INFLUENCE OF EXPECTED PREMIUM ON OPTIMAL REINSURANCE AND INVESTMENT STRATEGY

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ANIL GÜLVEREN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
FINANCIAL MATHEMATICS

JULY 2024

Approval of the thesis:

INFLUENCE OF EXPECTED PREMIUM ON OPTIMAL REINSURANCE AND INVESTMENT STRATEGY

submitted by **ANIL GÜLVEREN** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Financial Mathematics Department, Middle East Technical University** by,

Prof. Dr. A. Sevtap Selçuk-Kestel
Dean, Graduate School of **Applied Mathematics** ______________

Prof. Dr. A. Sevtap Selçuk-Kestel
Head of Department, **Financial Mathematics** ______________

Prof. Dr. A. Sevtap Selçuk-Kestel
Supervisor, **Actuarial Science, METU** ______________

Assist. Prof. Dr. Başak Bulut Karageyik
Co-supervisor, **Actuarial Science, Hacettepe University** ______________

Examining Committee Members:

Prof. Dr. Ömür Uğur
Scientific Computing, METU ______________

Prof. Dr. A. Sevtap Selçuk-Kestel
Actuarial Sciences, METU ______________

Assoc. Prof. Dr. Uğur Karabey
Actuarial Sciences, Hacettepe University ______________

Assist. Prof. Dr. Aslıhan Şentürk Acar
Actuarial Sciences, Hacettepe University ______________

Assist. Prof. Dr. Büşra Z. Temoçin
Actuarial Sciences, METU ______________

Date: ______________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: ANIL GÜLVEREN

Signature :

ABSTRACT

INFLUENCE OF EXPECTED PREMIUM ON OPTIMAL REINSURANCE AND INVESTMENT STRATEGY

Gülveren, Anıl
Ph.D., Department of Financial Mathematics
Supervisor : Prof. Dr. A. Sevtap Selçuk-Kestel
Co-Supervisor : Assist. Prof. Dr. Başak Bulut Karageyik

July 2024, 73 pages

This thesis investigates optimal investment and reinsurance strategies aimed at maximizing the expected utility of wealth. Building on foundational work in optimal investment and insurance strategy within a diffusion setup, we improve the investment model by incorporating an Ornstein-Uhlenbeck to model the instantaneous return rate of the risky asset. This approach provides a more detailed and realistic representation of market fluctuations, capturing both bull and bear markets compared to the one in literature. We propose two different utility functions, logarithmic utility and exponential utility, to define wealth process whose optimized function is developed using the Hamilton-Jacobi-Bellman (HJB) equation and derive closed-form expressions for the optimal investment and reinsurance decisions. In addition to developing the theoretical model, we conduct extensive sensitivity analysis to illustrate how different financial and insurance parameters influence the optimal strategies. The sensitivity analysis provides critical insights into how variations in parameters such as volatility, risk aversion, and time horizon impact the insurer's decisions. Additionally, we apply real-life insurance claims' severity and frequency data along with the market behaviour on a risky asset observations. Specifically, we find that the optimal investment strategy tends to increase in bullish markets and decrease with higher volatility, reflecting the insurer's need to balance risk and return. The optimal reinsurance strategy exhibits threshold behavior, remaining zero until a certain level of risk aversion is reached, beyond which it increases significantly. By incorporating realistic market dynamics and providing actionable insights, these advancements contribute to more effective risk management and strategic planning in volatile markets, providing a more robust and flexible framework for decision-making.

Keywords: Proportional reinsurance, Utility maximization, Hamilton–Jacobi–Bellman equation, Ornstein-Uhlenbeck Process

ÖZ

OPTİMUM REASÜRANS VE YATIRIM STRATEJİSİNDE BEKLENEN PRİM ETKİSİ

Gülveren, Anıl
Doktora, Finansal Matematik Bölümü
Tez Yöneticisi : Prof. Dr. A. Sevtap Selçuk-Kestel
Ortak Tez Yöneticisi : Yrd. Doç. Dr. Başak Bulut Karageyik

Temmuz 2024, 73 sayfa

Bu tez, servetin beklenen faydasını maksimuma çıkarmayı amaçlayan en uygun yatırım ve reasürans stratejilerini araştırmaktadır. Difüzyon yaklaşımında optimal yatırım ve sigorta stratejisi üzerine temel çalışmayı temel alarak, yatırım modelini, riskli varlığın anlık getiri oranını modellemek için bir Ornstein-Uhlenbeck süreci ekleyerek geliştirmekteyiz. Bu yaklaşım, literatürdeki modellere kıyasla hem yükselen hem de düşen piyasa koşullarını yakalayarak piyasa dalgalanmalarının daha ayrıntılı ve gerçekçi bir temsilini sağlamaktadır. Servet sürecini tanımlamak için logaritmik fayda ve üstel fayda olmak üzere iki farklı fayda fonksiyonu öneriyoruz, Hamilton-Jacobi-Bellman (HJB) denklemi kullanarak optimize edilen fonksiyonu geliştiriyoruz ve optimal yatırım ve reasürans kararları için kapalı form ifadeleri türetiyoruz. Teorik modeli geliştirmenin yanı sıra, farklı finansal ve sigorta parametrelerinin optimal stratejileri nasıl etkilediğini göstermek için kapsamlı duyarlılık analizi yapıyoruz. Duyarlılık analizi, oynaklık, riskten kaçınma ve zaman periyodu gibi parametrelerdeki değişikliklerin sigorta şirketinin kararlarını nasıl etkilediğine dair kritik iç görüleri ortaya koyuyoruz. Ayrıca, gerçek sigorta hasarlarının şiddeti ve sıklığı verilerini, riskli bir varlığın piyasa davranışıyla birlikte uyguluyoruz. Spesifik olarak, sigorta şirketinin risk ve getiriyi dengeleme ihtiyacını yansıtacak şekilde, optimal yatırım stratejisinin boğa piyasalarında artma eğiliminde olduğunu ve daha yüksek oynaklıkla birlikte düşme eğiliminde olduğunu gözlemliyoruz. Optimal reasürans stratejisi eşik davranışı sergiler, belirli bir riskten kaçınma düzeyine ulaşılıncaya kadar sıfır kalır ve bu düzeyin ötesinde önemli ölçüde artar. Gerçekçi piyasa dinamiklerini bir araya getirerek ve eyleme geçirilebilir öngörüler sağlayarak bu gelişmeler, değişken piyasalarda daha etkili risk yönetimine ve stratejik planlamaya katkıda bulunarak karar alma için daha sağlam ve esnek bir çerçeve sağlar.

Anahtar Kelimeler: Bölüşmeli Reasürans, Fayda maksimizasyonu, Hamilton–Jacobi–Bellman denklemi, Ornstein-Uhlenbeck süreci

To my family

ACKNOWLEDGMENTS

I would like to express my deepest appreciation to my thesis supervisor Prof. Dr. A. Sevtap SELÇUK-KESTEL for her patient guidance, enthusiastic encouragement and valuable advices during the development and preparation of this thesis. I will forever value her kind support and encouragement which have made great help in completing my studies.

I wish also to express my very great appreciation to my Co-supervisor Assist. Prof. Dr. Başak BULUT KARAGEYIK. She introduced me into reinsurance strategies and risk management area and made many insightful suggestions on my research and thesis, which are of great value to me.

I am very grateful to members of my Thesis Advisory Committee, Prof. Dr. Ömür UĞUR and Assoc. Prof. Dr. Uğur KARABEY for their constructive criticism and insightful guidance on improving my thesis. I greatly appreciate all examining committee members for their time and support.

I acknowledge all the support and help from the members of the Department of Financial Mathematics, Bilgi Yılmaz and many friends during my Ph.D. studies.

Finally, I would like to give my special thanks to my family for their love, support and understanding throughout my life.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

OU	Ornstein–Uhlenbeck Process
SDE	Stochastic Differential Equation
HJB	Hamilton-Jacobi-Bellman
GBM	Geometric Brownian Motion
W_t	Brownian Motion Process
$\mathcal{M}$	Moment Generating Function
$G(X)$	Claim Size Distribution Function
m_1	the First Moment of $G(X)$
m_2	the Second Moment of $G(X)$
X_i	Claim Amount
θ	Insurer's Premium Loading Factor
ξ	Reinsurer's Premium Loading Factor
c_I	Insurer's Premium
c_R	Reinsurer's Premium
U_t	Surplus Process
Z_t	Wealth Process
S_t^0	Risk-free Asset Price Process
S_t^1	Risky Asset Price Process
$u(.)$	Insurer's Utility Function
λ	Risk Aversion Parameter
$\psi(.)$	the Minimum Probability of Ruin
a.s.	Almost surely
s.t.	Subject to
VaR	Value-at-Risk
ES	Expected Shortfall

CHAPTER 1

INTRODUCTION

Insurance companies play a crucial role in the financial market by providing risk management services and financial protection to individuals and businesses. These companies are essential for maintaining the stability of the economy. Insurance companies should make strategic decisions regarding investments, risk management, and asset-liability management to meet their obligations to policyholders. These decisions become especially critical during unexpected crises that can threaten their financial stability, as maintaining solvency is vital.

One of the insurer's main strategies to manage risk is reinsurance. Reinsurance involves transferring some of the insurer's risk to another insurance entity, thereby reducing the potential impact of significant losses. While reinsurance is effective in reducing severe risks and ensuring the insurer can pay claims, it is also costly to buy reinsurance. The expense of reinsurance can diminish the insurer's profit margins. Thus, insurers should find a balance between transferring risk and maintaining profitability, which requires making dynamic decisions that consider the insurer's long-term sustainability and financial health.

Investment strategy is another key component of an insurer's financial management. The main goal of an insurer's investments is to generate returns that increase the insurer's capital. It also provides financial security for policyholders. Insurers can invest their premium income in various financial instruments, such as stocks, mutual fund, bonds or bank account to achieve higher returns and grow their assets. Well-planned investment strategies allow insurers to meet their obligations to policyholders and improve their competitive position in the market.

It is crucial for insurers to manage their financial risks effectively to maintain an optimal asset-liability balance. Insurer should ensure that their investment portfolios are aligned with their liability profiles to minimize the risk of mismatches considering interest rate risk and market volatility. Insurers can ensure that they have enough funds to meet their obligations to policyholder and survive economic shocks by aligning their assets and liabilities.

Most of the studies in literature focus on modeling risk processes and finding optimal investment strategies under various assumptions and constraints. To develop a strategy on the reinsurance agreement and the investment strategies, the literature offers various approaches. Stochastic models in developing optimal investment strategies, especially in understanding the behaviour of asset prices are commonly used, including the insurance and actuarial issues. Brownian motion with drift for general market trends and a geometric Brownian motion for the risky asset are introduced by Browne [2] whose work -explores an optimal investment strategy minimizing the probability of ruin. The study shows that the optimal strategy involves investing a fixed sum in the risky asset, regardless of the firm's surplus size, under exponential utility function.

Hipp et al. [11] also explore an investment strategy in a capital market index to minimize the probability of ruin risk under compound Poisson assumption with exponential claim size at which optimal strategy is determined using the Bellman equation with the existence of a smooth solution and a verification theorem also established. In contrast to Browne [2]'s findings, they demonstrate that the optimal investment strategy cannot remain constant under the compound Poisson process framework.

The extention of the paper [11] is made by incorporating a risk-free asset by Liu et al. [35]. The investment strategy is examined for different claim-size distributions and the optimal investment policy solved using Hamilton-Jacobi-Bellman (HJB) equation. Furthermore, they investigate the optimal strategy under conditions with no borrowing constraints and with the implementation of stop-loss insurance providing numerical solutions.

Taylor [31] was the first to consider how competition might affect an insurer's premium strategy. He studies how insurance prices changed in Australian insurance market and observes that when companies lower their price and make loss, they usually increase prices later to make a profit. He also finds that these companies often set their prices based on competitor's price, rather than their predicted claims distribution. Based on these findings, Taylor develops a simple model that uses both the law of demand and the distribution of claims. He examines two types of demand functions for setting insurance premiums: the exponential demand function and a constant price elasticity demand function. He shows that the future premium rates in the best underwriting approach increase with a higher discount rate for future profits. Additionally, if the discount rate is significantly high, the best strategy will not include any loss leaders. When price elasticity goes down, the negative effects of high premium rates, like less exposure, become smaller, and the benefits of low premium rates also decrease.

Emms et al. [8] extends Taylor [31]'s discrete-time deterministic model into a stochastic model in continuous time using optimal control theory. They assume that the market average premium follows a stochastic process using geometric Brownian motion. They also assume that the break-even premium rate is constant and the deterministic premium strategy is determined by $p/\bar{p} = k(t)$. They show that the optimal control $k(t)$ is bang-bang under an unconstrained model. This directly stems from the assumption that the insurer can force existing customers to pay the current premium rate. They modify the model to set the premium rate at the beginning of a policyholder's contract, as the optimal control depends on how much the insurer can raise the premium rate during the course of a policy.

Emms et al. [9] focus on the exponential function and consider the required rate of return on the capital involved in the insurance business. They explore the best strategies for setting premiums using two specific methods. The first method uses a linear function of the average market premium, while the second method employs a linear mix of the break-even premium and the average market premium.

Pantelous et al. [19] propose a stochastic demand function for an insurance company's business volume under a discrete-time model. They provide an endogenous analytical formula for the optimal premium strategy by using a linear discounted function for the wealth process of the insurer, particularly when the insurer is likely to lose market share.

Emms [10] examines the best pricing strategy for an insurance company to increase its wealth over a defined time period. The price of the company is influenced by the amount needed to cover insurance claims, as well as by competitors' premiums. He uses the Bellman equation to describe the best pricing strategy for maximizing the company's expected wealth at the end of the defined time period. He aims to maximize the expected total discounted utility of wealth, using a linear utility function.

Mao et al. [17] consider the mutual effect between price and investment strategy in the insurance industry. They assume that the average market price of insurance, the return on risky investments, and the insured loss are correlated stochastic processes. Their aim is to optimize the insurer's terminal wealth utility, and they use the Hamilton-Jacobi-Bellman (HJB) equation, which allows for the simultaneous determination of both optimal insurance pricing and investment strategies.

Asmussen et al. [1] introduce optimal premium problem and assume that the portfolio size depends on the premium and derive the optimal solution to minimize the probability of ruin. Zhou et al. [37] examine the optimal investment and premium strategy for a non-homogeneous compound Poisson process with changing intensity over time. Jiang et al. [13] derive the optimal investment and reinsurance strategy for a risk model with premium control, based on certain relationships between safety loading and claim rates. Using stochastic control theory, it aims to maximize the expected exponential utility of final wealth, and provides numerical illustrations to show how model parameters affect the strategy.

Schäl [27, 28] study methods to minimize the ruin probability using a variant of the Cramer-Lundberg model with exponentially distributed claims in a discrete time model. In terms of reinsurance aspects, the dynamic strategies for proportional reinsurance, deriving the optimal reinsurance strategies within both a diffusion framework under a classical risk model at which the reinsurance proportion to be adjusted continuously is explored by Schmidli [23]. Incorporating Cramer-Lundberg and HJB to propose and validate an optimal strategy, Schmidli [24] introduces a new approach to optimal control problems for reinsurance and investment strategies within a Black-Scholes-type market, providing a closed-form and analytical expression to lower the ruin probability whose results demonstrate that any increasing solution to the HJB equation is bounded and solves the optimization problem effectively. An extention of [2] and [23] by modifying their models to accommodate conditions with no borrowing and short selling constraint and buying quota share is proposed by Promislow et al. [21].

Eisenberg and Schimdli [26] study classical and diffusion models incorporating capital injections and reinsurance strategies. Their aim is to minimize the expected discounted capital injections over all admissible reinsurance strategies where the reinsurance parameter could be changed continuously.

Cao et al. [3] study optimal investment and insurance strategy in a diffusion setup to maximize exponential utility considering a proportional reinsurance strategy. They also show an optimal strategy aiming to maximize power utility with no safety loading assumption.

Another to be mentioned on the optimal investment and reinsurance policy is the work done by Zhang et al. [36] whose approach maximizes the expected exponential utility of the terminal wealth under a surplus process followed by a diffusion perturbed compound Poisson process with constraints on short selling and borrowing. They derive a closed form of solution neither short selling nor borrowing is allowed. They also provide numerical solutions.

Along with these studies concentrating on investment strategies, Mert and Selcuk-Kestel [18] propose premium share and expected costs of insurer and reinsurer for a stop loss agreement with the assumption on claims to follow a GBM.

However, these studies typically assume constant parameters on the risky asset dynamics except [18], which may not accurately reflect dynamic conditions. Moreover, many of these studies do not conduct thorough sensitivity analysis to understand how changes in market conditions could affect the optimal strategies.

Motivated by the work by Cao et al. [3] and Liang et al. [16], we aim to address and improved approach by considering the optimal investment and reinsurance problem with the goal of maximizing the expected utility of wealth. Contrary to Cao et al. [3] and Liang et al. [16], we improve the followings:

(i) Risky asset structure is taken as stochastic process (different than Cao et al. [3])

(ii) Logarithmic utility is also implemented which is not considered by Cao et al. [3] and Liang et al. [16]

(iii) Expected Value principle is considered under Ornstein-Uhlenbeck process (different than Liang et al. [16]

(iv) Sensitivity to parameters are performed under different values of the parameters (not considered by Cao et al. [3] and Liang et al. [16])

(v) Real-life insurance claims data is implemented to show the effectiveness of the proposed approach (not available in the literature)

The main approach in considering these gaps is related to nature of the reinsurance agreements with our analytical results we offer more realistic reinsurance and investment strategy in actuarial valuation using financial mathematical techniques. Using parameters estimated from real-life data, simulations are conducted to demonstrate the impact of reinsurance, providing new insights into optimal reinsurance and investment strategies. Furthermore, we also conduct a comprehensive sensitivity analysis on the optimal investment and reinsurance strategies to assess the effects of various model parameters on the insurer's decisions. These analyses provide valuable insights into the impact of changes in market conditions and parameter values on the optimal strategies. Our approach not only enhances the theoretical robustness of the model but also delivers practical guidance for insurers. It enables them to modify their investment and reinsurance strategies in order to better manage risks and utilize on opportunities in dynamic market environments.

The organization of the thesis is as follows: In Chapter 2, the core mathematical preliminaries are introduced for the latter chapters and the basic definitions and types of reinsurance policies are given. Chapter 3 introduces several assumptions and presents the surplus process with reinsurance and investment strategies providing a clear framework for understanding the wealth dynamic model used in this study. In Chapter 4, we formulate the optimization problem and study the optimal investment and reinsurance strategy that maximizes the expected utility of wealth. By using HJB equation, we obtain closed-form solutions for the optimal reinsurance and investment strategies. We illustrate some numerical results and sensitivity analyses in Chapter 5. This chapter demonstrates how different financial and insurance market parameters affect the optimal strategies. It also provides insights into the robustness of the findings. The findings of the thesis and discussion on the future research are given in Chapter 6.

CHAPTER 2

PRELIMINARIES

This chapter presents the basic foundation of the models used in the frame of the proposed approach. As this thesis combines the stochastic control and stochastic differential equations with insurance business, we define the concepts on Brownian motion, Ito's Process, Ornstein-Uhlenbeck Process and Stochastic Control problems. We also outline the concept of the Cramér-Lundberg model and its diffusion approximation. Additionally, reinsurance contracts and the premium requirements are given in this chapter.

Definition 2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function W_t of $t \geq 0$ that satisfies $W_0 = 0$ and that depends on ω. The following three properties are equivalent [29].

(i) For all $0 = t_0 < t_1 < \cdots < t_m$, the increments

$$W_{t_1} = W_{t_1} - W_{t_0},\, W_{t_2} - W_{t_1},\, \ldots,\, W_{t_m} - W_{t_{m-1}}$$

are independent and each of these increments is normally distributed with mean and variance given by

$$\mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0$$

and

$$\mathbb{V}[W_{t_{i+1}} - W_{t_i}] = t_{i+1} - t_i.$$

(ii) For all $0 = t_0 < t_1 < \cdots < t_m$, the random variables $W_{t_1}, W_{t_2}, \ldots, W_{t_m}$ are jointly normally distributed with means equal to zero and covariance matrix

$$\begin{bmatrix} \mathbb{E}[W_{t_1}^2] & \mathbb{E}[W_{t_1}W_{t_2}] & \cdots & \mathbb{E}[W_{t_1}W_{t_m})] \\ \mathbb{E}[W(_{t_2}W_{t_1}] & \mathbb{E}[W_{t_2}^2] & \cdots & \mathbb{E}[W_{t_2}W_{t_m}] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[W_{t_m}W_{t_1}] & \mathbb{E}[W_{t_m}W_{t_2}] & \cdots & \mathbb{E}[W_{t_m}^2] \end{bmatrix} = \begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_m \end{bmatrix}$$

(iii) For all $0 = t_0 < t_1 < \cdots < t_m$, the random variables $W_{t_1}, W_{t_2}, \ldots, W_{t_m}$ have the joint moment-generating function

$$\mathcal{M}(u_1, \ldots, u_m) = \mathbb{E}\left[\exp\left(u_1 W_{t_1} + \ldots + u_m W_{t_m}\right)\right].$$

If any of (i), (ii), or (iii) holds (and hence they all hold), then W_t, $t \geq 0$, is a Brownian motion.

Moreover, as time progresses, the available information changes. To represent this, the use of filtration and its relationship with Brownian motion are crucial for the subsequent analysis in the thesis.

Definition 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W_t, t \geq 0$. A filtration for the Brownian motion is a collection of σ-algebras $\mathcal{F}_t, t \geq 0$, satisfying [29]:

(i) (Information accumulates) For $0 \leq s < t$, every set in $\mathcal{F}_s$ is also in $\mathcal{F}_t$. In other words, there is at least as much information available at the later time $\mathcal{F}_t$ as there is at the earlier time $\mathcal{F}_s$.

(ii) (Adaptivity) For each $t \geq 0$, the Brownian motion W_t at time t is $\mathcal{F}_t$-measurable. In other words, the information available at time t is sufficient to evaluate the Brownian motion W_t at that time.

(iii) (Independence of future increments) For $0 \leq t < v$, the increment $W_v - W_t$ is independent of $\mathcal{F}_t$. In other words, any increment of the Brownian motion after time t is independent of the information available at time t.

Let $X_t, t \geq 0$, be a stochastic process. We say that X_t is adapted to the filtration $\mathcal{F}_t$ if for each $t \geq 0$ the random variable X_t is $\mathcal{F}_t$-measurable.

Definition 2.3 (Ito-Doeblin formula for Brownian Motion [29])**.** Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let W_t be a Brownian motion. Then, for every $T \geq 0$,

$$f(T,W_T) = f(0,W_0) + \int_0^T f_t(t,W_t)\,dt + \int_0^T f_x(t,W_t)\,dW_t \\ + \frac{1}{2}\int_0^T f_{xx}(t,W_t)\,dt.$$

Definition 2.4 ([29])**.** Let $W_t, t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t), t \geq 0$, be an associated filtration. An Ito process is a stochastic process of the form

$$X_t = X_0 + \int_0^t \delta_s\,dW_s + \int_0^t \theta_s\,ds, \tag{2.1}$$

where X_0 is nonrandom and δ_s and θ_s are adapted stochastic processes.

Lemma 2.1 ([29])**.** *The quadratic variation of the Ito process (2.1) is*

$$[X,X]_t = \int_0^t \delta_s^2\,ds.$$

Definition 2.5 ([29])**.** Let $X_t, t \geq 0$, be an Itô process as described in Definition 2.3, and let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Then, for every $T \geq 0$,

$$\begin{aligned} f(T,X_T) &= f(0,X_0) + \int_0^T f_t(t,X_t)\,dt \\ &\quad + \int_0^T f_x(t,X_t)\,dX_t + \frac{1}{2}\int_0^T f_{xx}(t,X_t)\,d[X,X]_t \\ &= f(0,X_0) \\ &\quad + \int_0^T \left(f_t(t,X_t) + f_x(t,X_t)\delta_t + \frac{1}{2} f_{xx}(t,X_t)\delta_t^2 \right) dt. \end{aligned}$$

2.1 Ornstein-Uhlenbeck Process

Introduced by Ornstein and Uhlenbeck [32] as a model for the velocity of Brownian particles (i.e., free particles moving in a gas) subject to friction, their work built on Einstein's study of Brownian motion, which considered frictionless movement. The Ornstein–Uhlenbeck process has numerous applications, including modeling interest rates in finance, as seen in Vasicek [33]. and Evans et. al. [6]. Simulation and estimation methods for this process are well-established, as discussed by Iacus [12].

Definition 2.6 ([14]). The Ornstein-Uhlenbeck process (OU process) is the solution of the differential equation

$$dX_t = -\alpha X_t dt + \sigma dW_t, \quad X_0 = \eta \tag{2.2}$$

where σ and α are positive constants and η is a random initial condition independent of W_t. This means that the OU process is a continuous process that fulfills

$$\int_{t_1}^{t_2} X_t(\omega)dt = \eta - \alpha \int_{t_1}^{t_2} X_s(\omega)ds + \sigma \int_{t_1}^{t_2} dW_s(\omega) \text{ a.s.}$$

The right-hand side of the equation is well-defined because X_s has continuous trajectories. Hence, the first integral exists in the Riemann sense and the second integral is simply a constant integrated with respect to W_t in the Itô sense.

Theorem 2.2 ([14]). *A unique solution to Equation 2.2 exists and can be written explicitly as*

$$X_t = e^{-\alpha t}\eta + \int_0^t e^{-\alpha(t-s)}\sigma dW_s$$

2.2 Stochastic Control Problems

A diffusion process $X = \{X_t\}_{t\geq 0}$ under a control $\alpha = \{\alpha_t\}_{t\geq 0}$ can be described by a stochastic differential equation (SDE) valued in $\mathbb{R}$:

$$dX_s = b(s, X_s, \alpha_s)ds + \sigma(s, X_s, \alpha_s)dW_s. \tag{2.2.1}$$

The control $\alpha = \{\alpha_t\}_{0\leq t\leq T}$ is a progressively measurable process, valued in $A \subset \mathbb{R}$. When this controlled SDE admits a unique strong solution starting from x at $s = t$, we then denote by $\{X_{t,x}, t \leq s \leq T\}$ this solution with almost surely continuous paths. For a constant control a and a function $\eta \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, define the infinitesimal generator of the controlled diffusion process by

$$L_a\eta = b(t, x, a)\eta_x + \frac{1}{2}\sigma^2(t, x, a)\eta_{xx},$$

where η_x (η_{xx} resp.) is the first (second resp.) order partial derivative of η with respect to x.

We fix a finite time horizon $T > 0$ with $0 < T < \infty$. Let $f : [0, T] \times \mathbb{R} \times A \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two measurable functions satisfying:

(i) g is lower-bounded, or

(ii) $|g(x)| \leq \kappa(1+|x|^2), \forall x \in \mathbb{R}$ for some constant κ independent of x.

For $(t,x) \in [0,T] \times \mathbb{R}$, denote by $A(t,x)$ the set of controls α such that

$\mathbb{E}\left[|f(s,X,\alpha)|\, ds\right] < \infty$. [20]

Definition 2.7 (Value function)**.** For all $(t,x) \in [0,T] \times \mathbb{R}$ and $\alpha \in A(t,x)$, the gain function, $J(t,x;\alpha)$, is defined as:

$$J(t,x;\alpha) = \mathbb{E}\left[\int_t^T f(s,X_{t,x},\alpha)ds + g(X_{T,x})\right].$$

The associated value function is

$$V(t,x) = \sup_{\alpha \in A(t,x)} J(t,x;\alpha).$$

Given an initial condition $(t,x) \in [0,T) \times \mathbb{R}$, we say that $\hat{\alpha} \in A(t,x)$ is an optimal control if $V(t,x) = J(t,x;\hat{\alpha})$. A control process α in the form $\alpha = a(s,X_{t,x})$ for some measurable function a from $[t,T] \times \mathbb{R}$ into A, is called a Markovian control.

Theorem 2.3 (HJB equation [20])**.** *Suppose functions b, σ, f, and g are uniformly continuous, and there exists a constant $\kappa > 0$ such that for $\varphi(t,x,a) = b(t,x,a)$, $\sigma(t,x,a)$, $f(t,x,a)$, $g(x)$, $\forall t \in [0,T]$, $x,y \in \mathbb{R}$, $a \in A$,*

$$|\varphi(t,x,a) - \varphi(t,y,a)| \leq \kappa|x-y| \quad \text{and} \quad |\varphi(t,0,a)| \leq \kappa.$$

If the value function $V \in C^{1,2}([0,T] \times \mathbb{R})$, then V is a solution of the following terminal value problem of a second-order partial differential equation, called the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} -\partial_t v(t,x) - H(t,x,v_x,v_{xx}) = 0, &\quad (t,x) \in [0,T) \times \mathbb{R}, \\ V(T,x) = g(x), &\quad x \in \mathbb{R}, \end{aligned} \tag{2.3}$$

where for $(t,x,p,M) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$H(t,x,p,M) = \sup_{a \in A}\left\{b(t,x,a)p + \frac{1}{2}\sigma^2(t,x,a)M + f(t,x,a)\right\}.$$

H is called the Hamiltonian of the associated control problem.

Theorem 2.4 (Verification theorem [20]). *Let $w \in C^{1,2}([0,T] \times \mathbb{R}))$ be a solution to Equation (2.3) satisfying a quadratic growth condition:*

$$|w(t,x)| \leq \kappa(1+|x|^2), \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

If there exists a measurable function $\alpha^*(t,x) : [0,T] \times \mathbb{R} \to A$ such that

$$\text{-}\partial_t w(t,s) - \sup_{\alpha \in A} H(t,x,w(t,x),w_{xx}(t,x)) = -\partial_t w - L\alpha^*(t,x)w(t,x) - f(t,x,\alpha^*(t,x))$$

and the stochastic differential equation

$$\partial_t \alpha^*(t,x) = 0,$$
$$dX_s = b(s,X_s,\alpha^*(s,X_s))ds + \sigma(s,X_s,\alpha^*(s,X_s))dW_s$$

admits a unique strong solution, denoted by $X^*(t,x)$ starting at $X = x$, and the process $\{\alpha^*(s,X^*(t,x)), t \leq s \leq T\} \in A(t,x)$, then $w = V$ on $[0,T] \times \mathbb{R}$, and α^* is an optimal Markovian control.

2.3 Cramér-Lundberg model and diffusion approximation

Considering the classical Cramér-Lundberg model [25], Surplus process is defined as

$$U_t = u_0 + ct - \sum_{i=1}^{N_t} X_i,$$

where u_0 is the initial surplus, the arrival process N_t is a Poisson process with constant intensity $\gamma > 0$, and random variables $\{X_i\}_{i=1}^{\infty}$ are strictly positive, independent and identically distributed claim sizes independent of N_t. We let $\{T_i\}_{i=1}^{\infty}$ denote claim occurrence times and for convenience we let $T_0 = 0$ and $G(X_i)$ denote the claim size distribution finite first and second moments $m_1 = \mathbb{E}(X_i) = \mu, m_2$, respectively [25]. $\{T_i\}_{i=1}^{\infty}$ and $G(X_i)$ are defined here to be used in reinsurance Section 2.5

The insurance premium, c, can be determined using different premium principles, such as the expected value principle, the variance principle and the standard deviation principle.

2.3.1 Diffusion approximation

The calculation of characteristics in the classical risk model is often challenging, leading to the search for approximations. One successful approach is the use of diffusion approximations. This method involves considering a series of classical risk models that weakly converge to a diffusion process. As the classical risk process features stationary and independent increments, a diffusion approximation is meaningful only if the diffusion process also exhibits stationary and independent increments. Consequently, the limiting process should to be a Brownian motion [25].

Let $C_t = \sum_{i=1}^{N_t} X_i$ and we can show that

$$\mathbb{E}(C_t) = \gamma t \mathbb{E}(X_i) = \gamma \mu t, \tag{2.4}$$

$$\text{Var}(C_t) = \gamma t \mathbb{E}(X_i^2) = \gamma(\mu^2 + \sigma^2)t. \tag{2.5}$$

Define sequence

$$C_t^{(n)} = \frac{C_{nt} - \gamma m_1 nt}{\sqrt{\gamma(\mu^2 + \sigma^2)n}} \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

where $C_{nt} = \sum_{i=1}^{N_{nt}} X_i$, then we have

$C_t^{(n)} \xrightarrow{d} W_t, \quad$ as $\quad n \to \infty$, where W_t is a standard Brownian motion.

Now we examine a sequence of risk processes

$$U_t^{(n)} = x + \frac{p^{(n)}nt - C_{nt}}{\sqrt{n}}$$

where $p^{(n)} = (1 + \theta^{(n)})\gamma\mu$.

$$\begin{aligned} \frac{p^{(n)}nt - C_{nt}}{\sqrt{n}} &= \frac{p^{(n)}nt - \gamma\mu nt}{\sqrt{n}} - \frac{C_{nt} - \gamma\mu nt}{\sqrt{n}} \\ &= \theta^{(n)}\gamma\mu\sqrt{n}t - \sqrt{gamma(\mu^2 + \sigma^2)}C_t^{(n)} \end{aligned}$$

Assume $\theta^{(n)}\sqrt{n} \to \tau$, as $n \to \infty$, where τ is a positive constant.

Subsequently, we obtain

$$U_t^{(n)} \xrightarrow{d} x + \tau\gamma\mu t - \sqrt{\gamma(\mu^2 + \sigma^2)}W_t, \quad \text{as} \quad n \to \infty.$$

Therefore, this sequence of risk processes weakly converges to a diffusion process. [7]

2.4 Premium Principles

The premium represents the payment made by a policyholder for obtaining complete or partial coverage against a specified risk, X, i.e. claims requirements (severity and frequency) which is characterized by the distribution of the random variable X. A premium principle takes the form $c = \varphi(X)$, where φ is a function mapping the risk to a numerical premium value. The premium principles presented in this section come from [5].

The **pure premium principle** sets

$$c_X = \mathbb{E}(X)$$

Thus, the pure premium is equal to the insurer's expected claims under the risk. From an insurer's perspective, the pure premium principle is not very appealing. The premium only covers the expected claims from the risk and does not include any extra charge for profit or unexpected high claims. An insurer using this method to calculate premiums probably will not stay in business for a long time.

The **expected value principle** sets

$$c_X = (1 + \theta)\mathbb{E}(X),$$

where $\theta > 0$ is referred to as the premium loading factor. The loading in the premium is $\theta\mathbb{E}(X)$. The expected value principle satisfies the non-negative loading property and the principle is additive since

$$(1 + \theta)\mathbb{E}(X_1 + X_2) = (1 + \theta)\mathbb{E}(X_1) + (1 + \theta)\mathbb{E}(X_2)$$

and is scale invariant since for $D = hX$,

$$\begin{aligned} c_d &= (1 + \theta)\mathbb{E}(D) = h(1 + \theta)\mathbb{E}(X) \\ c_d &= hc_x. \end{aligned}$$

The **variance principle** sets

$$c_X = \mathbb{E}(X) + \zeta\mathbb{V}(X),$$

where $\zeta > 0$. Thus, the loading in this premium is proportional to $\mathbb{V}(X)$. Since $\zeta > 0$, the variance principle has a non-negative loading. The principle is additive since $\mathbb{V}(X_1 + X_2) = \mathbb{V}(X_1) + \mathbb{V}(X_2)$ when X_1 and X_2 are independent. However, the variance principle is not scale invariant since for $D = hX$,

$$\begin{aligned} c_d &= \mathbb{E}(D) + \zeta\mathbb{V}(D) \\ &= h\mathbb{E}(X) + \zeta h^2\mathbb{V}(X) \\ c_d &\neq hc_x. \end{aligned}$$

The **standard deviation principle** sets

$$c_X = \mathbb{E}(X) + \zeta\mathbb{V}(X)^{1/2},$$

where $\zeta > 0$. Thus, under this premium principle, the loading is proportional to the standard deviation of X . Although the motivation for the standard deviation principle is the same as for the variance principle, these two principles have different properties. As $\zeta > 0$, the standard deviation principle has a non-negative loading and is scale invariant since for $D = hX$,

$$\begin{aligned} c_d &= \mathbb{E}(D) + \zeta\mathbb{V}(D)^{1/2} \\ &= h\mathbb{E}(X) + \zeta h\mathbb{V}(X)^{1/2} \\ c_d &= hc_x. \end{aligned}$$

The standard deviation principle is not additive since standard deviations are not additive.

The Expected Value principle provides a stronger method for calculating premiums. By adding a safety loading factor to the premium, it not only covers the expected claims but also gives a cushion against unexpected losses and ensures a profit for the insurer. This extra margin is vital for maintaining financial stability and staying competitive in the market. Insurers who use the Expected Value Principle are better prepared to manage variability in claims and can continue operating even when facing adverse claims situations. This principle is commonly used in various insurance products, such as life, health, and property insurance, because it balances fairness and financial prudence.

In contrast, the Variance and Standard Deviation principles bring more sophistication by considering the variability of losses. These principles provide a more detailed risk assessment by accounting for the spread of claims but also make the premium calculation process more complex. The Variance Principle adds a term proportional to the variance of the losses, while the Standard Deviation Principle uses the standard deviation, which is the square root of the variance. Both principles require precise estimation of these statistical measures, which can be difficult and resource-intensive. Insurers need detailed historical data and advanced statistical techniques to accurately estimate variance and standard deviation. Although these principles can lead to higher premiums that more accurately reflect the true risk, their complexity and the need for accurate data make them less straightforward to use compared to the Expected Value Principle. For these reasons, Expected Value Principle is studied as the premium valuation criteria in this thesis.

2.5 Reinsurance

Reinsurance is a fundamental concept in the insurance industry, playing a important role in managing risk. It involves transferring a portion of an insurer's risk to another insurance entity, referred to as the reinsurer. This transfer of risk helps insurers reduce the potential impact of significant losses, especially those arising from catastrophic events or unexpected large claims.

In a reinsurance agreement f where the assumption is $0 \leq f(X) \leq X$ for all $X \geq 0$, the direct insurer pays reinsurance premium to the reinsurer for a claim X, with the reinsurer covering $f(X)$ of the loss and the insurer retaining $R(X) = X - f(X)$. The reinsurance premium can be calculated under different premium principles, such as the expected value principle, as $(1+\xi)\mathbb{E}[f(X)]$, where $\xi \geq \theta$ is the loading factor for the reinsurer.

Reinsurance treaties are categorized into proportional and non-proportional types. Excess-of-loss and stop-loss are commonly used as non-proportional reinsurance policies. The subsequent section provides a brief modeling of these three reinsurance arrangements.

2.5.1 Proportional Reinsurance

Under a proportional reinsurance where $f(X) = qX$, the insurer can transfer a fraction q_t of the incoming claims to the reinsurer, where q_t satisfies $0 \leq q_t \leq 1$ for all t. In this agreement, the premium rate paid to the reinsurer is $(1+\xi)q_t\gamma m_1$ assuming that premium is calculated via expected value principle.

Let U_t represent the surplus process of the insurer at time t. With proportional reinsurance, surplus dynamic follows

$$dU_t^q = \gamma m_1[(1+\theta) - (1+\xi)q_t]\, dt - d\sum_{i=1}^{N_t}(1 - q_{T_i})X_i$$

2.5.2 Excess-of-loss Reinsurance

When the reinsurance agreement f is structured as an excess-of-loss policy, i.e,

$$f(X) = \begin{cases} 0, & \text{if } X \leq d \\ X - d, & \text{if } X > d, \end{cases}$$

the contract specifies a retention level d_t, where $d_t \geq 0$ is the control parameter chosen by the insurance company at time t. For an individual claim X_i, the reinsurer compensates the insurance company for the amount $\max\{0, X_i - d_{T_i}\}$, and the consequent liability for the insurer is $X_i \wedge d_{T_i}$, where $x \wedge y = \min\{x, y\}$. Figure 2.1 illustrates the Excess of Loss claim payments whose aggregated version refers to Stop Loss insurance.

Under Excess-of-loss Reinsurance, the dynamics of the insurer's surplus process is governed by

$$dU_t^d = [(\theta - \xi)\gamma m_1 + (1+\xi)\eta]dt - d\sum_{i=1}^{N_t} X_i \wedge d_{T_i}.$$

where

$$\eta = \mathbb{E}[X_i \wedge d].$$

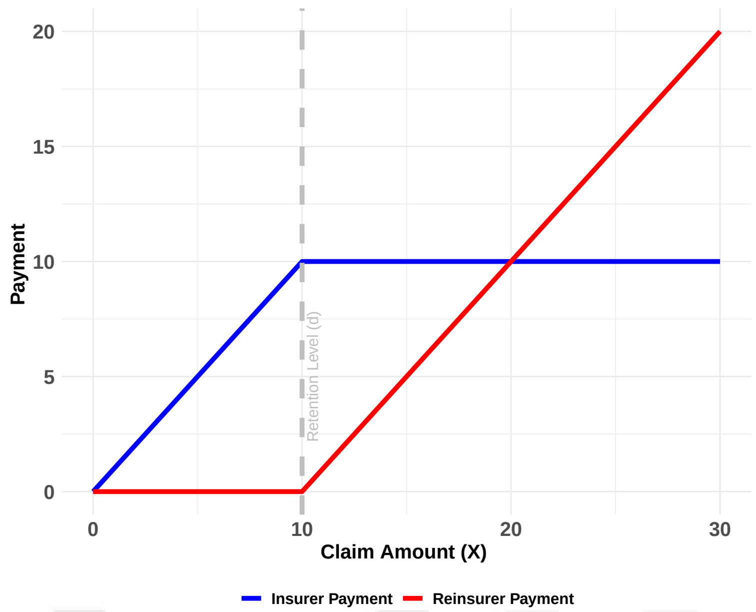


Figure 2.1: Illustration of Payment (TL Millions) under Excess of Loss Reinsurance

2.5.3 Stop-Loss Reinsurance

Similar to excess-of-loss reinsurance, a stop-loss reinsurance contract only obligates the reinsurer to pay if the losses incurred by the ceding company surpass a predetermined retention level g, $g \geq 0$. However, the stop-loss contract addresses aggregate risks and can limit the total amount of claim losses for which the insurance company is accountable for.

For the insurer's aggregate claim S, the reinsurance premium via expected value principle as follows

$$c_R = (1+\theta)\int_b^{\infty}(1-G(S))\,dS$$

where $G(S)$ denotes the aggregate claim size distribution function.

The reinsurer will compansate the insurer with $f(S) = (S-g)_+$ where $(S-g)_+ = max\{0, S-g\}$ and the net claim liability for the insurance company becomes $S \wedge g$.

Among those reinsurance agreement, we consider specifically Proportional reinsurance, due to its common use, easiness in interpretation and analytical practicality.
In the following chapter, we present our model settings in more detail, defining the core elements of the insurance aspect and the financial market model.

CHAPTER 3

MODEL ASSUMPTIONS

Reinsurance and investment strategies are critical components of the insurance industry, enabling insurers to manage risk and stabilize their financial performance. This chapter explores the dynamics of an insurance company's surplus under a proportional reinsurance contract within a stochastic framework and defines our assumptions. It should be pointed out that, from the leading two references, Cao et al. [3] assume that premium is calculated via expected value principle while Liang et al. [16] consider variance principle to determine premium under diffusion setting. Liang et al. [16] also assume that the instantaneous rate of investment return follows an Ornstein–Uhlenbeck process unlike [3] that assume the parameter on risky asset as constant.

3.1 Claim Process and Premium Calculation

We build upon the frameworks established by [21] and [3] to describe the claim process, X_t, which follows a stochastic process defined as

$$dX_t = adt - bdW_t^0, \tag{3.1}$$

where a is the average rate of claim losses, and b is the standard deviation of the claim loss. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a complete filtered probability space with a fixed terminal time $T > 0$ where W_t^0 denotes a standard Brownian motion on this space.

The premium rate c_I is determined via the expected value principle [5], a common approach in actuarial science, which is formulated as

$$c_I = (1+\theta)a \tag{3.2}$$

where $\theta > 0$ denotes the relative safety loading of the insurer. This premium calculation ensures that the insurer charges a premium rate that not only covers the anticipated claims but also includes a margin for profit and risk. It should be noted that the Variance Principle has been studied under a different setup in Liang et al. [16] due to the simplicity of analytical derivations which assures the boundaries to remain always positive. However, it should be mentioned that the most commonly used premium valuation in practice is based on Expected Value principle. In contrast to the Variance Principle, the Expected Value Principle involves more complex analytical challenges, requiring careful handling of boundary conditions.

3.2 Proportional Reinsurance and Admissible Strategies

We assume that the insurer has the option to purchase proportional reinsurance. This type of reinsurance allows the insurer to transfer a portion of their risk to the reinsurer. This method is widely used in the insurance industry because it helps insurers manage their risk exposure effectively.

Based on the same expected value principle, the premium, c_R, for the reinsurer is calculated as follows:

$$c_R = (1+\xi)q_t a, \tag{3.3}$$

where ξ is the safety loading for reinsurer. It should be noted that, the reinsurer typically imposes a higher safety loading ($\xi > \theta$) due to the higher costs associated with taking on ceded risk.

A strategy q_t is said to be admissible if q_t is an admissible adapted process which satisfies $\int_0^T q_s^2\, ds < \infty$ almost surely, for all $t \geq 0$.

As mentioned before, we consider proportional reinsurance agreement in this thesis because its mathematical formulation and analysis is simpler. Because of this simplicity, we can concentrate on the main aspects of maximizing utility without getting overwhelmed by the complexities of more complicated reinsurance structures. The other reason is that proportional reinsurance is widely used in the insurance industry. Its practical importance ensures that theoretical models and results are applicable and useful for real-world decision-making. We prefer to study strategies that have direct implications for industry practitioners.

3.3 Dynamics of Surplus Process

Based on approach proposed by Cao et al. [3], we model the dynamics of the surplus process, U_t, incorporating the effects of premiums, claims, and reinsurance costs as follows:

$$\begin{aligned} dU_t &= c_I dt - dX_t - c_R dt \\ &= (1+\theta)adt - a(1-q_t)dt + b(1-q_t)dW_t^0 - (1+\xi)q_t adt \\ &= (\theta - \xi q_t)adt + b(1-q_t)dW_t^0, \quad U_0 = u_0 \end{aligned} \tag{3.4}$$

In Equation (3.4), c_I and c_R represent the premium rates for the insurer and the reinsurer, respectively, and X_t denotes the claim process. It is assumed that initial surplus, u_0, is constant. This equation shows how premium income, claim payments, and reinsurance expenses affect the insurer's surplus over time.

3.4 Investment in Financial Market

In the financial market where the insurer operates, the surplus can be allocated into two types of assets: a risky asset, such as a stock or mutual fund, and a riskless asset, such as a bond or bank account. The riskless asset, S_t^0, is assumed to follow the dynamics:

$$dS_t^0 = rS_t^0 dt, \tag{3.5}$$

where r represents the continuously compound return rate, a positive constant indicating the guaranteed rate of growth per unit time.

Following the approach by Liang et al. [16], we model the dynamics of a risky asset, denoted by S_t^1, which is assumed to follow the process

$$dS_t^1 = k_t S_t^1 dt + \sigma S_t^1 dW_t^1, \quad \text{where } k_t = k + m_t, \tag{3.6}$$

where m_t is a solution to the equation

$$dm_t = \alpha m_t dt + \beta dW_t^2, \quad m_0 = m_0.$$

Here, m_0 is constant, $k > 0$ is the long-run mean growth rate of the asset, m_t follows an OU process which represents the effect of the market on the risky assets' price, $k+m_t$ is the instantaneous return rate of the risky asset and α, β are known constants. For the OU model, a bull market can be considered if m_t is substantially larger than 0. Otherwise, i.e., m_t is substantially less than 0, we consider a bear market (see [16]). We assume that both dW_t^1 and dW_t^2 are standard Brownian motions with a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and dW_t^0 is independent of dW_t^1 and dW_t^2. However, dW_t^1 and dW_t^2 are dependent with a correlation coefficient, ρ implying that $E[W_t^1 W_t^2] = \rho t$, where $\rho \in (-1,0) \cup (0,1)$. The dependence, represented by the correlation ρ, means that the random shocks affecting the returns of the risky asset (S_t^1) and the market effect process (m_t) are correlated. A positive correlation ($\rho > 0$) suggests that when the market influence increases, the return on the risky asset also tends to increase, and vice versa. Conversely, a negative correlation ($\rho < 0$) implies that when the market impact increases, the return on the risky asset tends to decrease. This correlation captures the relationship between market-wide influences and the asset's performance, which is crucial for modeling and understanding the joint dynamics of financial and economic variables. The perfect correlation is purposedly ignored in order to assure analytical solution.

3.5 Dynamics of Insurer's Wealth

Different than Liang et al. [16] and Cao et al. [3], we propose a dynamic wealth process whose base structure is inspired from their assumptions. Risk aspect of the proposed model considers the assumption on [3] whereas financial model part is inspired by [16]'s work. Both of these studies are modified and changed with respect to some actuarial and financial structures.

Let Z_t denote the wealth of the insurer at time t. The investment strategy, represented by a stochastic process Π_t, which represents the amount of money invested in the risky asset at time t. The remainder of insurer's wealth, $Z_t - \Pi_t$, is, consequently, invested in the riskless asset at time t. A strategy Π_t is said to be admissible if Π_t is an admissible adapted process which satisfies $\int_0^T \Pi_s^2\, ds < \infty$ almost surely, for all $T \geq 0$ with the set of all admissible strategies, $\mathcal{A}$.

Given the strategy (Π_t, q_t), we assume the dynamics of the wealth of the insurer, X_t, are generated by

$$\begin{aligned} dZ_t &= \frac{\Pi_t dS_t^1}{S_t^1} + \frac{(Z_t - \Pi_t)dS_t^0}{S_t^0} + c_0 dt - dC_t - c_1 dt \\ &= \left[rZ_t + (m_t + k - r)\Pi_t + (\theta - \xi q)a\right] dt + \Pi_t \sigma dW_t^1 + b(1-q)dW_t^0 \end{aligned} \tag{3.7}$$

where $Z_0 = z_0$ is the initial wealth. It should be noted that, the insurer's wealth comes out to be the function of both the investment and reinsurance strategies, the dynamics of the financial market, the claim process, the premium income, and the reinsurance costs. It should be noted that, Equation (3.7) illustrates the influence of reinsurance agreement parameter, as well as the loading factors regarding each parties and the loading factors associated to premium charged.

We define our optimization problem based on the dynamics of the insurer's wealth, Z_t, as described in Equation (3.7). It is important to highlight that our model for Z_t differs from the dynamics considered by Liang et al. [16] and Cao et al. [3] due to differences in premium calculation methods and financial market modeling.

In the next chapter, we explain our optimization problem in detail, focusing on the goal of maximizing the expected utility of terminal wealth. We also discuss the formulation and solution of the Hamilton-Jacobi-Bellman (HJB) equation, which is essential for determining the optimal investment and reinsurance strategies.

CHAPTER 4

OPTIMAL REINSURANCE AND INVESTMENT STRATEGY

In order to obtain an optimal reinsurance and investment strategy which satisfies the expectations of insurer and reinsurer, we need to evaluate the optimality in terms of the expected utilities with respect to each party. For this purpose, we define the insurer's utility function, $u(x)$ whose expected value enables us to define an objective function under certain conditions to be satisfied. The insurer aims to maximize their expected utility of the terminal wealth of the insurance portfolio, Z_T. Under the assumption on utility function of the insurer's wealth be a strictly increasing and concave, the optimization problem is proposed as

$$\max \mathbb{E}[u(Z_T)], \tag{4.1}$$

$$s.t. \begin{cases} dZ_t = [rZ_t + (m_t + k - r)\Pi_t + (\theta - \xi q)a]\, dt + \Pi_t \sigma dW_t^1 + b(1-q)dW_t^0 \\ dm_t = \alpha m_t dt + \beta dW_t^2, \quad m_0 = m_0 \\ (\Pi, q) \in \mathcal{A}. \end{cases} \tag{4.2}$$

It should be noted that the optimization problem is specific to the assumption made for this thesis. Here, the constraints assures the insurer's wealth proposed in Equation (4.2) to be satisfied with a stochastic behavior for m_t, which is the asset dynamics.

It gains importance to define an utility function for defining the preference of the insurer. The most commonly assumed utility functions in the literature are exponential and logarithmic depending on the structure of the defined problem. Based on the common choice in literature, we assume first that insurer has an exponential utility function with a constant absolute risk aversion parameter λ as (see [17])

$$u(z) = -\frac{1}{\lambda} e^{-\lambda z} \qquad \text{where } \lambda > 0.$$

The exponential utility function is often preferred due to its constant relative risk aversion property, which is well-suited for modeling investor behavior. However, in literature, the logarithmic utility function is generally avoided in these contexts for several reasons. First, logarithmic utility implies an investor's relative risk aversion decreases as wealth increases. This might not accurately represent the risk preferences of an insurer, who is focused on minimizing the probability of ruin. Second, the logarithmic utility does not consider wealth levels below zero, which can be unrealistic in situations where liabilities are greater than assets. Despite these considerations, as second we explore the logarithmic utility function in the Section 4.1. These two choices allow us to analyze the unique characteristics and implications of these selected utility functions within our framework, offering a different perspective on the insurer's decision-making process.

Then, the associated value function, $V(t, z, m)$, under the exponential utility function assumption is defined as

$$V(t, z, m) = \max \mathbb{E}\left[u(Z_T)|(Z_t, m_t) = (z, m)\right] \tag{4.3}$$

As given in Fleming and Soner [30], we observe that if the value function $V \in C^{1,2}$, then V satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\begin{aligned} V_t + \sup_{\Pi,q} \Big\{ &[rz + (m + k - r)\Pi + (\theta - \xi q)a]\, V_z + \left[\frac{1}{2}\Pi^2\sigma^2 + \frac{1}{2}b^2(1-q)^2\right] V_{zz} \\ &+ \alpha m V_m + \frac{1}{2}\beta^2 V_{mm} + \sigma\Pi\beta\rho V_{zm} \Big\} = 0, \end{aligned} \tag{4.4}$$

with the boundary condition:

$$V(T, z, m) = u(z) \tag{4.5}$$

and $V_t, V_z, V_{zz}, V_{zm}, V_m, V_{mm}$ are partial derivatives of V.

We propose the verification Theorem 4.1 which shows that the classical solution to the HJB equation yields the solution to the optimization problem.

Theorem 4.1. *Assume that $P \in C^{1,2}$ is the solution to Equation* (4.4) *with the boundary condition* (4.5)*. Then the optimal value function V of* (4.3) *and P coincide with each other.*

$$P(t, z, m) = V(t, z, m)$$

Let Π^*, q^* satisfy

$$\mathcal{A}^{\Pi^*,q^*} V(t, z, m) = 0$$

holds for all $(t, x, m) \in [0, T) \times R \times R$. $\mathcal{A}^{\Pi^*,q^*}$ represents the infinitesimal generator associated with the stochastic process under the control strategies Π^* and q^*.

Then, the strategy $(\Pi^*(t, Z_t^*, m_t), q^*(t, Z_t^*, m_t))$ is optimal, with

$$\max \mathbb{E}\left[u(Z_T^*)|(Z_t^*, m_t) = (z, m)\right] = V(t, z, m). \tag{4.6}$$

Here Z_t^* is the wealth process under the optimal strategy.

Proof. Let the strategy (Π, q) be any admissible control system, and (Π_s, q_s) be $\mathcal{F}(t)$-progressively measurable,

$$\mathcal{A}^{\Pi_s,q_s} V(t, z_s, m_s) = 0.$$

Similar method as Fleming and Soner [30], from Equation (4.5) and the Dynkin formula

$$\begin{aligned} P(t, z, m) &= \mathbb{E}\left[\int_t^T -\mathcal{A}^{\Pi_s,q_s} V(t, z_s, m_s)\, ds + u(V(T))\right] \\ &\leq \mathbb{E}\left[u(V(T))\right] \end{aligned}$$

The inequality $P(t, z, m) \leq \mathbb{E}\left[u(V(T))\right]$ demonstrates that the expected utility of any control strategy cannot exceed that of the optimal strategy, as the optimal value function $V(t, z, m)$ maximizes expected utility. This reflects the principle that optimal strategies, determined by maximizing the value function, yield the highest possible expected outcomes.

Taking conditional expectations given (t, x, m) on both sides of above equation and taking Equation (4.4) into consideration yields

$$P(t, z, m) \leq V(t, z, m)$$

When $(\Pi, q) = (\Pi^*, q^*)$, the inequality in the above formula turns into equality. Thus,

$$P(t, z, m) = V(t, z, m)$$

Therefore, the value function $V(t,z,m)$ given by Equation (4.3) coincides with the concave solution to the HJB equation, and the proof is complete. □

We can find the optimal investment strategy, Π^*, that maximizes the function in (4.4) by taking the derivative of the function with respect to Π and setting it to zero

$$(m+k-r)V_z + \Pi\sigma^2 V_{zz} + \sigma\beta\rho V_{zm} = 0 \tag{4.7}$$

Then, we obtain the optimal investment strategy as

$$\Pi^* = \frac{-(m+k-r)V_z - \sigma\beta\rho V_{zm}}{\sigma^2 V_{zz}}. \tag{4.8}$$

To seek the optimal reinsurance strategy, q^*, maximizing the function presented in Equation (4.4), we take derivatives of (4.4) with respect to q and set it to zero

$$-\xi a V_z - b^2(1-q)V_{zz} = 0. \tag{4.9}$$

This yields a solution for q^* as follows:

$$q^* = 1 + \frac{\xi a V_z}{b^2 V_{zz}}. \tag{4.10}$$

Note that $q^* < 1$. If $q^* \geq 0$, then q coincides with q^*. If $q^* \leq 0$, we simply let q^* be 0. These lead us to introduce the following two lemmas, which are used in the solution of (4.4):

Lemma 4.2. *Let* $\mathcal{A}_1 = \{(t,z,m) \in [0,T) \times R \times R : 0 < q_t^* < 1\}$. *Suppose that* V *is a solution to*

$$\begin{aligned} &V_t + [rz + (m+k-r)\Pi + (\theta - \xi q)a]\, V_z + \left[\frac{1}{2}\Pi^2\sigma^2 + \frac{1}{2}b^2(1-q)^2\right] V_{zz} \\ &+ \alpha m V_m + \frac{1}{2}\beta^2 V_{mm} + \sigma\Pi\beta\rho V_{zm} = 0, \end{aligned} \tag{4.11}$$

where

$$q = 1 + \frac{\xi a V_z}{b^2 V_{zz}}$$

and

$$\Pi = \frac{-(m+k-r)V_z - \sigma\beta\rho V_{zm}}{\sigma^2 V_{zz}},$$

$(t,z,m) \in \mathcal{A}_1$ *with* $V(T,z,m) = u(z)$ *and* $V(T,z,m)$ *is a concave and increasing function of* z. *Then* V *satisfies the HJB equation* (4.4) *with boundary condition Equation* (4.5) *in* $\mathcal{A}_1$.

Proof. It can be shown that the supremum of the HJB equation

$$V_t + \sup_{\Pi,q}\left\{[rz + (m+k-r)\Pi + (\theta - \xi q)a]\, V_z + \left[\frac{1}{2}\Pi^2\sigma^2 + \frac{1}{2}b^2(1-q)^2\right] V_{zz} + \alpha m V_m + \frac{1}{2}\beta^2 V_{mm} + \sigma\Pi\beta\rho V_{zm}\right\} = 0 \tag{4.12}$$

over q and Π in $\mathcal{A}_1$ is attained at q^* in

$$q^* = 1 + \frac{\xi a V_z}{b^2 V_{zz}} \tag{4.13}$$

and Π^* in

$$\Pi^* = \frac{-(m+k-r)V_z - \sigma\beta\rho V_{zm}}{\sigma^2 V_{zz}}. \tag{4.14}$$

Substituting them into Equation (4.12) yields

$$V_t + [rz + (m+k-r)\Pi + (\theta - \xi q)a]\, V_z + \left[\frac{1}{2}\Pi^2\sigma^2 + \frac{1}{2}b^2(1-q)^2\right] V_{zz} + \alpha m V_m + \frac{1}{2}\beta^2 V_{mm} + \sigma\Pi\beta\rho V_{zm} = 0. \tag{4.15}$$

□

Lemma 4.3. *Let* $\mathcal{A}_2 = \{(t,z,m) \in [0,T) \times R \times R : q_t^* < 0\}$. *Suppose that* V *is a solution to*

$$\begin{aligned} &V_t + [rz + (m+k-r)\Pi + \theta a]\, V_z + \left[\frac{1}{2}\Pi^2\sigma^2 + \frac{1}{2}b^2\right] V_{zz} \\ &+ \alpha m V_m + \frac{1}{2}\beta^2 V_{mm} + \sigma\Pi\beta\rho V_{zm} = 0, \end{aligned} \tag{4.16}$$

where

$$q = 0$$

and

$$\Pi = \frac{-(m+k-r)V_z - \sigma\beta\rho V_{zm}}{\sigma^2 V_{zz}},$$

$(t,z,m) \in \mathcal{A}_2$ *with* $V(T,z,m) = u(x)$ *and* $V(T,z,m)$ *is a concave and increasing function of* x. *Then* V *satisfies the HJB equation* (4.4) *with boundary condition Equation* (4.5) *in* $\mathcal{A}_2$.

Proof. It can be shown that the supremum of HJB equation

$$V_t + \sup_{\Pi,q}\left\{[rz + (m+k-r)\Pi + (\theta - \xi q)a]\,V_z + \left[\frac{1}{2}\Pi^2\sigma^2 + \frac{1}{2}b^2(1-q)^2\right]V_{zz} \right.$$
$$\left. +\alpha m V_m + \frac{1}{2}\beta^2 V_{mm} + \sigma\Pi\beta\rho V_{zm}\right\} = 0 \tag{4.17}$$

over q and Π in $\mathcal{A}_2$ is attained at $q = 0$ and Π^* in

$$\Pi^* = \frac{-(m+k-r)V_z - \sigma\beta\rho V_{zm}}{\sigma^2 V_{zz}}. \tag{4.18}$$

Substituting them into Equation (4.17) yields

$$\begin{aligned} &V_t + [rz + (m+k-r)\Pi + \theta a]\,V_z + \left[\frac{1}{2}\Pi^2\sigma^2 + \frac{1}{2}b^2\right]V_{zz} \\ &+ \alpha m V_m + \frac{1}{2}\beta^2 V_{mm} + \sigma\Pi\beta\rho V_{zm} = 0. \end{aligned} \tag{4.19}$$

□

We use the above two lemmas to find the solution to Equation (4.4) with boundary condition Equation (4.5). We are now looking for a candidate solution V of the ansatz form as follows

$$V(t,z,m) = -\frac{1}{\lambda}e^{-\lambda z e^{r(T-t)} + f(t,m)}.$$

Then, the first and second derivatives of V with respect to z and m are given by:

$$\begin{aligned} V_t &= V[\lambda z r e^{r(T-t)} + f_t], \\ V_z &= V[-\lambda e^{r(T-t)}], \\ V_{zz} &= V[\lambda^2 e^{2r(T-t)}], \\ V_{zm} &= V_{mz} = V[-\lambda e^{r(T-t)} f_m], \\ V_m &= V f_m, \\ V_{mm} &= V[f_m^2 + f_{mm}]. \end{aligned} \tag{4.20}$$

We can substitute these derivatives into the expressions of Π^* and q^*, to obtain their optimums given as:

$$\Pi^* = \frac{(m+k-r)}{\sigma^2\lambda}e^{-r(T-t)} + \frac{\beta\rho f_m}{\sigma\lambda}e^{-r(T-t)}, \tag{4.21}$$

$$q^* = 1 - \frac{\xi a}{b^2\lambda}e^{-r(T-t)}. \tag{4.22}$$

It should be noted that, q may come up two possible intervals which should be analyzed separately. Consistent with the literature [3], we need to determine the influence of q with respect to the dynamics among parameters of all processes involved in the optimization problem. These scenarios are as follows: Case 1 refers to $q \in (0, 1)$; whereas Case 2 considers $q < 0$.

Case 1:

$0 < q < 1 : \{\xi a \leq b^2\lambda \quad \text{or} \quad b^2\lambda < \xi a < b^2\lambda e^{rT}$ and $t < T-(ln(\xi a)-ln(b^2\lambda))/r\}$,

After substituting derivatives of V, Π^* and q^* into HJB equation (4.4) and simplifying algebraically the equation becomes:

$$\begin{aligned} f_t - \frac{(m+k-r)^2}{2\sigma^2} - \frac{(m+k-r)\beta\rho f_m}{\sigma} + (\xi-\theta)a\lambda e^{r(T-t)} - \frac{1}{2}\left(\frac{\xi a}{b}\right)^2 \\ + \frac{(\beta\rho f_m)^2}{2} + \alpha m f_m + \frac{\beta^2}{2}\left[f_m^2 + f_{mm}\right] = 0 \end{aligned} \tag{4.23}$$

with boundary condition $f(T, m) = 0$.

By using the methods of Rishel [22], we derive the following solution to Equation (4.23):

$$F(t, m) = G(t)m^2 + H(t)m + I(t) \tag{4.24}$$

where $G(t)$, $H(t)$ and $I(t)$ are the solutions to the following ordinary differential equations:

$$G^{'}(t) + \left[-2\beta^2\rho^2 + 2\beta^2\right]G^2(t) + \left[2\alpha - \frac{2\beta\rho}{\sigma}\right]G(t) - \frac{1}{2\sigma^2} = 0 \tag{4.25}$$

$$H^{'}(t) + \left((-2\beta^2\rho^2 G(t) + 2\beta^2 G(t) + \alpha - \frac{\beta\rho}{\sigma}\right)H(t) - \frac{(k-r)(2\sigma\beta\rho G(t)+1)}{\sigma^2} = 0 \tag{4.26}$$

and

$$\begin{aligned} I^{'}(t) + \frac{-\beta^2\rho^2+\beta^2}{2}H^2(t) - \frac{\beta\rho(k-r)}{\sigma}H(t) + \beta^2 G(t) + (\xi-\theta)a\lambda e^{r(T-t)} \\ - \frac{1}{2}\left(\frac{\xi a}{b}\right)^2 - \frac{(k-r)^2}{2\sigma^2} = 0 \end{aligned} \tag{4.27}$$

with the boundary conditions

$$G(T) = 0, \quad H(T) = 0, \quad I(T) = 0. \tag{4.28}$$

To derive Equations (4.29), (4.30), (4.31), we first substitute f_t, f_m, f_{mm} into Equation (4.23)

$$
\begin{aligned}
&G^{'}(t)m^2 + H^{'}(t)m + I^{'}(t) - \frac{m^2 + 2m(k-r) + (k-r)^2}{2\sigma^2} \\
&\quad - \frac{(m+k-r)\beta\rho[2G(t)m + H(t)]}{\sigma} + (\xi - \theta)a\lambda e^{r(T-t)} - \frac{1}{2}\left(\frac{\xi a}{b}\right)^2 \\
&\quad + \frac{(\beta\rho[2G(t)m + H(t)])^2}{2} + \alpha m[2G(t)m + H(t)] \\
&\quad + \frac{\beta^2}{2}\left[(2G(t)m + H(t))^2 + 2G(t)\right] = 0
\end{aligned}
$$

Grouping terms according to the powers of m leads to:

$$
\begin{aligned}
&\left[G^{'}(t) + \left[-2\beta^2\rho^2 + 2\beta^2\right]G^2(t) + \left[2\alpha - \frac{2\beta\rho}{\sigma}\right]G(t) - \frac{1}{2\sigma^2}\right]m^2 \\
&\quad + \left[H^{'}(t) + \left((-2\beta^2\rho^2 G(t) + 2\beta^2 G(t) + \alpha - \frac{\beta\rho}{\sigma}\right)H(t)\right. \\
&\quad \left.- \frac{(k-r)(2\sigma\beta\rho G(t) + 1)}{\sigma^2}\right]m \\
&\quad + I^{'}(t) + \frac{-\beta^2\rho^2 + \beta^2}{2}H^2(t) - \frac{\beta\rho(k-r)}{\sigma}H(t) + \beta^2 G(t) + (\xi - \theta)a\lambda e^{r(T-t)} \\
&\quad - \frac{1}{2}\left(\frac{\xi a}{b}\right)^2 - \frac{(k-r)^2}{2\sigma^2} = 0
\end{aligned}
$$

Then, we have:

$$
G^{'}(t) + \left[-2\beta^2\rho^2 + 2\beta^2\right]G^2(t) + \left[2\alpha - \frac{2\beta\rho}{\sigma}\right]G(t) - \frac{1}{2\sigma^2} = 0, \tag{4.29}
$$

$$
H^{'}(t) + \left((-2\beta^2\rho^2 G(t) + 2\beta^2 G(t) + \alpha - \frac{\beta\rho}{\sigma}\right)H(t) - \frac{(k-r)(2\sigma\beta\rho G(t) + 1)}{\sigma^2} = 0 \tag{4.30}
$$

$$
\begin{aligned}
I^{'}(t) &+ \frac{-\beta^2\rho^2 + \beta^2}{2}H^2(t) - \frac{\beta\rho(k-r)}{\sigma}H(t) + \beta^2 G(t) + (\xi - \theta)a\lambda e^{r(T-t)} \\
&- \frac{1}{2}\left(\frac{\xi a}{b}\right)^2 - \frac{(k-r)^2}{2\sigma^2} = 0
\end{aligned} \tag{4.31}
$$

Solutions to Equation (4.30) and (4.31) are just integrals since they are linear differential equations. Unlike them, Equation (4.30) takes the form of a Riccati equation. The Riccati equation is a first-order, nonlinear differential equation of the form:

$$
\frac{dy(t)}{dt} = a(t) + b(t)y(t) + c(t)y(t)^2
$$

where $y(t)$ is the unknown function, and $a(t)$, $b(t)$, and $c(t)$ are given functions of t. Here, Equation (4.30) includes nonlinear terms, specifically $\left[-2\beta^2\rho^2 + 2\beta^2\right]G^2(t)$. Therefore, we examine solutions to the Riccati equation for $G(t)$ given below.

We assume the following equations:

$$A = -2\beta^2\rho^2 + 2\beta^2, \qquad B = 2\alpha - \frac{2\beta\rho}{\sigma}, \qquad C = -\frac{1}{2\sigma^2}$$

Then Equation (4.29) becomes,

$$G^{'}t + AG^2(t) + BG(t) + C = 0$$

which is a normal Ricatti equation with the condition $B^2 - 4AC > 0$.

$$\left[2\alpha - \frac{2\beta\rho}{\sigma}\right]^2 + 4\frac{\beta^2(1-\rho^2)}{\sigma^2} > 0 \quad \text{since} \quad \rho \in (-1,0) \cup (0,1)$$

which is satisfied iff $\rho \neq |1|$.

Applying standard methods, the solution with the boundary condition $G(T) = 0$ as

$$G(t) = g_1 + \frac{e^{t\sqrt{B^2-4AC}}}{\frac{A}{\sqrt{B^2-4AC}}\left(e^{t\sqrt{B^2-4AC}} - e^{T\sqrt{B^2-4AC}}\right) - \frac{1}{g_1}e^{T\sqrt{B^2-4AC}}}. \tag{4.32}$$

where

$$g_1 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

With Equation (4.32), we have

$$H(t) = e^{\int_t^T y(s)\,ds}\left[-\int_t^T z(s)e^{-\int_s^T y(u)\,du}\,ds\right] \tag{4.33}$$

where

$$y(s) = -2\beta^2\rho^2G(s) + 2\beta^2G(s) + \alpha - \frac{\beta\rho}{\sigma}$$

and

$$z(s) = \frac{(k-r)(2\sigma\beta\rho G(s) + 1)}{\sigma^2}.$$

Similarly, we have

$$I(t) = \int_t^T \left[\frac{-\beta^2\rho^2 + \beta^2}{2}H^2(s) - \frac{\beta\rho(k-r)}{\sigma}H(s) + \beta^2G(s) + v(s)\right] ds \tag{4.34}$$

where

$$v(s) = (\xi - \theta)a\lambda e^{r(T-s)} - \frac{1}{2}\left(\frac{\xi a}{b}\right)^2 - \frac{(k-r)^2}{2\sigma^2}.$$

Moreover,

$$\Pi^* = \frac{(m+k-r)}{\sigma^2\lambda}e^{-r(T-t)} + \frac{\beta\rho\left[2G_tm_t + H_t\right]}{\sigma\lambda}e^{-r(T-t)},$$

and

$$q^* = 1 - \frac{\xi a}{b^2 \lambda} e^{-r(T-t)}.$$

Case 2:

$q < 0 : \{\xi a > b^2 \lambda e^{rT}$ or $b^2\lambda < \xi a < b^2 \lambda e^{rT}$ and $t > T - (ln(\xi a) - ln(b^2\lambda))/r\}$,

Considering case of $q_t < 0$, Lemma 4.3 implies that the solution to (4.4) can be obtained via (4.19). Substitute derivatives of V , Π^* and $q^* = 0$ into HJB equation (4.19) and simplifying algebraically, the equation becomes:

$$\begin{aligned} & f_t - \frac{(m+k-r)^2}{2\sigma^2} - \frac{(m+k-r)\beta\rho f_m}{\sigma} - \theta a \lambda e^{r(T-t)} + \frac{(\beta\rho f_m)^2}{2} \\ & + \alpha m f_m + \frac{\beta^2}{2}\left[f_m^2 + f_{mm}\right] = 0 \end{aligned} \tag{4.35}$$

with boundary condition $F(T, m) = 0$.

Using the methods of Rishel [22], we obtain the following solution of (4.35):

$$F(t, m) = L(t)m^2 + M(t)m + N(t) \tag{4.36}$$

where $L(t)$, $M(t)$ and $N(t)$ are solutions to the following ordinary differential equations:

$$L^{'}(t) + \left[-2\beta^2\rho^2 + 2\beta^2\right] L^2(t) + \left[2\alpha - \frac{2\beta\rho}{\sigma}\right] L(t) - \frac{1}{2\sigma^2} = 0, \tag{4.37}$$

$$M^{'}(t) + \left((-2\beta^2\rho^2 L(t) + 2\beta^2 L(t) + \alpha - \frac{\beta\rho}{\sigma}\right) M(t) - \frac{(k-r)(2\sigma\beta\rho L(t) + 1)}{\sigma^2} = 0 \tag{4.38}$$

and

$$N^{'}(t) + \frac{-\beta^2\rho^2 + \beta^2}{2} M^2(t) - \frac{\beta\rho(k-r)}{\sigma} M(t) + \beta^2 L(t) + -\theta a \lambda e^{r(T-t)} - \frac{(k-r)^2}{2\sigma^2} = 0 \tag{4.39}$$

with the boundary conditions

$$L(T) = 0, \quad M(T) = 0, \quad N(T) = 0. \tag{4.40}$$

To derive Equations (4.37), (4.38), (4.39), we first substitute f_t, f_m, f_{mm} into Equation (4.35) and solve it as we applied in the above approach. Solutions of Equation (4.38) and (4.39) are just integrals since they are linear differential equations. We examine solutions to Riccati equation for $L(t)$.

Let us assume:

$$A = -2\beta^2\rho^2 + 2\beta^2, \qquad B = 2\alpha - \frac{2\beta\rho}{\sigma}, \qquad C = -\frac{1}{2\sigma^2}$$

Then, Equation (4.37) becomes:

$$L^{'}t + AL^2(t) + BL(t) + C = 0$$

which is a standard Riccati equation with the condition $B^2 - 4AC > 0$.

Applying standard methods, we obtain the following solution to the Riccati equation with the boundary condition $L(T) = 0$:

$$L(t) = g_1 + \frac{e^{t\sqrt{B^2-4AC}}}{\frac{A}{\sqrt{B^2-4AC}}(e^{t\sqrt{B^2-4AC}} - e^{T\sqrt{B^2-4AC}}) - \frac{1}{g_1}e^{T\sqrt{B^2-4AC}}} \tag{4.41}$$

where

$$g_1 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

With Equation (4.41), we define $M(t)$ and $N(t)$ as follows

$$M(t) = e^{\int_t^T y(s)\,ds}\left[-\int_t^T z(s)e^{-\int_s^T y(u)\,du}\,ds\right] \tag{4.42}$$

where

$$y(s) = -2\beta^2\rho^2 L(s) + 2\beta^2 L(s) + \alpha - \frac{\beta\rho}{\sigma},$$

and

$$z(s) = \frac{(k-r)(2\sigma\beta\rho L(s) + 1)}{\sigma^2}.$$

$$N(t) = \int_t^T \left[\frac{-\beta^2\rho^2 + \beta^2}{2}M^2(s) - \frac{\beta\rho(k-r)}{\sigma}M(s) + \beta^2 L(s) + v(s)\right] ds \tag{4.43}$$

where

$$v(s) = -\theta a\lambda e^{r(T-s)} - \frac{(k-r)^2}{2\sigma^2}.$$

Moreover,

$$\Pi^* = \frac{(m+k-r)}{\sigma^2\lambda}e^{-r(T-t)} + \frac{\beta\rho\left[2L_t m_t + M_t\right]}{\sigma\lambda}e^{-r(T-t)}$$

and

$$q^* = 0.$$

The solutions derived above demonstrate how the HJB equation can be solved under certain conditions. These solutions provide an important foundation for determining optimal strategies and expressing the related value functions. The functions and their derivatives, which we have obtained, are key elements in more complex strategic decision-making processes. In this context, particularly under the assumption of an exponential utility function, a general theorem can be formulated that allows for the determination of optimal investment and insurance strategies. The exponential utility function has a constant absolute risk aversion property, where the risk aversion parameter λ stays the same no matter the level of wealth. This constancy makes it easier to analyze decision-making under uncertainty because the investor's or insurer's attitude towards risk does not change with their wealth. As a result, optimization problems become more straightforward.

Theorem 4.4. *Under the exponential utility function, there exist a classical solution $V \in C^{1,2}$ to (4.1) subject to (4.2). The solution V and corresponding optimal strategies are derived under four different cases (namely, C(I)-C(IV)) based on the conditions on the parameters.*

C(I): *If $\xi a \leq b^2\lambda$ then,*

$$\Pi^* = \frac{(m+k-r)}{\sigma^2\lambda}e^{-r(T-t)} + \frac{\beta\rho\left[2G_t m_t + H_t\right]}{\sigma\lambda}e^{-r(T-t)},$$

and

$$q^* = 1 - \frac{\xi a}{b^2\lambda}e^{-r(T-t)}$$

for any $t \in [0,T]$ and value function is

$$V(t,z,m) = -\frac{1}{\lambda}e^{-\lambda z e^{r(T-t)} + f(t,m)},$$

where

$$f(t,m) = G(t)m^2 + H(t)m + I(t),$$

C(II): *If $b^2\lambda < \xi a < b^2\lambda e^{rT}$ then,*

$$\Pi^* = \frac{(m+k-r)}{\sigma^2\lambda}e^{-r(T-t)} + \frac{\beta\rho\left[2G_t m_t + H_t\right]}{\sigma\lambda}e^{-r(T-t)},$$

and

$$q^* = 1 - \frac{\xi a}{b^2\lambda}e^{-r(T-t)}$$

for any $t < T - (ln(\xi a) - ln(b^2\lambda))/r$ and value function is

$$V(t,z,m) = -\frac{1}{\lambda}e^{-\lambda z e^{r(T-t)} + f(t,m)},$$

where

$$f(t,m) = G(t)m^2 + H(t)m + I(t),$$

In C(I) and C(II), the expressions, $G(t)$, $H(t)$ and $I(t)$ are given in (4.32), (4.33) and (4.34), respectively with terminal conditions (4.28),

C(III): *If $\xi a > b^2 \lambda e^{rT}$ then*

$$\Pi^* = \frac{(m+k-r)}{\sigma^2 \lambda} e^{-r(T-t)} + \frac{\beta\rho\,[2L_t m_t + M_t]}{\sigma\lambda} e^{-r(T-t)},$$

and

$$q^* = 0$$

for any $t \in [0,T]$ and value function is

$$V(t,z,m) = -\frac{1}{\lambda} e^{-\lambda z e^{r(T-t)} + F(t,m)},$$

where

$$F(t,m) = L(t)m^2 + M(t)m + N(t),$$

C(IV): *If $b^2\lambda < \xi a < b^2 \lambda e^{rT}$ then*

$$\Pi^* = \frac{(m+k-r)}{\sigma^2 \lambda} e^{-r(T-t)} + \frac{\beta\rho\,[2L_t m_t + M_t]}{\sigma\lambda} e^{-r(T-t)},$$

and

$$q^* = 0$$

and $t > T - (ln(\xi a) - ln(b^2\lambda))/r$ and value function is

$$V(t,z,m) = -\frac{1}{\lambda} e^{-\lambda z e^{r(T-t)} + F(t,m)},$$

where

$$F(t,m) = L(t)m^2 + M(t)m + N(t),$$

In the last two cases (C(III) and C(IV)), $L(t)$, $M(t)$ and $N(t)$ are expressed in (4.41), (4.42) and (4.43), respectively along with terminal conditions (4.64).

The inequality, $\xi a \leq b^2\lambda$, can be interpreted as a balance condition that must hold for the reinsurance agreement to be attractive or acceptable to the insurer. ξa represents the additional premium amount that the reinsurer requires to cover the expected loss and their profit margin. It is essentially the price the insurer pays to transfer risk to the reinsurer and $b^2\lambda$ represents the insurer's willingness to pay for risk transfer based on their risk aversion and the variability of the potential losses. It captures the value of the risk reduction from the insurer's perspective. If the inequality holds, it means the reinsurance deal is acceptable to the insurer because the cost of the reinsurance is justified by the reduction in risk. If the inequality does not hold, it suggests that the reinsurance premium is too high relative to the risk reduction benefit perceived by the insurer, and the insurer might consider it not worthwhile to purchase the reinsurance at that price.

4.1 Logarithmic Utility Function

The insurer assumes a logarithmic utility function as

$$u(z) = \ln z \qquad \text{where } z > 0.$$

Under the logarithmic utility function, we seek candidate solution of V form as follows to solve the HJB equation (4.4):

$$V(t, z, m) = \ln z + j(t, m) \tag{4.44}$$

the first and second derivatives of V with respect to z and m are given by:

$$\begin{aligned} V_t &= j_t, \\ V_z &= \frac{1}{z}, \\ V_{zz} &= -\frac{1}{z^2}, \\ V_{zm} &= V_{mz} = 0, \\ V_m &= j_m, \\ V_{mm} &= j_{mm}. \end{aligned} \tag{4.45}$$

We can substitute these derivatives into the expressions of

$$\Pi = \frac{-(m + k - r)V_z - \sigma\beta\rho V_{zm}}{\sigma^2 V_{zz}},$$

and

$$q^* = 1 + \frac{\xi a V_z}{b^2 V_{zz}},$$

to determine their optimal values as follows:

$$\Pi^* = \frac{(m+k-r)z}{\sigma^2}, \tag{4.46}$$

$$q^* = 1 - \frac{\xi a z}{b^2}. \tag{4.47}$$

Then, substitute the derivatives of V, Π^* and q^* into HJB equation (4.4) and simplifying algebraically the equation becomes:

$$j_t + r + \frac{(\theta-\xi)a}{z} + \frac{a^2\xi^2}{2b^2} + \frac{(k+m-r)^2}{2\sigma^2} + \alpha m j_m + \frac{1}{2}\beta^2 j_{mm} = 0 \tag{4.48}$$

with the boundary condition $j(T,m) = 0$.

Applying the approach of Rishel [22], we define

$$j(t,m) = P(t)m^2 + R(t)m + S(t) \tag{4.49}$$

where $P(t), R(t)$ and $S(t)$ are the solutions to the following ordinary differential equations:

$$P^{'}(t) + 2\alpha P(t) + \frac{1}{2\sigma^2} = 0, \tag{4.50}$$

$$R^{'}(t) + \alpha R(t) + \frac{k-r}{\sigma^2} = 0, \tag{4.51}$$

and

$$S'(t) + r + \frac{(\theta-\xi)a}{z} + \frac{a^2\xi^2}{2b^2} + \frac{(k-r)^2}{2\sigma^2} + \beta^2 P(t) = 0 \tag{4.52}$$

with boundary conditions

$$P(T) = 0, \quad R(T) = 0, \quad S(T) = 0. \tag{4.53}$$

To derive Equations (4.50),(4.51) and (4.52), we substitute j_t, j_m, j_{mm} into Equation (4.48)

$$\begin{aligned} r &+ \frac{(\theta-\xi)a}{z} + \frac{a^2\xi^2}{2b^2} + \frac{(k-r)^2}{2\sigma^2} + \frac{km}{\sigma^2} + \frac{m^2}{2\sigma^2} - \frac{mr}{\sigma^2} + 2m^2\alpha P(t) \\ &+ \beta^2 P(t) + m\alpha R(t) + m^2 P'(t) + mR'(t) + S'(t) = 0 \end{aligned} \tag{4.54}$$

Arranging the terms based on the powers of m leads to:

$$\begin{aligned} &\left[P^{'}(t) + 2\alpha P(t) + \frac{1}{2\sigma^2}\right] m^2 + \left[R^{'}(t) + \alpha R(t) + \frac{k-r}{\sigma^2}\right] m \\ &+ r + \frac{(\theta-\xi)a}{z} + \frac{a^2\xi^2}{2b^2} + \frac{(k-r)^2}{2\sigma^2} + \beta^2 P(t) + S'(t) = 0 \end{aligned} \tag{4.55}$$

Then, we have:

$$P^{'}(t) + 2\alpha P(t) + \frac{1}{2\sigma^2} = 0,$$

$$R^{'}(t) + \alpha R(t) + \frac{k-r}{\sigma^2} = 0,$$

and

$$S'(t) + r + \frac{(\theta-\xi)a}{z} + \frac{a^2\xi^2}{2b^2} + \frac{(k-r)^2}{2\sigma^2} + \beta^2 P(t) = 0$$

The solutions to Equations (4.50),(4.51) and (4.52) are given, respectively, as follows

$$P(t) = \frac{e^{2\alpha(T-t)}-1}{4\alpha\sigma^2}, \tag{4.56}$$

$$R(t) = \frac{e^{-\alpha t}\left(-e^{\alpha t}+e^{\alpha T}\right)(k-r)}{\alpha\sigma^2} \tag{4.57}$$

and

$$S(t) = \int_t^T \left[r + \frac{(\theta-\xi)a}{z} + \frac{a^2\xi^2}{2b^2} + \frac{(k-r)^2}{2\sigma^2} + \beta^2\frac{e^{2\alpha(T-t)}-1}{4\alpha\sigma^2}\right] ds \tag{4.58}$$

Since $S(t)$ includes z term, we can not derive closed-form solution for optimization problem. To simplify the solution to Equation (4.58), we consider the insurer and the reinsurer apply the same loading factors, i.e., $\xi = \theta > 0$. This assumption is also made by [5] under different reinsurance problem setup which is not directly related to our approach.

Then, Equation (4.48) turns into

$$j_t + r + \frac{a^2\xi^2}{2b^2} + \frac{(k+m-r)^2}{2\sigma^2} + \alpha m j_m + \frac{1}{2}\beta^2 j_{mm} = 0. \tag{4.59}$$

with the boundary condition $j(T,m) = 0$.

We define

$$j(t,m) = P(t)m^2 + R(t)m + S(t) \tag{4.60}$$

where $P(t), R(t)$ and $S(t)$ are the solutions to the following ordinary differential equations:

$$P^{'}(t) + 2\alpha P(t) + \frac{1}{2\sigma^2} = 0, \tag{4.61}$$

$$R^{'}(t) + \alpha R(t) + \frac{k-r}{\sigma^2} = 0, \tag{4.62}$$

and

$$S'(t) + r + \frac{a^2\xi^2}{2b^2} + \frac{(k-r)^2}{2\sigma^2} + \beta^2 P(t) = 0 \tag{4.63}$$

with boundary conditions

$$P(T) = 0, \quad R(T) = 0, \quad S(T) = 0. \tag{4.64}$$

The solutions to Equations (4.61), (4.62) and (4.63) are given, respectively, as follows

$$P(t) = \frac{e^{2\alpha(T-t)} - 1}{4\alpha\sigma^2}, \tag{4.65}$$

$$R(t) = \frac{e^{-\alpha t}\left(-e^{\alpha t} + e^{\alpha T}\right)(k - r)}{\alpha\sigma^2} \tag{4.66}$$

and

$$S(t) = \int_t^T \left[r + \frac{a^2\xi^2}{2b^2} + \frac{(k-r)^2}{2\sigma^2} + \beta^2 \frac{e^{2\alpha(T-t)} - 1}{4\alpha\sigma^2} \right] ds. \tag{4.67}$$

We define the following theorem with above findings but we skip boundaries of $q*$ here due to the focus of the thesis which regards the expected utility function as the primer approach.

Theorem 4.5. *Under the logaritmic utility function, there exist a classical solution $V \in C^{1,2}$ to (4.1) subject to (4.2) wtih the conditon that $\xi = \theta > 0$. Corresponding optimal strategies are derived as*

$$\Pi^* = \frac{(m + k - r)z}{\sigma^2}, \tag{4.68}$$

$$q^* = 1 - \frac{\xi a z}{b^2} \tag{4.69}$$

for any $t \in [0, T]$ and value function is

$$V(t, z, m) = lnz + j(t, m),$$

where

$$j(t, m) = P(t)m^2 + R(t)m + S(t),$$

and $P(t), R(t)$ and $S(t)$ are given in (4.65)*,* (4.66) *and* (4.67) *,respectively with terminal conditions* (4.64)*.*

In case, when loading factors are zero for both parties, i.e. $\xi = 0$ and $\theta = 0$, when the scenario of power utility function is examined under specific conditions as in Cao et al. [3], we similarly propose a detailed investigation under the pure premium principle, which aligns with these conditions. Considering pure premium principle, we define premium for insurer and reinsurer, respectively as

$$c_I = \mathbb{E}(X) = a, \quad C_R = \mathbb{E}(X_R) = q_t a.$$

Thus, dynamics of the wealth process, Z_t, is redefined as

$$\begin{aligned} dZ_t &= \frac{\Pi_t dS_t^1}{S_t^1} + \frac{(Z_t - \Pi_t)dS_t^0}{S_t^0} + c_0 dt - dC_t - c_1 dt \\ &= [rZ_t + (m_t + k - r)\Pi_t]\, dt + \Pi_t \sigma dW_t^1 + b(1-q)dW_t^0. \end{aligned} \tag{4.70}$$

After rearranging the wealth dynamics, proposed optimization problem redefined as (4.1) turns into

$$\max \mathbb{E}[u(Z_T)], \tag{4.71}$$

$$s.t. \begin{cases} dZ_t = [rZ_t + (m_t + k - r)\Pi_t]\, dt + \Pi_t \sigma dW_t^1 + b(1-q)dW_t^0 \\ dm_t = \alpha m_t dt + \beta dW_t^2, \quad m_0 = m_0 \\ (\Pi, q) \in \mathcal{A}. \end{cases} \tag{4.72}$$

Then, the associated value function, $V(t, z, m)$, is defined as

$$V(t, z, m) = \max \mathbb{E}\left[u(Z_T)|(Z_t, m_t) = (z, m)\right] \tag{4.73}$$

As given in Fleming and Soner [30], we observe that if the value function $V \in C^{1,2}$, then V satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\begin{aligned} V_t + \sup_{\Pi, q} \Big\{ & [rz + (m + k - r)\Pi]\, V_z + \left[\frac{1}{2}\Pi^2\sigma^2 + \frac{1}{2}b^2(1-q)^2\right] V_{zz} \\ & + \alpha m V_m + \frac{1}{2}\beta^2 V_{mm} + \sigma \Pi \beta \rho V_{zm} \Big\} = 0, \end{aligned} \tag{4.74}$$

with the boundary condition:

$$V(T, z, m) = u(z) \tag{4.75}$$

and $V_t, V_z, V_{zz}, V_{zm}, V_m, V_{mm}$ are partial derivatives of V.

The optimal investment strategy, Π^*, can be determined by maximizing the function in Equation (4.74). This is achieved by taking the derivative of the function with respect to Π and setting it to zero

$$(m + k - r)V_z + \Pi\sigma^2 V_{zz} + \sigma\beta\rho V_{zm} = 0 \tag{4.76}$$

Then, we obtain the optimal investment strategy as

$$\Pi^* = \frac{-(m + k - r)V_z - \sigma\beta\rho V_{zm}}{\sigma^2 V_{zz}}. \tag{4.77}$$

To seek the optimal reinsurance strategy, q^*, maximizing the function presented in Equation (4.74), we take derivatives of (4.74) with respect to q and set it to zero

$$-b^2(1-q)V_{zz} = 0. \tag{4.78}$$

This yields that $q^* = 1$.
Under logarithmic utility function, we look for candidate solution of ansatz form V as defined in Equation (4.44) with partial derivatives as defined in (4.45).
We can obtain Π^* as follows by integrating derivatives of V into Equation (4.77)

$$\Pi^* = \frac{(m+k-r)z}{\sigma^2}. \tag{4.79}$$

After substituting derivatives of V, Π^* and q^* into HJB equation (4.74) and simplifying algebraically the equation becomes:

$$J_t + r + \frac{(k+m-r)^2}{2\sigma^2} + \alpha m J_m + \frac{1}{2}\beta^2 J_{mm} = 0 \tag{4.80}$$

with the boundary condition $J(T,m) = 0$.
Using the method of Rishel [22], we define

$$J(t,m) = Q(t)m^2 + Y(t)m + Z(t) \tag{4.81}$$

where $Q(t), Y(t)$ and $Z(t)$ are the solutions to the following ordinary differential equations:

$$Q^{'}(t) + 2\alpha Q(t) + \frac{1}{2\sigma^2} = 0, \tag{4.82}$$

$$Y^{'}(t) + \alpha Y(t) + \frac{k-r}{\sigma^2} = 0, \tag{4.83}$$

and

$$Z'(t) + r + \frac{(k-r)^2}{2\sigma^2} + \beta^2 Z(t) = 0 \tag{4.84}$$

with boundary conditions

$$Q(T) = 0, \quad Y(T) = 0, \quad Z(T) = 0. \tag{4.85}$$

To derive Equations (4.82),(4.83) and (4.84), we substitute J_t, J_m, J_{mm} into Equation (4.80)

$$\begin{aligned} r + \frac{(k-r)^2}{2\sigma^2} + \frac{km}{\sigma^2} + \frac{m^2}{2\sigma^2} - \frac{mr}{\sigma^2} + 2m^2\alpha Q(t) \\ + \beta^2 Q(t) + m\alpha Y(t) + m^2 Q'(t) + mY'(t) + Z'(t) = 0 \end{aligned} \tag{4.86}$$

Grouping terms according to the powers of m leads to:

$$\left[Q^{'}(t) + 2\alpha Q(t) + \frac{1}{2\sigma^2}\right] m^2 + \left[Y^{'}(t) + \alpha Y(t) + \frac{k-r}{\sigma^2}\right] m \\ + r + \frac{(k-r)^2}{2\sigma^2} + \beta^2 Q(t) + Z'(t) = 0 \tag{4.87}$$

Then, we have:

$$Q^{'}(t) + 2\alpha Q(t) + \frac{1}{2\sigma^2} = 0,$$

$$Y^{'}(t) + \alpha Y(t) + \frac{k-r}{\sigma^2} = 0,$$

and

$$Z'(t) + r + \frac{(k-r)^2}{2\sigma^2} + \beta^2 Q(t) = 0$$

The solutions to Equations (4.82),(4.83) and (4.84) are given, respectively, as follows

$$Q(t) = \frac{e^{2\alpha(T-t)} - 1}{4\alpha\sigma^2}, \tag{4.88}$$

$$Y(t) = \frac{e^{-\alpha t}\left(-e^{\alpha t} + e^{\alpha T}\right)(k-r)}{\alpha\sigma^2} \tag{4.89}$$

and

$$Z(t) = \int_t^T \left[r + \frac{(k-r)^2}{2\sigma^2} + \beta^2 \frac{e^{2\alpha(T-t)} - 1}{4\alpha\sigma^2}\right] ds \tag{4.90}$$

Theorem 4.6. *Under the logarithmic utility function and pure premium principle ($\xi = \theta = 0$), there exist a classical solution $V \in C^{1,2}$ to (4.71) subject to (4.72). Corresponding optimal strategies are*

$$\Pi^* = \frac{(m+k-r)z}{\sigma^2} \qquad q^* = 1, \tag{4.91}$$

for any $t \in [0, T]$ and value function is

$$V(t, z, m) = lnz + J(t, m),$$

where

$$J(t, m) = Q(t)m^2 + Y(t)m + Z(t),$$

and $Q(t), Y(t)$ and $Z(t)$ are given in (4.88)*,* (4.89) *and* (4.90) *,respectively with terminal conditions* (4.85)*.*

After obtaining the analytical forms of the optimal strategies, we illustrate how various values of financial and insurance parameters affect the optimal strategies through sensitivity analysis.

CHAPTER 5

NUMERICAL ILLUSTRATIONS AND SENSITIVITY ANALYSIS

In this section, we present numerical results to demonstrate the practical applicability of the optimal strategies derived in the previous section. We first use initial parameter values sourced from existing literature for our sensitivity analysis. This approach allows us to assess the impact of model parameters on the insurer's decisions.

Table 5.1 presents the base parameter values used in our numerical examples and sensitivity analyses, with sources including the studies by Cheng et al. [4] and Li et al. [15]. The "Assumed" values in Table 5.1 chosen arbitrarily based on the expert opinions.

Table 5.1: Base Parameters for Sensitivity Analysis

Parameter	β	ρ	α	k	m_0	θ	ξ	σ	r	λ	a	b	t	T
Value	0.2	0.5	0.4	0.04	0.03	0.5	0.8	0.3	0.03	1	1.5	1	5	10
Source	Assumed					[4]		[15]						

We adhere to a systematic approach explained in detail in the following algorithm to ensure accuracy and comprehensiveness in our numerical illustrations of the proposed analytical derivations. Algorithm 1 gives the steps of the numerical evaluation of the analytical derivations, as well as the application of sensitivity analyses, whose results are summarized in Table 5.3 and Figures 5.1 - 5.5.

Algorithm 1 Optimal Portfolio Strategies and Sensitivity Analysis

Input: Define base parameters $t, T, \beta, \rho, \sigma, \alpha, k, r, \lambda, m, \xi, a, b$.

Initialize: Set base parameter values and define the functions $G(t)$, $H(t)$, $L(t)$, and $M(t)$.

for each parameter P in $\{\sigma, \alpha, \beta, k, r, \rho, \lambda, m, \xi, a, b\}$ **do**

  Define the range for P: P_{min} to P_{max} with step size ΔP.

  Perform sensitivity analysis for P:

  for $P \leftarrow P_{\text{min}}$ to P_{max} by ΔP **do**

    Update the parameter value to P.

    Recalculate the optimal portfolio strategy Π^* and the control variable q^* using the updated P.

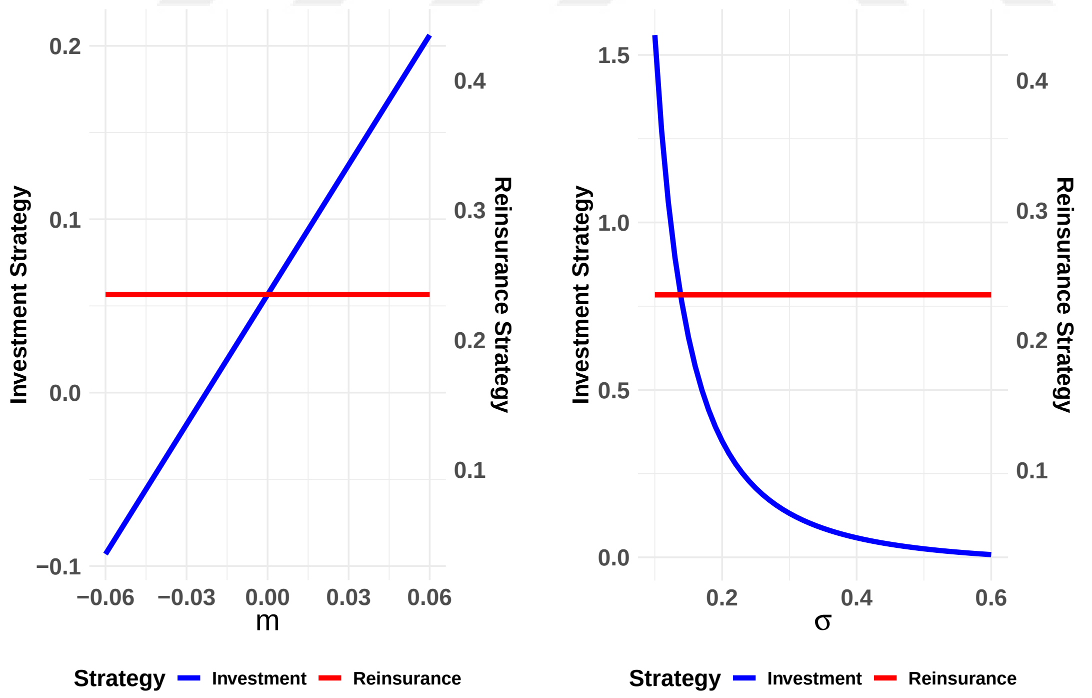
    Save the calculated Π^* and q^* values.

  end for

  Generate plots showing how Π^* and q^* vary with the parameter P.

end for

Output: Sensitivity analysis results and the corresponding plots for each parameter.

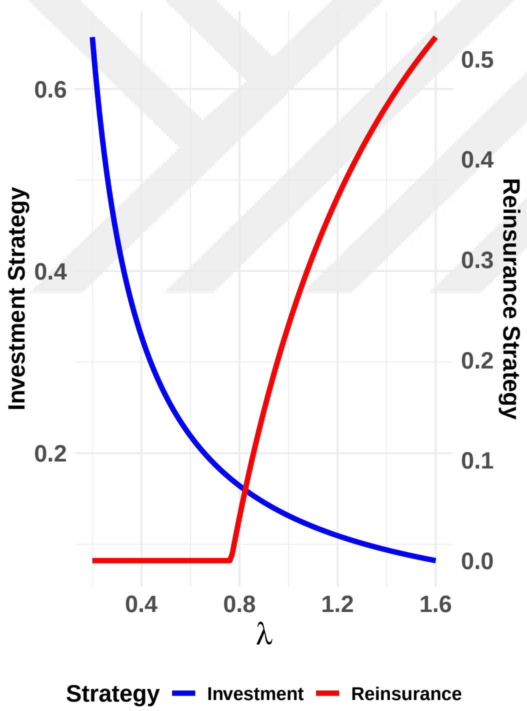


(a) Impact of m on the Optimal Strategy

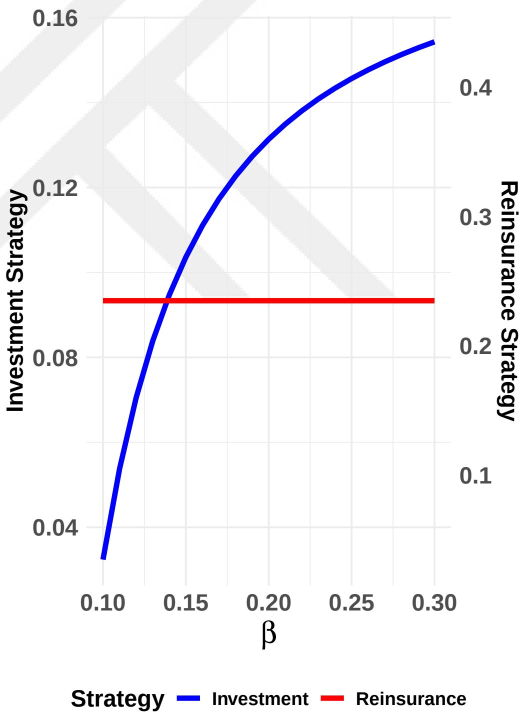
(b) Impact of σ on the Optimal Strategy

Figure 5.1: Comparisons on the Optimal Strategy with Variations in m and σ

We analyze the influence of the market's impact on the risky asset's price, m, and the volatility coefficient of the risky asset, σ on the optimal strategy. In Figure 5.1, it can be observed that an increase in the optimal investment strategy as, m,the market's impact on the risky asset's price rises. Since the instantaneous return rate of the risky asset is higher in a bull market ($m > 0$), the insurer finds the risky asset more attractive, deciding to invest more. Conversely, the optimal investment strategy suggests short-selling ($\Pi < 0$) in a bear market ($m < 0$), where prices of risky assets are expected to decline. By short-selling, the insurer can profit from the falling prices, buying back the assets at a lower price than they were sold for. On the other hand, as σ increases, indicating higher volatility and significant price fluctuations in the price of the risky asset, implying greater risk, the optimal investment strategy decreases and becomes more conservative to control this increased risk. To control this risk, it is necessary to allocate less surplus to the risky asset. Despite variations in both σ and m, the optimal reinsurance strategy remains unchanged, suggesting that these parameters do not affect the reinsurance strategy. This stability is attributed to our model's assumption of no correlation between the risk model and the financial market.



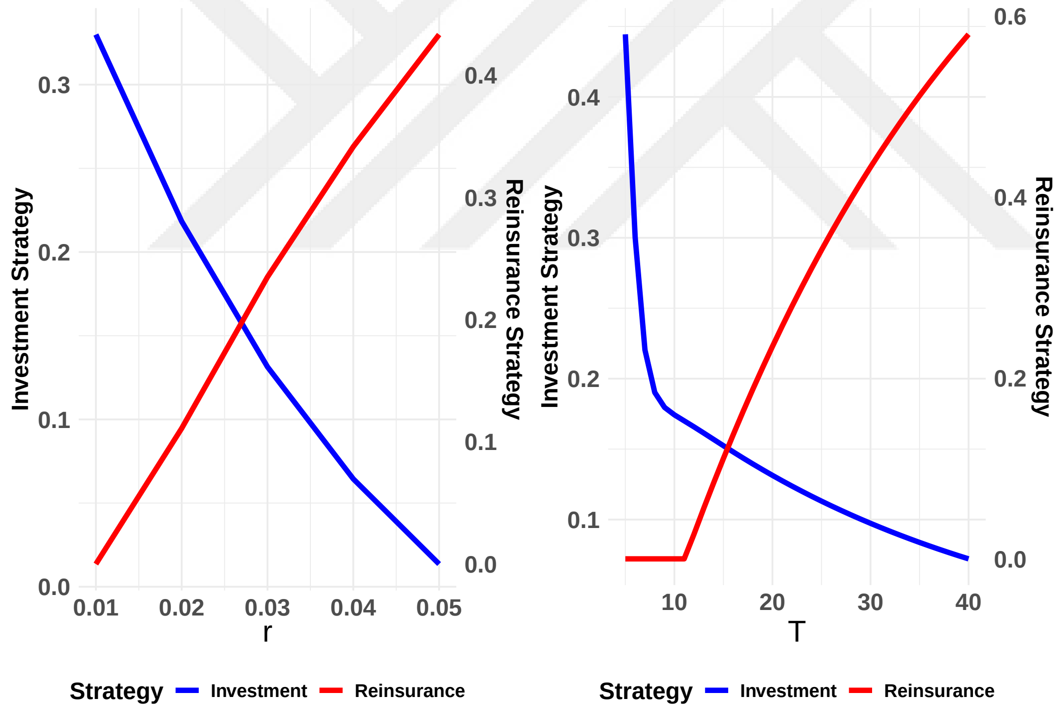
(a) Impact of λ on the Optimal Strategy



(b) Impact of β on the Optimal Strategy

Figure 5.2: Comparisons on the Optimal Strategy with Variations in λ and β

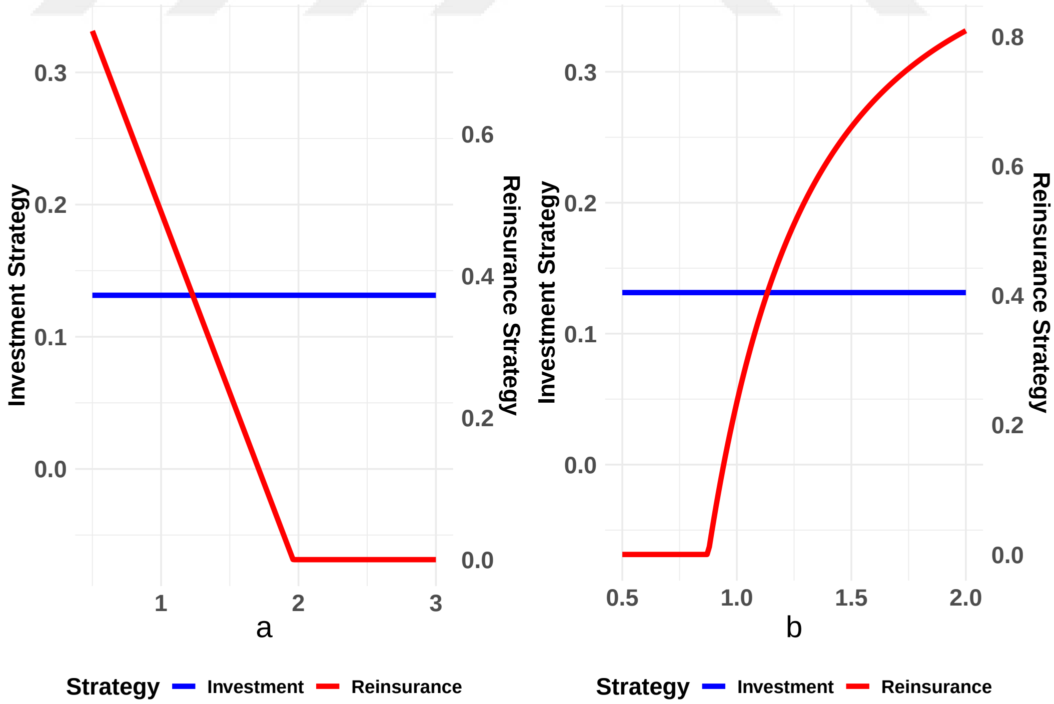
We notice from Figure 5.2 that the optimal investment strategy decreases as the relative risk aversion parameter, λ, increases. Consistent with the definition of risk aversion, insurers exhibiting higher relative risk aversion tend to invest more conservatively compared to those with lower levels. Meanwhile, we observe that the optimal reinsurance strategy remains zero until reaching a certain threshold, and then increases as the risk aversion parameter, λ, rises beyond that threshold. The larger λ indicates that the insurer is more risk-averse, prompting insurers to purchase reinsurance to cede the part of their potential losses to reinsurer. Regarding β, a higher value leads to higher values of the optimal investment strategy. The increase in β might also indicate a more dynamic and potentially rewarding market environment. If the correlation ρ between W_t^1 and W_t^2 is positive, this could mean that as the market effect m_t increases, so does the return on the risky asset. In such scenarios, the higher volatility introduced by a larger β could lead to more aggressive investment in the risky asset as investors seek to capitalize on the potential for higher returns. The analysis reveals that the optimal reinsurance strategy remains constant regardless of changes in β, suggesting that β variations do not affect the optimal reinsurance strategy.



(a) Impact of r on the Optimal Strategy (b) Impact of T on the Optimal Strategy

Figure 5.3: Comparisons on the Optimal Strategy with Variations in r and T

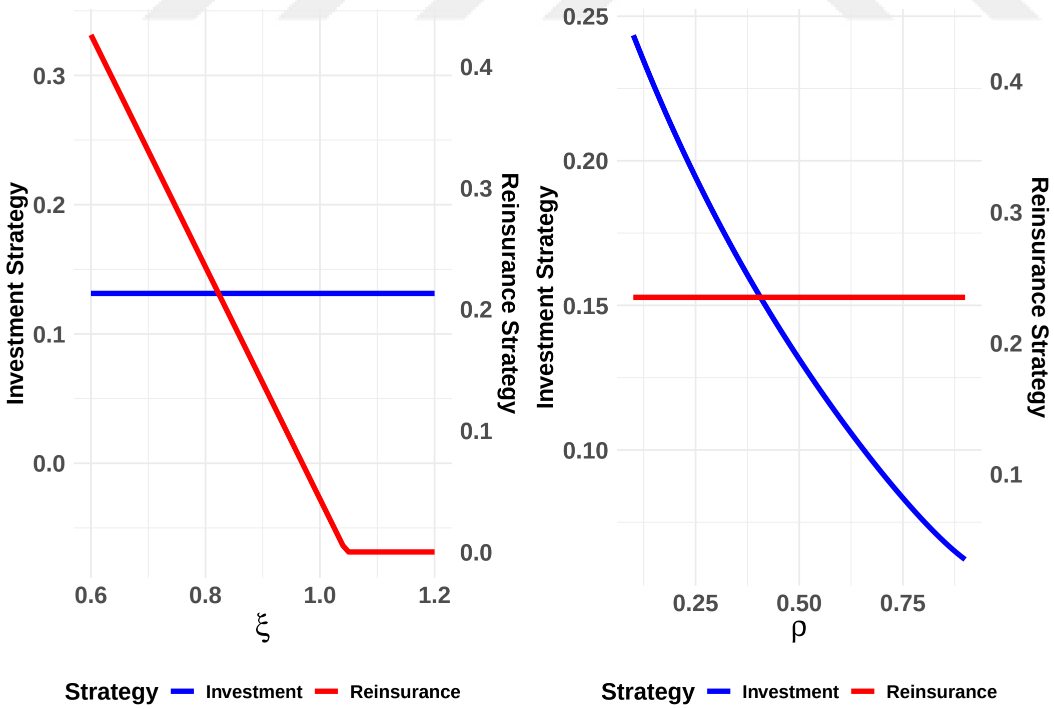
As next, the risk-free interest rate, r, is analyzed in Figure 5.3. When r increases, the optimal investment strategy decreases indicating that the insurer prefers investing its surplus in the risk-free asset, finding it a more convenient option due to the stable returns it offers despite fluctuating market conditions. Consequently, the insurer's preference shifts towards lower-risk investments, ensuring a stable return despite the changing market conditions. Moreover, as r increases, the insurer also raises the optimal reinsurance strategy. Higher returns from risk-free assets make the relative cost of reinsurance more acceptable. This situation allowing the additional income generated from higher interest rates can offset the cost of reinsurance premiums, making it more feasible to purchase greater coverage. Furthermore, as the time horizon T extends, the optimal investment strategy diminishes. With a longer time horizon, the present value of future returns decreases due to discounting. This makes the expected benefits of long-term investments less significant in today's terms, which results in a more cautious investment strategy. The optimal reinsurance strategy increases with a longer time horizon. This increase reflects the insurer's strategy to enhance protection over extended periods. Over longer periods, reinsurance costs can be spread more evenly, making it a cost-effective method to manage risk. Consequently, insurers might find it more affordable to secure higher levels of reinsurance to protect against long-term risks.



(a) Impact of a on the Optimal Strategy (b) Impact of b on the Optimal Strategy

Figure 5.4: Comparisons on the Optimal Strategy with Variations in a and b

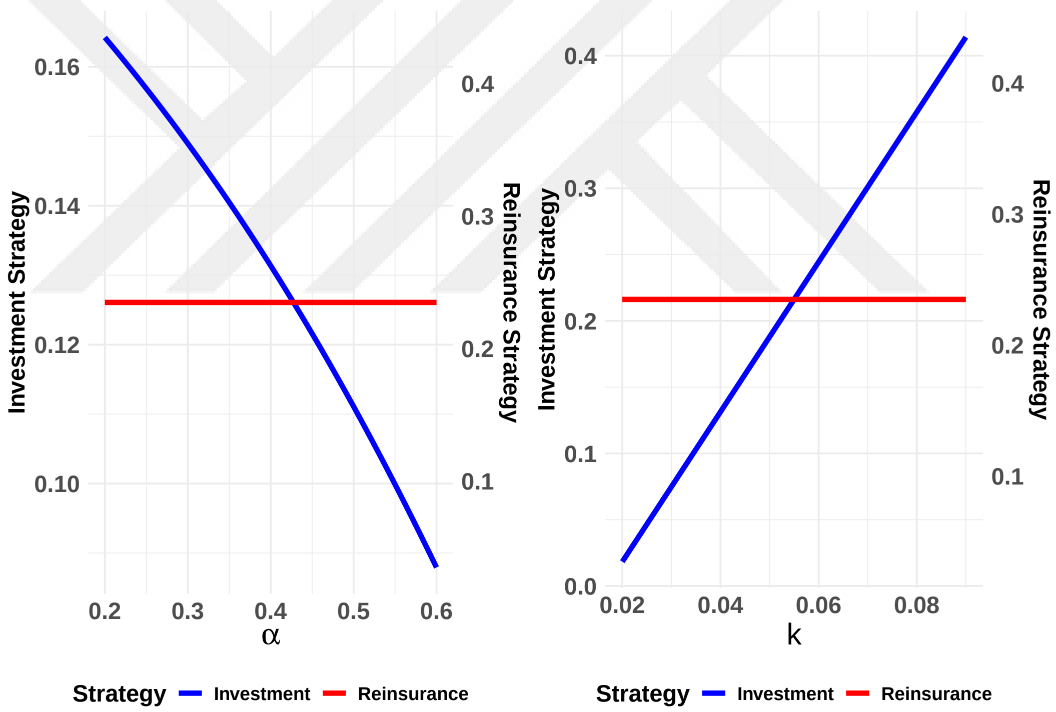
The influence of the average rate of claim loss, a, shown in Figure 5.4, is observed that an increase in a leads to a decrease in the optimal reinsurance strategy, and beyond a certain threshold, no reinsurance becomes optimal. As a rises, the cost of reinsurance premiums significantly increases. After a certain point, the cost of transferring risk to a reinsurer might become too high. Insurers may find it cheaper to keep the risk themselves instead of paying the high premiums for reinsurance. Also, when the average claim loss rate is rising, the reinsurance market may face capacity constraints. Limited availability of reinsurance or unfavorable market conditions might force insurers to retain more risk. Moreover, the optimal reinsurance strategy remains zero until a certain point and then exhibits a concave relation with the standard deviation of the claim loss rate, b, as shown in Figure 5.4. When b is low, implying minimal perceived risk, insurers may not to purchase reinsurance, as the expected claim variability falls within their risk tolerance and available capital. However, as the claims become riskier with higher b, insurers increasingly seek reinsurance to transfer a higher portion of the risk exposure to the reinsurer. Despite these fluctuations in a and b, the optimal investment strategy remains unchanged, suggesting that variations in these parameters do not impact the optimal investment decisions.



(a) Impact of ξ on Optimal Strategy (b) Impact of ρ on the Optimal Strategy

Figure 5.5: Comparisons on the Optimal Strategy with Variations in ξ and ρ

Having an important role in reinsurance premium, the safety loading, ξ, and the correlation, ρ, between the price of risky asset and the market are analyzed and shown in Figure 5.5. It can be noted that an increase in the reinsurance safety loading, ξ, results in a decrease in the optimal reinsurance strategy, which is consistent with the well-known law of demand, which states that higher prices lead to a lower quantity demand. As safety loading increases, the cost of purchasing reinsurance becomes higher, making it less attractive for insurers. Consequently, insurers choose to retain more risk themselves rather than incur the higher premiums associated with reinsurance coverage. In contrast, variations in ξ do not affect the optimal investment strategy. Regarding the characteristics of the dynamic market, as the correlation between the risky asset's price and the market's effect on that price (ρ) increases, the optimal investment strategy decreases. A high ρ value indicates that market shocks significantly impact the returns of the risky asset. This situation increases the volatility and creates more uncertainty about the asset's future price. Consequently, investors may reduce their investment in the asset to better manage their risk. Conversely, changes in ρ have no impact on the optimal reinsurance strategy.



(a) Impact of α on Optimal Strategy (b) Impact of k on the Optimal Strategy

Figure 5.6: Comparisons on the Optimal Strategy with Variations in α and k

We analyze the long-run average growth rate of a risky asset, k, and α in Figure 5.6. k indicates the expected return of the asset over a long period, providing investors with a benchmark for future performance. A higher k suggests that the asset is expected to yield higher returns over time, making it more attractive to investors. Conversely, a lower k indicates lower expected returns, leading to a more cautious investment approach. Essentially, k helps investors understand the long-term profitability and risk linked to the asset, thus influencing their investment choices accordingly. Conversely, higher α decreases optimal investment strategy less significantly. An increase in α means that the market effect m_t returns to its long-term average more quickly. This quicker return reduces the impact of fluctuations in m_t, resulting in a more stable but possibly less profitable situation for the risky asset. Changes in α or k have no impact on the optimal reinsurance strategy.

The impact of parameters on reinsurance and investment strategies explained in detail above is summarized in Table 5.2. A dash $(-)$ in Table 5.2 indicates that changes in the parameter do not affect the strategy.

Table 5.2: Influence of parameters on strategies

Strategy	**Parameter**	β	ρ	α	k	m	ξ	σ	r	λ	a	b	T
Reinsurance	**Increase**	$-$	$-$	$-$	$-$	$-$	$\downarrow$	$-$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$
Investment	**Increase**	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$-$	$\downarrow$	$\downarrow$	$\downarrow$	$-$	$-$	$\downarrow$

In order to illustrate the influence of the variations on these parameter on the optimum investment and optimum reinsurance strategy, we assign randomly increment on the parameter values to observe their impact. Table 5.3 summarizes the optimal values, Π^* and q^* with respect to the change in each parameter whose values are selected according to Table 5.1.

It can be seen from Table 5.3, as σ increases from 0.20 to 0.40, the optimal investment level at higher volatility in the risky asset prompts a shift toward a more conservative investment strategy. Similarly, an increase in β from 0.10 to 0.30 increases the optimal investment strategy from 0.0324 to 0.1543. This indicates that as β rises, investments in the risky asset are increased. Conversely, as m varies from -0.04 to 0.04, the optimal investment level adjusts from -0.0433 to 0.1563, showing a direct correlation between m and the investment levels in the risky asset. Despite these variations, the reinsurance strategy, q, consistently remains at 0.2348 across the different values of m, σ and β. As r increases from 0.01 to 0.05, the investment strategy decreases from 0.3299 to 0.0136, indicating that a higher risk-free rate encourages a

Table 5.3: Optimal strategies Π^* and q^* across different values of parameters

	σ			β			m		
	0.20	**0.30**	**0.40**	**0.10**	**0.20**	**0.30**	**-0.04**	**0.01**	**0.04**
Π^*	0.3472	0.1314	0.0583	0.0324	0.1314	0.1543	-0.0433	0.0815	0.1563
q^*	0.2348	0.2348	0.2348	0.2348	0.2348	0.2348	0.2348	0.2348	0.2348
	r			λ			T		
	0.01	**0.03**	**0.05**	**0.40**	**0.70**	**1.20**	**10.00**	**20.00**	**30.00**
Π^*	0.3299	0.1314	0.0136	0.3284	0.1877	0.1095	0.1743	0.1314	0.0973
q^*	0.0000	0.2348	0.4332	0.0000	0.0000	0.3624	0.0000	0.2348	0.4332
	ξ			a			b		
	0.60	**0.80**	**1.20**	**0.50**	**1.50**	**3.00**	**0.50**	**1.00**	**2.00**
Π^*	0.1314	0.1314	0.1314	0.1314	0.1314	0.1314	0.1314	0.1314	0.1314
q^*	0.4261	0.2348	0.0000	0.7449	0.2348	0.0000	0.0000	0.2348	0.8087
	ρ			α			k		
	0.25	**0.50**	**0.75**	**0.30**	**0.40**	**0.50**	**0.02**	**0.04**	**0.06**
Π^*	0.1943	0.1314	0.0834	0.1490	0.1314	0.1111	0.0183	0.1314	0.2444
q^*	0.2348	0.2348	0.2348	0.2348	0.2348	0.2348	0.2348	0.2348	0.2348

shift towards the risk-free asset as it becomes more attractive. Meanwhile, the reinsurance strategy shows a significant increase at $r = 0.05$ (0.4332), suggesting that higher risk-free rates might lead to an increased reliance on reinsurance.

Higher λ values (0.40 to 1.20) lead to a decrease in investment level from 0.3284 to 0.1095, reflecting a more conservative investment approach as risk aversion increases. The corresponding reinsurance strategy increases from 0 to 0.3624, demonstrating a higher dependence on reinsurance as the insurer's risk aversion grows. Similarly, as the time horizon extends from 10 to 30 years, the investment strategy decreases from 0.1743 to 0.0973, showing that a longer investment horizon leads to a more conservative investment strategy. This pattern shows the declining attractiveness of long-term and uncertain returns. Meanwhile, the reinsurance strategy increases from 0 to 0.4332, suggesting that a longer horizon also supports a higher optimal reinsurance level.

The reinsurance strategy, q^*, decreases from 0.4261 to 0, indicating that higher reinsurance safety loading reduces the optimal reinsurance level. It also decreases significantly from 0.7449 to 0 as a increases from 0.50 to 3, suggesting that larger claim sizes lead to a lower optimal reinsurance level. Conversely, the reinsurance strategy increases from 0 to 0.8087 as b increases from 0.50 to 2, indicating that a higher standard deviation of claims leads to a higher optimal reinsurance level. The investment strategy, Π, remains constant at 0.1314 across the different values of ξ, a and b.

For the correlation coefficient ρ, as it increases from 0.25 to 0.75, Π decreases from 0.1943 to 0.0834. This trend suggests that a higher correlation between the asset and liability risks leads to a more conservative investment strategy. However, the reinsurance strategy q remains constant at 0.2348 across different values of ρ, indicating that the correlation coefficient does not significantly impact the optimal reinsurance level. We notice a minor reduction in Π from 0.1490 to 0.1111 as α increases from 0.30 to 0.50, however, remains unchanged at 0.2348 across the different values of α, suggesting that α does not influence the reinsurance strategy. for the long-run mean growth rate of the asset, k, as it increases from 0.02 to 0.06, Π shows a significant increase from 0.0183 to 0.2444. This increase indicates that higher mean-reversion speeds prompt a more aggressive investment strategy in the risky asset. In contrast, the reinsurance strategy q remains constant at 0.2348, showing that the long-run mean growth rate of the asset does not affect the optimal reinsurance level.

CHAPTER 6

NUMERICAL SIMULATION USING REAL-LIFE DATA

We aim to work with real data to ensure that our simulations closely reflect actual market conditions and provide meaningful insights. Due to confidentiality concerns, sharing detailed data from insurance companies can be challenging. As a result, we conduct an analyses on the automobile insurance (Casco) and perform simulations using data from a non-life insurance company that operates in the Turkish market, which requires us to keep anonymous. The dataset utilized for this part consists of 100,511 insurance policies and 21,523 claims.

By analyzing this extensive dataset, we aim to accurately model the distribution of claim frequency and severity. The estimation process involved applying statistical techniques to assess the distribution of claim frequency and severity. Given the complexity and variability inherent in insurance claims, it was essential to choose a distribution model that appropriately captures the observed data patterns. After selecting the appropriate distribution, specifically, Poisson distribution for claim frequency and Gamma distribution for claim severity, we proceeded to estimate its parameters. These estimates are critical for understanding the risk profile and potential liabilities that the company may face in relation to Casco insurance. The descriptive statistics of claim frequency is given in Table 6.1. As shown in the table, the claim frequency have a relatively low mean and a right-skewed distribution, indicating that while most claims are of low frequency, there are occasional periods with a higher number of claims.

Table 6.1: Descriptive Statistics for Claim Frequency

Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum
0	0	0	0.2141	0	7

The histogram-density plot shows the distribution of claim frequency with the theoretical density curve, illustrating how well the chosen distribution fits the data. The P-P plot compares the cumulative probabilities of the observed and theoretical distributions, assessing their similarity. The Q-Q plot evaluates the alignment of quantiles between the data and the selected distribution to identify any deviations. Finally, the theoretical and empirical CDFs plot compares the cumulative distribution functions of the data and the fitted model, validating the suitability of the Poisson distribution for modeling claim frequency. The visual consistency in Figure 6.2 confirms that the Poisson model captures the characteristics of the observed data well.

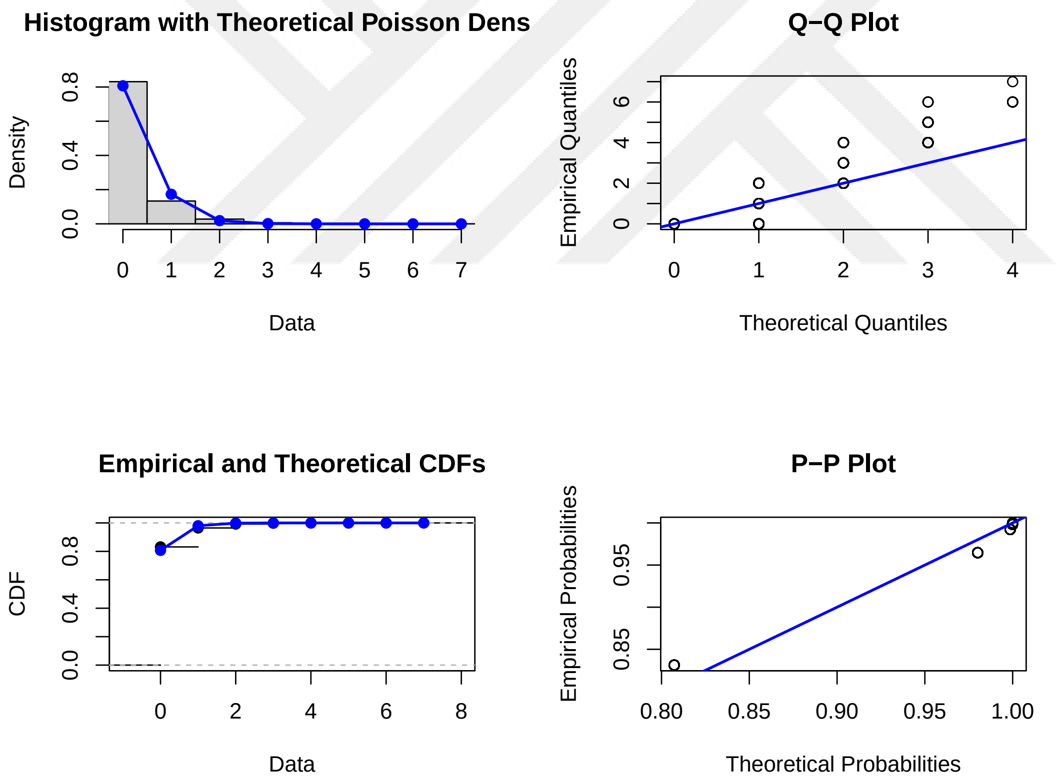


Figure 6.1: Histogram-density,P-P plot, Q-Q plot, and theoretical and empirical CDFs of Claim Frequency

Table 6.2 provides the descriptive statistics for claim severity. The data shows a minimum claim severity of 100 and a maximum of 6992, indicating a wide range of claim amounts. The mean claim severity is 1517.8, which is higher than the median of 992.8, suggesting a right-skewed distribution with a few high-value claims significantly affecting the average.

Table 6.2: Descriptive Statistics for Claim Severity

Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum
100	309	992.8	1517.8	2147	6992

The Gamma distribution effectively captures the characteristics of the claim severity data. The histogram-density plot shows a good alignment between the theoretical Gamma density and the observed frequency distribution, indicating suitability for modeling claim severity. The Q-Q plot confirms that the quantiles of the observed data closely match those of the Gamma distribution, capturing the skewness and tail behavior well. The P-P plot shows that the cumulative probabilities of the observed data and the Gamma distribution align closely, indicating strong agreement. Similarly, the theoretical and empirical CDFs closely follow each other, further validating the Gamma distribution's fit, especially in capturing the data's variability and right-skewed nature.

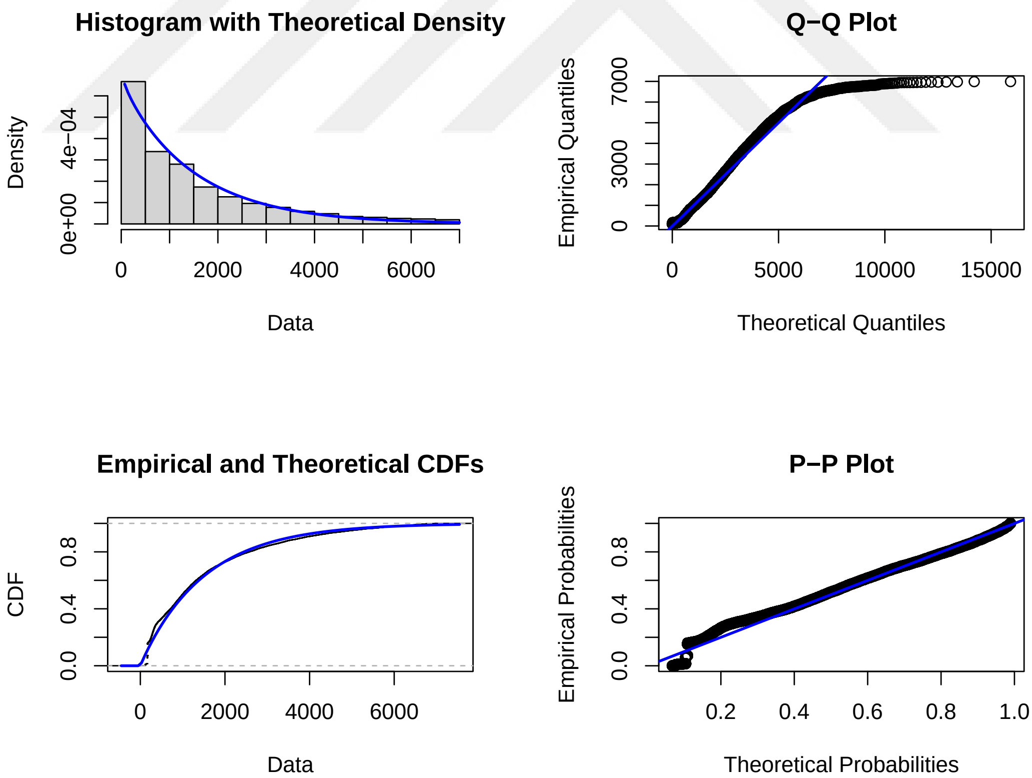


Figure 6.2: Histogram-density,P-P plot, Q-Q plot, and theoretical and empirical CDFs of Claim Severity

As next, we evaluate the investment process using modelling and behaviour of a risky asset. We choose Halkbank stock and implement its influence on investment strategy proposed in this thesis. Based on the datasets (claims and stock's price), we need to estimate the parameters of our proposed model whose values are presented in Table 6.3.

Table 6.3: Estimated Parameters from Real Life Data

Parameter	β	ρ	α	k	m_0	θ	ξ
Value	0.0106	0.7	0.0183	0.0303	0.0292	0.15	0.25
Parameter	σ	r	λ	a	b	t	T
Value	1.01	0.000342	0.01	1.41	65.98	0	10

This procedure is outlined in detail in Algorithm 2 which provides a systematic approach to modeling the stochastic processes that underlie the asset's price movements. By following this algorithm, the company can simulate future price scenarios and better manage the risks associated with its investment strategies.

Algorithm 2 Estimate Stock Return Parameters and Simulating Price Dynamics

1. Download the closing prices of Halkbank stock from Yahoo Finance for the years 2015-2016 [34].

2. Calculate the logarithmic return for each closing price:

$$\text{Log\_Return}_t = \log\left(\frac{P_{t+1}}{P_t}\right)$$

3. Decompose the seasonal trend via LOESS (STL) which is a robust method of time series decomposition often used in economic.

4. Apply the Euler-Maruyama method to discretize the logarithmic returns

5. Estimate the parameters of stochastic processes using maximum likelihood estimation.

6. Use the estimated parameters to back-transform the original stock prices.

7. Simulate the price series using these parameters.

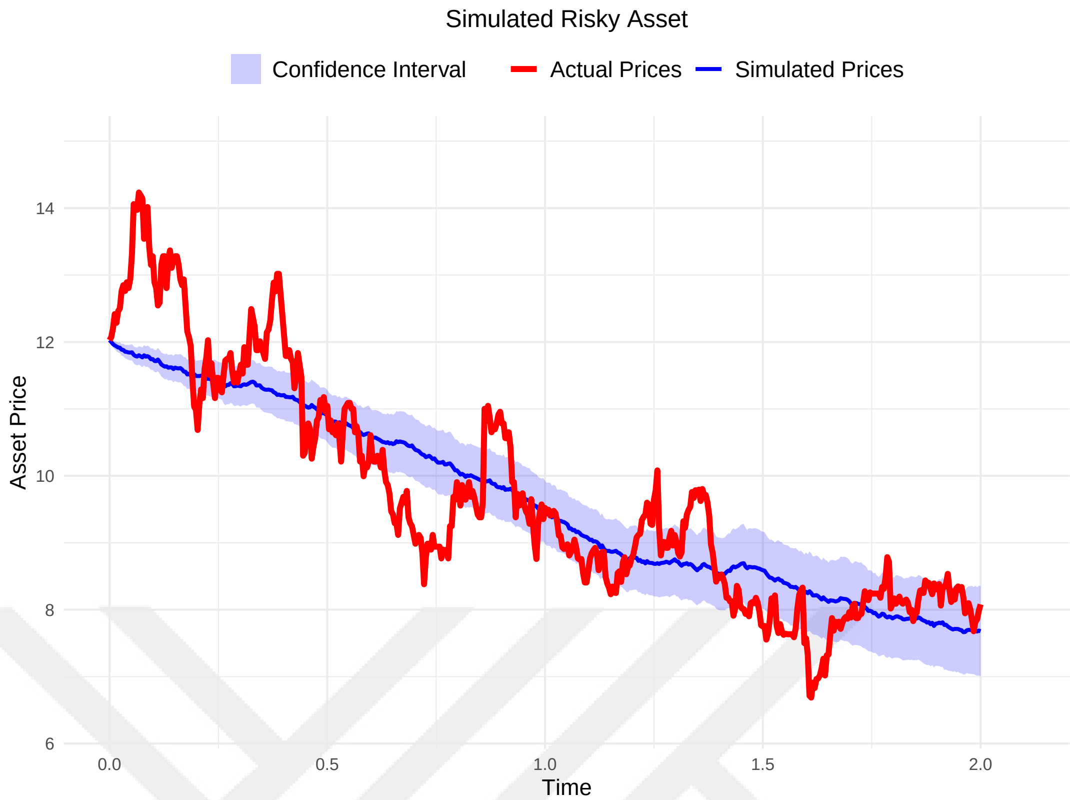


Figure 6.3: Simulated Halkbank Stock Price

In order to implement the influence of investment strategies, we choose robust asset from Turkish Stock market whose volatility is expected to be less compared to the other risky assets in BIST. Halkbank which is government trusted bank issues asset whose daily prices between 2015 and 2016 can be observed from Figure 6.3. Based on the data collected given time period (the most stable economy in Turkey) we assume the stock price dynamics follow as assumed stochastic process defined in Chapter 4:

$$dS_t^1 = k_t S_t^1 dt + \sigma S_t^1 dW_t^1, \quad \text{where } k_t = k + m_t,$$

where m_t is a solution to the equation

$$dm_t = \alpha m_t dt + \beta dW_t^2, \quad m_0 = m_0.$$

The parameters of this model are estimated and illustrated in Table 6.3.

After this step, we conduct 1000 simulations of the stochastic process to validate the accuracy of the estimated parameters. In each simulation, the evolution of prices are generated using the estimated parameters, and the results are compared with the actual stock prices within a 95% confidence interval. The Figure 6.3 illustrates the comparison between the simulated prices, generated using the estimated parameters, and the actual closing prices of Halkbank stocks. The close agreement between the simulated and real prices demonstrates that the model successfully captures the underlying dynamics of the stock. This validation provides confidence in the accuracy of the estimated parameters.

Given the reliability of these parameter estimates, we incorporated them into a wealth dynamics simulation. This simulation models the evolution of an investor's wealth over time, considering optimal investment and reinsurance strategies. The estimated parameters serve as key inputs in modeling the stochastic evolution of the investor's portfolio value.

In order to utilize the information depicted as a result of this thesis, we question how our approach is effective in reducing the ruin risk of each parties. For this purpose, we introduce the concept of ruin under the Classical Risk Model.

The time of ruin, denoted by

$$\tau = \inf\{t \geq 0 \mid U_t \leq 0\}$$

represents the first instance when the insurer's surplus drops to zero or below. The probability of ruin, denoted as $\psi(u)$, is defined as

$$\psi(u) = \mathbb{P}(\tau < \infty \mid u_0 = u).$$

To evaluate the risk of insolvency under these strategies, we define the time of ruin for the insurer's wealth as

$$\tau = \inf\{t \geq 0 \mid Z_t \leq 0\}$$

indicates the first time the insurer's wealth falls to zero or less. The probability of ruin, represented as $\psi(z_0)$, is defined as

$$\psi(z_0) = \inf_{\Pi, q \in \mathcal{A}} \mathbb{P}(\tau < \infty \mid Z_0 = z_0).$$

We examine the effects of reinsurance on the wealth dynamics of an insurance portfolio by conducting a large number of simulations. We conduct 10,000 simulations to evaluate the performance of the portfolio under two scenarios: (i) with reinsurance and (ii) without reinsurance.

The results of these simulations are quite significantly informative. In the scenario without reinsurance, while the average wealth tends to end higher (Figure 6.4), the portfolio faced insolvency in 10 out of the 10,000 simulations. This indicates a probability of ruin of 0.1%, suggesting that although there is potential for greater financial gains without reinsurance, there is also a notable risk of losing all capital, threatening the insurer's financial stability.

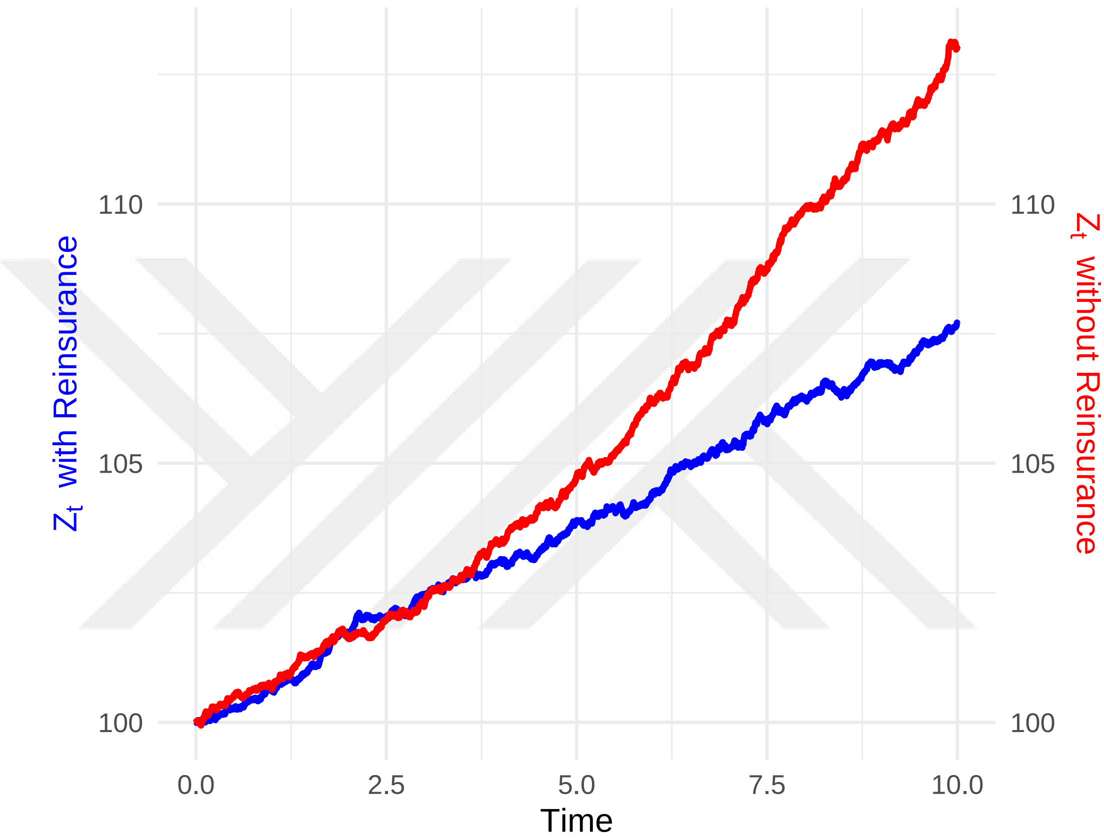


Figure 6.4: Simulation of Wealth Dynamics

In contrast, in the scenario with reinsurance, none of the 10,000 simulations resulted in insolvency, indicating a probability of ruin of 0%. This clear difference shows the protective role of reinsurance, which greatly reduces the risk of insolvency. By transferring some of the risk to a reinsurer, the insurance company can better handle large losses without exhausting its reserves. This finding is important because it demonstrates that reinsurance is a valuable risk management tool, helping to maintain the insurer's financial stability under challenging circumstances. Although the average wealth with reinsurance does not end as high as without reinsurance as shown in Figure 6.4, the portfolio is much more stable and secure, avoiding any scenarios of ruin.

As the impact of reinsurance is expected to reduce the probability of ruin, we observe that the implementation of our approach improves the robustness in the solvency of the company.

CHAPTER 7

CONCLUSION

This thesis investigates how an optimization set up improves the risk control for an insurance company, whose surplus process is modeled by a diffusion process. In order to maximize the expected utility of its wealth, the insurance company can purchase proportional reinsurance and invest its surplus in a financial market comprising both a risky asset and a risk-free asset. Our model is distinctive since the return rate of the risky asset follows an Ornstein-Uhlenbeck (OU) process, adding a layer of realism by allowing the return rate to vary over time, thus reflecting actual market conditions more accurately.

The primary motivation for this research is to fill in the gaps found in the current literature on optimal investment and reinsurance strategies. Although previous studies provide foundational insights, they often rely on simplifying assumptions, such as constant return rates. By employing the OU process to model the return rate of the risky asset, our study reflects the dynamic nature of financial markets, which can exhibit characteristics of both bull and bear markets. This approach aligns more closely with real-world scenarios and provides a more robust framework for decision-making in the insurance industry.

Under the criterion of maximizing the expected exponential utility, we develop the related HJB equation and derive closed-form expressions for optimal investment and reinsurance decisions. These expressions allow for straightforward computation and practical implementation, making them valuable tools for insurance companies seeking to optimize their financial strategies under varying market conditions. The inclusion of a logarithmic utility function expands the scope of utility analysis, offering insights into different risk preferences that were not previously explored under the Ornstein-Uhlenbeck process.

Numerical simulations are performed to support the theoretical findings by demonstrating how various financial and insurance parameters affect the optimal strategies. The sensitivity analysis provides several important insights:

(i) The optimal investment strategy increases with the market's influence on the risky asset's price, denoted by m, indicating a preference for higher investment in bullish markets. Conversely, high volatility, represented by σ, leads to a more conservative investment approach.

(ii) As the risk aversion parameter, λ, increases, the insurer's investment in risky assets decreases, reflecting a shift towards safer investments. The optimal reinsurance strategy remains at zero until risk aversion reaches a certain level, after which it increases significantly.

(iii) Higher risk-free interest rates, denoted by r, also lead to a more conservative investment strategy and an increased reliance on reinsurance.

The findings, from comparing scenarios with and without reinsurance based on the parameters obtained from real-life data, clearly show that reinsurance plays a crucial role in mitigating the risk of insolvency, allowing the insurer to better manage large losses and maintain a stable financial position. These results advocate for the strategic use of reinsurance as a risk management tool to enhance the robustness and long-term sustainability of insurance portfolios.

The results of our study are significant as they provide a novel approach to optimizing investment and reinsurance strategies by incorporating realistic market dynamics and providing actionable insights for insurers. Unlike previous models that rely on constant parameters in proportional reinsurance agreements, our approach reflects the fluctuating nature of financial markets, offering a more robust and flexible framework for decision-making. These improvements contribute not only to academic research but also provide practical benefits for the insurance industry, where effective risk management and strategic planning are crucial for ensuring financial stability and fostering growth.

In future research, it would be interesting to extend our model by incorporating a risk measure constraint such as Value-at-Risk (VaR), a standard risk measure widely used in the financial industry, or Expected Shortfall (ES), commonly employed in portfolio optimization for enhanced risk management. Integrating these measures would add further layers of risk control, providing a more comprehensive framework for insurers to manage their financial strategies effectively.

REFERENCES

[1] S. Asmussen, B. J. Christensen, and M. Taksar, Portfolio size as function of the premium: Modelling and optimization, Stochastics, 85(4), pp. 575–588, 2013.

[2] S. Browne, Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin, Mathematics of Operation Research, 20(4), pp. 937–958, 1995.

[3] Y. Cao and N. Wan, Optimal proportional reinsurance and investment based on hamilton–jacobi–bellman equation, Insurance: Mathematics and Economics, 45(2), pp. 157–162, 2009.

[4] G. Cheng and Y. Zhao, Optimal risk and dividend strategies with transaction costs and terminal value, Economic Modelling, 54, pp. 522–536, 2016.

[5] D. C. Dickson, *Insurance risk and ruin*, Cambridge University Press, Cambridge, 2016.

[6] L. T. Evans, S. P. Keef, and J. Okunev, Modelling real interest rates, Journal of Banking and Finance, 18(1), pp. 153–165, 1994.

[7] J. Grandell, A class of approximations of ruin probabilities, Scandinavian Actuarial Journal, supp.1, pp. 38–52, 1977.

[8] S. Haberman and P. Emms, Pricing general insurance using optimal control theory, ASTIN Bulletin, 35(2), pp. 427–453, 2005.

[9] S. Haberman and P. Emms, Dynamic pricing of general insurance in a competitive market, ASTIN Bulletin, 37(1), pp. 1–34, 2007.

[10] S. Haberman, I. Savoulli, and P. Emms, Optimal strategies for pricing general insurance, Insurance: Mathematics and Economics, 40(1), pp. 15–34, 2007.

[11] C. Hipp and M. Plum, Optimal investment for insurers, Insurance: Mathematics and Economics, 27(2), pp. 215–228, 2000.

[12] S. M. Iacus, *Simulation and inference for stochastic differential equations: with R examples*, Springer, New York, 2008.

[13] X. Jiang, K. C. Yuen, and M. Chen, Optimal investment and reinsurance with premium control, Journal of Industrial and Management Optimization, 16(6), pp. 2781–2797, 2020.

[14] I. Karatzas and S. Shreve, *Brownian motion and stochastic calculus*, Springer, Berlin, 1991.

[15] D. Li, X. Rong, and H. Zhao, Optimal investment problem for an insurer and a reinsurer under the proportional reinsurance model, WSEAS Transactions on Mathematics, 14(3), pp. 20–35, 2015.

[16] Z. Liang, K. Yuen, and J. Guo, Optimal proportional reinsurance and investment in a stock market with ornstein–uhlenbeck process, Insurance: Mathematics and Economics, 49(2), pp. 207–215, 2011.

[17] H. Mao, J. Carson, K. Ostaszewski, and Z. Wen, Optimal decision on dynamic insurance price and investment portfolio of an insurer, Insurance: Mathematics and Economics, 52(2), pp. 359–369, 2013.

[18] O. Mert and A. Selcuk-Kestel, Time dependent stop-loss reinsurance and exposure curves, Journal of Computational and Applied Mathematics, (389), p. 113348, 2021.

[19] A. Pantelous and E. Passalidou, Optimal premium pricing policy in a competitive insurance market environment, Annals of Actuarial Science, 7(2), pp. 175–191, 2013.

[20] H. Pham, *Continous-time Stochastic Control and Optimization with Financial Applications*, Springer-Verlag, Berlin, 2009.

[21] S. Promislow and V. Yound, Minimizing the probability of ruin when claims follow brownian motion with drift, North American Actuarial Journal, 9(3), pp. 110–128, 2005.

[22] R. Rishel, *Optimal Portfolio Management with Partial Observation and Power Utility Function*, In: Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of WH Fleming, Birkhäuser, Boston, MA, 1999.

[23] H. Schmidli, Optimal proportional reinsurance policies in a dynamic setting, Scandinavian Actuarial Journal, 2001(1), pp. 55–68, 2001.

[24] H. Schmidli, On minimizing the ruin probability by investment and reinsurance, The Annals of Applied Probability, 12(3), pp. 890–907, 2002.

[25] H. Schmidli, *Stochastic Control in Insurance*, Springer, London, 2008.

[26] H. Schmidli and J. Eisenberg, Optimal control of capital injections by reinsurance with a constant rate of interest, Journal of Applied Probability, 48(3), pp. 733–748, 2011.

[27] M. Schäl, Stochastic optimization for the ruin probability, Proceedings in Applied Mathematics and Mechanics, 3(1), pp. 17–19, 2003.

[28] M. Schäl, On discrete-time dynamic programming in insurance: Exponential utility and minimizing the ruin probability, Scandinavian Actuarial Journal, 2004(3), pp. 189–210, 2004.

[29] S. E. Shreve, *Stochastic calculus for finance II: Continuous-time models*, Springer, New York, 2004.

[30] H. Soner and W. Fleming, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, NY, 1993.

[31] G. Taylor, Underwriting strategy in competitive insurance environment, Insurance: Mathematics and Economics, 5(1), pp. 59–77, 1986.

[32] G. E. Uhlenbeck and L. S. Ornstein, On the theory of the brownian motion, Physical review, 36(5), pp. 823–841, 1930.

[33] O. Vasicek, An equilibrium characterization of the term structure, Journal of Financial Economics, 5(2), pp. 177–188, 1977.

[34] Yahoo Finance, `https://finance.yahoo.com`, accessed: 2024-08-20.

[35] H. Yang and C. Liu, Optimal investment for an insurer to minimize its probability of ruin, North American Actuarial Journal, 8(2), pp. 11–31, 2013.

[36] J. Zhang, S. Liu, and D. Kannan, Optimal investment and proportional reinsurance under no short-selling and no borrowing, Dynamic Systems and Applications, 20(2), pp. 205–222, 2011.

[37] M. Zhou, K. C. Yuen, and Y. C. C., Optimal investment and premium control in a nonlinear diffusion model, Acta Mathematicae Applicatae Sinica, 33(4), pp. 945–958, 2017.

CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Gülveren, Anıl
Nationality: Turkish (TC)

EDUCATION

Degree	Institution	Year of Graduation
M.S.	Institute of Applied Mathematics, METU	2016
B.S.	Department of Actuarial Science, Hacettepe University	2014
High School	Karşıyaka Anatolian High School	2008

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
3 Year	Willis Towers Watson	Senior Actuarial Analyst
1 Year	KPMG	Actuarial Manager
3 Year	Eureko Sigorta	Pricing Director