T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL

ON THE DIRECTED HAMILTON-WATERLOO PROBLEM WITH TWO CYCLE SIZES

FATİH YETGİN

A THESIS OF DOCTORATE
DEPARTMENT OF MATHEMATICS

ADVISOR: PROF. DR. SİBEL ÖZKAN

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T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

İKİ DÖNGÜ UZUNLUKLU YÖNLÜ HAMILTON-WATERLOO PROBLEMİ

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This thesis is dedicated to my late father,
whom I lost while working on this thesis,
and to my beloved mother.

ABSTRACT

This thesis studies the conditions for finding a 2-factorization of a graph G and handles the constructions used when finding the factors in this 2-factorization. Herein, a 2-factorization of a graph G is a partition of the edge set of G into 2-factors that are 2-regular spanning subgraphs of G. One of the important issues in the fields of Combinatorial Design Theory and Graph Theory is to explore the existence of a 2-factorization of a given graph. In addition, when introducing directed edges in the graph, 2-factorization problems become more challenging.

This thesis not only focuses on some 2-factorization problems involving the Oberwolfach Problem, the Hamilton-Waterloo Problem, and their directed versions but also gives some examples to make these problems more understandable. It also analyzes the necessary conditions for their solutions. First, the results from the literature on these problems are presented in detail. Then, the solutions to the Directed Hamilton-Waterloo Problem with two cycle sizes are given and the necessary constructions are developed.

The Directed Hamilton-Waterloo Problem asks for directed cycle factorizations of the complete symmetric digraph K_v^* into two non-isomorphic directed cycle factors. In the uniform version of the problem, factors consist of either directed m-cycles or n-cycles, and it is denoted by HWP$^*(v; m^r, n^s)$ where r and s are the number of factors of directed m-cycles and n-cycles, respectively, such that $r + s = v - 1$.

In this thesis, after introducing the Directed Hamilton-Waterloo Problem, which is a 2-factorization problem and our main focus, the necessary conditions for solving the problem are given. Also, the problem is completely solved for the factors with $(m, n) \in \{(4, 6), (4, 8), (4, 12), (4, 16), (6, 12), (8, 16)\}$. Furthermore, the problem is solved for $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$ when v is odd with a few possible exceptions. Solving this problem for certain m- and n-cycles means giving a solution to the problem for all possible values of r and s that satisfy the necessary conditions.

Finally, it is shown, with a few possible exceptions, that K_v^* can be factorized into some specific non-isomorphic factors, where these factors are uniform factors of K_v^* involving K_2^* or directed m-cycles, and directed m-cycles or $2m$-cycles for even m.

Keywords: **The Directed Hamilton-Waterloo Problem, 2-Factorizations, Directed Cycle Factorizations, Complete Symmetric Digraph.**

ÖZET

Bu tez bir G çizgesinin bir 2-faktörizasyonunu bulmak için gerekli şartları araştırmakta ve bu 2-faktörizasyondaki faktörleri bulurken kullanılan yapıları ele almaktadır. Burada G'nin bir 2-faktorizasyonu G'nin kenarlar kümesinin 2-faktörlere parçalanışıdır ve bir G çizgesinin bir 2-faktörü, G'nin 2-düzenli kapsayıcı bir alt çizgesidir. Kombinatoryal Tasarım Teorisi ve Çizge Teorisi alanlarındaki önemli konulardan biri, verilen bir çizgenin 2-faktorizasyonun varlığını araştırmaktır. Ek olarak, çizgenin kenarlarının yönlü olması şartı eklediğinde 2-faktorizasyon problemleri daha zor hale gelir.

Bu tez yanlızca Oberwolfach Problemi, Hamilton-Waterloo Problemi ve bu problemlerin yönlü versiyonlarını içeren bazı 2-faktorizasyon problemlerine odaklanmakla kalmayıp, aynı zamanda bu problemleri daha anlaşılır hale getirmek için bazı örnekler de vermektedir. Ayrıca, problemlerin çözümleri için gerekli şartlar incelenmiştir. Önce, ilgili problemler için literatürdeki sonuçlar ayrıntılı şekilde verilmiştir. Sonra, iki döngü uzunluklu Yönlü Hamilton-Waterloo probleminінin çözümleri ele alınmış ve bunun için gerekli yapılar geliştirilmiştir.

Yönlü Hamilton-Waterloo problemi, simetrik yönlü tam çizge K_v^*'ın iki izomorfik olmayan yönlü döngü faktörlere ayrışımını ister. Problemin tek tip versiyonunda, faktörler ya yönlü m-döngülerinden ya da n-döngülerinden oluşur ve o HWP$^*(v; m^r, n^s)$ ile gösterilir; burada r ve s sırasıyla $r + s = v - 1$ olacak şekilde yönlendirilmiş m-döngülerinin ve n-döngülerinin faktör sayısıdır.

Bu tezde, bir 2-faktorizasyon problemi ve bizim asıl odak noktamız olan Yönlü Hamilton-Waterloo Problemi tanıtıldıktan sonra, problemin çözümü için gerekli şartlar verilmiştir. Ayrıca problem, $(m, n) \in \{(4, 6), (4, 8), (4, 12), (4, 16), (6, 12), (8, 16)\}$ durumları için tamamen çözülmüştür. Dahası v'nin tek olduğu durumda ise $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$ için birkaç olası istisna dışında çözülmüştür. Bu problemi belirli m ve n döngüler için çözmek demek, gerekli koşulları sağlayan olası bütün r ve s değerleri için probleme bir çözüm vermek demektir.

Son olarak, K_v^*'ın izomorfik olmayan bazı özel faktörlere ayrışımının olabileceği birkaç olası istisna dışında gösterilmiştir: bu özel faktörler çift m değerleri için, K_v^*'ın K_2^* veya yönlü m-döngülerini, ve yönlü m-döngülerini veya $2m$-döngülerini içeren tek tip faktörleridir.

Anahtar Kelimeler: **Yönlü Hamilton-Waterloo Problemi, 2-Faktorizasyon, Yönlü Döngü Faktorizasyon, Tam Simetrik Yönlü Çizge.**

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LIST OF ABBREVIATIONS AND ACRONYMS

$V(G)$	:	Vertex set of a graph G
$E(G)$	:	Edge set of a graph G
C_n	:	Cycle of length n
K_n	:	Complete graph with order n
$K_{r,s}$	:	Complete bipartite graph
$K_{(x:y)}$	:	Complete equipartite graph having y parts of size x
G^*	:	Symmetric digraph
K_v^*	:	Complete symmetric digraph of order v
$K_{(x:y)}^*$	:	Complete symmetric equipartite digraph with y parts of size x
$\overrightarrow{X}(B;S)$	:	Directed Cayley graph on finite additive group B with connection set S
$\overrightarrow{C}_m$	:	Directed cycle of length m
$STS(v)$	:	Steiner triple system of order v
$KTS(v)$	:	Kirkman triple system of order v
$[a,b]$	:	The set of integers $\{a, a+1, \ldots, b\}$ for integers a and b with $a \leq b$

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1. INTRODUCTION

Combinatorial design theory and graph theory deal with the structural arrangements and relationships between some objects even if both theories offer different approaches for the solution. Therefore, they are intertwined areas of mathematics.

Graph theory is a branch of mathematics that deals with the representation of connections between objects. It studies the points (vertices) available in the graphs and the connection lines (edges) and their mathematical properties and connections. Moreover, graph theory not only analyzes these structures but also provides solutions in many practical areas through simple graphical models that are the schematic representation of complex problems. For the solution of the complex problems, one of the most popular ones is the Königsberg problem which plays a significant role in the bird of graph theory. It relies on such a question about a tour on the bridges in the Prussia city of Königsberg. This city consists of four small islands set apart by the Pregel river. There are totally seven bridges in this city, each of which connects 4 different islands to each other. The question is here: Is it possible to set off a walking tour anywhere in the city and return to the initial point by crossing each bridge only once?

This problem was a starting point for graph theory, thanks to the approach that Euler developed in the 18th century to analyze it. Euler represented the problem in a graph theory model via constructing a simple graph where each island is a vertex and bridges are edges connecting these islands. By analyzing this graph, Euler developed the basic concepts needed to determine whether it is possible to cross a bridge.

More importantly, the application of graph theory is quite vast and spans a lot of fields incorporating computer science, engineering, communication, bioinformatics, and social sciences to name just a few. An example is the examination of social media networks. Social media are large-scale, diverse networks of relationships amongst numerous users. These relationships manifest in all ways including friendships, followers, likes or shared content. Such networks can be mathematically modeled and analyzed using graph theory. Using the graph formulation, a social media network can be modeled to study relationships between users on such platform and how they interact with one another. Graph theory is helpful to solve different types of problems like discovering key people in a network, identifying groups with similar history and allegiances, tracing how information spreads, or even detecting spam accounts. Graph theory is also applied to problems in logistics, optimization and decision-making. Applied examples are everywhere; For instance, graph theory is being used in optimizing transportation networks or resource allocation.

Combinatorial design theory is an important area of mathematics. It deals with the study of combinatorial designs, which are structured arrangements of objects subject to certain constraints. These designs are widely used in areas as diverse as coding theory, cryptography, experimental design, and network optimization. A fundamental concept in combinatorial design theory is that of a block design, which consists of a finite set of elements (called points) and a collection of subsets of these elements (called blocks) that satisfy certain properties. Block designs can be represented using mathematical structures such as matrices, incidence matrices, or graphs.

The connection between combinatorial design theory and graph theory is quite strong. Combinatorial design theory also deals with organized objects, so many concepts from graph theory can be found in combinatorial design theory. For example, the relationships between the edges of a graph can be thought of as an arrangement in combinatorial design theory. Similarly, various structures in graph theory can also be found in combinatorial design theory, for example, various combinatorial structures can be modeled by graphs. Therefore, combinatorial design theory and graph theory are two complementary and closely related branches of mathematics.

The 2-factorization problem, which essentially studies the existence of a 2-factorization of a given graph (generally a complete graph K_v), is a problem at the intersection of graph theory and combinatorial design theory. This thesis presents the necessary conditions for finding a 2-factorization of a graph and its decomposition into certain special factors. Several 2-factorization problems and their directed versions are introduced, such as the Oberwolfach Problem and the Hamilton-Waterloo Problem. Then the necessary conditions for those are given. In particular, the Directed Hamilton-Waterloo Problem with detailed examples is presented and specific solutions for the uniform version of the problem are provided.

This introduction will initially introduce some basic graph terminology and concepts that will be used throughout this thesis, followed by a brief historical overview of well-known problems relevant to the findings presented in this thesis. Let us begin with the initial section where general definitions and notations are provided.

1.1. Basic Definitions and History

This section provides the basic definitions and notations used in this thesis. These are mostly standard and can be found in [1] and [2].

A graph G can be defined as a pair $(V(G), E(G))$ consisting of a vertex set $V(G)$, an edge set $E(G)$, where each edge in $E(G)$ is associated with two vertices (not necessarily distinct) called its endpoints. If an edge exists between vertex u and vertex v, this edge is denoted by $\{u, v\}$ where u and v are endpoints. If u and v are endpoints of an edge, u and v are called adjacent vertices or neighbors. The number of neighbours of a vertex v is called the degree of v and is denoted by $d_G(v)$. A graph whose vertices are all of degree k is said to be a k-regular graph.

A graph G is said to have multiple edges if there is more than one edge connecting two vertices. A simple graph is a graph having no loops or multiple edges where a loop is an edge whose endpoints are equal.

The complement, $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$. An independent set in a graph, also known as a stable set, refers to a set of pairwise nonadjacent vertices.

A path can be defined as a sequence of vertices such that the elements of the sequence are ordered, do not repeat, and each vertex in the sequence is adjacent to every other vertex. A sequence $v_0, v_1, \ldots, v_{n-1}$ is denoted by P_n and called the path graph of length $n-1$. A cycle is a path whose start vertex and end vertex are the same. A cycle graph of length n is denoted by $C_n : v_0, v_1, \ldots, v_{n-1}, v_0$ or $C_n = (v_0, v_1, \ldots, v_{n-1})$ where $\{v_0, v_1, \ldots, v_{n-1}\} \in V(C_n)$ and $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-1}, v_0\} \in E(C_n)$.

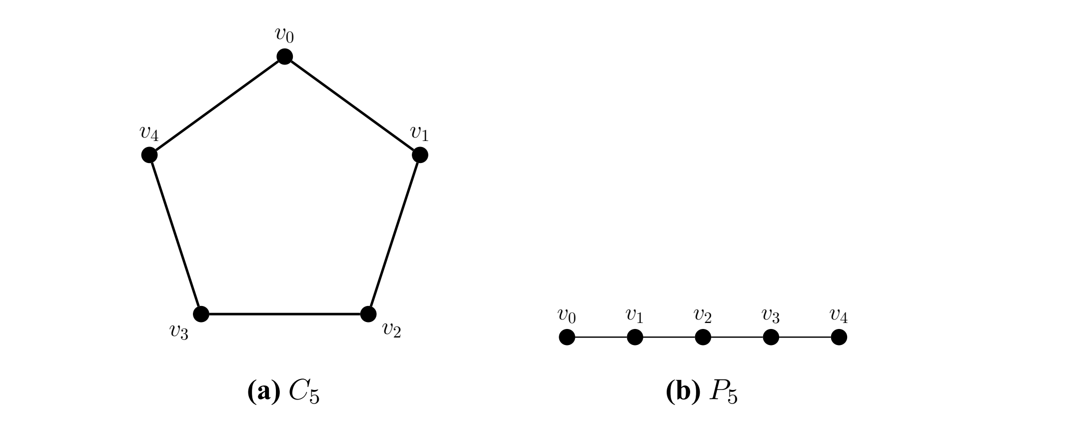
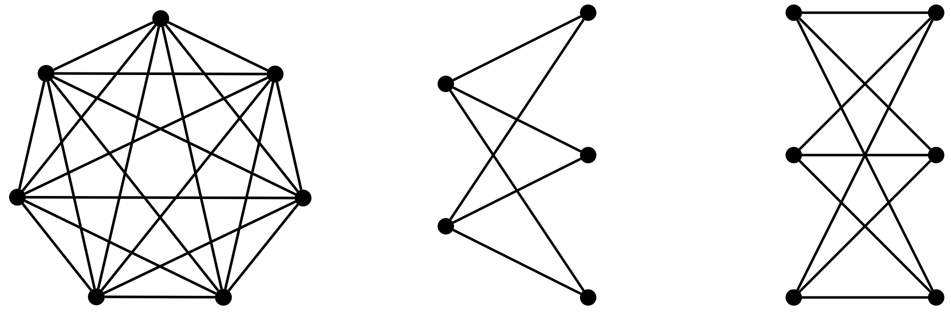


Figure 1.1: Cycle C_5 and Path P_5.

Let G be a graph, then a subgraph H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H is called a spanning subgraph of G if $V(H) = V(G)$, meaning that spanning subgraph contains all the vertices of G. If there is a path between every pair of vertices in G, this graph is called a connected graph.

A graph G is multipartite if vertex set of G consists of disjoint independent sets called partite sets of G. If the cardinalities of disjoint independent sets are equal, this graph is called an equipartite graph. When the number of disjoint independent sets is 2, this graph is called a bipartite graph.

A complete graph K_n is a simple graph in which there is an edge from every vertex of the graph to every other vertex. Note that $\overline{K}_n$ has n vertices and no edges. A complete bipartite graph is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have sizes r and s, $K_{r,s}$ denote a complete bipartite graph. We denote by $K_{(x:y)}$ a complete equipartite graph having y parts of size x each.



(a) K_7 **(b)** $K_{2,3}$ **(c)** $K_{(3:2)}$

Figure 1.2: A complete graph K_7, a complete bipartite graph $K_{2,3}$, and a complete equipartite graph $K_{(3:2)}$.

A directed graph (or digraph) D is a pair $(V(D), A(D))$ consisting of a vertex set $V(D)$, an arc set $A(D)$, and a function assigning each arc an ordered pair of vertices. The first vertex of the ordered pair is the tail of the arc, and the second is the head; together, they are the endpoints. Arcs are denoted by parentheses. We also use $(u, v)^*$ to denote the double arc which consists of (u, v) and (v, u).

For a simple graph D, D^* is used to denote the symmetric digraph with vertex set $V(D^*) = V(D)$ and arc set $A(D^*) = \bigcup_{\{x,y\}\in E(D)}\{(x, y), (y, x)\}$. Hence, K_v^* and $K_{(x:y)}^*$ respectively denote the complete symmetric digraph of order v and the complete symmetric equipartite digraph with y parts of size x.

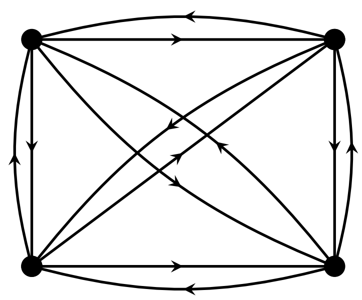


Figure 1.3: A complete symmetric digraph K_4^*.

$\vec{C}_m = (v_0, v_1, \ldots, v_{m-1})$ denotes a directed cycle of length m in a digraph D where $\{v_0, v_1, \ldots, v_{m-1}\} \in V(D)$ and $(v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_0) \in A(D)$. That is, there is an arc with tail v_i and head v_j if and only if v_j follows v_i in the vertex ordering where $i, j \in \mathbb{Z}_m$.

Example 1.1: The digraph D with vertex set $V = \{v_0, v_1, v_2, v_3\}$ and arc set $A(D) = \{(v_0, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_0), (v_3, v_1)\}$ is shown in the following figure. Also, $\vec{C}_4 = (v_0, v_1, v_2, v_3)$ and $\vec{C}_3 = (v_1, v_2, v_3)$ show a directed 4-cycle and a directed 3-cycle in D, respectively.

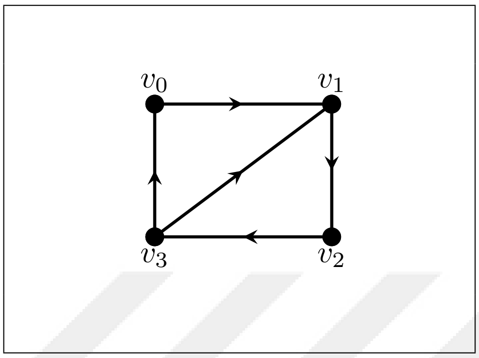


Figure 1.4: A Directed Graph Example.

Let B be a finite additive group and let S be a subset of B, where S does not contain the identity of B. The Directed Cayley graph $\vec{X}(B; S)$ on B with connection set S is a digraph with $V(\vec{X}(B; S)) = B$ and $A(\vec{X}(B; S)) = \{(x, y) : x, y \in B, y - x \in S\}$.

Example 1.2: The Directed Cayley graph $\vec{X}(\mathbb{Z}_9; \{-2, 1\})$ is drawn in Figure 1.5.

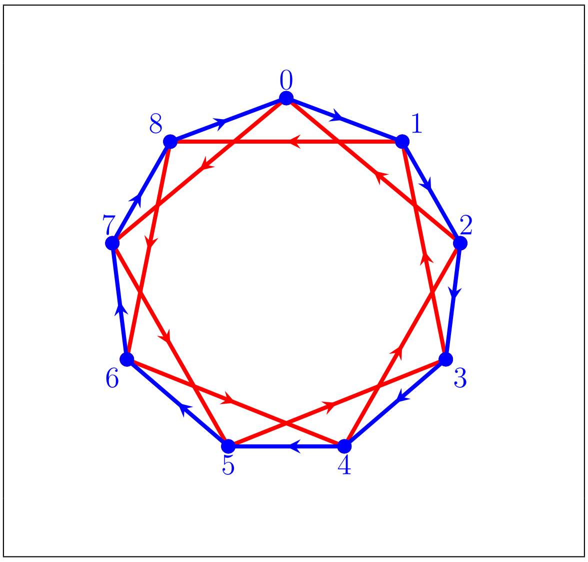


Figure 1.5: The Directed Cayley graph $\vec{X}(\mathbb{Z}_9; \{-2, 1\})$.

Let G and H be graphs. The wreath product of G and H, denoted by $G \wr H$, is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G \wr H) = \{\{(u_1, u_2), (v_1, v_2)\} :$

$\{u_1, v_1\} \in E(G)$, or $u_1 = v_1$ and $\{u_2, v_2\} \in E(H)\}$. That is $G \wr H$ obtained by replacing each vertex x of G with a copy of H, say H_x, and replacing each edge $\{x, y\}$ of G with the edges joining every vertex of H_x to every vertex of H_y.

In the case where D_1 and D_2 are both digraphs, then the $D_1 \wr D_2$ is the digraph obtained by replacing each vertex x of D_1 with a copy of D_2, say D_2^x, and replacing each arc (x, y) of D_1 by an arc pointing from every vertex of D_2^x to every vertex of D_2^y. For example, $K_x^* \wr \overline{K}_y \cong K_{(y:x)}^*$, $\overline{K}_x \wr K_y^* \cong xK_y^*$ and $\overline{K}_x \wr \overline{K}_y \cong \overline{K}_{xy}$.

For given three graphs G, H, and J, $(G \wr H) \wr J = G \wr (H \wr J)$, that is, the wreath product is associative (*p.* 185 of [3]). Although the wreath product is not commutativity in general, the isomorphism $G \wr H \cong H \wr G$ holds if both G and H are complete or both are completely disconnected. Note that, the above properties of the wreath product extend to digraphs.

Example 1.3: Figure 1.6 shows the wreath product of two digraphs.

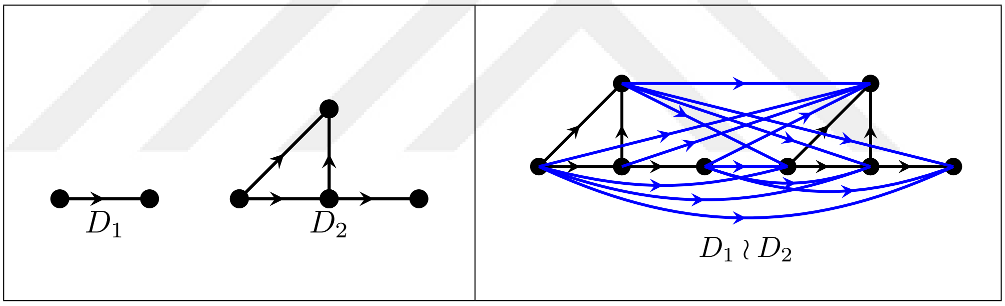


Figure 1.6: The wreath product of D_1 and D_2.

Let G be a graph and $G_0, G_1, \ldots, G_{k-1}$ be k vertex disjoint copies of G with $v_i \in V(G_i)$ for each $v \in V(G)$. Let $G[k]$ denote the graph with vertex set $V(G[k]) = V(G_0) \cup V(G_1) \cup \ldots \cup V(G_{k-1})$ and edge set $E(G[k]) = \{\{u_i, v_j\} : \{u, v\} \in E(G)$ and $0 \leq i, j \leq k-1\}$. For example, $K_2[m]$ is isomorpic to $K_{(m:2)}$.

Note that $G \wr \overline{K}_k \cong G[k]$. In this thesis, the notation $G[k]$ (which is called the power of a graph G) will be used frequently.

This definition can be extended to digraphs. Let D be a digraph and $D_0, D_1, \ldots, D_{k-1}$ be k vertex disjoint copies of D with $v_i \in V(D_i)$ for each $v \in V(D)$. Then, $D[k]$ has the vertex set $V(D[k]) = V(D_0) \cup V(D_1) \cup \cdots \cup V(D_{k-1})$ and arc set $A(D[k]) = \{(u_i, v_j) : (u, v) \in A(D)$ and $0 \leq i, j \leq k-1\}$.

Example 1.4: $\vec{C}_6$, $\vec{C}_6[2]$, C_6^* *and* $C_6^*[2]$ *can be drawn as follows:*

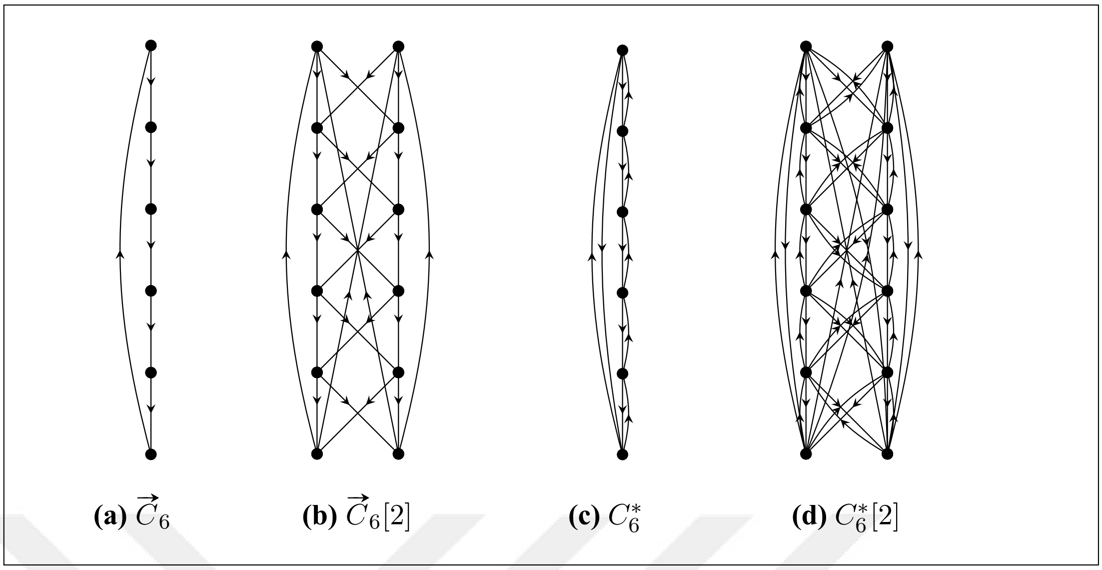


Figure 1.7: Power of Graph.

Let D be a digraph and $R(D)$ denote the digraph on the same vertex set as D but the arcs are taken in opposite directions.

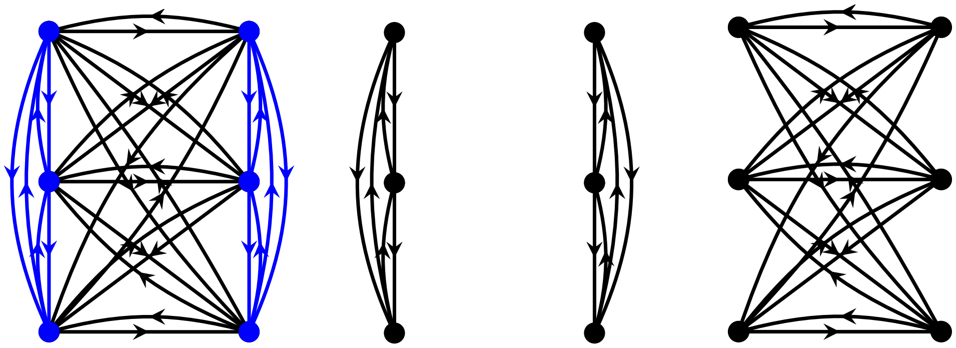
If G_1 and G_2 are two edge (arc)-disjoint graphs (digraphs) with $V(G_1) = V(G_2)$, then $G_1 \oplus G_2$ is used to denote the graph on the same vertex set with $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$ $(A(G_1 \oplus G_2) = A(G_1) \cup A(G_2))$. Also, we will denote the vertex disjoint union of α copies of G by αG.

Example 1.5: K_{mx}^* *is isomorphic to* $xK_m^* \oplus K_{(m:x)}^*$

Proof. Partition the vertices of K_{mx}^* into x sets of size m, represent each part of m vertices in K_{mx}^* with a single vertex and represent all double arcs between sets of size m as a single double arc, to get a K_x^*. Now, each vertex of K_x^* gives a K_m^*. Since there are x such vertices, xK_m^* is obtained from these vertices. There remain x sets of size m and double arcs connecting these sets. The graph $K_{(m:x)}^*$ is obtained from the remaining arcs. As a result, the following equation can be written.

$$K_{mx}^* \cong xK_m^* \oplus K_{(m:x)}^* \tag{1.1}$$

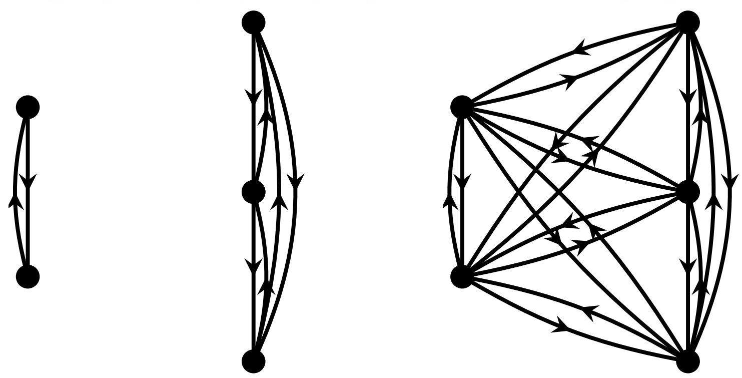
□



(a) K_6^* **(b)** $2K_3^*$ **(c)** $K_{(3:2)}^*$

Figure 1.8: K_6^* is isomorphic to $2K_3^* \oplus K_{(3:2)}^*$.

If G_1 and G_2 are two vertex-disjoint graphs, then the join of G_1 and G_2 denoted $G = G_1 \vee G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. This means that $G_1 \vee G_2$ is obtained by taking the union of G_1 and G_2, and using all possible edges between G_1 and G_2. If D_1 and D_2 are two vertex-disjoint digraphs, $D_1 \vee D_2$ is obtained by taking the union of D_1 and D_2, and using all possible arcs from D_1 to D_2 and from D_2 to D_1.

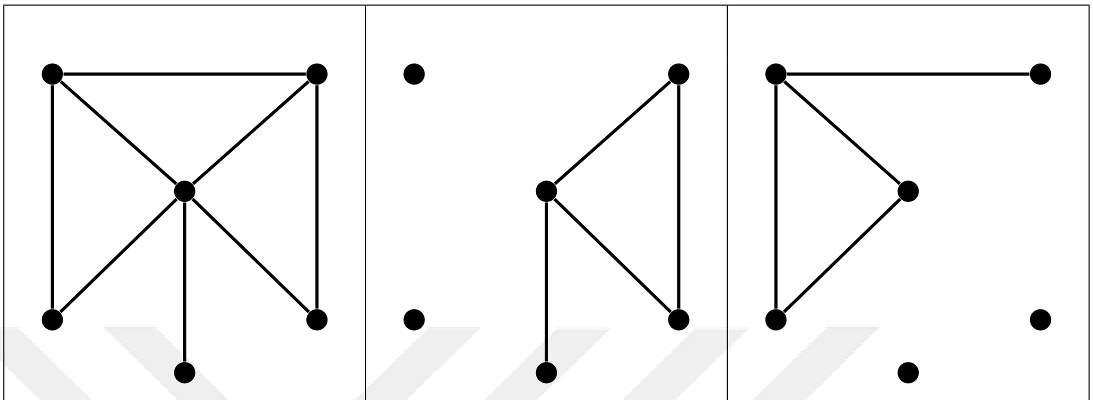


(a) D_1 and D_2 **(b)** $D_1 \vee D_2$

Figure 1.9: The join of D_1 and D_2.

An edge decomposition of a graph G is a set $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ of subgraphs of G such that the union of the edge sets of the subgraphs is equal to the edge set of the graph G and $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$. So, the edges of H_i are partitions the edges of G. Such a decomposition is called an $\{H_1, H_2, \ldots, H_k\}$-decomposition of G. In the case where each H_i is isomorphic to a graph such as H, this decomposition is called an H-decomposition. If each H_i is a cycle or a union of disjoint cycles, then this decomposition is called a cycle decomposition.

Note that if G has a $\{H_1, H_2, \ldots, H_k\}$-decomposition, then $G \wr \overline{K}_n$ has a $\{H_1 \wr \overline{K}_n, H_2 \wr \overline{K}_n, \ldots, H_k \wr \overline{K}_n\}$-decomposition [4].



(a) A graph G **(b)** A Subgraph H_1 **(c)** A Subgraph H_2

Figure 1.10: A $\{H_1, H_2\}$-decomposition of G.

Veblen proved that the necessary and sufficient condition for a graph to decompose into cycles is that the degree of each vertex of the graph must be even. If we think about a cycle decomposition, every vertex in a cycle is incident to two edges; so, the degree of every vertex in the graph has to be even.

Theorem 1.6: [5] A graph can be decomposed into cycles if and only if every vertex has an even degree.

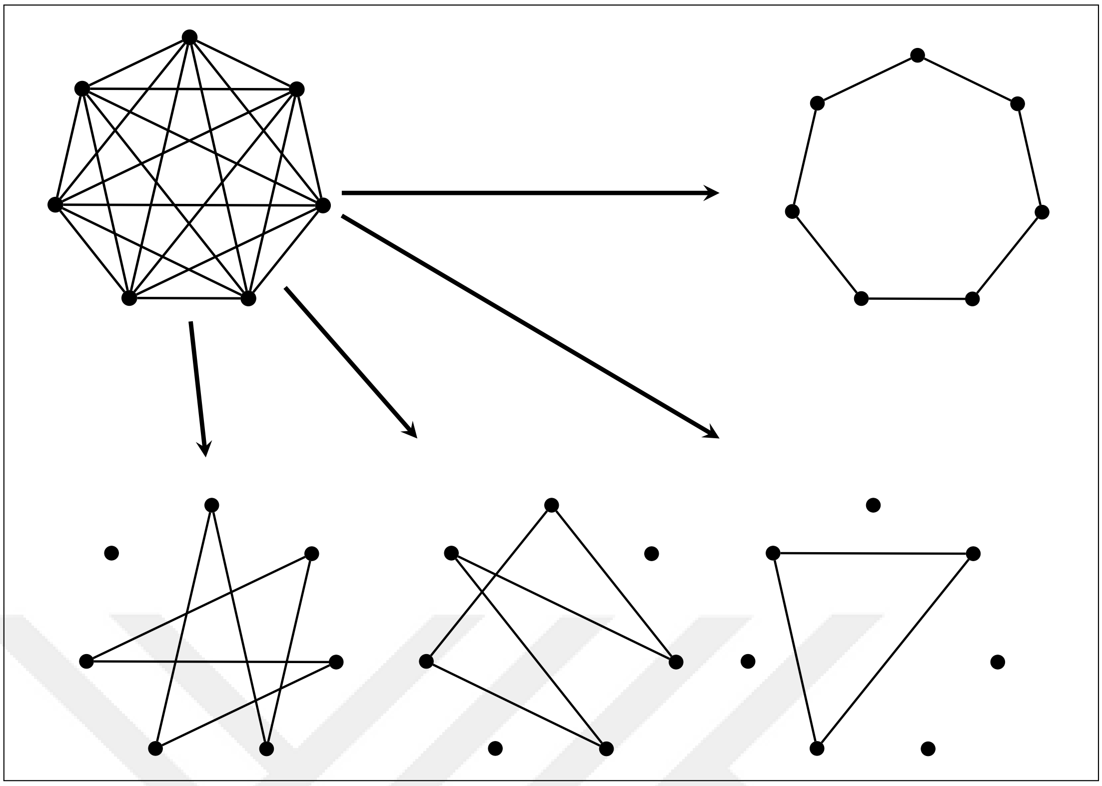


Figure 1.11: A $\{C_7, C_6, C_5, C_3\}$-decomposition of K_7.

Graph factorizations constitute an important part of graph decomposition problems, especially when each factor is of regular degree. A factor in a graph G is a spanning (not necessarily connected) subgraph of G. A k-factor of a graph G is a k-regular spanning subgraph of G. It is easy to see that a 1-factor is a perfect matching in a graph and a 2-factor is either an Hamilton cycle, which is a cycle that visits each vertex exactly, or union of cycles. If a 2-factor consists only of cycles (directed cycles) of length m, it is called a C_m-factor ($\overrightarrow{C}_m$-factor). Furthermore, in the special case where $m = 2$, this factor becomes a K_2-factor. In addition, if a 2-factor consists of directed 2-cycles, that factor is called a $\overrightarrow{C}_2$-factor or a K_2^*-factor. In this thesis, we will primarily use the K_2^*-factor notation.

According to Theorem 1.6, if n is even, K_n cannot be decomposed into cycles. In this case, the decomposition of $K_n - I$ is studied, where I is a 1-factor.

Example 1.7: A $\{C_6, C_3, C_3\}$-decomposition of $K_6 - I$ can be drawn as follows:

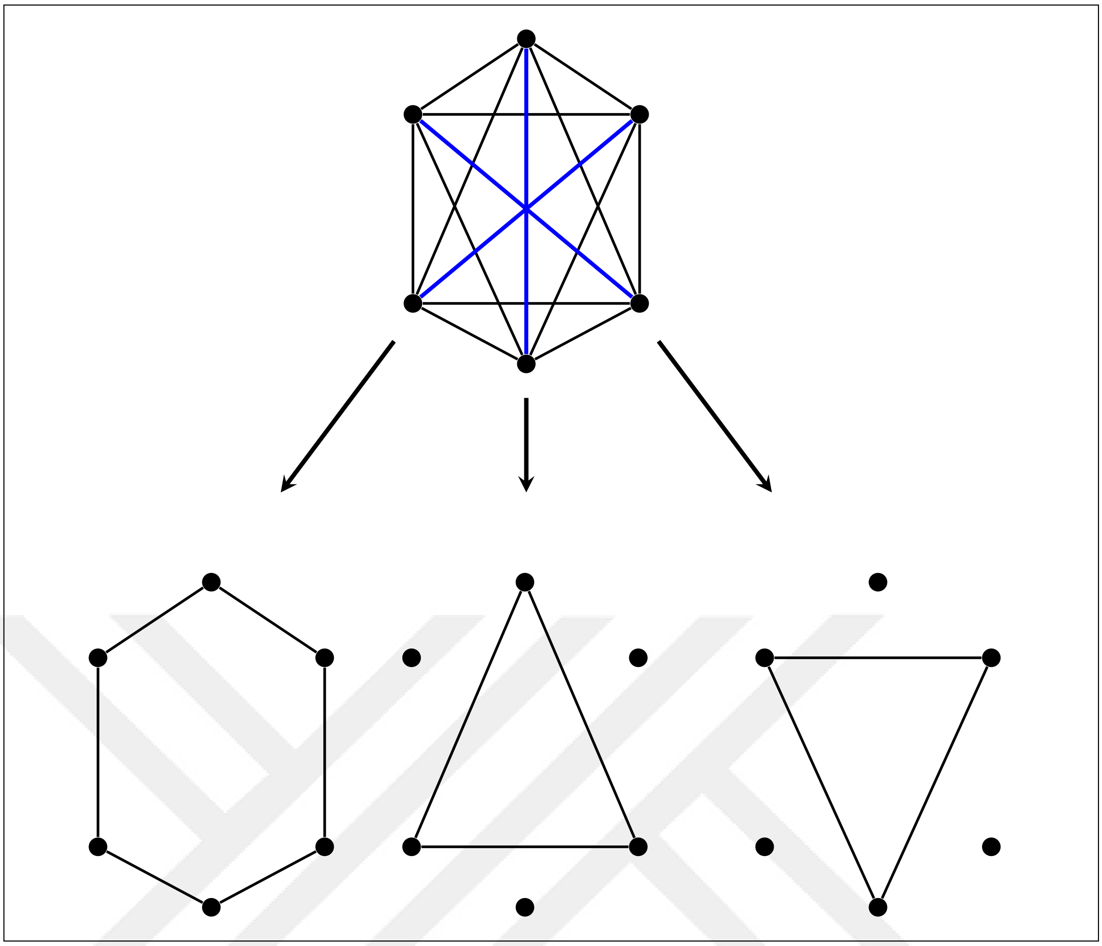


Figure 1.12: A $\{C_6, C_3, C_3\}$-decomposition of $K_6 - I$.

A k-factorization of a graph G is a partition of the edge set of G into k-factors; in other words, it is a decomposition of the edge set of G into edge-disjoint k-factors.

Example 1.8: There is a 2-factorization of K_5 (see Figure 1.13) and a 2-factorization $K_6 - I$ (see Figure 1.14).

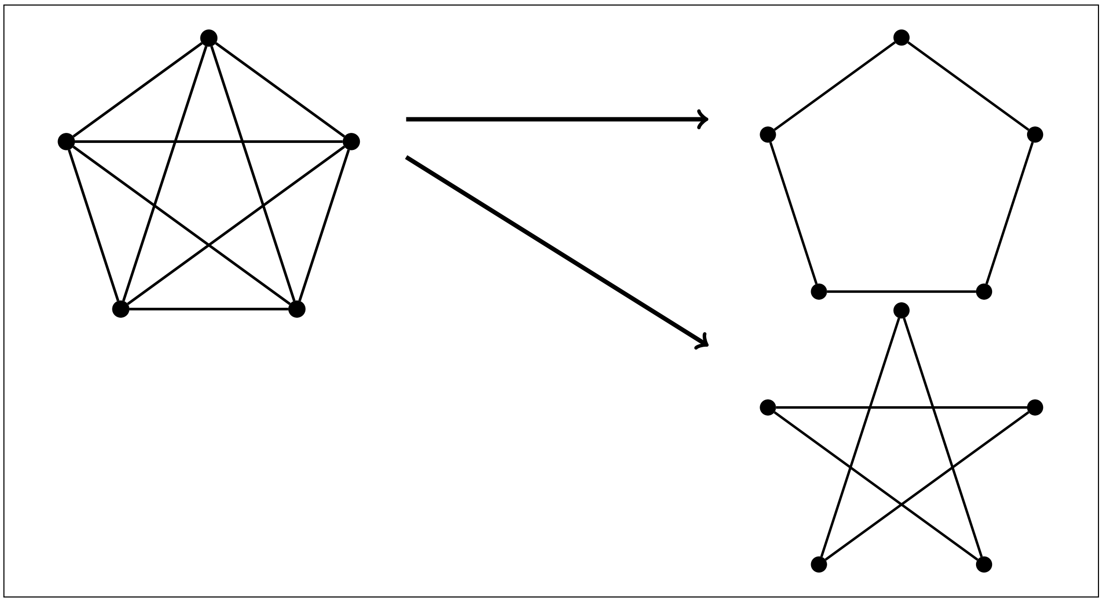


Figure 1.13: A 2-factorization of K_5.

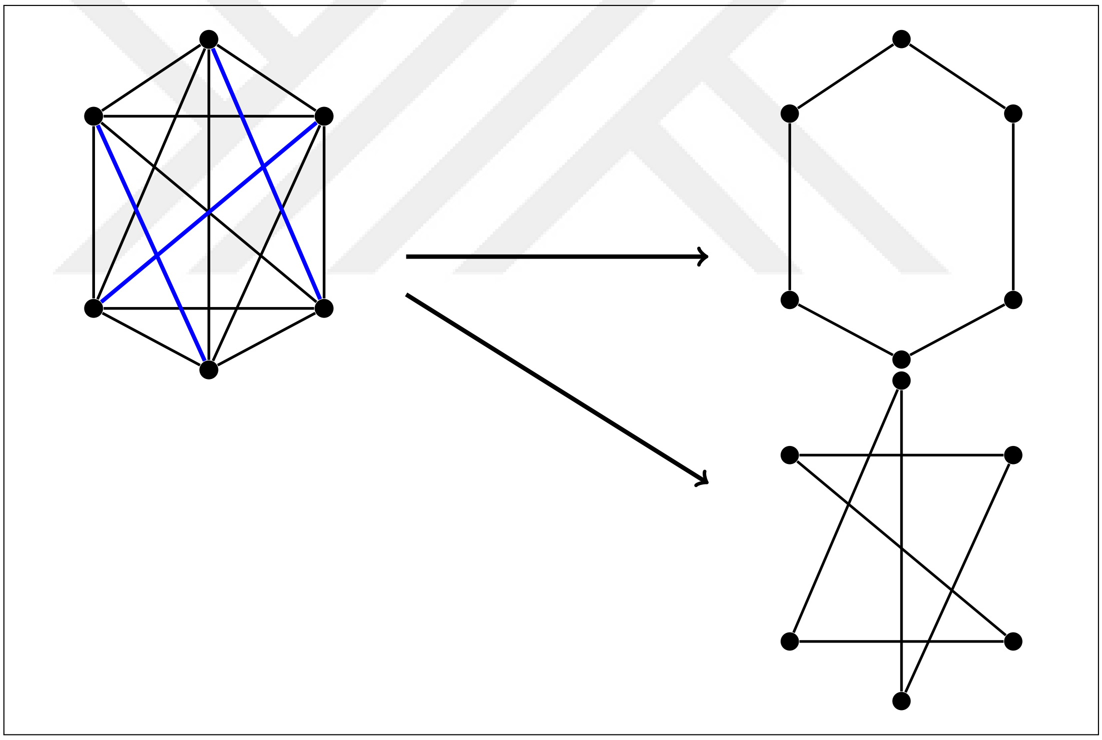


Figure 1.14: A 2-factorization of $K_6 - I$.

If a graph G can be decomposed into r_i factors isomorphic to the factor F_i for $i \in [1,t]$, then G is said to have an $\{F_1^{r_1}, F_2^{r_2}, \ldots, F_t^{r_t}\}$-factorization. When each F_i factor consists of only n_i-cycles for $i \in [1,t]$, then we will call the F_i factor as a C_{n_i}-factor and call this factorization as a $\left\{C_{n_1}^{r_1}, C_{n_2}^{r_2}, \ldots, C_{n_t}^{r_t}\right\}$-factorization where each r_i is the number of C_{n_i}-factors. This type of factorization can also be referred to as a cycle factorization. When all factors consist only of an n-cycle, this factorization is called $\{C_n^r\}$-factorization where r is the number of C_n-factors.

It is easy to see that there is an $H[k]$-factorization of $G[k]$ if the graph G has an H-factorization.

Example 1.9: A $\{F_1, F_2, F_3\}$-factorization of K_7 can be graphed as follows where F_1 and F_3 are Hamilton cycle (C_7-cycle) factors, and the factor F_2 consists of a C_3-cycle and a C_4-cycle:

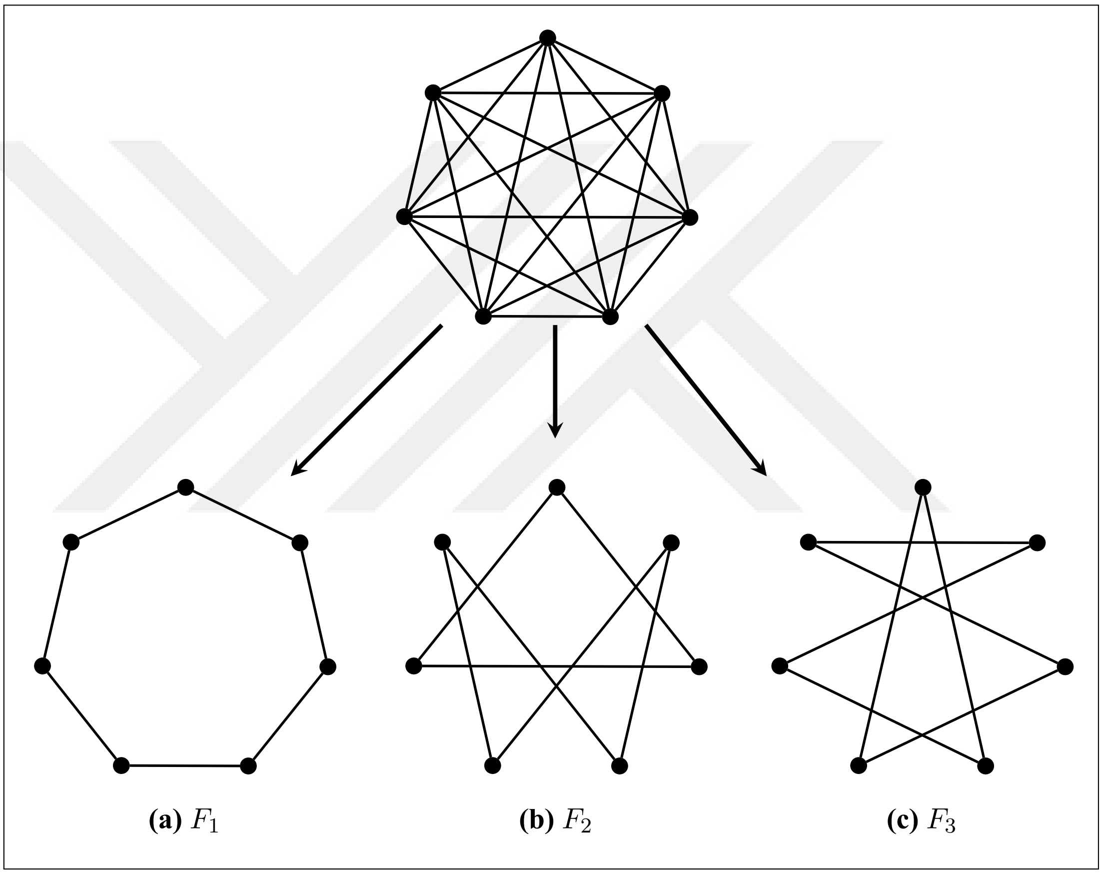


Figure 1.15: A $\{F_1, F_2, F_3\}$-factorization of K_7.

One of the oldest cycle decomposition problems is the problem of decomposing a complete graph into 3-cycles, which constitutes a triple system. Before proceeding to one of the most renowned instances, Kirkman's Schoolgirl Problem, let us first discuss the Steiner Triple Systems

A Steiner triple system of order v, denoted by $\mathrm{STS}(v)$, consists of a set S with v elements and a set T of three-element subsets of S, called triples, such that each distinct pair of S is contained in exactly one triple of T.

Example 1.10: STS(7) *exists for* $S = \{1,2,3,4,5,6,7\}$ *and* $T = \{\{1,2,4\},\{2,3,5\},$ $\{3,4,6\},\{4,5,7\},\{5,6,1\},\{6,7,2\},\{7,1,3\}\}$.

In the language of graph theory, the existence of STS(v) coincides with the decomposition of K_v into C_3. Kirkman provided necessary and sufficient conditions for a Steiner triple system to exist, as follows:

Theorem 1.11: [6] There exists a Steiner triple system of order $v \geq 3$ *if and only if* $v \equiv 1 \pmod 6$ *or* $v \equiv 3 \pmod 6$.

Kirkman's Schoolgirl Problem, it was proposed by Kirkman in 1850 and was solved a year later. It is stated as follows:

Fifteen girls will walk each day of the week. The girls will walk in five lines of three girls each. Every two girls must walk in the same line exactly once a week. Kirkman's Schoolgirl Problem asks whether this is possible.

A Kirkman triple system of order v consists of a v element set S, a collection T of triples from S, and a partition P of T, satisfying the conditions where each pair of distinct elements in S belongs to exactly one triple in T, and each element in S is contained in exactly one triple in each subset of P, and is denoted by KTS(v). The existence of KTS(15) means the existence of a solution to Kirkman's Schoolgirl Problem. The necessary and sufficient condition for the existence of KTS(v) was given by K. Ray-Chaudhuri and R.M. Wilson.

Theorem 1.12: [7] A Kirkman triple system of order v *exists if and only if* $v \equiv 3 \pmod 6$.

1.2. The Oberwolfach Problem

The Oberwolfach Problem is posed by Ringel (see [8]) as a seating arrangement problem at a meeting in Oberwolfach. Given a conference venue with k_i round tables, each of which has m_i seats for $i \in [1,t]$, it asks whether each participant of the conference (say v many for odd v) may sit next to (left or right) each other participant exactly once at the end of $\frac{v-1}{2}$ nights.

In graph theory language, the seating arrangement each night can be thought of as a cycle factor. Therefore, the number of factors matches the number of nights. Moreover, since it is required that at the end of $\frac{v-1}{2}$ nights each participant sits with every other participant exactly once, the problem becomes a problem of cycle factorization of K_v. So, it asks whether the complete graph K_v decomposes into isomorphic 2-factors where each 2-factor consists of k_i m_i-cycles for each $i \in [1, t]$ and $\sum_{i=1}^{t} k_i m_i = v$. This problem is denoted by OP$(m_1^{k_1}, m_2^{k_2}, \ldots, m_t^{k_t})$. If there is only one type of cycle, say of length m, in the factor, it can be denoted as OP(m^k), and this version of the problem is called the uniform Oberwolfach Problem. Its solution gives a $\{C_m^{\frac{v-1}{2}}\}$-factorization (or in short, a C_m-factorization) of K_v.

It is known that a necessary and sufficient condition for a graph to have a 2-factorization is that the graph must be even regular. Therefore, K_v does not have a 2-factorization when v is even. In this case, the existence of a 2-factorization of $K_v - I$ is searched [9], and this version of the problem is referred to as the spouse avoiding version. In addition, the number of factors is $\frac{v-2}{2}$ in this case.

Example 1.13: OP$(3, 4)$ has a solution. The existence of a solution to this problem means that in a conference with 7 participants, in a hall with round tables for 3 and 4 people, after 3 nights everyone can sit with each other absolutely once.

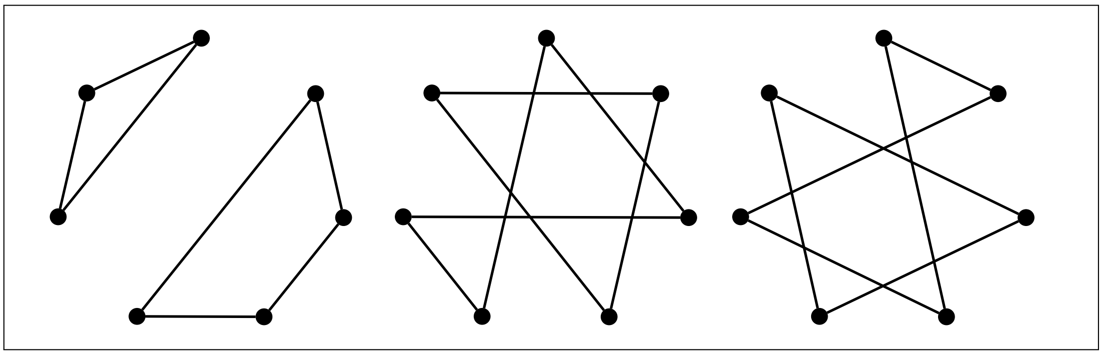


Figure 1.16: A solution to OP$(3, 4)$.

The uniform version of the Oberwolfach Problem has been mostly solved, see [4, 7, 10, 11]. But the general case of the Oberwolfach Problem is still open. It is known that OP(3^2), OP(3^4), OP$(4, 5)$ and OP$(3^2, 5)$ have no solution. In [12–16], it is shown that OP$(m_1^{k_1}, m_2^{k_2}, \ldots, m_t^{k_t})$ has a solution for all $n \leq 60$ with the above exceptions. There is a solution to the problem when the lengths of the cycles are all even [17, 18].

Liu [19] solved the Oberwalch problem for the complete equipartite graph $K_{(x:y)}$ with uniform cycle lengths. We will use this result to obtain a $\vec{C}_m$-factorization of $K^*_{(x:y)}$.

Theorem 1.14: [19] The complete equipartite graph $K_{(x:y)}$ has a C_m-factorization for $m \geq 3$ and $x \geq 2$ if and only if $m|xy$, $x(y-1)$ is even, m is even if $y = 2$ and $(x, y, m) \neq (2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)$.

The concept of factor and factorization can be applied to digraphs and one can consider the directed version of the Oberwolfach Problem.

1.2.1. The Directed Oberwolfach Problem

One generalization of Oberwolfach Problem may be to consider sitting on the right and sitting on the left of a participant as separate entities. To represent such a sitting, one has to use directed cycles which led us to work on directed graphs.

In the Directed Oberwolfach Problem, which was introduced in [20], the complete symmetric digraph K_v^* is decomposed into factors of directed cycles. Hence, the seating arrangement is done over $v-1$ nights. If the sizes of directed cycles are $m_1, m_2, \ldots, m_t$ and the number of each directed cycle m_i is k_i for $i \in [1,t]$ where $\sum_{i=1}^{t} k_i m_i = v$, the Directed Oberwolfach Problem is denoted by OP$^*(m_1^{k_1}, \ldots, m_t^{k_t})$. The Directed Uniform Oberwolfach Problem is denoted by OP$^*(m^k)$ where each 2-factor is composed of k directed m-cycles.

As the first result on the Directed Oberwolfach Problem, OP$^*(3^k)$ is solved with an exception for $v = 6$ by Bermond et al. [21]. In [22], Bennett and Zhang solved OP$^*(4^k)$ except for $v = 12$, and Adams and Bryant solved the remaining case OP$^*(4^3)$ [23].

In [24], Alspach et al. showed that K_v^* can be decomposed into $\vec{C}_m$ cycles with exceptions $(v, m) \neq (4, 4), (6, 3), (6, 6)$ if and only if $m|v(v-1)$. They studied the problem separately in cases where v and m are both even or have opposite parity.

Burgess and Sajna [20] studied the necessary and sufficient conditions for the Directed Oberwolfach Problem with cycles of length m. When m is even, they obtained a complete solution and presented a partial solution for odd cycle size. Also, they conjectured that K_{2m}^* admits a directed m-cycle factorization for odd m if and only if $m \geq 5$. In [25], Burgess et al. proved this conjecture for $m \leq 49$.

In [26], Shabani and Sajna proved that K_v^* has a $\{\vec{C}_2, \vec{C}_{v-2}\}$-factorization for $v \geq 5$ and obtained the necessary and sufficient conditions for K_v^* to admit a $\{\vec{C}_m, \vec{C}_{v-m}\}$-

factorization for $2 \leq m \leq v-2$ and for odd v. Also they showed that if $v \geq 5$ and $v \equiv 1, 3, \text{or } 7 \pmod 8$, then K_v^* has a $\{\vec{C}_2, \vec{C}_2, \ldots, \vec{C}_2, \vec{C}_3\}$-factorization.

Suzan Kadri and Mateja Sajna [27] gave a recursive construction to solve the directed Oberwolfach problem with variable cycle lengths. Using this construction, they showed that OP* (m_1, m_2) has a solution for all $2 \leq m_1 \leq m_2$ except when $(m_1, m_2) = (3, 3)$ and except possibly when $m_1 \in \{4, 6\}, m_2$ is even, and $m_1 + m_2 \geq 14$.

Alice Lacaze-Masmonteil proved that K_{2m}^* can be factorized into $\vec{C}_m$ where m is odd [28]. With this result, the Directed Oberwolfach Problem with tables of uniform length is completely solved.

The following theorem summarizes the previous results on the Directed Oberwolfach Problem that will be used in this thesis.

Theorem 1.15: [20–23, 25, 28–30] Let m and k be nonnegative integers. Then, OP*(m^k) *has a solution if and only if* $(m, k) \notin \{(3, 2), (4, 1), (6, 1)\}$.

The directed Oberwolfach problem for complete symmetric equipartite digraphs and uniform-length cycles was solved by Nevena Francetić and Mateja Šajna in [31] with a few possible exceptions.

1.3. The Hamilton-Waterloo Problem (HWP)

The Hamilton-Waterloo Problem is a generalization of the Oberwolfach Problem where there are two conference venues (one in Hamilton and one in Waterloo as one may guess) with different seating arrangements. In one of these venues, there are r round tables with m_i seats for each table for $i = 1, \ldots, r$. In the other venue, there are s round tables with n_i seats for each table for $i = 1, \ldots, s$. Participants can choose to sit in one of these rooms on some days and the other on other days. The total number of participants per night is the sum of the table sizes ($\sum_{i=1}^{r} m_i = \sum_{i=1}^{s} n_i = v$). In this problem, similar to the Oberwolfach problem, the question is whether each of the v participants sits next to each other exactly once after $\frac{v-1}{2}$ nights.

This time each 2-factor can be isomorphic to one of the given two 2-factors, say F_1 or F_2. The problem revolves around the existence of a $\{F_1^r, F_2^s\}$-factorization of K_v (or $K_v - I$ in the spouse avoiding version for even v). If F_1 consists of only m-cycles (each $m_i = m$ for $i = 1, \ldots, r$) and F_2 consists of only n-cycles (each $n_i = n$ for $i = 1, \ldots, s$), then the corresponding Hamilton-Waterloo Problem is called as the uniform Hamilton-Waterloo Problem, and it is denoted by HWP$(v; m^r, n^s)$ where r and s are the number of C_m-factors and C_n-factors where $r + s = \frac{v-1}{2}$, respectively. Having a solution to HWP$(v; m^r, n^s)$ means that K_v has a $\{C_m^r, C_n^s\}$-factorization. Solving the problem completely is to have a solution for all possible r and s. When $r = 0$ or $s = 0$, this problem reduces to the uniform Oberwolfach Problem.

Example 1.16: A solution to HWP$(9; 3^2, 9^2)$ *exists (see Figure 1.17).*

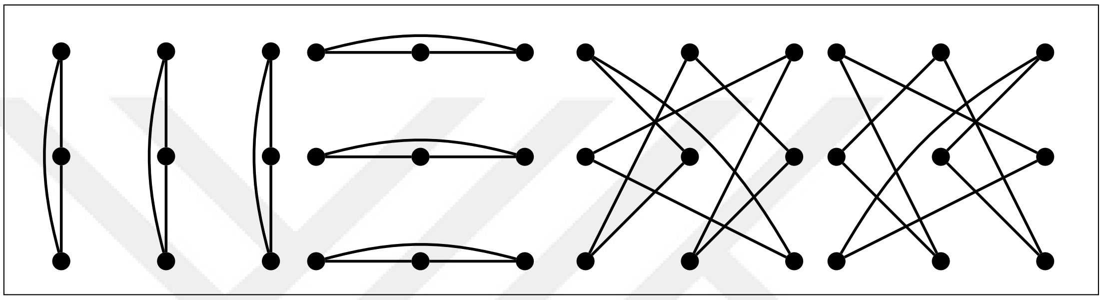


Figure 1.17: A $\{C_3^2, C_9^2\}$-factorization of K_9.

The necessary conditions for the existence of a solution to HWP$(v; m^r, n^s)$ are given by Adams et al. [32].

Lemma 1.17: [32] Let v, n, m, r and s be non-negative integers with $n, m \geq 3$. If there exists a solution to HWP$(v; m^r, n^s)$*, then*

1. *if $r > 0, v \equiv 0 \pmod{n}$,*
2. *if $s > 0, v \equiv 0 \pmod{m}$,*
3. $r + s = \lfloor \frac{v-1}{2} \rfloor$.

Initially, small cases such as $(m, n) \in \{(4, 6), (4, 8)$,$(4, 16), (8, 16), (3, 5), (3, 15), (5, 15)\}$ are studied and solved with a few exceptions by Adams et al. [32], and later the cases with variable cycle sizes are analyzed.

When m and n are odd, Burgess et al. [33, 34] solved the Hamilton-Waterloo problem for the cases where v is odd with some possible exceptions and proved the following theorem.

Theorem 1.18: [34] Let m, n and v be odd integers with $n \geq m \geq 3$, and let r and s be nonnegative integers. Then, HWP$(v; m^r, n^s)$ *has a solution if and only if $m, n | v$ and*

$r + s = \frac{v-1}{2}$, *except possibly when one of the following holds:*

1. $s = 1$;
2. $s = 3$ *and* $m \nmid n$;
3. $r = 1$, $m \nmid n$ *and* $mn \nmid v$;
4. $v = \frac{mn}{\gcd(m,n)}$.

In [17], the problem is solved in the case that both m and n are even and $v \equiv 0 \pmod 4$ except possibly when $r = 1$ or $s = 1$. When m and n are both even and $v \equiv 2 \pmod 4$, this problem is solved by R. Häggkvist in [18] whenever r and s are both even. Also, if m is even and $m|n$, the problem is completely solved in [35]. Burgess et al. gave an almost exact solution for $\mathrm{HWP}(v; m^r, n^s)$ when r and s are odd, with some possible exceptions [36], and summarized all results on this problem in the following theorem.

Theorem 1.19: [36] Let m *and* n *be integers with* $n > m \geq 2$, *and let* r *and* s *be positive integers.*

1. *If* $m|n$, *then there is a solution to* $\mathrm{HWP}(v; (2m)^r, (2n)^s)$ *if and only if* $2n|v$ *and* $r + s = \frac{v-2}{2}$.
2. *If* $m \nmid n$, *then there is a solution to* $\mathrm{HWP}(v; (2m)^r, (2n)^s)$ *if and only if* $2m$ *and* $2n$ *are both divisors of* v *and* $r + s = \frac{v-2}{2}$, *except possibly when at least one of the following holds:*
   (a) $s = 1$;
   (b) $s = 3, v \equiv 2 \pmod 4$ *and* $\gcd(m, n) = 1$;
   (c) $r = 1$ *and at least one of* m *or* n *is even;*
   (d) $r = 1, m$ *and* n *are odd, and either* $mn \nmid v$ *or* $v = 2mn$;
   (e) $v = \frac{2mn}{\gcd(m,n)} \equiv 2 \pmod 4$, *and* r *and* s *are odd.*

The situation in which m and n have different parities is the most challenging one. The case when $(m, n) = (3, 4)$ is completely solved [37–40]. When the parity of m and n is different, one of the cycle sizes is usually fixed. For instance, the cases $(m, n) \in \{(3, v), (3, 3x), (4, n)\}$ have been studied, see [40–43]. For more recent results on this problem, we refer the reader to [44–46].

Also, there exists an asymptotic solution (for sufficiently large v) [47] for the general form of the Oberwolfach and the Hamilton-Waterloo Problems. However, this asymptotic solution does not provide an explicit lower bound that guarantees the solvability of the problem. In [48], Traetta constructs solutions to the Oberwolfach Problem whenever F contains a cycle of length greater than an explicit lower bound.

1.3.1. The Directed Hamilton-Waterloo Problem (DHWP)

In the directed version of the Hamilton-Waterloo Problem, the question is whether v participants can each sit to the right of each other exactly once at the end of $v-1$ nights, given that the other conditions of the Hamilton-Waterloo Problem remain the same. Here K_v^* is decomposed into two types of directed 2-factors. If these factors consist of directed cycles of sizes m and n respectively, the notation $\mathrm{HWP}^*(v; m^r, n^s)$ is used to denote the Directed Uniform Hamilton-Waterloo Problem. Any of its solutions will be referred to as a $\{\vec{C}_m^r, \vec{C}_n^s\}$-factorization of K_v^*.

The necessary and sufficient conditions for the existence of a solution to the Directed Hamilton-Waterloo Problem can be given as follows:

Lemma 1.20: If $\mathrm{HWP}^*(v; m^r, n^s)$ *has a solution for nonnegative integers* r *and* s*, then the following statements hold:*

1. *if* $r > 0$*,* $v \equiv 0 \pmod{m}$*,*
2. *if* $s > 0$*,* $v \equiv 0 \pmod{n}$*,*
3. $r + s = v - 1$.

Example 1.21: There is a solution to $\mathrm{HWP}^*(8; 4^r, 8^s)$ *for* $r = 5$ *and* $s = 2$.

Proof. Let $r = 5$ and $s = 2$ be and $V(K_8^*) = \mathbb{Z}_8$. Define the directed 4-cycles as follows:

$$C_1^0 = (0,1,2,3) \qquad C_2^0 = (4,5,6,7)$$

$$C_1^1 = (0,3,2,1) \qquad C_2^1 = (4,7,6,5)$$

$$C_1^2 = (0,6,4,2) \qquad C_2^2 = (1,7,5,3)$$

$$C_1^3 = (0,2,4,6) \qquad C_2^3 = (1,3,5,7)$$

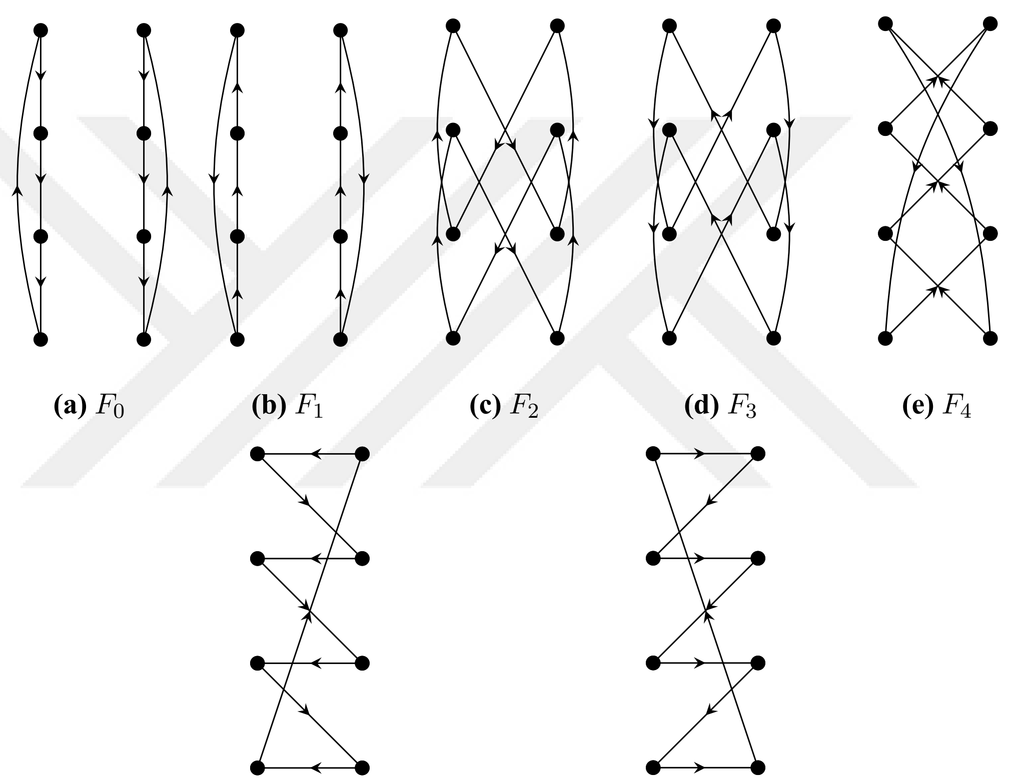
$$C_1^4 = (0,7,2,5) \qquad C_2^4 = (4,3,6,1),$$

and the directed 8-cycles as,

$$C_5 = (0, 5, 1, 6, 2, 7, 3, 4)$$

$$C_6 = (0, 4, 1, 5, 2, 6, 3, 7).$$

Note that each $F_i = \{C_1^i \cup C_2^i : i \in \mathbb{Z}_5\}$ is a $\overrightarrow{C}_4$-factors of K_8^* and also $C_5 = F_5$ and $C_6 = F_6$ are both $\overrightarrow{C}_8$-factors of K_8^*. So, $\mathcal{F} = \{F_0, F_1, F_2, F_3, F_4, F_5, F_6\}$ is a 2-factorization of K_8^*. This 2-factorization gives a solution to HWP$^*(8; 4^5, 8^2)$. $\square$



(a) F_0 **(b)** F_1 **(c)** F_2 **(d)** F_3 **(e)** F_4

(f) F_5 **(g)** F_6

Figure 1.18: A $\{\overrightarrow{C}_4^5, \overrightarrow{C}_8^2\}$-factorization of K_8^*.

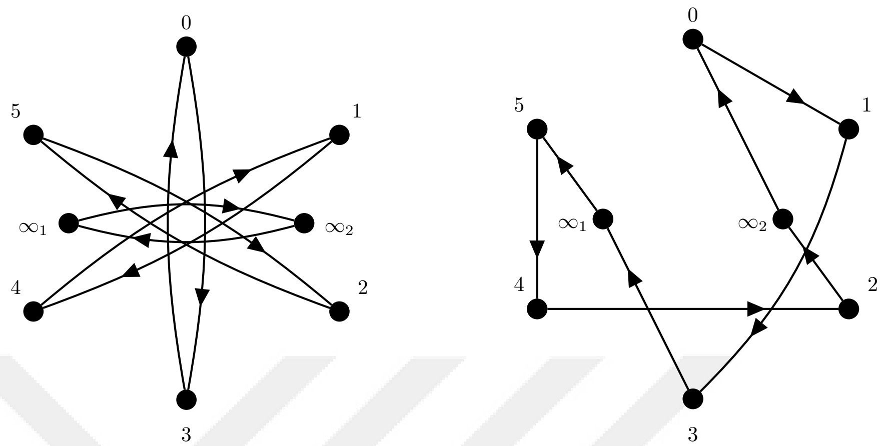
Since K_2^* can be considered as $\overrightarrow{C}_2$, the problem where one of the directed 2-factors is the K_2^*-factor can be included in the HWP$^*(v; 2^r, m^s)$.

Example 1.22: There exists a solution to HWP$^*(8; 2^r, 8^s)$ *for* $r \in \{1, 3\}$ *with* $r + s = 7$.

Proof. **Case 1** $(\boldsymbol{r = 1})$ **:** K_8^* can be represented as $K_6^* \vee K_2^*$, where the vertex set of K_2^* is $\{\infty_1, \infty_2\}$, and K_6^* can be represented as the circulant digraph $\overrightarrow{X}(\mathbb{Z}_6; S)$ with connection set $S = \{\pm 1, \pm 2, 3\}$. Let us define the permutation as $\rho = (0, 1, 2, 3, 4, 5)\,(\infty_1)$

(∞_2).

Define $F_0 = \{(\infty_1, \infty_2)^*, (0,3)^*, (1,4)^*, (2,5)^*\}$ factor of K_8^* as K_2^*-factor. Also, we define the directed 8-cycle as, $\overrightarrow{C} = (0, 1, 3, \infty_1, 5, 4, 2, \infty_2)$.



(a) Factor F_0 of K_8^* **(b)** The directed 8-cycle $\overrightarrow{C}$

Figure 1.19: Desired 2-factors of K_8^*.

Note that $\overrightarrow{C}$ is a $\overrightarrow{C}_8$-factor of K_8^*. For $i \in \mathbb{Z}_6$ and $j \in \{1, 2\}$, the arcs (i, ∞_j) and (∞_j, i) have differences of $+\infty_j$ and $-\infty_j$, respectively. Also, the difference of the arcs (∞_1, ∞_2) and (∞_2, ∞_1) remains undefined. So, it contains exactly one arc of each difference in the set $\{\pm 1, \pm 2, 3, \pm\infty_1, \pm\infty_2\}$.

Each element of $F_{i+1} = \{\rho^i(C) : i \in \mathbb{Z}_6\}$ is a $\overrightarrow{C}_8$-factor of K_8^*. Finally $\{F_0, F_1, F_2, F_3, F_4, F_5, F_6\}$ is a $\{(K_2^*)^1, \overrightarrow{C}_8^6\}$-factorization of K_8^*. Therefore, HWP$^*(8; 2^1, 8^6)$ have a solution.

Case 2 $(r = 3)$: Define

$$V(K_8^*) = \mathbb{Z}_4 \bigcup \{\infty_1, \infty_2, \infty_3, \infty_4\}$$

and

$$\overrightarrow{C} = (0, \infty_1, 3, \infty_4, 2, \infty_3, 1, \infty_2)$$

and

$$F_0 = \overrightarrow{C}, \quad F_i = \left\{\rho^i(F_0) : i \in \mathbb{Z}_4\right\}$$

where $\rho = (0, 1, 2, 3)\,(\infty_1)\,(\infty_2)\,(\infty_3)\,(\infty_4)$. Each F_i is $\overrightarrow{C}_8$-factor of K_8^*.

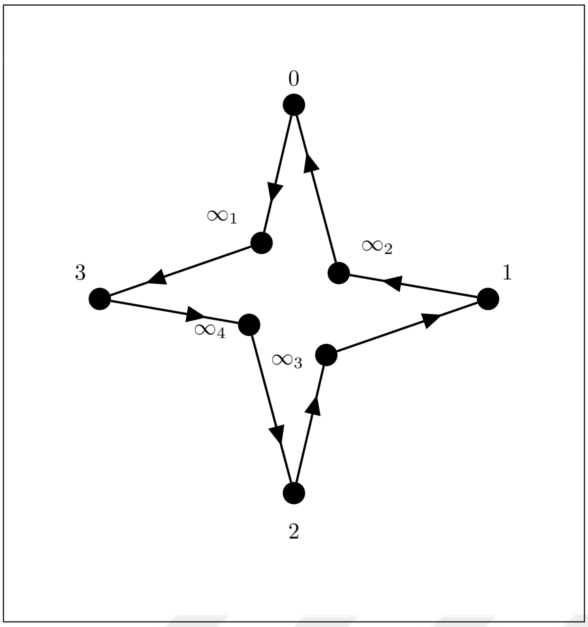


Figure 1.20: The factor F_0 of K_8^*.

Also, we define three K_2^*-factors: $F_4 = \{(\infty_1, \infty_2)^*, (\infty_3, \infty_4)^*, (0,3)^*, (1,2)^*\}$, $F_5 = \{(\infty_1, \infty_3)^*, (\infty_2, \infty_4)^*, (0,1)^*, (2,3)^*\}$, $F_6 = \{(\infty_1, \infty_4)^*, (\infty_2, \infty_3)^*, (0, 2)^*, (1,3)^*\}$. In Figure 1.21, K_2^*-factors of K_8^* are given. So, $\{F_0, F_1, F_2, F_3, F_4, F_5, F_6\}$ gives a $\{(K_2^*)^3, \overrightarrow{C}_8^4\}$-factorization of K_8^*. As a result, there is a solution for HWP*$(8; 2^3, 8^4)$. □

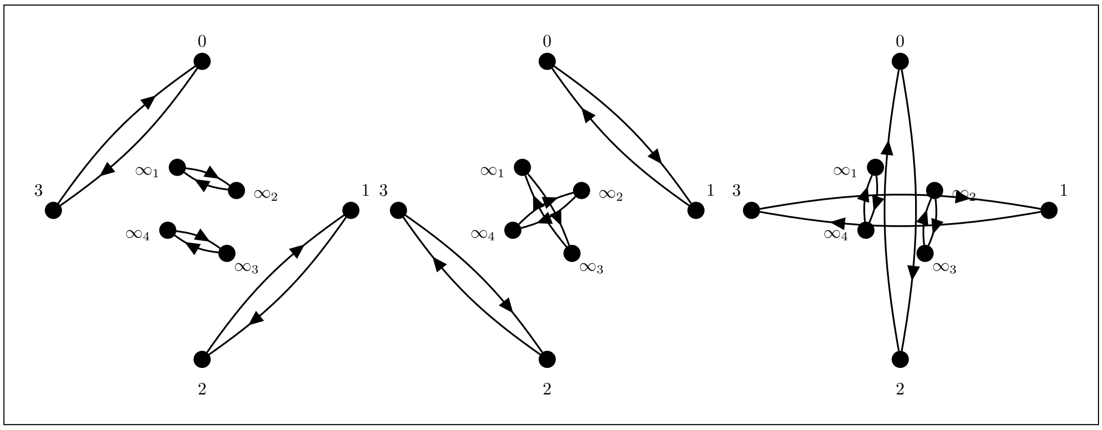


Figure 1.21: Desired K_2^*-factors of K_8^*.

2. PRELIMINARY RESULTS

In this section, both previous and new findings that serve as important results to advance the main conclusions of this thesis will be explained. Let us first start with the observation that is useful to reduce the number of cases when v is odd, and move to the preliminary results then. Given that there are no 2-factorizations of K_v for even v, we are unable to draw a similar conclusion when v is even and $m, n > 2$.

Observation 2.1: If $\mathrm{HWP}(v; m^r, n^s)$ *has a solution for some* r *and* s *, and* v *is odd, then* $\mathrm{HWP}^*(v; m^{2r}, n^{2s})$ *has a solution for the same* r *and* s*.*

A solution for $\mathrm{HWP}^*(v; m^{2r}, n^{2s})$ is obtained from a solution of $\mathrm{HWP}(v; m^r, n^s)$ by taking two copies of each 2-factor and replacing each edge $\{x, y\}$ with the arcs (x, y) and (y, x) in the two 2-factors. The following corollary is a consequence of this observation and Theorem 1.18.

Corollary 2.2: Let m, n*, and* v *be odd integers with* $n \geq m \geq 3$*, and let* r *and* s *be nonnegative integers. Then,* $\mathrm{HWP}^*(v; m^{2r}, n^{2s})$ *has a solution if and only if* m*,* $n|v$ *and* $r + s = v - 1$*, except possibly when one of the following holds:*

1. $s = 1$*;*
2. $s = 3$ *and* $m \nmid n$*;*
3. $r = 1$*,* $m \nmid n$ *and* $mn \nmid v$*;*
4. $v = \frac{mn}{\gcd(m,n)}$.

Similarly, an H^*-factorization of G^* is obtained from an H-factorization of G. The following proposition is useful for transferring the results of undirected graphs to symmetric digraphs. It states that if there is an H-factorization of the undirected graph G, then using this factorization one can get an H^*-factorization of G^*.

Proposition 2.3: Let G *be a graph and* H *be a subgraph of* G*. If* G *has an* H*-factorization, then* G^* *has an* H^**-factorization.*

It is known that K_{2x} has a 1-factorization [49]. Therefore, as a natural consequence of Proposition 2.3, the following proposition can be stated.

Proposition 2.4: The complete symmetric digraph K_{2x}^* *has a* K_2^**-factorization for every* *integer* $x \geq 1$.

The following Lemma 2.5 and Theorem 2.6 play a crucial role in the factorization of complete symmetric equipartite digraphs $K^*_{(x:y)}$ into directed m-cycles where $m|xy$.

Lemma 2.5: [20] Let $m \geq 4$ *be an even integer and* x *be a positive integer. Then,* $K^*_{(\frac{mx}{2}:2)}$ *has a* $\vec{C}_m$*-factorization.*

Theorem 2.6: [50] The complete symmetric equipartite digraph $K^*_{(x:y)}$ *has a* $\vec{C}_3$*-factorization if and only if* $3|xy$ *and* $(x, y) \neq (1, 6)$ *with possible exceptions* $(x, y) = (x, 6)$*, where* $x \notin \{x : x$ *is divisible by a prime less than* $17\}$.

Theorem 1.14 states that $K_{(x:y)}$ has a C_m-factorization with a few exceptions. We can use this result to show that $K^*_{(x:y)}$ has a $\vec{C}_m$-factorization. However, some of the exceptions in the undirected version do not exist in the symmetric case. A solution for these exceptions is provided. As a corollary of Lemma 2.5, Proposition 2.3, and Theorems 1.14 and 2.6, the following result can be given.

Lemma 2.7: The complete symmetric equipartite digraph $K^*_{(x:y)}$ *has a* $\vec{C}_m$*-factorization for* $m \geq 3$ *and* $x \geq 2$ *if* $m|xy$*,* $x(y-1)$ *is even,* m *is even when* $y = 2$*.*

Proof. Let $m|xy$, $x(y-1)$ be even, m be even when $y = 2$, and let $(x, y, m) \notin \{(2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)\}$. By Theorem 1.14, $K_{(x:y)}$ has a C_m-factorization and so, $K^*_{(x:y)}$ has a C^*_m-factorization by Proposition 2.3. Since each C^*_m has a $\vec{C}_m$-factorization, $K^*_{(x:y)}$ has a $\vec{C}_m$-factorization. Since $K^*_{(x:y)}$ has a $\vec{C}_m$-factorization for $(x, y, m) \in \{(2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)\}$ by Theorem 2.6 and Lemma 2.5, it is concluded that $K^*_{(x:y)}$ has a $\vec{C}_m$-factorization for $m \geq 3$ and $x \geq 2$ if $m|xy$, $x(y-1)$ is even, m is even when $y = 2$. □

Nevena Francetić and Mateja Šajna [31] provide the necessary and sufficient conditions for the existence of a $\vec{C}_m$-factorization of $K^*_{(x:y)}$. Furthermore, their result states the existence of a solution to the directed Oberwolfach problem for complete symmetric equipartite digraphs with uniform-length cycles.

Theorem 2.8: [31] Let x, y*, and* m *be integers greater than* 1*, and let* $g = \gcd(y, m)$*. Assume one of the following conditions holds:*

1. $x(y-1)$ *is even; or*

2. $g \notin \{1,3\}$; or

3. $g = 1$, and $y \equiv 0 \pmod 4$ or $y \equiv 0 \pmod 6$; or

4. $g = 3$, and if $y = 6$, then x is divisible by a prime $p \leq 37$.

Then the complete symmetric equipartite digraph $K^*_{(x:y)}$ has a $\vec{C}_m$-factorization if and only if $m|xy$ and m is even in case $y = 2$.

The necessary and sufficient condition for the existence of a 1-factorization of a complete equipartite graph $K_{(x:y)}$ is given by Hoffman and Rodger [51].

Theorem 2.9: [51] The complete equipartite graph $K_{(x:y)}$ has a 1-factorization if and only if xy is even.

The following lemma is a straightforward consequence of Proposition 2.3 and Theorem 2.9.

Lemma 2.10: The complete symmetric equipartite digraph $K^*_{(x:y)}$ has a K^*_2-factorization if and only if xy is even.

Häggkvist used $G[2]$ to build 2-factorizations that include even cycles [18]. Using Häggkvist Lemma, we assert that $C_m[2]$ can be decomposed into two C_{2m}-factors. This decomposition will be used to factorize $C^*_m[2]$ into K^*_2 factors and $\vec{C}_{2m}$ factors.

Lemma 2.11 (Häggkvist Lemma): Let G be a path or a cycle with n edges and let H be a 2-regular graph on $2n$ vertices with all components even cycle. Then $G[2] \cong G' \oplus G''$ where $G' \cong G'' \cong H$. Therefore, $G[2]$ has an H-decomposition.

The following two well-known results of Walecki imply that K_m (or, $K_m - F_m$ where F_m is a 1-factor of K_m defined in Lemma 2.13 and m is even) decomposes into Hamilton cycles. These results and Proposition 2.3 will be used to factorize K^*_m into symmetric Hamilton cycles in Section 4.

Lemma 2.12: [52] For all odd $m \geq 3$, K_m decomposes into $\left(\frac{m-1}{2}\right)$ Hamilton cycles.

Lemma 2.13: [52] Let F_m be a 1-factor of K_m with edge set $E(F_m) = \{\{0, m/2\}, \{i, m-i\} : 1 \leq i \leq (m/2)-1\}$. For all even $m \geq 4, K_m - F_m$ has an Hamilton cycle decomposition with prescribed cycles $\{\mathcal{C}, \sigma(\mathcal{C}), \sigma^2(\mathcal{C}) \ldots, \sigma^{\frac{m-4}{2}}(\mathcal{C})\}$ for $\sigma = (0)(1,2,3\ldots,m-2,m-1)$ where $\mathcal{C} = (0,1,2,m-1,3,m-2,\ldots,\frac{m}{2}-1,\frac{m}{2}+2, \frac{m}{2},\frac{m}{2}+1)$.

By Lemma 2.12, the decomposition of K_n for odd n is as follows:

$$K_n \cong \underbrace{C_n \oplus C_n \oplus C_n \oplus \ldots \oplus C_n}_{\frac{m-1}{2}} \tag{2.1}$$

Using by (2.1) and Proposition 2.3, we get $K_n^* \cong C_n^* \oplus C_n^* \oplus C_n^* \oplus \ldots \oplus C_n^*$.

By Lemma 2.13, K_m can decomposed as follows:

$$K_m \cong \underbrace{\mathcal{C} \oplus \mathcal{C} \oplus \mathcal{C} \oplus \ldots \oplus \mathcal{C}}_{\frac{m-2}{2}} \oplus F_m \tag{2.2}$$

The decomposition of K_m^* can be obtained by (2.2) and Proposition 2.3.

$$K_m^* \cong \mathcal{C}^* \oplus \mathcal{C}^* \oplus \mathcal{C}^* \oplus \ldots \oplus \mathcal{C}^* \oplus F_m^* \tag{2.3}$$

Lemmata 2.14 and 2.15 show the existence of the $\{C_m^r, C_{2m}^s\}$-factorization of the $C_m[2]$ and $(\mathcal{C} \oplus F_m)[2]$ for $r + s = 2$ and $r + s = 3$, respectively. They will be used to find a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization of the $C_m^*[2]$ for $r \in \{0, 2, 4\}$, $r + s = 4$ and a $\vec{C}_{2m}$-factorization of $\mathcal{C}^*[2] \oplus F_m^*[2]$ where $\mathcal{C}^*$ is the symmetric digraph obtained from $\mathcal{C}$, defined as in Lemma 2.13. Also, we use Γ_m and Γ_m^* to denote $\mathcal{C}[2] \oplus F_m[2]$ and $\mathcal{C}^*[2] \oplus F_m^*[2]$, respectively, for the rest of the thesis.

Lemma 2.14: [53] Let m be an integer with $m \geq 3$. Then $C_m[2]$ has a $\{C_m^r, C_{2m}^s\}$-factorization for nonnegative integers r and s with $r + s = 2$ except when m is odd and $r = 2$, and except possibly when m is even and $r = 1$.

Lemma 2.15: [53] Let $m \geq 4$ be an even integer and $\Gamma_m = \mathcal{C}[2] \oplus F_m[2]$ where the m-cycle $\mathcal{C}$ and the 1-factor F_m are the same as those defined in Lemma 2.13. Then Γ_m has a

1. *C_{2m}-factorization,*
2. *C_m-factorization when $m \equiv 0 \pmod 4$, and*
3. *$\{C_m^2, C_{2m}^1\}$-factorization when $m \equiv 2 \pmod 4$.*

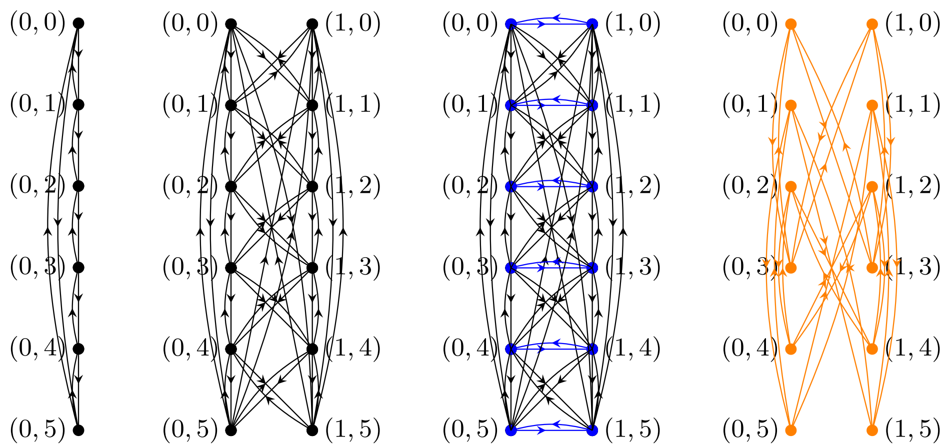
Let m be an even integer and the vertex set of K_{2m}^* be $\mathbb{Z}_{2m}$. Let I_{2m}^* be a K_2^*-factor of K_{2m}^* with $A(I_{2m}^*) = \{(i, m+i)^* : 0 \leq i \leq m-1\}$ and define the bijective function $f : \mathbb{Z}_{2m} \to \mathbb{Z}_2 \times \mathbb{Z}_m$ with

$$f(i) = \begin{cases} (0, i) & \text{if } i < m, \\ (1, i) & \text{if } i \geq m. \end{cases}$$

Then, $A(I_{2m}^*)$ can be restated as a set $\left\{((0,i),(1,i))^* : 0 \le i \le m-1\right\}$ on $\mathbb{Z}_2 \times \mathbb{Z}_m$ using this bijective function. We will represent $C_m^*[2]$ and $C_m^*[2] \oplus I_{2m}^*$ as the directed Cayley graphs $\vec{X}(\mathbb{Z}_2 \times \mathbb{Z}_m, S_1)$ and $\vec{X}(\mathbb{Z}_2 \times \mathbb{Z}_m,\, S_1 \cup \{(1,0)\})$, respectively, where $S_1 = \{(0,1),(1,1),(0,-1),(1,-1)\}$. Similarly, the graphs $\vec{C}_m[2]$ and $\vec{C}_m[2] \oplus I_{2m}^*$ are isomorphic to $\vec{X}(\mathbb{Z}_2 \times \mathbb{Z}_m, S_2)$ and $\vec{X}(\mathbb{Z}_2 \times \mathbb{Z}_m,\, S_2 \cup \{(1,0)\})$ where $S_2 = \{(0,1),(1,1)\}$, respectively.

Also, a factor F_m^* is defined as a K_2^*-factor of K_m^* with $A(F_m^*) = \{(0,m/2)^*, (i,m-i)^* : 1 \le i \le (m/2)-1\}$. The arc set of F_m^* which is denoted by $A(F_m^*)$, can be expressed as $\left\{((0,0),(0,m/2))^*, ((0,i),(0,m-i))^* : 1 \le i \le (m/2)-1\right\}$ using the above bijective function. Thus, the vertex set and the arc set of Γ_m^* can be represented as $V(\Gamma_m^*) = \mathbb{Z}_2 \times \mathbb{Z}_m$ and $A(\Gamma_m^*) = \bigcup_{j=0}^{m-1} \left\{((i,j),(i,j+1))^*, ((i,j),(i+1,j+1))^*\right\} \cup A(F_m^*)$ for $i = 0, 1$, respectively.

Example 2.16: C_6^*, $C_6^*[2]$, $C_6^*[2] \oplus I_{12}^*$ and $F_6^*[2]$ can be drawn as follows. Consider the graph $C_6^*[2] \oplus I_{12}^*$, in this graph the vertical arcs use the differences $(0,1)$ *and* $(0,-1)$. *The diagonal arcs use with the differences* $(1,1)$ *and* $(1,-1)$, *while the horizontal arcs employ the difference* $(1,0)$.



(a) C_6^* **(b)** $C_6^*[2]$ **(c)** $C_6^*[2] \oplus I_{12}^*$ **(d)** $F_6^*[2]$

Figure 2.1: Drawing some specific graphs and factors of K_{12}^*.

Using Lemmata 2.12 and 2.13, $K_{\frac{m}{2}}$ and $K_{\frac{m}{2}} - F_{\frac{m}{2}}$ factorize into $(\frac{m-2}{4})C_{\frac{m}{2}}$ cycles and $(\frac{m-4}{4})C_{\frac{m}{2}}$ cycles, respectively. Also, K_m is isomorphic to $K_{\frac{m}{2}}[2] \oplus I_m$. Hence, using Proposition 2.3, Lemmata 2.12 and 2.13, we will obtain a $\{(C_{\frac{m}{2}}^*[2])^{\frac{m-6}{4}}, C_{\frac{m}{2}}^*[2] \oplus I_m^*\}$-

factorization and a $\{(C^*_{\frac{m}{2}}[2])^{\frac{m-12}{4}}, C^*_{\frac{m}{2}}[2] \oplus I^*_m, \Gamma^*_{\frac{m}{2}}\}$-factorization of K^*_m depending on whether $m \equiv 0 \text{ or } 2 \pmod 4$, then use these factorizations to obtain a $\{(K^*_2)^r, \vec{C}^s_m\}$-factorization of K^*_{mx}. Thus, $C^*_{\frac{m}{2}}[2] \oplus I^*_m$ and $\Gamma^*_{\frac{m}{2}}$ must be factorized into K^*_2-factors and $\vec{C}_m$-factors. Furthermore, a $\{(K^*_2)^r, \vec{C}^s_{2m}\}$-factorization of $C^*_m[2]$ will need to be obtained in order to factorize K^*_{mx} into K^*_2-factors and $\vec{C}_m$-factors.

Lemma 2.17: Let $m \geq 4$ be an integer. Then $C^*_m[2]$ has a $\{(K^*_2)^r, \vec{C}^s_{2m}\}$-factorization for $r \in \{0, 2, 4\}$ and $r + s = 4$.

Proof. First, note that $C_m[2]$ has a decomposition into two C_{2m}-factors by Häggkvist Lemma (Lemma 2.11) and each C_{2m}-factor has a decomposition into two 1-factors, since $2m$ is even. $C_m[2] \cong C_{2m} \oplus C_{2m}$ and $C_{2m} \cong I \oplus I$ where I is a 1-factor of K_{2m}.

Case 1 $(r = 4)$: Decompose $C_m[2]$ into four 1-factors by using C_{2m}-factors. Then a K^*_2-factorization of $C^*_m[2]$ is obtained by Proposition 2.3.

Case 2 $(r = 2)$: Decompose $C_m[2]$ into one C_{2m} and two 1-factors. By Proposition 2.3, a $\{(K^*_2)^2, C^*_{2m}\}$-factorization of $C^*_m[2]$ is derived. Since C^*_{2m} has a $\vec{C}_{2m}$-factorization with two $\vec{C}_{2m}$-factors, a $\{(K^*_2)^2, \vec{C}^2_{2m}\}$-factorization of $C^*_m[2]$ is obtained.

Case 3 $(r = 0)$: Obtain a C^*_{2m}-factorization of $C^*_m[2]$ by Proposition 2.3. Since C^*_{2m} has a $\vec{C}_{2m}$-factorization with two $\vec{C}_{2m}$-factors, $C^*_m[2]$ has a $\vec{C}_{2m}$-factorization. □

Since I^*_{2m} and F^*_m are two K^*_2-factors of K^*_{2m}, the following result can be derived from Lemmata 2.15 and 2.17, and Proposition 2.3.

Corollary 2.18: Let $m \geq 4$ be an even integer. Then Γ^*_m has a $\{(K^*_2)^r, \vec{C}^s_{2m}\}$-factorization for $r \in \{0, 2, 4, 6\}$ and $r + s = 6$.

Proof. $F^*_m[2]$ decomposes into two K^*_2-factors. Therefore, Γ^*_m has a $\{(K^*_2)^r, \vec{C}^s_{2m}\}$-factorization for $r \in \{2, 4, 6\}$ with $r + s = 6$ by Lemma 2.17. Also, Γ^*_m has a $\vec{C}_{2m}$-factorization by Lemma 2.15 and Proposition 2.3. □

The following lemma is quite useful in solving the Directed Hamilton-Waterloo Problem for $n = 2$ and even m, when r is even.

Lemma 2.19: Let $m \geq 5$ be an integer. Then $C^*_m[2] \oplus I^*_{2m}$ has a $\{(K^*_2)^r, \vec{C}^s_{2m}\}$-factorization for $r \in \{0, 1, 3, 5\}$ and $r + s = 5$.

Proof. The cases $r \in \{1, 3, 5\}$ can be directly obtained from Lemma 2.17.

When $r = 0$, we will study the problem in two cases; m is odd or even.

Case 1 (odd $m \geq 5$) : Define five directed $2m$-cycles in $C_m^*[2] \oplus I_{2m}^*$ as follows:

$\vec{C}_{2m}^{(0)} = (v_0, v_1, \ldots, v_{2m-1})$ where $v_i = (\lfloor \frac{i}{m} \rfloor, i)$,

$\vec{C}_{2m}^{(1)} = (u_0, u_1, \ldots, u_{2m-1})$ where,

$$u_{2i} = \begin{cases} (0, 2i) & \text{if } 0 \leq i \leq \frac{m-1}{2}, \\ (0, -2i-1) & \text{if } \frac{m+1}{2} \leq i \leq m-1, \end{cases}$$

and

$$u_{2i+1} = \begin{cases} (1, 2i+1) & \text{if } 0 \leq i \leq \frac{m-3}{2}, \\ (1, -2i-2) & \text{if } \frac{m-1}{2} \leq i \leq m-1, \end{cases}$$

$C_{2m}^{(2)} = (x_{0,}, x_1, \ldots, x_{2m-1})$ where

$$x_i = \begin{cases} (0, m - \lfloor \frac{i}{2} \rfloor) & \text{if } i \equiv 0, 3 \pmod 4, \\ (1, m - \lfloor \frac{i}{2} \rfloor) & \text{if } i \equiv 1, 2 \pmod 4, \end{cases} \quad \text{for } 0 \leq i \leq 2m-3,$$

and $x_{2m-2} = (1, 1)$, $x_{2m-1} = (0, 1)$. Also, $C_{2m}^{(3)} = (y_{0,}, y_1, \ldots, y_{2m-1})$ where

$$y_i = u_i + (1, 2) \text{ for } 0 \leq i \leq m-3 \textit{ and } m+2 \leq i \leq 2m-1,$$

$$y_{m-2} = (1, 0), \; y_{m-1} = (0, 1), \; y_m = (1, 1), \; y_{m+1} = (0, 0).$$

Finally, $\vec{C}_{2m}^{(4)} = (z_0, z_1, \ldots, z_{2m-1})$ where

$$z_i = x_i + (1, 0) \text{ for } 3 \leq i \leq 2m-4,$$

$$z_0 = (2, 0), \; z_1 = (0, m), \; z_2 = (0, 0), \; z_{2m-1} = (1, 1), \; z_{2m-2} = (1, 2), \; z_{2m-3} = (0, 1).$$

Then, $\{\vec{C}_{2m}^{(0)}, \vec{C}_{2m}^{(1)}, \vec{C}_{2m}^{(2)}, \vec{C}_{2m}^{(3)}, \vec{C}_{2m}^{(4)}\}$ is a $\vec{C}_{2m}$-factorization of $C_m^*[2] \oplus I_{2m}^*$.

Case 2 (even $m \geq 6$) : Let $\vec{C}_{2m}^{(0)}$ be the same as in Case 1 and define the directed $2m$-cycles in $C_m^*[2] \oplus I_{2m}^*$ as follows:

$\vec{C}_{2m}^{(1)} = (x_0, x_1, \ldots, x_{2m-1})$ where $x_0 = (0, 0)$ and

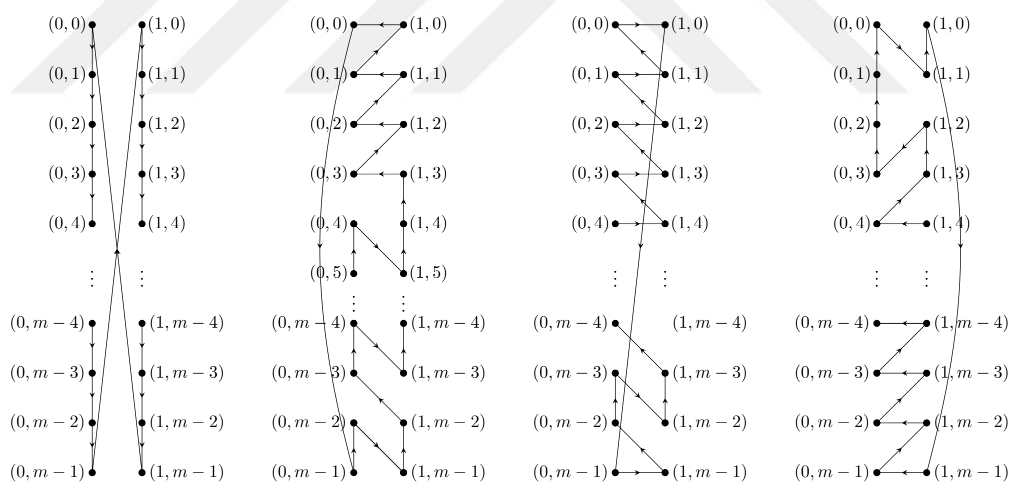
$$x_i = \begin{cases} (0, m - \lfloor \frac{i+2}{2} \rfloor) & \text{if } i \equiv 1, 2 \pmod 4, \\ (1, m - \lfloor \frac{i+2}{2} \rfloor + 1) & \text{if } i \equiv 0, 3 \pmod 4, \end{cases} \quad \text{for } 1 \leq i \leq 2m-8,$$

and $x_{2m-6+2i} = (0, 3-i)$ for $0 \le i \le 2$ and $x_{2m-7+2i} = (1, 3-i)$ for $0 \le i \le 3$. Also, $\vec{C}_{2m}^{(2)} = (u_0, u_1, \ldots, u_{2m-1})$ where $u_0 = (0,0)$, $u_1 = (1,0)$, $u_2 = (0, m-1)$ and

$$u_i = \begin{cases} \left(0, m - \lfloor \frac{i-1}{2} \rfloor - 1\right) & \text{if } i \equiv 0,1 \pmod 4, \\ \left(1, m - \lfloor \frac{i-1}{2} \rfloor\right) & \text{if } i \equiv 2,3 \pmod 4, \end{cases} \quad \text{for } 3 \le i \le 2m-9,$$

$$u_{2m-8+j} = \begin{cases} (0, 4 - \lfloor \frac{j}{2} \rfloor) & \text{if } j \equiv 0,2 \pmod 4, \\ (1, 4 - \lfloor \frac{j}{2} \rfloor) & \text{if } j \equiv 1,3 \pmod 4, \end{cases} \quad \text{for } 0 \le j \le 7, \text{and when } m = 6,$$

$u_3 = (1,5)$ and only the above piecewise function is used. Furthermore, $\vec{C}_{2m}^{(3)} = (y_0, y_1, \ldots, y_{2m-1})$ where $y_{2i+2} = (0, m-i)$ for $1 \le i \le m-4$, $y_{2i+1} = (1, m-i)$ for $1 \le i \le m-3$, $y_0 = (0,0)$, $y_1 = (1,1)$, $y_2 = (1,0)$, $y_{2m-4} = (1,2)$, $y_{2m-3} = (0,3)$, $y_{2m-2} = (0,2)$ and $y_{2m-1} = (0,1)$. Finally, $\vec{C}_{2m}^{(4)} = (z_0, z_1 \ldots, z_{2m-1})$ where $z_{9+2i} = (0, 4+i)$ for $1 \le i \le m-5$, $z_{10+2i} = (1, 4+i)$ for $0 \le i \le m-6$, $z_0 = (0,0)$, $z_1 = (1, m-1)$, $z_2 = (1,0)$, $z_3 = (0,1)$, $z_4 = (1,2)$, $z_5 = (1,1)$, $z_6 = (0,2)$, $z_7 = (1,3)$, $z_8 = (0,4)$, $z_9 = (0,3)$. Then $\left\{\vec{C}_{2m}^{(0)}, \vec{C}_{2m}^{(1)}, \vec{C}_{2m}^{(2)}, \vec{C}_{2m}^{(3)}, \vec{C}_{2m}^{(4)}\right\}$ is a $\vec{C}_{2m}$-factorization of $C_m^*[2] \oplus I_{2m}^*$. □



(a) $\vec{C}_{2m}^{(0)}$ **(b)** $\vec{C}_{2m}^{(1)}$ **(c)** $\vec{C}_{2m}^{(2)}$ **(d)** $\vec{C}_{2m}^{(3)}$

Figure 2.2: Four $2m$-cycles of $C_m^*[2] \oplus I_{2m}^*$.

By Lemma 2.14, $C_m[2]$ can be decomposed into two C_m-factors for even m. So, the following lemma is obtained similar to Lemma 2.17. Also, the following corollaries are obtained as a result of this lemma.

Lemma 2.20: Let $m \ge 4$ be an even integer. Then $C_m^*[2]$ has a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization for $r \in \{0,2,4\}$ and $r+s=4$.

Proof. Notice that $C_m[2]$ can be decomposed into two C_m-factors, as stated in Lemma 2.14. Moreover, each C_m-factor has a decomposition into two 1-factors, given that m is even.

Case 1 ($r = 4$) : Decompose $C_m[2]$ into four 1-factors by using C_m-factors. Then a K_2^*-factorization of $C_m^*[2]$ is obtained by Proposition 2.3.

Case 2 ($r = 2$) : Decompose $C_m[2]$ into one C_m and two 1-factors. By using Proposition 2.3, a $\{(K_2^*)^2, C_m^*\}$-factorization of $C_m^*[2]$ is obtained and also C_m^* has a $\vec{C}_m$-factorization with two $\vec{C}_m$-factors. So, we get a $\{(K_2^*)^2, \vec{C}_m^2\}$-factorization of $C_m^*[2]$.

Case 3 ($r = 0$) : Obtain a C_m^*-factorization of $C_m^*[2]$ by Proposition 2.3. Since C_m^* has a $\vec{C}_m$-factorization with two $\vec{C}_m$-factors, $C_m^*[2]$ has a $\vec{C}_m$-factorization. □

Because both I_{2m}^* and F_m^* are K_2^*-factors, the following corollaries can be derived from Lemma 2.20.

Corollary 2.21: Let $m \geq 4$ be an even integer. Then $C_m^*[2] \oplus I_{2m}^*$ has a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization for $r \in \{1, 3, 5\}$ and $r + s = 5$.

Corollary 2.22: Let $m \geq 4$ be an even integer. Then Γ_m^* has a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization for $r \in \{2, 4, 6\}$ and $r + s = 6$.

In this thesis, mainly recursive constructions are used to solve the Directed Hamilton-Waterloo Problem. To do this, graphs like K_{mx}^* and K_m^* are factorized into some special smaller graphs such as $C_m^*[2]$, $\vec{C}_m[2]$, $C_m^*[2] \oplus I_{2m}^*$, $\vec{C}_m[2] \oplus I_{2m}^*$ and similar ones, which are easier to factorize needed factors compared to graphs like K_{mx}^* and K_m^*. Therefore, $\vec{C}_m[2]$ and $\vec{C}_m[2] \oplus I_{2m}^*$ must be decomposed into the desired factors.

Lemma 2.23: For every even integer $m \geq 2$, $\vec{C}_m[2]$ has a $\vec{C}_m$-factorization and a $\vec{C}_{2m}$-factorization.

Proof. Let $m \geq 2$ be an integer. Let $\vec{C}_m^{(0)} = (v_0, v_1, \ldots, v_{m-1})$ be a directed m-cycle of $\vec{C}_m[2]$, where $v_i = (0, i)$ for $0 \leq i \leq m - 1$, and it can be checked that $F_1 = \vec{C}_m^{(0)} \cup (\vec{C}_m^{(0)} + (1, 0))$ is a directed m-cycle factor of $\vec{C}_m[2]$. Also, let $\vec{C}_m^{(1)} =$

$(u_0, u_1, \ldots, u_{m-1})$ be a directed m-cycle of $\vec{C}_m[2]$, where

$$u_i = \begin{cases} (0,i) & \text{if } i \text{ is even}, \\ (1,i) & \text{if } i \text{ is odd}, \end{cases}$$

for $0 \leq i \leq m-1$. It can also be checked that $F_2 = \vec{C}_m^{(1)} \cup (\vec{C}_m^{(1)} + (1,0))$ is a directed m-cycle factor of $\vec{C}_m[2]$. F_1 and F_2 are arc disjoint directed m-cycle factors of $\vec{C}_m[2]$. Thus $\{F_1, F_2\}$ is a $\vec{C}_m$-factorization of $\vec{C}_m[2]$.

Let $\vec{C}_{2m}^{(2)} = (v_0, v_1, \ldots, v_{2m-1})$ and $\vec{C}_{2m}^{(3)} = (w_0, w_1, \ldots, w_{2m-1})$ be directed $2m$-cycles of $\vec{C}_m[2]$, where

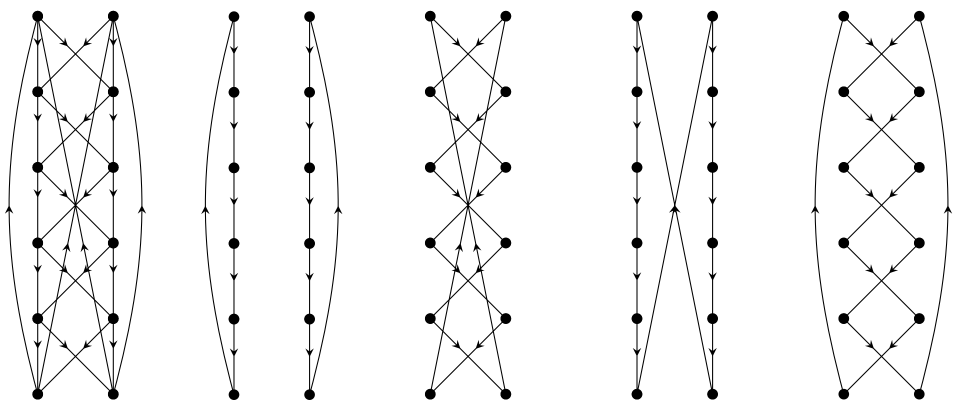
$$v_i = \begin{cases} (0,i) & \text{if } 0 \leq i \leq m-1, \\ (1,i) & \text{if } m \leq i \leq 2m-1, \end{cases}$$

and

$$w_i = \begin{cases} v_i & \text{if } i \text{ is even}, \\ v_i + (1,0) & \text{if } i \text{ is odd}. \end{cases}$$

It can be seen that $F_3 = \vec{C}_{2m}^{(2)}$ and $F_4 = \vec{C}_{2m}^{(3)}$ are arc disjoint directed $2m$-cycle factors of $\vec{C}_m[2]$. Thus $\{F_3, F_4\}$ is a $\vec{C}_{2m}$-factorization of $\vec{C}_m[2]$. □

Example 2.24: The following figure gives a $\vec{C}_6$-factorization and a $\vec{C}_{12}$-factorization of $\vec{C}_6[2]$, in turn. $\{F_1, F_2\}$-factorization is a $\vec{C}_6$-factorization of $\vec{C}_6[2]$ and $\{F_3, F_4\}$-factorization is a $\vec{C}_{12}$-factorization of $\vec{C}_6[2]$.



(a) $\vec{C}_6[2]$ **(b)** F_1 **(c)** F_2 **(d)** F_3 **(e)** F_4

Figure 2.3: Directed cycle factorizations of $\vec{C}_6[2]$.

Lemma 2.25: For every integer $m \geq 2$, $\vec{C}_m[2] \oplus I^*_{2m}$ has a $\{\vec{C}^1_m, \vec{C}^2_{2m}\}$-factorization.

Proof. The F_1 is defined as in Lemma 2.23; in this case, F_1 is a directed m-cycle factor of $\vec{C}_m[2] \oplus I^*_{2m}$. Also, let $\vec{C}^{(1)}_{2m} = (u_0, u_1, \ldots, u_{2m-1})$ be a directed $2m$-cycle of $\vec{C}_m[2] \oplus I^*_{2m}$, where $u_{2i} = (0, i)$, and $u_{2i+1} = (1, i)$ for $0 \leq i \leq m-1$. Similarly, it can be checked that $F_2 = \vec{C}^{(1)}_{2m}$ and $F_3 = \vec{C}^{(1)}_{2m} + (1, 0)$ are arc disjoint directed $2m$-cycle factors of $\vec{C}_m[2] \oplus I^*_{2m}$. Thus $\{F_1, F_2, F_3\}$ is a $\{\vec{C}^1_m, \vec{C}^2_{2m}\}$-factorization of $\vec{C}_m[2] \oplus I^*_{2m}$. □

The fourth section of this thesis provides the full solution of HWP$^*(v; m^r, (2m)^s)$ with a few possible exceptions when m is even. That is, the existence of a $\{\vec{C}^r_m, \vec{C}^s_{2m}\}$-factorization of K^*_{2mx} is studied for even m in that section. For this, first K^*_{2mx} is factorized into a K^*_{2m}-factor and $(2x-2)$ $K^*_{(m:2)}$-factors. Then, K^*_{2m} is factorized into a Γ^*_m-factor, a $C^*_m[2] \oplus I^*_{2m}$-factor and $C^*_m[2]$-factors. Consequently, a $\{\vec{C}^r_m, \vec{C}^s_{2m}\}$-factorization of these graphs is needed. Let us begin by considering a $\{\vec{C}^r_m, \vec{C}^s_{2m}\}$-factorization of $C^*_m[2] \oplus I^*_{2m}$, followed by a sequential examination of the desired factorizations of other graphs.

Lemma 2.26: Let $m \geq 4$ be an even integer. Then $C^*_m[2] \oplus I^*_{2m}$ has a $\{\vec{C}^r_m, \vec{C}^s_{2m}\}$-factorization for $r \in \{1, 3\}$ and $r + s = 5$.

Proof. **Case 1 ($r = 1$)** : Let F_1, F_2 and F_3 be equivalent to the F_1, F_2 and F_3 factors respectively, as stated in Lemma 2.25. Thus, F_1 is a directed m-cycle factor of $C^*_m[2] \oplus I^*_{2m}$. In addition, F_2 and F_3 are directed $2m$-cycle factors of $C^*_m[2] \oplus I^*_{2m}$, respectively.

Let $\vec{C}^{(2)}_{2m} = (x_0, x_1, \ldots, x_{2m-1})$ be a directed $2m$-cycle of $C^*_m[2] \oplus I^*_{2m}$, where $x_0 = (0, 0)$, $x_m = (1, 0)$, $x_{i+1} = (0, m-1-i)$ for $0 \leq i \leq m-2$ and $x_{j+1+m} = (1, m-1-j)$ for $0 \leq j \leq m-2$.

Let $\vec{C}^{(3)}_{2m} = (y_0, y_1, \ldots, y_{2m-1})$ be a directed $2m$-cycle of $C^*_m[2] \oplus I^*_{2m}$, where $y_m = (1, 0)$,

$$y_i = \begin{cases} (0, m-i) & \text{if } i \text{ is even,} \\ (1, m-i) & \text{if } i \text{ is odd,} \end{cases} \quad \text{for } 0 \leq i \leq m-1$$

and

$$y_i = \begin{cases} (1, 2m-i) & \text{if } i \text{ is even,} \\ (0, 2m-i) & \text{if } i \text{ is odd,} \end{cases} \quad \text{for } m+1 \le i \le 2m-1.$$

The factors $F_4 = \vec{C}_{2m}^{(2)}$ and $F_5 = \vec{C}_{2m}^{(3)}$ are arc disjoint directed $2m$-cycle factors of $C_m^*[2] \oplus I_{2m}^*$. Then, $\{F_1, F_2, F_3, F_4, F_5\}$ is a $\{\vec{C}_m^1, \vec{C}_{2m}^4\}$-factorization of $C_m^*[2] \oplus I_{2m}^*$.

Case 2 ($r = 3$) : Let F_1, F_2 and F_3 be the same as in Case 1. Using the arcs of $F_4 \bigcup F_5$, two new $\vec{C}_m$-factors can be obtained as follows:

The factor $F_4^{'} = R(F_1)$ is a $\vec{C}_m$-factor of $C_m^*[2] \oplus I_{2m}^*$. Let $\vec{C} = (y_0, y_1, \dots, y_{m-1})$ be a directed m-cycle of $C_m^*[2] \oplus I_{2m}^*$, where

$$y_i = \begin{cases} (0, i) & \text{if } i \text{ is even,} \\ (1, i) & \text{if } i \text{ is odd,} \end{cases} \quad \text{for } 0 \le i \le m-1.$$

It can be checked that $F_5^{'} = R(\vec{C}) \cup R(\vec{C} + (1,0))$ is a directed m-cycle factor of $C_m^*[2] \oplus I_{2m}^*$.

So, $\{F_1, F_2, F_3, F_4^{'}, F_5^{'}\}$ is a $\{\vec{C}_m^3, \vec{C}_{2m}^2\}$-factorization of $C_m^*[2] \oplus I_{2m}^*$. $\square$

Lemma 2.27: Let $m \ge 4$ be an even integer. Then $C_m^*[2]$ has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $r \in \{0, 2, 4\}$ and $r + s = 4$.

Proof. The cases $r \in \{0, 4\}$ are obtained by Lemmata 2.17 and 2.20. Let $\vec{C}_{2m}^{(1)} = (u_0, u_1, \dots, u_{2m-1})$ be a directed $2m$-cycle of $C_m^*[2]$, where

$$u_i = \begin{cases} (0, i) & \text{if } 0 \le i \le m-1, \\ (1, i) & \text{if } m \le i \le 2m-1. \end{cases}$$

And it can be checked that $F_1 = \vec{C}_{2m}^{(1)}$ is a $\vec{C}_{2m}$-factor of $C_m^*[2]$. Let $\vec{C}_{2m}^{(2)} = (v_0, v_1, \dots, v_{2m-1})$ be a directed $2m$-cycle of $C_m^*[2]$, where

$$v_i = \begin{cases} u_i & \text{if } i \text{ is even,} \\ u_i + (1,0) & \text{if } i \text{ is odd.} \end{cases}$$

The factor $F_2 = \vec{C}_{2m}^{(2)}$ is a $\vec{C}_{2m}$-factor of $C_m^*[2]$. Let $F_4^{'}$ and $F_5^{'}$ be the same as in Lemma 2.26. Then, $\{F_1, F_2, F_4^{'}, F_5^{'}\}$ is a $\{\vec{C}_m^2, \vec{C}_{2m}^2\}$-factorization of $C_m^*[2]$. $\square$

Example 2.28: The following figure gives a $\{\vec{C}_6^2, \vec{C}_{12}^2\}$-factorization of $C_6^*[2]$. The factors F_1 and F_2 are $\vec{C}_{12}$-factors of $C_6^*[2]$, and the factors F_3 and F_4 are $\vec{C}_6$-factors of $C_6^*[2]$.

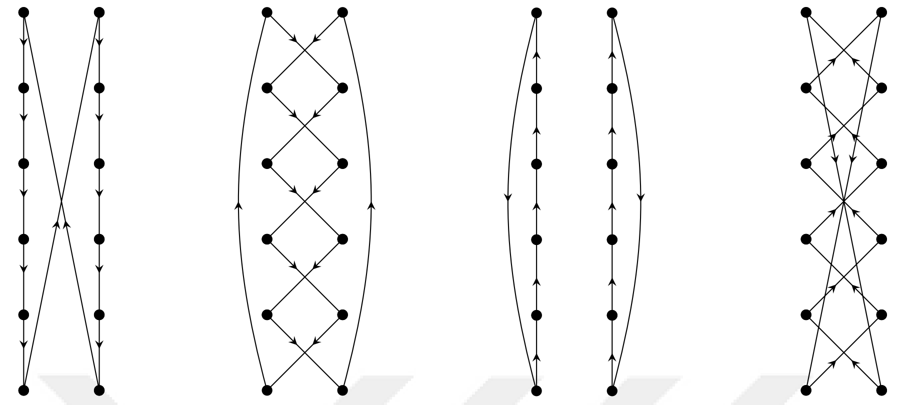


Figure 2.4: A $\{\vec{C}_6^2, \vec{C}_{12}^2\}$-factorization of $C_6^*[2]$.

Lemma 2.29: Let $m \geq 4$ be an even integer. Then $\Gamma_m^* = \mathcal{C}^*[2] \oplus F_m^*[2]$ has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $r \in \{0, 6\}$ and $r + s = 6$.

Proof. **Case 1 $(r = 0)$:** By Lemma 2.15 1. and Proposition 2.3, Γ_m^* has a $\vec{C}_{2m}$-factorization.

Case 2 $(r = 6)$: By Lemma 2.15 2. and Proposition 2.3, Γ_m^* has a $\vec{C}_m$-factorization for $m \equiv 0 \pmod 4$.

When $m \equiv 2 \pmod 4$, define the following m-cycles. Let $\vec{C}_m^{(0)}$ be the cycles $\vec{C}_m^{(0)}$, as stated in Lemma 2.23, then $\vec{C}_m^{(0)}$ is a directed m-cycle of Γ_m^*. We also define the directed m-cycles of Γ_m^* as shown below.

$\vec{C}_m^{(1)} = (x_0, x_1, \ldots x_{m-1})$ where $x_0 = (0, 0)$ and for $1 \leq i \leq m - 1$

$$x_i = \begin{cases} \left(\frac{1-(-1)^i}{2}, \frac{m}{2} - \lfloor \frac{i}{2} \rfloor\right) & \text{if } i \equiv 1, 2 \pmod 4, \\ \left(\frac{1-(-1)^i}{2}, \frac{m}{2} + \lfloor \frac{i}{2} \rfloor\right) & \text{if } i = 0, 3 \pmod 4. \end{cases}$$

$\vec{C}_m^{(2)} = (u_0, u_1, \ldots, u_{m-1})$ where $u_i = \begin{cases} (1, m-1-i) & \text{if } 0 \leq i \leq \frac{m}{2}, \\ (0, m-1-i) & \text{if } \frac{m}{2} + 1 \leq i \leq m - 1. \end{cases}$

$\vec{C}_m^{(3)} = (y_0, y_1, \ldots y_{m-1})$ where $y_0 = (0, 0)$, $y_1 = (0, \frac{m}{2})$, $y_2 = (1, \frac{m}{2} + 1)$, $y_3 =$

$(1, \frac{m}{2} - 1)$ and

$$y_i = \begin{cases} (1, \frac{m}{2} + (-1)^{i+1}\lfloor \frac{i}{2} \rfloor) & \text{if } i \equiv 0, 1 \pmod 4 \\ (0, \frac{m}{2} + (-1)^{i}\lfloor \frac{i}{2} \rfloor) & \text{if } i = 2, 3 \pmod 4 \end{cases} \quad \text{for} \quad 4 \leq i \leq m-1.$$

$\vec{C}_m^{(4)} = (z_0, z_1, \ldots, z_{m-1})$ where

$$z_i = \begin{cases} y_{m-i} + (1, 0) & \text{if } 1 \leq i \leq m-3 \\ y_{m-i} & \text{if } m-2 \leq i \leq m \end{cases}$$

Using the above m-cycles, the following m-cycle factors can be obtained. $F_0 = \vec{C}_m^{(0)} \cup (\vec{C}_m^{(0)} + (1, 0))$, $F_1 = \vec{C}_m^{(1)} \cup R(\vec{C}_m^{(1)} + (1, 0))$, $F_2 = R(F_1)$, $F_3 = \vec{C}_m^{(2)} \oplus (\vec{C}_m^{(2)} + (1, 0))$, $F_4 = \vec{C}_m^{(3)} \cup (\vec{C}_m^{(3)} + (1, 0))$ and $F_5 = \vec{C}_m^{(4)} \cup (\vec{C}_m^{(4)} + (1, 0))$. Then, $\{F_0, F_1, F_2, F_3, F_4, F_5\}$ is a $\vec{C}_m$-factorization of Γ_m^*. So, Γ_m^* has a $\vec{C}_m$-factorization for even $m \geq 4$. $\square$

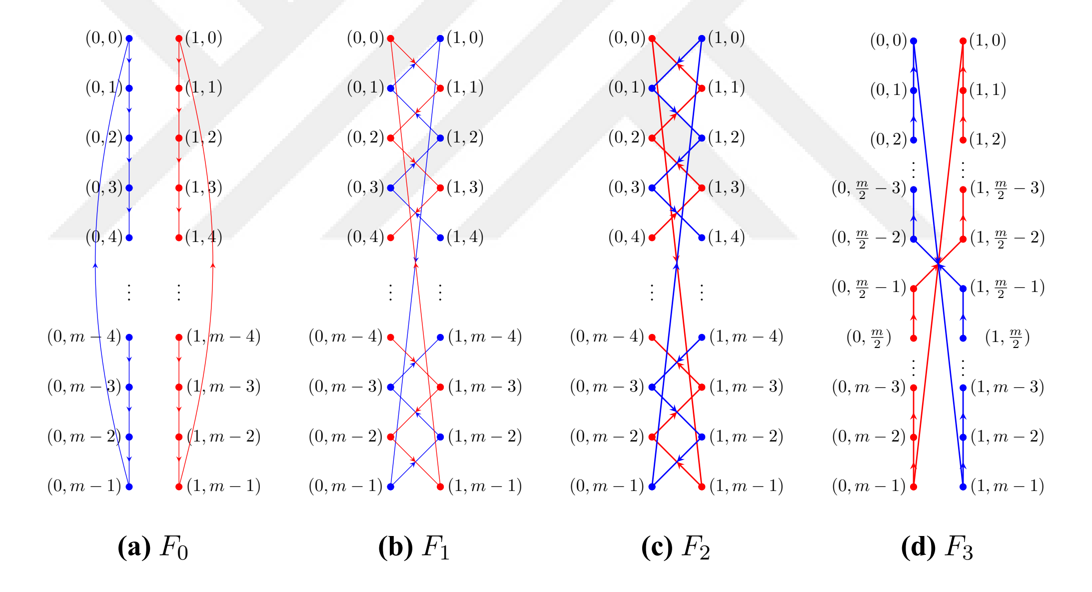


Figure 2.5: F_0, F_1, F_2, and F_3 factors of Γ_m^*.

3. DHWP WITH SMALL CYCLE SIZES

This section follow the lead of the first results for the undirected Hamilton-Waterloo Problem and provides solutions for cases with directed cycle sizes $\{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16),(3,5),(3,15),(5,15)\}$. Two different constructions depending on the parity of the cycle sizes are given. For even cycle sizes, using the construction in Lemma 3.2 and the preliminary lemmata required in the construction, HWP$^*(v;m^r,n^s)$ is solved for $(m,n)\in\{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16)\}$ with $r+s=v-1$. For odd cycle sizes, a new construction is given in the Lemma 3.7 when v is odd. Using this construction and the results required for this construction, we state that HWP$^*(v;m^r,n^s)$ has a solution for $(m,n)\in\{(3,5),(3,15),(5,15)\}$ for odd v with a few possible exceptions. Constructions given in Lemma 3.2 and Lemma 3.7 are general constructions and they can be used to solve the problem also for the other cycle sizes as long as the necessary small cases can be found.

3.1. Even Cycle Sizes

This section provides solutions to the Directed Hamilton-Waterloo Problem, with a special focus on cycles of even length. For this purpose, we introduce a property that will be used in our proofs before giving the general construction.

Let m be an integer and the vertex set of K_{2m} be $\mathbb{Z}_{2m}$. Let I_{2m} be a 1-factor of K_{2m} with $E(I_{2m})=\{\{i,m+i\}:0\le i\le m-1\}$. Note that $K_m[2]\cong K_{2m}-I_{2m}$ where vertex set of $K_m[2]$ is $\mathbb{Z}_{2m}$. According to Proposition 2.3, $K_m^*[2]$ is isomorphic to $K_{2m}^*-I_{2m}^*$ where I_{2m}^* is a K_2^*-factor of K_{2m}^* and has been defined previously.

The following lemma will be used in the first main construction of this section.

Lemma 3.1: $K_x^*[2]$ has a K_2^*-factorization for every integer $x\ge 2$.

Proof. Notice that $K_x^*[2]\cong K_{2x}^*-I_{2x}^*$. Using Kotzig's 1-factorization of K_{2x} and Proposition 2.3, a decomposition of $K_x^*[2]$ into $2x-2$ K_2^*-factors is obtained. □

The main construction is given in the following lemma, which is used to obtain solutions for the even cycle size cases.

Lemma 3.2: Let $m \geq 4$ and $n \geq 4$ be even and $h = \text{lcm}(m, n)$. If HWP$^*(h; m^{r'}, n^{s'})$ *has a solution for all nonnegative integers r', s' satisfying $r' + s' = h - 1$, then there is a solution to* HWP$^*(hx; m^r, n^s)$ *for all nonnegative integers r, s, and x with $r + s = hx - 1$.*

Proof. By (1.1), K^*_{hx} can be decomposed as follows:

$$K^*_{hx} \cong xK^*_h \oplus K^*_{(h:x)} \tag{3.1}$$

Since $K^*_{(h:x)} \cong K^*_x[h]$, $K^*_x[h]$ is isomorphic to $K^*_x[2][\frac{h}{2}]$ by the associativity of the power rule. Thus, by Lemma 3.1, $K^*_x[h]$ can be decomposed into factors each isomorphic to $K^*_2[\frac{h}{2}]$, and since $K^*_2[\frac{h}{2}] \cong K^*_{(\frac{h}{2}:2)}$, we have a decomposition of $K^*_x[h]$ into $(2x-2)$ $K^*_{(\frac{h}{2}:2)}$-factors.

$$K^*_{hx} \cong xK^*_h \oplus \underbrace{xK^*_{(\frac{h}{2}:2)} \oplus xK^*_{(\frac{h}{2}:2)} \oplus \ldots \oplus xK^*_{(\frac{h}{2}:2)}}_{2x-2} \tag{3.2}$$

Now, let F_0 be the K^*_h-factor and $F_1, F_2, \ldots, F_{2x-2}$ be the $K^*_{(\frac{h}{2}:2)}$-factors of K^*_{hx}. As a result, K^*_{hx} has a $\{F_0, F_1, F_2, \ldots, F_{2x-2}\}$-factorization. Since HWP$^*(h; m^{r'}, n^{s'})$ is assumed to have a solution for all nonnegative integers r' and s'; F_0 has a $\{\vec{C}^{r'}_m, \vec{C}^{s'}_n\}$-factorization for all nonnegative integers r' and s' where $r' + s' = h - 1$. Also, by Lemma 2.5 $K^*_{(\frac{h}{2}:2)}$ has a $\vec{C}_m$- and a $\vec{C}_n$-factorization for $m, n \geq 4$, so each F_j has a $\{\vec{C}^{\frac{h}{2}r_j}_m, \vec{C}^{\frac{h}{2}s_j}_n\}$-factorization for $j \in \{1, 2, \ldots, 2x-2\}$, where $r_j, s_j \in \{0, 1\}$ with $r_j + s_j = 1$. Those factorizations give us a $\{\vec{C}^r_m, \vec{C}^s_n\}$-factorization of K^*_{hx} where $r = r' + \frac{h}{2}\sum_{j=1}^{2x-2} r_j$ and $s = s' + \frac{h}{2}\sum_{j=1}^{2x-2} s_j$ with $r + s = r' + s' + \frac{h}{2}\sum_{j=1}^{2x-2}(r_j + s_j) = h - 1 + \frac{h}{2}(2x-2) = hx - 1$.

Since any nonnegative integer $0 \leq r \leq hx - 1$ can be written as $r = r' + \frac{h}{2}a$ for integers $0 \leq r' \leq h - 1$, $0 \leq a \leq 2x - 2$ and even h, a solution to HWP$^*(hx; m^r, n^s)$ exists for each $r \geq 0$ and $s \geq 0$ satisfying $r + s = hx - 1$. □

The following lemmata give the base blocks of our main construction in this section. The cases when $r = 0$ and $s = 0$ of the lemmata are obtained by Theorem 1.15 and the remaining factorizations for Lemma 3.3 and 3.4 are given in the Appendix.

Lemma 3.3: For nonnegative integers r and s, HWP$^*(8; 4^r, 8^s)$ *has a solution if and only if $r + s = 7$.*

Lemma 3.4: For nonnegative integers r and s, $\text{HWP}^*(12; m^r, n^s)$ *has a solution for* $(m,n) \in \{(4,6),(4,12),(6,12)\}$ *if and only if* $r+s=11$.

It is known that K^*_{2m} is isomorphic to $K^*_m[2] \oplus I^*_{2m}$ and also that $K_m - F_m$ factorize into C_m. Therefore, we can provide the following theorem for $(m,n) = (8,16)$, while the case of $(m,n) = (4,16)$ is given in the Appendix.

Lemma 3.5: For nonnegative integers r and s, $\text{HWP}^*(16; m^r, n^s)$ *has a solution for* $(m,n) \in \{(4,16),(8,16)\}$ *if and only if* $r+s=15$.

Proof. By Theorem 1.15, the cases when $r=0$ and $s=0$ are obtained.

Case 1 : For $(m,n) = (8,16)$, first assume that r is odd. One can see that $K^*_{16} \cong K^*_8[2] \oplus I^*_{16}$, and K^*_8 have a $\vec{C}_8$-factorization by Lemma 3.3. Then, one can factorize K^*_{16} into six $\vec{C}_8[2]$ and a single $\vec{C}_8[2] \oplus I^*_{16}$ factor. Also, each $\vec{C}_8[2]$ can be decomposed into two $\vec{C}_8$ or two $\vec{C}_{16}$-factors by Lemma 2.23. By Lemma 2.25, $\vec{C}_8[2] \oplus I^*_{16}$ has a $\{\vec{C}^1_8, \vec{C}^2_{16}\}$-factorization. Now, let r_0 and s_0 be nonnegative integers with $r_0 + s_0 = 6$. Decomposing r_0 $\vec{C}_8[2]$'s into $\vec{C}_8$-factors and remaining s_0 $\vec{C}_8[2]$'s into $\vec{C}_{16}$-factors, as well as Γ_8 into a $\{\vec{C}^1_8, \vec{C}^2_{16}\}$-factor gives us a $\{\vec{C}^{2r_0+1}_8, \vec{C}^{2s_0+2}_{16}\}$-factorization of K^*_{16}.

Since any odd integer r can be written as $r = 2r_0 + 1$ for a nonnegative integer r_0, $\text{HWP}^*(16; 8^r, 16^s)$ has a solution for odd r with $r + s = 2r_0 + 1 + 2s_0 + 2 = 2(r_0 + s_0) + 3 = 15$.

The solutions to the remaining even cases are listed in the Appendix.

Case 2 : For $(m,n) = (4,16)$, solutions to all cases are given in the Appendix except for $r=0$ and $s=0$. □

The previous lemmata establish the existence of solutions for the cases where v takes values 8, 12, and 16, and (m,n) belongs to the set $\{(4,8),(4,6),(4,12),(6,12),(4,16),(8,16)\}$. Now, using the main construction outlined in the Lemma 3.2, this result can be proved for $v = 8x$, $12x$, and $16x$, where $x = \text{lcm}(m,n)$.

Theorem 3.6: For nonnegative integers r and s, $\text{HWP}^*(v; m^r, n^s)$ *has a solution for* $(m,n) \in \{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16)\}$ *if and only if* $r+s=v-1$ *and* $\text{lcm}(m,n)|v$.

Proof. If a solution to $\text{HWP}^*(v; m^r, n^s)$ exists for $(m,n) \in \{(4,6),(4,8),(4,12),$

$(4,16),(6,12),(8,16)\}$, then by Lemma 1.20, $r+s=v-1$ must hold. Furthermore, since m and n divide v, it follows that $h = \text{lcm}(m,n)$ divides v.

For the sufficiency part, assume $h|v$ and $r+s=hx-1=v-1$ where x is a nonnegative integer.

For $(m,n)=(4,8)$, HWP$^*(8;4^{r_0},8^{s_0})$ has a solution for all nonnegative r_0 and s_0 with $r_0+s_0=7$ by Lemma 3.3. Then, HWP$^*(v;4^r,8^s)$ has a solution for $r+s=8x-1=v-1$ by Lemma 3.2.

For $(m,n)\in\{(4,6),(4,12),(6,12)\}$, HWP$^*(12;m^{r_1},n^{s_1})$ has a solution for all nonnegative r_1 and s_1 with $r_1+s_1=11$ by Lemma 3.4. Then, HWP$^*(v;m^r,n^s)$ has a solution by Lemma 3.2 for $(m,n)\in\{(4,6),(4,12),(6,12)\}$ with $r+s=12x-1=v-1$.

For $(m,n)\in\{(4,16),(8,16)\}$, HWP$^*(16;m^{r_2},n^{s_2})$ has a solution for all nonnegative r_2 and s_2 with $r_2+s_2=15$ by Lemma 3.5. Then, by Lemma 3.2, HWP$^*(v;m^r,n^s)$ has a solution for $(m,n)\in\{(4,16),(8,16)\}$ with $r+s=16x-1=v-1$. □

3.2. Odd Cycle Sizes

First, the following main construction for odd cycle sizes is given here, and using this construction it is shown that for $(m,n)\in\{(3,5),(3,15),(5,15)\}$, HWP$^*(v;m^r,n^s)$ has a solution for all nonnegative integers r and s satisfying $r+s=v-1$ with a few possible exceptions, where v is odd.

Lemma 3.7: Let $m\geq 3$ *and* $n\geq 3$ *be both odd,* $h=\text{lcm}(m,n)$ *and* $3|h$. *If* HWP$^*(h;m^{r'},n^{s'})$ *has a solution for all* r', s' *satisfying* $r'+s'=h-1$, *then there is a solution to* HWP$^*(hx;m^r,n^s)$ *for all nonnegative* r, s *and odd* x *satisfying* $r+s=hx-1$.

Proof. By (1.1), a decomposition of K^*_{hx} into a K^*_h and a $K^*_{(h:x)}$-factor is obtained. Since $K^*_{(h:x)}\cong K^*_x[h]$, it is seen that $K^*_x[h]\cong K^*_x[3][\frac{h}{3}]$.

It is clear that $K^*_x[3]$ is isomorphic to $K^*_{3x}-xK^*_3$. Since a Kirkman triple system of order $3x$ exists by Theorem 1.12, we have a C_3-factorization of K_{3x}. Then, a $C^*_3\cong K^*_3$-factorization of $K^*_{3x}-xK^*_3$ is obtained by Proposition 2.3. So, $K^*_x[3]$ has a decomposition into $\frac{3x-3}{2}$ K^*_3-factors. In $K^*_x[h]$, these K^*_3-factors form $K^*_{(\frac{h}{3}:3)}$-factors since $K_3^*[\frac{h}{3}] \cong K^*_{(\frac{h}{3}:3)}$.

$$K^*_{hx} \cong xK^*_h \oplus \underbrace{xK^*_{(\frac{h}{3}:3)} \oplus xK^*_{(\frac{h}{3}:3)} \oplus \ldots \oplus xK^*_{(\frac{h}{3}:3)}}_{\frac{3x-3}{2}} \tag{3.3}$$

Let F_0 be the K^*_h-factor and $F_1, F_2, \ldots, F_{\frac{3x-3}{2}}$ be the $K^*_{(\frac{h}{3}:3)}$-factors of K^*_{hx}. Since HWP$^*(h; m^{r'}, n^{s'})$ is assumed to have a solution for all nonnegative integers r' and s' where $r' + s' = h - 1$, F_0 has a $\{\overrightarrow{C}_m^{r'}, \overrightarrow{C}_n^{s'}\}$-factorization for all nonnegative integers r' and s' with $r' + s' = h - 1$. Also, $K^*_{(\frac{h}{3}:3)}$ has a $\overrightarrow{C}_m$-factorization and a $\overrightarrow{C}_n$-factorization by Lemma 2.7, so each F_j has a $\{\overrightarrow{C}_m^{\frac{2h}{3}r_j}, \overrightarrow{C}_n^{\frac{2h}{3}s_j}\}$-factorization for $j \in \{1, 2, \ldots, \frac{3x-3}{2}\}$, where $r_j, s_j \in \{0, 1\}$ with $r_j + s_j = 1$. These factorizations give us a $\{\overrightarrow{C}_m^r, \overrightarrow{C}_n^s\}$-factorization of K^*_{hx} where $r = r' + \sum_{i=0}^{\frac{3x-3}{2}} \frac{2h}{3} r_i$ and $s = s' + \sum_{i=0}^{\frac{3x-3}{2}} \frac{2h}{3} s_i$ with $r + s = r' + s' + \sum_{i=0}^{\frac{3x-3}{2}} \frac{2h}{3}(r_i + s_i) = h - 1 + hx - h = hx - 1$.

Since any nonnegative integer $0 \leq r \leq hx - 1$ can be written as $r = r' + \frac{2h}{3}a$ for integers $0 \leq r' \leq h - 1$ and $0 \leq a \leq \frac{3x-3}{2}$, a solution to HWP$^*(hx; m^r, n^s)$ exists for each $r \geq 0$ and $s \geq 0$ satisfying $r + s = hx - 1$. □

Lemma 3.8: For nonnegative integers r and s, HWP$^*(15; m^r, n^s)$ *has a solution for* $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$ *if and only if* $r + s = 14$ *except possibly for* $r \in \{11, 12, 13\}$ *when* $(m, n) = (3, 5)$ *and for* $r = 13$ *when* $(m, n) = (3, 15)$.

Proof. The cases when $r = 0$ and $s = 0$ can be obtained by Theorem 1.15. A solution to HWP$(15; m^{r_0}, n^{s_0})$ for $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$ with the exception $(m, n, r_0, s_0) = (3, 5, 6, 1)$ is given by Corollary 2.2. Thus, by Observation 2.1, we have a solution to HWP$^*(15; m^r, n^s)$ for $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$ with r and s are positive even integers except possibly when $(m, n, r, s) = (3, 5, 12, 2)$. The solutions for the odd cases are listed in the Appendix. □

Using Lemma 3.7 and Lemma 3.8, a solution to HWP$^*(v; m^r, n^s)$ can be given for all nonnegative integers r and s when $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$ satisfying $r + s = v - 1$ and odd $v > 15$ with a few possible exceptions.

Theorem 3.9: For all nonnegative integers r, s and odd $v > 15$, HWP$^*(v; m^r, n^s)$ *has a solution for* $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$ *if and only if* $r + s = v - 1$ *and* $15|v$ *except possibly* $s \in \{1, 2, 3\}$ *when* $(m, n) = (3, 5)$ *and* $s = 1$ *when* $(m, n) = (3, 15)$.

Proof. If a solution to HWP$^*(v; m^r, n^s)$ exists for $(m,n) \in \{(3,5),(3,15),(5,15)\}$, $r+s = v-1$ and lcm$(m,n) = 15|v$ by Lemma 1.20.

For the sufficiency part, assume $v = 15x$ and $r+s = 15x-1$ where $x > 1$ is an odd integer.

For $(m,n) = (5,15)$, HWP$^*(15; 5^{r_0}, 15^{s_0})$ has a solution for all nonnegative r_0 and s_0 with $r_0 + s_0 = 14$ by Lemma 3.8. Then, HWP$^*(v; 5^r, 15^s)$ has a solution for all nonnegative r and s with $r+s = 15x-1 = v-1$ by Lemma 3.7.

By (1.1), a decomposition of K^*_{15x} into a K^*_{15}-factor and $\frac{3x-3}{2}$ $K^*_{(5:3)}$-factors is achieved. By Lemma 3.8, K^*_{15} has a $\{\vec{C}^{r'}_m, \vec{C}^{s'}_n\}$-factorization for $(m,n) \in \{(3,5),(3,15)\}$ with $r' + s' = 14$ except possibly $r' \in \{11,12,13\}$ when $(m,n) = (3,5)$ and $r' = 13$ when $(m,n) = (3,15)$. Also, by Lemma 2.7, each $K^*_{(5:3)}$ has a $\{\vec{C}^{10r_j}_m, \vec{C}^{10s_j}_n\}$-factorization for $j \in \{1,2,\ldots,\frac{3x-3}{2}\}$, where $r_j, s_j \in \{0,1\}$ with $r_j + s_j = 1$.

Placing a $\vec{C}_m$-factorization on "a" of the $K^*_{(5:3)}$-factors, a $\vec{C}_n$-factorization on "b" of the $K^*_{(5:3)}$-factors for $0 \leq a, b \leq \frac{3x-3}{2}$ with $a + b = \frac{3x-3}{2}$, and taking a $\{\vec{C}^{r'}_m, \vec{C}^{s'}_n\}$-factorization of K^*_{15} give a $\{\vec{C}^{r'+10a}_m, \vec{C}^{s'+10b}_n\}$-factorization of K^*_{15x} for $(m,n) \in \{(3,5),(3,15)\}$. Let $r = r' + 10a$ and $s = s' + 10b$, then we have $r + s = r' + s' + 10(a+b) = 14 + 5(3x-3) = 15x - 1 = v - 1$ with $0 \leq r, s \leq 15x - 1$.

For $(m,n) = (3,5)$, the requested integer $r \in [0, 15x-5] \cup \{15x-1\}$ can be obtained from the sum of r' and $10a$ for $r' \in [0,14] \backslash \{11,12,13\}$ and $0 \leq a \leq \frac{3x-3}{2}$. Therefore, HWP$^*(15x; 3^r, 5^s)$ has a solution for all integers $0 \leq r, s \leq 15x-1$ with $r+s = 15x-1 = v-1$ except possibly when $s \in \{1,2,3\}$. Similarly, for $(m,n) = (3,15)$, since any nonnegative integer $r \in [0, 15x-1] \backslash \{15x-2\}$ can be written as $r = r' + 10a$ for $r' \in [0,14] \backslash \{13\}$ and $0 \leq a \leq \frac{3x-3}{2}$, a solution to HWP$^*(15x; 3^r, 15^s)$ exists for all integers $0 \leq r, s \leq 15x-1$ with $r+s = 15x-1 = v-1$ except possibly when $s = 1$. □

Now our results can be combined in the following main theorem.

Theorem 3.10: For nonnegative integers r and s, HWP$^*(v; m^r, n^s)$ *has a solution for*

1. $(m,n) \in \{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16)\}$ *when v is even,*
2. $(m,n) \in \{(3,5),(3,15),(5,15)\}$ *when v is odd*

if and only if $r + s = v - 1$ and lcm$(m,n)|v$ *except possibly $s \in \{1,2,3\}$ when*

$(m, n) = (3, 5)$*, and* $s = 1$ *when* $(m, n) = (3, 15)$.

4. DHWP INVOLVING EVEN CYCLE SIZES

In this section, the problem of decomposing K_v^* into two nonisomorphic factors, where these factors are uniform factors of K_v^* involving K_2^* or directed m-cycles, and directed m-cycles or $2m$-cycles for even m, will be studied. Section 4.1 is devoted to finding solutions to HWP$^*(v; 2^r, m^s)$ for even m with $r + s = v - 1$. Also a solution is denoted as a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization of K_v^*. In Section 4.2, we will concentrate on solving HWP$^*(v; m^r, (2m)^s)$ for even m with $r + s = v - 1$, which indicates the existence of a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization of K_v^* for the same values of r and s. Here are our main results:

Theorem 4.1: Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, HWP$^*(v; 2^r, m^s)$ *has a solution if $m|v$, $r + s = v - 1$, $s \neq 1$, $(r, v) \neq (0, 6)$, $(m, r, v) \neq (4, 0, 4)$, and one of the following conditions holds;*

1. *$m > 4$, $s \neq 3$ and $m \equiv 0 \pmod 4$,*
2. *$m > 4$, $\frac{v}{m}$ is even and $m \equiv 2 \pmod 4$,*
3. *$m > 4$, $\frac{v}{m}$ is odd, $s \neq 3$ and $m \equiv 2 \pmod 4$,*
4. *$m = 4$ and $v \equiv 0, 8, 16 \pmod{24}$,*
5. *$m = 4$, $v \equiv 12 \pmod{24}$ and $s \notin \{3, 5\}$,*
6. *$m = 4$, $v \equiv 4, 20 \pmod{24}$ and r is odd.*

Theorem 4.2: Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, HWP$^*(v; m^r, (2m)^s)$ *has a solution if and only if $m|v$, $r + s = v - 1$ and $v \geq 4$ except for $(s, v, m) \in \{(0, 4, 4), (0, 6, 3), (0, 6, 6)\}$, and except possibly when $s \in \{1, 3\}$.*

Recall that Γ_m^* is $\mathcal{C}^*[2] \oplus F_m^*[2]$. In order to solve HWP$^*(v; 2^r, m^s)$ for even m, a decomposition of K_{mx}^* into K_2^*-factors and $\vec{C}_m$-factors is required. To do this, we decompose K_{mx}^* into $C_m^*[2]$, $C_m^*[2] \oplus I_{2m}^*$, and Γ_m^*-factors. The desired factorizations of these graphs are obtained in Section 2. The following lemma, which will be employed when studying the $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization of K_{mx} in the case where x is even, provide the $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization of Γ_m^* for $m \equiv 2 \pmod 4$.

Lemma 4.3: Γ_m^* has a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization for $m \equiv 2 \pmod 4$ and $r \in \{1, 2, 3,$

$4, 6\}$ *with* $r + s = 6$.

Proof. The cases $r \in \{2, 4, 6\}$ are obtained by Corollary 2.22.

For $r = 1$, let $\vec{C}_m^{(0)}$ and $\vec{C}_m^{(1)}$ cycles be equivalent to the $\vec{C}_m^{(0)}$ and $\vec{C}_m^{(1)}$ cycles respectively, as stated in Lemma 2.23 and Lemma 2.29. Also, the following m-cycles are defined.

$\vec{C}_m^{(2)} = (u_0, u_1, \ldots, u_{m-1})$ where $u_i = \begin{cases} (0, i) & \text{if } i \text{ is even,} \\ (1, i) & \text{if } i \text{ is odd.} \end{cases}$

$\vec{C}_m^{(3)} = (z_0, z_1, \ldots, z_{m-1})$ where $z_m = (1, m-1)$, $z_{m-1} = (0, 0)$ and

$$z_{i=} \begin{cases} \left(0, \lfloor \frac{i}{2} \rfloor + 1\right) & \text{if } i \equiv 0 \pmod 4 \\ \left(1, \lceil \frac{i}{2} \rceil + 1\right) & \text{if } i \equiv 1 \pmod 4 \\ (0, m-i) & \text{if } i \equiv 2 \pmod 4 \\ (1, m-i) & \text{if } i \equiv 3 \pmod 4 \end{cases} \quad \text{for } 0 \leq i \leq \frac{m}{2}.$$

Let's choose the factor F_0 as isomorphic to $F_m^* \oplus (F_m^* + (1, 0))$, then F_0 becomes a K_2^*-factor. Using the above m-cycles, we obtain the following m-cycle factors: $F_1 = \vec{C}_m^{(0)} \cup (\vec{C}_m^{(0)} + (1, 0))$, $F_2 = R(F_1)$, $F_3 = \vec{C}_m^{(1)} \cup R(\vec{C}_m^{(1)} + (1, 0))$, $F_4 = \vec{C}_m^{(2)} \cup R(\vec{C}_m^{(2)} + (1, 0))$ and $F_5 = \vec{C}_m^{(3)} \cup (\vec{C}_m^{(3)} + (1, 0))$. Then, $\{F_0, F_1, F_2, F_3, F_4, F_5\}$ is a $\{(K_2^*)^1, \vec{C}_m^5\}$-factorization of Γ_m^*.

For $r = 3$, $F_1 \oplus F_2$ is a C_m^*-factor of Γ_m^* and has a factorization into two K_2^*-factors of Γ_m^* say F_1' and F_2'. Then $\left\{F_0, F_1', F_2', F_3, F_4, F_5\right\}$ is a $\{(K_2^*)^3, \vec{C}_m^3\}$-factorization of Γ_m^*. □

4.1. Solutions to HWP*$(v; 2^r, m^s)$

Now, solutions to the Directed Hamilton-Waterloo Problem can be given for K_2^* and $\vec{C}_m$ when m is even.

Theorem 4.4: Let r, s *be nonnegative integers, and let* $m \geq 6$ *be even. Then,* HWP*$(v; 2^r, m^s)$ *has a solution if and only if* $m|v$, $r + s = v - 1$ *and* $v \geq 6$ *except for* $s = 1$ *or* $(r, v) = (0, 6)$, *and except possibly when at least one of the following conditions holds;*

1. $s = 3$ *and* $m \equiv 0 \pmod 4$,
2. $s = 3$, $m \equiv 2 \pmod 4$ *and* $\frac{v}{m}$ *is odd.*

Proof. Take $(v-2)$ disjoint K_2^*-factors of K_v^*, say $H_1^*, H_2^*, \ldots, H_{v-2}^*$. It is obvious that $K_v^* - (H_1^* \oplus H_2^* \oplus \cdots \oplus H_{v-2}^*)$ is a K_2^*-factor in K_v^*. Thus, there is no $\{(K_2^*)^{v-2}, \vec{C}_m^1\}$-factorization of K_v^*. Therefore, it can be assumed that $s \neq 1$.

Since HWP$^*(v; n^r, m^s)$ has a solution for $r = 0$ except for $(v, m) = (6, 6)$ by Theorem 1.15, one may assume that $r \geq 1$.

Let $v = mx$ for a positive integer x. Partition the vertices of K_{mx}^* into $2x$ sets of size $\frac{m}{2}$, represent each part of $\frac{m}{2}$ vertices in K_{mx}^* with a single vertex and represent all double arcs between sets of size $\frac{m}{2}$ as a single double arc, to get a K_{2x}^*. By Proposition 2.4, K_{2x}^* has a decomposition into $(2x-1)$ K_2^*-factors. Then, construct a K_m^*-factor of K_{mx}^* from one of the K_2^*-factors, and a $K_{(\frac{m}{2}:2)}^*$-factor of K_{mx}^* from each of the remaining $(2x-2)$ K_2^*-factors. Then, K_{mx}^* can be factorized into a K_m^*-factor and $(2x-2)$ $K_{(\frac{m}{2}:2)}^*$-factors.

$$K_{mx}^* \cong xK_m^* \oplus \underbrace{xK_{(\frac{m}{2}:2)}^* \oplus xK_{(\frac{m}{2}:2)}^* \oplus \ldots \oplus xK_{(\frac{m}{2}:2)}^*}_{2x-2} \tag{4.1}$$

By Lemmata 2.10 and 2.5, $K_{(\frac{m}{2}:2)}^*$ decomposes into $\frac{m}{2}$ K_2^*-factors or $\frac{m}{2}$ $\vec{C}_m$-factors, respectively. As a result, we must decompose K_m^* into K_2^*-factors and $\vec{C}_m$-factors.

Case 1 (odd *r*) : By Lemma 2.13, factorize K_m into a F_m-factor and $(\frac{m-2}{2})$ C_m-factors. So, K_m^* can be factorized into a F_m^*-factor and $(\frac{m-2}{2})$ C_m^*-factors by Proposition 2.3.

Since C_m^* can be decomposed into two K_2^*-factors or two $\vec{C}_m$-factors for even m, K_m^* has a $\{(K_2^*)^{2r_1+1}, \vec{C}_m^{2s_1}\}$-factorization where $r_1 + s_1 = \frac{m-2}{2}$.

Since K_{mx}^* has a $\{K_m^*, (K_{(\frac{m}{2}:2)}^*)^{(2x-2)}\}$-factorization, placing a K_2^*-factorization on r_0 of the $K_{(\frac{m}{2}:2)}^*$ factors for r_0 even and $0 \leq r_0 \leq 2x-2$, a $\vec{C}_m$-factorization on s_0 of the $K_{(\frac{m}{2}:2)}^*$ where $r_0 + s_0 = 2x-2$, and taking a $\{(K_2^*)^{2r_1+1}, \vec{C}_m^{2s_1}\}$-factorization of K_m^* give a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization of K_{mx}^* where $r = \frac{m}{2}r_0 + 2r_1 + 1$ and $s = \frac{m}{2}s_0 + 2s_1$ with $r + s = \frac{m}{2}(r_0 + s_0) + 2(r_1 + s_1) + 1 = mx - 1 = v - 1$.

Since any nonnegative odd integer $1 \leq r \leq mx-1$ can be written as $r = \frac{m}{2}r_0 + 2r_1 + 1$ for integers $0 \leq r_0 \leq 2x-2$ and $0 \leq r_1 \leq \frac{m-2}{2}$, a solution to HWP$^*(v; 2^r, m^s)$ exists for each odd $r \geq 1$ and $s \geq 1$ satisfying $r + s = mx - 1 = v - 1$.

Case 2 (even *r*) :

(a) Assume $m \equiv 0 \pmod 4$. Therefore, $\frac{m}{2}$ is even. Each $K_{(\frac{m}{2}:2)}^*$ decompose into $\frac{m}{2}$ K_2^*-factors or $\frac{m}{2}$ $\vec{C}_m$-factors. For this reason we need a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization of

K^*_m for even r.

Also, $K^*_{\frac{m}{2}}$ can be factorized as $\bigoplus_{i=1}^{\frac{m-4}{4}} C^*_i \oplus F^*_{\frac{m}{2}}$ where each C^*_i is isomorphic to $C^*_{\frac{m}{2}}$. Then, $K^*_{\frac{m}{2}}[2] \cong \bigoplus_{i=1}^{\frac{m-4}{4}} C^*_i[2] \oplus F^*_{\frac{m}{2}}[2]$. Also, K^*_m is isomorphic to $K^*_{\frac{m}{2}}[2] \oplus I^*_m$. So, K^*_m can be factorized as follows:

$$K^*_m \cong K^*_{\frac{m}{2}}[2] \oplus I^*_m \cong \underbrace{C^*_{\frac{m}{2}}[2] \oplus C^*_{\frac{m}{2}}[2] \oplus C^*_{\frac{m}{2}}[2] \oplus \ldots \oplus C^*_{\frac{m}{2}}[2]}_{\frac{m-4}{4}} \oplus F^*_{\frac{m}{2}}[2] \oplus I^*_m \quad (4.2)$$

Therefore, K^*_m has a $\{(C^*_{\frac{m}{2}}[2])^{\frac{m-12}{4}}, C^*_{\frac{m}{2}} \oplus I^*_m, \Gamma^*_{\frac{m}{2}}\}$-factorization. By Lemma 2.17, each of $\frac{m-12}{4}$ $C^*_{\frac{m}{2}}[2]$-factors has a $\{(K^*_2)^{r_0}, \vec{C}^{s_0}_m\}$-factorization for $r_0 \in \{0, 2, 4\}$ and $r_0 + s_0 = 4$. By Lemma 2.19, $C^*_{\frac{m}{2}}[2] \oplus I^*_m$ has a $\{(K^*_2)^{r_1}, \vec{C}^{s_1}_m\}$-factorization for $r_1 \in \{0, 1, 3, 5\}$ and $r_1 + s_1 = 5$. By Corollary 2.18, $\Gamma^*_{\frac{m}{2}}$ has a $\{(K^*_2)^{r_2}, \vec{C}^{s_2}_m\}$-factorization for even m and $r_2 \in \{0, 2, 4, 6\}$ with $r_2 + s_2 = 6$. Those factorizations give a $\{(K^*_2)^{r'}, \vec{C}^{s'}_m\}$-factorization of K^*_m where $r' = (\frac{m-12}{4})r_0 + r_1 + r_2$ and $s' = (\frac{m-12}{4})s_0 + s_1 + s_2$ satisfying $r' + s' = (\frac{m-12}{4})4 + 5 + 6 = m - 1$ with $0 \leq r', s' \leq m - 1$. Choosing $r_1 = 0$ yields a $\{(K^*_2)^{r'}, \vec{C}^{s'}_m\}$-factorization of K^*_m for even r'. Since we cannot get $r_0 = 1, r_1 = 2$ or $r_2 = 3$ from the above factorizations, it can be seen that $r' = m - 4$ cannot be obtained.

Placing a K^*_2-factorization on r'' of the $K^*_{(\frac{m}{2}:2)}$-factors for $0 \leq r'' \leq 2x - 2$, a $\vec{C}_m$-factorization on s'' of the $K^*_{(\frac{m}{2}:2)}$ for $r'' + s'' = 2x - 2$, and taking a $\{(K^*_2)^{r'}, \vec{C}^{s'}_m\}$-factorization of K^*_m give a $\{(K^*_2)^{\frac{m}{2}r''+r'}, \vec{C}^{\frac{m}{2}r''+s'}_m\}$-factorization of K^*_{mx} where $\frac{m}{2}r'' + r'$ is even.

Any even integer $1 \leq r \leq mx - 1$ can be written as $r = \frac{m}{2}r'' + r'$ for integers $r' \in [0, m - 1]$ and $0 \leq r'' \leq 2x - 2$. Since $r' \neq m - 4$, a solution to HWP$^*(v; 2^r, m^s)$ exists for each even $r \geq 2$ except possibly $r = mx - 4 = v - 4$ and $s \geq 1$ satisfying $r + s = v - 1$.

(b) Assume $m \equiv 2 \pmod 4$. By Lemma 2.12, factorize K_n into $(\frac{n-1}{2})$ C_n-factors for odd n, and get a C^*_n-factorization of K^*_n by Proposition 2.3. Also, K^*_m can be factorized as $K^*_{\frac{m}{2}}[2] \oplus I^*_m$. Since $\frac{m}{2}$ is odd, K^*_m has a $\{(C^*_{\frac{m}{2}}[2])^{\frac{m-2}{4}}, I^*_m\}$-factorization.

$$K^*_m \cong K^*_{\frac{m}{2}}[2] \oplus I^*_m \cong \underbrace{C^*_{\frac{m}{2}}[2] \oplus C^*_{\frac{m}{2}}[2] \oplus C^*_{\frac{m}{2}}[2] \oplus \ldots \oplus C^*_{\frac{m}{2}}[2]}_{\frac{m-2}{4}} \oplus I^*_m \quad (4.3)$$

By Lemma 2.17, each of $C^*_{\frac{m}{2}}[2]$-factors has a $\{(K^*_2)^{r_0}, \vec{C}^{s_0}_m\}$-factorization for $r_0 \in$

$\{0, 2, 4\}$ and $r_0 + s_0 = 4$. By Lemma 2.19, $C^*_{\frac{m}{2}}[2] \oplus I^*_m$ has $\{(K^*_2)^{r_1}, \vec{C}^{s_1}_m\}$-factorization for $r_1 \in \{0, 1, 3, 5\}$ and $r_1 + s_1 = 5$.

Those factorizations give a $\{(K^*_2)^{r_2}, \vec{C}^{s_2}_m\}$-factorization of K^*_m for $r_2 = \frac{m-6}{4} r_0 + r_1$ and $s_2 = \frac{m-6}{4} s_0 + s_1$ with $r_2 + s_2 = m - 1$. Since we cannot get $r_0 = 1$ or $r_1 = 2$ from the above factorizations, it can be seen that $r_2 = m - 4$ cannot be obtained.

Placing a K^*_2-factorization on r' of the $K^*_{(\frac{m}{2}:2)}$ factors for $0 \leq r' \leq 2x - 2$ where we choose r' is even, a $\vec{C}_m$-factorization on s' of the $K^*_{(\frac{m}{2}:2)}$ with $r' + s' = 2x - 2$, and taking a $\{(K^*_2)^{r_2}, \vec{C}^{s_2}_m\}$-factorization of K^*_m give a $\{(K^*_2)^{\frac{m}{2}r'+r_2}, \vec{C}^{\frac{m}{2}s'+s_2}_m\}$-factorization of K^*_{mx} where $r = \frac{m}{2}r' + r_2$ and $s = \frac{m}{2}s' + s_2$. Also, the requested even integer $r \in [1, mx - 1]$ is obtained from the sum of $\frac{m}{2}r'$ and r_2 for integers $0 \leq r' \leq 2x - 2$ and $r_2 \in [0, m - 1]$. Since $r_2 \neq m - 4$, a solution to HWP$^*(v; 2^r, m^s)$ exists for even $r \geq 2$ except possibly $r = mx - 4 = v - 4$ and odd $s \geq 1$ satisfying $r + s = v - 1$.

If x is even, say $x = 2t$, factorize K^*_{mx} into a K^*_{2m}-factor and $(2t - 2)$ $K^*_{(m:2)}$-factors. $K^*_{(m:2)}$ has a K^*_2-factorization with m K^*_2-factors and a $\vec{C}_m$-factorization with m $\vec{C}_m$-factors by Lemmata 2.10 and 2.5, respectively. So, K^*_{2m} must be decomposed into K^*_2-factors and $\vec{C}_m$-factors. As before, K^*_{2m} can be factorized as $K^*_m[2] \oplus I^*_{2m}$. So, K^*_{2m} has a $\{(C^*_m[2])^{\frac{m-4}{2}}, I^*_{2m}, \Gamma^*_m\}$-factorization. By Lemma 2.20, each of $C^*_m[2]$-factors has a $\{(K^*_2)^{r_0}, \vec{C}^{s_0}_m\}$-factorization for $r_0 \in \{0, 2, 4\}$ and $r_0 + s_0 = 4$. By Corollary 2.21, $C^*_m[2] \oplus I^*_{2m}$ has a $\{(K^*_2)^{r_1}, \vec{C}^{s_1}_m\}$-factorization for $r_1 \in \{1, 3, 5\}$ and $r_1 + s_1 = 5$. By Lemma 4.3, Γ^*_m has a $\{(K^*_2)^{r_2}, \vec{C}^{s_2}_m\}$-factorization for $m \equiv 2 \pmod 4$ and $r_2 \in \{1, 2, 3, 4, 6\}$ with $r_2 + s_2 = 6$. Using these factorizations, a solution to the problem is obtained for $r = 2mt - 4 = mx - 4$ when $m \equiv 2 \pmod 4$ and even x. As a result, HWP$^*(v; 2^r, m^s)$ has a solution for $r = v - 4$ and even $\frac{v}{m}$ when $m \equiv 2 \pmod 4$. □

In Theorem 4.4, the necessary and sufficient conditions for the existence of a solution for HWP$^*(v; 2^r, m^s)$ are given for even $m \geq 6$. The construction in Theorem 4.4 is not valid when $m = 4$. Therefore, we also study the case of $m = 4$. To achieve this, it is imperative to factorize K^*_v into K^*_2-factors and $\vec{C}_4$-factors. To achieve this, K^*_v is initially factored into $C^*_4[2]$-factors and a $C^*_4[2] \oplus I^*_8$-factor or a K^*_{12}-factor and $K^*_{(4:3)}$-factors. Subsequently, we will decompose all the graphs in these factors into the desired factors. Now, let us commence with a $\{(K^*_2)^r, \vec{C}^s_4\}$-factorization of the $C^*_4[2] \oplus I^*_8$ for $r + s = 5$.

Lemma 4.5: $C_4^*[2] \oplus I_8^*$ *has a* $\{(K_2^*)^r, \overrightarrow{C}_4^s\}$*-factorization for* $r \in \{0,1,2,3,5\}$ *with* $r+s=5$.

Proof. $C_4^*[2] \oplus I_8^*$ is represented as the directed Cayley graph $\overrightarrow{X}(\mathbb{Z}_8, S)$ with connection set $S=\{\pm1,\pm3,4\}$.

For $r=0$, let a $\overrightarrow{C}_4$-factorization of $C_4^*[2] \oplus I_8^*$ be defined as follows:
$\mathcal{F}_1 = \Big\{[(0,1,2,3),(4,5,6,7)], [(0,3,2,1),(4,7,6,5)], [(0,5,1,4),(2,7,3,6)], [(0,4,3,7),(1,5,2,6)], [(0,7,2,5),(1,6,3,4)]\Big\}$.

For $r=2$, let a $\{(K_2^*)^2, \overrightarrow{C}_4^3\}$-factorization of $C_4^*[2] \oplus I_8^*$ be defined as follows:
$\mathcal{F}_2 = \Big\{[(0,4)^*,(1,5)^*,(2,6)^*,(3,7)^*], [(0,7)^*,(1,6)^*,(2,5)^*,(3,4)^*], [(0,1,2,3),(4,5,6,7))], [(0,3,6,5),(1,4,7,2)], [(0,5,4,1),(2,7,6,3)]\Big\}$.

The remaining cases are obtained from Corollary 2.21 for $m=4$. □

Lemma 4.6: K_{12}^* *has a* $\{(K_2^*)^r, \overrightarrow{C}_4^s\}$*-factorization for* $r \in \{0,1,2,3,4,5,7,9,11\}$ *with* $r+s=11$.

Proof. The cases $r=0$ and $r=11$ are obtained by Theorem 3.10 and Proposition 2.4, respectively. Since, $K_{12}-I$ has a C_4-factorization where I is a 1-factor of K_{12}, by Proposition 2.3, K_{12}^* can be factorized into five C_4^*-factors and one I^*-factor which is a K_2^*-factor of K_{12}^*. Also, C_4^* has a $\overrightarrow{C}_4$-factorization and K_2^*-factorization. So, a $\{(K_2^*)^r, \overrightarrow{C}_4^s\}$-factorization of K_{12}^* is obtained for $r \in \{1,3,5,7,9\}$ with $r+s=11$.

K_{12}^* is represented as the directed Cayley graph $\overrightarrow{X}(\mathbb{Z}_{12}, S)$ with connection set $S=\{\pm1,\pm2,\pm3,\pm4,\pm5,6\}$, and define the following factorizations of K_{12}^* for $r=2,4$, respectively.

$\mathcal{F}_1 = \Big\{[(0,6)^*,(1,7)^*,(2,8)^*,(3,9)^*,(4,10)^*,(5,11)^*], [(0,10)^*,(4,6)^*,(1,5)^*,(7,11)^*,(2,9)^*,(3,8)^*], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,6),(8,11,10,9)], [(0,3,1,8),(2,4,11,6),(5,9,7,10)], [(0,4,2,11),(1,6,8,10),(3,7,9,5)], [(0,5,8,7),(1,3,4,9),(2,10,6,11)], [(0,7,5,2),(1,10,8,4),(3,6,9,11)], [(0,8,6,1),(2,5,10,7),(3,11,9,4)], [(0,9,6,5),(1,11,4,8),(2,7,3,10)], [(0,11,1,9),(2,6,10,3),(4,7,8,5)]\Big\}$,

$\mathcal{F}_2 = \Big\{[(0,6)^*,(1,7)^*,(2,8)^*,(3,9)^*,(4,10)^*,(5,11)^*], [(0,10)^*,(4,6)^*,(1,5)^*,(7,11)^*,(2,9)^*,(3,8)^*], [(0,8)^*,(2,6)^*,(1,10)^*,(4,7)^*,(3,11)^*,(5,9)^*], [(0,1)^*,(2,3)^*,(4,5)^*,(6,7)^*,(8,9)^*,(10,11)^*], [(0,2,1,3),(4,8,11,9),(5,7,10,6)], [(0,3,10,5),(1,8,6,11),(2,4,9,7)], [(0,4,11,2),(1,6,10,9),(3,5,8,7)], [(0,5,6,9),(1,2,11,4),(3,7,8,10)], [(0,7,9,11),(1,4,3,6),(2,10,8,5)],

$[(0,9,10,7),(1,11,6,8),(2,5,3,4)],[(0,11,8,4),(1,9,6,3),(2,7,5,10)]\Big\}$.

Therefore, K_{12}^* has a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization for $r \in \{0,1,2,3,4,5,7,9,11\}$ with $r+s=11$. □

Lemma 4.7: $K_{(4:3)}^*$ has a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization for $r \in \{0,1,2,4,6,8\}$ with $r+s=8$.

Proof. The cases $r=0$ and $r=8$ are obtained by Lemmata 2.7 and 2.10, respectively. $K_{(4:3)}^*$ has a C_4^*-factorization by Lemma 2.7. Since C_4^* has a K_2^*-factorization and a $\vec{C}_4$-factorization, $K_{(4:3)}$ can be factorized into two K_2^*-factors and six $\vec{C}_4$-factors. Similarly, a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization of $K_{(4:3)}$ is obtained for $r \in \{4,6\}$ with $r+s=8$.

Finally, let $V(K_{(4:3)}^*) = \bigcup_{i=0}^{2}\{4i, 4i+1, 4i+2, 4i+3\}$ with the obvious vertex partition, and define the following factorization of $K_{(4:3)}^*$ for $r=1$:

$\mathcal{F}_1 = \Big\{[(0,4,2,5),(1,8,3,11),(6,9,7,10)],[(0,5,1,7),(2,9,4,11),(3,8,6,10)],[(0,7,1,9),$
$(2,4,3,10),(5,11,6,8)],[(0,8,1,10),(2,7,3,5),(4,9,6,11)],[(0,9,2,11),(1,5,3,6),$
$(4,10,7,8)],[(0,10,4,8),(1,11,5,9),(2,6,3,7)],[(0,11,3,4),(1,6,2,10),(5,8,7,9)],[(0,6)^*,$
$(1,4)^*,(2,8)^*,(3,9)^*,(5,10)^*,(7,11)^*]\Big\}$. □

Now we are going to analyze the solution of $\text{HWP}^*(v; 2^r, m^s)$ for $m=4$ in the following theorem.

Theorem 4.8: Let r, s be nonnegative integers. Then, $\text{HWP}^*(v; 2^r, 4^s)$ has a solution if and only if $r+s=v-1$ except for $s=1$ or $(r,v)=(0,4)$, and except possibly when at least one of the following conditions holds;

1. *$r \geq 2$ even and $v \equiv 4, 20 \pmod{24}$,*
2. *$s \in \{3,5\}$ and $v \equiv 12 \pmod{24}$.*

Proof. If you remove $(v-2)$ disjoint K_2^*-factors from K_v^*, then the remaining factor must be a K_2^*-factor in K_v^*. Thus, there is no $\{(K_2^*)^{v-2}, \vec{C}_4^1\}$-factorization of K_v^*. So, it can be assumed that $s \neq 1$.

Since $\text{HWP}^*(v; n^r, m^s)$ has a solution for $r=0$ except for $(v,m)=(4,4)$ by Theorem 1.15, $\text{HWP}^*(4; 2^r, 4^s)$ has no solution for $r=0$. So, one may assume that $r \geq 1$.

Case 1 ($v \equiv 0 \pmod{8}$) : Let $v=8k$ for a positive integer k. Note that, K_{8k}^* can be factorized as $K_{4k}^*[2] \oplus I_{8k}^*$. Also, $K_{4k}^*[2]$ can be factorized into $C_4^*[2]$-factors and a

$K_2^*[2]$-factor.

$$\begin{aligned} K_{8k}^* &\cong K_{4k}^*[2] \oplus I_{8k}^* \\ &\cong kC_4^*[2] \oplus kC_4^*[2] \oplus kC_4^*[2] \oplus \ldots \oplus kC_4^*[2] \oplus K_2^*[2] \oplus I_{8k}^* \end{aligned} \quad (4.4)$$

The graph $kC_4^*[2] \oplus I_{8k}^*$ can be considered as $(C_4^*[2] \oplus I_8^*)$-factor in K_{8k}^*. Therefore, K_{8k}^* has a $\{(C_4^*[2])^{2k-1}, I_8^*, K_2^*[2]\}$-factorization. Also, $C_4^*[2]$ has a $\{(K_2^*)^{r_0}, \vec{C}_4^{s_0}\}$-factorization for $r_0 \in \{0,2,4\}$ where $r_0 + s_0 = 4$ by Lemma 2.20. Since $K_2^*[2] = C_4^*$, $K_2^*[2]$ has a $\{(K_2^*)^{r_1}, \vec{C}_4^{s_1}\}$-factorization for $r_1 \in \{0,2\}$ and $r_1 + s_1 = 2$. By Lemma 4.5, $C_4^*[2] \oplus I_8^*$ has a $\{(K_2^*)^{r_2}, \vec{C}_4^{s_2}\}$-factorization for $r_2 \in \{0,1,2,3,5\}$ where $r_2 + s_2 = 5$. These factorizations give a $\left\{(K_2^*)^r, \vec{C}_4^s\right\}$-factorization of K_{8k}^* for $r \neq 8k-2$ with $r + s = 8k - 1$.

Then, HWP$^*(v; 2^r, 4^s)$ has a solution for $r + s = v - 1$, $s \neq 1$ and $v \equiv 0 \pmod 8$.

Case 2 ($v \equiv 4 \pmod 8$) : Let $v = 8k + 4$ for a nonnegative integer k.

(a) Assume r is odd. Partition the vertices of K_{8k+4}^* into $4k+2$ sets of size 2, represent each set of size 2 vertices in K_{8k+4}^* with a single vertex and represent all double arcs between sets of size 2 as a single double arc, to get a K_{4k+2}^*. By Proposition 2.4, K_{4k+2}^* has a decomposition into $4k+1$ K_2^*-factors. Construct a K_4^*-factor from one of the K_2^*-factors and a $K_{(2:2)}^*$-factor from each of the remaining $4k$ K_2^*-factors. Then, factorize K_{8k+4}^* into a K_4^*-factor and $(4k)$ $K_{(2:2)}^*$-factors.

$$K_{8k+4}^* \cong (2k+1)K_4^* \oplus \underbrace{(2k+1)K_{(2:2)}^* \oplus \ldots \oplus (2k+1)K_{(2:2)}^*}_{4k} \quad (4.5)$$

K_4^* has a decomposition into one K_2^* and two $\vec{C}_4$-factors or three K_2^*-factors, and $K_{(2:2)}^*$ has a $\{(K_2^*)^{r_0}, \vec{C}_4^{s_0}\}$-factorization for $r_0 \in \{0,2\}$ satisfying $r + s = 2$. So, K_{8k+4}^* has a $\{(K_2^*)^r, \vec{C}_4^s\}$-factorization for odd r. Therefore, HWP$^*(v; 2^r, 4^s)$ has a solution for odd r and $v \equiv 4 \pmod 8$.

(b) Starting with the assumption of r is even and $k \equiv 1 \pmod 3$, it follows that $v = 24l + 12$ for some nonnegative integer l.

Representing each part of 4 vertices in K_{24l+12}^* with a single vertex and all double arcs between parts of size 4 as a single double arc, a K_{6l+3}^* is derived. Since a Kirkman triple system exists for orders $6l+3$ by Theorem 1.12, we have a C_3-factorization of K_{6l+3}. Then, a C_3^*-factorization of K_{6l+3}^* is obtained by Proposition 2.3.

Construct a K_{12}^*-factor from one of the C_3^*-factors and $K_{(4:3)}^*$-factor from each of the remaining $3l$ C_3^*-factors. Then, get a $\{K_{12}^*, (K_{(4:3)}^*)^{3l}\}$-factorization of K_{24l+12}^*.

$$K_{24l+12}^* \cong (2l+1)K_{12}^* \oplus \underbrace{(2l+1)K_{(4:3)}^* \oplus \ldots \oplus (2l+1)K_{(4:3)}^*}_{3l}. \qquad (4.6)$$

By Lemma 4.6, K_{12}^* has a $\{(K_2^*)^{r_0}, \vec{C}_m^{s_0}\}$-factorization for $r_0 \in \{0,1,2,3,4,5,7,9,11\}$ with $r_0+s_0 = 11$. Also, $K_{(4:3)}^*$ has a $\{(K_2^*)^{r_1}, \vec{C}_4^{s_1}\}$-factorization by Lemma 4.7 for $r_1 \in \{0,1,2,4,6,8\}$ with $r_1+s_1 = 8$. Those factorizations give a $\{(K_2^*)^r, \vec{C}_m^s\}$-factorization of K_{24l+12}^* where $r = r_0 + ar_1$ and $s = s_0 + bs_1$ satisfying $r + s = 24l + 11 = v - 1$ with $1 \leq r, s \leq v - 1$ and $a + b = 3l$. The requested even $r \in [0, v-1]$ is derived except for $r = v - 6$ and $r = v - 4$, from the sum of r_0 and ar_1. Then, HWP$^*(v; 2^r, 4^s)$ has a solution for $r + s = v - 1$, $s \notin \{3, 5\}$ and $v \equiv 12 \pmod{24}$. □

Proving Theorem 4.1 was accomplished by proving Theorems 4.4 and 4.8.

4.2. Solutions to HWP$^*(v; m^r, (2m)^s)$

In this section, for even m, a solution to HWP$^*(v; m^r, (2m)^s)$ is provided for $r+s = v-1$ and except possibly when $s \in \{1, 3\}$.

Firstly, factorize K_{2mx}^* into a K_{2m}^*-factor and $(2x-2)$ $K_{(m:2)}^*$-factors. $K_{(m:2)}^*$ has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $r \in \{0, m\}$ and $r + s = m$. Using Lemma 2.13 and Proposition 2.3, a $\{(C_m^*[2])^{\frac{m-4}{2}}, I_{2m}^*, \Gamma_m^*\}$-factorization of K_{2m}^* is also obtained. Therefore, in order to factorize K_{2mx}^* into $\vec{C}_m$-factors and $\vec{C}_{2m}$-factors, Γ_m^*, $C_m^*[2] \oplus I_{2m}^*$ and $C_m^*[2]$ must be factorized into $\vec{C}_m$-factors and $\vec{C}_{2m}$-factors. The Lemmata 2.26, 2.27 and 2.29 explore the existence of a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization of these graphs for $r + s \in \{4, 5, 6\}$. Using these lemmata, the following theorem can be given.

Theorem 4.2: Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, HWP$^*(v; m^r, (2m)^s)$ *has a solution if and only if* $m|v$, $r + s = v - 1$ *and* $v \geq 4$ *except for* $(s, v, m) \in \{(0,4,4), (0,6,3), (0,6,6)\}$*, and except possibly when* $s \in \{1, 3\}$.

Proof. By Theorem 3.6, HWP$^*(v; 4^r, 8^s)$ has a solution for $r + s = v - 1$, so we may assume that $m \geq 6$. Furthermore, by Theorem 1.15, a solution to the HWP$^*(v; m^r, (2m)^s)$ exists for $r = s = 0$ and except for $(s, v, m) \in \{(0,4,4), (0,6,3), (0,6,6)\}$.

Factorize K^*_{2mx} into a K^*_{2m}-factor and $(2x-2)$ $K^*_{(m:2)}$-factors. By Lemma 2.7, $K^*_{(m:2)}$ decomposes into m $\vec{C}_m$-factors or m $\vec{C}_{2m}$-factors. So, K^*_{2m} must be decomposed into $\vec{C}_m$-factors and $\vec{C}_{2m}$-factors. As before, K^*_{2m} can be factorized as $K^*_m[2] \oplus I^*_{2m}$. Consequently, K^*_{2m} has a $\{(C^*_m[2])^{\frac{m-4}{2}}, I^*_{2m}, \Gamma^*_m\}$-factorization. By Lemma 2.27, each of $C^*_m[2]$-factors has a $\{\vec{C}^{r_0}_m, \vec{C}^{s_0}_{2m}\}$-factorization for $r_0 \in \{0,2,4\}$ and $r_0 + s_0 = 4$. By Lemmata 2.26 and 2.19, $C^*_m[2] \oplus I^*_{2m}$ has a $\{\vec{C}^{r_1}_m, \vec{C}^{s_1}_{2m}\}$-factorization for $r_1 \in \{0,1,3\}$ and $r_1 + s_1 = 5$. By Lemma 2.29, Γ^*_m has a $\{\vec{C}^{r_2}_m, \vec{C}^{s_2}_{2m}\}$-factorization for $r_2 \in \{0,6\}$ with $r_2 + s_2 = 6$. Those factorizations give a $\{\vec{C}^{r}_m, \vec{C}^{s}_{2m}\}$-factorization of K^*_{2m} where $r = (\frac{m-6}{2})r_0 + r_1 + r_2$ and $s = (\frac{m-6}{2})s_0 + s_1 + s_2$ satisfying $r + s = (\frac{m-6}{2})4 + 5 + 6 = 2m-1$ with $0 \leq r, s \leq 2m-1$ and $s \notin \{1,3\}$.

Placing a $\vec{C}_m$-factorization on r' of the $K^*_{(m:2)}$-factors for $0 \leq r' \leq 2x-2$, a $\vec{C}_{2m}$-factorization on s' of the $K^*_{(m:2)}$ for $r' + s' = 2x - 2$, and taking a $\{\vec{C}^{r}_m, \vec{C}^{s}_{2m}\}$-factorization of K^*_{2m} give a $\{\vec{C}^{mr'+r}_m, \vec{C}^{ms'+s}_{2m}\}$-factorization of K^*_{2mx}. Then, HWP$^*(v; m^r, 2m^s)$ has a solution except possibly when $s \in \{1,3\}$. $\square$

5. CONCLUSION

In this thesis, the Directed Hamilton-Waterloo Problem with two cycle sizes is studied. Although there are many results on the Hamilton-Waterloo Problem, to the best of our knowledge, our results are among the first findings for a directed version of the Hamilton-Waterloo Problem.

In Section 3, the cases $(m, n) \in \{(4,6),(4,8),(4,16),(8,16),(3,5),(3,15),(5,15)\}$ are analyzed, as was done in the first paper on the undirected Hamilton-Waterloo Problem by Adams et al. [32]. The problem has also been solved for the cases $(m, n) \in \{(4,12),(6,12)\}$. It has also been observed that if HWP$(v; m^r, n^s)$ has a solution for odd v, then HWP$^*(v; m^{2r}, n^{2s})$ has a solution for the same r and s as well. The constructions in Lemma 3.2 and Lemma 3.7 are actually general constructions, and they can also be used to solve the problem for the other cycle sizes as long as the necessary small cases can be found.

Also in the third section, by combining the results of Theorems 3.6 and 3.9, the following theorem is derived.

Theorem 3.10: For nonnegative integers r and s, HWP$^*(v; m^r, n^s)$ *has a solution for*

1. $(m, n) \in \{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16)\}$ *when v is even,*
2. $(m, n) \in \{(3,5),(3,15),(5,15)\}$ *when v is odd*

if and only if $r + s = v - 1$ and $\text{lcm}(m, n)|v$ *except possibly $s \in \{1,2,3\}$ when $(m, n) = (3,5)$ and $s = 1$ when $(m, n) = (3,15)$.*

If the exceptions in the cases $r \in \{11,12,13\}$ when $(m, n) = (3,5)$ and for $r = 13$ when $(m, n) = (3,15)$ in Lemma 3.8 are solved for $v = 15$, the possible exception $s \in \{1,2,3\}$ when $(m, n) = (3,5)$, and $s = 1$ when $(m, n) = (3,15)$ will be solved for $v = 15x$.

In Section 4, the focus was on establishing the existence and exploring all potential solutions for K_2^*-factors and directed m-cycles factors of K_v^*. In short, our research focused on finding the solutions to HWP$^*(v; 2^r, m^s)$ for even m with $r + s = v - 1$. This problem was addressed for values of m greater than 4 in Theorem 4.4 and it has been solved with a few possible exceptions. Furthermore, in Theorem 4.8 solutions for

HWP*$(v; 2^r, 4^s)$ are presented except for $r \in \{0, 2\}$ when $v = 4$ and except possibly when at least one of the following conditions holds;

1. $r \geq 2$ even and $v \equiv 4, 20 \pmod{24}$,
2. $s \in \{3, 5\}$ and $v \equiv 12 \pmod{24}$.

The following theorem can be stated as a consequence of Theorem 4.4 and 4.8.

Theorem 4.1: Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, HWP*$(v; 2^r, m^s)$ *has a solution if $m|v$, $r + s = v - 1$, $s \neq 1$, $(r, v) \neq (0, 6)$, $(m, r, v) \neq (4, 0, 4)$, and one of the following conditions holds;*

1. *$m > 4$, $s \neq 3$ and $m \equiv 0 \pmod{4}$,*
2. *$m > 4$, $\frac{v}{m}$ is even and $m \equiv 2 \pmod{4}$,*
3. *$m > 4$, $\frac{v}{m}$ is odd, $s \neq 3$ and $m \equiv 2 \pmod{4}$,*
4. *$m = 4$ and $v \equiv 0, 8, 16 \pmod{24}$,*
5. *$m = 4$, $v \equiv 12 \pmod{24}$ and $s \notin \{3, 5\}$,*
6. *$m = 4$, $v \equiv 4, 20 \pmod{24}$ and r is odd.*

Finally, we turned our attention to solving HWP*$(v; m^r, (2m)^s)$ for even values of m under the condition $r + s = v - 1$, successfully solving HWP*$(v; m^r, (2m)^s)$ except possibly for the case of $s \in \{1, 3\}$, as indicated in the following Theorem.

Theorem 4.2: Let r, s be nonnegative integers, and let $m \geq 4$ be even. Then, HWP*$(v; m^r, (2m)^s)$ *has a solution if and only if $m|v$, $r + s = v - 1$ and $v \geq 4$ except for $(s, v, m) \in \{(0, 4, 4), (0, 6, 3), (0, 6, 6)\}$, and except possibly when $s \in \{1, 3\}$.*

To solve the above problem, we examined the $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization of the graphs $C_m^*[2]$, $C_m^*[2] \oplus I_{2m}^*$, Γ_m^*, and $K_{(m:2)}^*$ for even m. If one of these graphs has a $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization for $s = 1$, the possible exceptions here are obtained. As future work, we can investigate the $\{\vec{C}_m^r, \vec{C}_{2m}^s\}$-factorization of these graphs for odd m as well, thus finding a solution to the he Directed Hamilton-Waterloo Problem with cycle lengths of different parities.

In this thesis, the Directed Hamilton-Waterloo Problem for small cycle lengths is studied first, following similar cases of the undirected Hamilton-Waterloo Problem. Here, the analyzed cases had cycle lengths that were either both odd or both even. In addition, the problem for the cycle lengths m and $2m$ is solved, where m is even.

Note that, if the directions of the arcs in the factors in the solution of HWP$^*(v; m^r, n^s)$ are removed, then solving HWP$^*(v; m^r, n^s)$ corresponds to factorizing the 2-fold K_v into r C_m-factors and s C_n-factors. So, a solution to HWP$^*(v; m^r, n^s)$ gives rise to a $\{C_m^r, C_n^s\}$-factorization of 2-fold K_v. But the converse is not necessarily true. So, our constructions have consequences on undirected factorizations as well.

As future research, one could explore situations where one of the cycle lengths remains constant, or more generally, cases where the parameters m and n can be arbitrary even or odd numbers. It may also be interesting to study the Directed Hamilton-Waterloo Problem in the case where a cycle is a Hamiltonian cycle.

Subsequently, it may be beneficial to study the cases with cycle lengths of different parities. Also, for the general cases we need to factorize $K_{\ell x}^*$ into $\vec{C}_m$-factors and $\vec{C}_n$-factors where $\ell = \text{lcm}(m, n)$. We can factorize $K_{\ell x}^*$ as follows:

$$K_{\ell x}^* \cong xK_\ell^* \oplus K_{(\ell:x)}^* \tag{5.1}$$

If one can factorize K_ℓ^* and $K_{(\ell:x)}^*$ into the desired $\vec{C}_m$-factors and $\vec{C}_n$-factors, the problem can be solved.

REFERENCES

[1] West D. B., (2001), “Introduction to Graph Theory”, 2st Edition, Pearson.

[2] Chartrand G., Lesniak L., Zhang P., (2010), “Graphs & digraphs”, 5th Edition, CRC press.

[3] Imrich W., Klavžar S., (2000), “Product graphs: structure and recognition”, John Wiley and Sons.

[4] Alspach B., Schellenberg P. J., Stinson D. R., Wagner D., (1989), “The Oberwolfach problem and factors of uniform odd length cycles”, Journal of Combinatorial Theory, Series A, 52(1), 20–43.

[5] Veblen O., (1912), “An application of modular equations in analysis situs”, The Annals of Mathematics, 14(1/4), 86–94.

[6] Kirkman T. P., (1847), “On a problem in combinations”, Cambridge and Dublin Mathematical Journal, 2, 191–204.

[7] Ray-Chaudhuri D. K., Wilson R. M., (1971), “Solution of Kirkman’s schoolgirl problem”, Proc. symp. pure Math, 19, 187–203.

[8] Guy R. K., (1971), “Unsolved combinatorial problems”, Proceedings on the Conference on Combinatorial Mathematics and its Applications, Oxford, 1967 D. J. A. Welsh, Ed.

[9] Huang C., Kotzig A., Rosa A., (1979), “On a variation of the Oberwolfach problem”, Discrete Mathematics, 27(3), 261–277.

[10] Alspach B., Häggkvist R., (1985), “Some observations on the Oberwolfach problem”, Journal of Graph Theory, 9(1), 177–187.

[11] Hoffman D. G., Schellenberg P. J., (1991), “The existence of C_k-factorizations of $K_{2n} - F$”, Discrete mathematics, 97(1-3), 243–250.

[12] Adams P., Bryant D., (2006), “Two-factorisations of complete graphs of orders fifteen and seventeen”, Australasian Journal of Combinatorics, 35(1), 113–118.

[13] Deza A., Franek F., Hua W., Meszka M., Rosa A., (2010), “Solutions to the Oberwolfach problem for orders 18 to 40”, Journal of Combinatorial Mathematics and Combinatorial Computing, 74, 95–102.

[14] Salassa F., Dragotto G., Traetta T., Buratti M., Della Croce F., (2021), “Merging combinatorial design and optimization: the Oberwolfach Problem”, Australasian Journal of Combinatorics, 79(1), 141–166.

[15] Franek F., Rosa A., (2000), “Two-factorizations of small complete graphs”, Journal of statistical planning and inference, 86(2), 435–442.

[16] Franek F., Holub J., Rosa A., (2004), “Two-factorizations of small complete graphs: II. The case of 13 vertices”, J. Combin. Math. Combin. Comput, 51, 89–94.

[17] Bryant D., Danziger P., (2011), “On bipartite 2-factorizations of $K_n - I$ and the Oberwolfach problem”, Journal of Graph Theory, 68(1), 22–37.

[18] Häggkvist R., (1985), “A lemma on cycle decompositions”, North-Holland Mathematics Studies, 115, 227–232.

[19] Liu J., (2003), “The equipartite Oberwolfach problem with uniform tables”, Journal of Combinatorial Theory, Series A, 101(1), 20–34.

[20] Burgess A., Šajna M., (2014), “On the directed Oberwolfach Problem with equal cycle lengths”, The Electronic Journal of Combinatorics, 21(1), 1–15.

[21] Bermond J. C., Germa A., Sotteau D., (1979), “Resolvable decomposition of K_n^*”, Journal of Combinatorial Theory, Series A, 26(2), 179–185.

[22] Bennett F., Zhang X., (1990), “Resolvable Mendelsohn designs with block size 4”, Aequationes Mathematicae, 40(1), 248–260.

[23] Adams P., Bryant D., (unpublished), “Resolvable directed cycle systems of all indices for cycle length 3 and 4”.

[24] Alspach B., Gavlas H., Šajna M., Verrall H., (2003), “Cycle decompositions IV: complete directed graphs and fixed length directed cycles”, Journal of Combinatorial Theory, Series A, 103(1), 165–208.

[25] Burgess A., Francetic N., Šajna M., (2018), “On the directed Oberwolfach Problem with equal cycle lengths: the odd case”, Australasian Journal of Combinatorics, 71(2), 272–292.

[26] Shabani E., Šajna M., (2020), “On the Directed Oberwolfach Problem with variable cycle lengths”, arXiv preprint arXiv:2009.08731, 1–15.

[27] Kadri S., Šajna M., (2023), “The directed Oberwolfach problem with variable cycle lengths: a recursive construction”, arXiv preprint arXiv:2309.12549.

[28] Lacaze-Masmonteil A., (2024), “Completing the solution of the directed Oberwolfach problem with cycles of equal length”, Journal of Combinatorial Designs, 32(1), 5–30.

[29] Abel R. J. R., Bennett F. E., Ge G., (2002), “Resolvable perfect Mendelsohn designs with block size five”, Discrete Math., 247, 1–12.

[30] Tillson T. W., (1980), “A Hamiltonian decomposition of K_{2m}^*, $2m \geq 8$”, Journal of Combinatorial Theory, Series B, 29(1), 68–74.

[31] Francetić N., Šajna M., (2023), “On the directed Oberwolfach problem for complete symmetric equipartite digraphs and uniform-length cycles”, Journal of Combinatorial Designs, 31(11), 604–641.

[32] Adams P., Billington E. J., Bryant D. E., El-Zanati S. I., (2002), “On the Hamilton–Waterloo problem”, Graphs and Combinatorics, 18, 31–51.

[33] Burgess A., Danziger P., Traetta T., (2017), “On the Hamilton–Waterloo problem with odd orders”, Journal of Combinatorial Designs, 25(6), 258–287.

[34] Burgess A., Danziger P., Traetta T., (2018), “On the Hamilton–Waterloo problem with odd cycle lengths”, Journal of Combinatorial Designs, 26(2), 51–83.

[35] Bryant D., Danziger P., Dean M., (2013), “On the Hamilton–Waterloo Problem for Bipartite 2-Factors”, Journal of Combinatorial Designs, 21(2), 60–80.

[36] Burgess A. C., Danziger P., Traetta T., (2019), “The Hamilton–Waterloo Problem with even cycle lengths”, Discrete Mathematics, 342(8), 2213–2222.

[37] Bonvicini S., Buratti M., (2018), “Octahedral, dicyclic and special linear solutions of some Hamilton–Waterloo problems”, Ars Mathematica Contemporanea, 14(1), 1–14.

[38] Danziger P., Quattrocchi G., Stevens B., (2009), “The Hamilton–Waterloo problem for cycle sizes 3 and 4”, Journal of Combinatorial Designs, 17(4), 342–352.

[39] Wang L., Chen F., Cao H., (2017), “The Hamilton–Waterloo problem for C_3-factors and C_n-factors”, Journal of Combinatorial Designs, 25(9), 385–418.

[40] Odabaşı U., Özkan S., (2016), “The Hamilton–Waterloo problem with C_4 and C_m factors”, Discrete Mathematics, 339(1), 263–269.

[41] Asplund J., Kamin D., Keranen M., Pastine A., Özkan S., (2016), “On the Hamilton–Waterloo problem with triangle factors and C_{3x}-factors”, Australasian Journal of Combinatorics, 64(3), 458–474.

[42] Lei H., Shen H., (2012), “The Hamilton–Waterloo problem for Hamilton cycles and triangle-factors”, Journal of Combinatorial Designs, 20(7), 305–316.

[43] Keranen M. S., Özkan S., (2013), “The Hamilton–Waterloo problem with 4-cycles and a single factor of n-cycles”, Graphs and Combinatorics, 29(6), 1827–1837.

[44] Burgess A., Danziger P., Pastine A., Traetta T., (2024), “Constructing uniform 2-factorizations via row-sum matrices: Solutions to the Hamilton–Waterloo problem”, Journal of Combinatorial Theory, Series A, 201, 105803.

[45] Burgess A., Danziger P., Traetta T., (2018), “On the Hamilton–Waterloo Problem with cycle lengths of distinct parities”, Discrete Mathematics, 341(6), 1636–1644.

[46] Keranen M. S., Pastine A., (2019), “On the Hamilton–Waterloo problem: the case of two cycles sizes of different parity”, Ars Mathematica Contemporanea, 17(2), 525.

[47] Glock S., Joos F., Kim J., Kühn D., Osthus D., (2021), “Resolution of the Oberwolfach problem”, Journal of the European Mathematical Society, 23(8), 2511–2547.

[48] Traetta T., (2023), “A constructive solution to op with a large cycle”, arXiv preprint arXiv:2306.12713.

[49] Reiss M., (1859), “Ueber eine Steinersche combinatorische Aufgabe, welche im 45sten Bande dieses Journals, Seite 181, gestellt worden ist.”, Journal für die reine und angewandte Mathematik, 56, 326–344.

[50] Bennett F., Wei R., Zhu L., (1997), “Resolvable Mendelsohn triple systems with equal sized holes”, Journal of Combinatorial Designs, 5(5), 329–340.

[51] Hoffman D. G., Rodger C. A., (1992), “The chromatic index of complete multipartite graphs”, Journal of Graph Theory, 16(2), 159–163.

[52] Lucas É., (1982), “Récréations mathématiques”, Gauthier-Villars.

[53] Odabaşı U., (2020), “Factorizations of complete graphs into cycles and 1-factors”, Contributions to Discrete Mathematics, 15(1), 80–89.

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CONFERENCE PRESENTATIONS

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Yetgin, F., Odabaşı, U., Özkan, S. (2022), "On the Directed Hamilton-Waterloo Problem with Uniform Cycle Sizes", ACA 2022 (Applications of Computer Algebra).

Yetgin, F., Odabaşı, U., Özkan, S. (2022), "On a Directed Cycle Factorization Problem", GTÜ Lisansüstü Araştırmalar Sempozyumu.

Yetgin, F., Odabaşı, U., Özkan, S. (2023), "On the Directed Hamilton-Waterloo Problem", WGT-Workshop on Graph Theory and Its Applications-XI.

APPENDIX

Let $V(K_8^*) = \mathbb{Z}_8, V(K_{12}^*) = \mathbb{Z}_{12}, V(K_{16}^*) = \mathbb{Z}_{16}$ and $V(K_{15}^*) = \mathbb{Z}_{15}$,

1. HWP*$(8; 4^1, 8^6)$,
   [(0,1,2,3),(4,5,6,7)], [(0,3,2,4,6,5,7,1)], [(0,2,1,3,5,4,7,6)], [(0,7,5,3,6,1,4,2)], [(0,6,4,3,1,7,2,5)], [(0,5,1,6,2,7,3,4)], [(0,4,1,5,2,6,3,7)]
2. HWP*$(8; 4^2, 8^5)$,
   [(0,2,7,6),(1,4,5,3)], [(0,5,6,7),(1,3,2,4)], [(0,1,7,2,6,3,5,4)], [(0,3,7,4,6,5,1,2)],[(0,4,2,5,7,3,6,1)], [(0,6,2,3,4,7,1,5)], [(0,7,5,2,1,6,4,3)]
3. HWP*$(8; 4^3, 8^4)$,
   [(0,1,2,3),(4,5,6,7)], [(0,2,4,6),(1,3,5,7)], [(0,3,2,1),(4,7,6,5)], [(0,4,1,5,2,6,3,7)], [(0,5,1,6,2,7,3,4)], [(0,6,4,3,1,7,2,5)], [(0,7,5,3,6,1,4,2)]
4. HWP*$(8; 4^4, 8^3)$,
   [(0,3,7,4),(1,2,6,5)], [(0,1,7,2),(3,5,4,6)], [(0,2,7,6),(1,4,5,3)], [(0,5,6,7),(1,3,2,4)], [(0,4,2,5,7,3,6,1)], [(0,6,2,3,4,7,1,5)], [(0,7,5,2,1,6,4,3)]
5. HWP*$(8; 4^5, 8^2)$,
   [(0,1,2,3),(4,5,6,7)], [(0,2,4,6),(1,3,5,7)], [(0,3,2,1),(4,7,6,5)], [(0,6,4,2),(1,7,5,3)], [(0,7,2,5),(4,3,6,1)], [(0,4,1,5,2,6,3,7)], [(0,5,1,6,2,7,3,4)]
6. HWP*$(8; 4^6, 8^1)$,
   [(0,2,1,3),(4,7,6,5)], [(0,3,7,4),(1,5,2,6)], [(0,4,1,6),(2,7,5,3)], [(0,5,7,1),(2,4,3,6)], [(0,6,4,2),(1,7,3,5)], [(0,7,2,5),(1,4,6,3)], [(0,1,2,3,4,5,6,7)]
7. HWP*$(12; 4^1, 6^{10})$,
   [(0,2,1,3,4,6),(5,7,8,10,9,11)], [(0,3,1,4,2,5),(6,8,11,9,7,10)], [(0,4,1,5,2,7),(3,9,6,11,10,8)], [(0,5,1,6,2,8),(3,10,4,11,7,9)][(0,6,1,7,2,9),(3,8,4,10,5,11)], [(0,7,1,8,2,10),(3,11,6,5,9,4)], [(0,8,6,3,2,4),(1,9,5,10,7,11)], [(0,9,1,10,2,11),(3,7,6,4,8,5)], [(0,10,3,6,9,2), (1,11,4,7,5,8)], [(0,11,2,6,10,1),(3,5,4,9,8,7)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)]
8. HWP*$(12; 4^2, 6^9)$,
   [(0,3,1,5,2,6),(4,7,10,8,11,9)], [(0,4,1,3,2,5),(6,8,10,9,7,11)], [0,5,1,6,2,8),(3,10,4,11,7,9)], [(0,6,1,7,2,9),(3,8,4,10,5,11)], [(0,7,1,9,6,10),(2,4,3,11,5,8)], [(0,8,1,11,2,7),(3,6,4,9,5,10)], [(0,9,1,10,2,11),(3,4,8,7,6,5)], [(0,10,7,3,9,2),(1,8,5,4,6,11)], [(0,11,4,2,10,1),(3,7,5,9,8,6)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)]
9. HWP*$(12; 4^3, 6^8)$,
   [(0,3,1,6,2,7),(4,8,10,5,11,9)], [(0,5,1,3,2,8),(4,6,10,9,7,11)], [(0,11,1,10,8,2),(3,4,7,6,5,9)], [(0,6,1,7,2,9),(3,8,4,11,5,10)], [(0,7,1,11,2,10),(3,9,6,8,5,4)], [(0,8,7,3,11,6),(1,9,5,2,4,10)], [(0,9,1,8,6,11),(2,5,3,7,10,4)], [(0,10,7,5,8,1),(2,11,3,6,4,9)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)]
10. HWP*$(12; 4^4, 6^7)$,
    [(0,3,1,7,2,8),(4,6,10,5,11,9)], [(0,6,1,3,2,9),(4,11,5,8,7,10)], [(0,7,1,8,4,10),(2,11,6,5,3,9)],[(0,8,6,4,3,11),(1,9,7,5,2,10)],[(0,9,5,10,8,1),(2,7,6,11,3,4)],[(0,10,7,3,6,2),(1,11,4,8,5,9)], [(0,11,1,10,3,7),(2,5,4,9,6,8)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,

2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,6),(2,4,7,11),(3,8,10,9)]

11. HWP*(12; $4^5, 6^6$),
[(0,3,1,8,4,2),(5,11,9,6,10,7)], [(0,7,1,9,2,8),(3,6,11,5,10,4)], [(0,8,5,9,1,10),(2,7,6,4,11,3)], [(0,9,7,10,1,11),(2,5,3,4,8,6)], [(0,10,5,2,11,1),(3,9,4,6,8,7)], [(0,11,6,5,4,9),(1,3,7,2,10,8)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,6),(2,4,7,11),(3,8,10,9)], [(0,6,1,7),(2,9,5,8),(3,11,4,10)]

12. HWP*(12; $4^6, 6^5$),
[(0,3,1,9,6,8),(2,7,10,4,11,5)], [(0,8,7,6,11,9),(1,3,4,2,5,10)], [(0,9,7,5,11,1),(2,10,8,6,4,3)], [(0,10,7,3,9,2),(1,11,6,5,4,8)], [(0,11,3,7,1,10),(2,8,5,9,4,6)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,6),(2,4,7,11),(3,8,10,9)], [(0,6,1,7),(2,9,5,8),(3,11,4,10)], [(0,7,2,11),(1,8,4,9),(3,6,10,5)]

13. HWP*(12; $4^7, 6^4$),
[(0,3,9,2,7,1),(4,11,5,10,8,6)], [(0,9,4,3,2,8),(1,10,7,6,5,11)], [(0,10,4,2,5,9),(1,11,6,8,7,3)], [(0,11,9,6,2,10),(1,3,7,5,4,8)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,6),(2,4,7,11),(3,8,10,9)], [(0,6,1,7),(2,9,5,8),(3,11,4,10)], [(0,7,2,11),(1,8,4,9),(3,6,10,5)], [0,8,5,2),(1,9,7,10),(3,4,6,11)]

14. HWP*(12; 4^8,6^3),
[(0,3,9,2,7,1),(4,11,5,10,8,6)], [(0,9,4,3,2,8),(1,10,7,6,5,11)], [(0,10,4,2,5,9),(1,11,6,8,7,3)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,11,10,9)], [(0,4,1,6),(2,11,9,7),(3,8,10,5)], [(0,5,1,7),(2,6,9,8),(3,11,4,10)], [(0,6,2,10),(1,9,5,8),(3,4,7,11)], [(0,7,9,11),(1,3,6,10),(2,4,8,5)], [(0,8,11,2),(1,5,4,9),(3,7,10,6)], [(0,11,7,5),(1,8,4,6),(2,9,3,10)]

15. HWP*(12; $4^9, 6^2$),
[(0,3,9,2,7,1),(4,11,5,10,8,6)], [(0,9,4,3,2,8),(1,10,7,6,5,11)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,7),(2,10,4,8),(3,6,11,9)], [(0,6,2,11),(1,9,7,3),(4,10,5,8)], [(0,7,11,2),(1,8,5,9),(3,4,6,10)], [(0,8,7,10),(1,3,11,6),(2,9,5,4)], [(0,10,9,6),(1,11,3,8),(2,4,7,5)], [(0,11,4,9),(1,6,8,10),(2,5,3,7)]

16. HWP*(12; $4^{10}, 6^1$),
[(0,3,9,2,7,1),(4,11,5,10,8,6)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,6),(8,11,10,9)], [(0,4,1,5),(2,6,8,10),(3,7,9,11)], [(0,5,1,6),(2,8,3,11),(4,9,7,10)], [(0,6,5,2),(1,9,3,10),(4,8,7,11)], [(0,7,2,10),(1,8,5,3),(4,6,11,9)], [(0,8,1,11),(2,5,9,6),(3,4,10,7)], [(0,9,1,7),(2,4,3,8),(5,11,6,10)], [(0,10,6,9),(1,3,2,11),(4,7,5,8)], [(0,11,7,8),(1,10,3,6),(2,9,5,4)]

17. HWP*(12; $4^1, 12^{10}$),
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,3,4,6,5,7,8,10,9,11)], [(0,3,1,4,2,5,8,6,9,7,11,10)],[(0,4,1,5,2,6,3,7,10,8,11,9)], [(0,5,1,6,2,4,10,3,11,7,9,8)], [(0,6,1,8,2,9,3,10,5,11,4,7)],[(0,7,1,9,2,8,3,5,10,4,11,6)], [(0,8,1,7,3,9,6,10,2,11,5,4)], [(0,9,4,8,5,3,6,11,1,10,7,2)],[(0,10,1,11,3,2,7,6,8,4,9,5)], [(0,11,2,10,6,4,3,8,7,5,9,1)]

18. HWP*(12; $4^2, 12^9$),
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,1,4,2,5,8,6,

9,7,11,10)], [(0,4,1,5,2,6,3,7,10,8,11,9)], [(0,5,1,6,2,7,3,10,9,8,4,11)], [(0,6,1,7,2, 9,5,10,3,11,4,8)], [(0,7,1,8,2,10,4,9,6,11,3,5)], [(0,8,1,9,2,11,6,5,3,4,10,7)], [(0,9,1 ,11,7,8,5,4,3,6,10,2)], [(0,10,6,8,3,9,4,7,5,11,2,1)], [(0,11,1,10,5,9,3,2,8,7,6,4)]

19. HWP*$(12; 4^3, 12^8)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,6,1),(7,10,2 ,8),(4,9,5,11)], [(0,4,1,5,2,6,3,7,8,11,10,9)], [(0,5,1,4,2,7,3,10,6,11,9,8)], [(0,6,2,1, 7,5,9,4,10,8,3,11)], [(0,7,1,6,5,8,4,11,3,9,2,10)], [(0,8,1,9,3,2,5,10,7,11,6,4)], [(0,9, 1,8,2,11,7,6,10,4,3,5)], [(0,10,3,1,11,2,9,6,8,5,4,7)], [(0,11,1,10,5,3,4,8,6,9,7,2)]

20. HWP*$(12; 4^4, 12^7)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,6,1),(7,10,2 ,8),(4,9,5,11)], [(0,4,1,10),(2,5,3,9),(11,7,6,8)], [(0,5,1,4,2,6,3,7,11,10,9,8)], [(0,6, 2,1,5,4,10,7,8,3,11,9)], [(0,7,1,6,4,11,2,9,3,10,8,5)], [(0,8,1,7,2,10,4,3,5,9,6,11)], [(0,9,1,11,6,10,5,8,4,7,3,2)], [(0,10,3,4,8,6,5,2,11,1,9,7)], [(0,11,3,1,8,2,7,5,10,6,9, 4)]

21. HWP*$(12; 4^5, 12^6)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,6,1),(7,10,2 ,8),(4,9,5,11)], [(0,4,1,10),(2,5,3,9),(11,7,6,8)], [(0,8,3,7),(9,6,11,2),(1,4,10,5)], [(0,5,2,1,6,3,4,7,11,10,9,8)], [(0,6,2,7,1,5,9,3,10,8,4,11)], [(0,7,2,6,4,8,5,10,3,11,1,9)], [(0,9,1,7,8,6,10,4,2,11,3,5)], [(0,10,7,5,8,1,11,6,9,4,3,2)], [(0,11,9,7,3,1,8,2,10,6, 5,4)]

22. HWP*$(12; 4^6, 12^5)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,6,1),(7,10,2 ,8),(4,9,5,11)], [(0,4,1,10),(2,5,3,9),(11,7,6,8)], [(0,8,3,7),(9,6,11,2),(1,4,10,5)], [(0,9,3,2),(1,5,10,8),(11,6,4,7)], [(0,5,2,1,6,3,4,11,10,9,7,8)], [(0,6,2,7,1,11,3,10,4, 8,5,9)], [(0,7,2,10,3,11,1,9,8,6,5,4)], [(0,10,6,9,1,7,3,5,8,4,2,11)], [(0,11,9,4,3,1,8, 2,6,10,7,5)]

23. HWP*$(12; 4^7, 12^4)$,
[(0,3,2,1),(4,7,6,9),(5,10,8,11)], [(0,4,8,3),(1,7,9,6),(2,5,11,10)], [(0,5,8,7),(1,9,2, 11),(3,6,4,10)], [(0,6,2,10),(1,11,8,5),(3,9,7,4)], [(0,7,5,4),(1,8,2,9),(3,10,6,11)], [(0,9,5,2),(1,6,3,8),(4,11,7,10)], [(0,10,9,8,),(1,4,2,7),(3,11,6,5)], [(0,11,4,1,10,5, 9,3,7,2,8,6)], [(0,8,4,9,11,2,6,10,7,3,1,5)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8, 10,1,3,5,7,11,9)]

24. HWP*$(12; 4^8, 12^3)$,
[(0,3,2,1),(4,7,5,9),(6,11,10,8)], [(0,4,1,6),(2,5,3,10),(7,9,8,11)], [(0,5,2,7),(1,10,4 ,8),(3,11,6,9)], [(0,6,4,2),(1,11,8,7),(3,9,5,10)], [(0,7,4,3),(1,8,5,11),(2,10,9,6)], [(0,9,2,8),(1,4,11,5),(3,7,10,6)], [(0,10,5,4),(1,9,7,6),(2,11,3,8)], [(0,11,4,10),(1,7,2, 9),(3,6,5,8)], [(0,8,4,9,11,2,6,10,7,3,1,5)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8, 10,1,3,5,7,11,9)]

25. HWP*$(12; 4^9, 12^2)$,
[(0,3,1,4),(2,5,8,6),(7,9,11,10)], [(0,4,1,5),(2,6,3,9),(7,10,8,11)], [(0,5,1,6),(2,7,3, 11),(4,10,9,8)], [(0,6,1,8),(2,10,3,7),(4,9,5,11)], [(0,7,5,10),(1,11,8,2),(3,6,9,4)], [(0,8,3,2),(1,7,4,11),(5,9,6,10)], [(0,9,7,1),(2,8,5,4),(3,10,6,11)], [(0,10,4,7),(1,9, 3,8),(2,11,6,5)], [(0,11,5,3),(1,10,2,9),(4,8,7,6)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0, 2,4,6,8,10,1,3,5,7,11,9)]

26. HWP*$(12; 4^{10}, 12^1)$,

[(0,2,1,3),(4,6,5,7),(8,10,9,11)], [(0,3,1,4),(2,5,8,11),(6,9,7,10)], [(0,4,1,5),(2,6,3,9),(7,11,10,8)], [(0,5,1,6),(2,4,10,7),(3,11,9,8)], [(0,6,2,7),(1,9,4,11),(3,8,5,10)], [(0,7,1,8),(2,11,6,10),(3,5,4,9)], [(0,8,1,10),(2,9,5,3),(4,7,6,11)], [(0,9,6,1),(2,10,4,8),(3,7,5,11)], [(0,10,5,2),(1,11,7,9),(3,6,8,4)], [(0,11,5,9),(1,7,3,10),(2,8,6,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)]

27. HWP*$(12; 6^1, 12^{10})$,
[(0,3,9,4,11,1),(2,5,10,8,7,6)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,1,3,5,4,6,8,10,7,11,9)], [(0,4,1,5,2,6,3,7,9,8,11,10)], [(0,5,1,4,2,7,3,6,10,9,11,8)], [(0,6,1,7,10,2,8,3,11,4,9,5)], [(0,7,1,6,9,2,10,4,8,5,11,3)], [(0,8,1,9,3,2,4,10,6,11,5,7)], [(0,9,7,2,11,6,5,3,10,1,8,4)], [(0,10,3,8,6,4,7,5,9,1,11,2)], [(0,11,7,4,3,1,10,5,8,2,9,6)]

28. HWP*$(12; 6^2, 12^9)$,
[(0,3,9,4,11,1),(2,5,10,8,7,6)], [(0,4,7,1,9,2),(3,10,6,11,8,5)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,1,3,5,4,6,8,10,7,11,9)], [(0,5,1,4,2,6,3,7,9,8,11,10)], [(0,6,1,5,2,4,9,11,7,10,3,8)], [(0,7,2,8,1,6,4,10,9,3,11,5)], [(0,8,2,7,3,6,9,1,10,5,11,4)], [(0,9,5,7,4,1,8,6,10,2,11,3)], [(0,10,1,11,6,5,8,4,3,2,9,7)], [(0,11,2,10,4,8,3,1,7,5,9,6)]

29. HWP*$(12; 6^3, 12^8)$,
[(0,3,9,4,11,1),(2,5,10,8,7,6)], [(0,4,7,1,9,2),(3,10,6,11,8,5)], [(0,6,1,11,3,8),(2,7,10,9,5,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,1,3,5,7,4,6,9,8,11,10)], [(0,5,1,4,3,2,6,8,10,7,11,9)], [(0,7,2,4,8,1,5,9,11,6,10,3)], [(0,8,2,9,3,6,5,11,4,10,1,7)], [(0,9,1,8,6,3,11,7,5,2,10,4)], [(0,10,2,11,5,8,4,9,7,3,1,6)], [(0,11,2,8,3,7,9,6,4,1,10,5)]

30. HWP*$(12; 6^4, 12^7)$,
[(0,3,9,4,11,1),(2,5,10,8,7,6)], [(0,4,7,1,9,2),(3,10,6,11,8,5)], [(0,6,1,11,3,8),(2,7,10,9,5,4)], [(0,9,1,8,2,10),(3,6,5,11,7,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,1,3,5,7,9,6,8,11,10,4)], [(0,5,1,4,6,3,2,8,10,7,11,9)], [(0,7,2,4,1,10,5,9,8,3,11,6)], [(0,8,4,9,11,2,6,10,3,1,7,5)], [(0,10,1,5,2,11,4,8,6,9,3,7)], [(0,11,5,8,1,6,4,10,2,9,7,3)]

31. HWP*$(12; 6^5, 12^6)$,
[(0,3,6,1,9,2),(4,11,7,10,8,5)], [(0,4,2,5,10,3),(1,11,8,7,6,9)], [(0,6,11,1,4,10),(2,7,5,3,9,8)], [(0,7,4,3,8,1),(2,10,9,5,11,6)], [(0,9,4,7,1,8),(2,11,3,10,6,5)],[(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)], [(0,5,8,3,11,10,2,1,7,9,6,4)], [(0,8,4,9,11,2,6,10,7,3,1,5)], [(0,10,4,8,11,5,1,6,3,2,9,7)], [(0,11,4,1,10,5,9,3,7,2,8,6)]

32. HWP*$(12; 6^6, 12^5)$,
[(0,3,2,5,1,8),(4,11,6,9,7,10)], [(0,4,2,7,5,10),(1,6,11,3,9,8)], [(0,6,1,9,5,2),(3,8,11,7,4,10)], [(0,7,6,5,11,1),(2,9,4,3,10,8)], [(0,9,2,10,6,3),(1,11,5,4,8,7)], [(0,10,9,1,4,7),(2,11,8,5,3,6)][(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)], [(0,5,8,3,11,10,2,1,7,9,6,4)], [(0,8,4,9,11,2,6,10,7,3,1,5)], [(0,11,4,1,10,5,9,3,7,2,8,6)]

33. HWP*$(12; 6^7, 12^4)$,
[(0,3,2,5,1,6),(4,7,10,9,8,11)], [(0,4,1,8,2,7),(3,9,5,10,6,11)], [(0,6,1,9,3,10),(2,8,5,4,11,7)], [(0,7,5,11,8,1),(2,9,4,10,3,6)], [(0,9,2,10,4,8),(1,11,6,5,3,7)], [(0,10,8,7,6,3),(1,4,2,11,5,9)], [(0,11,1,10,5,2),(3,8,6,9,7,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)],[(0,5,8,3,11,10,2,1,7,9,6,4)], [(0,8,4,9,11,2,6,10,7,3,1,5)]

34. HWP*$(12; 6^8, 12^3)$,
[(0,3,1,4,2,5),(6,9,7,10,8,11)], [(0,4,1,5,2,6),(3,10,9,11,8,7)], [(0,6,1,8,2,7),(3,9,4,10,5,11)], [(0,7,1,6,10,3),(2,11,4,8,5,9)], [(0,8,4,7,2,10),(1,9,5,3,6,11)], [(0,9,3,8,6,2),(1,10,7,4,11,5)], [(0,10,6,3,2,8),(1,11,7,5,4,9)], [(0,11,2,9,8,1),(3,7,6,5,10,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)], [(0,5,8,3,11,10,2,1,7,

9,6,4)]

35. HWP*$(12; 6^9, 12^2)$,
[(0,3,1,4,2,5),(6,9,7,10,8,11)], [(0,4,1,5,2,6),(3,8,7,9,11,10)], [(0,5,1,6,2,7),(3,10,9,4,11,8)], [(0,6,1,7,2,8),(3,11,4,10,5,9)], [(0,7,1,9,6,3),(2,11,5,10,4,8)], [(0,8,5,3,6,10),(1,11,7,4,9,2)],[(0,9,8,6,11,1),(2,10,7,5,4,3)], [(0,10,6,5,11,2),(1,8,4,7,3,9)], [(0,11,3,7,6,4),(1,10,2,9,5,8)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)]

36. HWP*$(12; 6^{10}, 12^1)$,
[(0,2,1,3,5,4),(6,8,7,9,11,10)], [(0,3,1,4,2,5),(6,9,7,10,8,11)], [(0,4,1,5,2,6),(3,7,11,8,10,9)], [(0,5,1,6,2,7),(3,10,4,11,9,8)], [(0,6,1,7,2,8),(3,9,4,10,5,11)], [(0,7,1,9,5,10),(2,11,4,8,6,3)], [(0,8,1,10,2,9),(3,11,7,4,6,5)], [(0,9,6,4,7,3),(1,11,5,8,2,10)], [(0,10,7,5,9,2),(1,8,4,3,6,11)], [(0,11,2,4,9,1),(3,8,5,7,6,10)], [(0,1,2,3,4,5,6,7,8,9,10,11)]

37. HWP*$(16; 8^2, 16^{13})$,
[(0,1,2,3,4,5,6,7),(8,9,10,11,12,13,14,15)], [(0,2,4,6,8,10,12,14),(1,3,5,7,9,11,15,13)],[(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,4,1,6,2,5,8,3,7,10,9,12,15,11,14,13)],[(0,5,1,7,3,2,6,4,8,11,9,14,10,13,15,12)], [(0,6,1,9,3,10,2,7,4,11,13,8,14,12,5,15)],[(0,7,1,10,3,6,5,2,15,14,8,13,11,4,12,9)], [(0,8,2,1,11,5,10,14,3,12,7,15,9,6,13,4)],[(0,9,4,2,8,12,3,13,6,14,1,15,7,11,10,5)], [(0,10,15,6,12,1,4,3,14,11,8,5,9,13,7,2)],[(0,11,3,15,2,13,5,12,4,14,9,1,8,7,6,10)], [(0,12,2,14,7,5,13,10,6,9,8,4,15,3,11,1)],[(0,13,3,1,12,6,11,2,9,15,5,14,4,10,7,8)], [(0,14,2,10,4,7,12,8,15,1,13,9,5,11,6,3)],[(0,15,4,13,12,10,8,1,5,3,9,2,11,7,14,6)]

38. HWP*$(16; 8^4, 16^{11})$,
[(0,6,12,15,5,14,7,1,),(2,9,4,3,13,10,8,11)], [(0,9,15,12,6,11,4,7),(1,10,2,5,13,8,14,3)],[(0,12,3,2,7,14,8,4),(1,11,10,5,15,6,13,9)], [(0,13,11,1,15,9,12,5),(2,14,6,10,3,7,4,8)],[(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)], [(0,2,4,6,8,10,12,14,1,3,5,7,9,11,15,13)],[(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,4,2,1,8,3,6,5,11,9,14,10,13,15,7,12)],[(0,5,2,6,1,9,3,10,7,11,13,4,15,14,12,8)], [(0,7,5,1,4,10,14,13,3,12,2,8,15,11,6,9)],[(0,8,1,7,15,3,11,14,4,13,5,12,9,6,2,10)], [(0,10,15,4,12,1,6,3,14,11,8,5,9,13,7,2)],[(0,11,3,15,2,13,1,12,4,14,9,5,8,7,10,6)], [(0,14,2,15,1,13,6,4,11,5,10,9,8,12,7,3)],[(0,15,8,13,12,10,4,1,5,3,9,2,11,7,6,14)]

39. HWP*$(16; 8^6, 16^9)$,
[(0,4,2,1,8,3,6,5),(7,11,10,13,9,12,15,14)], [(0,5,2,6,1,9,15,12),(3,10,7,14,8,11,13,4)],[(0,6,11,1,10,8,4,7),(2,14,12,5,13,15,9,3)], [(0,9,14,10,3,1,11,4),(2,5,15,7,12,6,13,8)],[(0,12,3,13,11,9,4,8),(1,15,5,14,6,10,2,7)], [(0,13,10,5,11,2,9,1),(3,7,4,15,6,12,8,14)],[(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)], [(0,2,4,6,8,10,12,14,1,3,5,7,9,11,15,13)],[(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,7,5,1,4,10,14,13,3,12,2,8,15,11,6,9)],[(0,8,1,7,15,3,11,14,4,13,5,12,9,6,2,10)], [(0,10,15,4,12,1,6,3,14,11,8,5,9,13,7,2)],[(0,11,3,15,2,13,1,12,4,14,9,5,8,7,10,6)], [(0,14,2,15,1,13,6,4,11,5,10,9,8,12,7,3)],[(0,15,8,13,12,10,4,1,5,3,9,2,11,7,6,14)]

40. HWP*$(16; 8^8, 16^7)$,
[(0,4,2,1,7,3,6,5),(8,11,9,12,15,14,10,13)], [(0,5,2,6,1,8,3,10),(4,7,11,13,15,12,9,14)],[(0,6,2,5,10,3,7,12),(1,11,14,8,4,13,9,15)], [(0,8,1,9,3,2,10,7),(4,15,5,14,12,6,13,11)],[(0,9,8,12,7,14,6,4),(1,10,2,15,3,13,5,11)], [(0,12,8,14,2,9,4,3),(1,13,6,11,10,5,15,7)],[(0,13,4,11,5,12,3,1),(2,14,7,15,9,6,10,8)], [(0,14,3,11,2,7,4,8),(1,15,6,12,5,13,10,9)],[(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)], [(0,2,4,6,8,10,12,14,1,

3,5,7,9,11,15,13)],[(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,7,5,1,4,10,14,13,3, 12,2,8,15,11,6,9)],[(0,10,15,4,12,1,6,3,14,11,8,5,9,13,7,2)], [(0,11,3,15,2,13,1,12, 4,14,9,5,8,7,10,6)],[(0,15,8,13,12,10,4,1,5,3,9,2,11,7,6,14)]

41. HWP*$(16; 8^{10}, 16^5)$,
[(0,4,2,1,6,3,7,5),(8,11,9,12,15,14,10,13)], [(0,5,1,4,3,2,6,9),(7,11,8,14,13,10,15, 12)],[(0,6,1,7,2,5,9,3),(4,10,14,12,8,15,11,13)], [(0,7,1,8,2,9,4,12),(3,6,13,15,5,14, 11,10)], [(0,8,1,9,6,2,10,7),(3,12,5,13,11,14,4,15)], [(0,9,1,10,2,7,14,8),(3,11,5,15, 4,13,6,12)],[(0,10,8,3,13,9,15,1),(2,14,7,4,11,6,5,12)], [(0,12,1,15,9,13,5,10),(2,8, 4,7,3,14,6,11)],[(0,13,7,15,6,10,5,2),(1,11,4,8,12,9,14,3)], [(0,14,2,15,7,12,6,4),(1, 13,3,10,9,8,5,11)],[(0,11,3,15,2,13,1,12,4,14,9,5,8,7,10,6)], [(0,15,8,13,12,10,4,1,5 ,3,9,2,11,7,6,14)],[(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,2,4,6,8,10,12,14,1, 3,5,7,9,11,15,13)],[(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)]

42. HWP*$(16; 8^{12}, 16^3)$,
[(0,4,1,5,2,6,3,7),(8,11,9,12,10,13,15,14)], [(0,5,1,4,2,7,3,6),(8,12,9,13,11,14,10, 15)],[(0,6,1,7,2,5,3,9),(4,8,13,10,14,12,15,11)], [(0,7,1,6,2,8,3,10),(4,11,5,14,13,9, 15,12)],[(0,8,1,9,2,10,3,12),(4,7,5,15,6,14,11,13)], [(0,9,1,8,2,11,3,14),(4,10,5,12, 6,13,7,15)],[(0,10,2,1,11,6,4,3),(5,13,8,15,9,14,7,12)], [(0,11,1,10,4,13,12,2),(3,15 ,7,6,5,9,8,14)],[(0,12,8,4,14,2,15,1),(3,13,5,11,7,10,6,9)], [(0,13,6,10,7,11,8,5),(1, 15,2,14,9,4,12,3)],[(0,14,6,11,10,9,5,8),(1,12,7,4,15,3,2,13)], [(0,15,5,10,8,7,14,4) ,(1,13,3,11,2,9,6,12)],[(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,2,4,6,8,10,12, 14,1,3,5,7,9,11,15,13)],[(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)]

43. HWP*$(16; 8^{14}, 16^1)$,
[(0,2,1,3,5,4,6,8),(7,9,11,10,12,14,13,15)], [(0,3,1,4,2,5,7,6),(8,10,9,12,15,13,11, 14)],[(0,4,1,5,2,6,3,7),(8,11,9,13,10,15,14,12)], [(0,5,1,6,2,4,3,9),(7,10,13,8,14,11, 15,12)],[(0,6,1,7,2,8,3,10),(4,9,14,5,15,11,13,12)], [(0,7,1,8,2,9,3,11),(4,12,5,13,6, 15,10,14)],[(0,8,1,9,2,7,3,13),(4,15,5,12,11,6,14,10)], [(0,9,1,10,2,11,3,12),(4,7,14 ,6,13,5,8,15)],[(0,10,1,11,2,12,3,14),(4,8,13,7,5,9,15,6)], [(0,11,1,12,2,13,9,5),(3, 15,8,6,10,7,4,14)],[(0,12,9,6,11,7,13,1),(2,15,3,8,4,10,5,14)], [(0,13,3,2,14,7,11,4) ,(1,15,9,8,5,10,6,12)],[(0,14,1,13,4,11,5,3),(2,10,8,12,6,9,7,15)], [(0,15,1,14,9,4, 13,2),(3,6,5,11,8,7,12,10)],[(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)]

44. HWP*$(16; 4^1, 16^{14})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,3,7,1,8,5,9,6,10,12,15,14, 11,13)],[(0,3,9,12,1,5,13,8,2,6,14,7,11,15,10,4)], [(0,4,12,2,1,11,7,5,8,14,9,13,10, 3,6,15)],[(0,5,12,4,1,15,3,11,2,14,6,8,7,10,13,9)], [(0,6,11,5,10,7,12,9,2,15,4,8,3, 13,1,14)],[(0,7,2,8,1,4,6,3,10,5,15,9,14,13,11,12)], [(0,8,4,15,13,2,12,14,1,3,5,7,9, 11,10,6)],[(0,9,15,8,11,14,12,10,2,5,3,1,6,13,4,7)], [(0,10,1,7,3,2,9,4,13,5,14,8,15, 6,12,11)],[(0,11,1,9,3,4,2,13,15,7,14,10,8,12,6,5)], [(0,12,3,8,6,1,13,7,15,11,9,5,4, 14,2,10)],[(0,13,3,14,5,11,4,9,8,10,15,1,12,7,6,2)], [(0,14,3,15,2,11,6,4,10,9,7,8, 13,12,5,1)],[(0,15,5,2,7,13,6,9,1,10,14,4,11,3,12,8)]

45. HWP*$(16; 4^2, 16^{13})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7), (9,11,13,15)], [(0,3,9,12,1,5,13,8,2,6,14,7,11,15,10,4)], [(0,4,12,2,1,11,7,5,8,14,9, 13,10,3,6,15)], [(0,5,12,4,1,15,3,11,2,14,6,8,7,10,13,9)], [(0,6,11,5,10,7,12,9,2,15, 4,8,3,13,1,14)], [(0,7,2,8,1,4,3,10,5,9,6,12,15,14,13,11)], [(0,8,4,2,7,3,12,5,1,9,14, 11,6,10,15,13)], [(0,9,15,8,11,14,12,10,2,5,3,1,6,13,4,7)], [(0,10,1,7,6,2,9,3,8,13,5, 14,4,15,11,12)], [(0,11,1,8,5,2,12,3,15,6,9,4,13,7,14,10)], [(0,12,6,1,10,8,15,2,11,

4,9,7,13,3,14,5)], [(0,13,2,10,6,3,4,14,1,12,11,9,5,15,7,8)], [(0,14,3,2,13,6,5,4,11, 10,9,8,12,7,15,1)], [(0,15,5,11,3,7,9,1,13,12,8,6,4,10,14,2)]

46. HWP*$(16; 4^3, 16^{12})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7), (9,11,13,15)], [(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,12,2,1,11,7,5,8, 14,9,13,10,3,6,15)], [(0,5,12,4,1,15,3,11,2,14,6,8,7,10,13,9)], [(0,6,11,5,10,7,12,9, 2,15,4,8,3,13,1,14)], [(0,7,3,2,8,4,9,1,10,5,11,6,12,15,14,13)], [(0,8,2,7,6,1,4,3,10, 9,14,5,15,13,12,11)], [(0,9,15,8,11,14,12,10,2,5,3,1,6,13,4,7)], [(0,10,1,7,8,5,14,2, 13,11,12,3,15,6,9,4)], [(0,11,1,8,6,2,9,3,12,7,15,10,14,4,13,5)], [(0,12,1,9,5,2,10,4, 14,11,15,7,13,6,3,8)], [(0,13,3,4,11,9,7,14,10,6,5,1,12,8,15,2)], [(0,14,3,7,11,10,8, 13,2,12,5,9,6,4,15,1)], [(0,15,5,4,2,11,3,14,1,13,7,9,8,12,6,10)]

47. HWP*$(16; 4^4, 16^{11})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7), (9,11,13,15)], [(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5) ,(3,14,9,13),(10,8,12,6)], [(0,5,12,4,1,15,3,11,2,14,6,8,7,10,13,9)], [(0,6,11,5,10,7, 12,9,2,15,4,8,3,13,1,14)], [(0,7,3,2,1,4,9,5,8,6,12,11,15,10,14,13)], [(0,8,2,7,6,1,9, 3,4,11,12,15,13,5,14,10)], [(0,9,15,8,11,14,12,10,2,5,3,1,6,13,4,7)], [(0,10,1,7,8,4, 2,13,6,9,14,11,3,12,5,15)], [(0,11,1,8,5,2,9,7,13,12,3,10,6,15,14,4)], [(0,12,1,10,3, 6,2,8,15,7,14,5,9,4,13,11)], [(0,13,7,9,8,14,1,12,2,11,10,4,3,15,6,5)], [(0,14,3,7,15, 1,13,2,10,5,11,9,6,4,12,8)], [(0,15,5,4,14,2,12,7,11,6,3,8,13,10,9,1)]

48. HWP*$(16; 4^5, 16^{10})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7), (9,11,13,15)],[(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5), (3,14,9,13),(10,8,12,6)], [(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5, 10,7,12,9,2,15,4,8,3,13,1,14)], [(0,7,3,1,4,2,5,8,6,9,14,11,12,15,13,10)], [(0,8,2,1, 6,3,4,7,9,5,15,10,14,13,12,11)], [(0,9,1,7,6,4,3,2,10,5,11,15,14,12,8,13)], [(0,10,1, 8,4,9,3,6,13,11,14,2,12,7,15,5)], [(0,11,2,7,8,5,3,12,1,13,4,14,10,9,6,15)], [(0,12,2, 9,4,11,3,10,6,8,15,7,13,5,14,1)], [(0,13,2,11,6,1,10,3,15,8,7,14,4,12,5,9)], [(0,14,5, 4,1,9,15,6,12,3,8,11,10,2,13,7)], [(0,15,1,12,10,4,13,6,5,2,14,3,7,11,9,8)]

49. HWP*$(16; 4^6, 16^9)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7), (9,11,13,15)], [(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5) ,(3,14,9,13),(10,8,12,6)], [(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11, 5),(1,7,12,9),(2,15,4,8),(3,13,10,14)], [(0,7,3,1,4,2,5,8,6,9,14,11,15,13,12,10)], [(0,8,3,2,1,6,4,7,11,12,5,10,9,15,14,13)], [(0,9,2,7,6,1,8,4,13,5,11,14,10,3,12,15)], [(0,10,1,9,3,4,11,2,12,7,8,13,6,15,5,14)], [(0,11,3,6,5,2,9,4,14,12,8,15,10,7,13,1)], [(0,12,1,10,2,11,6,13,4,3,15,7,14,5,9,8)], [(0,13,2,14,4,1,12,11,9,6,3,10,5,15,8,7)], [(0,14,1,13,7,15,6,12,2,10,4,9,5,3,8,11)], [(0,15,1,14,2,13,11,10,6,8,5,4,12,3,7,9)]

50. HWP*$(16; 4^7, 16^8)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7), (9,11,13,15)], [(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5) ,(3,14,9,13),(10,8,12,6)], [(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5),(1,7,12,9),(2,15,4,8),(3,13,10,14)], [(0,7,15,10),(1,12,2,14),(3,6,13,11),(4,9,8,5)], [(0,8,3,1,4,2,5,9,6,12,10,7,11,15,14,13)], [(0,9,2,1,6,3,4,7,8,13,12,11,14,10,5,15)], [(0,11,2,9,3,7,13,1,10,4,12,8,6,15,5,14)], [(0,12,1,9,4,11,10,2,13,7,14,5,3,15,6,8)], [(0,10,1,8,4,3,2,7,6,9,14,12,15,13,5,11)], [(0,13,4,14,2,11,9,15,8,7,3,12,5,10,6,1)

], [(0,14,11,6,4,1,13,2,12,3,10,9,5,8,15,7)], [(0,15,1,14,4,13,6,5,2,10,3,8,11,12,7,9)]

51. HWP*$(16;4^8,16^7)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)], [(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)], [(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5),(1,7,12,9),(2,15,4,8),(3,13,10,14)], [(0,7,15,10),(1,12,2,14),(3,6,13,11),(4,9,8,5)], [(0,8,15,7),(1,9,2,13),(3,10,4,12),(14,11,6,5)], [(0,9,3,1,4,2,5,8,6,12,10,7,11,15,14,13)], [(0,10,1,6,3,2,7,8,4,11,12,15,13,5,9,14)], [(0,11,2,1,8,3,4,13,7,14,12,5,10,6,9,15)], [(0,12,1,10,2,9,4,14,5,3,7,13,6,15,8,11)], [(0,13,2,12,8,7,3,15,6,4,1,14,10,5,11,9)], [(0,14,2,11,10,9,5,15,1,13,4,3,12,7,6,8)], [(0,15,5,2,10,3,8,13,12,11,14,4,7,9,6,1)]

52. HWP*$(16;4^9,16^6)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)], [(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)], [(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5),(1,7,12,9),(2,15,4,8),(3,13,10,14)], [(0,7,15,10),(1,12,2,14),(3,6,13,11),(4,9,8,5)], [(0,8,15,7),(1,9,2,13),(3,10,4,12),(14,11,6,5)], [(0,9,15,8),(1,6,12,10),(2,5,11,14),(7,13,4,3)], [(0,10,2,1,4,7,3,8,6,9,5,15,14,13,12,11)],[(0,11,2,7,6,1,8,3,4,13,5,10,9,14,12,15)], [(0,12,1,10,3,2,9,6,4,14,5,8,7,11,15,13)],[(0,13,6,3,15,1,14,10,7,8,4,2,11,12,5,9)], [(0,14,4,11,10,6,15,5,2,12,8,13,7,9,3,1)],[(0,15,6,8,11,9,4,1,13,2,10,5,3,12,7,14)]

53. HWP*$(16;4^{10},16^5)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)], [(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)], [(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5),(1,7,12,9),(2,15,4,8),(3,13,10,14)], [(0,7,15,10),(1,12,2,14),(3,6,13,11),(4,9,8,5)], [(0,8,15,7),(1,9,2,13),(3,10,4,12),(14,11,6,5)], [(0,9,15,8),(1,6,12,10),(2,5,11,14),(7,13,4,3)], [(0,10,5,9),(2,12,7,11),(3,15,1,8),(4,14,13,6)],[(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)], [(0,12,1,10,2,9,6,3,4,7,14,5,8,11,15,13)],[(0,13,2,11,10,7,6,9,14,12,15,5,3,8,4,1)], [(0,14,4,11,12,8,6,1,13,7,3,2,10,9,5,15)],[(0,15,6,8,7,9,4,13,5,2,1,14,10,3,12,11)]

54. HWP*$(16;4^{11},16^4)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,1,5),(3,6,4,8),(7,9,12,11),(10,14,13,15)], [(0,3,5,2),(1,6,8,7),(4,12,9,13),(10,15,11,14)], [(0,4,3,7),(1,8,2,15),(5,9,11,13),(6,10,12,14)], [(0,5,4,9),(1,12,2,8),(3,10,13,11),(6,14,7,15)], [(0,6,12,10),(1,3,11,5),(2,14,9,7),(4,13,8,15)], [(0,7,12,4),(1,15,3,9),(2,13,6,11),(5,14,8,10)], [(0,8,14,11),(1,9,4,10),(2,5,12,6),(3,15,7,13)], [(0,9,15,8),(1,7,5,11),(2,12,3,14),(4,6,13,10)], [(0,10,8,12),(1,14,3,13),(2,4,15,9),(5,7,11,6)],[(0,15,2,6),(1,11,4,14),(3,12,7,10),(5,13,9,8)], [(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)],[(0,12,1,10,2,9,6,3,4,7,14,5,8,11,15,13)], [(0,13,2,11,10,7,6,9,14,12,15,5,3,8,4,1)],[(0,14,4,11,12,8,6,1,13,7,3,2,10,9,5,15)]

55. HWP*$(16;4^{12},16^3)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,1,5),(3,6,4,8),(7,9,11,12),(10,14,13,15)], [(0,3,2,4),(1,6,5,7),(8,10,12,14),(9,15,11,13)], [(0,4,3,7),(1,8,2,5),(6,12,10,15),(9,13,11,14)], [(0,5,2,6,),(1,3,10,13),(4,9,12,11),(7,15,8,14)], [(0,6,1,9),(2,10,5,11),(3,13,4,14),(7,12,8,15)], [(0,7,2,12),(1,11,6,10),(3,14,4,13),(5,

15,9,8)], [(0,8,6,2),(1,13,10,9),(3,12,4,15),(5,14,11,7)], [(0,9,2,8),(1,12,3,15),(4,6,14,10),(5,13,7,11)], [(0,10,3,11),(1,15,2,14),(4,12,9,5),(6,8,7,13)],[(0,14,2,15),(1,7,10,8),(3,9,4,11),(5,12,6,13)], [(0,15,4,10),(1,14,6,11),(2,13,8,12),(3,5,9,7)], [(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)], [(0,12,1,10,2,9,6,3,4,7,14,5,8,11,15,13)], [(0,13,2,11,10,7,6,9,14,12,15,5,3,8,4,1)]

56. HWP*$(16; 4^{13}, 16^{2})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,1,5),(3,6,4,8),(7,9,11,12),(10,14,13,15)], [(0,3,2,4),(1,6,5,7),(8,10,9,13),(11,14,12,15)], [(0,4,1,7),(2,5,3,8),(6,12,10,15),(9,14,11,13)], [(0,5,1,8),(2,6,9,12),(3,10,13,11),(4,14,7,15)], [(0,6,1,9),(2,8,4,11),(3,12,14,10),(5,15,7,13)], [(0,7,2,10),(1,12,9,8),(3,15,5,14),(4,13,6,11)], [(0,8,6,2),(1,14,9,15),(3,5,13,7),(4,12,11,10)], [(0,9,4,15),(1,13,2,14),(3,7,10,12),(5,11,6,8)], [(0,10,5,12),(1,15,3,11),(2,13,4,9),(6,14,8,7)],[(0,13,1,11),(2,15,8,14),(3,9,5,4),(6,10,7,12)], [(0,14,4,6),(1,3,13,10),(2,12,8,15),(5,9,7,11)], [(0,15,9,1),(2,11,7,5),(3,14,6,13),(4,10,8,12)], [(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)], [(0,12,1,10,2,9,6,3,4,7,14,5,8,11,15,13)]

57. HWP*$(16; 4^{14}, 16^{1})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,1,5),(3,4,6,8),(7,9,11,12),(10,14,13,15)], [(0,3,2,4),(1,6,5,7),(8,10,9,13),(11,14,12,15)], [(0,4,1,7),(2,5,3,6),(8,12,9,14),(10,15,13,11)], [(0,5,1,8),(2,6,3,7),(4,14,9,15),(10,12,11,13)], [(0,6,1,9),(2,8,4,10),(3,12,14,11),(5,13,7,15)], [(0,7,3,10),(1,11,2,9),(4,12,6,13),(5,15,8,14)], [(0,8,1,12),(2,10,5,14),(3,9,7,13),(4,11,15,6)], [(0,9,12,2),(1,3,8,15),(4,13,5,11),(6,14,10,7)], [(0,10,13,1),(2,15,9,8),(3,14,6,12),(4,7,11,5)],[(0,12,8,6),(1,10,3,11),(2,13,9,5),(4,15,7,14)], [(0,13,6,11),(1,15,2,14),(3,5,9,4),(7,12,10,8)], [(0,14,3,15),(1,13,2,12),(4,9,6,10),(5,8,11,7)], [(0,15,3,13),(1,14,7,10),(2,11,6,9),(4,8,5,12)], [(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)]

58. HWP*$(15; 3^{1}, 15^{13})$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4,8,10,6,12,14,1,3,5,7,9,11,13)],[(0,3,9,1,8,4,6,14,7,5,10,13,2,12,11)], [(0,4,7,1,9,3,2,14,13,6,5,8,11,12,10)], [(0,5,13,1,4,10,2,9,14,3,6,11,7,12,8)], [(0,6,1,5,2,3,7,4,9,8,13,11,14,10,12)], [(0,7,2,1,6,3,8,5,9,13,12,4,11,10,14)], [(0,8,1,7,3,10,4,2,13,9,12,5,14,11,6)], [(0,9,2,5,1,10,3,11,4,14,6,13,8,12,7)], [(0,10,1,11,2,6,4,3,14,8,7,13,5,12,9)], [(0,11,1,12,2,7,6,10,8,14,9,5,4,13,3)], [(0,12,1,14,2,10,5,6,9,7,11,8,3,13,4)], [(0,13,10,7,14,5,11,3,12,6,2,8,9,4,1)], [(0,14,4,12,3,1,13,7,10,9,6,8,2,11,5)]

59. HWP*$(15; 3^{3}, 15^{11})$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)], [(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7,1,9,3,2,14,13,6,5,8,11,12,10)], [(0,5,13,1,4,10,2,9,14,3,6,11,7,12,8)], [(0,6,1,3,7,2,5,4,8,9,13,11,14,10,12)], [(0,7,5,1,6,2,3,8,12,9,4,13,10,14,11)], [(0,8,1,5,2,6,3,13,4,11,10,9,12,7,14)], [(0,9,1,7,6,4,12,2,10,3,11,5,14,8,13)], [(0,10,1,11,2,7,9,5,12,6,13,8,14,4,3)], [(0,11,1,10,4,2,8,3,14,6,9,7,13,12,5)], [(0,12,4,9,6,14,2,13,7,10,8,5,11,3,1)], [(0,13,3,12,1,14,5,9,2,11,8,4,6,10,7)], [(0,14,9,8,7,11,4,1,13,2,12,3,10,5,6)]

60. HWP*$(15; 3^{5}, 15^{9})$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)], [(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13),(5,11,8),(10,12,6)], [(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1,3,2,5,4,8,9,7,13,10,14,11,12)], [(0,7,1,5,2,3,8,4,6,13,11,14,9,12,10)], [(0,8,1,6,2,7,5,

9,13,12,3,14,10,4,11)], [(0,9,1,7,2,6,3,11,10,5,12,4,13,8,14)], [(0,10,2,8,3,7,11,1, 13,4,12,9,5,14,6)], [(0,11,2,10,3,12,1,14,4,9,8,13,7,6,5)], [(0,12,2,11,4,3,13,1,10, 7,9,6,14,5,8)], [(0,13,6,9,4,2,12,7,14,3,10,8,11,5,1)], [(0,14,8,12,5,6,4,1,11,7,10,9, 2,13,3)]

61. HWP*$(15; 3^7, 15^7)$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7), (9,11,13)], [(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13), (5,11,8),(10,12,6)], [(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1),(2,13,10) ,(3,8,14),(4,11,5),(7,9,12)], [(0,7,11),(2,6,5),(1,14,9),(4,12,10),(3,13,8)], [(0,8,1,3, 2,5,6,4,9,13,7,10,14,11,12)],[(0,9,2,3,7,1,5,8,12,4,6,13,11,14,10)], [(0,10,3,11,1,6, 2,7,5,14,4,13,12,9,8)],[(0,11,2,8,4,1,10,7,13,3,12,5,9,6,14)], [(0,12,2,11,4,3,10,8,9, 7,14,5,1,13,6)],[(0,13,1,7,2,12,3,14,6,9,4,8,11,10,5)], [(0,14,8,13,4,2,10,9,5,12,1, 11,7,6,3)]

62. HWP*$(15; 3^9, 15^5)$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7), (9,11,13)], [(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13), (5,11,8),(10,12,6)], [(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1),(2,13,10) ,(3,8,14),(4,11,5),(7,9,12)], [(0,7,11),(2,6,5),(1,14,9),(4,12,10),(3,13,8)], [(0,8,12) ,(1,13,4),(2,10,3),(7,14,11),(5,6,9)], [(0,9,8),(2,5,12),(11,4,13),(3,14,6),(1,10,7)], [(0,10,5,1,3,7,2,8,11,12,4,9,13,6,14)], [(0,11,1,5,8,9,2,7,10,14,4,6,13,12,3)], [(0,12, 1,7,13,3,10,8,4,2,11,14,5,9,6)], [(0,13,7,5,14,8,1,6,2,12,9,4,3,11,10)], [(0,14,10,9,7 ,6,4,8,13,1,11,2,3,12,5)]

63. HWP*$(15; 3^{11}, 15^3)$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7), (9,11,13)], [(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13), (5,11,8),(10,12,6)], [(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1),(2,13,10) ,(3,8,14),(4,11,5),(7,9,12)], [(0,7,11),(2,6,5),(1,14,9),(4,12,10),(3,13,8)], [(0,8,12) ,(1,13,4),(2,10,3),(7,14,11),(5,6,9)], [(0,9,8),(2,5,12),(11,4,13),(3,14,6),(1,10,7)], [(0,10,14),(12,4,3),(2,8,11),(7,5,9),(1,6,13)], [(0,11,10),(1,5,8),(2,12,9),(3,7,13), (4,6,14)], [(0,12,1,3,10,9,6,4,8,13,7,2,11,14,5)], [(0,13,12,5,14,10,8,9,4,2,3,11,1, 7,6)], [(0,14,8,4,9,13,6,2,7,10,5,1,11,12,3)]

64. HWP*$(15; 5^1, 15^{13})$,
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,4,10,14,12,1,3,5,7,9,11,13)], [(0,3,9,1,8,2,4,14,7,5,10,13,6,12,11)], [(0,4,7,1,9,3,2,14,13,5,11,8,10,12,6)], [(0,5,13,1,4,2,10,9,14,3,6,11,7,12,8)], [(0,6,1,5,2,7,3,8,4,9,13,12,14,11,10)], [(0,7,2,1,6,3,10,4,5,8,13,11,14,9,12)], [(0,8,1,7,4,3,11,2,12,9,6,13,10,5,14)], [(0,9,2,5,1,10,3,13,4,11,6,14,8,12,7)], [(0,10,1,11,3,7,6,2,13,8,14,4,12,5,9)], [(0,11,1,12,2,6,9,4,13,7,10,8,3,14,5)], [(0,12,10,6,5,4,8,11,9,7,13,3,1,14,2)], [(0,13,9,8,7,14,6,10,2,11,5,3,12,4,1)], [(0,14,1,13,2,9,10,7,11,4,6,8,5,12,3)]

65. HWP*$(15; 5^3, 15^{11})$,
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11, 13)],[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9,3,2,14,13,5,11,8,10, 12,6)],[(0,5,13,1,4,2,10,9,14,3,6,11,7,12,8)], [(0,6,1,5,3,7,2,9,4,10,8,13,12,14, 11)],[(0,7,3,1,6,2,5,4,11,14,9,13,8,12,10)], [(0,8,1,10,2,6,3,5,7,14,4,13,11,9,12)], [(0,9,2,1,8,3,11,4,5,12,7,13,10,6,14)], [(0,10,1,11,2,7,5,8,4,12,3,14,6,13,9)], [(0,11,1,12,2,13,4,8,14,5,9,7,6,10,3)], [(0,12,5,10,4,1,14,7,9,6,8,11,3,13,2)],

[(0,13,3,8,2,12,4,6,9,10,7,11,5,14,1)], [(0,14,2,11,6,4,3,12,9,8,7,10,5,1,13)]

66. HWP*$(15; 5^5, 15^9)$,
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,1,4,2,5,7,3,8,11,14,9,13,12,10)],[(0,7,2,6,3,1,5,4,12,14,11,9,10,8,13)], [(0,8,1,6,2,7,9,12,3,13,10,4,11,5,14)],[(0,9,2,10,1,8,4,3,6,14,5,12,7,13,11)], [(0,10,2,9,3,5,1,12,4,13,8,14,7,11,6)], [(0,11,1,10,3,14,2,12,5,9,7,6,13,4,8)], [(0,12,9,4,5,10,6,8,7,14,1,11,2,13,3)], [(0,13,1,14,3,7,10,5,11,4,6,9,8,12,2)], [(0,14,4,1,13,9,6,10,7,5,8,2,11,3,12)]

67. HWP*$(15; 5^7, 15^7)$,
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,13,11,2),(1,10,3,7,14),(4,8,12,5,9)],[(0,7,6,2,11),(1,14,3,8,4),(13,10,5,12,9)], [(0,8,1,4,2,5,7,3,13,9,12,14,11,6,10)], [(0,9,2,6,1,5,4,3,12,10,7,13,8,11,14)], [(0,10,1,6,3,5,8,14,2,9,7,11,4,13,12)], [(0,11,1,8,7,5,10,2,13,4,12,3,14,9,6)], [(0,12,2,7,10,8,13,1,11,5,14,4,6,9,3)], [(0,13,3,6,14,5,1,12,7,2,10,4,11,9,8)], [(0,14,7,9,10,6,8,2,12,4,5,11,3,1,13)],

68. HWP*$(15; 5^9, 15^5)$,
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,13,11,2),(1,10,3,7,14),(4,8,12,5,9)],[(0,7,6,2,11),(1,14,3,8,4),(13,10,5,12,9)], [(0,8,13,3,12),(1,11,4,6,10),(2,5,14,7,9)],[(0,9,10,6,8),(1,13,12,3,5),(2,7,11,14,4)], [(0,10,2,6,1,4,3,13,8,7,5,11,9,12,14)],[(0,11,1,5,4,12,7,10,8,14,2,13,9,3,6)], [(0,12,4,13,1,8,2,9,6,14,11,5,10,7,3)], [(0,13,4,11,3,14,5,8,1,6,9,7,2,12,10)], [(0,14,9,8,11,6,3,1,12,2,10,4,5,7,13)]

69. HWP*$(15; 5^{11}, 15^3)$,
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,2,7,3),(1,12,4,5,10),(8,13,11,14,9)],[(0,7,9,4,13),(1,8,12,3,5),(2,10,6,14,11)], [(0,8,4,2,11),(1,14,7,6,3),(5,12,9,13,10)],[(0,9,6,13,12),(1,10,7,11,4),(2,5,14,3,8], [(0,12,5,9,2),(1,13,3,7,14),(4,8,11,6,10)],[(0,14,4,6,8),(1,11,5,7,13),(2,9,10,3,12)], [(0,10,2,6,1,4,3,13,8,7,5,11,9,12,14)],[(0,11,1,5,4,12,7,10,8,14,2,13,9,3,6)], [(0,13,4,11,3,14,5,8,1,6,9,7,2,12,10)]

70. HWP*$(15; 5^{13}, 15^1)$,
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,13,11,2),(1,10,3,7,14),(4,8,12,5,9)],[(0,7,2,11,14),(1,6,9,12,3),(4,13,10,5,8)], [(0,8,13,3,12),(1,11,4,6,10),(2,5,14,7,9)],[(0,9,6,2,10),(1,8,7,11,5),(3,13,12,14,4)], [(0,10,4,11,3),(1,14,5,7,6),(2,12,9,13,8)],[(0,12,4,1,13),(2,9,7,5,10),(3,8,11,6,14)], [(0,13,4,5,11),(1,12,2,6,8),(3,14,9,10,7)],[(0,14,11,9,8),(1,4,2,7,13),(3,5,12,10,6)], [(0,11,1,5,4,12,7,10,8,14,2,13,9,3,6)]

71. HWP*$(15; 3^1, 5^{13})$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,1,4,7),(3,14,5,9,8),(6,12,11,13,10)],[(0,3,9,1,8),(2,11,5,6,13),(4,12,14,7,10)], [(0,4,2,9,14),(1,6,10,7,12),(3,13,8,5,11)],[(0,5,14,13,1),(2,6,4,10,12),(3,7,11,8,9)], [(0,6,1,13,11),(2,10,3,12,7),(4,8,

14,9,5)],[(0,7,1,14,10),(2,5,8,13,6),(3,11,4,9,12)], [(0,8,1,5,12),(2,3,10,14,11),(4, 6,9,7,13)],[(0,9,13,12,5),(1,11,10,8,4),(2,7,14,6,3)], [(0,10,5,1,3),(9,2,8,11,12),(4, 13,7,6,14)],[(0,11,14,2,13),(1,12,6,8,10),(3,5,7,9,4)], [(0,12,8,7,4),(1,10,2,14,3), (5,13,9,11,6)],[(0,13,5,10,9),(1,7,3,6,11),(2,12,4,14,8)], [(0,14,1,9,6),(2,4,11,7,5), (3,8,12,10,13)]

72. **HWP*$(15; 3^3, 5^{11})$,**
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7), (9,11,13)], [(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,2,9,14),(1,6,10,7, 12),(3,13,8,5,11)], [(0,5,14,13,1),(2,6,4,10,12),(3,7,11,8,9)], [(0,6,1,13,11),(2,10, 3,12,7),(4,8,14,9,5)], [(0,7,1,14,10),(2,5,8,13,6),(3,11,4,9,12)], [(0,8,1,5,12),(2,3, 10,14,11),(4,6,9,7,13)], [(0,9,13,12,5),(1,11,10,8,4),(2,7,14,6,3)], [(0,10,5,1,3),(9, 2,8,11,12),(4,13,7,6,14)], [(0,11,14,2,13),(1,7,10,4,3),(5,9,8,12,6)], [(0,12,8,3,6), (1,4,11,7,9),(2,14,5,13,10)], [(0,13,3,14,8),(1,10,9,6,11),(2,12,4,7,5)], [(0,14,3,8, 7),(1,9,4,12,10),(2,11,5,6,13)]

73. **HWP*$(15; 3^5, 5^9)$,**
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7), (9,11,13)], [(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13), (5,11,8),(10,12,6)], [(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1,13,11), (2,10,3,12,7),(4,8,14,9,5)], [(0,7,1,14,10),(2,5,8,13,6),(3,11,4,9,12)], [(0,8,1,5,12), (2,3,10,14,11),(4,6,9,7,13)][(0,9,13,12,5),(1,11,10,8,4),(2,7,14,6,3)], [(0,10,5,1,3), (9,2,8,11,12),(4,13,7,6,14)], [(0,11,7,5,14),(1,10,4,2,12),(3,8,9,6,13)], [(0,12,2,13, 1),(3,7,10,9,8),(4,11,14,5,6)], [(0,13,10,2,6),(1,7,11,5,9),(3,14,8,12,4)], [(0,14,3, 13,8),(1,6,5,2,11),(4,12,10,7,9)]

74. **HWP*$(15; 3^7, 5^7)$,**
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7), (9,11,13)], [(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13), (5,11,8),(10,12,6)], [(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1),(2,13,10) ,(3,8,14),(4,11,5),(7,9,12)], [(0,7,11),(2,6,5),(1,14,9),(4,12,10),(3,13,8)], [(0,8,1,5 ,12),(2,3,10,14,11),(4,6,9,7,13)], [(0,9,13,12,5),(1,11,10,8,4),(2,7,14,6,3)], [(0,10,5 ,1,3),(9,2,8,11,12),(4,13,7,6,14)], [(0,11,1,13,6),(2,5,14,8,12),(3,7,10,9,4)], [(0,12, 3,14,10),(1,7,5,8,13),(2,11,4,9,6)], [(0,13,3,11,14),(1,10,7,2,12),(4,8,9,5,6)], [(0,14 ,5,9,8),(1,6,13,11,7),(2,10,3,12,4)]

75. **HWP*$(15; 3^9, 5^5)$,**
[(0,1,3),(2,6,7),(4,5,8),(9,13,12),(10,11,14)], [(0,2,10),(1,6,8),(3,13,7),(4,12,14), (5,11,9)], [(0,7,13),(1,8,6),(2,9,12),(3,10,14),(4,11,5)], [(0,9,8),(1,2,11),(3,14,12), (4,13,10),(5,7,6)], [(0,10,5),(1,4,3),(2,13,8),(6,14,9),(7,11,12)], [(0,11,6),(1,12,13) ,(2,8,14),(3,4,9),(5,10,7)], [(0,12,4),(1,14,5),(2,3,6),(7,10,8),(9,11,13)], [(0,13,14) ,(1,5,12),(2,7,4),(3,8,11),(6,9,10)], [(0,14,11),(1,13,4),(2,12,10),(3,5,6),(7,8,9)], [(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)] , [(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,13,11,2),(1,10,3,7,14),(4,8,12,5,9)], [(0,8,13,3,12),(1,11,4,6,10),(2,5,14,7,9)]