DISTRIBUTED GROUP CONSENSUS IN TIME-VARYING NETWORKS

by

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ABSTRACT

DISTRIBUTED GROUP CONSENSUS IN TIME-VARYING NETWORKS

In past decades, the problem of achieving a common value in a distributed network, or in other words, the distributed consensus problem, has drawn significant attention from researchers due to its rapidly increasing application area, such as mobile robots, clock synchronization, orientation and flight formation of unmanned air vehicles, smart power grids, and rendezvous problem. These applications involve the design and analysis of distributed consensus algorithms regarding the stability and convergence properties of the algorithm. Depending on the topology of the network, consensus can be achieved in the sense of all agents achieving a common state, where complete agreement is beheld, which is a well-studied problem in the literature, as well as consensus can be achieved in the sense of agents form clusters, where the full agreement does not happen. However, we may still claim a partial agreement state has been reached in terms of multiple equilibrium states. This thesis analyzes the multi-equilibria consensus problem for multi-agent networks under switching topologies in discrete time. The main contributions of the thesis are to give sufficient conditions for a network to achieve multi-equilibria consensus under dynamically changing topologies and to express the number of equilibria points in terms of the network's topological properties. It is also demonstrated that the proposed definition of the reduced graph and graph reduction method reduces the size of the network while preserving some critical structural and topological properties that are used to determine the number of clusters forming in the steady-state.

ÖZET

DEĞİŞKEN İLİNGELİ AĞLARDA DAĞITIK TAKIM ONAYLAŞIMI

Çok etmenli dağıtık bir ağda ortak bir değere ulaşmak şeklinde tanımlanabilecek dağıtık onaylaşım problemi, uygulama alanının hızlı şekilde genişlemesiyle beraber birçok araştırmacının dikkatini çekmiş bir konudur. Dağıtık onaylaşım problemi, gezgin robotlar, saat senkronizasyonu, insansız have araçlarının yönelim ve uçuş düzeni, akıllı güç şebekeleri ve randevu problemi gibi birçok alanda potansiyel ve uygulama bulmaktadır. Bu uygulamalar dağıtık onaylaşım algoritmalarının kararlılık ve yakınsama analizlerinin yanı sıra dağıtık onaylaşım algoritması tasarımını da içerir. Ağın ilingesel yapısına bağlı olarak onaylaşım, literatürde çokça çalışıldığı üzere ağdaki tüm etmenlerin tek bir değere ulaşmaları ve dolayısıyla ağda tam anlaşma sağlanması şeklinde gerçekleşebileceği gibi aynı zamanda ağdaki belirli etmenlerin takımlar oluşturarak ağda kısmi bir anlaşma sağlanması suretiyle birden çok takımın kendi içerilerinde sabit olacak şekilde farklı değerlere ulaşmaları suretiyle de gerçekleşebilir. Bu tezde ayrık zamanda değişken ilingeli çok etmenli ağlarda çoklu denge noktalı takım onaylaşımı problemi ele alınmıştır. Çalışmanın temel katkısı, ağın ilingesinin zamanla değiştiği durumda takım onaylaşımına ulaşılabilmesi için yeter koşullar ortaya koyması, ve takım onaylaşımı durumunda ağda kaç farklı değerin oluşacağını ağın ilingesel özelliklerinden çıkarımla söylemesidir. Ayrıca indirgenmiş çizge ve ölçeklenmiş çizge yöntemleriyle çizgenin boyutu azaltılırken ağda kaç farklı takımın oluşacağını belirleyen yapısal ve ilingesel birtakım anahtar özelliklerin korunduğu gösterilmiş ve ağın irdelenmesinde bu özellikler kullanılmıştır.

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LIST OF SYMBOLS

$\mathcal{A}$	The adjacency matrix
D	The weighted in-degree matrix
$\mathcal{E}$	The set of edges
$\mathcal{G}$	A directed graph
$\mathcal{G}_{p,i}$	The ith primary layer subgraph
$\mathcal{G}_{p,i}$	The ith secondary layer subgraph
I	Identity matrix
$\mathcal{I}$	The index set
$Im(W)$	Image of a matrix W
$Ker(W)$	Kernel of a matrix W
l_p	The number of primary layer subgraphs
l_s	The number of secondary layer subgraphs
$\mathcal{L}$	The Laplacian matrix
L	The weighted Laplacian matrix
$\mathcal{N}_i$	The neighborhood of v_i
n_p	The total number of nodes in the primary layer subgraphs
n_s	The total number of nodes in the secondary layer subgraphs
$\mathbb{R}$	Real numbers
u	Right eigenvector of a matrix
$U(a,b)$	Uniform distribution between a and b
$\mathcal{V}$	The set of vertices
$\mathcal{V}_{p,i}$	The set of vertices belonging to ith primary layer subgraph
$\mathcal{V}_{s,i}$	The set of vertices belonging to ith secondary layer subgraph
w^T	Left eigenvector of a matrix
w_{ij}	The (i,j)th element of W matrix
W_p	Primary layer weighting matrix
W_{sp}	Primary layer to secondary layer weighting matrix
W_s	Secondary layer weighting matrix

$\overline{W}$	Reduced weighting matrix
$x(k)$	The state vector
$\overline{x}(k)$	Reduced state vector
$x_i(k)$	The state of node v_i at time step k
$x_p(k)$	The state vector of primary layer subgraphs at time step k
$x_s(k)$	The state vector of secondary layer subgraphs at time step k
x_p^*	The equilibrium point primary layer dynamics
x_s^*	The equilibrium point of secondary layer dynamics
$\mathbb{Z}_+$	Positive integers
λ	Eigenvalue of a matrix
$\rho(.)$	Spectral radius of a matrix
$\Phi(.,.)$	State transition matrix
$\mathbb{0}_n$	0 vector of size n
$0_{n\times m}$	$n \times m$ matrix consisting of all zeros
$\mathbb{1}_n$	1 vector of size n
$\|.\|$	Cardinality of a set
$\|\|.\|\|$	Infinity norm of a vector or a matrix

LIST OF ACRONYMS/ABBREVIATIONS

BIBS	Bounded-Input Bounded-State
LTV	Linear Time-Varying

1. INTRODUCTION

In multi-agent systems, consensus, namely achieving a common value, is one of the fundamental problems that has attracted substantial attention in contemporary research. With the quick development of technology, especially with the increasing processing capacity over the decades, consensus problems have become essential in many systems. Some specific applications include path and motion planning for mobile robots, clock synchronization in time-sensitive civilian and military areas, orientation, flight formation of unmanned air vehicles, and electric power systems.

Centralized control and decision-making theory is an approach under the assumption that a single central decision-maker has access to complete information of the system. However, centralized control and decision-making theory became infeasible or impractical to apply to many real-world systems due to the increasing size and complexity of such systems, limitations on communication between agents of the system, and constraints on energy usage or response time of the system. For these reasons, the decentralization of such systems is indispensable, as more than one decision maker coexists in the system, with access to partial information of the system. In this context, centralized consensus is easier than distributed consensus since a single decision-maker is authorized to update the state of every agent so that synchronization and consensus are achieved. On the contrary, distributed consensus, where the system consists of independent components communicating through a distributed protocol, has its challenges, such as guaranteeing the existence of a consensus state, convergence rate to the consensus state if it exists, or maintaining a consensus state under communication interrupts. From now on, distributed consensus will be referred to simply as consensus.

In the literature, there is a substantial amount of work on consensus problems where the agents in a multi-agent network converge to a single steady-state value; in other words, a complete agreement is achieved among all the participants in the network. In the situation where a partial agreement is achieved instead of a complete agreement, we may still claim consensus is achieved in terms of multiple equilibrium points. The latter case is called cluster (or group) consensus in the literature and also drew significant attention. Both single-equilibrium and multi-equilibria consensus problems can be studied in various aspects depending on the topology and properties of the multi-agent dynamics under consideration, such as linearity/non-linearity of the dynamics, continuous time-discrete time system dynamics, autonomous-non autonomous system dynamics, order of the system dynamics, the existence of communication delays, time variance/invariance, the directionality of information exchange, etc [1–11]. Some representative lines of research are discussed below.

Jadbabaie et al. [1] consider averaging-based discrete time and continuous time update schemes where information exchange between agents in the network is bidirectional, i.e., the graph representation of the multi-agent network is undirected. The authors analyze the problem in both leaderless and leader-following cases under time-invariant and dynamic interaction topologies. By constructing a graph-theoretic framework for the problem, they state that for a leaderless network, consensus is achieved if non-empty, bounded time intervals exist such that, during each interval, the graph is strongly connected. This result is then extended to the leader-following case, showing that each agent should be linked to the leader frequently enough for consensus to be achieved.

Ren and Beard [2] aim to explore the minimum requisites for achieving consensus when the information exchange is allowed to be unidirectional. A significant contribution of their study is demonstrating that a nonnegative matrix sharing the exact positive row sums possesses its spectral radius as a simple eigenvalue only if the directed graph formed by this matrix exhibits a spanning tree. This differs from the Perron-Frobenius theorem for nonnegative matrices, which focuses solely on irreducible matrices, i.e., those with strongly connected graphs. Additionally, if this matrix includes positive diagonal entries, it shows that its row sum represents the unique eigenvalue with the maximum modulus. These results show that consensus under time-invariant interaction topology and fixed weighting coefficients can be attained asymptotically if the graph representing the multi-agent network contains a spanning tree for discrete time and continuous time first-order consensus schemes. Furthermore, they extend their results to dynamic interaction topologies and state that consensus can be attained asymptotically if the union of interaction graphs contains a spanning tree.

In [3], Olfati-Saber and Murray propose two linear consensus algorithms, one for zero communication delay and another for nonzero communication delay cases. They analyze the average-consensus problem using the proposed protocols. Utilizing algebraic graph theory, they prove that asymptotic convergence is established for time-invariant networks, provided that the graph representing the multi-agent network is strongly connected and balanced. In other words, the in-degree and out-degree of each node in the graph are equal. Using Lyapunov theory, they provide a similar condition for networks under dynamic interaction topologies. A final contribution of the paper is the analysis of the convergence of the proposed protocol for undirected networks with communication time delays. They set an upper bound for the time delays to ensure the system achieves consensus. Additionally, they derive the result that trade-offs exist between performance-robustness and high performance-low communication cost.

Ren studies leader-following consensus in directed networks with time-varying reference state in [5]. The author proposes multiple linear consensus protocols for the network in three different cases: in the first, the reference state is available to every other agent in the network; in the second, the reference state is available only to a portion of the agents in the network, and in the last case, a separation between state values of each agent in the network is introduced. For each case, it is shown that the proposed algorithms solve the consensus problem asymptotically if and only if the directed graph has a directed spanning tree. Subsequently, the author applies the results to the multi-vehicle rendezvous problem and numerically demonstrates them.

The aforementioned studies primarily focused on the single-equilibrium consensus problem, analyzing it from multiple perspectives. However, agents naturally form multiple groups or clusters in many real-world networks to fulfill specific tasks. In such cases, when the network is not strongly connected or lacks a directed spanning tree, a cluster (multi-equilibria) consensus state emerges. Real-life examples of this phenomenon include physical scenarios like cluster synchronization of coupled chaotic oscillators [12] and cluster bifurcation [13], as well as natural occurrences such as team hunting by predators, obstacle avoidance in animal herds [14], or specific tasks that require clustering, such as the cooperative searching of autonomous vehicles for multiple objects [15]. In addition, the formation of personal or political opinions among individuals in a society is another example of a network where cluster consensus is observed [16,17]. When analyzing such networks, most studies assume prior knowledge of the network topology or a former organization of the network structure.

Yu et al. [7] study cluster consensus problem under dynamic interaction topologies with and without communication delays. Authors develop a method, so-called multi-tree-form transformation, where the graph is partitioned into multiple subgraphs, each is a tree on its own, and the union of all the nodes in these partitions gives the set of nodes of the graph itself. Then, they use this transformation to reduce the order of the system and show that the consensus protocol solves the cluster consensus problem if and only if all the solutions of the reduced order system are asymptotically stable.

In [8], Safavi and Khan analyze the stability and convergence properties of first-order linear time-varying systems using infinite products of sub-stochastic matrices and slice notation they introduce. Using the slice notation, they state sufficiency conditions for the asymptotic stability of an autonomous system and then apply the results to a multi-agent network under dynamic interaction topologies where leader-following consensus is considered. They show that an infinite product of sub-stochastic matrices converges to zero. They define leader-following dynamics to be autonomous dynamics. Therefore, they conclude that each one of the agents converges to the anchor's state as an infinite product converges, and the anchor utterly governs the dynamics.

Erkan et al. [9] define two novel concepts, primary layer subgraph, and secondary layer subgraph, to analyze the cluster consensus problem. Instead of imposing restrictive assumptions regarding prior knowledge or arrangement of the network, they employ primary and secondary layer detection algorithms to partition the given network in a way that does not rely on any prior information. Using these two concepts, they demonstrate that the number of clusters equals the sum of the number of primary and secondary layer subgraphs, providing a framework to determine the number of equilibrium points without prior knowledge of the network. This is in contrast to previous research in the literature, where clusters are predetermined. The authors further investigate scenarios where communication lags exist within the system. Their findings demonstrate that these delays, as long as they remain limited, do not impact the number of clusters determined by the primary and secondary layer subgraphs.

In [11], Develer and Akar investigate the problem of cluster consensus in continuous-time multi-agent systems of first and second order, taking into account the presence or absence of delays. The authors extend the results to continuous time and second-order dynamics by building upon the methodology presented in [9]. In this extension, they present necessity and sufficiency conditions for coupling strengths and communication delays, determining the requirements for the multi-agent network to achieve cluster consensus, where the number of clusters is identified as in [9]. Their results are extended to higher order systems in [18].

1.1. Motivation of the Thesis

In most real-time applications, systems either inherit complex dynamics leading to cluster consensus phenomena, or the system's objective requires clustering; therefore, multi-equilibria cluster consensus has become crucial in modeling, analyzing, and controlling multi-agent systems. Despite the remarkable work in the study of consensus protocols, there is a lack of research in the analysis of the multi-equilibria cluster consensus problem. Regarding the technological breakneck, one may expect the number of agents participating in multi-agent networks to proliferate, parallel to the growth in processing power and production. Therefore, as networks begin to become more complex, the need for distributed computation increases as well, whereas, on the contrary, the overall knowledge of the networks deteriorates. Most of the contemporary literature on the topic, as seen in [1–3,5,8], focuses on single-equilibrium consensus. A lesser attempt is made for multi-equilibria consensus, as in [7,9,11]. For multi-equilibria consensus, [9] provides a novel framework, introducing a graph partitioning method that makes it possible to determine the number of clusters without former knowledge of the network. Instead, an algorithm is proposed that performs a partitioning on the graph to determine the number of clusters. However, these studies make limited reference to time-varying networks. For time-varying networks, most of the work in the literature sets tight conditions or does not include multi-equilibria cases. Hereby, the primary motivation of this thesis is to provide a framework for the analysis of multi-agent networks where multiple clusters exist, in the sense of stating conditions for the stability of the network, as well as giving results on the number of clusters formed in the network without an apriori knowledge of the network, where the network is not assumed to be time-invariant. From a comprehensive perspective, this thesis aims to study the multi-equilibria consensus problem for time-varying networks and to determine the steady-state behavior of such networks from the topological properties of the network.

1.2. Contribution of the Thesis

The main contributions of the thesis can be stated as follows:

- Using primary and secondary layer subgraph concepts and the partition arising from these subgraphs, the dynamics of a first-order linear time-varying system are analyzed under assumptions on network topology, and convergence results are given, along with the number of clusters forming in the steady-state, which is determined without former partitioning of the network.
- Reduced graph notation is used to analyze multi-agent networks' convergence and cluster formation. Graph reduction methods are introduced to the formerly given framework, and results are presented. The stability and convergence of the

reduced network are analyzed, and the number of clusters formed in the steady-state for the reduced network is given.

- Sufficient conditions on the reduced graph of a network to achieve multi-equilibria consensus are presented.

1.3. Organization of the Thesis

The rest of the thesis is organized in the following manner:

In Chapter 2, graph theoretic preliminaries are given along with necessary mathematical basics. Then, the discrete-time consensus protocol is described, and consensus and multi-equilibria consensus are defined. A well-known result in the literature on the classical consensus problem is mentioned. Using the primary and secondary layer subgraph partition, primary and secondary layer dynamics are obtained, and a theorem on multi-equilibria cluster consensus that will be frequently visited is detailed.

In Chapter 3, based on the partition made in Chapter 2, dynamics arising from the partition are analyzed in terms of stability and convergence, where the dynamics are no longer time-invariant. Under certain assumptions on network structure, convergence is shown, and the number of clusters is determined.

In Chapter 4, the reduced graph is introduced, and a mathematical framework for graph reduction is presented. Based on the spectral properties of a given graph, spectral analysis for the reduced graph is implemented. Results on the stability and convergence of the reduced graph are presented from the relation between the original and the reduced graph. Then, some sufficient conditions for a time-varying network to achieve multi-equilibria consensus are presented, along with the number of clusters formed in the steady-state. The conditions are presented using reduced graph notation.

Lastly, in Chapter 5, concluding remarks are given, and future work based on this thesis is discussed.

2. DISCRETE-TIME DISTRIBUTED CONSENSUS ALGORITHM

In this chapter, we introduce the fundamental principles of graph theory and explain how multi-agent networks can be represented using graph-theoretical notation. We present the discrete-time distributed consensus algorithm, which will be discussed throughout the thesis, and we define the multi-equilibria consensus problem in multi-agent systems. Then, we analyze multi-equilibria consensus in networks where the weighting matrix has semisimple eigenvalues; namely, the graph representing the network does not have a spanning tree. We focus on fixed topology networks where the weighting matrix is time-invariant.

2.1. Graph Theory and Representation of Networks

A distributed system involving multiple agents can be symbolized by a directed graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the graph comprises vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ indexed by the set $\mathcal{I} = \{1, 2, \cdots, n\}$ and a collection of edges representing connections between pairs, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Information flow between pairs is denoted by (v_i, v_j) where the agent j receives information from agent i. The adjacency matrix of $\mathcal{G}$ is a non-negative matrix, represented as $\mathcal{A} = [a_{ij}]$ where $a_{ij} = 0$ if (v_j, v_i) does not belong to $\mathcal{E}$ and $a_{ij} > 0$ otherwise. The weighted in-degree matrix of $\mathcal{G}$ is a non-negative, diagonal matrix, denoted by $D = [d_{ij}]$ where $d_{ij} = 0$ when $i \neq j$ and $d_{ii} > 0$ otherwise. When $d_{ii} > 0$, it equals the number of edges that end up at node v_i. $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}$, which is defined as $\mathcal{L} = D - \mathcal{A}$ and L is the weighted Laplacian matrix of $\mathcal{G}$, which is defined as $L = D - W$ where W is the weighting matrix of the graph. For the rest of the thesis, the term degree matrix will be used in place of the weighted in-degree matrix.

The neighborhood of a vertex v_i comprises the set of vertices adjacent to v_i, which can be defined mathematically as $\mathcal{N}_i = \{v_j : (v_j, v_i) \in \mathcal{E}\}$. A directed path between two vertices v_i and v_j is defined as a finite sequence of vertices $v_i, \ldots, v_j$ such that $(v_i, v_{i+1}) \in \mathcal{E}$, $i = 1, \ldots, j-1$. The set of reachable vertices by v_i is denoted by S_i and defined as the set of vertices such that a directed path exists between v_i and each of them.

A graph is labeled as *strongly connected* when a directed path connects any pair of vertices. A directed graph is said to have a *spanning tree* if there exists at least one vertex v_i such that $S_i = \mathcal{V}$ with v_i being referred to as the *root* of the spanning tree.

If graph topology is time-dependent, the network can be represented by a dynamical graph $\mathcal{G}(k) = (\mathcal{V}(k), \mathcal{E}(k))$. $\mathcal{V}(k)$ and $\mathcal{E}(k)$ are respectively the set of vertices and the set of edges of the graph at time step k.

Definition 1. *A subgraph of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, such that $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E} \cap (\mathcal{V}' \times \mathcal{V}')$.*

Example 1. *A graph with 6 vertices and 6 edges is shown in Figure 2.1(a). By removing vertices 3 and 4 and the edges that either start or end at these vertices, we end up with the graph in Figure 2.1(b). We deleted vertices and edges to obtain the graph in Figure 2.1(b); therefore, the conditions given in Definition 1 are satisfied, which implies the graph in Figure 2.1(b) is a subgraph of the graph in Figure 2.1(a).*

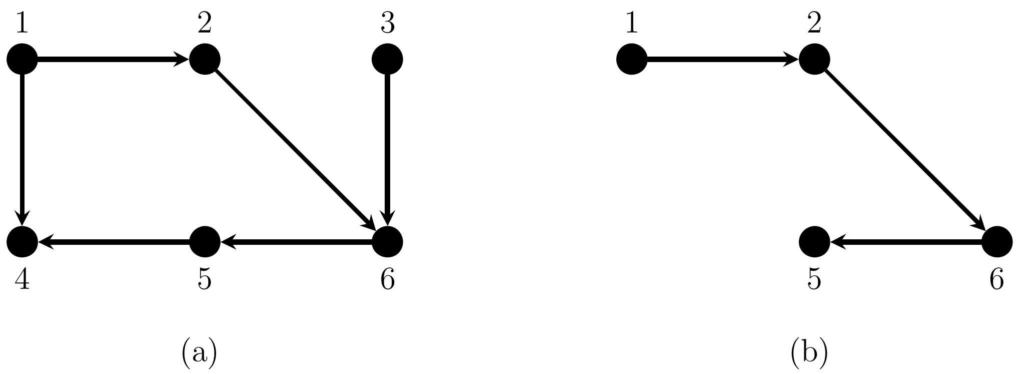

Figure 2.1. (a) Graph representation of a network consisting of 6 agents, (b) A subgraph of the given graph.

Definition 2. *[9](Primary layer subgraphs) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph under consideration. There exist l_p ($l_p \geq 1$) subsets in the vertex set $\mathcal{V}$ such that each subset $\mathcal{V}_{p,i}$, $i = 1, \ldots, l_p$, is the largest possible subset that has a spanning tree for its subgraph $\mathcal{G}_{p,i}$ and for all $v_a \in \mathcal{V}_{p,i}$ and $v_b \notin \mathcal{V}_{p,i}$, we have $(v_b, v_a) \notin \mathcal{E}$. It is said that $\mathcal{G}_{p,i}$ ($i = 1, \ldots, l_p$) are the primary layer subgraphs of $\mathcal{G}$ where the number of primary layer subgraphs is denoted by l_p.*

Definition 3. *[9](Secondary layer subgraphs) Let $\overline{\mathcal{V}}$ be the set which consists of vertices that are not in the primary layer subgraphs, i.e., $\overline{\mathcal{V}} = \mathcal{V} \setminus \bigcup_{i=1}^{l_p} \mathcal{V}_{p,i}$. Then there exist l_s subsets in $\overline{\mathcal{V}}$ such that each subset $\mathcal{V}_{s,i}$, $i = 1, \ldots, l_s$, has a spanning tree for its subgraph $\mathcal{G}_{s,i}$ and there exists exactly one vertex $v_a \in \mathcal{V}_{s,i}$ which satisfies the following*

(i) *For all $v_b \in \mathcal{V}_{s,i} \setminus v_a$ and $v_c \in \mathcal{V} \setminus \mathcal{V}_{s,i}$, we have $(v_c, v_b) \notin \mathcal{E}$.*

(ii) *There exist at least two vertices in two different subgraphs (either primary or secondary layer) v_d and v_e such that $(v_d, v_a) \in \mathcal{E}$ and $(v_e, v_a) \in \mathcal{E}$.*

(iii) *v_a is the root of the spanning tree in $\mathcal{V}_{s,i}$.*

It is said that $\mathcal{G}_{s,i}$ ($i = 1, \ldots, l_s$) are the secondary layer subgraphs of $\mathcal{G}$ where the number of secondary layer subgraphs is denoted by l_s. The subsets $\mathcal{V}_{s,i}$, $i = 1, \ldots, l_s$ are defined as the vertex sets of $\mathcal{G}_{s,i}$.

Example 2. *For the graph given in Figure 2.1(a), vertices 1 and 2 form a primary layer subgraph with v_1 being the root of the spanning tree for $\mathcal{G}_{p,1}$. If vertex 6 is included in $\mathcal{G}_{p,1}$, then vertex 3 should also be included since any vertex in a primary layer subgraph does not receive information from a vertex that does not belong to that subgraph. However, if vertex 3 is included, there is no spanning tree for that subgraph. Therefore, vertices 1 and 2 form a primary layer subgraph. With the same reasoning, including vertex 4 is also not possible. Figure 2.2 illustrates the partitioning of the provided graph into primary and secondary layer subgraphs.*

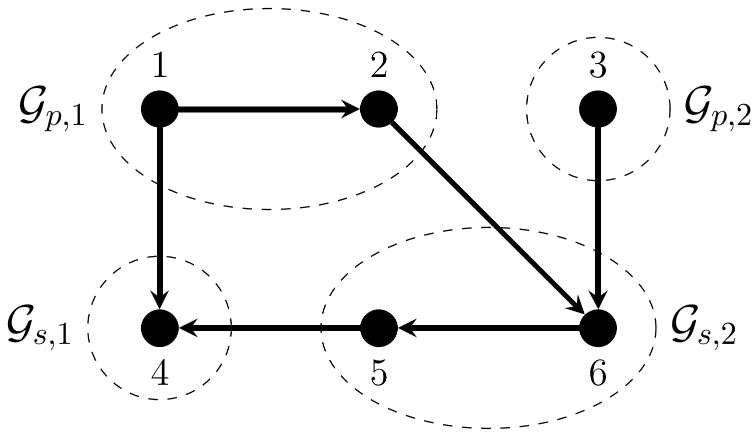

Figure 2.2. Partitioning of the network given in Figure 2.1 in terms of primary and secondary layer subgraphs.

2.2. Discrete-Time Consensus Algorithm

Consider a multi-agent system comprising $n > 1$ ($n \in \mathbb{Z}_+$) agents, whose interaction network is defined by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $x_i \in \mathbb{R}$ denote the state value of the ith agent, and w_{ij} represent the non-negative averaging coefficient. Thus, the dynamics of the ith agent can be expressed as

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j\in\mathcal{N}_i} w_{ij}x_j(k),\ i \in \mathcal{I}. \tag{2.1}$$

Equation (2.1) can be expressed in matrix form as

$$x(k+1) = Wx(k). \tag{2.2}$$

Here $x(k) = [x_1(k) \dots x_n(k)]^T \in \mathbb{R}^n$ denotes the state vector and $W = [w_{ij}] \in \mathbb{R}^{n\times n}$ is a matrix satisfying Assumption 1. W is called the weighting matrix of the graph representing the multi-agent network. $\mathcal{G}(W) = (\mathcal{V}, \mathcal{E})$ signifies the graph associated with the weighting matrix W.

Assumption 1. *The averaging coefficients are assumed to satisfy*

(i) $w_{ii} > 0, \forall i \in \mathcal{I}$,

(ii) $w_{ij} > 0$ *if* $(v_j, v_i) \in \mathcal{E}$ *and* $w_{ij} = 0$ *if* $(v_j, v_i) \notin \mathcal{E}$ *for all* $i, j \in \mathcal{I}$ *and* $i \neq j$,

(iii) $\sum_{j=1}^{n} w_{ij} = 1, \forall i \in \mathcal{I}$.

Definition 4. *A matrix $W \in \mathbb{R}^{n\times n}$ is called row-stochastic if the following hold.*

(i) All entries of W are nonnegative,

(ii) Each row of W sums up to 1.

Throughout the thesis, the term stochastic will be used in place of row-stochastic. Assumption 1 states that the network matrix is row-stochastic and ensures that at each time step, the state of an agent resides at the convex hull of the neighbor agent states, and each agent in the network receives information from its neighbors with positive weight.

From the Perron-Frobenius theorem and its extension on non-negative matrices, and by utilizing the Gershgorin circle theorem, the largest eigenvalue of a stochastic matrix is equal to 1 in magnitude, and as a result, all eigenvalues of a stochastic matrix resides in the closed unit disk [19]. Therefore, Assumption 1 guarantees that the spectral radius of W, $\rho(W) = 1$, and all other eigenvalues of W are strictly less than 1 in magnitude.

For networks with varying topology, the system given in (2.2) becomes

$$x(k+1) = W(k)x(k), \tag{2.3}$$

where $W(k) = [w_{ij}(k)] \in \mathbb{R}^{n\times n}$ are row-stochastic matrices that satisfy Assumption 1 for all k.

Before defining multi-equilibria consensus, we may first define the conventional consensus problem.

Definition 5. *(Consensus) It is said that the network represented by* (2.1) *converges to consensus equilibria state if there exists a constant c such that*

$$\lim_{k\to\infty} x_i(k) = c, \; i = 1, \ldots, n \tag{2.4}$$

for any arbitrary initial condition $[x_1(0), \ldots, x_n(0)]^T \in \mathbb{R}^n$ and any arbitrary selection of weighting coefficients w_{ij} that adhere to Assumption 1.

From [2], if a nonnegative matrix $W \in \mathbb{R}^{n\times n}$ has the same positive constant row sums given by $\lambda > 0$, then λ is an eigenvalue of W with an associated eigenvector $\mathbb{1}_n$ and $\rho(W) = \lambda$. In addition, the eigenvalue λ of W has an algebraic multiplicity equal to one if and only if the graph associated with W has a spanning tree. Furthermore, if the network represented by (2.1) converges to consensus, $\lambda = 1$ is a simple eigenvalue of W.

The solution to the system, with dynamics given by (2.2) is $x(k) = W^k x(0)$. Consider the Jordan form of W, $W = PJP^{-1}$. If $\lambda = 1$ is a simple eigenvalue, we have the partition

$$W = PJP^{-1} = P \begin{bmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & * \end{bmatrix} P^{-1}. \tag{2.5}$$

Then,

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} PJ^kP^{-1}x(0) = P \begin{bmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{bmatrix} P^{-1}x(0) \tag{2.6}$$

$$= \frac{uw^T}{w^Tu}x(0) = x^*, \tag{2.7}$$

where u and w^T are right and left eigenvectors corresponding to $\lambda = 1$, respectively. Then, the protocol given in (2.1) solves the consensus problem given in Definition 5 if and only if the associated graph $\mathcal{G}(W)$ has a spanning tree [2].

The conventional consensus problem is well analyzed in the literature, and powerful results related to the topology of the network are introduced. However, when the graph does not have a spanning tree, it does not necessarily mean the system would diverge. Suppose $\mathcal{G}(W)$ does not have a spanning tree. Then, $\lambda = 1$ is not a simple eigenvalue of W. Therefore, $\operatorname{rank}(\lim_{k\to\infty} W^k) = \operatorname{rank}(\lim_{k\to\infty} J^k) > 1$, which implies that the protocol given in (2.2) cannot solve the consensus problem defined in Definition 5. Yet, multiple groups can be formed in the steady-state when consensus is not achieved, and this case is analyzed under the multi-equilibria consensus problem whose definition is given below.

Definition 6. *[9](Multi-equilibria consensus) It is said that the network represented by* (2.1) *converges to* K *consensus equilibria states if there exist* K *distinct constants* c_l*, and* K *non-empty sets* S_l*,* $l = 1, \cdots, K$*, such that*

$$\bigcup_{l=1}^{K} S_l = \mathcal{V},\ \ S_l \cap S_m = \emptyset,\ \ \textit{for } l \neq m,\ \ \textit{and } l, m = 1, \ldots, K \tag{2.8}$$

and for the set S_l*, we have*

$$\lim_{k\to\infty} x_i(k) = c_l,\ \forall v_i \in S_l,\ \ i = 1, \ldots, n \tag{2.9}$$

for arbitrary initial condition $[x_1(0), \ldots, x_n(0)]^T \in \mathbb{R}^n$ *and arbitrary choice of averaging (weighting) coefficients* w_{ij}*, satisfying Assumption 1.*

The following example demonstrates the difference between conventional consensus and multi-equilibria consensus, where a difference in the topological properties of the network results in different steady-state behavior. Two networks, one containing a spanning tree and one that does not contain a spanning tree are considered to show the difference.

Example 3. *Consider the two networks given in Figure 2.3.*

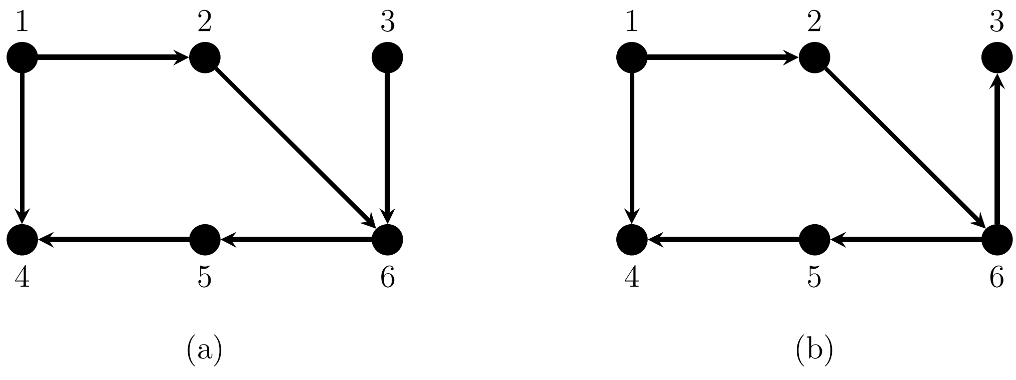

Figure 2.3. (a) Graph with 2 primary, 2 secondary-layer subgraphs, (b) Graph containing a spanning tree.

Suppose the weighting matrices are given as

$$W_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0.5 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0.1 & 0.2 & 0 & 0 & 0.7 \end{bmatrix}, \quad W_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.2 & 0 & 0 & 0.5 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0.3 & 0 & 0 & 0 & 0.7 \end{bmatrix}. \tag{2.10}$$

Notice that W_a is in block lower triangular form, whereas no permutation matrix can bring W_b to block lower triangular form. Therefore, we can conclude that W_b is irreducible, which implies $\mathcal{G}(W_b)$ contains a spanning tree. Then, consensus is guaranteed. However, W_a is reducible, which means $\mathcal{G}(W_a)$ does not contain a spanning tree, which leads the network to achieve consensus in the sense of conventional definition. However, the system reaches a multi-equilibria consensus. Figure 2.4 shows the simulation of networks.

This example demonstrates that the spanning tree guarantees the convergence to a single value and the lack of a spanning tree gives rise to a multi-equilibrium state.

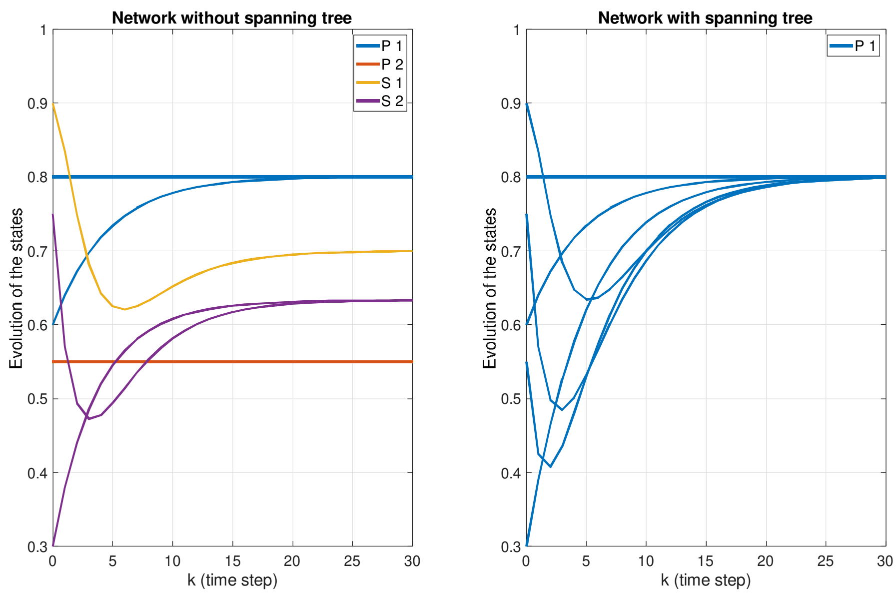

Figure 2.4. Simulation of networks with and without a spanning tree.

2.3. Analysis of Multi-Equilibria Consensus for Fixed Topology Networks

If the graph $\mathcal{G}(W)$ does not have a spanning tree, the eigenvalue with the largest modulus is no longer simple but, instead, semi-simple. Therefore, the consensus problem defined in Definition 5 cannot be solved by the protocol (2.2). We cannot find unique left and right eigenvectors to characterize a single-equilibria convergence.

A matrix $W \in \mathbb{R}^{n\times n}$ is said to be a *reducible matrix* if there exists a permutation matrix P such that $\mathrm{P}^T W \mathrm{P}$ is block lower triangular. Otherwise, W is said to be an *irreducible matrix*. In [20], a matrix W is irreducible or 1-reducible if and only if $\mathcal{G}(W)$ contains a spanning tree. From Perron–Frobenius theorem [21], if a nonnegative matrix $W \in \mathbb{R}^{n\times n}$ is irreducible, then each of the following is true.

(i) $r = \rho(W)$ and $r > 0$,
(ii) Algebraic multiplicity of r is 1,
(iii) There exists an eigenvector $u > 0$, such that $Wu = ru$.

Now, we can state the following lemma.

Lemma 1. *[9] After certain transformations, any weighting matrix W satisfying Assumption 1 can be expressed in the form*

$$W = \begin{bmatrix} W_{1,1} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & W_{l_p,l_p} & 0 & \dots & 0 \\ W_{l_p+1,1} & \dots & \dots & W_{l_p+1,l_p+1} & \dots & W_{l_p+1,l_p+l_s} \\ \vdots & & & & \ddots & \vdots \\ W_{l_p+l_s,1} & \dots & \dots & \dots & \dots & W_{l_p+l_s,l_p+l_s} \end{bmatrix}. \tag{2.11}$$

The first l_p blocks correspond to the primary layer subgraphs and the rest to the secondary layer subgraphs.

Graphs containing spanning trees consist of only one primary layer subgraph: the graph itself. Therefore, they remain unaffected under the partitioning given in (2.11). For graphs consisting of more than 1 primary and secondary layer subgraphs, this partition brings the network matrix to a block lower triangular form. Using the partition obtained, we can rewrite the dynamics given by (2.2) as

$$\begin{bmatrix} x_p(k+1) \\ x_s(k+1) \end{bmatrix} = \begin{bmatrix} W_p & 0 \\ W_{sp} & W_s \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_s(k) \end{bmatrix}. \tag{2.12}$$

Here, $x_p(k) \in \mathbb{R}^{n_p}$ and $x_s(k) \in \mathbb{R}^{n_s}$ are the state variables of the agents in the network that belong to primary and secondary layer subgraphs respectively, and weight matrices are

$$W_p = \begin{bmatrix} W_{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & W_{l_p,l_p} \end{bmatrix}_{n_p \times n_p}, \tag{2.13}$$

$$W_{sp} = \begin{bmatrix} W_{l_p+1,1} & \dots & W_{l_p+1,l_p} \\ \vdots & \ddots & \vdots \\ W_{l_p+l_s,1} & \dots & W_{l_p+l_s,l_p} \end{bmatrix}_{n_s \times n_p}, \tag{2.14}$$

$$W_s = \begin{bmatrix} {}_{l_p+1,l_p+1} & \dots & W_{l_p+1,l_p+l_s} \\ \vdots & \ddots & \vdots \\ W_{l_p+l_s,1} & \dots & W_{l_p+l_s,l_p+l_s} \end{bmatrix}_{n_s \times n_s}, \tag{2.15}$$

where n_p and n_s are the total number of nodes in primary and secondary layer subgraphs, respectively. Also note that if the network topology varies over time steps, (2.12) becomes

$$\begin{bmatrix} x_p(k+1) \\ x_s(k+1) \end{bmatrix} = \begin{bmatrix} W_p(k) & 0 \\ W_{sp}(k) & W_s(k) \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_s(k) \end{bmatrix}. \tag{2.16}$$

Therefore, we may divide the system dynamics into two: primary layer dynamics and secondary layer dynamics, which can be expressed as

$$x_p(k+1) = W_p(k) x_p(k), \tag{2.17}$$

$$x_s(k+1) = W_{sp}(k) x_p(k) + W_s(k) x_s(k). \tag{2.18}$$

The system represented by (2.17) is autonomous, whereas the system described by (2.18) is non-autonomous with $x_p(k)$ acting as an input. Therefore, a homogeneous linear difference equation governs primary layer dynamics, and a non-homogeneous linear difference equation governs secondary layer dynamics.

Under the assumption that the network topology is fixed over time steps, we direct our focus to (2.12). The stability of this system can be shown easily since the system matrix is fixed [9].

Definition 7. *(Sub-stochastic matrix) Let $W = [w_{ij}] \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. W is said to be sub-stochastic if $\sum_{j=1}^{n} w_{ij} \leq 1$ for all $i = 1, \ldots, n$. If the equality holds for all i, W is said to be stochastic. W is said to be strictly sub-stochastic if $\sum_{j=1}^{n} w_{ij} < 1$ for all $i = 1, \ldots, n$.*

From Assumption 1, it follows that W_s is a sub-stochastic matrix. Furthermore, it is shown in [9] that W_s has its eigenvalues inside the unit circle, therefore, W has exactly l_p eigenvalues at $\lambda = 1$, and the rest lie inside the unit circle. To further gain insight into the convergence of the system, consider the Jordan form of W, $W = PJP^{-1}$. Notice that we can express P and P^{-1} in terms of right and left-eigenspaces of W as

$$P = \Big[u_1, \ldots, u_n\Big], \qquad P^{-1} = \Big[w_1, \ldots, w_n\Big]^T. \tag{2.19}$$

Let l_p and l_s denote the number of primary and secondary layer subgraphs, respectively. Without loss of generality, assume J is in the form

$$J = \begin{bmatrix} I_{l_p\times l_p} & 0_{l_p\times(n-l_p)} \\ 0_{(n-l_p)\times l_p} & T_{(n-l_p)\times(n-l_p)} \end{bmatrix}, \tag{2.20}$$

for some T matrix. We know that $\rho(T) < 1$. Therefore, we have

$$\lim_{k\to\infty} J^k = \lim_{k\to\infty} \begin{bmatrix} I_{l_p\times l_p} & 0_{l_p\times(n-l_p)} \\ 0_{(n-l_p)\times l_p} & T_{(n-l_p)\times(n-l_p)} \end{bmatrix}^k = \begin{bmatrix} I_{l_p\times l_p} & 0_{l_p\times(n-l_p)} \\ 0_{(n-l_p)\times l_p} & 0_{(n-l_p)\times(n-l_p)} \end{bmatrix}. \tag{2.21}$$

Then, considering the limit

$$\lim_{k\to\infty} W^k = \lim_{k\to\infty}(PJP^{-1})^k = P(\lim_{k\to\infty} J^k)P^{-1} = P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \sum_{i=1}^{l_p} \frac{u_i w_i^T}{w_i^T u_i}, \qquad (2.22)$$

where u_i and w_i^T are linearly independent right and left eigenvectors associated with $\lambda = 1$ respectively.

Besides the stability, the number of equilibria states is another topic. The subsequent lemma provides the count of equilibrium states for a multi-equilibria consensus, with dynamics described in (2.1).

Lemma 2. *[9] The protocol given in* (2.2) *solves the multi-equilibria consensus problem defined in Definition 6, under Assumption 1, where the number of equilibrium states for a network with graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is given by* $(l_p + l_s)$ *where* l_p *and* l_s *represent the counts of primary and secondary layer subgraphs, respectively.*

The following example illustrates the number of equilibrium states formed in multi-equilibria consensus for a network that does not contain a spanning tree.

Example 4. *Consider the network given in Figure 2.5. The evolution of the state values is shown in Figure 2.6 for randomly selected initial state vector and weighting coefficients,* w_{ij}*, that satisfy Assumption 1. The network converges to 5 clusters, 3 of which correspond to primary layer subgraphs, and the remaining 2 correspond to secondary layer subgraphs. This demonstrates the result that the network converges to* $K = l_p + l_s = 5$ *clusters.*

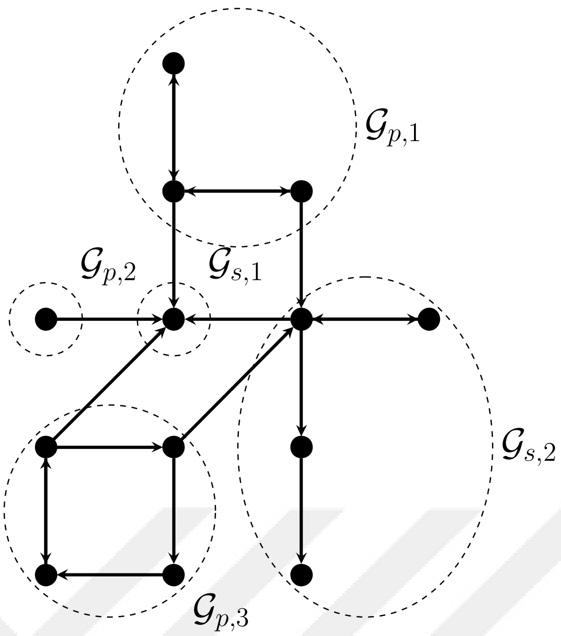

Figure 2.5. A graph consisting of 13 nodes and 19 edges.

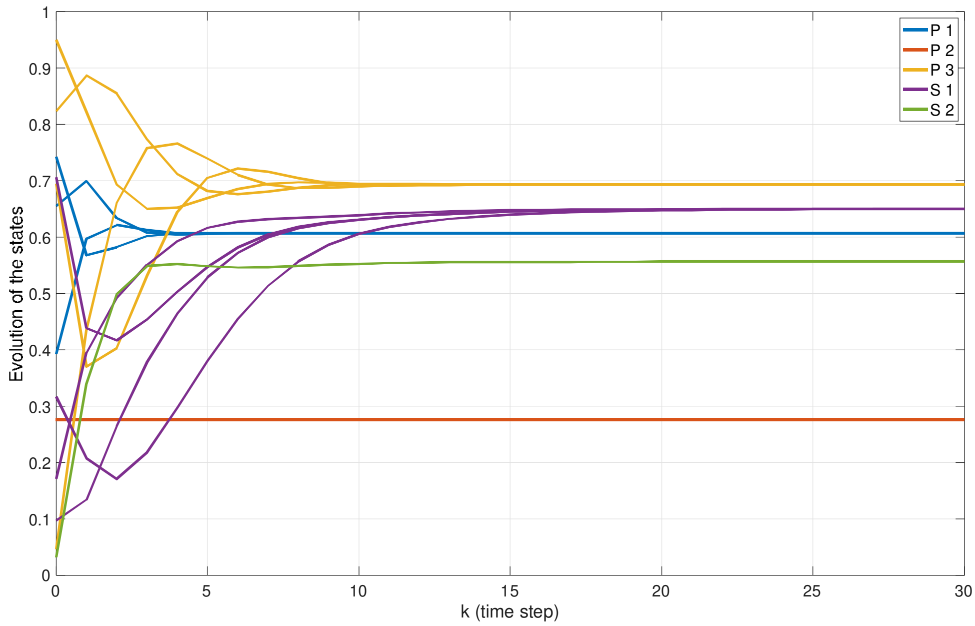

Figure 2.6. Simulation of network given in Figure 2.5.

2.4. Summary of the Chapter

In this chapter, we have developed basic mathematical preliminaries, such as graph theoretic definitions, representation of networks using graphs, and partitioning of graphs using novel concepts of primary and secondary layer subgraph notations. We also stated the conventional consensus and multi-equilibria consensus problems for the discrete-time consensus algorithm, along with mathematical and graph theoretical preliminaries. For the classical consensus problem, we gave basic yet powerful results which are well-known in the literature. Then, we presented the first-order dynamics in discrete time and, with the help of nonnegative matrix theory, presented primary results on stability and convergence of first-order dynamics with extended results on multi-equilibria consensus and determining the number of clusters. This chapter has focused on fixed topology networks. In the next chapter, we are going to analyze varying topology networks with first-order dynamics in discrete time.

3. VARYING TOPOLOGY NETWORKS

This chapter considers the stability and convergence of linear time-varying systems. We impose assumptions on the primary and secondary layer dynamics and analyze the solution of the system given by time-varying dynamics under these assumptions. Section 3.1 is divided into three subsections where we start from autonomous secondary layer dynamics and end up at the complete dynamics of the system. In each subsection, we provide the necessary mathematical analysis to develop our results for the subsequent section. We impose assumptions on $W_s(k)$ and $W_{sp}(k)$, and carry an analysis from the solution of the system given by linear time-varying (LTV) dynamics. Then, finally, in Section 3.2, we offer detailed examples illustrating prior results.

3.1. Linear Time-Varying Dynamics

Lemma 2, a powerful statement on the stability and convergence of the multi-agent system, is no longer applicable when the dynamics of the system are evolving with time. Although the states are not necessarily unbounded, they may not be convergent. In case they are convergent, the number of clusters formed in the steady-state is now dependent on a set of different topologies and weighting coefficients from which the weighting matrix is switching at each time step. Recall that time-varying dynamics are given in (2.16). Before analyzing primary layer and secondary layer dynamics, we focus on simpler dynamics to build the tools we need to analyze the complete system dynamics.

Consider the dynamics given by

$$x_s(k+1) = W_s(k)x_s(k), \tag{3.1}$$

$$x_s(k+1) = W_s(k)x_s(k) + W_{sp}(k)x_p^*. \tag{3.2}$$

Equation (3.1) describes an autonomous system. Relating to secondary layer dynamics, system (3.1) can be viewed as a particular case of system (2.18) where the states of the nodes in primary layer subgraphs are all zero. Equation (3.2), on the other hand, describes a non-autonomous system with fixed input. But in a similar fashion, system (3.2) can be thought of as system (2.18) with the states of the nodes in primary layer subgraphs initially in equilibrium. Therefore, systems (3.1) and (3.2) both correspond to the system (2.16) where the state values of the nodes in primary layer subgraphs are static, or dynamic with admissible initial conditions and weighting matrices $0_{n_s \times n_s}$ and $I_{n_s \times n_s}$ respectively.

From [22], the solution of the system (2.18) is

$$x_s(k) = \Phi(k, k_0)x_s(k_0) + \sum_{i=k_0}^{k} \Phi(k, i+1)W_{sp}(i)x_p(i), \tag{3.3}$$

where $\Phi(k, k_0)$ is the state transition matrix given by

$$\Phi(k, k_0) = W_s(k-1)W_s(k-2)\dots W_s(k_0), \tag{3.4}$$

$$\Phi(i, i) = I. \tag{3.5}$$

The result above expresses the solution of the dynamical system (2.16); therefore, it is also applicable to systems (3.1) and (3.2).

We shall first have a glimpse at stability theory before proceeding with analyzing the system's stability and convergence. Note that a point x_s^* is called an equilibrium point of (3.1) if

$$x_s^* = W_s(k)x_s^*, \tag{3.6}$$

for all k. The equilibrium points of a system represent the state values at which the system stays once it reaches. To characterize all equilibrium points of (3.1), we may solve (3.6). Recall that (3.6) is an eigenvalue-eigenvector equation with eigenvalue $\lambda = 1$. Then, equilibrium points of (3.1) are the eigenvectors of $W_s(k)$ corresponding to eigenvalue $\lambda = 1$. However, $W_s(k)$ does not have any eigenvalue at 1. Then, $W_s(k) - I$ is full rank, which means the null space of $W_s(k) - I$ is empty. The only vector that satisfies $(W_s(k) - I)x_s^* = 0$ is zero vector. Thus, the origin is the only equilibrium point of (3.1). Based on equilibrium points, stability can now be defined.

Definition 8. *[23] The equilibrium point* $x_s^* = \mathbb{0}_{n_s}$ *of* (3.1) *is said to be asymptotically stable if*

(i) for every $\epsilon > 0$*, there exists a* $\delta(\epsilon) > 0$ *such that* $||\Phi(k, k_0)|| < \epsilon$ *for all* $k \geq k_0$ *whenever* $||x_s(k_0)|| < \delta(\epsilon)$ *and*

(ii) there exists an $\eta > 0$ *such that* $\lim_{k \to \infty} \Phi(k, k_0) = 0$ *whenever* $||x_s(k_0)|| < \eta$.

The first condition of Definition 8 states that the equilibrium point of (3.1) is stable if the state stays arbitrarily close to the equilibrium point whenever the initial state is chosen close enough to the equilibrium point. The second condition imposes that the equilibrium point of (3.1) is asymptotically stable if the state is attracted to the equilibrium point whenever it is sufficiently close to it. If, for all initial conditions, the state is attracted to the equilibrium point, then the equilibrium point of (3.1) is called *globally asymptotically stable*. If an equilibrium point is not stable, then it is unstable. We will call the systems with no unstable equilibrium points stable for convenience. From [23], if the equilibrium point of (3.1) is asymptotically stable, then it is globally asymptotically stable.

3.1.1. No Input Dynamics

It follows from (3.3) that for the system (3.1), the solution is given by

$$x_s(k) = \Phi(k, 0)x_s(0) = W_s(k-1)W_s(k-2)\dots W_s(0)x_s(0). \tag{3.7}$$

$W_s(k)$ is sub-stochastic for all k. From Definition 3, diagonal elements of $W_s(k)$ are always less than one. Therefore, not a single element of $W_s(k)$ can be 1, even though the row sums can be 1. Therefore, under Assumption 1, $W_s(k)$ is a sub-stochastic matrix with diagonal elements strictly less than 1.

Suppose $A, B \in \mathbb{R}^{n_s \times n_s}$ are two secondary layer system matrices satisfying Assumption 1. Let $r_i(A)$ denote ith row sum of a matrix,

$$r_i(A) = \sum_{j=1}^{n_s} A_{ij}. \tag{3.8}$$

Then, consider the ith row sum of matrix $C = AB$, $r_i(C)$, which is

$$r_i(C) = \sum_{j=1}^{n_s} C_{ij} = \sum_{j=1}^{n_s} (AB)_{ij}. \tag{3.9}$$

Recall i, jth element of C is given by

$$C_{ij} = \sum_{k=1}^{n_s} A_{ik} B_{kj}. \tag{3.10}$$

Plugging (3.10) into (3.9) gives

$$r_i(C) = \sum_{j=1}^{n_s} \sum_{k=1}^{n_s} A_{ik} B_{kj}. \tag{3.11}$$

We can rearrange the sum as.

$$r_i(C) = \sum_{k=1}^{n_s} A_{ik} \sum_{j=1}^{n_s} B_{kj} = \sum_{k=1}^{n_s} A_{ik} r_k(B). \tag{3.12}$$

Trivially, if ith row of A does not sum up to 1, ith row of C also does not sum up to 1,

$$\sum_{k=1}^{n_s} A_{ik} r_k(B) \leq \sum_{k=1}^{n_s} A_{ik} < 1. \tag{3.13}$$

There exists at least one k such that $r_k(B) < 1$. Then,

$$r_i(C) = \sum_{k=1}^{n_s} A_{ik} r_k(B) = 1 \tag{3.14}$$

is satisfied only if $A_{ik} = 0$ whenever $r_k(B) < 1$. Therefore, if v_k receives information from primary layer subgraphs in $\mathcal{G}(B)$, v_i should not receive information from v_k in $\mathcal{G}(A)$ to attain $r_i(C) = 1$. Therefore $r_i(C) = 1$ imposes two conditions,

(i) $r_i(A) = 1$ and

(ii) if $(v_j, v_i) \in \mathcal{E}_A$, then $r_j(B) = 1$.

Then, if there exists an edge starting from v_j in A, AB would contain at least two rows with row sums less than 1. If A does not contain such an edge, v_j is a secondary layer subgraph on its own. Then A should have another row with a row sum less than 1 because secondary layer subgraphs are contained in the reachable set of at least one primary layer subgraph, which implies there exists a secondary layer subgraph root with a row sum less than 1. In each case, the number of rows with a row sum less than 1 increases at each multiplication. Then, at most in n_s steps, there are n_s rows with row sums less than 1. Therefore, we state the following result.

Lemma 3. *Under Assumption 1, the system* (3.1) *is asymptotically stable.*

Proof. For any k, $||W_s(k)W_s(k-1)\dots W_s(k-n_s+1)||_\infty \leq c < 1$ for some scalar c. Then,

$$\lim_{k\to\infty} ||\Phi(k,0)||_\infty = \lim_{k\to\infty} ||\prod_{i=0}^{k} W_s(i)||_\infty \leq \lim_{k\to\infty} c^k = 0. \tag{3.15}$$

Therefore, the solution given by (3.7) approaches a 0-norm solution, i.e., the system defined by (3.1) is asymptotically stable. □

All the norms we consider will be infinity norms, therefore for convenience, we will use the notation $||.||$ to denote the infinity norm for the rest of the thesis.

3.1.2. Fixed Input Dynamics

The preceding subsection analyzes the internal stability of an autonomous system. The system's dynamic description was independent of an external input; therefore, the system's response was solely dependent on its intrinsic properties. When the system under consideration is non-autonomous, we consider the system's external stability, namely, how the system responds when a bounded input is present. First, we will give the definition of bounded-input bounded-state stability.

Definition 9. *[24] If there is a scalar M such that the nonzero input satisfies $||u(k)|| \leq M$ for every k, then the input is said to be bounded. If for every bounded input, and for arbitrary initial conditions, there exists a scalar $0 \leq \delta(M, x_0)$ such that state satisfies $||x|| \leq \delta$, then the system is bounded-input bounded-state (BIBS) stable.*

Recall that asymptotic stability requires convergence to an equilibrium point. On the other hand, BIBS stability is not defined based on equilibrium points or convergence to equilibrium points. Yet, for a linear system to be BIBS stable, asymptotic stability is required.

Lemma 4. *[22] Consider the system with the state equation*

$$x(k+1) = A(k)x(k) + B(k)u(k). \tag{3.16}$$

If the dynamics $x(k+1) = A(k)x(k)$ *is asymptotically stable and* $B(.)$ *is bounded, then the system* (3.16) *is BIBS stable.*

Lemma 3 states (3.2) is asymptotically stable. Row sums of $W_{sp}(k)$ are strictly less than 1; therefore, $W_{sp}(k)$ is bounded. Therefore, the system (3.2) is BIBS stable.

Under Lemma 4, we guarantee that the states under dynamics (3.2) stay bounded for a bounded input. Since our input is fixed and hence bounded, states are bounded. However, Lemma 4 does not guarantee convergence of the states. An example of a bounded yet divergent case is when states oscillate.

For the rest of the thesis, the networks we consider are presumed to satisfy the following assumption for the primary layer subgraphs.

Assumption 2. *The number of primary layer subgraphs and the nodes in each primary layer subgraph is time-invariant.*

For analyzing the convergence of system (3.2), we impose a further assumption on the weighting matrix.

Assumption 3. $(I - W_s(k))^{-1} W_{sp}(k) = W^*$ *for some fixed matrix* W^* *and for all* $k \geq 0$.

Lemma 5. *Under Assumptions 1, 2 and 3, the system* (3.2) *is convergent.*

Proof. It follows from (3.3) that solution for the system (3.2) is given by

$$x_s(k) = \Phi(k,0)x_s(0) + \sum_{i=0}^{k-1} \Phi(k,i+1)W_{sp}(i)x_p^*. \tag{3.17}$$

Lemma 3 shows that $\lim_{k\to\infty} \Phi(k,0) = 0_{n_s \times n_s}$. Therefore,

$$\lim_{k\to\infty} x_s(k) = \lim_{k\to\infty} \sum_{i=0}^{k-1} \Phi(k,i+1)W_{sp}(i)x_p^*. \tag{3.18}$$

Using Assumption 3 and rewriting the equation,

$$\lim_{k\to\infty} x_s(k) = \lim_{k\to\infty} \sum_{i=0}^{k-1} \Phi(k, i+1)(I - W_s(i))W^* x_p^*. \tag{3.19}$$

Recall $\Phi(k, i+1) = W_s(k-1)W_s(k-2)\dots W_s(i+1)$. Then, $\Phi(k, i+1)W_s(i) = \Phi(k, i)$, therefore,

$$\begin{aligned}
\lim_{k\to\infty} x_s(k) &= \lim_{k\to\infty} \sum_{i=0}^{k-1} \Phi(k, i+1)(I - W_s(i))W^* x_p^* \\
&= \lim_{k\to\infty} \left[\sum_{i=0}^{k-1} \Phi(k, i+1) - \sum_{i=0}^{k-1} \Phi(k, i) \right] W^* x_p^* \\
&= \lim_{k\to\infty} \left[\sum_{i=0}^{k-1} \Phi(k, i+1) - \sum_{i=-1}^{k-2} \Phi(k, i+1) \right] W^* x_p^* \\
&= \lim_{k\to\infty} [\Phi(k, k) - \Phi(k, 0)] W^* x_p^* \\
&= \lim_{k\to\infty} [I - \Phi(k, 0)] W^* x_p^* \\
&= W^* x_p^*.
\end{aligned} \tag{3.20}$$

Therefore, the solution given by (3.17) converges to a fixed point in the limit. □

3.1.3. Complete Dynamics

In previous subsections, we considered special cases of (2.18), where the input was zero or it was fixed. However, notice that the input to system (2.18) is the state of the system (2.17); therefore, according to dynamics described by (2.16), input to (2.18) is not fixed or zero. Consequently, we analyze complete dynamics given by (2.18) using the results from previous subsections. From Lemma 4, the system is BIBS stable. Notice that BIBS stability is independent of the choice of bounded input, whether it is fixed or not, but it solely depends on system matrices.

It follows from (3.3) that for the system (2.18), the solution is given by

$$x_s(k) = \Phi(k, 0)x_s(0) + \sum_{i=0}^{k-1} \Phi(k, i+1) W_{sp}(i) x_p(i). \tag{3.21}$$

Previous analysis show that $\lim_{k\to\infty} \Phi(k,0) = 0_{n_s \times n_s}$, therefore, in the limit,

$$\lim_{k\to\infty} x_s(k) = \lim_{k\to\infty} \sum_{i=0}^{k-1} \Phi(k,i+1) W_{sp}(i) x_p(i). \tag{3.22}$$

Under Assumption 3,

$$\begin{aligned}
\lim_{k\to\infty} x_s(k) &= \lim_{k\to\infty} \sum_{i=0}^{k-1} \Phi(k,i+1)(I - W_s(i)) W^* x_p(i) \\
&= \lim_{k\to\infty} \left[\sum_{i=0}^{k-1} \Phi(k,i+1) W^* x_p(i) - \sum_{i=0}^{k-1} \Phi(k,i) W^* x_p(i) \right] \\
&= \lim_{k\to\infty} \left[\left(\sum_{i=1}^{k-1} \Phi(k,i) W^* (x_p(i-1) - x_p(i)) \right) + W^* x_p(k-1) - \Phi(k,0) W^* x_p(0) \right] \\
&= \lim_{k\to\infty} \left[\left(\sum_{i=1}^{k-1} \Phi(k,i) W^* (\Phi_p(i-1,0) - \Phi_p(i,0)) x_p(0) \right) + W^* x_p(k-1) \right].
\end{aligned} \tag{3.23}$$

where $\Phi_p(k,k_0)$ is the state transition matrix for the primary layer dynamics, given by

$$\Phi_p(k,k_0) = W_p(k-1) W_p(k-2) \dots W_p(k_0), \tag{3.24}$$

$$\Phi_p(i,i) = I. \tag{3.25}$$

Lemma 6. *Under Assumption 2, 1-eigenspace of $W_p(k)$ is time-invariant.*

Proof. For all k, $W_p(k)$ can be partitioned as

$$W_p(k) = \begin{bmatrix} W_{p_1}(k) & 0 & \dots & 0 \\ 0 & W_{p_2}(k) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & W_{p_{l_p}}(k) \end{bmatrix}, \tag{3.26}$$

where $W_{p_i}(k)$ is a square matrix of size $|\mathcal{V}_{p_i}(k)|$ for all $i = 1, \dots, l_p$. Assumption 2 imposes that the above partitioning is time-invariant. This translates as for all k, the size and ordering of each block $W_{p_i}(k)$ is the same. Then, we can construct 1-eigenspace of $W_p(k)$ by finding 1-eigenspaces of each $W_{p_i}(k)$ separately. Notice $\mathbb{1}_{|\mathcal{V}_{p_i}(k)|}$, a vector of size $|\mathcal{V}_{p_i}(k)|$ consisting of all 1s, is an eigenvector of $W_{p_i}(k)$. Then, padding zeros elsewhere and $\mathbb{1}_{|\mathcal{V}_{p_i}(k)|}$ at corresponding rows, we obtain an eigenvector of $W_p(k)$. This method can be iterated to construct 1-eigenspace of $W_p(k)$ from 1-eigenspaces of separate blocks. Since this partitioning is time-invariant, 1-eigenspace of each block is time-invariant, it consists of a vector of all 1s, and the length of the vector is fixed. The multiplicity of the eigenvalue $\lambda = 1$ is l_p, which is also fixed under Assumption 2. Then, 1-eigenspace of $W_p(k)$ is time-invariant. □

Any matrix can be expressed in terms of its eigenvalues and generalized eigenspace, which follows directly from Jordan normal form theorem. Perron-Frobenius theorem for reducible matrices states all eigenvectors of $W_p(k)$ related to $\lambda = 1$ are generalized eigenvectors of rank 1 [25]. Then, let

$$W_p^* = \sum_{i=1}^{l_p} u_i w_i^T, \tag{3.27}$$

$$W_{tr}(k) = \sum_{j=l_p+1}^{n_p} \lambda_j(k) u_j(k) w_j^T(k), \tag{3.28}$$

where u_i and w_i^T are respectively left and right eigenvectors associated with the eigenvalue $\lambda = 1$, and let u_j and w_j^T are respectively left and right generalized eigenvectors associated with the eigenvalue $\lambda_j \neq 1$. Without loss of generality, suppose $w_i^T u_i = w_j^T u_j = 1$ for all i and j. From Lemma 6, W_p^* is time-invariant. Therefore, under Assumption 2, we have

$$\begin{aligned} W_p(k) &= \sum_{i=1}^{n_p} \lambda_i(k) u_i(k) w_i^T(k) \\ &= \sum_{i=1}^{l_p} u_i w_i^T + \sum_{j=l_p+1}^{n_p} \lambda_j(k) u_j(k) w_j^T(k) \\ &= W_p^* + W_{tr}(k). \end{aligned} \tag{3.29}$$

Note that this representation is equivalent to Jordan normal form of $W_p(k)$.

Lemma 7. *Under Assumption 2, $W_p^* W_{tr}(k) = 0$ for all k, where W_p^* and $W_{tr}(k)$ are defined as in* (3.27) *and* (3.28)*.*

Proof. For any matrix A, suppose u is a right eigenvector associated with λ_1, and w^T is a left eigenvector associated with λ_2. We have $Au = \lambda_1 u$ and $w^T A = \lambda_2 w^T$. Then, we can write

$$\lambda_2(w^T u) = (\lambda_2 w^T)u = (w^T A)u = w^T(Au) = w^T \lambda_1 u = \lambda_1(w^T u). \tag{3.30}$$

Therefore, either $\lambda_1 = \lambda_2$ or $w^T u = 0$. Then, for any distinct i and k, eigenvectors corresponding to the distinct eigenvalues λ_i and λ_k satisfy $u_i w_i^T u_k w_k^T = 0$, which implies $W_p^* W_{tr}(k) = 0$. □

Then, we can state the following proposition for primary layer dynamics.

Lemma 8. *Under Assumptions 1 and 2, the system* (2.17) *is convergent.*

Proof. If we express $W_p(k)$ in the form (3.29), using Lemma 7, we obtain

$$\begin{aligned}
\lim_{k\to\infty} \prod_{i=0}^{k} W_p(i) &= \prod_{i=0}^{k} \left(W_p^* + W_{tr}(i)\right) \\
&= \lim_{k\to\infty} W_p^{*k} + \lim_{k\to\infty} \prod_{i=0}^{k} W_{tr}(i) \\
&= \lim_{k\to\infty} W_p^{*k} + \lim_{k\to\infty} \prod_{i=0}^{k} \sum_{j=l_p+1}^{n_p} \lambda_j(i) v_j(i) w_j^T(i) \\
&= \lim_{k\to\infty} W_p^{*k} + \lim_{k\to\infty} \sum_{j=l_p+1}^{n_p} \prod_{i=0}^{k} \lambda_j(i) v_j(i) w_j^T(i). \qquad (3.31)
\end{aligned}$$

Since $\lambda_j(i) < 1$, $\lim_{k\to\infty} \prod_{i=0}^{k} \lambda_j(i) v_j(i) w_j^T(i) = 0$, so,

$$\lim_{k\to\infty} \prod_{i=0}^{k} W_p(i) = \lim_{k\to\infty} W_p^{*k}. \qquad (3.32)$$

Notice all eigenvalues of W_p^* are either 0 or 1. Then, this implies W_p^* is an idempotent matrix, i.e. $W_p^{*2} = W_p^*$. Then,

$$\lim_{k\to\infty} \Phi_p(k,0) = \lim_{k\to\infty} W_p^{*k} = W_p^*, \qquad (3.33)$$

which implies,

$$\lim_{k\to\infty} x_p(k) = \lim_{k\to\infty} \Phi_p(k,0) x_p(0) = W_p^* x_p(0). \qquad (3.34)$$

Therefore, state values of primary layer agents converge to a fixed point in the limit. □

Now consider the state transition matrix for consecutive time steps $i-1$ and i,

$$\Phi_p(i-1,0) = W_p(i-2)\dots W_p(0) = W_p^* + \prod_{j=0}^{i-2} W_{tr}(j), \qquad (3.35)$$

$$\Phi_p(i,0) = W_p(i-1)\dots W_p(0) = W_p^* + \prod_{j=0}^{i-1} W_{tr}(j). \tag{3.36}$$

Therefore, for the norm of the difference of given state transition matrices, we get

$$\Phi_p(i-1,0) - \Phi_p(i,0) = \prod_{j=0}^{i-2} W_{tr}(j) - \prod_{j=0}^{i-1} W_{tr}(j), \tag{3.37}$$

$$||\Phi_p(i-1,0) - \Phi_p(i,0)|| \leq c^{i-1} + c^i, \tag{3.38}$$

for some $c < 1$.

Theorem 1. *Under Assumptions 1, 2, and 3, the system* (2.16) *is convergent.*

Proof. From Lemma 8, x_p^* exists and equal to $x_p^* = W_p^* x_p(0)$. This gives the proof for the subsystem (2.17). Consider the equation

$$W^* \mathbb{1}_{n_p} = (I - W_s(k))^{-1} W_{sp}(k) \mathbb{1}_{n_p}. \tag{3.39}$$

Notice $W_{sp}(k)\mathbb{1}_{n_p} = \mathbb{1}_{n_s} - W_s(k)\mathbb{1}_{n_s} = (I - W_s(k))\mathbb{1}_{n_s}$. Row sums of $W_{sp}(k)$ and $W_s(k)$ add up to 1. Then,

$$W^* \mathbb{1}_{n_p} = (I - W_s(k))^{-1}(I - W_s(k))\mathbb{1}_{n_s} = \mathbb{1}_{n_s}. \tag{3.40}$$

Also, notice $(I - W_s(k))^{-1} = \sum_{i=0}^{\infty} W_s(k)^i$ is nonnegative. W_{sp} is also nonnegative, then, W^* is a nonnegative, row-stochastic matrix, thus $||W^*|| = 1$.

For admissible k and i, we have $||\Phi(k,i)|| \leq b^{k-i}$ for some $b < 1$. From (3.38), $||\Phi_p(i-1,0) - \Phi_p(i,0)|| \leq (c^{i-1} + c^i)$. Then, let $d = \max\{b, c\} < 1$ to write

$$||\Phi(k,i)|| \leq d^{k-i}, \quad ||\Phi_p(i-1,0) - \Phi_p(i,0)|| \leq (d^{i-1} + d^i). \tag{3.41}$$

Now consider the norm of the limit of the solution of the system, given by (3.23). Notice $W^* x_p(k-1)$ is convergent. Then,

$$\begin{aligned}
||\sum_{i=1}^{k-1} \Phi(k,i) W^* (\Phi_p(i-1,0) - \Phi_p(i,0))|| &\leq \sum_{i=1}^{k-1} ||\Phi(k,i) W^* (\Phi_p(i-1,0) - \Phi_p(i,0))|| \\
&\leq \sum_{i=1}^{k-1} ||\Phi(k,i)|| \; ||W^*|| \; ||(\Phi_p(i-1,0) - \Phi_p(i,0))|| \\
&\leq \sum_{i=1}^{k-1} d^{k-i}(d^{i-1} + d^i) = \sum_{i=1}^{k-1} (d^{k-1} + d^k).
\end{aligned} \tag{3.42}$$

In the limit, we have

$$\lim_{k\to\infty}\sum_{i=1}^{k-1}(d^{k-1}+d^{k}) = \lim_{k\to\infty}(k-1)(d^{k-1}+d^{k}) = 0. \tag{3.43}$$

Then, (3.23) becomes

$$\begin{aligned}\lim_{k\to\infty} x_s(k) &= \lim_{k\to\infty}\left[\left(\sum_{i=1}^{k-1}\Phi(k,i)W^*(\Phi_p(i-1,0)-\Phi_p(i,0))\right)x_p(0)+W^*x_p(k-1)\right]\\ &= \lim_{k\to\infty} W^*x_p(k-1)\\ &= W^*x_p^*. \end{aligned}\tag{3.44}$$

Therefore, the system (2.16) converges to a fixed point in the limit. □

Assumption 3 is a rather restrictive assumption that requires a particular condition to be satisfied at each time step. Now, let's consider that the weighting matrix satisfies the following assumption, a relaxed version of Assumption 3.

Assumption 4. $\lim_{k\to\infty}(I-W_s(k))^{-1}W_{sp}(k) = W^*$.

From Assumption 4, we can write

$$\lim_{k\to\infty}||W^*-\widehat{W}(k)|| = 0, \tag{3.45}$$

where $\widehat{W}(k) = (I-W_s(k))^{-1}W_{sp}(k)$. Now, we can express $\widehat{W}(k) = W^*-B(k)$ with B satisfying

$$\lim_{k\to\infty} B(k) = 0_{n_s\times n_p}, \tag{3.46}$$

since any convergence requires an element-wise convergence. Then the limit of the solution of the system (2.18) becomes

$$\begin{aligned}\lim_{k\to\infty} x_s(k) &= \lim_{k\to\infty}\sum_{i=0}^{k-1}\Phi(k,i+1)(I-W_s(i))\widehat{W}(i)x_p(i)\\ &= \lim_{k\to\infty}\left[\sum_{i=0}^{k-1}\Phi(k,i+1)\widehat{W}(i)x_p(i)-\sum_{i=0}^{k-1}\Phi(k,i)\widehat{W}(i)x_p(i)\right]\\ &= \lim_{k\to\infty}\left[\left(\sum_{i=1}^{k-1}\Phi(k,i)(\widehat{W}(i-1)x_p(i-1)-\widehat{W}(i)x_p(i))\right)+\widehat{W}(k-1)x_p(k-1)\right]\end{aligned}$$

$$= \lim_{k\to\infty} \left(\sum_{i=1}^{k-1} \Phi(k,i)(\widehat{W}(i-1)\Phi_p(i-1,0) - \widehat{W}(i)\Phi_p(i,0)) \right) x_p(0)$$
$$+ \lim_{k\to\infty} \widehat{W}(k-1)x_p(k-1). \tag{3.47}$$

Theorem 2. *Under Assumptions 1, 2, and 4 the system* (2.16) *is convergent.*

Proof. Recall that from Lemma 8, x_p^* exists and equal to $x_p^* = W_p^* x_p(0)$. This gives the proof for the subsystem (2.17). For the subsystem (2.18), utilizing $||\widehat{W}(k)|| = 1$, a similar approach can be followed as in the proof of Proposition 1. Before considering the limit, we first write

$$\begin{aligned} &\widehat{W}(i-1)\Phi_p(i-1,0) - \widehat{W}(i)\Phi_p(i,0) \\ =&\widehat{W}(i-1)\Phi_p(i-1,0) - \widehat{W}(i-1)\Phi_p(i,0) + \widehat{W}(i-1)\Phi_p(i,0) - \widehat{W}(i)\Phi_p(i,0) \\ =&\widehat{W}(i-1)(\Phi_p(i-1,0) - \Phi_p(i,0)) + \Phi_p(i,0)(\widehat{W}(i-1) - \widehat{W}(i)). \end{aligned} \tag{3.48}$$

Then, the norm of the above expression is upper bounded by

$$||\widehat{W}(i-1)||\ ||\Phi_p(i-1,0) - \Phi_p(i,0)|| + ||\Phi_p(i,0)||\ ||\widehat{W}(i-1) - \widehat{W}(i)||. \tag{3.49}$$

Recall $||\widehat{W}(k)|| = 1$ and $||\Phi_p(k,0)|| = 1$. Then, the upper bound is

$$||\Phi_p(i-1,0) - \Phi_p(i,0)|| + ||\widehat{W}(i-1) - \widehat{W}(i)||. \tag{3.50}$$

Now, considering the norm of (3.47), for the first term, we have

$$\begin{aligned} &\lim_{k\to\infty} ||\sum_{i=1}^{k-1} \Phi(k,i)(\widehat{W}(i-1)\Phi_p(i-1,0) - \widehat{W}(i)\Phi_p(i,0))|| \\ \le &\lim_{k\to\infty} \sum_{i=1}^{k-1} ||\Phi(k,i)(\widehat{W}(i-1)\Phi_p(i-1,0) - \widehat{W}(i)\Phi_p(i,0))|| \\ \le &\lim_{k\to\infty} \sum_{i=1}^{k-1} ||\Phi(k,i)||\ ||(\widehat{W}(i-1)\Phi_p(i-1,0) - \widehat{W}(i)\Phi_p(i,0))|| \\ \le &\lim_{k\to\infty} \sum_{i=1}^{k-1} (d^{k-i})(||\Phi_p(i-1,0) - \Phi_p(i,0)|| + ||\widehat{W}(i-1) - \widehat{W}(i)||) \\ \le &\lim_{k\to\infty} \sum_{i=1}^{k-1} (d^{k-i})(d^{i-1} + d^i + ||\widehat{W}(i-1) - \widehat{W}(i)||) \\ = &\lim_{k\to\infty} \sum_{i=1}^{k-1} d^{k-1} + d^k + d^{k-i}||\widehat{W}(i-1) - \widehat{W}(i)|| \end{aligned}$$

$$= \lim_{k\to\infty} \sum_{i=1}^{k-1} d^{k-i} ||\widehat{W}(i-1) - \widehat{W}(i)||, \tag{3.51}$$

for some $d < 1$. Since the sequence $\widehat{W}(i)$ is a Cauchy sequence, there exists a positive integer M such that for all $i > M$, $||\widehat{W}(i-1) - \widehat{W}(i)|| < d^i$ is satisfied. Then,

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{i=1}^{k-1} d^{k-i} ||\widehat{W}(i-1) - \widehat{W}(i)|| \\
&= \lim_{k\to\infty} \sum_{i=1}^{M} d^{k-i} ||\widehat{W}(i-1) - \widehat{W}(i)|| + \lim_{k\to\infty} \sum_{i=M+1}^{k-1} d^{k-i} ||\widehat{W}(i-1) - \widehat{W}(i)|| \\
&= \lim_{k\to\infty} \sum_{i=M+1}^{k-1} d^{k-i} ||\widehat{W}(i-1) - \widehat{W}(i)|| \\
&\leq \lim_{k\to\infty} \sum_{i=M+1}^{k-1} d^{k-i} d^i \\
&= \lim_{k\to\infty} \sum_{i=M+1}^{k-1} d^k = 0.
\end{aligned} \tag{3.52}$$

Then, (3.47) under Assumption 4 becomes

$$\lim_{k\to\infty} x_s(k) = \lim_{k\to\infty} \widehat{W}(k-1) x_p(k-1) = W^* x_p^*. \tag{3.53}$$

Therefore, the system (2.16) converges to a fixed point in the limit. □

3.2. Numerical Examples

In this section, we give numerical examples to demonstrate the results of Theorems 1 and 2. We then present an example where the conditions for Theorem 2 are not satisfied, yet consensus is achieved, to show that Assumptions 2 and 4 are sufficient but not necessary for multi-equilibria consensus.

Example 5. *Consider the graph given in Figure 3.1.*

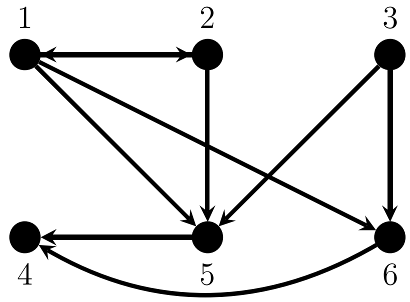

Figure 3.1. A directed graph consisting of 6 vertices and 9 edges.

Suppose the weight matrix and initial state values for the network are given as

$$W(k) = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2+c & c & 0.8-2c \\ \frac{0.8}{3}+a & \frac{0.8}{3} & \frac{0.8}{3}-a & 0 & 0.2 & 0 \\ 0.4+b & 0 & 0.4-b & 0 & 0 & 0.2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0.80 \\ 0.30 \\ 0.20 \\ 0.06 \\ 0.42 \\ 0.70 \end{bmatrix}, \quad (3.54)$$

where a*,* b *and* c *are uniform random variables with* $a \sim U(0, 0.5^{0.1k+2})$*,* $b \sim U(0, 0.3^{0.1k+2})$ *and* $c \sim U(0, 0.4^{0.2k+0.5})$*. Notice that as* k *approaches to infinity,* a*,* b *and* c *approach to 0 therefore,* $\mathcal{G}(W(k))$ *approaches to the graph given in Figure 3.2.*

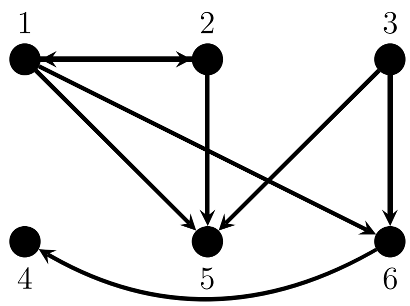

Figure 3.2. The directed graph to which $\mathcal{G}(W(k))$ approaches as k approaches infinity.

Nodes v_1, v_2, and v_3 belong to primary layer subgraphs, and the rest of the nodes belong to the secondary layer subgraphs for all k, so

$$W_p(k) = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.55}$$

$$W_{sp}(k) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{0.8}{3} + a & \frac{0.8}{3} & \frac{0.8}{3} - a \\ \frac{2}{5} + b & 0 & \frac{2}{5} - b \end{bmatrix}, \tag{3.56}$$

$$W_s(k) = \begin{bmatrix} \frac{1}{5} + c & c & \frac{4}{5} - 2c \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}. \tag{3.57}$$

Then, $(I - W_s(k))^{-1} W_{sp}(k)$ becomes

$$(I - W_s(k))^{-1} W_{sp}(k) = \begin{bmatrix} \frac{50bc - 25ac - 20b + 40/3c - 8}{20c - 16} & \frac{4c}{12 - 15c} & \frac{-50bc + 25ac + 20b + 40/3c - 8}{20c - 16} \\ \frac{5}{4}a + \frac{1}{3} & \frac{1}{3} & \frac{1}{3} - \frac{5}{4}a \\ \frac{5}{4}b + \frac{1}{2} & 0 & \frac{1}{2} - \frac{5}{4}b \end{bmatrix}. \tag{3.58}$$

W^* can be found as

$$W^* = \lim_{k \to \infty} (I - W_s(k))^{-1} W_{sp}(k) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}. \tag{3.59}$$

Therefore, Assumption 4 is satisfied. Notice that $W_p(k)$ is time-invariant. Assumption 2 is also satisfied. According to Theorem 2, convergence is guaranteed. Figure 3.3 demonstrates the simulation of the network with weighting matrix $W(k)$. It can be observed that the system converges into 4 clusters: 2 clusters for primary layer subgraphs and 2 clusters for secondary layer subgraphs. Therefore, the results are verified.

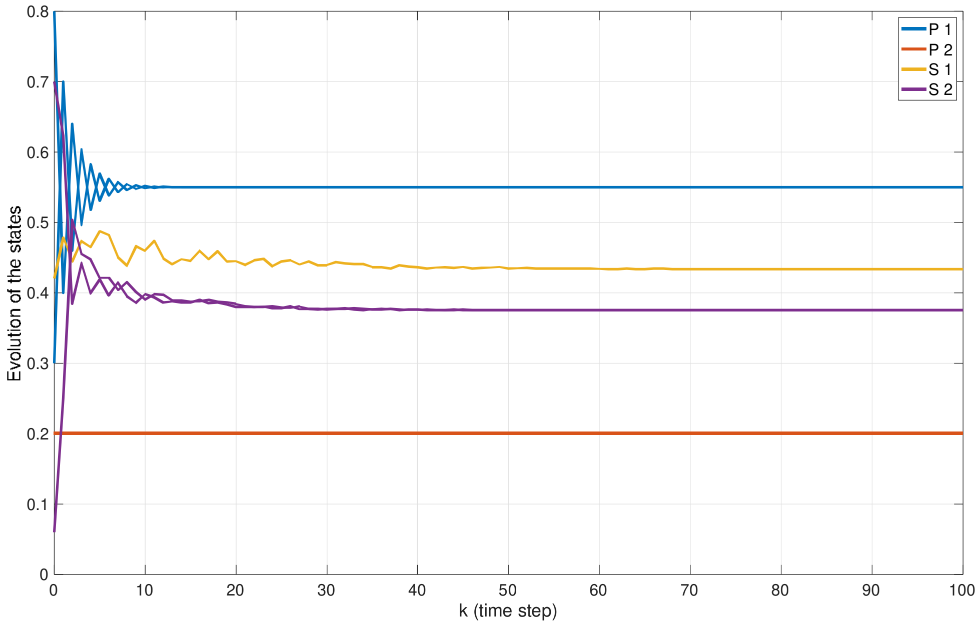

Figure 3.3. Simulation results for the network given in Figure 3.1.

Example 6. *Consider the graph given in Figure 3.4. The graph consists of 3 primary layer subgraphs with* $\mathcal{V}_{p_1} = \{v_1, v_2, v_3, v_4\}$, $\mathcal{V}_{p_2} = \{v_5, v_6, v_7, v_8\}$, $\mathcal{V}_{p_3} = \{v_9, v_{10}, v_{11}\}$, *and 5 secondary layer subgraphs with* $\mathcal{V}_{s_1} = \{v_{12}, v_{13}, v_{14}, v_{15}\}$, $\mathcal{V}_{s_2} = \{v_{16}, v_{17}, v_{18}\}$, $\mathcal{V}_{s_3} = \{v_{19}, v_{20}, v_{21}, v_{22}\}$, $\mathcal{V}_{s_4} = \{v_{23}, v_{24}\}$ *and* $\mathcal{V}_{s_5} = \{v_{25}\}$. *Suppose all averaging coefficients are chosen as* $w_{ij}(k) = 1/|\mathcal{N}_i|$ *where* $|\mathcal{N}_i|$ *is the size of the set of neighbors of agent* i. *Let the initial state value of node* v_i *be* $x_i(0) = \frac{10ln(2i)}{i}$. *Consider the following cases.*

Case A: *Weights corresponding to the agents in groups* $\mathcal{V}_{p_1}, \mathcal{V}_{p_3}, \mathcal{V}_{s_5}$ *change randomly.*
Case B: *Weights corresponding to the agent 12 are changing according to the following rule:*

$$w_{12j}(k) = \begin{cases} 0.2 + 0.1\sin(\pi k/10), & for\ j = 9 \\ \frac{0.8 - 0.1\sin(\pi k/10)}{3}, & for\ j = \{3, 4, 12\} \\ 0, & otherwise. \end{cases} \tag{3.60}$$

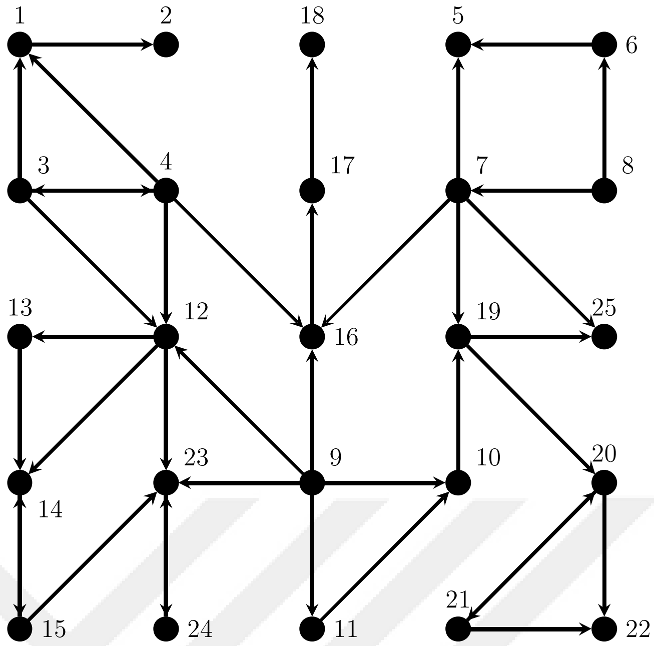

Figure 3.4. A directed graph consisting of 25 vertices and 39 edges.

For Case A, the matrix $(I - W_s(k))^{-1}W_{sp}(k)$ is found to be

$$W^* = \begin{bmatrix} 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 2/9 & 2/9 & 0 & 0 & 0 & 0 & 5/9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & 0 & 0 & 1/4 & 0 \end{bmatrix}. \tag{3.61}$$

Therefore, Assumption 3 is satisfied, i.e. $(I - W_s(k))^{-1}W_{sp}(k)$ is fixed for all k. Notice that the weights within V_{p_1} are changing, but this does not affect the primary layer subgraph partitioning, so Assumption 2 is also satisfied. According to Theorem 1, convergence is guaranteed. Figure 3.5 shows the simulation of the network for the case described in Case A. As observed from Figure 3.5, there are 8 clusters in equilibrium. Therefore, the simulation is consistent with Theorem 1.

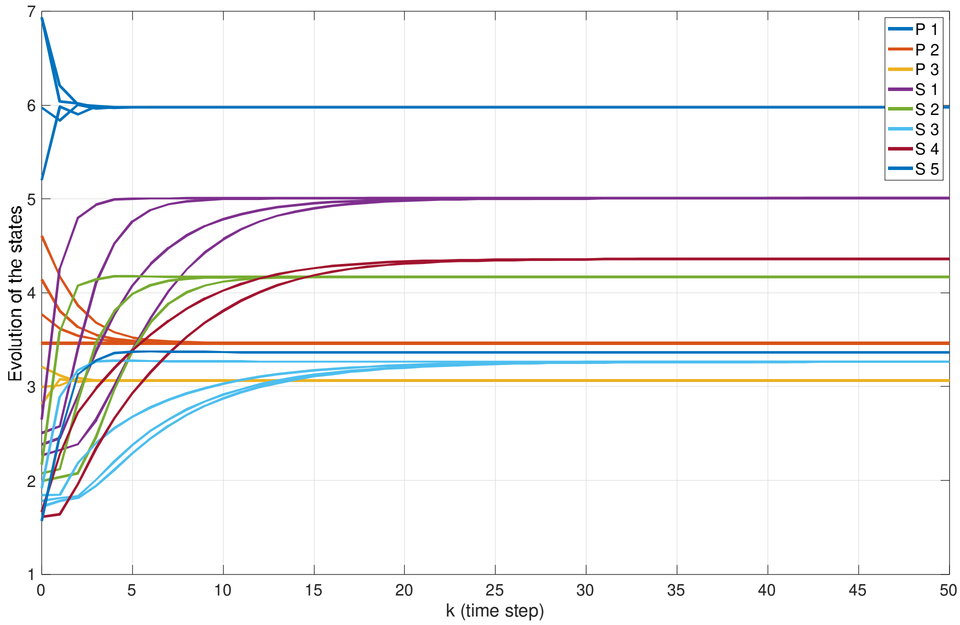

Figure 3.5. Simulation results for the network described in case A.

For Case B, the matrix $(I - W_s(k))^{-1}W_{sp}(k)$ *is found to be*

$$W^* = \begin{bmatrix} 0 & 0 & s_1 & s_1 & 0 & 0 & 0 & 0 & s_2 & 0 & 0 \\ 0 & 0 & s_1 & s_1 & 0 & 0 & 0 & 0 & s_2 & 0 & 0 \\ 0 & 0 & s_1 & s_1 & 0 & 0 & 0 & 0 & s_2 & 0 & 0 \\ 0 & 0 & s_1 & s_1 & 0 & 0 & 0 & 0 & s_2 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & \frac{2s_1}{3} & \frac{2s_1}{3} & 0 & 0 & 0 & 0 & 2s_2 + \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2s_1}{3} & \frac{2s_1}{3} & 0 & 0 & 0 & 0 & 2s_2 + \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & 0 & 0 & 1/4 & 0 \end{bmatrix}, \tag{3.62}$$

where $s = 0.1sin(\pi k/10)$, $s_1 = \dfrac{4/15 - s}{s + 22}$ *and* $s_2 = \dfrac{s + 1/5}{s + 22}$. *Therefore, Assumption 3 is not satisfied, i.e.* $(I - W_s(k))^{-1}W_{sp}(k)$ *is not fixed for all* k. *Figure 3.6 shows the simulation of the network for the case described in Case B.*

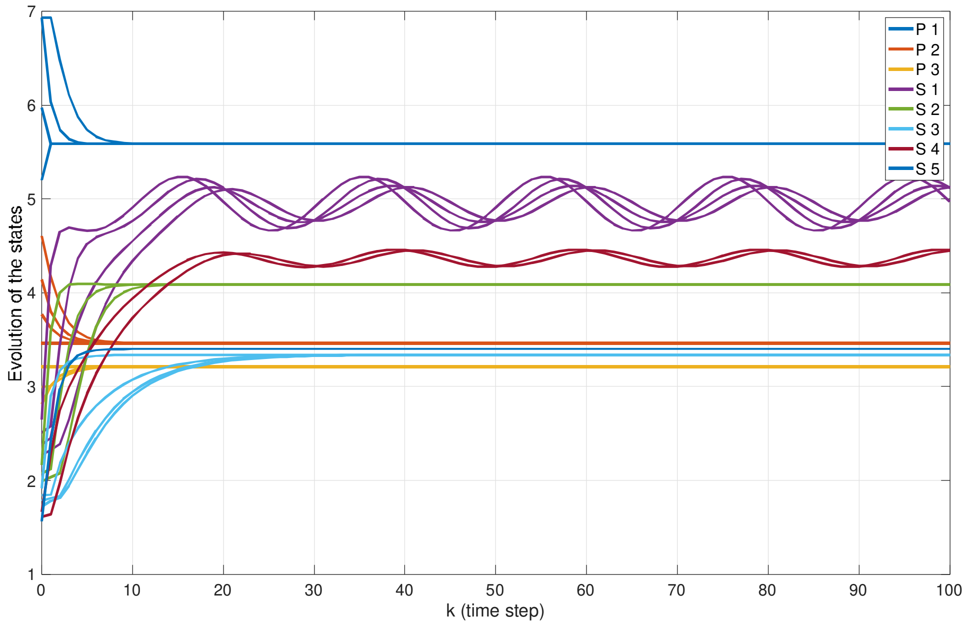

Figure 3.6. Simulation results for the network described in case B.

From Figure 3.5, we can see that primary layer dynamics converge, however secondary layer dynamics does not. Notice $(I - W_s(k))^{-1}W_{sp}(k)$ *is periodic with period* $N = 20$. *Therefore, secondary layer states reside in the convex hull of primary layer states yet exhibit oscillations arising from the structure of* $(I - W_s(k))^{-1}W_{sp}(k)$.

Example 7. *Consider the two graphs given in Figure 3.7. Graphs differ only by a single edge. The graph in Figure 3.7(a) has a directed edge from node* v_2 *to* v_4*, whereas the graph in Figure 3.7(b) has a directed edge from node* v_1 *to* v_4 *instead. Also, note that both graphs have the same partitioning in terms of primary and secondary layer subgraphs. Both consists of 2 primary layer subgraphs with* $\mathcal{V}_{p_1} = \{v_1, v_2\}$, $\mathcal{V}_{p_2} = \{v_3\}$, *and 2 secondary layer subgraphs with* $\mathcal{V}_{s_1} = \{v_4, v_5\}$ *and* $\mathcal{V}_{s_2} = \{v_6\}$.

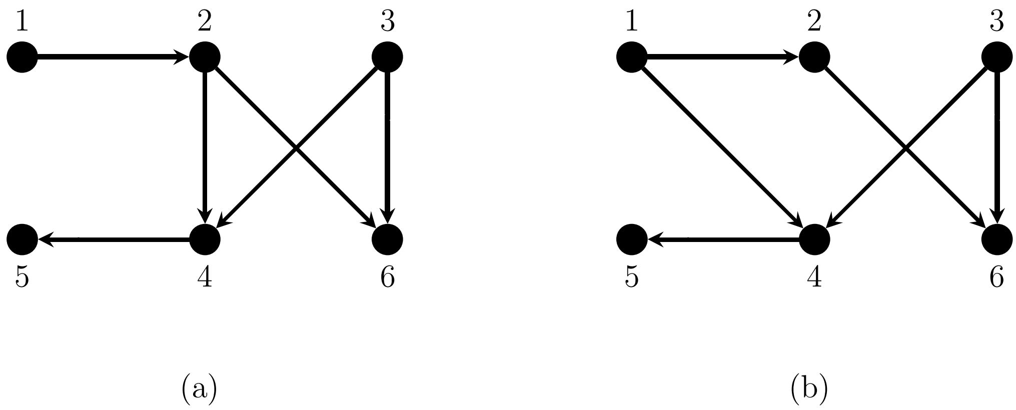

Figure 3.7. Two directed graphs consisting of 6 vertices and 6 edges.

Suppose the networks' weight matrix and initial state values are given to be

$$W_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{3}{10} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5} & 0 \\ 0 & \frac{2}{5} & \frac{1}{5} & 0 & 0 & \frac{2}{5} \end{bmatrix}, W_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{3}{10} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5} & 0 \\ 0 & \frac{2}{5} & \frac{1}{5} & 0 & 0 & \frac{2}{5} \end{bmatrix}, x(0) = \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{10} \\ \frac{3}{10} \\ \frac{3}{5} \\ \frac{1}{2} \end{bmatrix}. \quad (3.63)$$

Suppose the system switches between two networks. For odd time steps, the weight matrix is W_a; for even time steps, the weight matrix is W_b. Then, we have the following

$$(I - W_s(k))^{-1} W_{sp}(k) = \begin{cases} W_1, & \textit{if } k \textit{ is odd} \\ W_2, & \textit{if } k \textit{ is even.} \end{cases} \quad (3.64)$$

where

$$W_1 = \begin{bmatrix} 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad W_2 = \begin{bmatrix} \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}. \quad (3.65)$$

Therefore, the preliminaries for Theorem 2 are not satisfied. Figure 3.8 shows the simulation of the network for the described scenario.

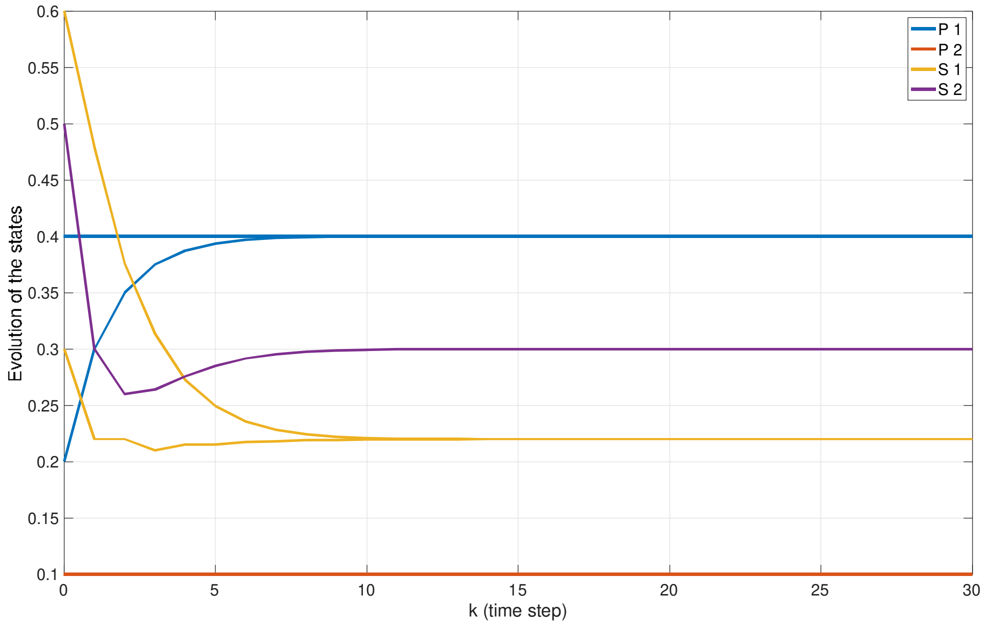

Figure 3.8. Simulation results for the switching system.

Although Theorem 2 is not applicable, consensus is achieved in the network. Notice that W_1 *and* W_2 *are equal except for the first two rows of columns 1 and 2. These entries correspond to how nodes* v_1 *and* v_2 *affect the states of the nodes* v_4 *and* v_5*. Considering* W_1*,* v_4 *and* v_5 *are affected by nodes* v_2 *and* v_3*. Where for* W_2*,* v_4 *and* v_5 *are affected by nodes* v_1 *and* v_3*. Notice that the nodes* v_1 *and* v_2 *belong to the same primary layer subgraph, and primary layer subgraphs converge to a single point, consensus is achieved. As a consequence, as long as the weights are equal, which node from a primary layer subgraph gives information does not affect the steady-state of the secondary layer subgraphs. This example demonstrates that Theorem 2 gives a sufficient condition, but not necessary.*

3.3. Summary of the Chapter

In this chapter, topology-varying networks are analyzed. We started analyzing secondary layer dynamics under three cases. First, we analyzed autonomous dynamics where no input is present. After showing stability and convergence of autonomous dynamics, we focused on a non-autonomous system with fixed input. Then, we investigated non-autonomous dynamics along with primary layer dynamics and presented a sufficient condition for a system to achieve multi-equilibria consensus, and then relaxed the assumption to show a stronger condition. The results of this chapter are supported by numerical simulations at the end of the chapter. In the next chapter, we are going to develop graph reduction tools required to analyze networks in lower dimensions.

4. GRAPH REDUCTION AND DYNAMICS IN REDUCED ORDER

In this chapter, we provide a graph reduction technique to reduce the size of the graph. Graph reduction preserves some structural properties of the original graph. We analyze such properties and relate them to the ones of the original graph. Then, we link the dynamics of two systems, the original system and the reduced-order system, and analyze their convergence behavior. Section 4.1 briefly defines the reduced graph and gives a mathematical framework for graph reduction. Spectral properties of the reduced graph are analyzed, and conditions under which graph reduction preserves the spectral properties of two graphs are presented. Then, in Section 4.2, the convergence and equilibrium behavior of the reduced-order system is analyzed, and the results are related to the convergence and equilibrium behavior of the original high-order system. Finally in Section 4.3, we analyze primary and secondary layer dynamics in the lower dimension, which are obtained by graph reduction. Then, we propose sample topologies that satisfy the assumptions, hence guaranteeing consensus.

4.1. Reduced Graph and Reduced-Order System Representation

We first start with giving the definition of the reduced graph.

Definition 10. *[26](Reduced Graph) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph consisting of l_p primary and l_s secondary layer subgraphs. Let $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$ denote the primary layer subgraphs for $i = 1, \ldots, l_p$ and the secondary layer subgraphs for $i = l_p + 1, \ldots, l_p + l_s$. Let $W = [w_{ij}]$ be a weighting matrix corresponding to $\mathcal{G}$. Then $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ is called the reduced graph of $\mathcal{G}$ with the reduced weighting matrix $\overline{W} = [\overline{w}_{ij}]$ where $\overline{\mathcal{V}} = \{\overline{v}_1, \ldots, \overline{v}_{l_p+l_s}\}$,*

$$(\overline{v}_i, \overline{v}_j) = \begin{cases} \in \overline{\mathcal{E}}, \ \textit{if } \exists a \in \mathcal{V}_i, \ b \in \mathcal{V}_j \ \textit{such that } (a,b) \in \mathcal{E} \\ \notin \mathcal{E}, \ \textit{otherwise.} \end{cases} \tag{4.1}$$

and $\overline{w}_{ij} = \sum_{k \in \mathcal{V}_i} \sum_{l \in \mathcal{V}_j} w_{kl}$ *for* $i \neq j$, *and* $\overline{w}_{ii} = 1 - \sum_{\substack{j=1 \\ j \neq i}}^{l_p + l_s} \overline{w}_{ij}$.

Remark 1. *From Definition 10, the vertices in the primary layer subgraphs do not receive information from the rest of the graph. Therefore, the weighting matrix of the reduced graph is of the form*

$$\overline{W} = \begin{bmatrix} I_{l_p} & 0_{l_p \times l_s} \\ \overline{W}_{sp} & \overline{W}_s \end{bmatrix}. \tag{4.2}$$

Suppose we partition the nodes into K non-overlapping sets, $\{\mathcal{V}_1, \ldots, \mathcal{V}_K\}$, where each partition corresponds to a primary or secondary layer subgraph. Let $P \in \mathrm{R}^{n \times K}$ be the matrix whose rows are partition indicator vectors,

$$P_{i,j} = \begin{cases} 1, & \text{if } v_i \in \mathcal{V}_j \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Here, $K = l_p + l_s$ is the size of the reduced graph, $n = n_p + n_s$ is the size of the original graph, v_j is the jth node of the original graph, and $\mathcal{V}_i$ is the ith subgraph. Then, if we construct such a matrix with graph partitioning according to primary and secondary layer subgraphs, the P matrix inherits some special properties. For a_i, $i = 1, \ldots, K$, let $diag(a_i)$ denote the diagonal matrix $A \in \mathbb{R}^{K \times K}$ with a_i at ith diagonal entry and zero elsewhere.

Lemma 9. *A matrix constructed according to* (4.3), *with primary and secondary layer partitioning, satisfies the below statements.*

(i) $P^T P = diag(|\mathcal{V}_i|)$,

(ii) $\overline{W} = P^T W P - P^T P + I$.

Proof (i). Directly follows from the definition of the P matrix. Since no vertex can belong to two or more partitions of vertex set V, each row of P has a single nonzero element. Let p_i denote the columns of P, i.e. $P = \left[p_1, p_2, \ldots, p_K\right]$. Then, $p_i^T p_j \neq 0$ if and only if $i = j$. For the (i,i)th element of $P^T P$, we have $(P^T P)_{ij} = p_i^T p_i$. Notice that p_i^T has $|\mathcal{V}_i|$ nonzero entries, then, $p_i^T p_i = ||p_i^T||^2 = |\mathcal{V}_i|$. Then,

$$(P^T P)_{ij} = \begin{cases} |\mathcal{V}_i|, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

$$= \text{diag}(|\mathcal{V}_i|).$$

□

Proof (ii). For the (i,j)th element of the product $P^T W$, we have

$$(P^T W)_{ij} = \sum_{k \in \mathcal{V}_i} w_{kj}. \tag{4.5}$$

Then, multiplying this matrix by P gives the weights of information $\overline{v}_i$ of the reduced graph received from $\overline{v}_j$ of the reduced graph. However, self-weights do not match since

$$(P^T W P)_{ij} = \sum_{k \in \mathcal{V}_i} \sum_{j \in \mathcal{V}_j} w_{kl}. \tag{4.6}$$

One may notice that PWP^T is defined the same as Definition 10 for $i \neq j$. It can be followed from the above sum notation that for diagonal entries, PWP^T is the sum of all weights within that layer. This value corresponds to the number of nodes for primary layer subgraphs and the number of nodes minus $\overline{w}_{ii}$ for secondary layer subgraphs. Therefore, subtracting a diagonal matrix with the number of each subgraph containing minus one at its diagonal entries would give us a reduced weighting matrix. Therefore,

$$\begin{aligned} \overline{W} &= PWP^T - diag(|\mathcal{V}_i| - 1) \\ &= P^T W P - diag(|\mathcal{V}_i|) + I \\ &= P^T W P - P^T P + I. \end{aligned}$$

□

4.1.1. Relation to the Laplacian Matrix

Before delving into the analysis of the reduced graph, first consider the continuous-time analog of system (2.3), given as

$$\dot{x}(t) = -L(t)x(t). \tag{4.7}$$

From the Taylor series expansion, using a single derivative term, one can write

$$x(t + \Delta t) = x(t) + \dot{x}(t)\Delta t, \tag{4.8}$$

and by letting time step equal to 1, i.e. $\Delta t = 1$, and replacing time variable t by discrete time variable k, we get

$$\begin{aligned} x(k+1) &= x(k) - L(k)x(k) \\ &= (I - L(k))x(k) \\ &= W(k)x(k), \end{aligned} \tag{4.9}$$

which is (2.3) itself. It is shown in [27] that a reduced-order model for the network with dynamics given by (4.7) can be obtained via the projection $L_r(t) = P^T L(t) P$ where L_r is the reduced Laplacian and P is a matrix constructed according to (4.3). Note that $P^T P$ is a diagonal matrix with sizes of clusters at its diagonal entries. A similar approach is followed in [28] to obtain reduced-order dynamics.

This projection is called a Galerkin projection, where the trajectories of the state variable are supposed to be contained in a lower dimensional subspace [29]. Accordingly, the reduced-order initial condition becomes $\overline{x}(0) = P^+ x_0$. Equivalently, we can write $P\overline{x}(0) = x(0)$, since $P^+ P = I$. With these relations, we state the following result.

Lemma 10. *The reduced weighting matrix $\overline{W}$ of $\mathcal{G}(W)$, given by Definition 10 and the reduced weighting matrix W_r associated with the reduced Laplacian L_r, are equal, i.e. $\overline{W} = W_r$.*

Proof. From Lemma 9, we have $\overline{W} = P^T W P - P^T P + I$. Then, using the relation between the Laplacian matrix and the weight matrix,

$$\overline{W} = I + P^T W P - P^T P$$

$$= I + P^T(W - I)P$$
$$= I - P^T L P$$
$$= I - L_r = W_r. \qquad \square$$

Therefore, we are now equipped with the necessary mathematical tools to analyze the properties of the reduced graph. Note that the reduced graph, $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$, is a graph with $|\overline{\mathcal{V}}| = K$ nodes with $K \leq n$ where there are K partitions $\{\mathcal{V}_1, \ldots, \mathcal{V}_K\}$ of $\mathcal{V}$ such that each partition corresponds to a single node in the coarse graph.

4.1.2. Spectral Analysis of Reduced Graph

We can determine the response of a stable system by analyzing its spectra. Recall the equilibrium points of the system (2.2) correspond to 1-eigenspace of the weighting matrix. Therefore, if a system is time-invariant, checking the eigenvectors with eigenvalues $\lambda = 1$ is enough to talk about the equilibrium states. In the time-variant case, further assumptions can be made. Yet it is clear that 1-eigenspace of the system matrix is linked to equilibrium states. Therefore, analyzing the spectra of the reduced weighting matrix is a primer for studying the steady-state behavior of the reduced system.

Constructing the 1-eigenspace of a nonnegative, reducible, row-stochastic matrix by selecting steady-state distributions for each primary layer subgraph is meaningful in the sense of connecting the spectra and steady-state behavior. Recall from (2.22) that as k approaches infinity, W^k can be expressed as a linear combination of the product of its right and left eigenvectors associated with $\lambda = 1$. Therefore, there is a natural relation between the weighting matrix's spectra and the system's steady-state behavior, which was aforementioned. Now, we move on to analyze the spectra of the reduced weighting matrix.

Unit eigenvalues of W are zero eigenvalues of L. Notice that the eigenvectors associated with zero eigenvalues of L are contained in the columns of P. Letting $Im(P)$ denote the column space or the image of matrix P, we can conclude

$$Lv = 0 \implies v \in Im(P). \tag{4.10}$$

This result is useful since we can construct the reduced weighting matrix from the reduced Laplacian.

Lemma 11. *Suppose* $(1, u)$ *is an eigenpair of* W*. Then,* $(1, P^{+}u)$ *is an eigenpair of* $\overline{W}$*. Similarly, if* $(1, \overline{u})$ *is an eigenpair of* $\overline{W}$*,* $(1, P\overline{u})$ *is an eigenpair of* W*.*

Proof. PP^{+} is an orthogonal projection onto $Im(P)$. Then, $u \in Im(P)$. $PP^{+}v$ is then parallel to u. If $(1, u)$ is an eigenpair of W, $(0, u)$ is an eigenpair of L. Then, $LPP^{+}u = 0$, which implies $P^{T}LPP^{+}u = \overline{L}P^{+}u = 0$. Therefore, $(0, P^{+}u)$ is an eigenpair of $\overline{L}$, and $(1, P^{+}u)$ is an eigenpair of $\overline{W}$.

P is full rank. Therefore, the kernel of P is empty. So, either we have $P\overline{u}$ as an eigenvector of L with eigenvalue 0, or we have $LP\overline{u} \neq 0$ and $P^{T}(LP\overline{u}) = 0$. Suppose $LP\overline{u} \neq 0$ and $P^{T}(LP\overline{u}) = 0$. Let $y = LP\overline{u}$, with nonzero y and $P^{T}y = 0$. Then, y is in the kernel of P^{T}, therefore, orthogonal to the row-space of P^{T}, i.e., orthogonal to the column-space of P. We have

$$y \in Im(LP), \quad y \perp Im(P) \implies y \in \mathcal{K}er(P) \tag{4.11}$$

which is a contradiction. Therefore, $P\overline{u}$ is an eigenvector of L with eigenvalue 0, and $(1, P\overline{u})$ is an eigenpair of W. □

Remark 1 is enough to say that both W and $\overline{W}$ have l_p eigenvalues with $\lambda = 1$. Lemma 11 relate 1-eigenspaces of W and $\overline{W}$. Therefore, the steady-state behavior of the high-order and reduced-order systems can be related.

4.2. Steady-State Behaviour of Reduced-Order System

Consider the dynamics (2.2) and the corresponding reduced dynamics

$$\overline{x}(k+1) = \overline{W}\,\overline{x}(k). \tag{4.12}$$

If initial conditions satisfy $P\overline{x}(0) = x(0)$, then the trajectories of both systems converge to common clusters. Notice that the two systems' state vectors are not the same size. Therefore, two states cannot converge to the same point. However, the motivation for graph reduction was to find a subspace containing the state variable's trajectories. Therefore, the reduced-order model is representative of the original model in a lower dimensional subspace, where both system's states carry common information, like convergence and the number of clusters in steady-state. The following examples are given to demonstrate this statement.

Example 8. *Consider the network given in Figure 4.1. The graph consists of 2 primary layer subgraphs with* $\mathcal{V}_{p_1} = \{v_1, v_2\}$, $\mathcal{V}_{p_2} = \{v_3\}$ *and 2 secondary layer subgraphs with* $\mathcal{V}_{s_1} = \{v_4, v_5\}$, $\mathcal{V}_{s_2} = \{v_6\}$*. Therefore, the reduced graph consists of 4 nodes. Weighting matrix* W*, indicator matrix* P*, and resultant reduced weighting matrix* $\overline{W}$ *are given as*

$$W = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{10} & \frac{7}{10} & 0 \\ \frac{3}{10} & 0 & \frac{1}{2} & 0 & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & 0 & 0 & \frac{2}{5} \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{W} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{2}{5} & 0 & \frac{2}{5} \end{bmatrix}. \tag{4.13}$$

Initial conditions that satisfy $P\overline{x}(0) = x(0)$ *are selected as*

$$\overline{x}(0) = [1, 2, 3, 4]^T, \;\; x(0) = [1, 1, 2, 3, 3, 4]^T. \tag{4.14}$$

Figure 4.2 depicts the evolution of the states of the original and reduced network. For each high-order and reduced-order model, the system converges to $l_p + l_s$*, namely 4 clusters, corresponding to the total number of primary and secondary layer subgraphs.*

Figure 4.3 depicts the evolution of the states of the original and reduced network when initial conditions that do not satisfy $P\overline{x}(0) = x(0)$ *are selected as*

$$\overline{x}(0) = [1, 2, 3, 4]^T, \ \ x(0) = [1, 1.6, 2.2, 2.8, 3.4, 4]^T. \tag{4.15}$$

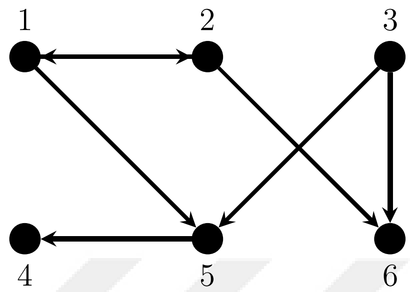

Figure 4.1. A directed graph consisting of 6 vertices and 7 edges.

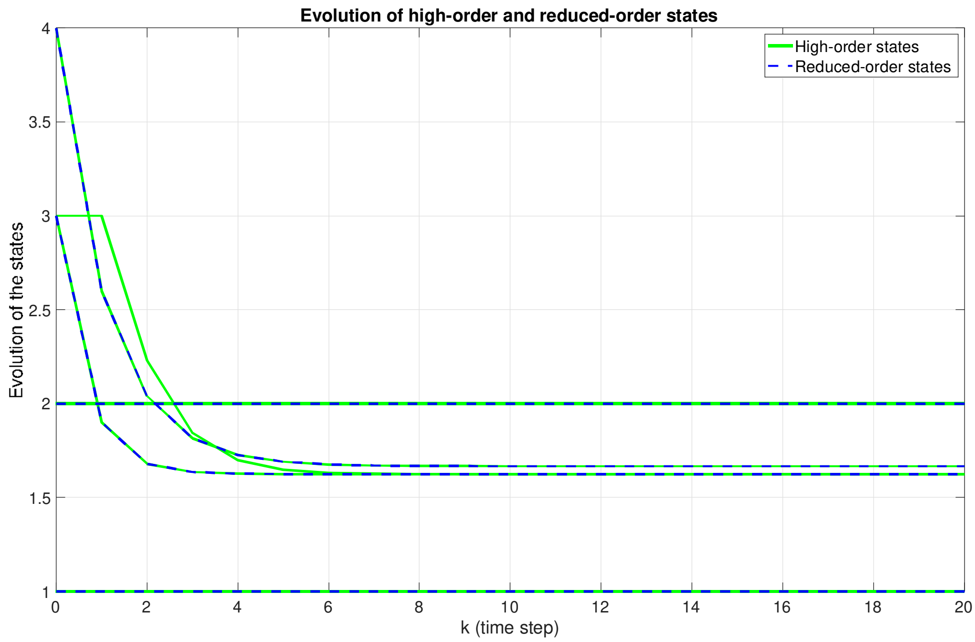

Figure 4.2. Simulation results for network given in Figure 4.1 with initial condition criteria satisfied.

For each high-order and reduced-order model, the system converges to $l_p + l_s$ *clusters, however, clusters formed in steady-state are not identical. The image of* P *consists of its column vectors. These 4 vectors span a subspace of the original state-space. Recall that the motivation behind constructing a Galerkin projection-based reduced-order model was to find a lower-dimensional subspace that contains the trajectories of the* *state variable. Whenever $P\overline{x}(0) = x(0)$ is satisfied, $x(0)$ is contained in that lower dimensional subspace. Otherwise, the initial condition is not contained in that subspace, therefore, state trajectories are not also necessarily contained in that subspace.*

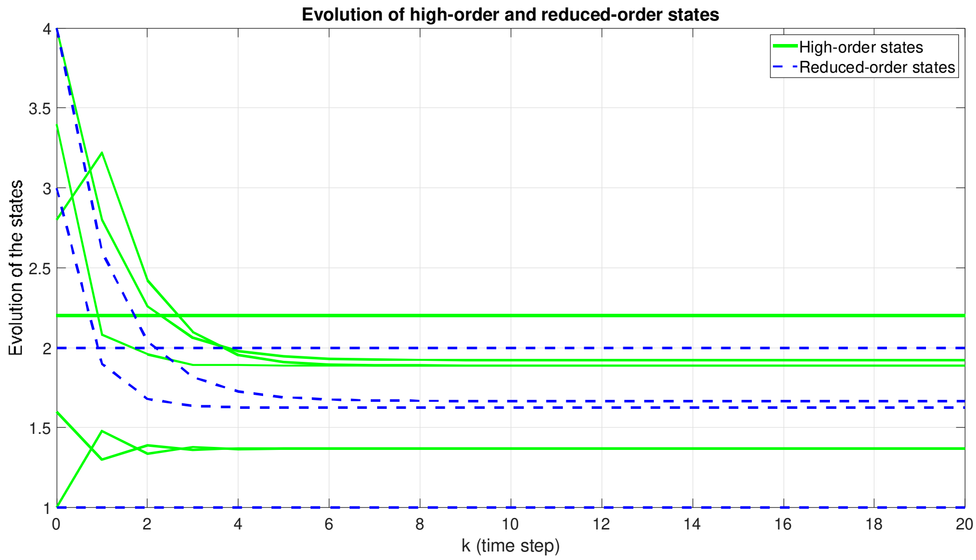

Figure 4.3. Simulation results for network given in Figure 4.1 with initial condition criteria not satisfied.

Example 9. *Consider the network given in Figure 3.4. The graph consists of 3 primary layer subgraphs with $\mathcal{V}_{p1} = \{v_1, v_2, v_3, v_4\}$, $\mathcal{V}_{p2} = \{v_5, v_6, v_7, v_8\}$, $\mathcal{V}_{p3} = \{v_9, v_{10}, v_{11}\}$, and 5 secondary layer subgraphs with $\mathcal{V}_{s_1} = \{v_{12}, v_{13}, v_{14}, v_{15}\}$, $\mathcal{V}_{s_2} = \{v_{16}, v_{17}, v_{18}\}$, $\mathcal{V}_{s_3} = \{v_{19}, v_{20}, v_{21}, v_{22}\}$, $\mathcal{V}_{s_4} = \{v_{23}, v_{24}\}$ and $\mathcal{V}_{s_5} = \{v_{25}\}$. Therefore, the reduced graph consists of 8 nodes. All averaging coefficients are chosen as $w_{ij} = 1/|\mathcal{N}_i|$ where $|\mathcal{N}_i|$ is the size of the set of neighbors of agent i and indicator matrix P is constructed according to primary and secondary layer subgraphs.*

Initial conditions are selected as

$$\overline{x}(0) = [4, 5, 8, 6, 3, 7, 1, 2]^T, \;\; x(0) = P\overline{x}(0). \tag{4.16}$$

Figure 4.4 depicts the evolution of the states of the original and reduced network. For each high-order and reduced-order model, the system converges to $l_p + l_s$, namely 8 clusters, corresponding to the total number of primary and secondary layer subgraphs.

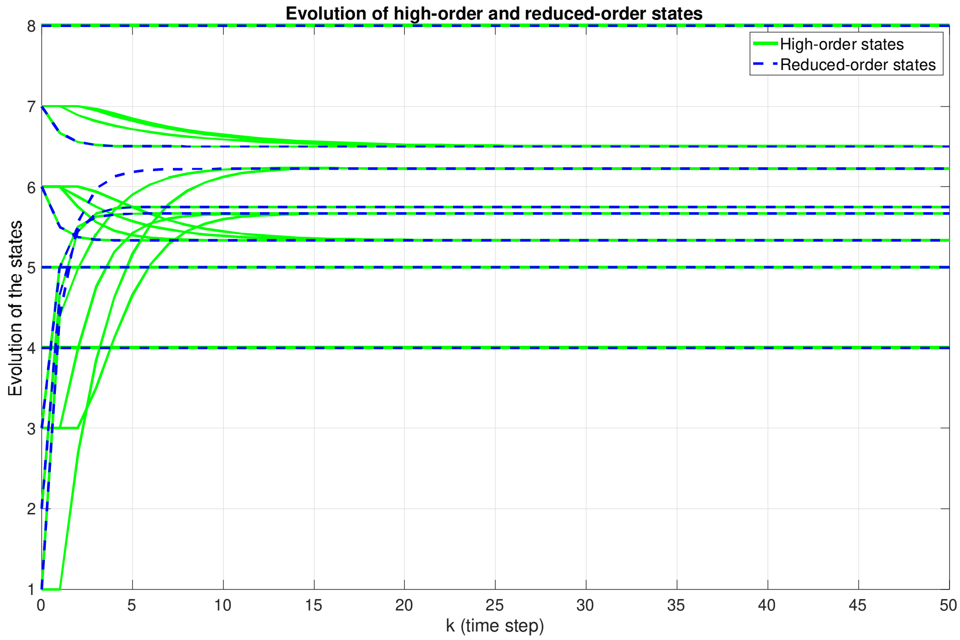

Figure 4.4. Simulation results for network given in Figure 3.4.

The preceding section stated that the equilibrium points of the system (2.2), or (4.12) can be characterized by 1-eigenspaces of their weighting matrices. More explicitly, equilibrium points of such systems can be written as a linear combination of the right eigenvectors associated with $\lambda = 1$. Lemma 11 linked the 1-eigenspaces of W and $\overline{W}$. Then, a relationship between the steady-state equilibrium of the high-order system and the reduced-order system directly follows.

Lemma 12. *Suppose x^* is an equilibrium point of the system* (2.2)*. Then, P^+x^* is an equilibrium point of the reduced-order system* (4.12)*. Similarly, suppose $\overline{x}^*$ is an equilibrium point of the reduced-order system* (4.12)*. Then, $P\overline{x}^*$ is an equilibrium point of the system* (2.2)*.*

The number of clusters the system (4.12) converges in the steady-state is $l_p + l_s$. The reduced-order system is of size $l_p + l_s$. Therefore, the reduced system converges to $l_p + l_s$ equilibria consensus for initial conditions satisfying $P\overline{x}(0) = x(0)$. Lemma 2 states that for arbitrary initial condition, a system converges to $l_p + l_s$ equilibria consensus, which is enough to conclude that the reduced system converges to $l_p + l_s$ equilibria consensus for any arbitrary initial condition since it converges to $l_p + l_s$

equilibria consensus for at least one initial condition. Note that by picking specific initial conditions, it is possible to make the system converge to $\bar{l} < l_p + l_s$ equilibria consensus; however, the number of clusters is upper bounded by $l_p + l_s$. For example, if the initial condition is selected to be identical for all agents, the system would be in equilibrium, meaning there would be a single cluster in the equilibrium. Yet, this is a particular condition that is not likely to be encountered. For the general case, number of clusters is $l_p + l_s$. Therefore, we can state the following lemma.

Lemma 13. *The number of clusters formed in the steady-state of system* (2.2) *is equal to the number of clusters formed in the steady-state of system* (4.12)*. Therefore, both* (2.2) *and its reduced system* (4.12) *converge to same number of clusters.*

4.3. Analysis of Varying Topology Networks in Lower Dimension

Consider primary and secondary layer dynamics of the reduced-order model given by (4.12) separately. From (4.2), we obtain

$$\overline{x}_p(k+1) = \overline{x}_p(k), \tag{4.17}$$

$$\overline{x}_s(k+1) = \overline{W}_s(k)\overline{x}_s(k) + \overline{W}_{sp}(k)\overline{x}_p(k). \tag{4.18}$$

Notice that system (4.17) is in equilibria for any initial condition, namely, all points in $\mathbb{R}^{l_p}$ are stable equilibrium points of (4.17). For system (4.18), analysis conducted in Section 3.1 is applicable. The following assumption is the restatement of Assumption 4 in terms of reduced matrices.

Assumption 5. $\lim_{k\to\infty}(I - \overline{W}_s(k))^{-1}\overline{W}_{sp}(k) = \overline{W}^*$.

Theorem 3. *Under Assumptions 1, 2 and 5, the system* (4.12) *is convergent. The number of clusters formed in the steady-state is $K = l_p + l_s$ for arbitrary initial conditions.*

Proof. For subsystem (4.17), $\overline{x}_p^*$ exists and equal to $\overline{x}_p(0)$. The limit of the state of the subsystem (4.18) is given by

$$\lim_{k\to\infty}\left(\sum_{i=1}^{k-1}\overline{\Phi}(k,i)(\widehat{W}(i-1)\overline{\Phi}_p(i-1,0)-\widehat{W}(i)\overline{\Phi}_p(i,0))\overline{x}_p(0)\right)$$
$$+\lim_{k\to\infty}\widehat{W}(k-1)\overline{x}_p(k-1), \tag{4.19}$$

where $\widehat{W}(k)=(I-\overline{W}_s(k))^{-1}\overline{W}_{sp}(k)$. For the state-transition matrix of primary layer dynamics, we have

$$\overline{\Phi}_p(i,0)=\prod_{k=0}^{i}\overline{W}_p(k)=\prod_{k=0}^{i}I_{l_p\times l_p}=I_{l_p\times l_p}. \tag{4.20}$$

Then the limit becomes

$$\lim_{k\to\infty}\overline{x}_s(k)=\lim_{k\to\infty}\left(\sum_{i=1}^{k-1}\overline{\Phi}(k,i)(\widehat{W}(i-1)-\widehat{W}(i))\overline{x}_p(0)\right)$$
$$+\lim_{k\to\infty}\widehat{W}(k-1)\overline{x}_p(k-1). \tag{4.21}$$

Recall $\widehat{W}(k)$ is a Cauchy sequence. We can find a positive integer M such that for all i, we have $||\widehat{W}(i-1)-\widehat{W}(i)||<d^i$. Therefore, for some $d<1$, the norm of the first term can be expressed as

$$\lim_{k\to\infty}||\sum_{i=1}^{k-1}\overline{\Phi}(k,i)\widehat{W}(i-1)-\widehat{W}(i))||\leq\lim_{k\to\infty}\sum_{i=1}^{k-1}d^{k-i}d^i=\sum_{i=1}^{k-1}d^k=0. \tag{4.22}$$

Then, we can express the limit of the state vector as

$$\lim_{k\to\infty}\overline{x}_s(k)=\lim_{k\to\infty}(\overline{W}^*-B(k))\overline{x}_p(k-1)=\lim_{k\to\infty}\overline{W}^*\overline{x}_p(k-1)=\overline{W}^*\overline{x}_p^*. \tag{4.23}$$

Therefore, the system (4.12) converges to a fixed point in the limit. For an arbitrary initial condition, $\overline{x}_p^*=\overline{x}_p(0)$ stays at l_p distinct clusters. Also, observe $\overline{W}^*$ has l_s rows. If no two rows of $\overline{W}^*$ are identical, l_s clusters form in secondary layer dynamics. However, under certain specific conditions, fewer clusters may arise. When the initial conditions of two or more distinct primary layer subgraphs are the same, fewer clusters may arise from primary layer dynamics. Similarly, if two or more rows of $\overline{W}^*$ are identical, fewer clusters will be observed amongst secondary layer subgraphs. Although these cases do not lead the system to achieve l_p+l_s clusters, they are very particular and out of common for most of the networks. For networks under arbitrary initial conditions, the network converges to $K=l_p+l_s$ consensus equilibria states. □

Although Theorem 3 gives sufficient conditions, Assumption 5 also seems to be necessary. To see this, consider the case when Assumption 5 is not satisfied. Let $\widehat{W}(k) = (I - \overline{W}_s(k))^{-1}\overline{W}_{sp}(k)$ and we can write

$$\widehat{W}(k) = \overline{W}^* + B(k), \tag{4.24}$$

for some fixed, row stochastic matrix $\overline{W}^*$. For arbitrary $\overline{W}^*$, $B(k)$ has zero row sums since both $\overline{W}^*$ and $\widehat{W}(k)$ are row-stochastic. Then, the limit of the state of the subsystem (4.18) is given by

$$\begin{aligned} \lim_{k\to\infty} \overline{x}_s(k) &= \lim_{k\to\infty} (\overline{W}^* - B(k))\overline{x}_p(0) \\ &= \lim_{k\to\infty} \overline{W}^*\overline{x}_p(0) - \lim_{k\to\infty} B(k)\overline{x}_p(0) \\ &= \overline{W}^*\overline{x}_p^* + \lim_{k\to\infty} B(k)\overline{x}_p^*. \end{aligned} \tag{4.25}$$

Now consider

$$\lim_{k\to\infty} (\overline{x}_s(k+1) - \overline{x}_s(k)) = \lim_{k\to\infty} (B(k+1) - B(k))\overline{x}_p^*. \tag{4.26}$$

We assumed $B(k)$ is not convergent. Therefore, the above limit is zero if and only if $\overline{x}_p^*$ is in the nullspace of $\lim_{k\to\infty} B(k+1) - B(k)$. Since $B(k+1) - B(k)$ has zero row sums, $\overline{x}_p^*$ should have zero elements at some entries or consist of all 1s. In the former case, primary layer dynamics giving zero states in the steady-state means we have zero initial conditions for some nodes. In the latter case, either there is a single primary layer subgraph, which implies that there are no secondary layer subgraphs, or all primaries converge to the same point due to coinciding initial conditions, which is also a particular case in which all secondary layer subgraphs also converge to that common value since the convex hull of primaries consists only of that specific vector. Therefore, there is no such matrix $B(k)$ for arbitrary initial conditions, which means there is no consensus.

The network topology given in Example 7 had x_p^* contained in the nullspace of $\lim_{k\to\infty} B(k+1) - B(k)$, which is because two nodes from the same primary layer subgraph were periodically giving the same amount of information to a secondary layer subgraph in turns. However, considering the reduced graph, only one node represents each subgraph, this case is eliminated. Hence, Theorem 3 gives a more general result, which is illustrated in the following example.

Example 10. *Recall Example 7. The reduced graph associated with the graphs given in Figure 3.7 is shown in Figure 4.5.*

For the weighting matrices introduced in Example 7, the reduced weighting matrix for both systems is

$$\overline{W} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.2 & 0.3 & 0.5 & 0 \\ 0.4 & 0.2 & 0 & 0.4 \end{bmatrix} \tag{4.27}$$

With weight matrix $\overline{W}$, we have

$$(I - \overline{W}_s(k))^{-1}\overline{W}_{sp}(k) = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}. \tag{4.28}$$

Therefore, the preliminaries for Theorem 3 are satisfied. For the initial condition $\overline{x}(0) = [0.4, 0.1, 0.6, 0.5]^T$, the simulation of the network is shown in Figure 4.6.

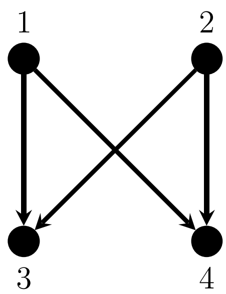

Figure 4.5. Reduced graph consisting of 4 nodes and 4 edges.

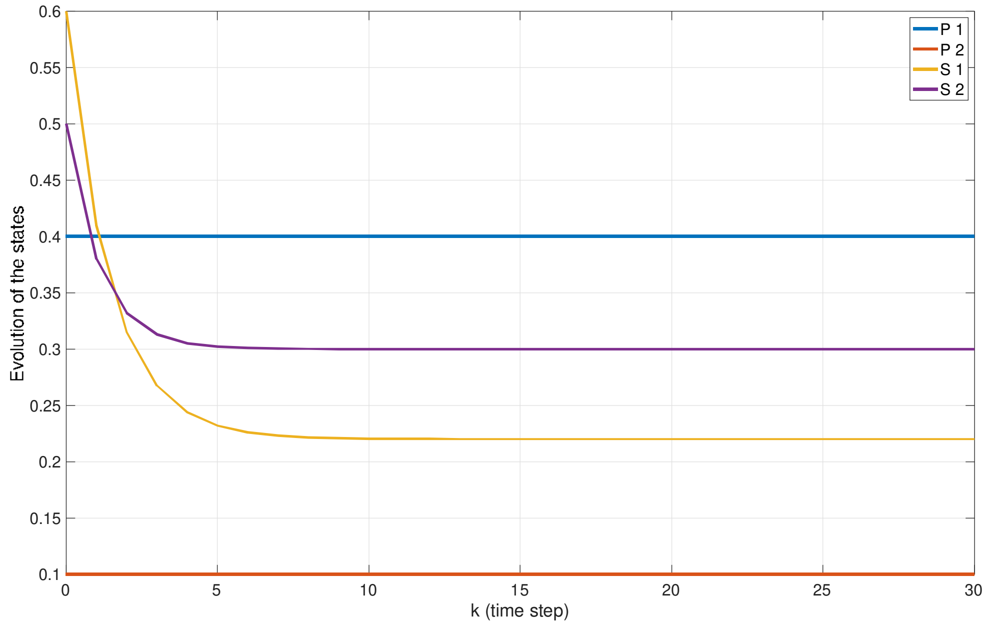

Figure 4.6. Simulation results for the switching system.

As expected, consensus is achieved in the network. For the system described in Example 7, $\widehat{W}(k) = (I - W_s(k))^{-1}W_{sp}(k)$ *was*

$$(I - W_s(k))^{-1}W_{sp}(k) = \begin{cases} W_1, & \textit{if } k \textit{ is odd} \\ W_2, & \textit{if } k \textit{ is even.} \end{cases} \tag{4.29}$$

where

$$W_1 = \begin{bmatrix} 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad W_2 = \begin{bmatrix} \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}. \tag{4.30}$$

Then, from $\widehat{W}(k) = W^* + B(k)$*, admissible choices for* W^* *and* B *can be*

$$W^* = \begin{bmatrix} 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad B(k) = \begin{bmatrix} (-1)^{k+1}\frac{2}{5} & (-1)^{k}\frac{2}{5} & 0 \\ (-1)^{k+1}\frac{2}{5} & (-1)^{k}\frac{2}{5} & 0 \\ (-1)^{k}\frac{2}{3} & (-1)^{k+1}\frac{2}{3} & 0 \end{bmatrix}. \tag{4.31}$$

Recall the steady-state value for nodes in primary layer subgraphs were $x_p^* = [0.4, 0.4, 0.1]^T$*. Then,* $[B(k+1) - B(k)]x_p^*(k)$ *is*

$$\begin{bmatrix} (-1)^{k+1}\frac{4}{5} & (-1)^{k}\frac{4}{5} & 0 \\ (-1)^{k+1}\frac{4}{5} & (-1)^{k}\frac{4}{5} & 0 \\ (-1)^{k}\frac{4}{3} & (-1)^{k+1}\frac{4}{3} & 0 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.4 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.32}$$

However, such a case cannot be encountered when dealing with reduced graphs since all nodes represent a distinct subgraph. This example illustrates the results of Theorem 3, showing that it gives a more inclusive statement compared to Theorem 2.

4.4. Convergence Guaranteed Topology

Consider a network that satisfies Assumption 2, together with the assumption below.

Assumption 6. *The roots of secondary layer subgraphs receive information with fixed weights, i.e., the rows in the weight matrix corresponding to the roots of secondary layers are fixed.*

Assumption 2 implies that $\overline{W}_p$ is time-invariant when such a network is reduced. Recall that in the reduced weighting matrix, $\overline{W}_p$ is the identity matrix. Since primary layer partitioning remains time-invariant, $\overline{W}_p$ is also time-invariant.

Only nodes in secondary layer subgraphs that can receive information from nodes in primary layer subgraphs are the roots of secondary layer subgraphs. Then, Assumption 6 implies that $\overline{W}_{sp}$ is time-invariant. For roots that receive from secondary layer subgraphs, corresponding elements in W_s are also time-invariant. From the definition of the reduced graph, diagonal entries of W_s were picked such that row sums are 1. Therefore, $\overline{W}_s$ is also time-invariant. From the above discussion, if Assumptions 2 and 6 are satisfied, we can state the reduced weight matrix is fixed. $\overline{W}$ is in the following form

$$\overline{W} = \begin{bmatrix} I_{l_p} & 0 \\ \overline{W}_{sp} & \overline{W}_s \end{bmatrix}. \tag{4.33}$$

Since there are l_p eigenvalues at $\lambda = 1$, which are semisimple, there are l_p linearly independent eigenvectors associated with these unit eigenvalues. Suppose each such eigenvector is denoted by

$$\overline{u}_i = \begin{bmatrix} \overline{u}_{pi} \\ \overline{u}_{si} \end{bmatrix}, \quad i \in \{1, \ldots, l_p\}. \tag{4.34}$$

The eigenvalue-eigenvector relation gives us two equations,

$$\overline{u}_p = \overline{u}_p, \tag{4.35}$$

$$\overline{W}_{sp}\overline{u}_p + \overline{W}_s\overline{u}_s = \overline{u}_s. \tag{4.36}$$

Trivially, we can pick $\overline{u}_{pi} = e_i$, where e_i is the standard basis vector with one at ith entry and zero elsewhere. For (4.36), we have $(I - \overline{W}_s)^{-1}\overline{W}_{sp}\overline{u}_p = \overline{u}_s$. Notice that $\overline{u}_p$ and $\overline{u}_s$ are time-invariant. Therefore, 1-eigenspace of the reduced weight matrix is time-invariant, which implies that convergence is guaranteed.

4.5. Summary of the Chapter

In this chapter, graph reduction and the spectral analysis of the resulting reduced graph are analyzed. First, we introduced the reduced graph and defined the reduced graph using the network. Then, we analyzed the resulting low-order model and gave a mathematical projection-based perspective for studying the properties of the reduced graph and linking these properties to the properties of the original graph. With the results in spectral properties of the reduced graph, we analyzed the equilibrium behavior of the reduced network along with the original network to relate the equilibrium of the two systems. We then stated that the number of clusters formed in the equilibrium of the two systems is equal. Using the graph reduction technique, we presented some sufficient conditions for multi-equilibria consensus in terms of the reduced graph. Then, we presented a sample network topology that guarantees the conditions are met and, therefore, guarantees convergence.

5. CONCLUSION

This thesis discussed the multi-equilibria consensus problem for multi-agent systems with first-order dynamics in discrete time under varying network topologies. The thesis contributes to providing sufficient conditions for time-varying networks to reach multi-equilibria consensus.

The first part of the thesis reviews the basics and preliminaries of graph theory, along with a statement of the problem and the model. The second part provides a stability and convergence analysis of networks with fixed topologies. In the third part, stability and convergence analysis of networks with varying topologies is conducted under two assumptions, corresponding to two sufficient conditions for the dynamics to converge to a multi-equilibria consensus state. The reduced graph concept is introduced in the last part, and a similar analysis of stability and convergence of networks with varying topologies via graph reduction is given. Using these results, some sufficient conditions are presented.

The extension of this thesis for future studies can be as follows,

- Showing different network topologies that satisfy sufficient conditions,
- Extending the results to time-varying networks with time-delays,
- Extending the results to hybrid networks,
- Extending the results to higher-order systems,
- Extending the results to nonlinear dynamics.

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