DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

STATISTICAL INFERENCES IN STRESS-STRENGTH RELIABILITY MODELS

by
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İZMİR

STATISTICAL INFERENCES IN STRESS-STRENGTH RELIABILITY MODELS

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M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**STATISTICAL INFERENCES IN STRESS-STRENGTH RELIABILITY MODELS**" completed by **ECEM YAZGAN** under supervision of **PROF. DR. SELMA GÜRLER** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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STATISTICAL INFERENCES IN STRESS-STRENGTH RELIABILITY MODELS

ABSTRACT

This study considers the reliability of the stress-strength model in the presence of the fuzziness when the stress and strength variables have weighted exponential distribution with the common shape parameter. It obtains the mean remaining strength and fuzzy mean remaining strength for weighted exponential distribution in order to calculate how long a component can live on average under stress. The comparative simulation results with conventional and fuzzy approaches are presented to observe the effect of parameter changes on the models using the maximum likelihood method for different sample sizes. Also, the developed models are applied to two real data and generate estimation results for two approaches.

Keywords: Stress-Strength model, fuzzy reliability, mean remaining strength, weighted exponential, maximum likelihood

STRES-DAYANIKLILIK GÜVENİLİRLİK MODELLERİNDE İSTATİSTİKSEL ÇIKARSAMALAR

ÖZ

Bu çalışma, stres ve dayanıklılık değişkenlerinin ortak şekil parametresi ile ağırlıklı üstel dağılıma sahip olduğu durumlarda, bulanıklığın varlığında stres-dayanıklılık modelinin güvenilirliğini ele almaktadır. Bir bileşenin ortalama olarak stres altında ne kadar süre yaşayabileceğini hesaplamak için ağırlıklı üstel dağılım için ortalama kalan dayanıklılığı ve bulanık ortalama kalan dayanıklılığı elde eder. Farklı örneklem büyüklükleri için en çok olabilirlik yöntemi kullanılarak modellerde parametre değişikliklerinin etkisini gözlemlemek için geleneksel ve bulanık yaklaşımlarla karşılaştırmalı simülasyon sonuçları sunulmuştur. Ayrıca geliştirilen modeller iki gerçek veriye uygulanmış ve iki yaklaşım için tahmin sonuçları elde edilmiştir.

Anahtar kelimeler: Stres-Dayanıklılık modeli, bulanık güvenilirlik, ortalama geriye kalan dayanıklılık, ağırlıklı üstel dağılım, en çok olabilirlik

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CHAPTER ONE
INTRODUCTION

The stress-strength model is one of the important tasks in medicine, engineering and quality control. It attends to reliability of a component with strength X that is subject to the random stress Y. In the conventional stress-strength analysis, a component, an equipment or a system fail if the stress exceeds the strength, i.e., $X < Y$. In fact, in stress-strength reliability modeling, $R = P(X > Y)$ is considered as the probability that the system has the strength enough to overcome the stress imposed on it. The stress-strength idea was arisen with the work of et al. (1956) and developed by Birnbaum & et al. (1958). The term stress-strength was first introduced by Church & Harris (1970). The problem of estimating the reliability R has been widely discussed by many authors. Awad & Gharraf (1986) considered the estimation of $P(X > Y)$, when X and Y are two independent but not identically distributed Burr random variables. Kundu & Gupta (2005) studied the maximum likelihood (ML) estimator of R, when X and Y are two independent random variables from generalized exponential distribution assuming that the common scale parameter is known. Kundu & Gupta (2006) also estimated R for Weibull distribution. Rezaei et al. (2010) considered the estimation of $P(X > Y)$ for generalized Pareto distribution. For more examples, see Pak et al. (2014), Al-Mutairi et al. (2013), Eryılmaz (2010), Asgharzadeh et al. (2013), Makhdoom (2012), Singh et al. (2015), Mokhlis et al. (2017) and Iranmanesh et al. (2018). For the comprehensive review studies of the stress-strength modeling, see Bhattacharyya & Johnson (1974) and Kotz & Pensky (2003).

Besides, some researchers would like to know that how long the component can still be reliable. When the life of a component is defined as strength to failure, researhers can decide on the useful life by using the information of the remaining strength to failure. Guess et al. (2005) showed that how to use mean residual life (MRL) function for analysis of the tensile strength of medium density fiberboard (MDF) data. Gurler (2013) considered the MRS for the simple stress-strength models and their *k-out-of-n:F*, parallel and series structures. Gurler & Bairamov (2009) studied the MRS for a *k-out-*

of-n:F, parallel and series systems with exchangeable components in stress-strength setup.

The conventional stress-strength reliability is considered as exact values. However, sometimes the survivors can not be reported precisely under some unexpected situations that can be occured by misdetection of failures by a user, by an inattentive records or measurements, etc. In order to overcome this problem, several researchers paid regard to apply the fuzzy set theory which have been introduced by Zadeh (1978). Huang (1995) introduced a methodology for the reliability modeling in the presence of fuzziness and extended conventional reliability into fuzzy reliability by using the notion of probability measure of the fuzzy event. Cai (1996) presented an overview on the on the application of fuzzy methodology by representing the failure in terms of fuzzy sets. For more examples regarding the system/component reliability using fuzzy concept can be found in Huang (1996, 1997). Huang et al. (2006) studied on estimates of the parameters and the reliability function of multi-parameter lifetime distributions representing the lifetime data with fuzzy measures. Li & Kapur (2013) proposed fuzzy reliability measures treating success and failure as a fuzzy state. For other results, related to fuzziness and reliability one can see Huang et al. (2013) and Hussian & Amin (2017) and references therein. As a recent work which is more relevant to our study, Eryilmaz & Tütüncü (2015) introduced the fuzzy stress-strength reliability for a single unit and extended the concept to multicomponent systems by following the findings of Huang (1995). They defined some properties of fuzzy reliability using a fuzzy membership function which is related to the value of $(x - y)$.

Among many lifetime distributions in the literature, exponential distribution has an important role in reliability analysis. Gupta & Kundu (2009) revised the exponential model using the method of Azzalini (1985) and purposed a new class of weighted exponential (WE) ditribution. It can be used as an alternative model for positively skewed data to the other different weighted versions of the exponential distribution such as gamma, Weibull and generalized exponential distributions. Recently, Farahani & Khorram (2014) investigated the classical and Bayesian estimation of parameters of the WE distribution. Dey et al. (2015) considered the reliability characteristics and various properties with several estimation techniques for estimating the unknown parameters of WE distribution. In the context of stress-strength models, Makhdoom (2012) obtained the stress-stregth reliability using two independent WE distributed random variables with common scale parameter. In literature, estimation of the reliability characteristics in a stress-strength setup is usually studied with a conventional model. To the best our knowledge, there has been no previous work handling the reliability and the mean remaining strength (MRS) models in fuzzy concept, where the distributions of stress and strength variables are WE with common shape parameter. In this research, we consider the estimation of stress-strength reliability in the presence of the fuzziness when the random variables X and Y follow WE distribution. In particular, we extend the study of Eryilmaz & Tütüncü (2015). The rest of the thesis is organized as follows: in the following some properties of WE distribution are presented. Section 2 contains a brief explanation of the fuzzy theory and its effect on stress-strength reliability model. Section 3 discusses the conventional and fuzzy approaches for MRS. In Section 4, the ML estimation of the reliability and the fuzzy reliability for WE distribution are derived assuming that the random stress Y and strength X variables are independent with the common shape parameter. In Section 5, the simulation study and a real data example for the methodology are provided. Finally in Section 6, the concluding remarks are given.

1.1 Weighted Exponential Distribution

By applying Azzalini's method to the exponential distribution, a new class of weighted exponential distribution was obtained by Gupta & Kundu (2009). They observed that the shapes of the probability density functions of the WE distribution and the other generalization of the exponential distribution are alike. Also, the concluded that in some circumstances parameter WE distribution is more suitable to use rather than Weibull, gamma or generalized exponential distribution. Recently, Farahani & Khorram (2014) investigated the classical Bayesian estimation of parameters of the WE distribution. Dey et al. (2015) considered the reliability characteristics and various properties with several estimation techniques for estimating the unknown parameters of WE distribution.

1.1.1 Some Properties of WE Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from WE distribution with the shape parameter α and scale parameter λ. Then the cumulative distribution function (cdf) is

$$F_x(x; \alpha, \lambda) = 1 + \frac{e^{-\lambda x}(e^{-\alpha\lambda x} - \alpha - 1)}{\alpha}, \qquad x > 0, \tag{1.1}$$

and the probability density function (pdf) is,

$$f_x(x; \alpha, \lambda) = \frac{\alpha + 1}{\alpha}\lambda e^{-\lambda x}(1 - e^{-\alpha\lambda x}), \qquad x > 0, \tag{1.2}$$

where the shape parameter $\alpha > 0$ and the scale parameter $\lambda > 0$. Also the corresponding hazard rate function is

$$h_x(x; \alpha, \lambda) = \frac{\lambda(\alpha + 1)(1 - e^{-\alpha\lambda x})}{\alpha + 1 - e^{-\alpha\lambda x}}, \qquad x > 0. \tag{1.3}$$

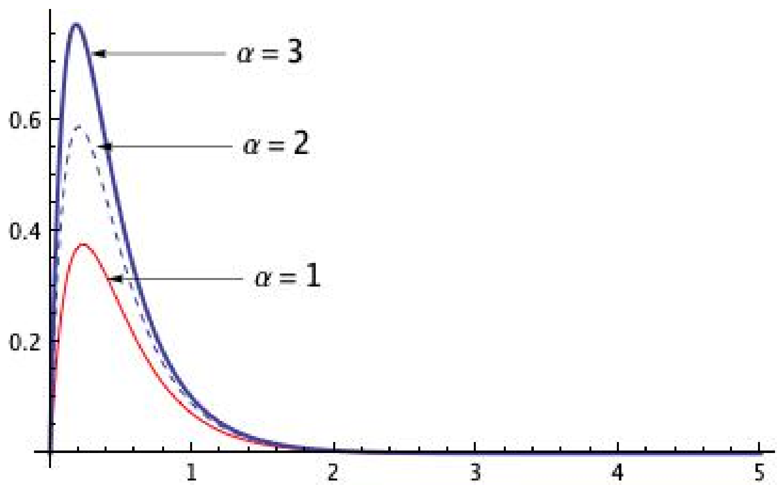


Figure 1.1 The shapes of pdf for WE with different α values, when $\lambda = 3$.

Remark that, as $\alpha \to 0$, $WE(\alpha, \lambda)$ converges to gamma distribution with shape parameter 2 and scale parameter λ and as $\alpha \to \infty$, it converges to exponential distribution with scale parameter λ. The shapes of the pdf of WE distribution for different values of α are given in Figure Figure 1.1. Prekopa (1973) indicated that if the density function is log concave on (a, b), then the reliability function $\bar{F}(x)$ is also log concave on (a, b). Since the hazard rate is defined by $h(x) = -\bar{F}(x)'/\bar{F}(x)$, it must be said that $\bar{F}(x)$ is log concave if and only if hazard rate function is monotone increasing. As shown in the Figure 1.1, since $f_x(x; \alpha, \lambda)$ is always log concave, $h_x(x; \alpha, \lambda)$ will be an increasing function for all $\alpha > 0$, which means that it is convenient for modeling lifetime of a structure.

CHAPTER TWO
RELIABILITY IN A STRESS-STRENGTH MODEL WITH FUZZY CONCEPT

2.1 The Reliability of a Stress-Strength Model

The reliability concept showed up with mathematical meaning after World War 1. By connecting reliability and operational safety of one, two and four engine airplanes, it was measured as the number of accidents per hours. In Word War 2, while trying to develop the V-1 missile, considered theory stated that systems are functioning if and only if all the components of this system are functioning. After that, better metarials and designs were used and tried to achieve a higher system reliability. Still many industries are doing a lot of work about the analysis of risk and reliability. The reliability of a component or a system is its capability to meet its requirements for some specified lifetime. It can be considered as a probabiltiy that an item will not fail to perform its desired function. To show the reliability function, it is good to mention time to failure of an item. By the time to failure of an item, we mean time elapsing from when the item is put into operation until it fails for the first time. Thus, it is natural to interpret the time to failure as a random variable T.

We will assume that the time to failure T is continuously distributed with the pdf $f(t)$ and the cdf $F(t)$ where

$$F(t) = P(T \leq t) = \int_0^t f(u)du. \tag{2.1}$$

Therefore, the reliability function of an item can be defined by;

$$R(t) = 1 - F(t) = 1 - P(T \leq t) = P(T > t) = \int_t^\infty f(u)du. \tag{2.2}$$

In other words, one can say that R(t) is the probability that the item does not fail in the time interval (0,t] which means item survives.

In the seventies of the 20th century, researchers established a connection between the reliability and the stress-strength analysis. In a stress-strength model, a component fails if the applied stress exceeds the component's strength. According to them, values of stress can be computed deterministically given the set of initial values. Church & Harris (1970) provide an example of a missile flight where the initial values of the stress correspond to propulsive force, angles of elevation, atmospheric conditions etc. This can be described as reliability of a component in terms of random variable X representing ”strength” and the Y representing ”stress”. So that the reliability is defined as the probability of the stregth exceeding the stress i.e. $P(X > Y)$. In addition to this, since the nature of stress and strength are not related each other, it was accepted that X and Y are independent variables. Let X and Y are two independent random strength and stress variables with cdf $F_X(x)$ and $F_Y(y)$, respectively. Then the reliability of stress strength model is

$$R = P(Y < X) = \iint_{y<x} dF_X(x)dF_Y(y). \tag{2.3}$$

Example: Consider the case when X and Y are independent exponential random variables with pdfs $f_X(x|\lambda)$ and $f_Y(y|\theta)$. Let $X \sim Exp(\lambda)$ and $Y \sim Exp(\theta)$.
The exponential distribution cdf and pdf; $F(x) = 1 - e^{-\lambda x}$, $f(x) = \lambda e^{-\lambda x}$.
Then we have,

$$R = P(Y < X) = \int_0^\infty \int_y^\infty \lambda.e^{-\lambda x}.\theta.e^{-\theta y}\, dx\, dy \tag{2.4}$$

$$= \int_0^\infty e^{-\lambda y}.\theta.e^{-\theta y}\, dy \tag{2.5}$$

$$= \int_0^\infty \theta.e^{-y(\lambda+\theta)}\, dy \tag{2.6}$$

$$= \frac{\theta}{\lambda+\theta}. \tag{2.7}$$

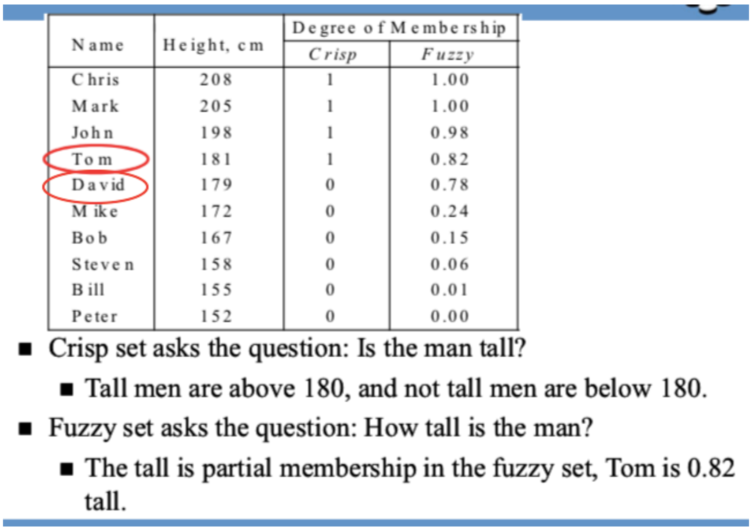
Recently, stress-strength models have received considerable attention. In particular, the reliability of a component which is usually concerned with the probability $P(X > Y)$ in a single stress-strength model is of great interest. Examples of such results and references can be found in Hanagal (1999), Greco & Ventura

(2011), Domma & Giordano (2013). At the same time, researchers have been studying on broader and more realistic models. Nadarajah & Kotz (2006) considered when X and Y follow bivariate exponential distribution, Jose (2011) discussed the applications of Marshall-Olkin Family of distributions in reliability theory. It is also possible to obtain some recent results in the multi-component stress-strength set up, see for example, Eryılmaz (2008), Kantam & Rao (2010) and Rao et al. (2015), Eryılmaz & İşçioğlu (2011), Dey et al. (2017), Kayal et al. (2020). While collecting data, wrong results or rough estimates can be reached and this may have been fuzziness of the environment or negligence of the observes. To avoid such problems, Huang (1995) investigated reliability of a system in the presence of fuzziness atteched to operating time. In system reliability, the event, say A, $A = (T > t)$ is considered where T is time to failure see in Eq(2.1). Therefore, it's characteristic function is

$$C_A(x,t) = \begin{cases} 0, & x \leq t \\ 1, & x > t. \end{cases}$$

It is clear that the charateristic function is equal only two values 0 and 1. In other words if x is in A, the system is functioning and $C_A(x,t) = 1$. If we take the set as a real interval, $[0,1]$, the characteristic function $C_A(x,t)$ can be extended into membership function $\mu_{\widetilde{A}}(x,t)$ where $\widetilde{A}$ is now fuzzy event i.e. $\widetilde{A} = (T \succ t)$. The terms fuzzily bigger than and fuzzily less than will be used for such notations. It may be good to explain that, the main idea of the fuzzy logic were introduced by Zadeh (1996). It is basically, to capture the uncertainty of the human thinking and to express it with appropriate mathematical tools. Unlike computers, the human reasoning is not binary where everything is either true (1) or false (0). Actually this set is an extension of a classical set. In classical set, for example $A = \{x|x > 6\}$ any number either belongs to A (1) or not (0). But fuzzy sets permit membership function valued in the interval $[0,1]$.

As in the following tall man example, by using fuzzy set theory more reliable or more sensitive results are achieved. In reliability context, the only considering is whether X is greater than Y or not. However, if we examine X is how bigger than Y, we can get more reliable informations. To achieve this, Eryilmaz & Tütüncü (2015) followed the



Name	Height, cm	Degree of Membership	
		Crisp	Fuzzy
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00

- Crisp set asks the question: Is the man tall?
  - Tall men are above 180, and not tall men are below 180.
- Fuzzy set asks the question: How tall is the man?
  - The tall is partial membership in the fuzzy set, Tom is 0.82 tall.

Figure 2.1 Crisp and fuzzy sets of short, average and tall men.

idea of Huang (1995) and by using an appropriate membership function they examined the stress strength reliability in the presence of fuzziness and formed the fuzzy stress strength reliability. For the conventional event $(X > Y)$, the reliability function can be written as

$$R = P(Y < X) = \iint_{y<x} C_A(x, y) dF_X(x) dF_Y(y), \tag{2.8}$$

where a characteristic function $C_A : X \to \{0, 1\}$ which can be described as

$$C_A(x, y) = \begin{cases} 0, & x \le y, \\ 1, & x > y, \end{cases} \tag{2.9}$$

and $A = \{x : x > y\}$ is a subset of X, as well. It is obvious that, characteristic fucntion can be resulted in only two values, 0 and 1. It implies that, for example, if the stress Y exceeds the strength X, where $C_A(x, y) = 0$, a system or equipment is corrupted and no longer works.

For the fuzzy event $(X \succ Y)$, Eryilmaz & Tütüncü (2015) have defined fuzzy stress strength reliability as given below

$$\widetilde{R} = R_F = P(Y \prec X) = \iint_{y<x} \mu_{\widetilde{A}}(x, y) dF_X(x) dF_Y(y). \tag{2.10}$$

A fuzzy subset $\widetilde{A}$ of X is considered by extending the characteristic function to a membership function which is $\mu_{\widetilde{A}} : X \rightarrow [0, 1]$. To obtain X is how bigger than Y, this function can be choosen as a function of the difference between values of X and Y. Therefore the corresponding membership function is

$$\mu_{\widetilde{A(y)}}(x) = \begin{cases} 0, & \text{if } y \geq x \\ g(x-y), & \text{if } y < x, \end{cases} \tag{2.11}$$

and g is an increasing function. So that, for $X = x$ and $Y = y$, with an increase in the values of $x - y$, the system becomes more sensitive and gives more reliable results.

Example(Eryilmaz & Tütüncü (2015)): Let $X \sim Exp(\lambda)$ and $Y \sim Exp(\theta)$. The membership function defined in Eq(2.11) can then be defined as;

$$\mu_{A(y)}(x) = \begin{cases} 0, & \text{if } y \geq x \\ 1 - e^{-k(x-y)}, & \text{if } y < x. \end{cases} \tag{2.12}$$

Therefore the Fuzzy Stress-Strength Reliability is given by;

$$R_F = P(Y \prec X) = \int_0^\infty \int_y^\infty \left(1 - e^{-k(x-y)}\right) \lambda e^{-\lambda x} \theta e^{-\theta y} \, dx \, dy \tag{2.13}$$

$$= \left(1 - \frac{\lambda}{\lambda + k}\right) \left(\frac{\theta}{\lambda + \theta}\right) = \left(\frac{k}{\lambda + k}\right) R. \tag{2.14}$$

It is clear that $R_F < R$ and as $k \rightarrow \infty$, the membership function reduces to the characteristic function which means the fuzziness disappears. Therefore, it can be said that fuzzy reliability R_F tends to the traditional reliability R.

2.1.1 Reliability for WE Distribution in a Stress-Strength Setup

Let X and Y be two independent random variables with $WE(\alpha, \lambda)$ and $WE(\alpha, \beta)$, respectively. The cdf of weighted exponential distribution can be found in Eq(1.1).

Then the conventional reliability can be written as

$$\begin{aligned} R &= P(Y < X) \\ &= \int_0^\infty \int_y^\infty \frac{\alpha+1}{\alpha}\lambda \mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\alpha\lambda x})\frac{\alpha+1}{\alpha}\beta \mathrm{e}^{-\beta y}(1-\mathrm{e}^{-\alpha\beta y})\, dx\, dy \\ &= \frac{\beta^2\left((1+\alpha)\beta+(3+\alpha(3+\alpha))\lambda\right)}{(\beta+\lambda)(\beta+\alpha\beta+\lambda)(\beta+\lambda+\alpha\lambda)}. \end{aligned} \tag{2.15}$$

If the membership function defined in Eq.(2.12) is used for constant $k > 0$, the fuzzy reliability is given by

$$\begin{aligned} R_F &= P(Y \prec X) \\ &= \int_0^\infty \int_y^\infty (1-e^{-k(x-y)})\frac{\alpha+1}{\alpha}\lambda \mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\alpha\lambda x}) \\ &\times \frac{\alpha+1}{\alpha}\beta \mathrm{e}^{-\beta y}(1-\mathrm{e}^{-\alpha\beta y})\, dx\, dy \\ &= k\beta^2 \\ &\times \frac{[k(1+\alpha)\beta+k(3+\alpha(3+\alpha))\lambda+(2+\alpha)\lambda((1+\alpha)\beta+(2+\alpha(2+\alpha))\lambda)]}{(k+\lambda)(k+\lambda+\alpha\lambda)(\beta+\lambda)(\beta+\alpha\beta+\lambda)(\beta+\lambda+\alpha\lambda)} \\ &= \frac{k^2}{(k+\lambda)(k+\lambda+\alpha\lambda)}R+S, \end{aligned} \tag{2.16}$$

where R is conventional reliability and

$$S = \frac{k\beta^2(2+\alpha)\lambda\left((1+\alpha)\beta+(2+\alpha(2+\alpha))\lambda\right)}{(k+\lambda)(k+\lambda+\alpha\lambda)(\beta+\lambda)(\beta+\alpha\beta+\lambda)(\beta+\lambda+\alpha\lambda)}. \tag{2.17}$$

It can be seen that when the k value increases fuzzy reliability R_F tends to the conventional reliability R.

CHAPTER THREE
MEAN REMAINING STRENGTH WITH FUZZY CONCEPT

3.1 Mean Remaining Strength in a Stress-Strength Setup

As it was already pointed out, the stress-strength models may interest the probability of the strength exceeding the stress. In addition to that, the remaining lifetime of a component can be examined. The life of a component can be defined as the strength to failure. Gurler (2013) considered the mean remaining strength, MRS, for the systems that consist of n independent components under some structures when the stress and the strength are independent random variables. For some recent studies on MRS, see Bairamov et al. (2015), Pakdaman et al. (2019) and Kızılaslan (2019). The MRS of a component under the stress Y, denoted by ϕ^Y, is the expected value of the remaining strength which is given as below

$$\phi^Y = E(X - Y|X > Y). \tag{3.1}$$

If we assume that a component has survived up to stress Y which is independent of the strength X, the reliability function of the conditonal random variable $X - Y|X > Y$ can be expressed as

$$P(X - Y > x|X > Y) = \frac{P(X > Y + x)}{P(X > Y)} = \frac{\int_0^\infty \int_{x+y}^\infty dF_X(x)dF_Y(y)}{\int_0^\infty \int_y^\infty dF_X(x)dF_Y(y)}. \tag{3.2}$$

Therefore, conditioning on $Y = y$, the mean remaining strenght of X, i.e., ϕ^Y, is given by

$$\phi^Y = \int_0^\infty \left[\frac{\int_0^\infty \int_{x+y}^\infty dF_X(x)dF_Y(y)}{\int_0^\infty \int_y^\infty dF_X(x)dF_Y(y)} \right] dx. \tag{3.3}$$

Similarly, we may also compute the fuzzy MRS which can be denoted as ϕ_F^Y. By using membership function defined in Eq(2.11), the fuzzy reliability function of a conditional random variable is

$$P(X - Y \succ x | X \succ Y) = \frac{P(X \succ Y + x)}{P(X \succ Y)} = \frac{\int_0^\infty \int_{x+y}^\infty g(x-y) dF_X(x) dF_Y(y)}{\int_0^\infty \int_y^\infty g(x-y) dF_X(x) dF_Y(y)}. \tag{3.4}$$

Therefore, the conditioning on $Y = y$, the fuzzy MRS of X, i.e., ϕ_F^Y, is given by

$$\phi_F^Y = \int_0^\infty \left[\frac{\int_0^\infty \int_{x+y}^\infty g(x-y) dF_Y(y)}{\int_0^\infty \int_y^\infty g(x-y) dF_Y(y)} \right] dx. \tag{3.5}$$

3.1.1 MRS for WE Distribution in a Stress-Strength Setup

Let X and Y be two independent random variables with $WE(\alpha, \lambda)$ and $WE(\alpha, \beta)$, respectively. The cdf of weighted exponential distribution can be found in Eq(1.1). Then MRS for weighted exponential distribution can be written as follows

$$\phi^Y = \frac{(2+\alpha)((1+\alpha)\beta + (2+\alpha(2+\alpha))\lambda)}{(1+\alpha)\lambda((1+\alpha)\beta + (3+\alpha(3+\alpha))\lambda)}. \tag{3.6}$$

We can also compute fuzzy mean remaining strength which can be denoted by ϕ_F^Y. By using membership function defined in Eq(2.11), the fuzzy reliability function of a conditional random variable is

$$P(X - Y \succ x | X \succ Y) = \frac{P(X \succ Y + x)}{P(X \succ Y)} = \frac{\int_0^\infty \int_{x+y}^\infty g(x-y) dF_X(x) dF_Y(y)}{\int_0^\infty \int_y^\infty g(x-y) dF_X(x) dF_Y(y)}. \tag{3.7}$$

Therefore, conditioning on $Y = y$, the fuzzy mean remaining strenght of X, i.e., ϕ_F^Y, is given by

$$\phi_F^Y = \int_0^\infty \left[\frac{\int_0^\infty \int_{x+y}^\infty g(x-y) dF_Y(y)}{\int_0^\infty \int_y^\infty g(x-y) dF_Y(y)} \right] dx. \tag{3.8}$$

If we choose membership function as in the Eq(2.12), the fuzzy MRS for weighted exponential distribution can be written as follows

$$\phi_F^Y = \frac{1}{(k\alpha(k(1+\alpha)\beta + k(3+\alpha(3+\alpha))\lambda + (2+\alpha)\lambda((1+\alpha)\beta + (2+\alpha(2+\alpha))\lambda)}\cdot$$
$$\left[\frac{(1+\alpha)\lambda(k+\lambda)(\beta+\alpha\beta+\lambda)}{(k+\lambda+\alpha\lambda)} - \frac{(1+\alpha)^2\lambda(k+\lambda+\alpha\lambda)(\beta+\lambda+\alpha\lambda)}{(k+\lambda)} + \right.$$
$$\left.\frac{(k+\lambda)(k+\lambda+\alpha\lambda)((1+\alpha)^3(\beta+\lambda+\alpha\lambda) - (\beta+\alpha\beta+\lambda))(\beta+\alpha\beta+\lambda)}{\lambda(\alpha+1)}\right] \quad (3.9)$$

CHAPTER FOUR

ESTIMATION OF RELIABILITY AND MRS WITH WE DISTRIBUTION

The maximum likelihood is the commonly used procedure to estimate reliability. Comprehensive descriptions about MLE can be found in researches of Casella & Berger (2021), Lehmann & Casella (1998), Scott & Nowak (2004).

To compute MLE of R and R_F, we need to have the MLEs of α, λ, β. Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_m)$ be two independent random samples from $WE(\alpha, \lambda)$ and $WE(\alpha, \beta)$, respectively. Therefore, the log-likelihood function of the observed sample is

$$
\begin{aligned}
l(\alpha, \beta, \lambda) \;=\; & (n+m)\log\left(\frac{\alpha+1}{\alpha}\right) + n\log\lambda + m\log\beta - \lambda\sum_{i=1}^{n} x_i - \beta\sum_{j=1}^{m} y_j \\
& + \sum_{i=1}^{n}\log(1-\mathrm{e}^{-\alpha\lambda x_i}) + \sum_{j=1}^{m}\log(1-\mathrm{e}^{-\alpha\beta y_j}). \qquad (4.1)
\end{aligned}
$$

The MLE of α, λ and β; say $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\beta}$, can be obtained as the solutions of

$$\frac{\partial l}{\partial \alpha} = -\frac{m+n}{\alpha+\alpha^2} + \sum_{i=1}^{n}\frac{\mathrm{e}^{-\alpha\lambda x_i}\lambda x_i}{1-\mathrm{e}^{-\alpha\lambda x_i}} + \sum_{i=1}^{n}\frac{\mathrm{e}^{-\alpha\beta y_j}\beta y_j}{1-\mathrm{e}^{-\alpha\beta y_j}} = 0, \qquad (4.2)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{\mathrm{e}^{-\alpha\lambda x_i}\alpha x_i}{1-\mathrm{e}^{-\alpha\lambda x_i}} = 0, \qquad (4.3)$$

$$\frac{\partial l}{\partial \beta} = \frac{m}{\beta} - \sum_{j=1}^{m} y_j + \sum_{j=1}^{m}\frac{\mathrm{e}^{-\alpha\beta y_j}\alpha y_j}{1-\mathrm{e}^{-\alpha\beta y_j}} = 0. \qquad (4.4)$$

Since $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\beta}$ are fixed point solutions of these non-linear equations, they can be obtained by applying an iterative method. A simple function *nlm* from the statistical software R can be used to find the solution of these equations. Therefore due to the invariance property of the MLEs, the MLE of R and R_F are computed to be

$$\hat{R} = \frac{\hat{\beta}^2\left((1+\hat{\alpha})\hat{\beta} + (3+\hat{\alpha}(3+\hat{\alpha}))\hat{\lambda}\right)}{(\hat{\beta}+\hat{\lambda})(\hat{\beta}+\hat{\alpha}\hat{\beta}+\hat{\lambda})(\hat{\beta}+\hat{\lambda}+\hat{\alpha}\hat{\lambda})}, \qquad (4.5)$$

$$\hat{R}_F = \frac{k^2}{(k+\hat{\lambda})(k+\hat{\lambda}+\hat{\alpha}\hat{\lambda})}\hat{R} + \hat{S}. \qquad (4.6)$$

Also based on the invariance property of MLEs, by substituting $\hat{\alpha}, \hat{\lambda}, \hat{\beta}$ in the equations Eq(4.2), the MLE of ϕ^Y, ϕ_F^Y, denoted as $\hat{\phi}^Y$, $\hat{\phi}_F^Y$, can be computed.

CHAPTER FIVE
SIMULATION STUDY AND REAL DATA EXAMPLE

5.1 Simulation Study

In this section, we conduct a Monte Carlo simulation study to obtain the numerical solutions for the ML estimates of conventional and fuzzy reliability with WE distribution using the R-package called ’nleqslv’. Also, the values of MRS and fuzzy MRS are investigated. The bias and mean squared error (MSE) values are obtained using the equations given respectively as below:

$$bias(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta), \tag{5.1}$$

$$MSE(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2, \tag{5.2}$$

where, $\hat{\theta}$ is the estimator of θ parameter and $\hat{\theta}_i$ represents the estimation of parameter in i^{th} repetition for $i = 1, \ldots, N$. The study is designed with $N = 10,000$ repetitions for different values of α and k where k is the constant in the membership function. Table 5.1 shows the mean values of the estimations of conventional reliability, fuzzy reliability, MRS, and fuzzy MRS under $X \sim WE(\alpha, 0.25)$ and $Y \sim WE(\alpha, 0.55)$. It is observed that when the k value increases, fuzzy reliability and fuzzy MRS values tend to the conventional values which means that the uncertainity disappears. Also, the estimated values of the MRS in the fuzzy model is greater than the value in the conventional model. Table 5.2 gives the bias and the MSE values for the conventional reliability, fuzzy reliability, MRS, and fuzzy MRS when the uncertainity is very high, i.e. $k = 1$. As expected, the values of bias and MSE are decreasing when the sample sizes (n, m) are increasing for each parameter. The smallest bias values for all calculations and the smallest MSE values for fuzzy reliability, MRS, and fuzzy MRS are obtained when $\alpha = 5$. Also, it is observed that the smallest MSE values for conventional reliability are obtained when $\alpha = 1$.

Table 5.1 The estimated values of R, R_F, ϕ^Y and ϕ_F^Y for $(n, m) = (20, 20)$

$(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\lambda})$	**k**	$\widehat{\mathbf{R}}$	$\widehat{\mathbf{R}}_{\mathbf{F}}$	$\widehat{\boldsymbol{\phi}}^{\mathbf{Y}}$	$\widehat{\boldsymbol{\phi}}_F^{\mathbf{Y}}$
(0.2, 0.55, 0.25)	1	0.7422	0.6529	5.8601	6.5173
	10	0.7422	0.7333	5.8601	5.9279
	100	0.7422	0.7413	5.8601	5.8668
(1, 0.55, 0.25)	1	0.7441	0.6400	5.1855	5.8628
	10	0.7441	0.7337	5.1855	5.2565
	100	0.7441	0.7431	5.1855	5.1925
(5, 0.55, 0.25)	1	0.7242	0.5929	4.1776	4.8912
	10	0.7242	0.7101	4.1776	4.2567
	100	0.7242	0.7228	4.1776	4.1854

Table 5.2 Bias and MSE values for $\hat{R}$, $\hat{R}_F$, $\hat{\phi}^Y$ and $\hat{\phi}_F^Y$, when $k = 1$

$\alpha = 0.2, \beta = 0.55, \lambda = 0.25$	**Bias**				**MSE**			
(n,m)	$\widehat{\mathbf{R}}$	$\widehat{\mathbf{R}}_F$	$\hat{\boldsymbol{\phi}}^{\mathbf{Y}}$	$\hat{\boldsymbol{\phi}}_F^{\mathbf{Y}}$	$\widehat{\mathbf{R}}$	$\widehat{\mathbf{R}}_F$	$\hat{\boldsymbol{\phi}}^{\mathbf{Y}}$	$\hat{\boldsymbol{\phi}}_F^{\mathbf{Y}}$
(5,5)	-0.0592	-0.0746	-0.5899	-0.5308	0.0238	0.0296	4.3491	4.2626
(10,10)	-0.0375	-0.0458	-0.2428	-0.1895	0.0108	0.0134	2.0703	2.0354
(15,15)	-0.0298	-0.0357	-0.1489	-0.1032	0.0072	0.0089	1.3958	1.3739
(20,20)	-0.0253	-0.0295	-0.0720	-0.0313	0.0053	0.0065	1.0371	1.0227
(60,60)	-0.0127	-0.0140	0.0264	0.0495	0.0016	0.0019	0.3575	0.3556
(120,120)	-0.0078	-0.0087	0.0262	0.0421	0.0008	0.0009	0.1840	0.1832
$\alpha = 1, \beta = 0.55, \lambda = 0.25$	**Bias**				**MSE**			
(5,5)	-0.0382	-0.1451	0.3044	1.0027	0.0167	0.0394	3.6500	4.6098
(10,10)	-0.0254	-0.1297	0.2950	0.9825	0.0087	0.0267	1.8695	2.7837
(15,15)	-0.0195	-0.1232	0.2511	0.9279	0.0058	0.0218	1.2399	2.0678
(20,20)	-0.0161	-0.0162	0.2381	0.2823	0.0045	0.0055	0.9460	0.9921
(60,60)	-0.0072	-0.0073	0.1022	0.1219	0.0015	0.0018	0.3105	0.3255
(120,120)	-0.0034	-0.0033	0.0486	0.0572	0.0007	0.0009	0.1540	0.1605
$\alpha = 5, \beta = 0.55, \lambda = 0.25$	**Bias**				**MSE**			
(5,5)	-0.0218	-0.0208	0.2340	0.2367	0.0177	0.0201	3.0230	3.1655
(10,10)	-0.0109	-0.0106	0.0967	0.0928	0.0097	0.0112	1.5123	1.6029
(15,15)	-0.0071	-0.0069	0.0557	0.0488	0.0066	0.0079	1.0494	1.1221
(20,20)	-0.0061	-0.0060	0.0174	0.0084	0.0053	0.0062	0.7926	0.8532
(60,60)	-0.0042	-0.0047	-0.0091	-0.0062	0.0019	0.0022	0.2759	0.3050
(120,120)	-0.0033	-0.0038	-0.0046	-0.0061	0.0009	0.0010	0.1411	0.1574

5.2 Real Data

In this section, we present two real data examples to illustrate the proposed models for WE distribution.

5.2.1 Example 1

We used the data set originally introduced by Xia et al. (2009) to illustrate the mentioned models by fitting the WE distribution. The original data consists of the breaking strengths of jute fibers at four different gauge lengths. We analysed the breaking strengths of jute fibres with 10 mm and 20 mm considering X and Y, respectively. Also, Saraçoğlu et al. (2012), Hassan & Alohali (2018) used the same pair of the data set to estimate of R for exponential distribution. The data sets are given in below:

Data $(\mathbf{X})$: 693.73, 704.66, 323.83, 778.17, 123.06, 637.66, 383.43, 151.48, 108.94, 50.16, 671.49, 183.16, 257.44, 727.23, 291.27, 101.15, 376.42, 163.40, 141.38, 700.74, 262.90, 353.24, 422.11, 43.93, 590.48, 212.13, 303.90, 506.60, 530.55, 177.25.

Data $(\mathbf{Y})$: 71.46, 419.02, 284.64, 585.57, 456.60, 113.85, 187.85, 688.16, 662.66, 45.58, 578.62, 756.70, 594.29, 166.49, 99.72, 707.36, 765.14, 187.13, 145.96, 350.70, 547.44, 116.99, 375.81, 581.60, 119.86, 48.01, 200.16, 36.75, 244.53, 83.55.

According to the Kolmogrov-Smirnov goodness of fit test results for the data sets X and Y, they follow the WE distribution where the p values are 0.9096 and 0.4014, respectively. Therefore, we can say that given data follow WE distribution.The ML estimates of the parameters are found for X as $\hat{\alpha} = 2.9382, \hat{\lambda} = 0.0034$, ($X \sim WE(2.9382, 0.0034)$), and for Y as $\hat{\alpha} = 2.9382, \hat{\beta} = 0.0037$, ($Y \sim WE(2.9382, 0.0037)$). Based on the ML estimates of parameters, the estimated values of reliability and MRS for both conventional and fuzzy approaches are reported in Table 5.3.

Table 5.3 The estimated values of reliability and MRS for both conventional and fuzzy approaches ($k = 1$)

	$\hat{\mathbf{R}}$	$\hat{\phi}^{\mathbf{Y}}$
Conventional Approach	0.5327	313.5988
Fuzzy Approach	0.5313	314.4258

5.2.2 Example 2

As a second example, we used the data set originally reported by Lawless (2011) to practice how proposed models can be used in real life. The original data sets consist of failure times, in minutes, for two types of electrical insulation in an experiment in which the insulation is subjected to a continuously increasing voltage stress. Also, Mokhlis et al. (2017) used the same data to estimate of R for negative exponential distribution. The data sets are given in below

Data $(\mathbf{X})$: 219.3, 79.4, 86.0, 150.2, 21.7, 18.5, 121.9, 40.5, 147.1, 35.1, 42.3, 48.7.

Data $(\mathbf{Y})$: 21.8, 70.7, 24.4, 138.6, 151.9, 75.3, 12.3, 95.5, 98.1, 43.2, 28.6, 46.9.

Table 5.4 The estimated values of reliability and MRS for both conventional and fuzzy approaches ($k = 1$)

	$\hat{\mathbf{R}}$	$\hat{\phi}^{\mathbf{Y}}$
Conventional Approach	0.5836	64.4165
Fuzzy Approach	0.5770	65.1344

According to the Kolmogrov-Smirnov goodness of fit test results for the data sets X and Y, p values are found that 0.7783 and 0.9817, respectively. It is clear from the results that given data follow WE distribution.The ML estimates of the parameters are found for X as $\hat{\alpha} = 0.00002, \hat{\lambda} = 0.0237$, ($X \sim WE(0.00002, 0.0237)$), and for Y as $\hat{\alpha} = 0.00002, \hat{\beta} = 0.0297$, ($Y \sim WE(0.00002, 0.0297)$). Based on the ML estimates of parameters, the estimated values of reliability and MRS for both conventional and fuzzy approaches are reported in Table 5.4.

CHAPTER SIX
CONCLUSION

This research presents the stress-strength reliability and mean remaining strength under conventional and fuzzy approaches when the stress and strength random variables are independent and follow WE distribution with the common shape parameter. Based on Monte Carlo simulation study, the reliability and MRS are estimated for different k values. Moreover, bias and MSE values are obtained for different sample sizes and different shape parameters. In addition, the proposed models are illustrated with two real data sets which are followed by WE distribution. It is observed that the models with fuzzy approach are sensitive and the estimated fuzzy MRS is found higher than the conventional case. Also, it can be said that different membership functions for fuzzy case will give different measures for the fuzzy reliability and MRS.

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