}{(1 + \frac{\kappa r^2}{4})^2} \frac{P^i P^k}{P^0} \\ & - \frac{a(1 + \Phi)}{(1 + \frac{\kappa r^2}{4})} \left[\frac{(1 + \frac{\kappa r^2}{4})^2}{a^4} \partial_i \Psi P^0 + 2 \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) P^i \right. \\ & \left. + 2 \left(\partial_k \Phi - \frac{1}{2} \frac{\kappa x_k}{(1 + \frac{\kappa r^2}{4})} \right) \frac{P^i P^k}{P^0} - \left(\partial_i \Phi - \frac{1}{2} \frac{\kappa x_k}{(1 + \frac{\kappa r^2}{4})} \right) \frac{P^i P^i}{P^0} \right]. \end{aligned} \quad (6.28)$$

Once the energy-momentum four vector is used for P^μ components, Equation (6.28) can be written as

$$\begin{aligned} \frac{dp^i}{dt} = & \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) p^i + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_k \Phi \frac{p^i p^k}{E} - \frac{1}{2a^2} \kappa x_k \frac{p^i p^k}{E} \\ & - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Psi E - 2 \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) p^i - 2 \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_k \Phi \frac{p^i p^k}{E} \\ & + \frac{1}{a^2} \kappa x_k \frac{p^i p^k}{E} + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Phi \frac{p^2}{E} - \frac{1}{2a^2} \kappa x_k \frac{p^i p^k}{E}. \end{aligned} \quad (6.29)$$

Here, the similar terms cancel each other and Equation (6.29) reduces to

$$\begin{aligned} \frac{dp^i}{dt} = & - \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) p^i - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_k \Phi \frac{p^i p^k}{E} - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Psi E \\ & + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Phi \frac{p^2}{E} + \frac{1}{2a^2} \kappa x_k \frac{p^i p^k}{E} - \frac{1}{2a^2} \kappa x_i \frac{p^2}{E}. \end{aligned} \quad (6.30)$$

Since

$$\frac{dp}{dt} = \frac{d}{dt} \sqrt{\delta_{ij} p^i p^j} = \delta_{ij} \frac{p^i}{p} \frac{dp^j}{dt}, \quad (6.31)$$

using Equation (6.30) one can write

$$\begin{aligned} \frac{dp}{dt} = & \delta_{ij} \frac{p^i}{p} \left[- \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) p^j - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_k \Phi \frac{p^j p^k}{E} - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_j \Psi E \right. \\ & \left. + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_j \Phi \frac{p^2}{E} + \frac{1}{2a^2} \kappa x_k \frac{p^j p^k}{E} - \frac{1}{2a^2} \kappa x_j \frac{p^2}{E} \right]. \end{aligned} \quad (6.32)$$

By multiplying the terms in the parentheses with the factor outside in Equation (6.32), one obtains

$$\begin{aligned} \frac{dp}{dt} = & - \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) p - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_k \Phi \frac{p^2 \hat{p}^k}{E} - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_j \Psi E \hat{p}^j \\ & + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_j \Phi \frac{p^2 \hat{p}^j}{E} + \frac{1}{2a^2} \kappa x_k \frac{p^2 \hat{p}^k}{E} - \frac{1}{2a^2} \kappa x_j \frac{p^2 \hat{p}^j}{E}. \end{aligned} \quad (6.33)$$

Again the similar terms cancel and one has

$$\frac{dp}{dt} = - \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) p - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Psi E \hat{p}^i. \quad (6.34)$$

For the last term in the Boltzmann equation (Equation (6.2)), one needs to find

$$\frac{d\hat{p}^i}{dt} = \frac{1}{p} \frac{dp^i}{dt} - \frac{p^i}{p^2} \frac{dp}{dt}. \quad (6.35)$$

In order to do this, Equation (6.30) and Equation (6.34) are used to get

$$\begin{aligned} \frac{d\hat{p}^i}{dt} = & - \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) \hat{p}^i - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_k \Phi \frac{\hat{p}^i p^k}{E} - \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Psi \frac{E}{p} \\ & + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Phi \frac{p}{E} + \frac{1}{2a^2} \kappa x_k \frac{\hat{p}^i p^k}{E} - \frac{1}{2a^2} \kappa x_i \frac{p}{E} \\ & + \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) \hat{p}^i + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_k \Psi \frac{E}{p} \hat{p}^i \hat{p}^k. \end{aligned} \quad (6.36)$$

After some algebraic manipulations, Equation (6.36) can be put in the following form

$$\frac{d\hat{p}^i}{dt} = \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \frac{E}{p} (\delta^{ik} - \hat{p}^i \hat{p}^k) \left(\partial_k \Phi \frac{p^2}{E^2} - \partial_k \Psi - \frac{1}{2} \frac{\kappa x_k}{(1 + \frac{\kappa r^2}{4})} \right). \quad (6.37)$$

6.2.1. The Collisionless Boltzmann Equation for Massive Particles

In this section, we collect all the terms obtained in the previous section and put them together to get the collisionless Boltzmann equation (by collisionless, it is meant that $C[f]$ on the right hand side of Equation (6.2) is zero) for massive particles

$$\begin{aligned} \frac{df}{dt} = & \frac{\partial f}{\partial t} + (1 + \Psi - \Phi) \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \frac{\partial f}{\partial x^i} \hat{p}^i \\ & - \frac{\partial f}{\partial p} \left[\left(\frac{\bar{H}}{a} + \dot{\Phi} \right) p + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Psi p^i \right] \\ & + \frac{\partial f}{\partial \hat{p}^i} \frac{(1 + \frac{\kappa r^2}{4})}{a^2} (\delta^{ik} - \hat{p}^i \hat{p}^k) \left[\partial_k \Phi - \partial_k \Psi - \frac{1}{2} \frac{\kappa x_k}{(1 + \frac{\kappa r^2}{4})} \right]. \end{aligned} \quad (6.38)$$

Here, $\frac{\partial f}{\partial x^i}$ and $\frac{\partial f}{\partial \hat{p}^i}$ are first order terms. So, in order to write the collisionless Boltzmann equation upto first order for massive particles, one should ignore the terms higher than the first order in Equation (6.38) and the result is

$$\begin{aligned} \frac{df}{dt} = & \frac{\partial f}{\partial t} + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \frac{\partial f}{\partial x^i} \frac{p}{E} \hat{p}^i \\ & - \frac{\partial f}{\partial p} p \left(\frac{\bar{H}}{a} + \dot{\Phi} + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Psi \frac{E}{p} \hat{p}^i \right). \end{aligned} \quad (6.39)$$

6.2.2. The Collisionless Boltzmann Equation for Radiation

A similar collisionless Boltzmann equation in Equation (6.39) can be written for relativistic particles such as photons and neutrinos. The only difference compared the massive particle case is that the particle momentum is equal to the energy.

$$\begin{aligned} \frac{df}{dt} = & \frac{\partial f}{\partial t} + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \frac{\partial f}{\partial x^i} \hat{p}^i \\ & - \frac{\partial f}{\partial p} p \left(\frac{\bar{H}}{a} + \dot{\Phi} + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Psi \hat{p}^i \right). \end{aligned} \quad (6.40)$$

The distribution function including the first order temperature deviation from the average, can be written as

$$f(\vec{x}, p, \hat{p}, t) = \left\{ \exp \left[\frac{p}{T(t)(1 + \Theta(x, \hat{p}, t))} \right] - 1 \right\}^{-1}, \quad (6.41)$$

where $\Theta = \frac{dT}{T}$. It is simply the ratio of the temperature fluctuation to the average temperature and depends on the position in space, time and the direction of observation, not the momentum itself.

At this point, one can expand the distribution function in Equation (6.41) around the average temperature like

$$\begin{aligned} f(\vec{x}, \vec{p}, t) &= \frac{1}{\exp \left[\frac{p}{T(t)} \right] - 1} + \frac{\partial}{\partial T} \left\{ \frac{1}{\exp \left[\frac{p}{T(t)} \right] - 1} \right\} T(t) \Theta(\vec{x}, \hat{p}, t) \\ &= f^{(0)}(p, t) - p \frac{\partial f^{(0)}}{\partial p} \Theta(\vec{x}, \hat{p}, t). \end{aligned} \quad (6.42)$$

In the second step of Equation (6.42), we have used the relation $-T \frac{\partial f^{(0)}}{\partial T} = p \frac{\partial f^{(0)}}{\partial p}$ and $f^{(0)} \equiv \{\exp[p/T(t)] - 1\}^{-1}$ which is, by definition, the zeroth order distribution function with zero chemical potential. After Equation (6.42) is inserted in Equation (6.40) and the zeroth order terms are ignored, we are left with

$$\begin{aligned} \left. \frac{df}{dt} \right|_{\text{1st order}} &= -p \frac{\partial}{\partial t} \left(\frac{\partial f^{(0)}}{\partial p} \Theta \right) - p \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \frac{\partial f^{(0)}}{\partial p} \partial_i \Theta \hat{p}^i \\ &+ \frac{\bar{H}}{a} \Theta p \frac{\partial}{\partial p} \left(p \frac{\partial f^{(0)}}{\partial p} \right) - p \frac{\partial f^{(0)}}{\partial p} \left(\dot{\Phi} + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Psi \hat{p}^i \right). \end{aligned} \quad (6.43)$$

For the first term on the right hand side of Equation (6.43), one can take the derivative by applying chain rule like

$$\begin{aligned}
 -p \frac{\partial}{\partial t} \left(\frac{\partial f^{(0)}}{\partial p} \Theta \right) &= -p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \Theta}{\partial t} - p \Theta \frac{dT}{dt} \frac{\partial^2 f^{(0)}}{\partial T \partial p} \\
 &= -p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \Theta}{\partial t} + p \Theta \frac{dT/dt}{T} \frac{\partial}{\partial p} \left(p \frac{\partial f^{(0)}}{\partial p} \right).
 \end{aligned} \tag{6.44}$$

In the second line of this equation, we have again used $-T \frac{\partial f^{(0)}}{\partial T} = p \frac{\partial f^{(0)}}{\partial p}$. The last term in Equation (6.44) cancels the third term on the right hand side of Equation (6.43) because $\frac{dT/dt}{T} = -\bar{H}/a$. Finally, the first order collisionless Boltzmann equation for photons and neutrinos is obtained as

$$\left. \frac{df}{dt} \right|_{\text{1st order}} = -p \frac{\partial f^{(0)}}{\partial p} \left[\dot{\Theta} + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Theta \hat{p}^i + \dot{\Phi} + \frac{(1 + \frac{\kappa r^2}{4})}{a^2} \partial_i \Psi \hat{p}^i \right]. \tag{6.45}$$

7. COSMOLOGICAL APPLICATIONS

In the previous chapters, the first order metric tensor written in the Newtonian gauge has been used to obtain the first order field equation and the Boltzmann equations in homogeneous and inhomogeneous spaces for collisionless relativistic and massive particles. Up to now, we have not been interested in the collision terms but in order to study the BBN, the CMB and the LSS formation, one needs to consider collisions and reactions between different kinds of particles. For instance, for the following reaction

$$1 + 2 \leftrightarrow 3 + 4 \quad (7.1)$$

in which the momentum and the energy must be conserved

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4; \quad E_1(\vec{p}_1) + E_2(\vec{p}_2) = E_3(\vec{p}_3) + E_4(\vec{p}_4), \quad (7.2)$$

the collision term is given by

$$\begin{aligned} C[f_1] = & \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} \\ & \times (2\pi)^4 \delta_D^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \delta_D^{(1)}(E_1 + E_2 - E_3 - E_4) |\mathcal{M}|^2 \\ & \times \{f_3 f_4 [1 \pm f_1] [1 \pm f_2] - f_1 f_2 [1 \pm f_3] [1 \pm f_4]\}. \end{aligned} \quad (7.3)$$

From this point forward, we follow the notation in Modern Cosmology [79] by Scott Dodelson. Here, Equation (7.3) needs some explanation. In the collision term for particle 1, the integrals in the first line are there to take all collision scenarios among the particles with different momenta into consideration. In the second line, while the delta functions guarantee that momentum and energy are conserved, $|\mathcal{M}|$ is the scattering amplitude. Finally, in the last line distribution functions determine the probability of being at any of accessible states for each particle. Plus and minus signs in the square brackets represent the Bose enhancement and the Pauli blocking, respectively. Using the collision term in the Boltzmann equation for homogeneous space (Equation 6.18)

gives

$$\begin{aligned}
a^{-3} \frac{d(n_1 a^3)}{dt} &= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 2E_4} \\
&\times (2\pi)^4 \delta_D^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \delta_D^{(1)}(E_1 + E_2 - E_3 - E_4) |\mathcal{M}|^2 \quad (7.4) \\
&\times \{f_3 f_4 [1 \pm f_1] [1 \pm f_2] - f_1 f_2 [1 \pm f_3] [1 \pm f_4]\}.
\end{aligned}$$

At this point, to simplify Equation (7.4) the quantum corrections in the distribution functions and the square brackets in the last line can be ignored. Writing the distribution function as

$$f(E) \rightarrow e^{(\mu-E)/T}, \quad (7.5)$$

the last line of Equation (7.4) turns into (since $E_1 + E_2 = E_3 + E_4$)

$$\begin{aligned}
&f_3 f_4 [1 \pm f_1] [1 \pm f_2] - f_1 f_2 [1 \pm f_3] [1 \pm f_4] \\
&\rightarrow e^{-(E_1+E_2)/T} \{e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T}\}. \quad (7.6)
\end{aligned}$$

By definition the number density for any kind of particle s is

$$n_s \equiv g_s e^{\mu_s/T} \int \frac{d^3 p}{(2\pi)^3} e^{-E_s(p)/T}, \quad (7.7)$$

where g_s and μ are the degeneracy number and the chemical potential, respectively. Since the chemical potential is independent of momentum, it can be taken outside the integral. The number densities without incorporating the chemical potential (denoted by $n_s^{(0)}$) for relativistic and non-relativistic cases are

$$n_s^{(0)} \equiv g_s \int \frac{d^3 p}{(2\pi)^3} e^{-E_s(p)/T} = \begin{cases} g_s \left(\frac{m_s T}{2\pi}\right)^{3/2} e^{-m_s/T} & \text{if } m_s \gg T, \\ g_s \frac{T^3}{\pi^2} & \text{if } m_s \ll T. \end{cases} \quad (7.8)$$

Defining the following as the thermally averaged cross-section

$$\begin{aligned}
\langle \sigma v \rangle &\equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 2E_4} e^{-(E_1+E_2)/T} \\
&\times (2\pi)^4 \delta_D^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \delta_D^{(1)}(E_1 + E_2 - E_3 - E_4) |\mathcal{M}|^2, \quad (7.9)
\end{aligned}$$

and using the number density definitions in Equation (7.7) and Equation (7.8) make

it possible to write Equation (7.4) as

$$a^{-3} \frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right). \quad (7.10)$$

In case of equilibrium, since the left hand side of Equation (7.10) is zero (no change in $n_1 a^3$ over time), the right hand side should be equal zero too and so

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}. \quad (7.11)$$

Finding the collision term of photons with electrons (Compton scattering) is a long calculation. So, at this point we will not go into details (the details can be found in Modern Cosmology [79] by Scott Dodelson) and it is given by

$$C[f(\vec{p})] = -p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b], \quad (7.12)$$

where n_e , σ_T and $\vec{v}_b$ are the number density of electrons, the Thomson cross section and the baryon velocity, respectively. One can use Equation (6.45) and Equation (7.12) to write the Boltzmann equation for photons

$$\dot{\Theta} + \frac{\left(1 + \frac{\kappa r^2}{4}\right)}{a^2} \partial_i \Theta \hat{p}^i + \dot{\Phi} + \frac{\left(1 + \frac{\kappa r^2}{4}\right)}{a^2} \partial_i \Psi \hat{p}^i = n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b]. \quad (7.13)$$

With the flat space approximation and the Fourier transformed Θ , Ψ and Φ , Equation (7.13) can be written in conformal time as ($'$ denotes the conformal time derivative, k is the wave number and tilde represents the Fourier transformation of a function)

$$\tilde{\Theta}' + ik\mu\tilde{\Theta} + \tilde{\Phi}' + ik\mu\tilde{\Psi} = -\tau' [\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b], \quad (7.14)$$

where

$$\mu \equiv \frac{\vec{k} \cdot \hat{p}}{k}, \quad (7.15)$$

and

$$\tau' \equiv \frac{d\tau}{d\eta} = -n_e \sigma_T a^2. \quad (7.16)$$

With Equation (7.16), one can define the optical depth

$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} d\eta' n_e \sigma_T a^2. \quad (7.17)$$

For the Boltzmann equation for dark matter particles, there is no need for the collision term because the dark matter particles do not interact with any other form of matter and radiation. Since the relativistic energy is

$$E = \sqrt{p^2 + m^2} \quad (7.18)$$

and so

$$\frac{\partial f}{\partial p} p = \frac{\partial f}{\partial E} \frac{p^2}{E}, \quad (7.19)$$

Equation (6.39) turns into

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{(1 + \kappa r^2/4)}{a^2} \frac{\partial f}{\partial x^i} \frac{p}{E} \hat{p}^i - \frac{\partial f}{\partial E} \left(\frac{\bar{H}}{a} \frac{p^2}{E} + \dot{\Phi} \frac{p^2}{E} + \frac{(1 + \kappa r^2/4)}{a^2} \partial_i \Psi p \hat{p}^i \right). \quad (7.20)$$

Finding the zeroth moment of this equation is straightforward and it can be found as

$$\frac{\partial n_{dm}}{\partial t} + \frac{(1 + \kappa r^2/4)}{a^2} \frac{\partial}{\partial x^i} (n_{dm} v^i) + 3 \left(\frac{\bar{H}}{a} + \dot{\Phi} \right) n_{dm} = 0. \quad (7.21)$$

Ignoring the zeroth order part, which we already know,

$$\frac{\partial \bar{n}_{dm}}{\partial t} + 3 \frac{\bar{H}}{a} \bar{n}_{dm} = 0, \quad (7.22)$$

one can get the first order part like

$$\frac{\partial \delta_{dm}}{\partial t} + \frac{(1 + \kappa r^2/4)}{a^2} \frac{\partial v^i}{\partial x^i} + 3 \dot{\Phi} = 0, \quad (7.23)$$

where we have used

$$n_{dm} = \bar{n}_{dm} [1 + \delta_{dm}(\vec{x}, t)]. \quad (7.24)$$

Also, the first moment of Equation (7.20), which is needed for the velocity of dark matter, is

$$\frac{\partial}{\partial t} (n_{dm} v^j) + 4 \frac{\bar{H}}{a} n_{dm} v^j + n_{dm} \frac{(1 + \kappa r^2/4)}{a^2} \frac{\partial \Psi}{\partial x^j} = 0. \quad (7.25)$$

Inserting the zeroth order part, (Equation (7.22)), reduces Equation (7.25) to

$$\frac{\partial v^j}{\partial t} + \frac{\bar{H}}{a} v^j + \frac{(1 + \kappa r^2/4)}{a^2} \frac{\partial \Psi}{\partial x^j} = 0. \quad (7.26)$$

Again, the Fourier transformed versions of Equation (7.23) and Equation (7.26) after

the flat space assumption can be written in conformal time as

$$\tilde{\delta}'_{dm} + ik\tilde{v}_{dm} + 3\tilde{\Phi}' = 0, \quad (7.27)$$

$$\tilde{v}'_{dm} + \frac{a'}{a}\tilde{v}_{dm} + ik\tilde{\Psi} = 0. \quad (7.28)$$

For the Boltzmann equations for baryons, the same procedure as in the dark matter case can be followed. The only difference is that baryons will interact with photons. So, the collision term should be determined but again it is a long calculation to give here. The derivation can be found in Modern Cosmology [79] by Scott Dodelson and the equations for baryons are

$$\tilde{\delta}'_b + ik\tilde{v}_b + 3\tilde{\Phi}' = 0 \quad (7.29)$$

and

$$\tilde{v}'_b + \frac{a'}{a}\tilde{v}_b + ik\tilde{\Psi} = \tau' \frac{4p_\gamma}{3p_b} \left[3i\tilde{\Theta}_1 + \tilde{v}_b \right], \quad (7.30)$$

where the dipole for temperature fluctuations is

$$\Theta_1 \equiv i \int_{-1}^1 \frac{d\mu}{2} \mu \Theta(\mu). \quad (7.31)$$

In general, for any moment of $\Theta = \delta T/T$, the following expression, in which P_l are Legendre polynomials, will be used throughout this paper

$$\Theta_l = \frac{1}{(-i)^l} \int_{-1}^1 \frac{d\mu}{2} P_l(\mu) \Theta(\mu). \quad (7.32)$$

7.1. Big Bang Nucleosynthesis

The Big Bang Nucleosynthesis is one of the main areas for the very early universe which has to be studied in detail. At that time, the universe consisted of some relativistic particles such as photons, neutrinos and electrons, and some non-relativistic ones such as protons and neutrons. Protons and neutrons are the essential ones responsible for the nucleosynthesis and their ratio is what we need in order to trace the light nuclei abundances. However, since free neutrons are not stable and they decay into protons

over time, we first should focus on the deuteron production. So, the reaction that has to be considered is



Since we look for the equilibrium and the chemical potential for photons is zero, i.e., $n_\gamma^{(0)} = n_\gamma$, the right hand side of Equation (7.10) gives

$$\frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}}. \quad (7.34)$$

Using Equation (7.8) for the right hand side of Equation (7.34) yields

$$\frac{n_D}{n_n n_p} = \frac{3}{4} \left(\frac{2\pi m_D}{m_n m_p T} \right)^{3/2} e^{(m_n + m_p - m_D)/T}. \quad (7.35)$$

At this point, one can ignore the small mass difference between the mass of a deuteron and the summation of the rest masses of a proton and a neutron in the pre-factor. However, this difference in the argument of the exponential must be kept. Hence, Equation (7.35) turns into

$$\frac{n_D}{n_n n_p} = \frac{3}{4} \left(\frac{4\pi}{m_p T} \right)^{3/2} e^{B_D/T}, \quad (7.36)$$

where B_D is the binding energy of deuteron, $B_D = 2.22$ MeV. In general, the binding energy for a nucleus N with atomic number Z and mass number A is given by the formula

$$B_N \equiv Zm_p + (A - Z)m_n - m_N. \quad (7.37)$$

Again, disregarding the small mass difference between proton and neutron and using the baryon to photon ratio derived for our theoretical model

$$\eta_b \equiv \frac{n_b}{n_\gamma} = 3.04 \times 10^{-8} \alpha h^2 \Omega_{b0}, \quad (7.38)$$

we can rewrite Equation (7.36) as (since the photon number density is proportional to T^3)

$$\frac{n_D}{n_b} \sim \eta_b \left(\frac{T}{m_p} \right)^{3/2} e^{B_D/T}. \quad (7.39)$$

In order to roughly determine the temperature at which the nucleosynthesis takes place one should consider Equation (7.39). In equilibrium, the deuteron to baryon ratio is

order of unity, so one can take the natural logarithm of both sides of Equation (7.39) to get

$$\ln(\eta_b) + \frac{3}{2} \ln(T_{nuc}/m_p) \sim -\frac{B_D}{T_{nuc}}. \quad (7.40)$$

One can solve this equation numerically once the exact value of η_b is known but for now we can deduce an approximate value for T_{nuc} by $\eta_b = 3.04 \times 10^{-8} \alpha h^2 \Omega_{b0}$. With the assumptions $\alpha = 1$, $h = 0.7$ and $\Omega_{b0} = 0.2$, Equation (7.40) yields $T_{nuc} \simeq 0.065$ MeV.

7.1.1. Neutron Abundance and Helium Fraction

For the neutron abundance, the starting point is the proton to neutron ratio in terms of their number densities calculated according to Equation (7.8) and it is

$$\frac{n_p^{(0)}}{n_n^{(0)}} = \frac{e^{-m_p/T} \int dp p^2 e^{-p^2/2m_p T}}{e^{-m_n/T} \int dp p^2 e^{-p^2/2m_n T}} = \frac{g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} e^{-m_p/T}}{g_n \left(\frac{m_n T}{2\pi}\right)^{3/2} e^{-m_n/T}}. \quad (7.41)$$

Again, ignoring the small mass difference in the prefactor results in

$$\frac{n_p^{(0)}}{n_n^{(0)}} = e^{\mathcal{Q}/T}, \quad (7.42)$$

where $\mathcal{Q} \equiv m_n - m_p = 1.293$ MeV. Since the neutron to proton decay reaction is

$$n \leftrightarrow p + e^- + \bar{\nu}_e, \quad (7.43)$$

and for leptons $n_l = n_l^{(0)}$, Equation (7.10) gives

$$a^{-3} \frac{d(n_n a^3)}{dt} = n_l^{(0)} \langle \sigma v \rangle \left(\frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n \right). \quad (7.44)$$

After defining the ratio of neutrons to the total number of neutrons and protons as

$$X_n \equiv \frac{n_n}{n_n + n_p}, \quad (7.45)$$

and $n_l^{(0)} \langle \sigma v \rangle$ as λ_{np} , Equation (7.44) reduces to

$$\frac{dX_n}{dt} = \lambda_{np} [(1 - X_n) e^{-\mathcal{Q}/T} - X_n]. \quad (7.46)$$

Now, in order to simplify this expression, one can define a new variable like

$$x \equiv \frac{Q}{T}. \quad (7.47)$$

Since $\dot{x} = -x\dot{T}/T$ and $T \propto a^{-1}$,

$$\frac{\dot{T}}{T} = -\frac{\bar{H}}{a} = -\sqrt{\frac{8\pi\rho}{3\alpha\phi_0}}. \quad (7.48)$$

The decay process takes place in the radiation dominated era, so ρ in Equation (7.48) is the energy density of relativistic particles given by

$$\begin{aligned} \rho &= \frac{\pi^2}{30} T^4 \left(\sum_{s=\text{bosons}} g_s + \frac{7}{8} \sum_{s=\text{fermions}} g_s \right) \\ &\equiv g_* \frac{\pi^2}{30} T^4, \end{aligned} \quad (7.49)$$

where g_s is the number of degrees of freedom for each type of particle ($g_\gamma = 2, g_\nu = 6$ and $g_{e^-} = g_{e^+} = 2$) and g_* is the effective number of degrees of freedom ($g_* \simeq 10.75$). With the new variable x and Equation (7.48), Equation (7.46) turns out to be

$$\frac{dX_n}{dx} = \frac{ax\lambda_{np}}{\bar{H}(x=1)} [e^{-x} - X_n(1 + e^{-x})], \quad (7.50)$$

where λ_{np} [80, 81] is

$$\lambda_{np} = \frac{255}{\tau_n x^5} (12 + 6x + x^2). \quad (7.51)$$

Here, τ_n is the mean life of neutron and one can take the numerical integration of Equation (7.50) to plot the temporal evolution of neutron abundance as can be checked in Figure 7.1. The vertical axis of this graph is the ratio of the number of neutrons to the total number of protons and neutrons. In our model there is factor of $\sqrt{\alpha}$ in the right hand side of Equation (7.50) which is absent in the original equation based on the standard model of cosmology. Since Equation (7.50) could not be solved analytically, the plots for different values of α have been given in Figure 7.1. The ratios freeze out at a temperature approximately 0.1 MeV and for $\alpha = 1$, $X_n \simeq 0.12$ in our model. Below this temperature, there are two kinds of processes that must be taken into consideration, the neutron decay and the deuteron production. As we have seen previously, the deuteron production starts at $T_{nuc} \simeq 0.065$ MeV. Therefore, as

the temperature drops from 0.1 MeV to 0.065 MeV, the neutron decay continues to take place. We can use the following time-temperature relation to account the decay process

$$t = 132.4\sqrt{\alpha} \left(\frac{0.1 \text{ MeV}}{T} \right)^2 \text{ second.} \quad (7.52)$$

Since $\tau_n = 885.7 \pm 0.8 \text{ s}$, the decay rate for $\alpha = 1$ is $\exp[-(132.4/885.7)(0.1/0.065)^2] = 0.70$. So, the neutron abundance just before the nucleosynthesis starts is

$$X_n(T_{nuc}) \simeq 0.084. \quad (7.53)$$

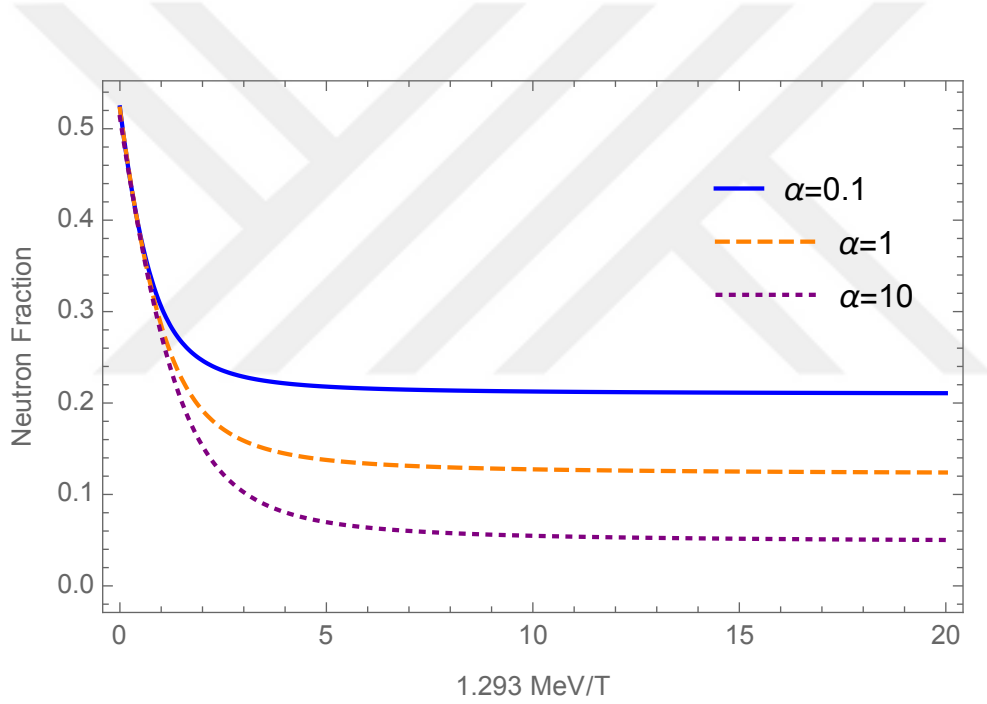


Figure 7.1. Neutron fraction over time. The vertical and the horizontal axes are

$$X_n = n_n/(n_n + n_p) \text{ and } Q/T \text{ respectively.}$$

For every helium nucleus, 2 neutrons and 2 protons are needed. So, the number of helium nuclei produced is half of the number of neutrons. However, the mass of a helium nucleus is four times the mass of a proton or neutron, so the mass fraction of helium is

$$Y_P = 2X_n(T_{nuc}). \quad (7.54)$$

7.2. Recombination

Determining the redshift or the scale factor of the recombination and decoupling epochs is very important for the study of the CMB. When electrons come together with protons to form neutral hydrogen atoms, photons decoupled from electrons travel freely in space from the time of decoupling to us today. So, the following reaction is the one that has to be taken into consideration to study the recombination



For the equilibrium (in light of Equation (7.10)), one gets

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}, \quad (7.56)$$

however a more accurate approach requires tracking the evolution of the electron number density over time and solving the Boltzmann equation in Equation (7.10) for the reaction in Equation (7.55). With the following definition for the fraction of free electrons (assuming the universe is neutral in electric charge that means the number of electrons and the number of protons are equal)

$$X_e \equiv \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}, \quad (7.57)$$

using Equation (7.8) for the right hand side of Equation (7.56) yields the Saha equation (again, we ignore the small mass difference between hydrogen atom and proton in the prefactor)

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left[\left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-(m_e + m_p - m_H)/T} \right]. \quad (7.58)$$

The binding energy of hydrogen atom is $\zeta \equiv m_e + m_p - m_H = 13.6$ eV in the argument of the exponential. Since $n_e + n_H = \eta_b n_\gamma = 3.04 \times 10^{-8} (\alpha h^2 \Omega_{b0}) (\pi^2/45) T^3 \sim 10^{-10} T^3$, the right hand side of Equation (7.58) is of order $10^{10} (m_e/T)^{3/2} \simeq 10^{16}$ at a temperature that is of order 13.6 eV. Therefore, X_e must be very close to 1 so that Equation (7.58) is satisfied, which means that almost all electrons are free and the recombination occurs at a much lower temperature. A more proper way to track the number of free electrons

is solving the following Boltzmann equation

$$\begin{aligned} a^{-3} \frac{d(n_e a^3)}{dt} &= n_e^{(0)} n_p^{(0)} \langle \sigma v \rangle \left(\frac{n_H}{n_H^{(0)}} - \frac{n_e^2}{n_e^{(0)} n_p^{(0)}} \right) \\ &= n_b \langle \sigma v \rangle \left[(1 - X_e) \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\zeta/T} - X_e^2 n_b \right]. \end{aligned} \quad (7.59)$$

The number of electrons is equal to $n_b X_e$ and the baryon number is conserved in the production of hydrogen atoms. So, in the left hand side of Equation (7.59), n_b can be taken outside the derivative and the equation reduces to

$$\frac{dX_e}{dt} = [(1 - X_e) \xi - X_e^2 n_b \gamma^{(2)}], \quad (7.60)$$

where

$$\xi \equiv \langle \sigma v \rangle \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\zeta/T} \quad (7.61)$$

and

$$\gamma^{(2)} \equiv \langle \sigma v \rangle. \quad (7.62)$$

Here, the superscript (2) denotes the recombination for the first excited state. This is because an electron recombined with a proton at the ground state will emit a photon that is able to ionize another hydrogen atom and so the net effect for the recombination process will be zero. The thermally averaged cross section for the first excited state is given by

$$\gamma^{(2)} = 9.78 \frac{\alpha_{QED}^2}{m_e^2} \left(\frac{\zeta}{T} \right)^{1/2} \ln \left(\frac{\zeta}{T} \right). \quad (7.63)$$

In Equation (7.63), α_{QED} is not the coupling parameter in our model but the fine structure constant in the quantum electrodynamics.

The results of the numerical integrations of the Saha approximation (Equation (7.58)) have been plotted in Figure 7.2 for $h = 0.7$, $\Omega_{b0} = 0.2$ and different α values. As it can be seen, the redshift of the recombination is found $1800 > z_* > 1100$ in our model for some reasonably chosen α values.

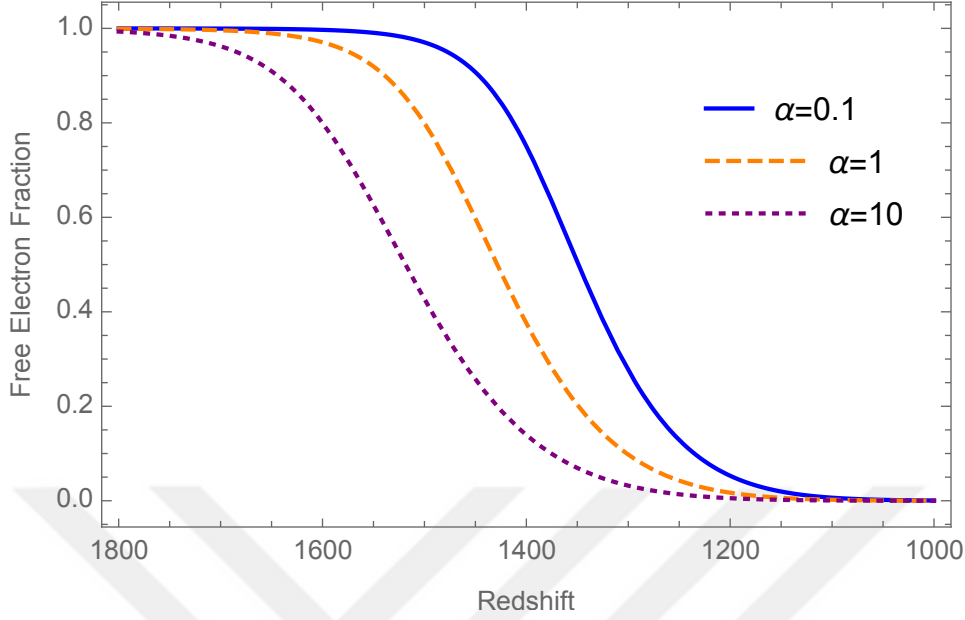


Figure 7.2. The curves represent the Saha approximation to the temporal evolution of the free electron fraction for various α values.

Now, we can proceed to determine the decoupling era as a function of redshift (or the scale factor). It is the era in which the Compton scattering rate of photons from electrons is smaller than the expansion rate. We ignore some relativistic effects and use the Thomson cross section that is $\sigma_T = 0.665 \times 10^{-24} \text{ cm}^2$. Thus, the scattering rate is

$$n_e \sigma_T = X_e n_b \sigma_T = 7.477 \times 10^{-30} X_e \Omega_{b0} \alpha h^2 a^{-3} \text{ cm}^{-1}, \quad (7.64)$$

where we have used $m_p n_b / \rho_{cr} = \Omega_{b0} a^{-3}$ since the baryon number is equal to the sum of free protons and hydrogen atoms (helium abundance is disregarded). The ratio to the expansion rate is

$$\frac{n_e \sigma_T}{\dot{a}/a} = 0.0692 X_e \Omega_{b0} \alpha h a^{-3/2} \Omega_{m0}^{-1/2} (1 + a_{eq}/a)^{-1/2}, \quad (7.65)$$

where we have used

$$\dot{a} = \bar{H} = \bar{H}_0 \Omega_{m0}^{1/2} a^{-1/2} (1 + a_{eq}/a)^{1/2}. \quad (7.66)$$

In order to get a crude estimation for the redshift of the decoupling era from Equation (7.65), one can make the right hand side equal to 1 and solve it for the scale factor. X_e

on the right is also a function of the scale factor and we have no analytically obtained expression for it. However, one can extract a function from Figure 7.2 for X_e and then use it in Equation (7.65). Following this procedure for the plot of $\alpha = 1$ in Figure 7.2 yields the redshift of decoupling era around ≈ 1200 . This means that, as expected, the recombination period is followed by the decoupling process.

7.3. Solutions to Radiation and Matter Perturbations

Up to now, we have used the zeroth order Boltzmann equations to study the nucleosynthesis in the early universe and the recombination process. In this section, however, we will focus on the perturbed Boltzmann equations since the CMB and the LSS are what we observe as perturbations in the radiation and the matter densities, respectively. We need to know their temporal evolution during periods dominated by both radiation and matter.

7.3.1. Solutions to Tightly Coupled Radiation Perturbations

Radiation perturbations in the early radiation dominated era are given by the solutions to the following equations. The first two of them are gained after contracting Equation (7.14) with the zeroth and the first order Legendre polynomials (from this moment on, we omit the tildes that denote the Fourier transformations of the functions). Θ_0 and Θ_1 are the monopole and dipole terms, respectively. The third equation is the velocity equation for the baryon-photon fluid. The equations, in this order, are

$$\Theta'_0 + k\Theta_1 = -\Phi', \quad (7.67)$$

$$\Theta'_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Phi + \tau' \left[\Theta_1 - \frac{iv_b}{3} \right] \quad (7.68)$$

and

$$v_b = -3i\Theta_1 + \frac{R}{\tau'} \left[v'_b + \frac{a'}{a}v_b - ik\Phi \right]. \quad (7.69)$$

Before photons were able to move freely in space, they were tightly coupled to electrons until the period of decoupling. Thanks to the free streaming of photons, we are able to observe radiation perturbations as acoustic peaks at certain angular scales today. So, we need the radiation perturbation solutions in this epoch. The derivation in the standard model is the same as in our model. So, the detailed derivation will not be given here but it can be checked in [79] and the equation to be solved is

$$\Theta_0'' + \frac{a'}{a} \frac{R}{1+R} \Theta_0' + k^2 c_s^2 \Theta_0 = S(k, \eta). \quad (7.70)$$

The source term $S(k, \eta)$ and the sound speed c_s are respectively given by

$$S(k, \eta) = \frac{k^2}{3} \Phi - \frac{a'}{a} \frac{R}{1+R} \Phi' - \Phi'', \quad (7.71)$$

and

$$c_s(\eta) = \sqrt{\frac{1}{3(1+R(\eta))}}, \quad (7.72)$$

where

$$R = \frac{3\rho_b}{4\rho_r}. \quad (7.73)$$

The last two terms in the source can be ignored in comparison with the first one because they are of order $1/\eta^2$ while the first term is of order k^2 and we are interested in the scales smaller than the comoving Hubble radius $(a\bar{H})^{-1}$ (throughout this section, the scales smaller than the comoving Hubble radius will be called as the sub-horizon scales and the larger ones as the super-horizon scales).

In order to solve Equation (7.70) in the deep radiation dominated era, one needs the field equation (dark and baryonic matter density perturbations can be ignored compared to radiation perturbations for this period of time). In this era, the source term is dependent on Θ_0 (Equation (5.40), we have used the fact that $\Psi = -\Phi$) by the relation

$$\Phi'' + k^2 \Phi = -\frac{32\pi}{3\beta\phi_0} a^4 \bar{\rho}_r \Theta_0. \quad (7.74)$$

Using the Friedmann-like equation (Equation (3.23)) in Equation (7.74), one can

rewrite it as

$$\begin{aligned}
\Phi'' + k^2\Phi &= -4\epsilon a^2 \left(\frac{8\pi}{3\alpha\phi_0} a^2 \bar{\rho}_r \right) \Theta_0 \\
&= -4\epsilon a^2 \bar{H}^2 \Theta_0 \\
&= -4\epsilon a_{eq} \bar{H}_0^2 \Theta_0,
\end{aligned} \tag{7.75}$$

where $\epsilon \equiv \alpha/\beta$ which is the ratio of two parameters resulting from the JBD coupling parameter ω for the zeroth and first order calculations. Also, we have used $a^2 \bar{H}^2 = a_{eq} \bar{H}_0^2$ in the second line for the radiation dominated epoch. Ignoring the first term on the left hand side of Equation (7.75) and using it in Equation (7.70) (here we also set $R = 0$ in the damping term because of $\rho_r \gg \rho_b$ for this era and there is a more effective damping process called diffusion damping, which will be introduced later, in comparison with this term) yields

$$\Theta_0'' + \left(k^2 c_s^2 + \frac{4}{3} \epsilon a^2 \bar{H}^2 \right) \Theta_0 = 0. \tag{7.76}$$

This equation can be arranged after taking $c_s = 1/\sqrt{3}$ like

$$\Theta_0'' + \frac{k^2}{3} \left(1 + \frac{4\epsilon a_{eq} \bar{H}_0^2}{k^2} \right) \Theta_0 = 0. \tag{7.77}$$

The solution to Equation (7.77) is

$$\Theta_0(\eta) = \Theta_0(\eta_i) \cos[\omega(\eta - \eta_i)], \tag{7.78}$$

Here, the other solution $\sin(\omega\eta)$ has been omitted because of the initial conditions, subscript i in η_i stands for initial and the frequency ω is defined as

$$\omega = \frac{k}{\sqrt{3}} \left(1 + \frac{4\epsilon a_{eq} \bar{H}_0^2}{k^2} \right)^{\frac{1}{2}} \simeq \frac{k}{\sqrt{3}}. \tag{7.79}$$

We have just obtained that the monopole term for radiation perturbations is oscillating. This is expected because of radiation pressure that creates repulsion while gravity attracts. In addition, we assume that radiation behaves like a perfect fluid because of its coupling to electrons via Compton scattering. So, the mean free paths of tightly coupled photons are too short and only the monopole and dipole terms are important. In the next subsection, however, we will abandon this perfect fluid assumption.

tion and calculate the diffusion damping scale for radiation perturbations. They are reduced by a factor of e^{-k^2/k_D^2} because of diffusion damping (or Silk damping). Thus, at some point in the radiation dominated era, the source term should have been determined by dark matter perturbations. When this happens, potential perturbations are independent of radiation perturbations and Equation (7.70) turns into an inhomogeneous differential equation due to the source term. Now, the solution to Equation (7.70), which is the solution of the monopole, can be found as

$$\begin{aligned} \Theta_0(\vec{k}, \eta) &= \Theta_0(\vec{k}, \eta_i) \cos \{k r_s(\eta)\} \\ &\quad - \frac{k}{\sqrt{3}} \int_{\eta_t}^{\eta} d\tilde{\eta} \Psi(\vec{k}, \tilde{\eta}) \sin \{k [r_s(\eta) - r_s(\tilde{\eta})]\} \quad \text{if } \eta > \eta_t. \end{aligned} \quad (7.80)$$

Here, we have introduced the comoving sound horizon $r_s(\eta) = \int_{\eta_i}^{\eta} c_s d\eta$ and η_t (subscript t denotes transition) that is the conformal time at which the transition from radiation perturbations to dark matter perturbations for the source term occurs. For the dipole, the monopole solution can be used in Equation (7.67) by ignoring the conformal time derivative of Φ to get

$$\begin{aligned} \Theta_1(\vec{k}, \eta) &= \frac{1}{\sqrt{3}} \Theta_0(\vec{k}, \eta_i) \sin \{k r_s(\eta)\} \\ &\quad + \frac{k}{3} \int_{\eta_t}^{\eta} d\tilde{\eta} \Psi(\vec{k}, \tilde{\eta}) \cos \{k [r_s(\eta) - r_s(\tilde{\eta})]\} \quad \text{if } \eta > \eta_t. \end{aligned} \quad (7.81)$$

The terms in the first lines of the monopole and dipole solutions, which are the initial inhomogeneities, will be damped (with a factor of e^{-k^2/k_D^2}). However, the integral terms in the second lines grow once the potential perturbations start to be specified by dark matter perturbations. So, these terms are responsible for generating the peaks in the CMB at certain angular scales. Incorporating the diffusion damping factor when obtaining the general solutions (via the Green's function method) to the monopole and dipole results in

$$\begin{aligned} \Theta_0(\vec{k}, \eta) &= \Theta_0(\vec{k}, \eta_i) \cos \{k r_s(\eta)\} e^{-k^2/k_D^2(\eta)} \\ &\quad - \frac{k}{\sqrt{3}} \int_{\eta_t}^{\eta} d\tilde{\eta} \Psi(\vec{k}, \tilde{\eta}) e^{-k^2 \left(\frac{1}{k_D^2(\eta)} - \frac{1}{k_D^2(\tilde{\eta})} \right)} \sin \{k [r_s(\eta) - r_s(\tilde{\eta})]\} \quad \text{if } \eta > \eta_t \end{aligned} \quad (7.82)$$

and

$$\begin{aligned} \Theta_1(\vec{k}, \eta) = & \frac{1}{\sqrt{3}} \Theta_0(\vec{k}, \eta_i) \sin \{k r_s(\eta)\} e^{-k^2/k_D^2(\eta)} \\ & + \frac{k}{3} \int_{\eta_t}^{\eta} d\tilde{\eta} \Psi(\vec{k}, \tilde{\eta}) e^{-k^2 \left(\frac{1}{k_D^2(\eta)} - \frac{1}{k_D^2(\tilde{\eta})} \right)} \cos \{k [r_s(\eta) - r_s(\tilde{\eta})]\} \quad \text{if } \eta > \eta_t. \end{aligned} \quad (7.83)$$

7.3.2. Diffusion Damping

As we have mentioned before, the oscillating nature of the solutions to the monopole and dipole have been obtained based on the assumption that radiation behaves like a perfect fluid. However, this is not the case because of the mean free paths of scattering photons that could be large enough to make radiation perturbations to diffuse. Hence, the amplitude of radiation perturbations will be damped. To account this fact, one should look for the imaginary part of the photon-baryon fluid velocity and the detailed derivation can be found in [79]. Derivation of the damping scale is the same as in our model, the only difference is the derivative of the optical depth τ' . In our model, it is defined as $\tau' = -n_e \sigma_T a^2$ whereas in the standard model it is $\tau' = -n_e \sigma_T a$. So, in our model, the damping scale is given by

$$\frac{1}{k_D^2(\eta)} \equiv \int_{\eta_i}^{\eta} \frac{d\tilde{\eta}}{6(1+R)n_e \sigma_T a^2(\tilde{\eta})} \left(\frac{R^2}{1+R} + \frac{8}{9} \right). \quad (7.84)$$

We can calculate the denominator of the integrand (set $R = 0$ again) as

$$n_e \sigma_T a^2 = 2.3 \times 10^{-5} \alpha X_e \Omega_{b0} h^2 a^{-1} (1 - Y_P/2) \text{ Mpc}^{-1}. \quad (7.85)$$

Here, the free electron density has been written as $n_e = X_e (\Omega_{b0} \rho_{cr0}/m_p)(1 - Y_P/2)$. Since Y_P is the mass fraction of helium, the ratio of the helium number to the baryon number is $Y_P/4$. Also, every helium atom contains two electron. So, when finding the free electron density, the baryon number should be multiplied by $(1 - Y_P/2)$. Assuming $R = 0$ and plugging Equation (7.85) in Equation (7.84) yields

$$\frac{1}{k_D^2(y)} = 3.86 \times 10^7 (\alpha X_e \Omega_{b0} h^2)^{-1} (1 - Y_P/2)^{-1} (\Omega_{m0} h^2)^{-1/2} a_{eq}^{1/2} [(1+y)^{1/2} - 1] \text{ Mpc}^2. \quad (7.86)$$

At this point, we have defined a new variable $y = a/a_{eq}$. Since, in our model $\Omega_{m0} \simeq 1$ (there is no dark energy in our model and $\Omega_{r0} \ll \Omega_{m0}$), we are left with

$$\frac{1}{k_D^2(y)} = 3.86 \times 10^7 (\alpha X_e \Omega_{b0} h^3)^{-1} (1 - Y_P/2)^{-1} a_{eq}^{1/2} [(1 + y)^{1/2} - 1] \text{ Mpc}^2. \quad (7.87)$$

In order to simplify this expression, one can write a_{eq} in terms of the other parameters of our model like ($\rho_{r0} \simeq 4.64 \times 10^{-31} \text{ kg/m}^3$)

$$a_{eq} \simeq \frac{\rho_r(\eta_0)}{\rho_{cr}(\eta_0)} = 4.64 \times 10^{-31} \frac{8\pi}{3\alpha\phi_0 \bar{H}_0^2} = \frac{2.5 \times 10^{-5}}{\alpha h^2}, \quad (7.88)$$

then Equation (7.87) reduces in Mpc^2 to

$$\frac{1}{k_D^2(y)} = 1.93 \times 10^5 (\alpha^{3/2} X_e \Omega_{b0} h^4)^{-1} (1 - Y_P/2)^{-1} [(1 + y)^{1/2} - 1] \text{ Mpc}^2 \quad (7.89)$$

and in $\bar{H}_0^{-2}$ to

$$\frac{1}{k_D^2(y)} = 2.15 \times 10^{-2} (\alpha^{3/2} X_e \Omega_{b0} h^2)^{-1} (1 - Y_P/2)^{-1} [(1 + y)^{1/2} - 1] \bar{H}_0^{-2}, \quad (7.90)$$

where

$$\bar{H}_0^{-1} \simeq 3000 h^{-1} \text{ Mpc}. \quad (7.91)$$

As is seen in Figure 7.3, the damping scale for $h = 0.7$, $\Omega_{b0} = 0.2$ and $\alpha = 1$ at the time of decoupling ($y = a/a_{eq} \approx 20$ since $a_{eq} \approx 5 \times 10^{-5}$ and $a_* \approx 10^{-3}$ for $h = 0.7$, $\Omega_{b0} = 0.2$ and $\alpha = 1$) is about 4000 Mpc. Especially, the comoving distance scales smaller than this will be damped accordingly. The comoving distance from the decoupling era to the present is about $60\bar{H}_0^{-1} \simeq 257143 \text{ Mpc}$. Thus, the angular scales corresponding to l values greater than $k(\eta_0 - \eta_*) = 257143/4000 \simeq 64$ are the ones that are significantly damped. Almost all of these l values are the ones at which we expect to observe the CMB anisotropy peaks. However, as we have seen in the previous section, even though the oscillating terms in the solutions to the monopole and dipole (Equation (7.82) and Equation (7.83)) are damped, the integral terms could be responsible for the acoustic peaks since the potential Ψ (studied in the upcoming sections) grows with time when it starts to be designated by dark matter perturbations.

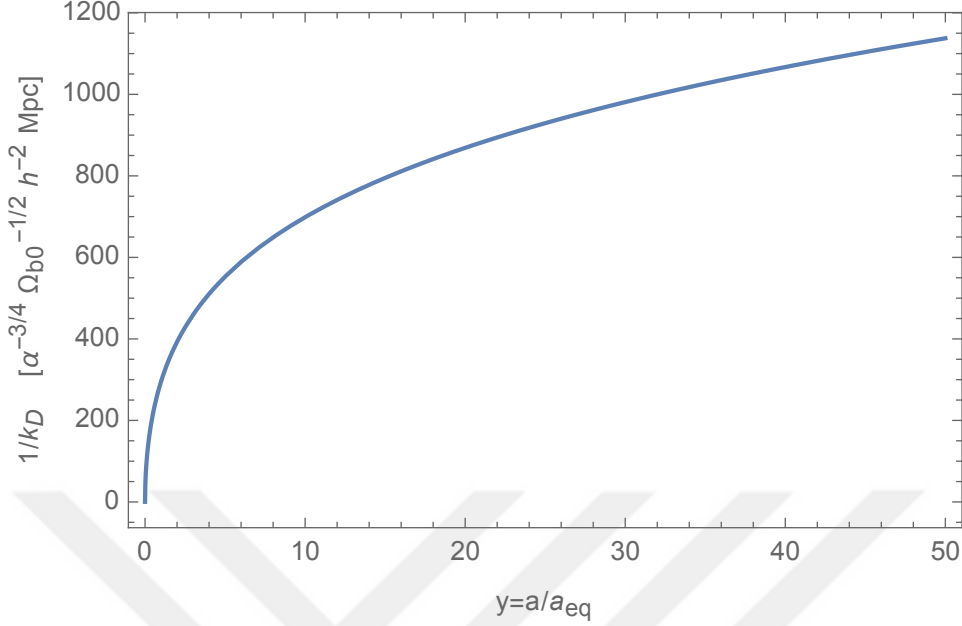


Figure 7.3. The graph shows the damping scale as a function of $y = a/a_{eq}$.

7.3.3. Dark Matter Perturbations in the Early Radiation Era

The dark matter perturbations are responsible for the LSS formation we observe today. So, their temporal evolution should be obtained as if they grow with time in order to promote the formation of galaxies and cluster of galaxies, etc. We start with the Boltzmann equations for dark matter, Equation (7.27) and Equation (7.28), and again the tildes are omitted. Inserting the conformal derivative of the velocity (Equation (7.28)) into Equation (7.27) yields

$$\delta''_{dm} - \frac{a'}{a} i k v_{dm} + k^2 \Psi + 3\Phi'' = 0. \quad (7.92)$$

Now, for the velocity in this equation, Equation (7.28) can be used to obtain

$$\delta''_{dm} + \frac{a'}{a} \delta'_{dm} + 3 \frac{a'}{a} \Phi' + k^2 \Psi + 3\Phi'' = 0. \quad (7.93)$$

Since $\Phi = -\Psi$, Equation (7.93) can be arranged as

$$\delta''_{dm} + \frac{a'}{a} \delta'_{dm} = -3\Phi'' + k^2 \Phi - 3 \frac{a'}{a} \Phi'. \quad (7.94)$$

The right hand side of the equation can be considered as a source term and one should look for the homogeneous solutions and the particular solution for this equation. To

get the homogeneous solutions, the equation to be solved is

$$\delta_{dm}'' + \frac{a'}{a} \delta_{dm}' = 0. \quad (7.95)$$

Since a'/a is $a_{eq}^{1/2} \bar{H}_0$ for the epoch dominated by radiation, Equation (7.95) turns into

$$\delta_{dm}'' + a_{eq}^{1/2} \bar{H}_0 \delta_{dm}' = 0. \quad (7.96)$$

Thus, the homogeneous solutions are

$$s_1 = C \quad (7.97)$$

and

$$s_2 = D \exp[-a_{eq}^{1/2} \bar{H}_0 (\eta - \eta_i)], \quad (7.98)$$

where C and D are constants. The general homogeneous solution is their summation and it is

$$\delta_{dm}(k, \eta) = C + D \exp[-a_{eq}^{1/2} \bar{H}_0 (\eta - \eta_i)]. \quad (7.99)$$

To get the particular solution, which is the integral of the source term $S(k, \eta)$ weighted by the Green's function

$$\int_{\eta_i}^{\eta} d\tilde{\eta} S(k, \tilde{\eta}) \left[\frac{s_1(\eta) s_2(\tilde{\eta}) - s_1(\tilde{\eta}) s_2(\eta)}{\dot{s}_1(\tilde{\eta}) s_2(\tilde{\eta}) - s_1(\tilde{\eta}) \dot{s}_2(\tilde{\eta})} \right], \quad (7.100)$$

one can use Equation (7.97) and Equation (7.98) and it is straightforward to obtain

$$\frac{1}{a_{eq}^{1/2} \bar{H}_0} \int_{\eta_i}^{\eta} d\tilde{\eta} S(k, \tilde{\eta}) \{1 - \exp[a_{eq}^{1/2} \bar{H}_0 (\tilde{\eta} - \eta)]\}, \quad (7.101)$$

where

$$S(k, \tilde{\eta}) = -3\Phi'' + k^2\Phi - 3\frac{a'}{a}\Phi'. \quad (7.102)$$

Here, we again ignore the first and third terms in the source and use Equation (7.75) and Equation (7.78) in Equation (7.101) to get

$$-4\epsilon a_{eq}^{1/2} \bar{H}_0 \Theta_0(\vec{k}, \eta_i) \int_{\eta_i}^{\eta} d\tilde{\eta} \cos\left(k/\sqrt{3} (\tilde{\eta} - \eta_i)\right) \{1 - \exp[a_{eq}^{1/2} \bar{H}_0 (\tilde{\eta} - \eta)]\}. \quad (7.103)$$

The result of this integral, which is the particular solution, is

$$\frac{4\epsilon\Theta_0(\vec{k}, \eta_i)}{1 + k^2(a_{eq}\bar{H}_0^2)^{-1}/3} \left[\cos\left(\frac{k}{\sqrt{3}}(\eta - \eta_i)\right) - \frac{1}{k(a_{eq}^{1/2}\bar{H}_0)^{-1}/\sqrt{3}} \sin\left(\frac{k}{\sqrt{3}}(\eta - \eta_i)\right) \right]. \quad (7.104)$$

The sine mode in the particular solution is smaller than the cosine mode by an order of $1/[k(a_{eq}^{1/2}\bar{H}_0)^{-1}/\sqrt{3}]$. So, one can disregard the sine mode and the exponential decaying homogeneous solution. Then, the solution turns out to be

$$\delta_{dm}(k, \eta) = C + \frac{4\epsilon\Theta_0(\vec{k}, \eta_i)e^{-k^2/k_D^2}}{1 + k^2(a_{eq}\bar{H}_0^2)^{-1}/3} \cos\left(\frac{k}{\sqrt{3}}(\eta - \eta_i)\right). \quad (7.105)$$

In Equation (7.105), the factor e^{-k^2/k_D^2} has been inserted manually since the monopole term is damped because of the diffusion damping. In addition, as anticipated, constant dark matter perturbations in the early radiation epoch are accompanied by an oscillating term due to oscillating radiation perturbations. This second oscillating term gets smaller for the smaller scales because of the factor of $k^2(a_{eq}\bar{H}_0^2)^{-1}$ in the denominator. Here, $(a_{eq}\bar{H}_0^2)^{-1/2}$ is the comoving Hubble radius of the radiation era.

7.3.4. Dark Matter Perturbations in the Late Radiation Era

As we have seen, radiation perturbations decay substantially because of the diffusion damping process. So, for all the scales at some point in time potential perturbations get the most of the contribution from dark matter perturbations. Even in the radiation era, Φ and Ψ may not be determined by Θ_0 because $\bar{\rho}_r\Theta_0$ could be smaller than $\bar{\rho}_{dm}\delta_{dm}$ because of $\Theta_0 \ll \delta_{dm}$ for some damped scales. For some larger ones this occurs in the matter era because $\bar{\rho}_r < \bar{\rho}_{dm}$. Here, it will be in the interest of the reader to clarify the technical jargon to avoid some confusion in the future. The scales on which the mentioned transition occurs in the radiation era will be called small scales, and the scales on which it occurs in the matter era will be called large scales. Now, we need to know the evolution of dark matter perturbations on these scales in the late radiation era and the matter era. To start with, the first order field equation is

$$\Phi'' + k^2\Phi = -\frac{4\pi}{3\beta\phi_0}a^4\delta\rho_{dm} = -\frac{\epsilon}{2}a^2\left(\frac{8\pi}{3\alpha\phi}\bar{\rho}_{dm}\right)\delta_{dm} = -\frac{\epsilon}{2}\frac{a^2\bar{H}^2}{(1 + a_{eq}/a)}\delta_{dm}. \quad (7.106)$$

At this point, we switch back to variable $y = a/a_{eq}$ and the conformal time derivative can be written in terms of this variable as

$$\frac{d}{d\eta} = \frac{dy}{d\eta} \frac{d}{dy} = \frac{d}{d\eta} \left(\frac{a}{a_{eq}} \right) \frac{d}{dy} = \frac{a'}{a_{eq}} \frac{d}{dy} = \frac{a^2}{a_{eq}} \frac{da}{dt} \frac{d}{dy} = y^2 a_{eq} \bar{H} \frac{d}{dy}. \quad (7.107)$$

Thus, the two Boltzmann equations for dark matter perturbations (Equation (7.27) and Equation (7.28)) and Equation (7.106) turn into

$$\frac{d\delta_{dm}}{dy} + \frac{ikv_{dm}}{y^2 a_{eq} \bar{H}} = -3 \frac{d\Phi}{dy}, \quad (7.108)$$

$$\frac{dv_{dm}}{dy} + \frac{v_{dm}}{y} = \frac{ik\Phi}{y^2 a_{eq} \bar{H}}, \quad (7.109)$$

and

$$k^2 \Phi = -\frac{\epsilon}{2} \frac{y^2 a_{eq}^2 \bar{H}^2}{(1 + 1/y)} \delta_{dm} = -\frac{\epsilon}{2} \frac{y^3}{(y + 1)} a_{eq}^2 \bar{H}^2 \delta_{dm}. \quad (7.110)$$

When writing Equation (7.110), we have neglected Φ'' term compared to $k^2 \Phi$. Taking derivative of Equation (7.108) with respect to y one more time

$$\frac{d^2 \delta_{dm}}{dy^2} + \frac{ik}{y^2 a_{eq} \bar{H}} \frac{dv_{dm}}{dy} - \frac{(1 + \frac{3}{2}y)ikv_{dm}}{y^3(1 + y)a_{eq} \bar{H}} = -3 \frac{d^2 \Phi}{dy^2}, \quad (7.111)$$

and for $\frac{dv_{dm}}{dy}$ in Equation (7.111), using Equation (7.109) gives

$$\frac{d^2 \delta_{dm}}{dy^2} + \frac{ik}{y^2 a_{eq} \bar{H}} \left(-\frac{v_{dm}}{y} + \frac{ik\Phi}{y^2 a_{eq} \bar{H}} \right) - \frac{(1 + \frac{3}{2}y)ikv_{dm}}{y^3(1 + y)a_{eq} \bar{H}} = -3 \frac{d^2 \Phi}{dy^2}, \quad (7.112)$$

which can be arranged as

$$\frac{d^2 \delta_{dm}}{dy^2} - \frac{(2 + \frac{5}{2}y)ikv_{dm}}{y^3(1 + y)a_{eq} \bar{H}} = -3 \frac{d^2 \Phi}{dy^2} + \frac{k^2 \Phi}{y^4 a_{eq}^2 \bar{H}^2}. \quad (7.113)$$

For the scales, which we are interested in, the second term on the right is larger than the first one by a factor of $k^2/(a\bar{H})^2$. Hence, it can be disregarded. In addition, since (Equation (7.110))

$$\frac{k^2 \Phi}{y^4 a_{eq}^2 \bar{H}^2} = -\frac{\epsilon}{2y(y + 1)} \delta_{dm}, \quad (7.114)$$

Equation (7.113) reduces to

$$\frac{d^2 \delta_{dm}}{dy^2} - \frac{(2 + \frac{5}{2}y)ikv_{dm}}{y^3(1 + y)a_{eq} \bar{H}} = -\frac{\epsilon}{2y(y + 1)} \delta_{dm}. \quad (7.115)$$

Finally, using Equation (7.108) for $(ikv_{dm})/(y^2 a_{eq} \bar{H})$ by ignoring the potential term which is smaller compared to the dark matter perturbation results in (like the Meszaros equation)

$$\frac{d^2 \delta_{dm}}{dy^2} + \frac{(2 + \frac{5}{2}y)}{y(1+y)} \frac{d\delta_{dm}}{dy} + \frac{\epsilon}{2y(1+y)} \delta_{dm} = 0. \quad (7.116)$$

The exact solution of this equation is not as simple as we can use it in Equation (7.106). However, with the approximation $y = a/a_{eq} \ll 1$ for the radiation dominated epoch it reduces to

$$\frac{d^2 \delta_{dm}}{dy^2} + \frac{2}{y} \frac{d\delta_{dm}}{dy} + \frac{\epsilon}{2y} \delta_{dm} = 0, \quad (7.117)$$

and the solution is given by

$$\delta_{dm}(k, y) = A \frac{I_1 \left[\sqrt{2|\epsilon|y} \right]}{\sqrt{|\epsilon|y/2}} + B \frac{K_1 \left[\sqrt{2|\epsilon|y} \right]}{\sqrt{|\epsilon|y/2}}. \quad (7.118)$$

These are the first order modified Bessel functions of the first and second kind. We are interested only in growing solutions and the second kind solution K_1 , which is a decaying one, blows up at the transition (at $y = y_t$, where subscript t stands for transition), so it is ignored. Since

$$\lim_{y \rightarrow y_t} \frac{I_1 \left[\sqrt{2|\epsilon|(y - y_t)} \right]}{\sqrt{|\epsilon|(y - y_t)/2}} = 1, \quad (7.119)$$

we are left with

$$\delta_{dm}(k, y) = \delta_{dm}(k, y_t) \frac{I_1 \left[\sqrt{2|\epsilon|(y - y_t)} \right]}{\sqrt{|\epsilon|(y - y_t)/2}} \quad \text{if } 1 \gg y > y_t. \quad (7.120)$$

This solution shows that dark matter perturbations for the scales on which the transition occurs in the radiation era, grow exponentially.

7.3.5. Dark Matter Perturbations in the Dark Matter Era

Equation (7.116) can be solved for $y = a/a_{eq} \gg 1$ to get a solution to dark matter perturbations in the late dark matter dominated era. In this limit, Equation (7.116)

becomes

$$\frac{d^2 \delta_{dm}}{dy^2} + \frac{5}{2y} \frac{d\delta_{dm}}{dy} + \frac{\epsilon}{2y^2} \delta_{dm} = 0. \quad (7.121)$$

The general solution to this equation is

$$\delta_{dm}(k, y) = Ay^n + By^m, \quad (7.122)$$

where

$$n = \frac{1}{4} (-3 + \sqrt{9 - 8\epsilon}) \text{ and } m = \frac{1}{4} (-3 - \sqrt{9 - 8\epsilon}). \quad (7.123)$$

We expect ϵ to be a negative parameter, so n will give a growing solution, which is needed for the formation of LSS. Thus, for large scales, dark matter perturbations are given by

$$\delta_{dm}(k, y) = \delta_{dm}(k, y_t) \left(\frac{y}{y_t} \right)^n \quad \text{if } y > y_t \gg 1. \quad (7.124)$$

For small scales, although the transition takes place in the radiation era and the evolution of δ_{dm} is given by Equation (7.120), this will no longer be true once $y \gg 1$. At some point, the exponential growth will be followed by the power law growth in the matter era.

7.4. The Matter Power Spectrum

In the early universe, potential perturbations are determined by radiation perturbations and it is known from Equation (7.75) that the expression showing the relation between them is

$$\Phi(k, \eta_i) = -\frac{4\epsilon}{k^2} \bar{H}_0^2 a_{eq} \Theta_0(k, \eta_i). \quad (7.125)$$

As time passes the oscillating radiation perturbations start to be reduced in amplitude because of the Silk damping (diffusion damping) and at some point in the radiation or matter era (while for some small scales this takes place in the radiation dominated era, for some large scales it occurs in the matter dominated era), they start to be determined by dark matter perturbations. After the transition potential perturbations

for small scales are given by

$$\begin{aligned}\Phi(k, \eta) &= -\frac{\epsilon}{2k^2} \frac{a^2 \bar{H}^2}{\left(1 + \frac{1}{y}\right)} \delta_{dm}(k, \eta) = -\frac{\epsilon}{2k^2} \bar{H}_0^2 a_{eq} y \delta_{dm}(k, \eta) \\ &= -\frac{\epsilon}{2k^2} \bar{H}_0^2 a_{eq} y \delta_{dm}(k, y_t) \frac{I_1 \left[\sqrt{2|\epsilon|(y - y_t)} \right]}{\sqrt{|\epsilon|(y - y_t)}/2} \quad \text{if } 1 \gg y > y_t.\end{aligned}\tag{7.126}$$

In the first line of Equation (7.126), Equation (5.40) is written for the case potential perturbations are dominated by dark matter perturbations. Then, in the second line, the temporal evolution equation of $\delta_{dm}(k, y)$ has been plugged from Equation (7.120). As is stated, this expression is valid when $1 \gg y > y_t(k)$. Here, $y_t(k)$ is the moment at which the transition takes place and it is a function of wavenumber k because for each wavenumber the transition period is different, earlier for smaller scales and later for larger ones (earlier for bigger wavenumbers, later for smaller wavenumbers). Damped potential perturbations in the early universe (because of damped radiation perturbations) can be made equal to potential perturbations at $y_t(k)$ (here we disregard the oscillating nature of the initial potential perturbations and focus on the amplitude of them) like

$$-\frac{4\epsilon}{k^2} \bar{H}_0^2 a_{eq} \Theta_0(k, y_i) e^{-k^2/k_D^2(y_t)} = -\frac{\epsilon}{2k^2} \bar{H}_0^2 a_{eq} y_t \delta_{dm}(k, y_t).\tag{7.127}$$

From Equation (7.127), it is obvious that

$$\delta_{dm}(k, y_t) = 8\Theta_0(k, y_i) \frac{e^{-k^2/k_D^2(y_t)}}{y_t}.\tag{7.128}$$

In Equation (7.105), one can see that dark matter perturbations are nearly constant when $y < y_t$ because the amplitude of oscillating second term is reduced by a factor of $1/[1 + k^2(a_{eq} \bar{H}_0^2)^{-1}/3]$ and e^{-k^2/k_D^2} relative the constant first term. Hence, after ignoring the second term in Equation (7.105) one can assume $\delta_{dm}(k, y_i) = \delta_{dm}(k, y_t)$ and we are left with

$$\delta_{dm}(k, y_i) = 8\Theta_0(k, y_i) \frac{e^{-k^2/k_D^2(y_t)}}{y_t}.\tag{7.129}$$

If there was a trivial relation between the initial radiation and matter perturbations like $\delta_{dm}(k, y_i)/\Theta_0(k, y_i) = \text{constant}$ (as in the case of adiabatic or isocurvature perturbations), then the factor of $e^{-k^2/k_D^2(y_t)}/y_t$ will be constant although this constant

value would be reached at different $y_t(k)$ values for different scales. Unfortunately, at this point we do not have the necessary information on this subject. In the inflationary cosmology, initial conditions for perturbations are related to the inflaton field. However, in our model there should be no mechanism like inflationary expansion. The first reason is we do not need this kind of expansion in order to solve the horizon and flatness problems mentioned previously. The second reason if we had inflationary expansion just after the birth of the universe, we would never be able to observe the scales taken beyond the comoving Hubble horizon because in our model the comoving Hubble horizon is constant in the radiation dominated era and getting smaller in the matter dominated era. Therefore, there is a need for a theoretical model that can produce the initial perturbations to replace the inflation theory. If we had such a model, $y_t(k)$ for each k mode could be obtained. For now, we prefer to stay away from speculation and content ourselves with giving the equations of how these perturbations will evolve over time. Using Equation (7.128) in Equation (7.126) results in

$$\Phi(k, y) = -\frac{4\epsilon}{k^2} \bar{H}_0^2 a_{eq} \Theta_0(k, y_i) e^{-k^2/k_D^2(y_t)} \frac{y}{y_t} \frac{I_1 \left[\sqrt{2|\epsilon|(y - y_t)} \right]}{\sqrt{|\epsilon|(y - y_t)/2}} \quad \text{if } 1 \gg y > y_t. \quad (7.130)$$

It has been also found in Equation (7.124) that the growth of dark matter perturbations turns from exponential to power law at some point in time. Unlike $y_t(k)$, this moment in time is not the function of k because it is not about the Silk damping, which depends on the scale, but about the change in the background equations. So, it is the same for all scales. We have solved Equation (7.116) for $y \ll 1$ and $y \gg 1$ but not for $y \sim 1$. So, how δ_{dm} behaves when $y \sim 1$ is not known because an analytically exact solution to Equation (7.116) has not been obtained. For now, to get an analytically approximate answer is the way to go and the crude approximation we make is to assume that δ_{dm} evolves until $y \sim 1$ according to Equation (7.120) and after $y \sim 1$ according to Equation (7.124). Therefore, the potential perturbations for $y > 1$ are given by

$$\Phi(k, y) = -\frac{4\epsilon}{k^2} \bar{H}_0^2 a_{eq} \Theta_0(k, y_i) \frac{e^{-k^2/k_D^2(y_t)}}{y_t} \frac{I_1 \left[\sqrt{2|\epsilon|(1 - y_t)} \right]}{\sqrt{|\epsilon|(1 - y_t)/2}} y^{1+n} \quad \text{if } y > 1 \quad (7.131)$$

or in terms of the initial potential perturbations by

$$\Phi(k, y) = \Phi(k, y_i) \frac{e^{-k^2/k_D^2(y_t)} I_1 \left[\sqrt{2|\epsilon|(1-y_t)} \right]}{y_t \sqrt{|\epsilon|(1-y_t)/2}} y^{1+n} \quad \text{if } y > 1. \quad (7.132)$$

In order to get the amplitudes of dark matter perturbations today, one can take advantage of Equation (5.40) and Equation (7.132). From Equation (5.40), potential perturbations determined by dark matter for today's universe is

$$\Phi(k, y_0) = -\frac{\epsilon}{2k^2} \bar{H}_0^2 a_{eq} y_0 \delta_{dm}(k, y_0) = -\frac{\epsilon}{2k^2} \bar{H}_0^2 \delta_{dm}(k, y_0). \quad (7.133)$$

Then, Equation (7.131) and Equation (7.133) can be made equal at $y = y_0$ like

$$-\frac{4\epsilon}{k^2} \bar{H}_0^2 a_{eq} \Theta_0(k, y_i) \frac{e^{-k^2/k_D^2(y_t)} I_1 \left[\sqrt{2|\epsilon|(1-y_t)} \right]}{y_t \sqrt{|\epsilon|(1-y_t)/2}} y_0^{1+n} = -\frac{\epsilon}{2k^2} \bar{H}_0^2 \delta_{dm}(k, y_0) \quad (7.134)$$

in order to get

$$\begin{aligned} \delta_{dm}(k, y_0) &= 8\Theta_0(k, y_i) \frac{e^{-k^2/k_D^2(y_t)} I_1 \left[\sqrt{2|\epsilon|(1-y_t)} \right]}{y_t \sqrt{|\epsilon|(1-y_t)/2}} y_0^n \\ &= 8\Theta_0(k, y_i) \frac{e^{-k^2/k_D^2(y_t)} I_1 \left[\sqrt{2|\epsilon|(1-y_t)} \right]}{y_t \sqrt{|\epsilon|(1-y_t)/2}} a_{eq}^{-n}. \end{aligned} \quad (7.135)$$

This is the relation for small scales that connects the initial radiation perturbations to today's dark matter perturbations. For large scales, a similar expression, except the function of exponential growth from $y = y_t$ to $y = 1$, can be written as

$$\delta_{dm}(k, y_0) = 8\Theta_0(k, y_i) \frac{e^{-k^2/k_D^2(y_t)}}{y_t} \left(\frac{y_0}{y_t} \right)^n. \quad (7.136)$$

7.5. The Cosmic Microwave Background

We have given the solutions to the tightly-coupled limit of radiation perturbations in the Boltzmann equations. What we observe today as the CMB are the moments of photons streaming freely in space from the era of decoupling to the present. For this reason, the era of decoupling is called the last scattering surface for photons. Since it is not an exact moment in time but a period of time, the last scattering surface may

differ for the CMB photons. At the time of decoupling, we are interested only in the contributions of each k mode to the monopole and dipole terms because the mean free path of photons are so small. In other words, higher moments can not survive because of the Compton scattering. However, for the CMB anisotropy spectrum today, one needs a solution for $\Theta_l(k, \eta_0)$. The derivation of the solution will not be given here but the details can be found in [79]. We can use the following same expression derived in the standard cosmology with a simple exception since the Boltzmann equations for our model are the same. The exception comes with the definition of the optical depth formula as mentioned earlier and the solution to $\Theta_l(k, \eta_0)$ is

$$\begin{aligned} \Theta_l(k, \eta_0) = & \int_{\eta_i}^{\eta_0} d\eta g(\eta) [\Theta_0(k, \eta) + \Psi(k, \eta)] j_l[k(\eta_0 - \eta)] \\ & - \frac{i}{k} \int_{\eta_i}^{\eta_0} d\eta g(\eta) v_b(k, \eta) \frac{d}{d\eta} j_l[k(\eta_0 - \eta)] \\ & + \int_{\eta_i}^{\eta_0} d\eta e^{-\tau} [\Psi'(k, \eta) - \Phi'(k, \eta)] j_l[k(\eta_0 - \eta)]. \end{aligned} \quad (7.137)$$

Here, $g(\eta)$ is the visibility function and it can be considered as the probability of the time at which a CMB photon scattered last. By definition, it is

$$g(\eta) = -\tau'(\eta) e^{-\tau(\eta)}. \quad (7.138)$$

To remind the reader, $\tau' = -n_e \sigma_T a^2$ and $\tau(\eta) = \int_{\eta}^{\eta_0} d\eta' n_e \sigma_T a^2$. The visibility function is very small in the early and late universe but peaks at the decoupling era. It is so small in the early universe because $\tau(\eta)$ is large in the argument of the exponential and it is so small in the late universe because $\tau'(\eta)$ is small. Also, since the sum of all probabilities should be one, it satisfies

$$\int_{\eta_i}^{\eta_0} d\eta g(\eta) = 1. \quad (7.139)$$

In Equation (7.137), j_l is the l th order spherical Bessel function and it determines the amount of contribution of any wavenumber k to the l th moment of radiation perturbations. The wavenumber that makes the main contribution to the l th moment is $k \sim l/(\eta_0 - \eta_*)$ (here, η_* is the conformal time of the decoupling) because $j_l(x)$ gets its maximum value when $l \sim x$. Furthermore, from Equation (7.69) $v_b = -3i\Theta_1$ (since $\tau'(\eta)$ is large before the decoupling, the second term can be disregarded) and

the expression

$$\frac{dj_l(x)}{dx} = j_{l-1}(x) - \frac{l+1}{x} j_l(x) \quad (7.140)$$

to get rid of the derivative of the Bessel function can be plugged in the second line of Equation (7.137). The result is (we have also used $\Psi'(k, \eta) = -\Phi'(k, \eta)$ in the third line)

$$\begin{aligned} \Theta_l(k, \eta_0) = & \int_{\eta_i}^{\eta_0} d\eta g(\eta) [\Theta_0(k, \eta) + \Psi(k, \eta)] j_l[k(\eta_0 - \eta)] \\ & + 3 \int_{\eta_i}^{\eta_0} d\eta g(\eta) \Theta_1(k, \eta) \left(j_{l-1}[k(\eta_0 - \eta)] - (l+1) \frac{j_l[k(\eta_0 - \eta)]}{k(\eta_0 - \eta)} \right) \\ & + 2 \int_{\eta_i}^{\eta_0} d\eta e^{-\tau} \Psi'(k, \eta) j_l[k(\eta_0 - \eta)]. \end{aligned} \quad (7.141)$$

The last line of this equation is known as the Integrated Sachs-Wolfe (ISW) effect and it can be considered as the effect of changing gravitational potential to the CMB photons on their way to us. If the potential is constant like the case in the standard cosmology (in the matter dominated era of the standard cosmology, this potential is constant if one ignores the late dark energy dominated epoch), there is no contribution resulting from this term. However, in our approach, Ψ is not constant but increasing as the power of the scale factor in the matter dominated era. So, the last term in Equation (7.141) contributes without depending on the existence of dark energy and the derivative of Ψ can be calculated as

$$\begin{aligned} \Psi'(k, \eta) = & \frac{d}{d\eta} \Psi(k, \eta) = a^2 \bar{H} \frac{d}{da} \Psi(k, a) = a^2 \bar{H} \frac{d}{da} \left(\frac{\epsilon}{2k^2} \bar{H}^2 \delta_{dm}(k, \eta_0) a^{2+n} \right) \\ = & a^2 \bar{H} \frac{d}{da} \left(\frac{\epsilon}{2k^2} \bar{H}_0^2 \delta_{dm}(k, \eta_0) a^{1+n} \right) \\ = & a^2 \bar{H} \left(\frac{\epsilon(1+n)}{2k^2} \bar{H}_0^2 \delta_{dm}(k, \eta_0) a^n \right) \\ = & \frac{\epsilon(1+n)}{2k^2} \bar{H}^3 \delta_{dm}(k, \eta_0) a^{3+n}. \end{aligned} \quad (7.142)$$

In the first line of Equation (7.142), we have used $a^2(d/dt) = a^2\dot{a}(d/da)$ in the place of conformal derivative $d/d\eta$ and written $\Psi(k, \eta)$ in terms of $\delta_{dm}(k, \eta_0)$. Then, in the second line, since we are interested in only the potential for the matter dominated era,

$\bar{H}_0^2 a^{1+n}$ is used for $\bar{H}^2 a^{2+n}$ because $\bar{H}^2 \simeq \bar{H}_0^2 \Omega_m / a$ and $\Omega_m \simeq 1$. The reason for this is that $e^{-\tau}$ in the integrand is so small before the decoupling (so nearly no contribution to the integral before decoupling) and almost equal to 1 after the decoupling and photons decouples from electrons in the matter dominated era. Hence, the last term in Equation (7.141) can be stated as

$$\begin{aligned}
 2 \int_{\eta_*}^{\eta_0} d\eta e^{-\tau} \Psi'(k, \eta) j_l[k(\eta_0 - \eta)] &= \frac{\epsilon(1+n)}{2k^2} \delta_{dm}(k, \eta_0) \int_{t_*}^{t_0} \frac{dt}{a^2} \bar{H}^3 a^{3+n} e^{-\tau} j_l[k(\eta_0 - \eta)] \\
 &= \frac{\epsilon(1+n)}{2k^2} \delta_{dm}(k, \eta_0) \int_{a_*}^1 da \bar{H}^2 a^{1+n} e^{-\tau} j_l[k(\eta_0 - \eta)] \\
 &= \frac{\epsilon(1+n)}{2k^2} \delta_{dm}(k, \eta_0) \bar{H}_0^2 \int_{a_*}^1 da a^n e^{-\tau} j_l[k(\eta_0 - \eta)]
 \end{aligned} \tag{7.143}$$

Without having an idea about the mechanism generating the initial perturbations, it is not possible to get $y_t(k)$ from Equation (7.129) for each wavenumber. In addition, we do not know a_{eq} and a_* exactly due to their dependence on the parameter α and h (Equation (7.88) and Figure 7.2). In this case, determining that the temperature fluctuations resulting in the CMB anisotropies are the oscillating initial radiation perturbations or the driven ones right after the transition y_t at which the temperature fluctuations start to be determined by dark matter perturbations is a challenge for us. For some of large scales these could be the initial perturbations that survive until $y = y_*$ (in our model Silk damping is very effective in reducing the amplitudes of the initial perturbations because of the large $1/k_D^2$ factor at y_*) whereas for some smaller scales they could be the ones originating from collapsing dark matter inhomogeneities. In Equation (7.141), for the contributions of scales on which the initial perturbations reach at the decoupling era without being damped, to the monopole and dipole terms one can use

$$\Theta_0(\vec{k}, \eta) = \Theta_0(\vec{k}, \eta_i) \cos \{kr_s(\eta)\} e^{-k^2/k_D^2(\eta)} \tag{7.144}$$

and

$$\Theta_1(\vec{k}, \eta) = \frac{1}{\sqrt{3}} \Theta_0(\vec{k}, \eta_i) \sin \{kr_s(\eta)\} e^{-k^2/k_D^2(\eta)}. \tag{7.145}$$

For the rest of the scales, the second terms in Equation (7.82) and Equation (7.83) will be dominant at the decoupling era and the contributions to the monopole and dipole term are given by

$$\Theta_0(\vec{k}, \eta) = \frac{-\epsilon}{2\sqrt{3}k} \bar{H}_0^2 \delta_{dm}(k, \eta_0) \int_{\eta_t}^{\eta} d\tilde{\eta} a^{1+n}(\tilde{\eta}) e^{-k^2 \left(\frac{1}{k_D^2(\eta)} - \frac{1}{k_D^2(\tilde{\eta})} \right)} \sin \{k [r_s(\eta) - r_s(\tilde{\eta})]\} \quad (7.146)$$

and

$$\Theta_1(\vec{k}, \eta) = \frac{\epsilon}{6k} \bar{H}_0^2 \delta_{dm}(k, \eta_0) \int_{\eta_t}^{\eta} d\tilde{\eta} a^{1+n}(\tilde{\eta}) e^{-k^2 \left(\frac{1}{k_D^2(\eta)} - \frac{1}{k_D^2(\tilde{\eta})} \right)} \cos \{k [r_s(\eta) - r_s(\tilde{\eta})]\}. \quad (7.147)$$

In these last two equations, the potentials in the integrands of Equation (7.82) and Equation (7.83) have been written in terms of $\delta_{dm}(k, \eta_0)$. Since the decoupling era is quite after $a = a_{eq}$ and potential perturbations grow as the power of the scale factor, multiplying $\delta_{dm}(k, \eta_0)$ by the factor of a^{1+n} gives the potentials at the decoupling.

8. CONCLUSION

In order to build the approach presented in this thesis, one of the main motivations is the relation indicated in Equation (1.9). It is a simple but an important relation because of the connection between the gravitational potential and the scalar field $\phi = 1/G$ in the JBD theory. More significantly, however, it is the idea that the scalar field could be related to the components of the cosmological metric as it does as a potential term in other circumstances in the general theory of relativity. Since there is a time dependent gravitational potential of the matter distribution related to the scalar field, it seems obvious that our space-time intervals must be relative among coordinate systems when this field changes with time. So, based on this, we have made use of the connection in Equation (1.9) to select the JBD scalar field proportional to $1/a^2$. Then, the FLRW metric has been coordinate transformed to embed the concept of relative space-time in the new metric tensor. Once the metric is constructed, it is straightforward to do all basic calculations like finding the Ricci tensor components, the Ricci scalar and the field equation.

Solutions for the flatness and the horizon problems have naturally come into the picture without any further modification apart from the one mentioned above. Lastly, it has been shown that the luminosity distance vs. red-shift data of Type Ia supernovae is in agreement with the prediction of our approach. The best-fit values for the Hubble constant for all spatial scenarios are the same and $\bar{H}_0 = 70.37$ km/s/Mpc. The curvature density parameter and the JBD constant are found as $\Omega_{\kappa 0} = -1.21 \times 10^{-7}$ and $\omega = -3/2(1 + 0.55) \approx -2.32$ for a closed universe. For an open universe they are $\Omega_{\kappa 0} = 9.04 \times 10^{-7}$ and $\omega = -3/2(1 + 0.18) \approx -1.77$. Although the scale factor for the matter dominated universe is still given by $a \sim t^{2/3}$ and there is no accelerating expansion in our comoving time, the combined effect of the contraction in time and the expansion in space intervals makes supernovae to be observed dimmer in our comoving coordinates. Among them, the last one is the most important success of this new approach in comparison with the standard model. There may be no need for the

cosmological constant or other forms of dark energy in this model to conform with the supernovae data. Embedding the time dependent scalar field into the JBD theory in a particular way can solve three major problems of the standard model of cosmology.

As the next step, after adding perturbations to the scalar field and the metric, the field equation was found. Considering that this approach is the weak field limit of the theory, we obtained the scalar wave equation as expected. When we ignored the changes over time, the equation reduced to the Poisson-like equation. However, at this point, we noticed that the JBD parameter changed its behaviour in comparison with the zeroth order solutions. It has been seen that the JBD parameter is less than $-3/2$ in the zeroth order calculations, but then we have realized that this parameter must be greater than $-3/2$ due to the requirement that gravity must be an attractive force in the Poisson-like equation, which has been acquired in the first order calculations. This resembles the phenomenon that is frequently encountered in the literature in some versions of the JBD theory. This is the parameter of the theory that behaves differently in cosmological and solar system applications. Although we have no idea why this occurs, we stated that gravity can be repulsive on large scales and attractive on small scales. Similar situations, and therefore the possibility that gravity depends on scale, have also been mentioned in the literature.

Then, the procedure to obtain the Boltzmann equations for the metric tensor is given in detail. Although these equations are given for flat space in the literature, in our calculations these equations have been found for the most general metric form, that is, for curved space. If we ignore spatial curvature, which is the approach we applied in our calculations, the only difference between the Boltzmann equations in the standard cosmology and the ones we obtained was the definition of optical depth. This is due to the difference in the definition of time in the coordinate transformed metric tensor. Although this coordinate transformation in the metric does not seem to make a significant difference for the Boltzmann equations, it is of great importance due to the method we use to obtain the field equation.

Once equipped with all the necessary equations, the neutron abundance, the helium fraction, and the redshift of the recombination era, which are some applications of our approach in cosmology, have been worked out. The results were expressed depending on the parameters of our model. Some of the results that could not be given analytically were solved numerically and then their plots were presented.

Finally, it was found out how radiation and matter perturbations evolve in eras when the expansion of the universe was dictated by its varied dominant energy content. We obtained solutions that give acoustic oscillations for radiation perturbations in the deep radiation era and growing dark matter perturbations in the late radiation and matter eras, which are crucial to attain the LSS. It is difficult to predict what the shape of the matter power spectrum and the CMB anisotropy spectrum will be because there is no place for the inflation theory in our model and at this stage we do not have another theory that gives the shape of the initial perturbations. We evade to speculate on this issue, we only gave the equations that determine how these perturbations will evolve over time. There are two reasons why there is no room for the inflation theory in our model. Firstly, the flatness and horizon problems that the inflation theory claims to solve do not appear in the approach we have developed. The second reason is that even if we tried to integrate the inflation theory into our model, it would be pointless because the inflationary expansion at the very beginning of the universe would take almost all the scales of perturbations outside the comoving Hubble horizon. In our model, the comoving Hubble horizon is constant in the radiation era and shrinks in the matter era, so we would not be able to see perturbations on these scales.

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APPENDIX A: VARIATION OF THE ACTION

In this Appendix, the steps of obtaining the first term in the variation of the JBD action with respect to the scalar field are given. Although our model is quite different from the JBD theory, the action is the same and given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R - \omega g^{\mu\nu}(\phi) \frac{\partial_\mu \phi \partial_\nu \phi}{\phi} + 16\pi \mathcal{L}_M(g_{\mu\nu}(\phi), \psi_i) \right).$$

The difference comes with the fact that the metric tensor is the function of the scalar field. So, the variation for some of the terms can be done with respect to $g^{\mu\nu}$ and then multiplied by $dg^{\mu\nu}/d\phi$. Even though taking variance of the JBD action with respect to the inverse of the metric tensor is the standard procedure, it is cumbersome and not straightforward. So, we will only focus on this one because the variation with respect to the scalar field is easy to operate. Let us begin with writing the following

$$\delta S = \frac{1}{16\pi} \int d^4x \left(\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \frac{dg^{\mu\nu}}{d\phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta(\partial_\mu \phi) \right).$$

Since the Ricci scalar is the contraction of the Ricci tensor with the inverse of the metric, the action becomes

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R_{\mu\nu} g^{\mu\nu} - \omega g^{\mu\nu} \frac{\partial_\mu \phi \partial_\nu \phi}{\phi} + 16\pi \mathcal{L}_M \right). \quad (\text{A.1})$$

For the first term in the variation of the action, one can write

$$\begin{aligned} \int d^4x \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \frac{dg^{\mu\nu}}{d\phi} \delta\phi &= \int d^4x \left[\delta\sqrt{-g} \phi R_{\mu\nu} g^{\mu\nu} + \sqrt{-g} \phi \delta R_{\mu\nu} g^{\mu\nu} + \sqrt{-g} \phi R_{\mu\nu} \delta g^{\mu\nu} \right. \\ &\quad \left. - \delta\sqrt{-g} \omega g^{\mu\nu} \frac{\partial_\mu \phi \partial_\nu \phi}{\phi} - \sqrt{-g} \omega \delta g^{\mu\nu} \frac{\partial_\mu \phi \partial_\nu \phi}{\phi} \right] \\ &\quad + \delta \int d^4x (\sqrt{-g} 16\pi \mathcal{L}_M). \end{aligned} \quad (\text{A.2})$$

Now, there are two variations we have to find. One is $\delta\sqrt{-g}$, and the other one is $\delta R_{\mu\nu}$. Let us begin with the first one and write the following relation

$$\ln(\det M) = \text{Tr}(\ln M), \quad (\text{A.3})$$

where M is a square matrix with nonzero determinant. Taking variance of this yields

$$\frac{1}{\det M} \delta(\det M) = \text{Tr}(M^{-1} \delta M). \quad (\text{A.4})$$

After replacing matrix M with the metric $g_{\mu\nu}$ and the determinant of the matrix with the determinant of the metric, which is g , Equation (A.4) can be written as

$$\delta g = g(g^{\mu\nu} \delta g_{\mu\nu}). \quad (\text{A.5})$$

Since we are looking for the variation with respect to $g^{\mu\nu}$, we should raise sub indices of $\delta g_{\mu\nu}$. To be able to do that we will consider the Kronecker delta which is constant. Taking variance of the relation $g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu^\nu$ gives

$$\delta g_{\mu\sigma} = -g_{\sigma\nu} g_{\mu\sigma} \delta g^{\sigma\nu}. \quad (\text{A.6})$$

By using Equation (A.6) in Equation (A.5), Equation (A.5) can be arranged as

$$\delta g = -g(g_{\mu\nu} \delta g^{\mu\nu}). \quad (\text{A.7})$$

Now if we turn back to

$$\delta \sqrt{-g} = -\frac{1}{2} \frac{\delta g}{\sqrt{-g}}, \quad (\text{A.8})$$

and substitute Equation (A.7) into this, we obtain

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \quad (\text{A.9})$$

For the variance of the Ricci scalar, we start with the relation

$$\delta R_{\mu\nu} = \delta R_{\mu\sigma\nu}^\sigma = \delta(\partial_\sigma \Gamma_{\nu\mu}^\sigma + \Gamma_{\sigma\lambda}^\sigma \Gamma_{\nu\mu}^\lambda - \partial_\nu \Gamma_{\sigma\mu}^\sigma - \Gamma_{\nu\lambda}^\sigma \Gamma_{\sigma\mu}^\lambda), \quad (\text{A.10})$$

and so

$$\delta R_{\mu\sigma\nu}^\sigma = \partial_\sigma \delta \Gamma_{\nu\mu}^\sigma + \delta \Gamma_{\sigma\lambda}^\sigma \Gamma_{\nu\mu}^\lambda + \Gamma_{\sigma\lambda}^\sigma \delta \Gamma_{\nu\mu}^\lambda - \partial_\nu \delta \Gamma_{\sigma\mu}^\sigma - \delta \Gamma_{\nu\lambda}^\sigma \Gamma_{\sigma\mu}^\lambda - \Gamma_{\nu\lambda}^\sigma \delta \Gamma_{\sigma\mu}^\lambda. \quad (\text{A.11})$$

Since covariant derivatives of the variations of the Christoffel symbols are

$$\nabla_\sigma \delta \Gamma_{\nu\mu}^\sigma = \partial_\sigma \delta \Gamma_{\nu\mu}^\sigma + \Gamma_{\sigma\lambda}^\sigma \delta \Gamma_{\nu\mu}^\lambda - \Gamma_{\sigma\nu}^\lambda \delta \Gamma_{\lambda\mu}^\sigma - \Gamma_{\sigma\mu}^\lambda \delta \Gamma_{\nu\lambda}^\sigma \quad (\text{A.12})$$

and

$$\nabla_\nu \delta \Gamma_{\sigma\mu}^\sigma = \partial_\nu \delta \Gamma_{\sigma\mu}^\sigma + \Gamma_{\nu\lambda}^\sigma \delta \Gamma_{\sigma\mu}^\lambda - \Gamma_{\nu\sigma}^\lambda \delta \Gamma_{\lambda\mu}^\sigma - \Gamma_{\nu\mu}^\lambda \delta \Gamma_{\sigma\lambda}^\sigma, \quad (\text{A.13})$$

the variation of the Riemann tensor, then, can be written as

$$\delta R_{\mu\sigma\nu}^{\sigma} = \nabla_{\sigma}\delta\Gamma_{\nu\mu}^{\sigma} - \nabla_{\nu}\delta\Gamma_{\sigma\mu}^{\sigma}. \quad (\text{A.14})$$

With this, the second term in the parenthesis in Equation (A.2) turns into

$$\begin{aligned} \int dx^4 \sqrt{-g} \phi g^{\mu\nu} \delta R_{\mu\nu} &= \int dx^4 \sqrt{-g} \phi g^{\mu\nu} (\nabla_{\sigma}\delta\Gamma_{\nu\mu}^{\sigma} - \nabla_{\nu}\delta\Gamma_{\sigma\mu}^{\sigma}) \\ &= \int dx^4 \sqrt{-g} \phi \nabla_{\lambda} (g^{\mu\nu} \delta\Gamma_{\nu\mu}^{\lambda} - g^{\mu\lambda} \delta\Gamma_{\sigma\mu}^{\sigma}). \end{aligned} \quad (\text{A.15})$$

Then, regarding how the Christoffel symbol is defined, one can obtain

$$\delta\Gamma_{\nu\mu}^{\lambda} = \frac{1}{2} g^{\lambda\rho} (\partial_{\nu}\delta g_{\rho\mu} + \partial_{\mu}\delta g_{\nu\rho} - \partial_{\rho}\delta g_{\mu\nu}) \quad (\text{A.16})$$

and

$$\delta\Gamma_{\nu\mu}^{\lambda} = \frac{1}{2} g^{\lambda\rho} (\nabla_{\nu}\delta g_{\rho\mu} + \nabla_{\mu}\delta g_{\nu\rho} - \nabla_{\rho}\delta g_{\mu\nu}). \quad (\text{A.17})$$

Now, we will consider the first term (for the second one, the same procedure should be followed) in the parenthesis in Equation (A.15), which is given by

$$g^{\mu\nu} \delta\Gamma_{\nu\mu}^{\lambda} = \frac{1}{2} (g^{\mu\nu} g^{\lambda\rho} \nabla_{\nu} \delta g_{\rho\mu} + g^{\mu\nu} g^{\lambda\rho} \nabla_{\mu} \delta g_{\nu\rho} - g^{\mu\nu} g^{\lambda\rho} \nabla_{\rho} \delta g_{\mu\nu}). \quad (\text{A.18})$$

By using Equation (A.6), this equation can be arranged as

$$\begin{aligned} g^{\mu\nu} \delta\Gamma_{\nu\mu}^{\lambda} &= \frac{1}{2} (-g^{\mu\nu} g^{\lambda\rho} g_{\rho\lambda} g_{\mu\nu} \nabla_{\nu} \delta g^{\nu\lambda} - g^{\mu\nu} g^{\lambda\rho} g_{\mu\nu} g_{\rho\lambda} \nabla_{\mu} \delta g^{\mu\lambda} \\ &\quad + g^{\mu\nu} g^{\lambda\rho} g_{\mu\nu} g_{\rho\lambda} \nabla_{\rho} \delta g^{\mu\nu}) \end{aligned} \quad (\text{A.19})$$

to get

$$g^{\mu\nu} \delta\Gamma_{\nu\mu}^{\lambda} = \frac{1}{2} (-\nabla_{\nu} \delta g^{\nu\lambda} - \nabla_{\mu} \delta g^{\mu\lambda} + g_{\mu\nu} \nabla^{\lambda} \delta g^{\mu\nu}). \quad (\text{A.20})$$

And for the second term, applying the same procedure gives

$$g^{\mu\lambda} \delta\Gamma_{\sigma\mu}^{\sigma} = \frac{1}{2} (-\nabla_{\sigma} \delta g^{\sigma\lambda} - g_{\rho\sigma} \nabla^{\lambda} \delta g^{\sigma\rho} + \nabla_{\rho} \delta g^{\rho\lambda}). \quad (\text{A.21})$$

By taking the difference between the two

$$g^{\mu\nu} \delta\Gamma_{\nu\mu}^{\lambda} - g^{\mu\lambda} \delta\Gamma_{\sigma\mu}^{\sigma} = g_{\mu\nu} \nabla^{\lambda} \delta g^{\mu\nu} - \nabla_{\mu} \delta g^{\mu\lambda}, \quad (\text{A.22})$$

we can write Equation (A.15) as

$$\begin{aligned}
& \int d^4x \sqrt{-g} \phi \nabla_\lambda (g^{\mu\nu} \delta \Gamma_{\nu\mu}^\lambda - g^{\mu\lambda} \delta \Gamma_{\sigma\mu}^\sigma) \\
&= \int d^4x \sqrt{-g} \phi \nabla_\lambda (g_{\mu\nu} \nabla^\lambda \delta g^{\mu\nu} - \nabla_\mu \delta g^{\mu\lambda}) \\
&= - \int d^4x \sqrt{-g} \nabla_\lambda \phi (g_{\mu\nu} \nabla^\lambda \delta g^{\mu\nu} - \nabla_\mu \delta g^{\mu\lambda}) \\
&= \int d^4x \sqrt{-g} (g_{\mu\nu} \nabla_\lambda \nabla^\lambda \phi \delta g^{\mu\nu} - \nabla_\mu \nabla_\lambda \phi \delta g^{\mu\lambda}) \\
&= \int d^4x \sqrt{-g} (g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi - \nabla_\mu \nabla_\nu \phi) \delta g^{\mu\nu}.
\end{aligned} \tag{A.23}$$

In the second line of Equation (A.23), we have used the equality in Equation (A.22). In the third line, the method of integration by parts has been applied. Also the volume element of a covariant divergence has been set equal to the boundary contribution at infinity, which can be adjusted as zero, thanks to Stokes' theorem. In the fourth line, again integration by parts method and Stokes' theorem have been applied. In the last line, some simple index manipulations have been made to make it look nicer. Substituting Equation (A.9) and Equation (A.23) into Equation (A.2) leads

$$\begin{aligned}
\int d^4x \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \frac{dg^{\mu\nu}}{d\phi} \delta\phi &= \int d^4x \sqrt{-g} \delta g^{\mu\nu} \left[\phi R_{\mu\nu} + (g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \partial_\beta \phi - \nabla_\mu \partial_\nu \phi) \right. \\
&\quad \left. - \frac{1}{2} \phi R g_{\mu\nu} + \frac{1}{2} \omega g_{\mu\nu} g^{\alpha\beta} \frac{\partial_\alpha \phi \partial_\beta \phi}{\phi} \right. \\
&\quad \left. - \omega \frac{\partial_\mu \phi \partial_\nu \phi}{\phi} - 8\pi T_{\mu\nu} \right],
\end{aligned} \tag{A.24}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}.$$