THE STAR-TRIANGLE RELATION AND THE STAR-STAR RELATION AS INTEGRABILITY CONDITIONS

by

Mustafa Mullahasanoğlu

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ABSTRACT

THE STAR-TRIANGLE RELATION AND THE STAR-STAR RELATION AS INTEGRABILITY CONDITIONS

Integrable models are significant examples for our understanding and they are elementary tools to get into physical reality. Their crucial roles are beyond physics and any investigation for the integrability of a model shares many interesting intersections with mathematics. One of the most famous examples of integrable models is the two-dimensional Ising model which is the simplification of ferromagnetic material. The question will be interested in in this thesis is what the star-triangle relation and the star-star relation are as integrability conditions for two-dimensional lattice spin models in statistical mechanics.

The integrability conditions are particular equations that describe essential steps to be satisfied by the physical expressions such as Boltzmann weights in this case. Last decade, many non-trivial solutions to the star-triangle relation (STR) and the star-star relation (SSR) have been obtained and the subject is realized to be that it has many more connections with the fields of mathematical physics and even pure mathematics.

The recently studied solutions are in terms of special functions such as lens elliptic hypergeometric gamma function, hyperbolic hypergeometric gamma function, and so on. In this thesis, there will be a discussion the explicit expressions of the solutions of lens hyperbolic hypergeometric gamma model.

Also we get a journey in the relations of solutions to integrability conditions to other fields in the literature.

ÖZET

İNTEGRALLENEBİLİRLİK KOŞULLARI OLARAK YILDIZ-ÜÇGEN İLİŞKİSİ VE YILDIZ-YILDIZ İLİŞKİSİ

İntegrallenebilir modeller anlayışımız açısından önemli örneklerdir ve fiziksel gerçekliğe ulaşmak için temel araçlardır. Önemli rolleri fiziğin ötesindedir ve bir modelin integre edilebilirliğine yönelik herhangi bir araştırma, matematikle pek çok ilginç kesişimi paylaşır. İntegrallenebilir modellerin en ünlü örneklerinden biri ferromanyetik malzemenin basitleştirilmesi olan iki boyutlu Ising modelidir. Bu tezde ilgilenilecek soru, istatistiksel mekanikte iki boyutlu örgü spin modelleri için integrallenebilirlik koşulları olarak yıldız-üçgen ilişkisi ve yıldız-yıldız ilişkisinin ne olduğudur.

İntegrallenebilirlik koşulları, bu durumda Boltzmann ağırlıkları gibi fiziksel ifadelerle yerine getirilmesi gereken temel adımları tanımlayan özel denklemlerdir. Son on yılda yıldız-üçgen ilişkisi ve yıldız-yıldız ilişkisine yönelik pek çok basit olmayan çözüm elde edilmiş ve konunun matematiksel fizik ve hatta saf matematik alanlarıyla çok daha fazla bağlantısı olduğu anlaşılmıştır.

Yakın zamanda incelenen çözümler, lens eliptik hipergeometrik gama fonksiyonu, hiperbolik hipergeometrik gama fonksiyonu vb. gibi özel fonksiyonlar türündedir. Bu tezde lens hiperbolik hipergeometrik gama modelinin çözümlerinin açık ifadeleri tartışılacaktır.

Ayrıca integrallenebilirlik koşullarının çözümlerinin literatürdeki diğer alanlarla ilişkilerinde bir yolculuğa çıkıyoruz.

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LIST OF SYMBOLS

E_k	Energy of a microstate
F	Free energy
H	Applied magnetic field
k_B	Boltzmann constant
m	Discrete spin variable
M	Magnetization
N	Number of particles in the lattice
$\mathcal{N}$	Number of supersymmetries
p_k	Probability of state k
q_j	Rapidity parameters
$R_{t_{ij}t_{kl}}$	Boltzmann weight for an interaction round a face
$\mathbb{R}$	Real numbers
$S_b^3/\mathbb{Z}_r$	Squashed lens space
T	Temperature
U	Internal energy
$U_q(sl(2))$	q-deformed universal enveloping algebra of $sl(2)$
$W(\sigma_i, \sigma_j)$	Boltzmann weight for an edge interaction
x	Continuous spin variable
Z	Partition function
$\mathbb{Z}$	Integers
α	Spectral parameters
β	Inverse temperature
η	Crossing parameter
σ	Spin multiplet contains both continuous and discrete spins

LIST OF ACRONYMS/ABBREVIATIONS

IRF	Interaction round a face
SSR	Star-star relation
STR	Star-triangle relation

1. INTRODUCTION

The Ising model [1] recently passed 100 years old. It was only a model for the simplification and interpretation of ferromagnetic materials. After every decade spent on it, it is understood that it is not just original for expanding our perception of different physical phenomena but also sharing connections with interesting branches of condensed matter, high energy physics, mathematical physics, and even pure mathematics.

Even nowadays we witness that the model gets richer and promises much more insights. Such an enormous field of research gained its popularity with its drawbacks and the most fundamental problem is that there does not appear a solution at first sight and its solution is not trivial. Even its simplest version in some dimensions is very difficult, it is getting very hard to overcome those inconveniences when a model is attached with extra properties such as an external magnetic field in a two-dimensional Ising model. In particular, the three-dimensional Ising model does not respond too much to understanding inquiry and attempts to solve it.

The problem is to solve the model to understand its physical observables. The revolutionary idea comes with the integrability property of a dynamical system that suggests focusing on the conserved quantities of the model. There are of course different definitions of integrability but, in this thesis, we will investigate a search to find solutions to the STR and the SSR in which they propose infinitely many conserved quantities together with spectral parameters. the STR and the SSR work as local conditions and lead to global invariances which is the commutation relation of the transfer matrices in this case.

After Onsager's famous solution [2, 3], there are developed different approaches and perspectives to construct the exact solution of the two-dimensional Ising model. In his seminal work, Onsager exactly calculates the partition function and obtains the critical point for the phase transition by the use of the star-triangle relation [2] which is the simplest and most distinguished version of the Yang-Baxter equation. It is then studied that the star-triangle relation is common and essential for the integrability and exact solvability of the Fateev-Zamolodchikov model [4], Kashiwara-Miwa model [5], critical Potts model [6, 7], etc.

Another interesting integrable model obtained by solving the star-triangle relation using hyperbolic hypergeometric gamma functions was investigated by Faddeev and Volkov [8, 9]. It is then called the Faddeev-Volkov model. It is shown that the free energy of the Faddeev-Volkov model can be calculated exactly and it is also presented in the thermodynamic limit in [10].

When the elliptic hypergeometric gamma function solution of the star-triangle relation by Bazhanov and Sergeev [11], see also [12], it is called the master solution since it is shown that all discussed models can be obtained in the limit of this novel solution. This master solution respects the positivity of Boltzmann weights which describe interaction between the next-neighbour continuous spins with two temperature-like parameters. In the root of unity or at the fixed value of one of the parameters to zero temperature, it is shown that all discussed models can be obtained from this master solution of the star-triangle relation.

Spiridonov [13], see also [14], realized that the star-triangle relation of the Bazhanov-Sergeev model is the duality of four-dimensional $\mathcal{N} = 1$ gauge theories. Then finally via supersymmetric gauge theories, Yamazaki [15] introduced the most general, lens elliptic hypergeometric gamma, solution for the star-star relation. This new solution can be reduced to the Bazhanov-Sergeev model by taking a special case of the discrete parameter since this most general model contains both continuous and discrete spins.

The star-star relation [16] takes place in the developments discussed above since it is the fundamental statement that one can find the star-star relation if there is a star-triangle relation for a model. However, Baxter shows that there are models in which the Boltzmann weight satisfies the STR but does not satisfy the star-triangle relation. In [15, 17] the multi-spin solutions for the STR using elliptic hypergeometric gamma functions are also studied.

A detailed and exhaustive review of the subject can be seen in [18, 19] and also the thesis [20], especially for the recent developments in the field, and diverse studies and different examples can be found references therein. The investigation of the star-triangle relation in the transfer matrix method and the solution of the two-dimensional Ising model are given in detail in [21].

From different perspectives, we touch on some of them, solutions to the star-triangle relation (Yang-Baxter equation) [13, 14], [23–30] or star-star relation [31–34] for spin lattice models have remarkable intersections with various fields of research such as exact results in supersymmetric gauge theories [35–48] and their mathematical structures interacting with diverse concepts in mathematics, see, e.g., [49–52], knot theory [53], pentagon identities [53–61], Bailey pairs [25], [62–66], quantum algebras [30], [67–69], etc.

2. BASIC NOTIONS OF STATISTICAL PHYSICS

One of the central problems of the statistical mechanics is a dynamics of many-particle system. In both aspects of classical mechanics and quantum mechanics, the equilibrium states of matter with many distinguished particles play crucial roles in understanding the dynamics. Statistical mechanics establish much more fundamental theories and concepts to study macroscopic behaviors of matter consisting of microscopic elements. To build efficient techniques we develop models that simplify physical realities into systems designed with many symmetrical constraints. One can say that statistical physics brings simplified models to describe physical complexities in an approximate way.

Fundamental concepts of statistical physics can be found in various books and an extended version of this chapter can be found [21, 22] in detail and Wikipedia with related content is much more rich.

2.1. Canonical Ensemble and Partition Function

In this thesis, we will construct the model, Ising model, to investigate magnetic materials. The macroscopic property of a magnetic material is the main search and, to do so, the exact solutions and detailed expressions of the Ising model are required. The Ising model is built in any dimension with individual particles possessing spin properties.

We consider the Ising model as a whole system and it is not separated into subsystems in our consideration. It only interchanges energy with an environment that can be said to be a heat bath and the system always contains the same number of particles. This picture is called a canonical ensemble and the most prominent property of a canonical ensemble is that the physical observables and expressions should be taken into account for the whole structure of the model at once. Therefore, the system can be studied as the microstate, a particular state of the macroscopic model. When the model is provided with all possible states, one can write the probability of the microstate in a certain energy E_k of the model

$$p_k = \frac{1}{Z}e^{-\beta E_k}, \tag{2.1}$$

where $\beta = 1/k_B T$ is called inverse temperature with k_B Boltzmann constant and T absolute temperature, Z and exponential $W = e^{-\beta E_k}$ stand for the normalization factor and Boltzmann weight, respectively. However, the unity of the sum over all probabilities gives rise to

$$Z = \sum_k e^{-\beta E_k}, \tag{2.2}$$

which is called the partition function. For a given temperature, the partition function counts all microstates of the model. The partition function is a useful tool for describing thermodynamic variables such as entropy. As an example, one can reach the internal energy of the system by calculating the expected value of microscopic energies

$$U = \sum_k E_k p_k = \sum_k E_k \left(\frac{1}{Z}e^{-\beta E_k}\right) = \frac{1}{Z}\left(-\frac{\partial}{\partial \beta}\sum_k e^{-\beta E_k}\right) = -\frac{\partial \ln Z}{\partial \beta}. \tag{2.3}$$

Another notion for canonical ensembles is Helmholtz-free energy which is the thermodynamic potential of closed system. The Legendre transformation relates to internal energy and free energy. Free energy is a state function and is written as

$$F = -\frac{1}{\beta}\ln Z, \tag{2.4}$$

where it reaches its minimum value at equilibrium and its change during the thermodynamic process is the maximum value of the work that can be achieved by the dynamical system under constant T.

2.2. Phase Transitions and Critical Points

The change in the state of the physical system can be sharp and can be distinguished by its physical properties. Such instantaneous changes are called phase transitions. The daily life example is boiling water. The density of water suddenly decreases and a new form of water becomes vapor. It is known that this change is caused due to the change in thermodynamic variable temperature. In normal conditions, this behavior occurs at 100°C, and such a special singularity is called a critical point. So if a keen difference happens for a property of a dynamical system at a certain case of a thermodynamic variable, it means that phase transition happens at a critical point. For state functions, the situation becomes the non-analyticity of a function for a specific value of a thermodynamic quantity.

The phase transition interestingly occurs for the magnetization M property of magnetized ferromagnetic materials. The Ising model is therefore a simplified physical system for the investigation of such strange behavior of magnetic materials. However, the Ising model is adjusted to the canonical ensemble and so is in contact with a heat bath at a fixed temperature to preserve thermal equilibrium. The thermodynamic quantity is the magnetic field $\vec{H}$ enforced externally. The magnetization of the ferromagnetic material at zero magnetic fields is not zero $+M_0$ and when the direction of the applied magnetic field is reversed like $-\vec{H}$ the magnetization will change suddenly to its negative $-M_0$. Therefore this kind of discontinuous change in the magnetization implies the phase transition.

On the other hand, the critical point is related to the temperature. The temperature is said to be fixed but the flip of the spontaneous magnetization occurs at some interval of the temperature from 0 to T_c (the Curie point). That is, the critical temperature stands for the upper bound of the spontaneous magnetization, and the spontaneous magnetization loses its discontinuity. That is, the state function of the magnetization $M(H,T)$ is continuous out of the interval restricted with critical points. One can also write magnetization as a partial derivative of the free energy with respect to H

$$M(H,T) = -\frac{\partial F}{\partial H}, \tag{2.5}$$

where the order of phase transition comes out and when the magnetization has discontinuity it is called a first-order phase transition. Since the discontinuity at an order of derivative, where the previous derivations are continuous of the free energy determines the order of phase transition.

3. THE TWO-DIMENSIONAL SQUARE LATTICE SPIN MODEL

3.1. Definition of the Model

The two-dimensional square lattice model will be introduced and constructed. It can be called an Ising-like model since the spins sitting at vertices interact only with the nearest neighbor through edges. The spins standing as elementary particles for the ferromagnetic property of the model will be considered in two types. That is continuous spin variables x and discrete spin variables m take values from $\mathbb{R}$ or its subsets and from $\mathbb{Z}$ or its subsets, respectively. However, each spin site contains both kinds of spins at a vertex, and the multiplet $\sigma_i = (x_i, m_i)$ taking values from $\mathbb{R} \times \mathbb{Z}$ or its subsets is used. It is necessary to consider discrete and continuous spin values in the same set since spin variables live all together in Boltzmann weights.

The geometry of a two-dimensional edge interacting lattice spin model takes the form of a two-dimensional torus since periodic boundary conditions are imposed on a square lattice like the Ising model [21].

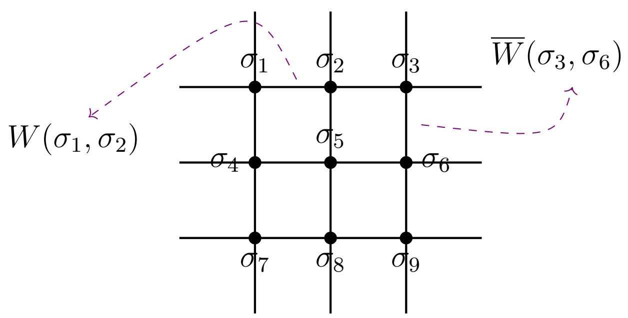

Figure 3.1. The two-dimensional square lattice spin model.

A pictorial description of the lattice spin model is given in Figure 3.1. and spin multiplets consisting of continuous and discrete spin variables are located at vertices. Each distinguishable spin interacts anisotropically through edges and the model has two kinds of Boltzmann weights. Boltzmann weights of horizontal and vertical interactions are denoted as $W(\sigma_i, \sigma_j)$ and $\overline{W}(\sigma_i, \sigma_j)$, respectively.

When the summation and integration are taken over all spin states, the corresponding partition function becomes

$$Z = \sum \int \prod_{<i,j>} W(\sigma_i, \sigma_j) \prod_{<k,l>} \overline{W}(\sigma_k, \sigma_l) \prod_{n=1}^{N} S(\sigma_n), \tag{3.1}$$

where $< i, j >$ restricts the multiplication of Boltzmann weights for only nearest neighbor spin sites and $S(\sigma_n)$ is a self-interaction for all spins in which the total number of spins is N.

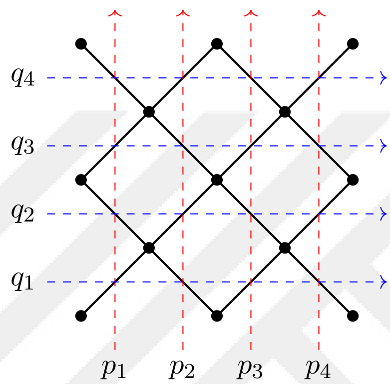

Figure 3.2. Rapidity lines on the rotated square lattice.

If one also wishes to determine every interaction and make Boltzmann weights distinguishable, the way to accomplish it is by introducing rapidity lines as depicted in Figure 3.2. The transfer matrix method is applied by rotating 45° the square lattice and identifying transfer matrices by horizontal rapidity lines. The rapidity parameters q_i and p_j are real-valued and label every Boltzmann weight of an edge interaction.

Then, it can be realized that the horizontal and vertical interactions are determined through the rapidity lines. The difference of the edge interaction turns into the intersection types of the rapidity lines. In this case horizontal and vertical edges become the rapidity lines incoming from the same side and opposite sides, respectively, as shown in Figure 3.3. That is, if the rapidity lines come to the edge from the same face of the square lattice it is the horizontal edge interaction and it will be indexed $W_{p_1 q_1}(\sigma_i, \sigma_j)$. However, if rapidity lines come from different faces of the lattice, cross at the same edges, and go to opposite faces, the Boltzmann weight at this edge is vertical and denoted as $\overline{W}_{p_1 q_2}(\sigma_i, \sigma_j)$.

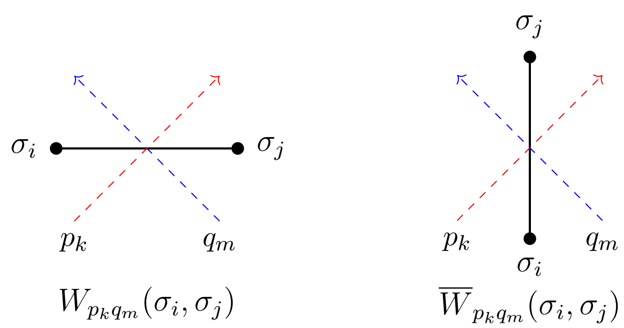

Figure 3.3. Anisotropic Boltzmann weights.

Both horizontal and vertical Boltzmann weights $W_{p_k q_m}(\sigma_i, \sigma_j)$ and $\overline{W}_{p_k q_m}(\sigma_i, \sigma_j)$ are the functions of the rapidity parameters as associated but their dependence is restricted to the difference of the rapidity parameters $q_m - p_k$. This constraint reduces parameters and allows us to introduce the spectral parameter α. That is, Boltzmann weights in terms of spectral parameters take the following form $W_\alpha(\sigma_i, \sigma_j)$ and $\overline{W}_\beta(\sigma_i, \sigma_j)$.

Boltzmann weights of most of the known two-dimensional integrable lattice spin models depend on the difference of rapidity parameters but there are also exceptions for this difference property such as the Chiral Potts model [7]. However, the most general lens elliptic model [70] has the difference property of rapidity parameters but it is studied that a particular limit to the "master" model gives the Chiral Potts model.

However, an additional constraint will be considered and this restriction helps us to identify horizontal and vertical Bolztamnn weights to each other. That is, the vertical Boltzmann weight can be written in terms of the horizontal Boltzmann weight and vice versa

$$\overline{W}_\alpha = W_{\eta-\alpha}, \tag{3.2}$$

where positive valued η is a particular value for each lattice spin model and it is called the crossing parameter.

3.2. Transfer Matrix

Constructing transfer matrix [21] is the method to solve the lattice spin models and the significant step is the derivation of their commutation relations. It is pretty easy to solve the one-dimensional Ising model by the transfer matrix method but there appears necessity of the local conditions on Boltzmann weights. Both the STR and the SSR are local conditions for the commutation relations of transfer matrices.

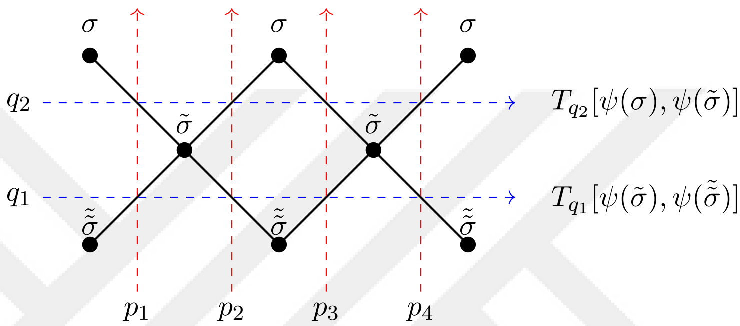

Figure 3.4. The transfer matrices.

Let us assume that a function can be written for all possible states of every row of rapidity lines as depicted in Figure 3.4. So each transfer matrix can be labeled with a rapidity parameter and defined as a multiplication of all Boltzmann weights that have the same rapidity parameter.

$$T_{q_i}[\psi(\sigma), \psi(\tilde{\sigma})] = \prod_{j=1}^{L} W_{p_j q_i}(\sigma_j, \tilde{\sigma}_j)\overline{W}_{p_j q_i}(\sigma_{j+1}, \tilde{\sigma}_j), \tag{3.3}$$

where L stands for the total number of spins at the q_i row and those spins at the row are presented with ψ. Then the partition function turns into the multiplication of all transfer matrices and can be written as

$$Z = \sum \int \prod_{<ij>} T_{q_i}[\psi(\sigma), \psi(\tilde{\sigma})] T_{q_j}[\psi(\tilde{\sigma}), \psi(\tilde{\tilde{\sigma}})]. \tag{3.4}$$

When the summation and the integration over spins are computed the partition function becomes in the trace form. So the problem reduces to the commutation property of the transfer matrices

$$T_{q_1}[\psi(\tilde{\sigma}), \psi(\tilde{\tilde{\sigma}})] T_{q_2}[\psi(\sigma), \psi(\tilde{\sigma})] = T_{q_2}[\psi(\sigma), \psi(\tilde{\sigma})] T_{q_1}[\psi(\tilde{\sigma}), \psi(\tilde{\tilde{\sigma}})]. \tag{3.5}$$

where the commutation relation can be seen as the flip of rows with different rapidity lines. The local conditions which are the STR and the SSR are the tools to flip rows step by step from the case as pictured in Figure 3.4. to the case as pictured in Figure

3.5. or from the left-hand side of the commutation Equation (3.5) into the right-hand side.

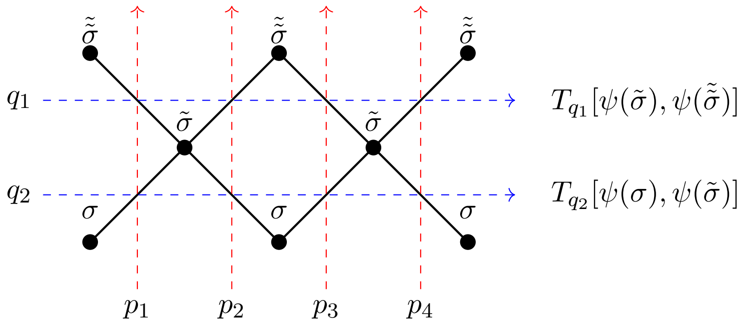

Figure 3.5. Commuted transfer matrices.

Finally, if Boltzmann weights satisfy the STR or the STR, it is sufficient for the commutation relation. Therefore these local sufficient conditions are called integrability conditions of the lattice spin models in statistical mechanics.

3.3. The Inversion Relation

The integrable models discussed in this thesis have also an inversion relation [71, 72] property which is depicted in Figure 3.6. and Figure 3.7.

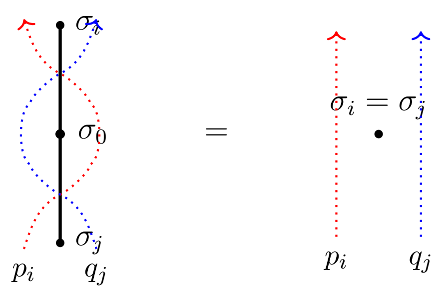

Figure 3.6. The first inversion relation.

The STR implies the first inversion relation as pictured in Figure 3.6.

$$W_\alpha(\sigma_i, \sigma_j)W_{-\alpha}(\sigma_i, \sigma_j) = 1, \tag{3.6}$$

and the second inversion relation as pictured in Figure 3.7.

$$\begin{aligned} \sum_{m_0} \int dx_0 S(\sigma_0) W_{\eta-\alpha}(\sigma_i, \sigma_0) W_{\eta+\alpha}(\sigma_0, \sigma_j) \\ = \frac{1}{S(\sigma_i)}(\delta(\sigma_i + \sigma_j) + \delta(\sigma_i - \sigma_j)). \end{aligned} \tag{3.7}$$

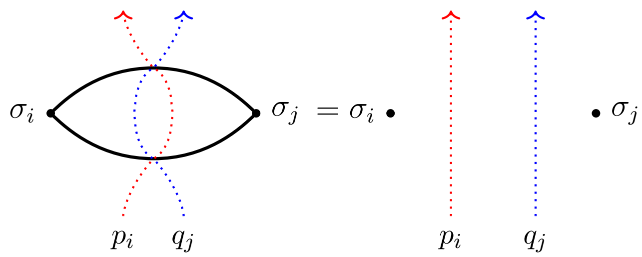

Figure 3.7. The second inversion relation.

One can use the inversion relation and the STR to approach the solution of the integrable model in statistical mechanics. In the thermodynamic limit when the number of particles goes to infinity $N \to \infty$, it can be shown that

$$\lim_{N\to\infty} N^{-1} \log Z = 0. \tag{3.8}$$

where the free energy of bulk for a model vanishes and see [71–73] for more details.

The inversion relation [52,74,75] can be obtained by reducing the STR and this reduction corresponds to the chiral symmetry breaking in the level of partition functions of supersymmetric gauge theories. This reduction is performed for four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories and the three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories.

4. INTEGRABILITY PROPERTIES AND BEYOND

4.1. The Star-Triangle Relation

The problem of the exact solution of the two-dimensional lattice spin model has reduced the commutation relation of the transfer matrices and then the pictorial process shows that it corresponds to the flip of the rows. The pictorial proof of the flip of transfer matrices [21] is given with the help of the star-triangle relation. At this stage, the evolution of the partition function reduces to the solution to the STR in terms of certain Boltzmann weights $W(\sigma_i, \sigma_j)$ and $\overline{W}(\sigma_i, \sigma_j)$.

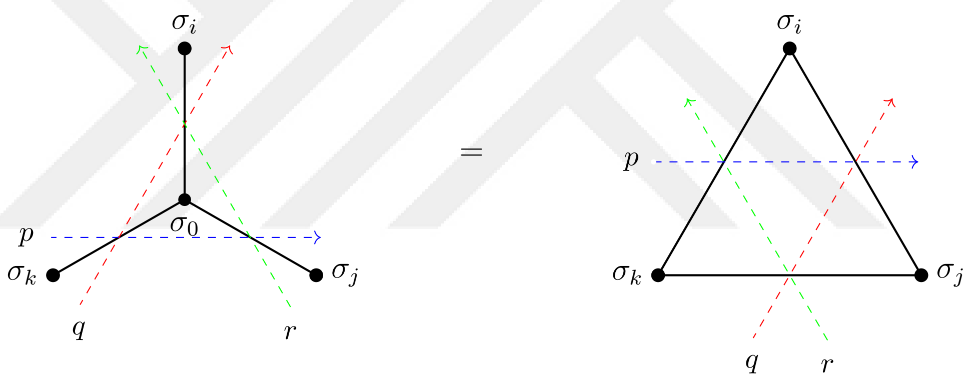

Figure 4.1. The star-triangle relation.

We note that there is a necessity of two types of the STR if $W(\sigma_i, \sigma_j) \neq W(\sigma_j, \sigma_i)$. The other version of the STR is written in terms of reflected Boltzmann weights. However, the models we describe have the reflection symmetry that is, the order of spins in the Boltzmann weight is not important. Under additional constraint with crossing parameter, the mathematical expression of the STR as shown in Figure 4.1. can be written as

$$\begin{aligned} &\sum_{m_0} \int dx_0 S(\sigma_0) W_\alpha(\sigma_1, \sigma_0) W_\beta(\sigma_2, \sigma_0) W_\gamma(\sigma_3, \sigma_0) \\ &\qquad = \mathcal{R}(\alpha, \beta, \gamma) W_{\eta-\alpha}(\sigma_1, \sigma_2) W_{\eta-\beta}(\sigma_1, \sigma_3) W_{\eta-\gamma}(\sigma_2, \sigma_3), \end{aligned} \tag{4.1}$$

where $\mathcal{R}(\alpha, \beta, \gamma)$ is a spin-independent function [76] that can be distributed to Boltzmann weights to normalize, and the constant crossing parameter in terms of spectral parameters is $\eta = \alpha + \beta + \gamma$. Equation (4.1) are evaluated over the center spin $\sigma_0 = (x_0, m_0)$ as shown in the LHS of the case as depicted in Figure 4.1.

4.1.1. Example 1: Lens Hyperbolic Hypergeometric Solution

As an example, we will investigate solutions to the STR in terms of lens hyperbolic hypergeometric gamma functions, see Appendix A.1 for the definitions and notations, [26] and the associated Bolztamnn weight is the following

$$W_{\alpha_i,\beta_i}(\sigma_i,\sigma_j)=\gamma_h(-\alpha_i\pm x_i\pm x_j,-\beta_i\pm m_i\pm m_j;\omega_1,\omega_2), \tag{4.2}$$

where ω_1,ω_2 are introduced as temperature-like parameters and α_i and β_i are continuous and discrete spectral parameters, respectively. The normalization factor and the definition of the Boltzmann weight in different notations can be seen in [18, 26]. In this form, the Bolztamnn weight is not normalized and there appears the spin-independent function

$$R(\alpha_i,\beta_i)=\prod_{i=1}^{3}\gamma_h(-2\alpha_i,-2\beta_i;\omega_1,\omega_2). \tag{4.3}$$

There is also the self-interaction contribution

$$S(\sigma_0)=\frac{1}{\gamma_h(\pm 2x_0,\pm 2m_0;\omega_1,\omega_2)}. \tag{4.4}$$

The integral identity satisfyng the lens hyperbolic hypergeometric solution to the STR is

$$\begin{aligned}\sum_{m=0}^{[r/2]}\epsilon(m)\int_{-\infty}^{\infty}\frac{\prod_{i=1}^{6}\gamma_h(a_i\pm x,u_i\pm m;\omega_1,\omega_2)}{\gamma_h(\pm 2x,\pm 2m;\omega_1,\omega_2)}\frac{dz}{2r\sqrt{-\omega_1\omega_2}}&\\ =\prod_{1\leq i<j\leq 6}\gamma_h(a_i+a_j,u_i+u_j;\omega_1,\omega_2),&\end{aligned} \tag{4.5}$$

where there are the balancing conditions $\sum_{i=1}^{6}a_i=\omega_1+\omega_2$ and $\sum_{i=1}^{6}u_i=r$. The function $\epsilon(y)$ is defined as $\epsilon(0)=\epsilon(\lfloor r/2\rfloor)=1$ and $\epsilon(y)=2$ for all other values. The details of the balancing conditions can be seen in [26] since we use modularity property for discrete variables.

The solution to the STR can be seen by introducing variables $a_i=-\alpha_i+x_i$ and $a_{i+3}=-\alpha_i-x_i$ with the condition $u_i=-\beta_i+m_i$ and $u_{i+3}=-\beta_i-m_i$ for $i=1,2,3$ in the integral identity Equation (4.5). Afterward, it is obvious that the integral identity Equation (4.5) takes the form of the STR Equation (4.1).

The lens hyperbolic hypergeometric gamma function solution to the STR can be reduced to generalization of the Faddeev-Volkov model [13] by fixing $r=1$.

4.1.2. Example 2: Lens Hyperbolic Hypergeometric Solution

The integral identity Equation (4.5) is reduced by the asymptotic properties of the lens hyperbolic hypergeometric gamma function given in Appendix A.1 and the reduction process is called gauge symmetry breaking in the supersymmetric gauge theory side. Also after some manipulations the reduced integral identity [30] takes the following form

$$\begin{aligned}\sum_{m=0}^{[r/2]} \epsilon(m) \int_{-\infty}^{\infty} \prod_{i=1}^{3} \gamma_h(a_i - x, u_i - m; \omega_1, \omega_2) \gamma_h(b_i + x, v_i + m; \omega_1, \omega_2) \frac{dx}{r\sqrt{-\omega_1 \omega_2}} \\ = \prod_{i,j=1}^{3} \gamma_h(a_i + b_j, u_i + v_j; \omega_1, \omega_2),\end{aligned} \tag{4.6}$$

where the balancing conditions are $\sum_{i=1}^{3} a_i + b_i = \omega_1 + \omega_2$ and $\sum_{i=1}^{3} u_i + v_i = r$.

When the following change of variables are applied

$$\begin{aligned} a_i &= -\alpha_i + x_i, \qquad & b_i &= -\alpha_i - x_i, \qquad & i &= 1,2,3, \\ u_i &= -\beta_i + m_i, \qquad & v_i &= -\beta_i - m_i, \qquad & i &= 1,2,3,, \end{aligned} \tag{4.7}$$

another solution to the STR is obtained and the corresponding Boltzmann weight is

$$\begin{aligned} W_{\alpha_i,\beta_i}(\sigma_i, \sigma_j) = \gamma_h(-\alpha_i + x_i - x_j, -\beta_i + m_i - m_j; \omega_1, \omega_2) \\ \times \gamma_h(-\alpha_i - x_i + x_j, -\beta_i - m_i + m_j; \omega_1, \omega_2) \ , \end{aligned} \tag{4.8}$$

The self-interaction term Equation (4.4) vanishes while gauge symmetry breaking process but the spin-independent function stays the same Equation (4.3).

4.1.3. Yang-Baxter Equation for IRF Models

It is known that each edge interaction model has an associated IRF model. The existence of the solutions to the star-triangle relation implies the Yang-Baxter equation for IRF models. So the integrability condition for IRF model is satisfied. However, we mention a Boltzmann weight of a face interaction in terms of four edge interactions, that is, instead of an interaction of four spins altogether, the factorized quartic spin interactions with a central spin will be considered. factorization means that four separate Boltzmann weights of four spins with the central spin will be considered as one IRF.

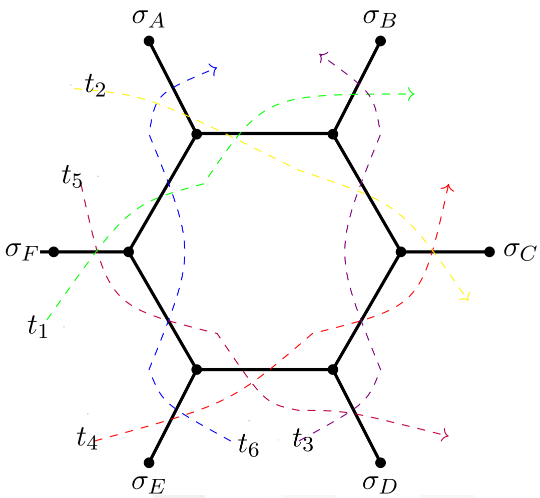

Figure 4.2. Hexagonal lattice with rapidity lines.

We derive the Yang-Baxter equation for IRF from the hexagon lattice as shown in Figure 4.2. In the derivation of two sides of the Yang-Baxter equation, we apply the STR for particular spin sites in order. For the left-hand side (the right-hand side) of the Yang-Baxter equation, we first apply the STR to obtain a triangle from the stars with the central spins on odd sites σ_1, σ_3, and σ_5 (on even sites σ_2, σ_4 and σ_6). In both cases, there will appear a triangle at the center of the figure and it will be converted to the star by the STR. Finally, we acquire the Yang-Baxter equation as pictured in Figure 4.3. It is important to keep rapidity lines properly during the derivation of the Yang-Baxter equation for IRF models.

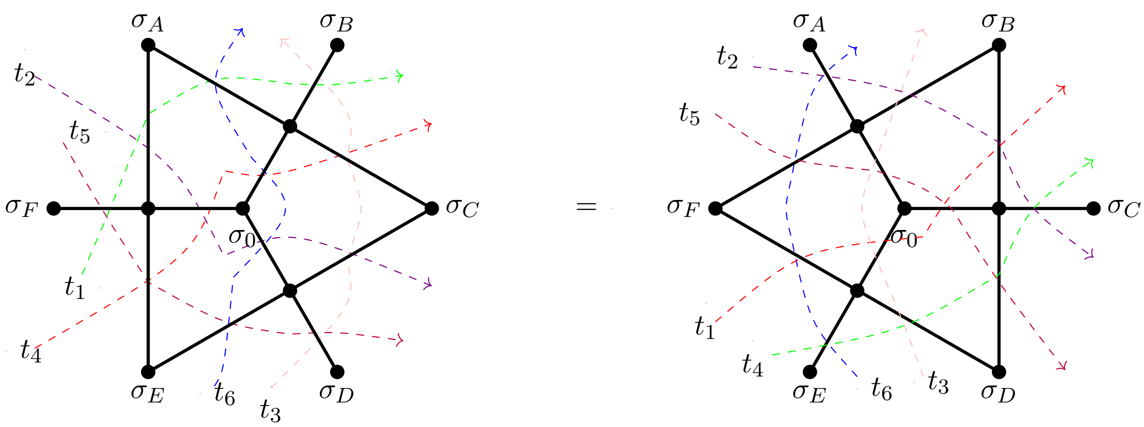

Figure 4.3. The Yang-Baxter equation for IRF models.

The pictorial proof is given by transformations and similar steps can be followed in mathematical expression. To write the Yang-Baxter equation, we define the factorized Bolztamnn weight for IRF

$$R_{t_{34}t_{21}}\begin{pmatrix}\sigma_4 & \sigma_3\\ \sigma_1 & \sigma_2\end{pmatrix} = \sum_{m_0}\int dx_0 W_{t_{32}}\left(\sigma_1,\sigma_0\right)W_{t_{24}}\left(\sigma_0,\sigma_2\right)W_{t_{41}}\left(\sigma_3,\sigma_0\right)W_{t_{13}}\left(\sigma_0,\sigma_4\right), \tag{4.9}$$

where t_{ij} is the shorthand notation of two separate rapidity lines t_i and t_j intersecting at a given Boltzmann weight.

Then the figurative representation of the Yang-Baxter equation as depicted in Figure 4.3. can be written as the following mathematical form

$$\begin{aligned}&\sum_{m_0}\int dx_0\ R_{t_{25}t_{41}}\begin{pmatrix}\sigma_A & \sigma_0\\ \sigma_F & \sigma_E\end{pmatrix} R_{t_{63}t_{25}}\begin{pmatrix}\sigma_0 & \sigma_C\\ \sigma_E & \sigma_D\end{pmatrix} R_{t_{41}t_{63}}\begin{pmatrix}\sigma_A & \sigma_B\\ \sigma_0 & \sigma_C\end{pmatrix}\\ &\quad= \sum_{m_0}\int dx_0\ R_{t_{41}t_{63}}\begin{pmatrix}\sigma_F & \sigma_0\\ \sigma_E & \sigma_D\end{pmatrix} R_{t_{63}t_{25}}\begin{pmatrix}\sigma_A & \sigma_B\\ \sigma_F & \sigma_0\end{pmatrix} R_{t_{25}t_{41}}\begin{pmatrix}\sigma_B & \sigma_C\\ \sigma_0 & \sigma_D\end{pmatrix}.\end{aligned} \tag{4.10}$$

If we leave the construction behind, the IRF model and its Yang-Baxter equation [77] can be figured out as Figure 4.4.

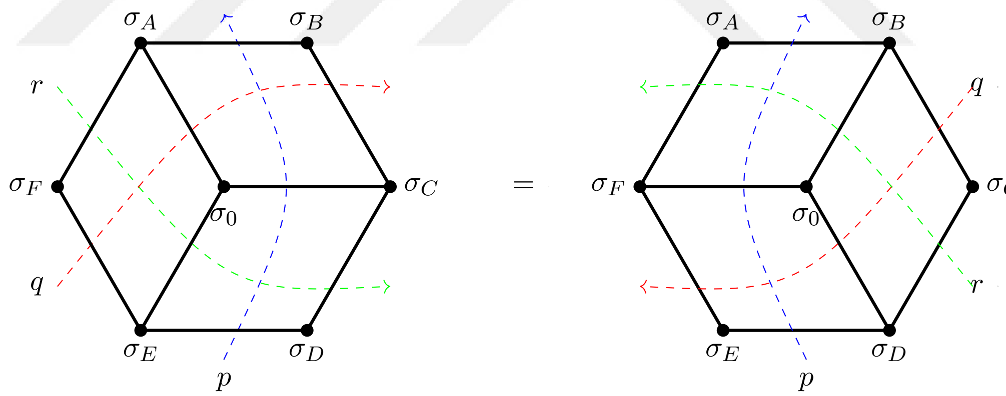

Figure 4.4. The corresponding star-triangle relation for IRF models.

An explicit expression of the Yang-Baxter equation of the IRF model as shown in Figure 4.4. is

$$\begin{aligned}&\sum_{m_0}\int dx_0\ R_{qr}\begin{pmatrix}\sigma_A & \sigma_0\\ \sigma_F & \sigma_E\end{pmatrix} R_{pr}\begin{pmatrix}\sigma_0 & \sigma_C\\ \sigma_E & \sigma_D\end{pmatrix} R_{pq}\begin{pmatrix}\sigma_A & \sigma_B\\ \sigma_0 & \sigma_C\end{pmatrix}\\ &\quad= \sum_{m_0}\int dx_0\ R_{pq}\begin{pmatrix}\sigma_F & \sigma_0\\ \sigma_E & \sigma_D\end{pmatrix} R_{pr}\begin{pmatrix}\sigma_A & \sigma_B\\ \sigma_F & \sigma_0\end{pmatrix} R_{qr}\begin{pmatrix}\sigma_B & \sigma_C\\ \sigma_0 & \sigma_D\end{pmatrix},\end{aligned} \tag{4.11}$$

where the central spins are summed and integrated over both sides.

4.1.4. Bailey Pairs

Constructing Bailey pairs of [78,79] hypergeometric beta integral identities which are studied as the STR provides obtaining the vertex models. When a vertex model is considered the spin sites are the edges of the square lattice and they interact at the vertex of the intersection of the four spins. The Yang-Baxter equation for the vertex model is straightforward by the Coxeter relations [25] when the Bailey pairs are constructed.

Definition 4.1. *Two functions $\alpha(x,m;t,p)$ and $\beta(x,m;t,p)$, where $x,t \in \mathbb{C}$ and $m,p \in \mathbb{Z}$, form an integral hyperbolic hypergeometric Bailey pair with respect to t and p if the functions satisfy*

$$\beta(z,m;t,p) = M(t,p)_{z,m;x,j}\alpha(x,j;t,p) \; , \tag{4.12}$$

where $M(t,p)_{z,m;x,j}$ is an operator integrating over $x \in \mathbb{C}$ and summing over $j \in \mathbb{Z}$, which also called an integral-sum operator.

We also assume an operator $D(s,q;y,l;x,k)$ with continuous $s,y,x \in \mathbb{C}$ and discrete variables $q,l,k \in \mathbb{Z}$, such that the operator attaches the new variables to functions in which it satisfies the relation

$$D(s,q;y,l;x,k)D(-s,-q;y,l;x,k) = 1 \; , \quad \& \quad D(0,0;y,l;x,k) = 1 \; . \tag{4.13}$$

Suppose that the operators M and D satisfy the "star-triangle relation"

$$\begin{gathered} M(s,q)_{w,k;z,m}D(s+t,q+p;y,l;z,m)M(t,p)_{z,m;x,j} = \\ D(t,p;y,l;w,k)M(s+t,q+p)_{w,k;x,j}D(s,q;y,l;x,j), \end{gathered} \tag{4.14}$$

where $w \in \mathbb{C}$ and $k \in \mathbb{Z}$.

When one acquires a particular Bailey pair by using M and D operators, the Bailey lemma argues that it reveals infinitely many Bailey pairs.

Lemma 4.2 (Bailey Lemma)**.** *When $\alpha(x,m;t,p)$ and $\beta(x,m;t,p)$ form an integral hyperbolic hypergeometric Bailey pair with respect to t and p, the novel function as a part of the sequences of functions $\beta'(x,k;t+s,p+q)$ and reparametrized function $\alpha'(x,k;t+s,p+q)$ defined by*

$$\alpha'(x,k;t+s,p+q) = D(s,q;y,l;x,k)\alpha(x,k;t,p),$$
$$\beta'(x,k;t+s,p+q) = D(-t,-p;y,l;x,k) \times M(s,q)_{x,k;z,m}D(s+t,p+q;y,l;z,m)\beta(z,m;t,p), \tag{4.15}$$

form an integral hyperbolic hypergeometric Bailey pair with respect to the new parameters $t+s$ and $p+q$.

Proof. Recall the definition of Bailey pairs Equation (4.12) for the functions $\alpha'(x,j;t+s,p+q)$ and $\beta'(x,k;t+s,p+q)$

$$\beta'(w,k;t+s,p+q) = M(t+s,p+q)_{w,k;x,j}\alpha'(x,j;t+s,p+q)\ , \tag{4.16}$$

and substitute the functions into the Equation (4.12) defining a Bailey pair

$$D(-t,-p;y,l;w,k)M(s,q)_{w,k;z,m}D(s+t,p+q;y,l;z,m)\beta(z,m;t,p) = M(s+t,p+q)_{w,k;x,j}D(s,q;y,l;x,j)\alpha(x,j;t,p). \tag{4.17}$$

The proof is easily completed by the use of the properties of the D operator and then the problem ends up with the STR

$$M(s,q)_{w,k;z,m}D(s+t,q+p;y,l;z,m)M(t,p)_{z,m;x,j} = D(t,p;y,l;w,k)M(s+t,q+p)_{w,k;x,j}D(s,q;y,l;x,j), \tag{4.18}$$

which is assumed to be true in Equation (4.14). □

It is understood that the Bailey pairs can be constructed for the STR and the construction in terms of elliptic, hyperbolic, and trigonometric hypergeometric functions are studied in various fields [53,63]. From the statistical mechanics' point of view, Bailey pair construction provides vertex integrable models [25,66].

4.1.5. Constructing Bailey Pairs for the Example 1

We will construct the operators satisfying (4.14) in terms of the lens hyperbolic hypergeometric gamma functions. We also note that there is not any systematic way to construct Bailey pairs and related operators. We first introduce the following operator

$$D(t,p;y,l;w,k) = \gamma_h(t+y\pm w+\omega\rho, p\pm k+r\sigma+l;\omega_1,\omega_2) \times \gamma_h(t-y\pm w+\omega\rho, r-(p\pm k+r\sigma+l);\omega_1,\omega_2)\ , \tag{4.19}$$

and check that the operator satisfies the following conditions

$$D(t,p;y,l;w,k)D(-t,-p;y,l;w,k)=1\ , \qquad D(0,0;y,l;w,k)=1, \tag{4.20}$$

where the characteristic properties of lens hyperbolic hypergeometric gamma functions are used.

Then we construct the integral sum operator

$$\begin{aligned} M(t,p)_{z,m;x,j} =& \frac{1}{C(t,p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t+z\pm x, m-p\pm j;\omega_1,\omega_2) \\ & \times \gamma_h(-t-z\pm x,-m-p\pm j;\omega_1,\omega_2) \frac{[\mathrm{d}_j x]}{2r\sqrt{-\omega_1\omega_2}}\ , \end{aligned} \tag{4.21}$$

where the measure is defined as

$$[\mathrm{d}_j x] = \frac{\epsilon(j)\mathrm{d}x}{\gamma_h(\pm 2x, \pm 2j;\omega_1,\omega_2)}\ , \tag{4.22}$$

and the spin-independent function is distributed as

$$C(t,p) =\gamma_h(-2t,-2p;\omega_1,\omega_2). \tag{4.23}$$

One can show that the operators satisfy (4.14) with the help of the integral identity Equation (4.5) under the following change of variables

$$\begin{aligned} a_{1,2} &= -s\pm w, & a_3 &= s+t+y+\omega\rho\ , \\ a_4 &= s+t-y+\omega(1-\rho), & a_{5,6} &= -t\pm x\ , \\ u_{1,2} &= -q\pm k, & u_3 &= q+p+l+r\sigma\ , \\ u_4 &= q+p-l+r(1-\sigma), & u_{5,6} &= -p\pm m\ . \end{aligned} \tag{4.24}$$

The detailed proof for this construction can be seen in [66].

4.1.6. Constructing Bailey Pairs for the Example 2

Similarly, we introduce operators to construct the Bailey pairs. It can be shown that the operators satisfy the STR (4.14) by the use of integral identity Equation (4.6). The first operator is the following

$$\begin{aligned} D(t,p;y,l;w,k) =& \gamma_h(t+y+w+\omega\rho, p+k+r\sigma+l;\omega_1,\omega_2) \\ & \times \gamma_h(t-y-w+\omega\rho, r-(p-k+r\sigma+l);\omega_1,\omega_2), \end{aligned} \tag{4.25}$$

where satisfies the conditions Equation (4.20). The integral sum operator takes the following form

$$
\begin{aligned}
M(t,p)_{z,m;x,j} =& \frac{1}{C(t,p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t+z+x, m-p+j; \omega_1, \omega_2) \\
& \times \gamma_h(-t-z-x, -m-p-j; \omega_1, \omega_2) \frac{\epsilon(j)\mathrm{d}x}{2r\sqrt{-\omega_1\omega_2}} \,,
\end{aligned}
\tag{4.26}
$$

where there is no necessity to redefine measure due to the absence of the self-interaction term but the spin-independent function is the same as Equation (4.23).

To use the integral identity Equation (4.6) for the proof of the STR Equation (4.14) of the operators, the parameters will be re-defined as

$$
\begin{aligned}
a_1, b_1 &= -s \pm w \,, & a_2 &= s+t+y+\omega\rho \,, \\
b_2 &= s+t-y+\omega(1-\rho) \,, & a_3, b_3 &= -t \pm x \,, \\
u_1, v_1 &= -q \pm k \,, & u_2 &= q+p+l+r\sigma \,, \\
v_2 &= q+p-l+r(1-\sigma) \,, & u_3, v_3 &= -p \pm m \,.
\end{aligned}
\tag{4.27}
$$

The calculations are straightforward as presented in [66] but constructing the operators is not systematic.

4.1.7. Relations to the Quantum Groups

It is necessary to use the star-triangle relation of the Faddeev-Volkov model [82] to prove how the orthogonality and completeness relations are satisfied by the Clebsh-Gordan coefficients for the self-dual series of $U_q(sl(2))$ [67,83]. From the gauge theory aspect, the same integral identity is also studied as the duality of three dimensional $\mathcal{N}=2$ supersymmetric gauge theories on squashed three-sphere S_b^3 [83]. In [67], it is also shown that there needs to be a bit of a general version (additional free parameter takes two values) of the STR of the Faddeev-Volkov model to show the same discussion for the quantum-deformed superalgebra $U_q(osp(1|2))$.

Later, the most general case with arbitrary parameter r is studied for the three-dimensional $\mathcal{N}=2$ supersymmetric gauge theories on lens space $S_b^3/\mathbb{Z}_r$. The integral identity studied as a STR is the equality of the partition function of dual supersymmetric gauge theories on $S_b^3/\mathbb{Z}_r$. The relation directly appears that one can find a solution to the STR [27,30] for the $U_q(osp(1|2))$ [67,84,85]. A special case $r=2$ of the same integral identity is for the three-dimensional $\mathcal{N}=2$ supersymmetric gauge theories on S_b^3/Z_2 (or topologically $\mathbb{RP}^3$). So this particular case of the integral identity is a tool to prove the orthogonality and completeness relation of the Clebsch-Gordan coefficients for the self-dual continuous series of $U_q(osp(1|2))$.

It is also good to remark that the relation between the STR and quantum algebra extends the relation with the Liouville field theory due to the deep connection between the two-dimensional conformal field theory and representation theory. In other words, there is an equivalence between the representations of the Liouville theory and the unitary representations of the modular double of quantum algebra, see for more details [67, 83, 85]. Therefore the STR should be considered with its role in Liouville field theories.

4.1.8. The Pentagon Identity

The mathematical expression of the basic 2-3 Pachner move [86] for triangulated three-dimensional manifolds is described by the pentagon identity [54, 55] written as

$$\mathcal{B}\mathcal{B}\mathcal{B} = \mathcal{B}\mathcal{B}. \tag{4.28}$$

Some of the integral identities [30, 60] studied as solutions to the STR are also solutions to the pentagon identity. Its interpretation takes place as a map in *3d–3d* correspondence [87, 88]. In this correspondence, a topological invariant of corresponding 3-manifolds is constructed via the three-dimensional $\mathcal{N}=2$ supersymmetric gauge theories. The integral identity takes the following form

$$\begin{aligned} &\sum_m \int dx \prod_{i=1}^{3} \mathcal{B}(a_i - x, u_i - m; b_i + x, v_i + m) \\ &= \mathcal{B}(a_1 + b_2, u_1 + v_2; a_2 + b_3, u_2 + v_3)\mathcal{B}(a_1 + b_3, u_1 + v_3; a_2 + b_1, u_2 + v_1). \end{aligned} \tag{4.29}$$

In the literature, Euler's gamma function (rational) [53, 55, 60, 89], trigonometric [56–58], hyperbolic hypergeometric gamma function [59, 90], and lens hyperbolic hypergeometric gamma function [30, 91] solutions to the pentagon identity is presented. Constructing Bailey pairs for the pentagon identity is also studied in [53, 65, 66, 92].

4.2. The Star-Star Relation

The commutation relation Equation (3.5) of the transfer matrices can be also satisfied by the STR as shown in Figure 4.5. which is the local condition on Boltzmann weights like the STR. So the STR is an integrability condition for the lattice spin models in statistical mechanics. It is obvious that the STR can be proven by the use of the star-triangle relation if Boltzmann weights already satisfy the STR. However, the four-state three-layer Zamolodchikov model [16] is an example of a situation in which the Boltzmann weights satisfy the STR but do not solve the STR.

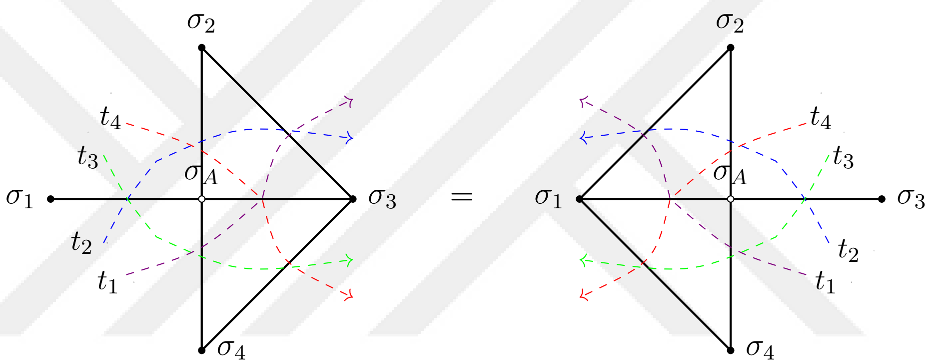

Figure 4.5. The star-star relation.

The mathematical form of the star-star relation can be seen as the following due to the flip of the spectral parameters standing on opposite sides

$$R_{t_{34}t_{21}}\begin{pmatrix} & \sigma_1 & \\ \sigma_2 & & \sigma_3 \\ & \sigma_4 & \end{pmatrix} = R_{t_{21}t_{34}}\begin{pmatrix} & \sigma_1 & \\ \sigma_2 & & \sigma_3 \\ & \sigma_4 & \end{pmatrix}, \tag{4.30}$$

where each side can be written in terms of a Boltzmann weight of an IRF model with additional edge interactions

$$R_{t_{34}t_{21}}\begin{pmatrix} & \sigma_1 & \\ \sigma_2 & & \sigma_3 \\ & \sigma_4 & \end{pmatrix} = W_{t_{21}}(\sigma_2,\sigma_3)W_{t_{34}}(\sigma_3,\sigma_4)R_{t_{34}t_{21}}\begin{pmatrix} \sigma_4 & \sigma_3 \\ \sigma_1 & \sigma_2 \end{pmatrix}. \tag{4.31}$$

The STR has also a condition on spectral parameters and can be derived easily while deriving the STR [16]. Then, the star-star relation is

$$R_{t_{34}t_{21}}\begin{pmatrix} \sigma_4 & \sigma_3 \\ \sigma_1 & \sigma_2 \end{pmatrix} = \frac{W_{t_{21}}(\sigma_1,\sigma_2)W_{t_{34}}(\sigma_1,\sigma_4)}{W_{t_{21}}(\sigma_2,\sigma_3)W_{t_{34}}(\sigma_3,\sigma_4)}R_{t_{21}t_{34}}\begin{pmatrix} \sigma_4 & \sigma_3 \\ \sigma_1 & \sigma_2 \end{pmatrix}, \tag{4.32}$$

where fractions will be eliminated during the application for the flip of transfer matrices due to the inversion relation.

4.2.1. Example 1: Lens Hyperbolic Hypergeometric Solution

The STR implies the STR but there is also a double integral method introduced in [93] to obtain the integral identity satisfying the star-star relation. The STR of the integrable model [26] is studied in [34] and the integral identity is the following

$$\begin{aligned}\sum_{m=0}^{[r/2]}\epsilon(m)\int_{-\infty}^{\infty}\frac{\prod_{i=1}^{8}\gamma_h(a_i\pm x,u_i\pm m;\omega_1,\omega_2)}{\gamma_h(\pm 2x,\pm 2m;\omega_1,\omega_2)}\frac{dx}{r\sqrt{-\omega_1\omega_2}}\\ =\frac{\prod_{1\le i<j\le 4}\gamma_h(a_i+a_j,u_i+u_j;\omega_1,\omega_2)}{\prod_{5\le i<j\le 8}\gamma_h(\tilde{a}_i+\tilde{a}_j,\tilde{u}_i+\tilde{u}_j;\omega_1,\omega_2)}\\ \times\sum_{y=0}^{[r/2]}\epsilon(y)\int_{-\infty}^{\infty}\frac{\prod_{i=1}^{8}\gamma_h(\tilde{a}_i\pm z,\tilde{u}_i\pm y;\omega_1,\omega_2)}{\gamma_h(\pm 2z,\pm 2y;\omega_1,\omega_2)}\frac{dz}{r\sqrt{-\omega_1\omega_2}}\,,\end{aligned} \tag{4.33}$$

where the balancing conditions are the following $\sum_{i=1}^{8}a_i=2(\omega_1+\omega_2)$, $\sum_{i=1}^{8}u_i=0$, and tilde variables defined as

$$\begin{aligned}\tilde{a}_i=a_i+s,\quad \tilde{u}_i=u_i+p,\quad \text{if}\quad i=1,2,3,4\,,\\ \tilde{a}_i=a_i-s,\quad \tilde{u}_i=u_i-p,\quad \text{if}\quad i=5,6,7,8,\end{aligned} \tag{4.34}$$

where the parameters are

$$\begin{aligned}s&=\frac{1}{2}\left(\omega_1+\omega_2-\sum_{i=1}^{4}a_i\right)=\frac{1}{2}\left(-\omega_1-\omega_2+\sum_{i=5}^{8}a_i\right)\\ p&=-\frac{1}{2}\left(\sum_{i=1}^{4}u_i\right)=\frac{1}{2}\left(\sum_{i=5}^{8}u_i\right)\,.\end{aligned} \tag{4.35}$$

The integral identity for the case of $r=1$ is also studied in [95].

4.2.2. Example 2: Lens Hyperbolic Hypergeometric Solution

The star-star relation of the discrete generalized of the Faddeev-Volkov model which is Example 2 Equation (4.6) is obtained in [33] by the double integral method and it is shown that the integral identity Equation (4.33) can be also reduced to the same STR by breaking the gauge symmetry in [34]

$$\sum_{y=0}^{[r/2]} \epsilon(y) \int_{-\infty}^{\infty} \gamma_h(a_i - z, u_i - y; \omega_1, \omega_2)\gamma_h(b_i + z, v_i + y; \omega_1, \omega_2)\frac{dz}{r\sqrt{-\omega_1\omega_2}}$$
$$= \frac{\prod_{i,j=1}^{2} \gamma_h(a_i + b_j, u_i + v_j; \omega_1, \omega_2)}{\prod_{i,j=4}^{4} \gamma_h(\tilde{a}_i + \tilde{b}_j, \tilde{u}_i + \tilde{v}_j; \omega_1, \omega_2)} \tag{4.36}$$
$$\times \sum_{m=0}^{[r/2]} \epsilon(m) \int_{-\infty}^{\infty} \gamma_h(\tilde{a}_i - x, \tilde{u}_i - m; \omega_1, \omega_2)\gamma_h(\tilde{b}_i + x, \tilde{v}_i + m; \omega_1, \omega_2)\frac{dx}{r\sqrt{-\omega_1\omega_2}},$$

where the balancing conditions become $\sum_{i=1}^{4} a_i + b_i = 2(\omega_1 + \omega_2)$ and $\sum_{i=1}^{4} u_i + v_i = 2r$, and the following redefinitions are used

$$\begin{aligned} &\tilde{a}_i = a_i + s, \quad \tilde{b}_i = b_i + s, \quad \tilde{u}_i = u_i + p, \quad \tilde{v}_i = v_i + p, \quad \text{if} \quad i = 1, 2\,, \\ &\tilde{a}_i = a_i - s, \quad \tilde{b}_i = b_i - s, \quad \tilde{u}_i = u_i - p, \quad \tilde{v}_i = v_i - p, \quad \text{if} \quad i = 3, 4\,, \end{aligned} \tag{4.37}$$

with the parameters

$$\begin{aligned} s &= \frac{1}{2}(\omega_1 + \omega_2 - a_1 - a_2 - b_1 - b_2) = \frac{1}{2}(-\omega_1 - \omega_2 + a_3 + a_4 + b_3 + b_4)\,, \\ p &= -\frac{1}{2}(u_1 + u_2 + v_1 + v_2) = \frac{1}{2}(u_3 + u_4 + v_3 + v_4)\,. \end{aligned} \tag{4.38}$$

The $r = 1$ case of the integral identity Equation (4.36) is the STR of the Faddeev-Volkov model [95].

4.2.3. Yang-Baxter Equation for IRF Models

The existence of the star-star relation implies the Yang-Baxter equation as pictured in Figure 4.6. of an IRF model. The equality can be proven by the use of the STR at four steps from the left-hand side to the right-hand side.

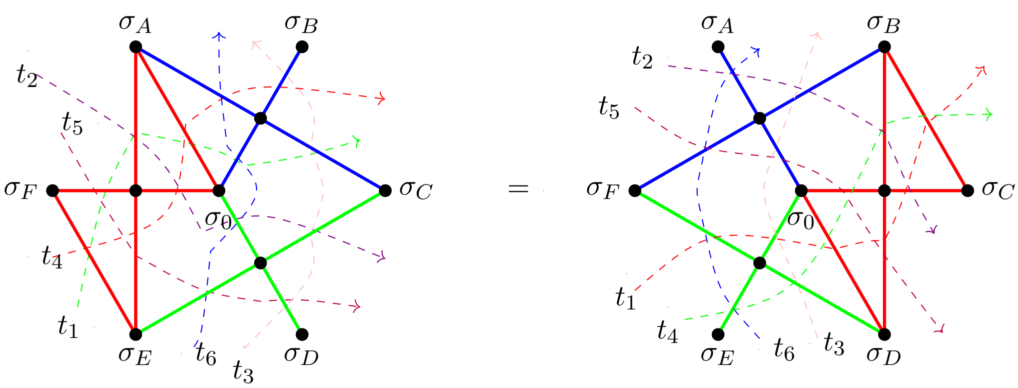

Figure 4.6. The Yang-Baxter equation for IRF models.

The mathematical form of the Yang-Baxter equation of IRF model is

$$\sum_{m_0\in\mathbb{Z}}\int dx_0 \;\; R_{t_{25}t_{41}}\begin{pmatrix} & \sigma_A & \\ \sigma_F & & \sigma_0 \\ & \sigma_E & \end{pmatrix} R_{t_{63}t_{25}}\begin{pmatrix} \sigma_0 & \sigma_C \\ \sigma_E & \sigma_D \end{pmatrix} R_{t_{41}t_{63}}\begin{pmatrix} \sigma_A & \sigma_B \\ \sigma_0 & \sigma_C \end{pmatrix}$$
$$= \sum_{m_0\in\mathbb{Z}}\int dx_0 \;\; R_{t_{41}t_{63}}\begin{pmatrix} \sigma_F & \sigma_0 \\ \sigma_E & \sigma_D \end{pmatrix} R_{t_{63}t_{25}}\begin{pmatrix} \sigma_A & \sigma_B \\ \sigma_F & \sigma_0 \end{pmatrix} R_{t_{25}t_{41}}\begin{pmatrix} & \sigma_B & \\ \sigma_0 & & \sigma_C \\ & \sigma_D & \end{pmatrix}, \tag{4.39}$$

where additional Boltzmann weights for the edge interactions are considered in the left (right) part of the left (right) side with the help of the definition Equation (4.31).

4.2.4. Bailey Pairs

We continue to discuss Bailey pairs constructed from the initial Bailey pair. If we remember an integral-sum operator $M(t,p)_{z,m;x,j}$ acts on a sequence of functions $f_j(x)$, the definition of Bailey pairs (4.12) allows to introduce $\alpha(x,j;t,p) = \delta_{jn}\delta(x-u)$ with novel parameters $n\in\mathbb{Z}$, $u\in\mathbb{C}$.

So, $\beta(z,m;t,p)$ becomes

$$\begin{aligned}\beta(z,m;t,p) &= M(t,p)_{z,m;x,j}\delta_{jn}\delta(x-u)\\ &:= M(t,p;z,m;u,n),\end{aligned} \tag{4.40}$$

and still is part of Bailey pairs with $\alpha(x,j;t,p)$. Then, we can also construct secondary Bailey pairs with the Bailey lemma,

$$\begin{aligned}\alpha(x,k;t+s;p+q) &= D(s,q;y,l;x,k)\alpha(x,k;t,p),\\ \beta(x,k;t+s;p+q) &= D(-t,-p;y,l;x,k)\\ &\quad\times M(s,q)_{x,k;z,m}D(s+t,p+q;y,l;z,m)\beta(z,m;t,p),\end{aligned} \tag{4.41}$$

where at this level, it is proven that the Bailey pairs yield the STR

$$\begin{aligned}&M(s,q)_{w,k;z,m}D(s+t,p+q;y,l;z,m)M(t,p;z,m;u,n)\\ &\quad= D(t,p;y,l;w,k)M(s+t,p+q;w,k;u,n)D(s,q;y,l,u,n).\end{aligned} \tag{4.42}$$

However, if we go further and work with second Bailey pairs (4.42) $\tilde{\alpha}(z,m;s,q)$ and $\tilde{\beta}(w,k;s,q)$ which have similar definition

$$\begin{aligned}\tilde{\alpha}(z,m;s,q) &= D(s+t,p+q;y,l;z,m)M(t,p;z,m;u,n),\\ \tilde{\beta}(w,k;s,q) &= D(t,p;y,l;w,k)M(s+t,p+q;w,k;u,n)D(s,q;y,l,u,n),\end{aligned} \tag{4.43}$$

respect the Bailey pair construction with the parameters $s \in \mathbb{C}$, $q \in \mathbb{Z}$. When the Bailey Lemma is applied once more, we obtain the following

$$\begin{aligned}\tilde{\alpha}'(z,m;s+c,q+d) =& D(c,d;a,b;z,m)D(s+t,p+q;y,l;z,m)M(t,p;z,m;u,n),\\ \tilde{\beta}'(x,j;s+c,q+d) =& D(-s,-q;a,b;x,j)M(c,d)_{x,j;w,k}D(s+c,q+d;a,b;w,k)\\ &\times D(t,p;y,l;w,k)M(s+t,p+q;w,k;u,n)D(s,q;y,l,u,n),\end{aligned} \tag{4.44}$$

where $a, c \in \mathbb{C}$ and $b, d \in \mathbb{Z}$ are arbitrary parameters. Then it replaced in the definition

$$\tilde{\beta}'(x,j;s+c,q+d) = M(s+c,q+d)_{x,j;z,m}\tilde{\alpha}'(z,m;s+c,q+d), \tag{4.45}$$

reveals the star-star relation

$$\begin{aligned}&M(c,d)_{x,j;w,k}D(s+c,q+d;a,b;w,k)D(t,p;y,l;w,k)M(s+t,p+q;w,k;u,n)\\ &= D(-s,-q;y,l,u,n)D(s,q;a,b;x,j)\\ &\times M(s+c,q+d)_{x,j;z,m}D(c,d;a,b;z,m)D(s+t,p+q;y,l;z,m)M(t,p;z,m;u,n)\,.\end{aligned} \tag{4.46}$$

So, the first Bailey pair gives the STR, and the second Bailey pair with the particular case of the first ones yields the STR.

4.2.5. Constructing Bailey Pairs for the Example 1

The Bailey pair construction for the star-star relation Equation (4.33) is based on the operators that we used for constructing Bailey pairs of the STR Equation (4.33). However, we need to adjust the Bailey pairs as indicated in Equation (4.40) to acquire the STR Equation (4.46) and we define the following additional operator

$$\begin{aligned}M(t,p;z,m;u,n) =& \frac{\Delta_n^u}{C(t,p)}\gamma_h(-t+z\pm u, m-p\pm n;\omega_1,\omega_2)\\ &\times \gamma_h(-t-z\pm u,-m-p\pm n;\omega_1,\omega_2)\end{aligned} \tag{4.47}$$

where and $C(t,p)$ is Equation (4.23) and the integral measure is eliminated but stays the following contribution term

$$\Delta_n^u = \frac{\epsilon(n)}{2r\sqrt{-\omega_1\omega_2}}\frac{1}{\gamma_h(\pm 2u,\pm 2n;\omega_1,\omega_2)}. \tag{4.48}$$

Then the change of variables is applied as

$$\begin{aligned} a_{1,2} &= -c \pm x\,, \quad a_3 = s + c + a + \omega\rho\,, \quad a_4 = s + c - a + \omega(1-\rho)\,,\\ a_5 &= t + y + \omega\rho\,, \quad a_6 = t - y + \omega(1-\rho)\,, \quad a_{7,8} = -(s+t) \pm u,\\ u_{1,2} &= -d \pm j\,, \quad u_3 = q + d + b + r\sigma\,, \quad u_4 = q + d - b + r(1-\sigma),\\ u_5 &= p + l + r\sigma\,, \quad u_6 = p - l + r(1-\sigma)\,, \quad u_{7,8} = -(p+q) \pm n\,, \end{aligned} \tag{4.49}$$

whereby the use of this change of variables and the integral identity Equation (4.33), one can see that the operators Equation (4.19), Equation (4.21) and Equation (4.47) satisfy the STR Equation (4.46).

4.2.6. Constructing Bailey Pairs for the Example 2

Similarly, we need to define the following operator to show that the Bailey pair construction for the STR Equation (4.46) can be achieved for the integral identity Equation (4.36) by using operators of the STR Equation (4.36)

$$\begin{aligned} M(t,p;z,m;x,j) =& \frac{1}{C(t,p)} \frac{\epsilon(n)}{r\sqrt{-\omega_1\omega_2}}\\ & \times \gamma_h(-t+z+x, m-p+j; \omega_1, \omega_2)\\ & \times \gamma_h(-t-z-x, -m-p-j; \omega_1, \omega_2)\,, \end{aligned} \tag{4.50}$$

The STR Equation (4.46) is satisfied by using integral identity Equation (4.6) under following re-parametrizations

$$\begin{aligned} a_1, b_1 &= -c \pm x\,, \quad a_2 = s + c + a + \omega\rho\,, \quad b_2 = s + c - a + \omega(1-\rho)\,,\\ a_3 &= t + y + \omega\rho\,, \quad b_3 = t - y + \omega(1-\rho)\,, \quad a_4, b_4 = -(s+t) \pm u\,,\\ u_1, v_1 &= -d \pm j\,, \quad u_2 = q + d + b + r\sigma\,, \quad v_2 = q + d - b + r(1-\sigma)\,,\\ u_3 &= p + l + r\sigma\,, \quad v_3 = p - l + r(1-\sigma)\,, \quad u_4, v_4 = -(p+q) \pm n\,. \end{aligned} \tag{4.51}$$

4.2.7. $W(E_7)$ Transformation

The star-star relation is a kind of generation for the $W(E_7)$ Weyl group symmetry transformations for the function that presents the interaction of four spins with the central spin or the IRF.

From the hypergeometric special functions aspect, the lens elliptic hypergeometric gamma function solution [70, 95]n of the star-star solution (the A_n elliptic hypergeometric sum/integral transformation formula) is analog but the generalization of the Euler-Gauss hypergeometric function.

However, its hyperbolic hypergeometric version [34, 95] which can be seen as dimensionaşl reduction in supersymmetric gauge theory understanding is also analog for the Euler–Gauss hypergeometric ${}_2F_1$-function. This parallel construction is also interesting in that the functions have the property of the solution for a difference equation. In the hyperbolic case for the supersymmetric gauge theories on the three-dimensional squashed spheres, the function stands as the extended version of the analog.

A similar parallelism for symmetry transformation appears between the complex Euler gamma solution [95] of the STR and a general complex analog of the four-term Bailey transformation for non-terminating hypergeometric ${}_9F_8$-series.

4.2.8. Painleve Equation

The discussion on the Painleve equation is related to the previous $W(E_7)$ transformation property of the star-star relation. The lens elliptic hypergeometric function solution of the STR ensures the Weyl group $W(E_7)$ $W(E_7)$ invariance of the τ-functions and it is studied also for four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories on $S^3/\mathbb{Z}_r \times S^1$ [36].

Recently, in [96, 97], the τ-functions are studied in detail for the elliptic discrete Painleve equation of type $E_8^{(1)}$ and it is generalized to lens τ-function with free parameter r by the use of elliptic beta integral which is the most general solution of the STR for the lattice model consisting continuous and discrete spins, see [23, 32, 36] for details on the relation with supersymmetric gauge theories and the mathematical proof of the identities. In the generalization, a bilinear Hirota-type equation for the τ-function is proposed and its solutions are investigated by the lens elliptic beta integrals. However, it is still an open research field on what kind of corresponding generalized elliptic discrete Painleve equation is studied by the novel elliptic hypergeometric τ-function.

Another perspective proposed in [52] can be found between hypergeometric solutions to the q-Painleve equation [98, 99] which is invariant under a particular Weyl symmetry and the star-triangle relation [25, 26, 94] which is satisfied in terms of trigonometric hypergeometric functions.

When the developments are explored in the relation between the Painleve equation and integrable models [52, 97] associated with supersymmetric gauge theories, various connections and problems could appear and some of them are the Hamiltonian form of the discrete Painleve equation with new τ-function and the geometric aspects of these equations along the lines of Sakai's classification [100].

5. SEIBERG DUALITIES AND QUIVER DIAGRAMS

5.1. The Three-Dimensional Seiberg Dualities

The duality of two different theories represents the same observables in a particular case. If we remember the Kramers-Wannier duality [21], the free energy of two different lattice spin models is the same. In the case of Seiberg duality, [101], the partition functions of two supersymmetric gauge theories possessing distinguished physical observables in the ultraviolet energy states are equal when the two theories are adjusted to the same infrared fixed point. There is no exact proof of the Seiberg duality but the equality of partition functions of two theories at the same energy level is strong evidence.

The duality in three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories is argued in [102, 103] and studied for sphere partition functions (e.g. [104]), squashed sphere partition functions (e.g. [47,48,105]), superconformal indices (e.g. [46,56,58,106]), lens partition functions (e.g. [36–38]) and so on.

The integral identity Equation (4.5) is obtained by dimensional reduction, e.g. see [26, 36, 48]. The idea is the following: the four-dimensional $\mathcal{N} = 1$ lens index on $S_b^3/\mathbb{Z}_r \times S^1$ is reduced to the three-dimensional $\mathcal{N} = 2$ lens partition function on $S_b^3/\mathbb{Z}_r$ by geometrically shrinking the circle S^1 to zero.

The integral identity Equation (4.5) for a fixed value of $r = 1$ is studied as the duality of the three-dimensional gauge theories on the squashed sphere in [83, 107] and the integral identity is mathematically proven in [108].

From the ingredients perspective, the integral identity Equation (4.5) is the equality of the partition functions of the dual three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on lens space $S_b^3/\mathbb{Z}_r$. On the left-hand side, the theory has the gauge symmetry group $SU(2)$ and flavor symmetry group $SU(6)$. While the vector multiplet (denominator of the integral identity Equation (4.5) or the self-interaction term Equation (4.4)) transforms under the adjoint representation of the gauge group, the chiral multiplets (numerator of integrand Equation (4.5) or Boltzmann weights for the edge interactions) transform under the fundamental representation of both the gauge group and the flavor group. On the other hand, the dual theory, on the right-hand side, does not have gauge symmetry and contains fifteen chiral multiplets (all of the spin-independent functions and the Bolztamnn weights for edge interactions) in the totally antisymmetric tensor representation of the flavor group.

The latter solution to the star-triangle relation Equation (4.6) is obtained by breaking gauge symmetry [13, 27] from $SU(2)$ to $U(1)$ while reducing the flavor group to $SU(3)_L \times SU(3)_R$. In this situation, chiral multiplets (integrand or Boltzmann weight for edge interactions) belonging to the $SU(3)_L$ and $SU(3)_R$ transform in the fundamental representation of the gauge group and in the anti-fundamental representation, respectively. In the dual theory, there is no gauge symmetry and chiral multiplets (whole part of the left-hand side in STR) transform in the fundamental representation of the flavor group $SU(3)_L \times SU(3)_R$.

5.2. Gauge symmetry breaking

There have been discussed different techniques to obtain various solutions to the star-triangle and star-star relation. One of the examples is the dimensional reduction for supersymmetric gauge theories. For example, one can obtain lens hyperbolic gamma function solutions when the four-dimensional $\mathcal{N} = 1$ supersymmetric dual theories are reduced to three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories.

There is also a distinguished method in supersymmetric gauge theories to obtain solutions to the STR or the STR. It is called the gauge symmetry breaking [13, 30]. It is used to acquire the integral identity that stands for the duality of the supersymmetric theories with $U(1)$ gauge symmetry from the equality of partition functions of supersymmetric theories with $SU(2)$ gauge symmetry. Hence the idea of the gauge symmetry breaking method is the reduction of the gauge symmetry from the $SU(2)$ gauge group to the $U(1)$ gauge group.

We apply the procedure to ensure that the application of the gauge symmetry breaking allows us to reach another solution of the relations for lattice spin models in statistical mechanics such as studied for the STR [13,30] and the star-star relation [34].

From a statistical mechanics point of view, the Boltzmann weight Equation (4.2) of the integrable model [26] given above as an example is broken and turns into Boltzmann weight Equation (4.8) of the different lattice model [30].

Therefore, it is interesting that the gauge symmetry-breaking method plays a role in the gauge/YBE correspondence. The flavor symmetry group will be also broken during the gauge symmetry breaking.

5.3. Quiver Gauge Theory

We study that the three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories can be associated with a quiver diagram for a quiver gauge theory. It will be observed that the quiver diagrams can be identified as the relations for the lattice spin models. Then the dictionary will be established and the notions will be mapped from quiver gauge theory to the integrable lattice models in statistical mechanics. That is, the lattice structure is the same for the supersymmetric $\mathcal{N} = 2$ quiver gauge theories and integrable models. Seiberg dualities will correspond to the STR and the SSR. This equivalency is called the gauge/YBE correspondence.

The identifications [15, 19] follow as vertices, edges, and interactions in statistical mechanics for loops, arrows, and bifundamental matter in quiver gauge theory, respectively. In the quiver diagram, see [80,81] for more details, gauge groups, and the bifundamental matter content will be cited on the vertices and the edges, respectively.

The arrow will not be indicated in this discussion since its role is to specify fundamental (outgoing arrow) and antifundamental (incoming arrow) representations. We note that the superpotential terms are not defined and it will be on the face of each square in the lattice if it is discussed. The Boltzmann weight for nearest-neighbor edge interaction and the self-interaction will be identified to the chiral multiplets and vector multiplets contributions, respectively. So, in the context of gauge/YBE correspondence, therefore the partition function of the integrable model will be equivalent to the lens partition function of the supersymmetric quiver gauge theory.

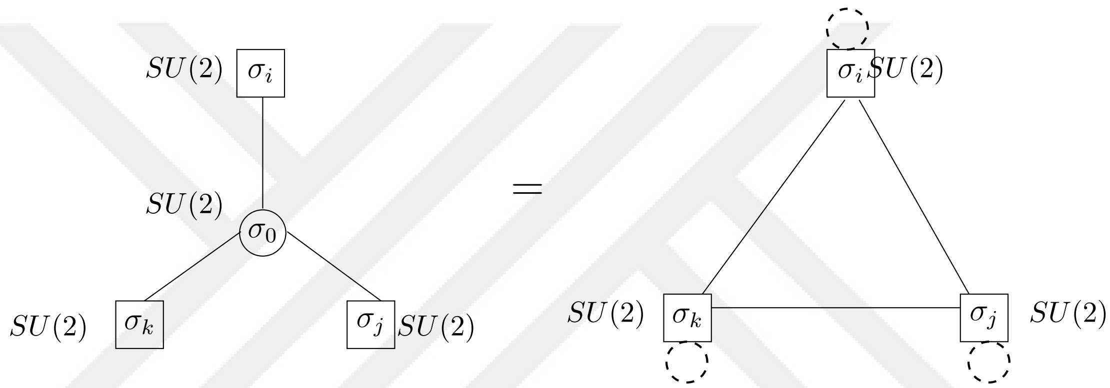

Figure 5.1. Seiberg duality as the star-triangle relation

The zig-zag paths corresponding to the spectral lines are not indicated since they are the same but the explicit expressions detailed discussions can be found in [19].

After introducing the dictionary, we study the STR pictured in Figure 5.1. as the quiver diagram in the supersymmetric gauge theory's side. The gauge symmetry group is placed at the centered circle of the left-hand side of the STR and the flavor symmetry group is broken from $SU(6)$ group to $SU(2) \times SU(2) \times SU(2)$ by adding a certain superpotential. The dashed circles at the right-hand side stand for the remaining mesons (or spin-independent contributions in the STR Equation (4.1)) transform under the adjoint representation of the $SU(2)$ flavor symmetries.

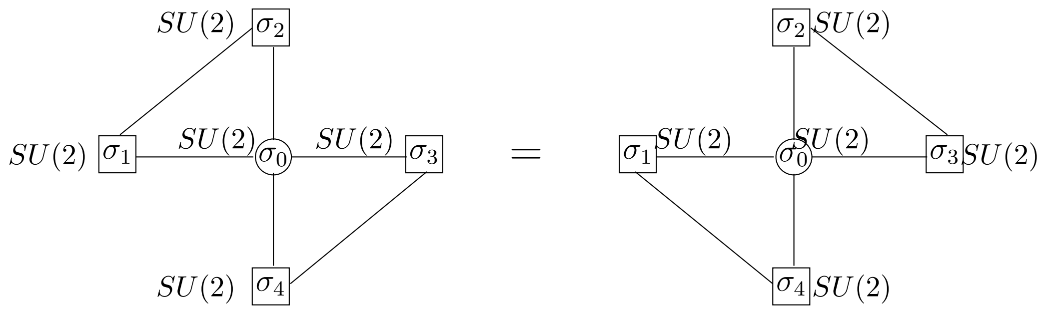

Figure 5.2. Seiberg duality as the star-star relation.

However, in the STR, we have a gauge group symmetry on both sides and they are pictured at the center of the star as shown in Figure 5.3 In this case, the flavor symmetry group $SU(8)$ is broken into $SU(2) \times SU(2) \times SU(2) \times SU(2)$ by adding a superpotential from the supersymmetry perspective.

The similar derivation of the Yang-Baxter equation for IRF models can be studied in the quiver gauge theories [25] and the identified representation of the Figure 4.3. is depicted in the Figure 5.3. Also, the circles and the boxes stand for the gauge symmetry groups of a quiver gauge theory and the flavor symmetry group as pictured in the STR and the SSR for the dualities. As presented as integrability conditions on the statistical mechanics' side, the STR also reveals the identity of quiver gauge theories, see [15, 19] for more details.

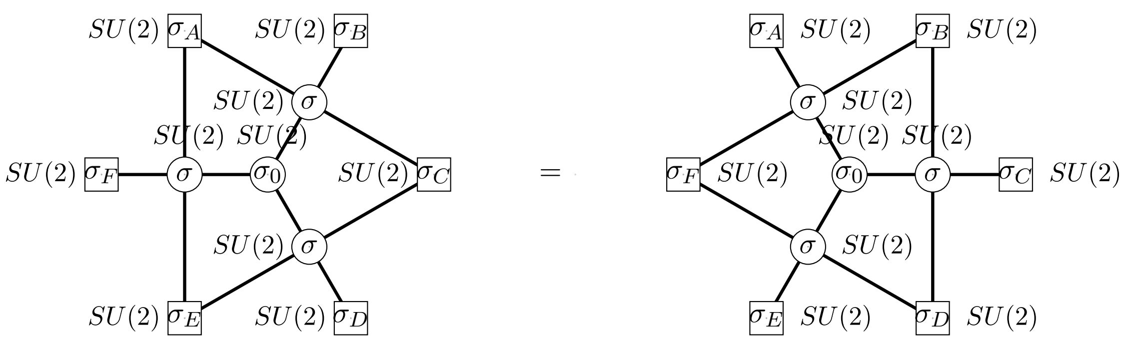

Figure 5.3. The quiver diagram for the YBE for IRF models.

More details and many diverse connections of the gauge/YBE correspondence are studied in the quiver gauge theory perspective, see e.g. [14, 25, 26, 31, 70] and [18, 19] references therein.

6. FUTURE RESEARCH AND REFLECTIONS

6.1. Quantum Groups

It has been discussed that the star-triangle relation of the Faddeev-Volkov model [82] and its a bit of generalization is a tool to study the orthogonality and completeness relations for the Clebsh-Gordan coefficients for the self-dual series of $U_q(sl(2))$ [67, 83] and the quantum-deformed superalgebra $U_q(osp(1|2))$, respectively. However, the question arises for arbitrary parameter r in the solution to the STR with lens (or parafermionic) hyperbolic hypergeometric gamma functions. That is, it would be more promising when the relation between the integral identity of the more general solution of the Faddeev-Volkov model and its corresponding quantum algebra gets connected.

A similar problem can be investigated between the STR and quantum algebra, that is, the STR can play what kind of roles in quantum algebra?

6.2. Knot Theory

The integral form of knot invariant [109–112] is studied in terms of the quantum dilogarithm function and the invariance in three dimensions comes from the quantum dilogarithm function solutions to the pentagon identity related to the Heisenberg double. In the same work of Hikami, it is also shown that the Yang-Baxter equation is constructed by the use of pentagon identity. It seems parallel to develop a general structure by the use of a more general lens hyperbolic hypergeometric gamma function solution of pentagon identities in which the same integral identity is already studied as the STR.

This could be another approach to studying knot invariants via techniques of statistical mechanics, the review lecture is presented in [113] by Wu. The third Reidemester move corresponds to the star-triangle relation but, even for given examples, it could be interesting to see the corresponding move and meaning in knot theory for the star-star relation. The same map between solvable models interacting through edge, vertex, and IRF models and knot invariants can be studied by the use of novel integrable models.

6.3. Higher-Spin Interacting Non-Planar Models from Integrable Models

6.3.1. The Star-Square Relation

One of the identities that helps to construct a non-integrable lattice model is the star-square relation. The star-square relation as shown in Figure 6.1. [114, 115] is studied for different inquiries such as the equivalence between the Ising model and the 8-vertex model [116, 117]. However, the first attempt is presented as solutions to the star-square relation from the three-dimensional duality of supersymmetric gauge theories and it could be generalized to multi-spin solutions to the star-square relation in terms of elliptic hypergeometric gamma functions.

The essential idea is that the existence of the integrable models and the star-square relation maps the integrable lattice spin models in statistical mechanics to another non-planar and non-integrable lattice spin model consists of four-spin interactions $\widetilde{V}_{\sum_{i=1}^4 \alpha_i}(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ and diagonal-neighbor and next-neighbor two-spin interactions $V_{\alpha_i+\alpha_j}(\sigma_i, \sigma_j)$. The star-square relation is the pictures as the four nearest-neighbor interactions on the star side and four nearest-neighbors (dotted lines), two diagonal-neighbors (dashed lines), and one quadruple interaction (broken circle) at the right-hand side as shown in Figure 6.1. and it can be expressed in the following form

$$\sum_{m_0} \int dx_0 S(\sigma_0) \prod_{i=1}^{4} W_{\alpha_i}(\sigma_i, \sigma_0) = \widetilde{V}_{\sum_{i=1}^4 \alpha_i}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \prod_{1 \leq i < j \leq 4} V_{\alpha_i+\alpha_j}(\sigma_i, \sigma_j). \quad (6.1)$$

There could be different types of solutions to the star-square relation since the solution presented in [118] has non-integrable Boltzmann weights as well. Therefore solutions to the star square relation only in terms of Boltzmann weights of an integrable model can be seen as further research.

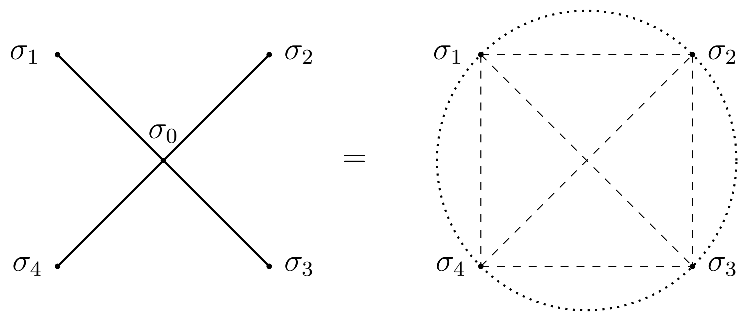

Figure 6.1. The star-square relation.

Together with the inversion relation the star-square relation is one of the main tools for the vertex formulation [119] of the Bazhanov–Baxter model [120] for the proof of the tetrahedron equation [121] which is the intergability condition for corresponding model [122].

6.3.2. The Generalized Star-Triangle Relation

The generalized star-triangle relation [123,124] which reveals a non-planar lattice spin model from an integrable model. The generalized star-triangle relation [115] depicted in Figure 6.2. consists of three nearest neighbor interactions at the left and four interactions which are three nearest neighbors (dashed lines) and one triple interaction (broken circle) at the right.

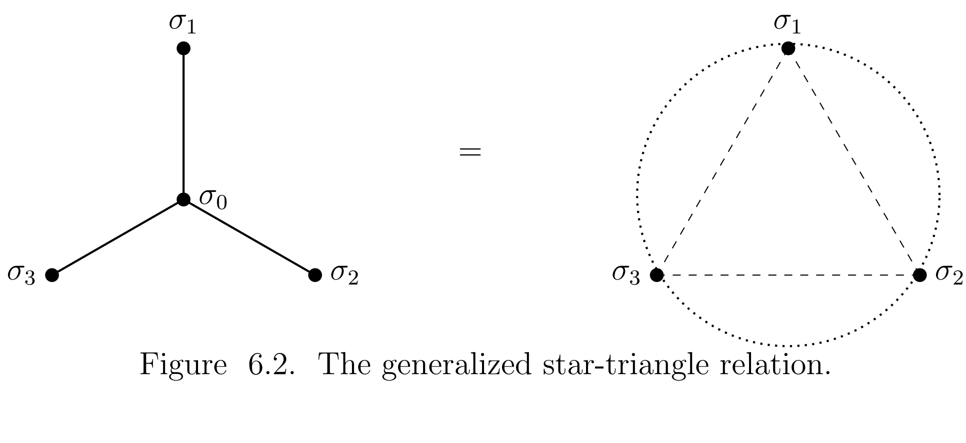

Figure 6.2. The generalized star-triangle relation.

The generalized star-triangle relation can be expressed as

$$\sum_{m_0}\int dx_0 S(\sigma_0)\prod_{i=1}^{3}W_{\alpha_i}(\sigma_i,\sigma_0)=\overline{V}_{\sum_{i=1}^{3}\alpha_i}(\sigma_1,\sigma_2,\sigma_3)\prod_{1\leq i<j\leq 3}V_{\alpha_i+\alpha_j}(\sigma_i,\sigma_j)\prod_{i=1}^{3}\overline{S}(\sigma_i), \tag{6.2}$$

where $\overline{V}_{\sum_{i=1}^{3}\alpha_i}(\sigma_1,\sigma_2,\sigma_3)$ is the Boltzmann weight for the triple interaction and $\overline{S}$ is self-interaction term appearing at the right-hand-side.

6.4. Decorating integrable lattice spin models

We consider decoration -or iteration- transformation [123] depicted in Figure 6.3. for lattice spin models in statistical mechanics. The decoration transformation is a map between a spin system consisting of two outer spins interacting with a central spin to a two-spin system with a single interaction. It is a tool to acquire solutions for decorated models since it relates the partition functions of the solved model and its decorated version up to some coefficient. As a future direction, it could be interesting to work on the solutions to the decoration transformation to decorate integrable lattice spin models obtained via the gauge/YBE correspondence.

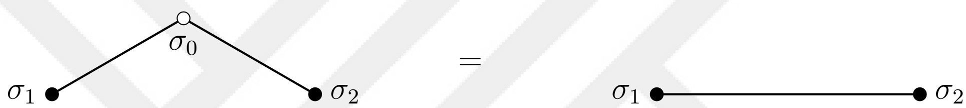

Figure 6.3. The decoration transformation.

In [125], a similar idea is discussed as the triangle identity which describes non-integrable lattice spin models but the decoration transformation itself can be solved by Boltzmann weights of integrable lattice spin models. The transformation has the following mathematical expression

$$\sum_{m_0} \int dx_0 \, S(\sigma_0) W_\alpha(\sigma_1, \sigma_0) W_\beta(\sigma_2, \sigma_0) = \mathcal{R}(\alpha, \beta) W_{\alpha+\beta}(\sigma_1, \sigma_2), \tag{6.3}$$

where $\mathcal{R}(\alpha, \beta)$ and $S(\sigma_0)$ stand for a spin-independent function and self-interaction term, respectively, and α, β are the spectral parameters without the condition crossing parameter. Integration for continuous spin variables x_0 and summation for discrete spin variables m_0 in the left-hand side of Equation (6.3) are evaluated over the center spin $\sigma_0 = (x_0, m_0)$ as shown its elimination as shown in Figure 6.3.

We note that the decoration transformation can be obtained by reducing spins from the star-triangle relation and the reduction is the decrease of the number of flavors from the supersymmetric gauge theories aspect.

7. CONCLUSION

In this thesis, we study the star-triangle relation and the star-star relation as integrability conditions in two-dimensional lattice spin models in statistical mechanics. The integrability conditions are described in how they appear as essential tools during the attempt to solve the system in the transfer matrix method.

It is understood that when a Boltzmann weight satisfies the star-triangle relation or the star-star relation the new edge interacting integrable model is explored. Then it is shown that the edge interaction reveals the novel vertex and IRF models by constructing Bailey pairs and deriving integral identity to prove the Yang-Baxter equation for both, respectively. For the construction of these three types of the lattice spin models and solutions to their integrability conditions are studied in terms of lens hyperbolic hypergeometric gamma functions.

Also, various research fields and associated connections are presented to explore the broad web of the studies when the integrability conditions are taken into the center. Some of the connections such as quantum groups, Painleve equations, knot theory, pentagon identity, Bailey pairs, and supersymmetric gauge theories are discussed.

Recent developments are summarized and referred to the associated original works. Finally, as it is discussed, novel solutions to the integrability conditions in terms of lens elliptic hypergeometric gamma, lens hyperbolic hypergeometric gamma, and rational gamma functions bring up many open directions.

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APPENDIX A: SPECIAL FUNCTIONS AND NOTATIONS

A.1. Lens Hyperbolic Hypergeometric Gamma Function

The main special function that we utilize is the hyperbolic hypergeometric gamma function. It is a variant of Faddeev's non-compact quantum dilogarithm [108, 126]

$$\gamma^{(2)}(z;\omega_1,\omega_2) = e^{\frac{\pi i}{2}B_{2,2}(z;\omega_1,\omega_2)} \frac{(e^{2\pi i \frac{z}{\omega_2}}\tilde{q};\tilde{q})_\infty}{(e^{2\pi i \frac{z}{\omega_1}};q)_\infty}\ , \tag{A.1}$$

where $(z;q)_\infty = \prod_{i=0}^{\infty}(1-zq^i)$ is called the q-Pochhammer symbol, $\tilde{q} = e^{2\pi i\omega_1/\omega_2}$ and $q = e^{-2\pi i\omega_2/\omega_1}$ with the complex variables ω_1, ω_2.

The second Bernoulli polynomial that we work with while calculating gauge symmetry breaking and parameter reduction with the use of asymptotic relations of hyperbolic hypergeometric gamma functions is

$$B_{2,2}(z;\omega_1,\omega_2) = \frac{z^2}{\omega_1\omega_2} - \frac{z}{\omega_1} - \frac{z}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}. \tag{A.2}$$

We introduce one of the several integral representations for the hyperbolic hypergeometric gamma function, see, e.g. [127, 128],

$$\gamma^{(2)}(z;\omega_1,\omega_2) = \exp\left(-\int_0^\infty \frac{dx}{x}\left[\frac{\sinh x(2z-\omega_1-\omega_2)}{2\sinh(x\omega_1)\sinh(x\omega_2)} - \frac{2z-\omega_1-\omega_2}{2x\omega_1\omega_2}\right]\right), \tag{A.3}$$

where $Re(\omega_1), Re(\omega_2) > 0$ and $Re(\omega_1+\omega_2) > Re(z) > 0$.

The hyperbolic hypergeometric gamma function has characteristic properties and some of them are

$$\begin{aligned}
&\text{Symmetry:} && \gamma^{(2)}(z;\omega_1,\omega_2) = \gamma^{(2)}(z;\omega_2,\omega_1)\\
&\text{Reflection:} && \gamma^{(2)}(z;\omega_1,\omega_2)\gamma^{(2)}(\omega_1+\omega_2-z;\omega_1,\omega_2) = 1\\
&\text{Scaling:} && \gamma^{(2)}(z;\omega_1,\omega_2) = \gamma^{(2)}(uz;u\omega_1,u\omega_2)\\
&\text{Conjugation:} && \gamma^{(2)}(z;\omega_1,\omega_2)^* = \gamma^{(2)}(z^*;\omega_2^*,\omega_1^*)\\
&\text{Difference equation:} && \frac{\gamma^{(2)}(z+\omega_1;\omega_1,\omega_2)}{\gamma^{(2)}(z;\omega_1,\omega_2)} = 2\sin\left(\frac{\pi z}{\omega_2}\right),\ (\omega_1 \leftrightarrow \omega_2)
\end{aligned} \tag{A.4}$$

We also introduce the asymptotic behavior of the function to use for the gauge symmetry-breaking

$$\lim_{z\to\infty} e^{\frac{\pi i}{2}B_{2,2}(z;\omega_1,\omega_2)}\gamma^{(2)}(z;\omega_1,\omega_2) = 1 \text{ for } \arg\omega_2 + \pi > \arg z > \arg\omega_1$$
$$\lim_{z\to\infty} e^{-\frac{\pi i}{2}B_{2,2}(z;\omega_1,\omega_2)}\gamma^{(2)}(z;\omega_1,\omega_2) = 1 \text{ for } \arg\omega_2 > \arg z > \arg\omega_1 - \pi\ , \tag{A.5}$$

where $\mathrm{Im}(\frac{\omega_1}{\omega_2}) > 0$.

Another asymptotic relation reduces the hyperbolic gamma function to the Euler gamma function

$$\lim_{\omega_2\to\infty}\left(\frac{\omega_2}{2\pi\omega_1}\right)^{\frac{z}{\omega_2}-\frac{1}{2}}\gamma^{(2)}(z;\omega_1,\omega_2) = \frac{\Gamma(z/\omega_1)}{\sqrt{2\pi}}. \tag{A.6}$$

Since it is simpler to present integral identities, we introduce another version of hyperbolic hypergeometric gamma function [26] also related to the improved double sine function [39]

$$\gamma_h(z,y;\omega_1,\omega_2) = \gamma^{(2)}(-iz - i\omega_1 y; -i\omega_1 r, -i\omega) \times \gamma^{(2)}(-iz - i\omega_2(r-y); -i\omega_2 r, -i\omega), \tag{A.7}$$

where $\omega := \omega_1 + \omega_2$ shorthand notation is used and we will carry out it in the thesis.

The reflection property of the function is

$$\gamma_h(z,y;\omega_1,\omega_2)\gamma_h(\omega - z, r - y;\omega_1,\omega_2) = 1, \tag{A.8}$$

and we will also use the following notation

$$\gamma_h(\pm z, \pm y;\omega_1,\omega_2) = \gamma_h(z,y;\omega_1,\omega_2)\gamma_h(-z,-y;\omega_1,\omega_2)\ . \tag{A.9}$$