EFFECTS OF NON-ABELIAN MAGNETIC FIELDS ON PAIR PRODUCTION IN FLAT AND CURVED SPACES

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ABSTRACT

EFFECTS OF NON-ABELIAN MAGNETIC FIELDS ON PAIR PRODUCTION IN FLAT AND CURVED SPACES

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Our objective in this thesis is to compute the pair production rates for both bosons and fermions under the influence of non-abelian gauge fields on the manifolds $\mathbb{R}^{3,1} \equiv \mathbb{R}^2 \times \mathbb{R}^{1,1}$ and $S^2 \times \mathbb{R}^{1,1}$. We will compare the pair production rates of the spherical cases with the flat ones, and also compare the non-abelian cases with the abelian ones to see effects of both curvature and non-abelian field strength on the pair production. We first review the pair production process, i.e. the so-called Schwinger effect using the path integral formalism for bosonic spin-0 i.e. scalar fields and for fermions, spin-$1/2$ i.e. spinor fields, and also subsequently review the recent results obtained in the literature on $\mathbb{R}^{3,1}$ and $S^2 \times \mathbb{R}^{1,1}$ with abelian orthogonal uniform electric and magnetic fields. We then move on to generalize these results by the inclusion of a uniform non-abelian magnetic field due to an external $SU(2)$ gauge field. In doing so, we find the opportunity to compare the pair production rates on $\mathbb{R}^{3,1}$ and $S^2 \times \mathbb{R}^{1,1}$ with non-abelian field switched on, and also compare its influence to previously obtained results without the non-abelian field. Novel effects of the presence of the uniform non-abelian magnetic field together with the effects of constant positive curvature of the S^2-submanifold are emphasized.

Keywords: Non-abelian Gauge Theory, Pair Production, Schwinger Effect, Landau Problem, Dirac Operator on Curved Spaces

ÖZ

DÜZ VE EĞRİ UZAYLARDA DEĞİŞMELİ OLMAYAN ALANLARIN ÇİFT ÜRETİMİ ÜZERİNDEKİ ETKİSİ

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Bu tezin amacı $\mathbb{R}^{3,1} \equiv \mathbb{R}^2 \times \mathbb{R}^{1,1}$ ve $S^2 \times \mathbb{R}^{1,1}$ tipi manifoldların ve değişmeli olmayan alanların, bozon ve fermiyonlardaki çift üretimi üzerindeki etksini araştırmaktır. Hem küresel ve düz geometrilerdeki hem de değişmeli ve değişmeli olmayan alanların etkisi altındaki çift üretimi oranlanarak geometri ve değişmeli olmayan alanların çift üretimi üzerindeki etkisi görülecektir. İlk olarak Feynman'ın yol integrasyonu metodu kullanılarak bozonik (spin-0) ve fermiyonik (spin-$1/2$) parçacıklarındaki çift üretimi olayı (Schwinger mekanizması) yeniden değerlendirip, daha sonra literatüre yeni katılmış olan $\mathbb{R}^{3,1}$ ve $S^2 \times \mathbb{R}^{1,1}$ tipi manifoldlar üzerinde değişmeli ve değişmeli olmayan manyetik alanlar bulunduğu durumlardaki çift üretimi değerlendirilecektir. Ardından, bu sonuçlar $SU(2)$ ayar alanından oluşan değişmeli olmayan manyetik alanlar için genellenecektir. Bu genelleme yapılırken, $\mathbb{R}^{3,1}$ ve $S^2 \times \mathbb{R}^{1,1}$ üzerindeki çift üretimleri değişmeli olmayan maynetik alan etkisi varken karşılaştırma ve aynı zamanda değişmeli olmayan alanların olduğu durumla, literatürdeki değişmeli olmayan alanların olmadığı durumu karşılaştırma fırsatı elde edilecektir. Tekdüze değişmeli olmayan alanların S^2 altmanifoldu ile birlikte getirdiği yeni etkiler üzerinde de durulacaktır.

Anahtar Kelimeler: Değişmeli Olmayan Ayar Teorisi, Çift Üretimi, Schwinger Efekti, Landau Problemi, Eğik Uzayda Dirac Operatörü

To my family

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TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

$\eta_{\mu\nu} = (-1, 1, 1, 1)$	Is our sign convention.
2D	2 Dimensional
3D	3 Dimensional
p.p.p	Pair production probability
IRR	Irreducible representation

CHAPTER 1

INTRODUCTION

Pair production under constant, strong electric fields, otherwise known as Schwinger effect has become important area of research since the original calculation done by J. Schwinger [1]. One important feature of Schwinger effect is that the pair production exists for any electric field value, however for small electric field values pair production rate is greatly diminished since the pair production probability is proportional to $e^{-\pi m^2/E}$. Because of this reason, up until now, Schwinger effect is not observed in experiments, since the electric field strength required for the observations are larger than our current technological means. This effect is also examined for time dependent electric fields [2, 3, 4, 5]. Under the influence of these types of electric fields, pair production probability increases but not sufficiently so that it allows a direct experimental verification. Schwinger mechanism is also investigated for inhomogeneous fields [6, 7], but currently no experimental verification can be done in these setups either. Dual effect of the creation of monopole anti-monopole pairs under the influence of magnetic fields is discussed in the early literature [8]. There have also been the discussion of pair production in AdS_2 space [9] as well as in the deSitter Space dS_2 [10, 11]

Chapter 2 of this thesis will serve as a review, in which we will discuss the one-loop effective action for scalar and spinor fields under the influence of background gauge field. We will calculate the vacuum to vacuum transition amplitude, and subsequently we will derive the probability for vacuum to vacuum transition. Subtracting this probability from the total probability 1 gives us the probability for pair production. We will present the calculations of the one-loop effective action for scalar and spinor fields under the influence of a constant electric field and reproduce Schwinger's original result.

Subsequently, we review the effects of uniform abelian magnetic field and constant positive curvature on pair production following the recent article [12]. Here we consider manifolds of the type $\mathbb{R}^2 \times \mathbb{R}^{1,1} \equiv \mathbb{R}^{3,1}$ and $S^2 \times \mathbb{R}^{1,1}$. In addition to a uniform electric field in the spatial direction in the subspace $\mathbb{R}^{1,1}$ a uniform magnetic field will be present on $\mathbb{R}^2$ and S^2, the magnetic field in the latter will be provided by a Dirac monopole. Evaluating the effective action on $\mathbb{R}^4$ and $\mathbb{R}^2 \times S^2$, we Wick rotate $\mathbb{R}^2 \to \mathbb{R}^{1,1}$ to obtain the pair production rates.

In Chapter 3, we shift our attention to the effects of non-abelian magnetic fields on pair production on flat surfaces. Here our approach will be similar to the case with purely the abelian field. Essentially we will consider a case with uniform magnetic fields with both abelian and non-abelian parts. We will first handle the calculation for the scalar fields. This task will be facilitated by using the result of the spectrum of the gauged Laplacian provided in [13], we will nevertheless reproduce the results for completeness. Calculating the pair production rates gives us novel physics and allows us to comment on the effects of the non-abelian magnetic field on pair production. These results will be discussed in detail in Section 3.1.2. This will be done in two ways. First, at different values of the non-abelian magnetic field strength, we will plot and inspect the profiles of a function which is proportional to the effective action; Secondly, we will form and examine the profile of a function which allows us to compare the situation with non-abelian field present to that in the absence of the non-abelian magnetic field.

In order to proceed to discuss the pair production for spinor fields with non-abelian field switched on, we need to first address the problem of finding the spectrum of the gauged Dirac operator, $\not{D}^2$, on this setting. We take up and handle this task in Section 3.2.1 and determine the spectrum of $\not{D}^2$ using methods similar to those used in the discussion of the Jaynes-Cummings model [14]. To the best of our knowledge these results are all new and provide a novel contribution to the literature. Here we also determine the eigenstates of the operator $\not{D}^2$, which will be implicitly used in the calculation of the pair production rates. Employing these results, we proceed to compute the pair production rates for spinor fields and illustrate our results by considering the profile of function constructed in the same spirit as in the case of scalar fields. These results are presented in Section 3.2.2.

In Chapter 4, we focus on the problem of computation of pair production rates on the manifold $S^2 \times \mathbb{R}^{1,1}$. Here in addition to the uniform electric field in the spatial direction of the subspace $\mathbb{R}^{1,1}$, and a uniform magnetic field composed of an abelian part due to a Dirac monopole placed at the center of S^2 and a non-abelian part due to a $SU(2)$ gauge field is present. In order to evaluate the pair production rates for scalar and spinor fields on this manifold, we need the spectrum of both the gauged Laplacian, D^2, and that of the gauged Dirac operator, $\not{D}^2$, respectively. Incidentally the spectrum of D^2 is determined in a previous work [13] within the context of generalizing the Landau problem on S^2 by switching on a non-abelian uniform gauge field. We review the result obtained in [13] in detail and adopt it to solve the gauged Dirac operator on S^2 in this context. This result is also new to the best of our knowledge.

With the spectrum of D^2 and $\not{D}^2$ available to us, we follow the approach given in [12] and reviewed in Chapter 2, to compute the pair production rates on $\mathbb{R}^{1,1} \times S^2$. For this purpose, we start our considerations on the Euclidean manifold $\mathbb{R}^2 \times S^2$ with another magnetic field perpendicular to the $\mathbb{R}^2$ plane, and Wick rotate $\mathbb{R}^2$ to $\mathbb{R}^{1,1}$ and this latter magnetic field to a uniform electric field ($B \to iE$) at an appropriate stage of our calculations. Our objectives are two fold in studying this problem, one is to find out how the constant positive curvature of S^2 affects the pair production rates for scalar and spinor particles, and the second is to examine the direct effect of the non-abelian gauge fields on the pair production. In order to reach our first goal we form a function, which is designed to give the ratio for relative pair production rates on $\mathbb{R}^{1,1} \times S^2$ to that on $\mathbb{R}^{1,1} \times \mathbb{R}^2$. We plot the profile of this function to inspect the effects of curvature in pair production. To meet our second objective, we inspect this ratio for various different values of the non-abelian gauge field strength. Noting that this is not completely independent of the effect of the curvature, we also compare the pair production rates on $S^2 \times \mathbb{R}^{1,1}$ by forming a function which is the ratio of pair production rates at non-zero values of non-abelian field strength to that with only non-vanishing abelian magnetic fields.

In Chapter 5, we give a summary of our findings presented in this thesis, and indicate a few directions for future research.

CHAPTER 2

PAIR PRODUCTION IN FLAT AND CURVED SPACES: A REVIEW

In the presence of classical electric fields in vacuum, there is the possibility of pair production. This is known in the literature as the Schwinger effect, i.e. production of a charged pair of a particle and its anti-particle, first calculated by Julian Schwinger in his seminal article in 1951 [1]. Schwinger's result gives us the amplitude and hence the probability for production of pairs of particles and anti-particles per unit volume of the Minkowski space. This vacuum effect occurs in the presence of electric fields. In other words it does not occur if only a pure magnetic field is applied to the vacuum. Nevertheless, it is possible to contemplate situations in which both an electric field and a magnetic field is present, a particular treatable case being mutually perpendicular uniform electric and magnetic fields on the Minkowski space.

In this chapter we will present a review of the pair production process for scalar and spinor fields under the influence of an external classical electromagnetic field, directed perpendicular to an already existing uniform electric field. For this purpose, we first start with the review of the Schwinger effect, following the article of Holstein [15], and the lecture notes of Fradkin [16] on part integral techniques; first for the scalar and subsequently for the spinor fields. Computing the vacuum to vacuum transition probability and subtracting it from the total probability 1 gives us the pair production probability in each case. We will follow the computation given in [12] to discuss the pair production rates in the presence of a uniform magnetic field given in the direction perpendicular to the electric field.

In Subsection 2.5.3 we will move on to computing the pair production problem under perpendicular electric and magnetic fields on $\mathbb{R}^{1,1} \times S^2$ [12]. To examine this problem we will start our discussion on the Euclidean space $\mathbb{R}^2 \times S^2$ with a uniform magnetic field perpendicular to R^2 and another perpendicular to surface of S^2, the latter one is provided via a Dirac monopole placed at the center of S^2. By Wick rotating $\mathbb{R}^2$ to $\mathbb{R}^{1,1}$ and the magnetic field perpendicular to $\mathbb{R}^2$ to an electric field in the spatial direction in $\mathbb{R}^{1,1}$ we will calculate the pair production rates for scalar and spinor fields. Subsequently we inspect how the curvature affects the pair production by looking at by evaluating and plotting an appropriate function which is a direct measure of the relative pair production rates.

2.1 Transition Amplitude for Complex Scalar Fields

We start our discussion by giving the action for the complex scalar field under the influence of an external electromagnetic field $A_\mu(x)$ in $3+1$ dimensions. This action can be written in the following form:

$$\begin{aligned} S[\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*] &= \int d^4x\,\mathcal{L}(\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*)\,, \\ &= \int d^4x\,(D_\mu\phi^*)(D^\mu\phi) - m^2\phi\phi^*\,, \end{aligned} \tag{2.1}$$

Here ϕ and ϕ^* indicates the complex scalar fields, and $\mathcal{L}[\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*]$ indicates the lagrangian density functional. Here, we have used the usual definition for the covariant derivative, $D_\mu \equiv \partial_\mu - iA_\mu$, and we work with the units in which the charge of the scalar field is unity ($e = 1$). Integrating this expression by parts gives

$$\begin{aligned} S[\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*] &= \int d^4x\,(\partial_\mu - iA_\mu)\phi^*(D_\mu\phi) - m^2\phi\phi^*, \\ &= \int d^4x\,(\partial_\mu\phi^*)(D_\mu\phi) - i\phi^*A_\mu(D_\mu\phi) - m^2\phi\phi^*, \\ &= \int d^4x\,\partial_\mu(\phi^*D_\mu\phi) - \phi^*(\partial_\mu D^\mu\phi) - i\phi^*A_\mu(D_\mu\phi) - m^2\phi\phi^*, \\ &= -\int d^4x\,\phi^*(D^2+m^2)\phi, \end{aligned} \tag{2.2}$$

where in passing from third line to the final line in Eq. 2.2 we have assumed that $|\phi|$ vanishes sufficiently fast at infinity, so that the total derivative term (surface term) vanishes. Also in the last line of (2.2) we have introduced the notation $D^2 \equiv D_\mu D^\mu$.

In quantum mechanics, the transition amplitude i.e. the propagator between the states at positions $q(t_a)$ and $q(t_b)$, between the states at an initial time, t_a and the states at a final time, t_b, can be written as as a path integral [17],

$$D(q_a, t_a; q_b, t_b) = \int \mathcal{D}q \, e^{\frac{i}{\hbar} \int dt L(q, \dot{q})}, \tag{2.3}$$

where $L(q, \dot{q})$ is the lagrangian of the corresponding classical system, whose quantum mechanical transition amplitude between initial and final states is being sought for.

The integration measure $\mathcal{D}q$ is given as

$$\mathcal{D}q = \lim_{N \to \infty} \prod_{i=1}^{N} \frac{dq_i}{\sqrt{2\pi\hbar}}, \tag{2.4}$$

and it essentially allows to take into account all possible paths from the initial to the final configuration, as opposed to only the classically favored path. A short summary of the path integrals in quantum mechanics, sufficient for our purposes, is given in Appendix A. We want to consider the vacuum to vacuum transition amplitude, that is the amplitude for initial and final states to be vacuum state. To do this we must take the limits, $t_a \to -\infty$ and $t_b \to \infty$ [15]. Taking this limit allows us to, effectively, only consider the vacuum (ground) state, since at sufficiently large times the system can be assumed to be in the ground state.

We can pass from quantum mechanics to quantum field theory and write the vacuum to vacuum transition amplitude for complex scalar fields by making this following changes.

$$\begin{aligned} q(t) &\to \phi(x), \phi^*(x), \\ \int dt L(q, \dot{q}) &\to \int d^4x \mathcal{L}[\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*]. \end{aligned} \tag{2.5}$$

Denoting the vacuum state as $|\Omega\rangle$, we have, for the vacuum to vacuum transition amplitude is given as

$$\langle \Omega | \Omega \rangle = \int \mathcal{D}\phi \mathcal{D}\phi^* e^{i \int d^4x \mathcal{L}[\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*]}. \tag{2.6}$$

In particular, for the vacuum to vacuum transition amplitude for the complex scalar fields with a classical background gauge field, we may proceed to insert the action from (2.2) into (2.6) to get

$$\langle \Omega | \Omega \rangle = \int \mathcal{D}\phi \mathcal{D}\phi^* e^{-\int d^4x \, \phi^* (D^2 + m^2) \phi}. \tag{2.7}$$

Let us note that this transition amplitude encodes the information or the amplitude for the possible production of particle anti-particle pairs corresponding to the field ϕ and ϕ^*. We will see how and under which circumstances this gives a non-zero amplitude.

Our first task is to calculate the integral (2.7). Let us use the notation $\mathcal{O} \equiv (D^2 + m^2)$ for convenience. Also it is useful to separate the field ϕ into its real and imaginary parts as

$$\begin{aligned} \phi(x) &= \phi_1(x) + i\phi_2(x), \\ \phi^*(x) &= \phi_1(x) - i\phi_2(x). \end{aligned} \tag{2.8}$$

(2.8) allows us to rewrite the integration measure as,

$$\mathcal{D}\phi\mathcal{D}\phi^* = \mathcal{D}\phi_1\mathcal{D}\phi_2, \tag{2.9}$$

and the vacuum to vacuum transition amplitude in (2.7) as

$$\begin{aligned} \langle\Omega|\Omega\rangle &= \int \mathcal{D}\phi^* \int \mathcal{D}\phi \, e^{-\int d^4x \, \phi^*\mathcal{O}\phi}, \\ &= \int \mathcal{D}\phi_1 \int \mathcal{D}\phi_2 \, e^{-\int d^4x \, [\phi_1 - i\phi_2]\mathcal{O}[\phi_1 + i\phi_2]}, \\ &= \int \mathcal{D}\phi_1 \int \mathcal{D}\phi_2 \, e^{-\int d^4x \, \phi_1\mathcal{O}\phi_1 + \phi_2\mathcal{O}\phi_2 + i\phi_1\mathcal{O}\phi_2 - i\phi_2\mathcal{O}\phi_1}, \\ &= \int \mathcal{D}\phi_1 \int \mathcal{D}\phi_2 \, e^{-\int d^4x \, \phi_1\mathcal{O}\phi_1 + \phi_2\mathcal{O}\phi_2}, \\ &\equiv \left[\int \mathcal{D}\Phi \, e^{-\int d^4x \, \Phi\mathcal{O}\Phi} \right]^2. \end{aligned} \tag{2.10}$$

From the last line of (2.10) it is clear that the transition amplitude is just the square of the transition amplitude for a real scalar field, say Φ, which we can proceed to calculate as follows:

$$\langle\Omega|\Omega\rangle_\Phi = \int \mathcal{D}\Phi \, e^{-\int d^4x \, \Phi\mathcal{O}\Phi}. \tag{2.11}$$

We can expand Φ in terms of the eigenstates of the operator $\mathcal{O}$, which forms a complete and orthonormal set. Without loss of generality we may proceed in a notation in which the eigenbasis of $\mathcal{O}$ is discrete (while continuous spectrum can also be treated on equal footing) and write

$$\Phi = \sum_n a_n \varphi_n(x), \tag{2.12}$$

where the basis $\{\varphi_n\}$ satisfy

$$\mathcal{O}\varphi_n(x) = \lambda_n \varphi_n(x),$$
$$\int d^4x\, \varphi_m(x)\varphi_n(x) = \delta_{mn}, \tag{2.13}$$

with λ_n denoting the eigenvalues, i.e the spectrum of $\mathcal{O}$. The measure for the path integral in (2.11) then takes the form

$$\mathcal{D}\Phi = \lim_{N\to\infty} \prod_{i=1}^{N} \frac{da_i}{\sqrt{2\pi\hbar}}, \tag{2.14}$$

since $\mathcal{D}\Phi$ indicates the integration over all field configurations and integrating with respect to all the a_n will therefore yield the desired result. For the transition amplitude in (2.11) we therefore have,

$$\begin{aligned}\langle\Omega|\Omega\rangle_\Phi &= \int \lim_{N\to\infty} \prod_{i=1}^{N} \frac{da_i}{\sqrt{2\pi\hbar}} e^{-\int d^4x \sum_{n,m} a_n\varphi_n \mathcal{O} a_m\varphi_m},\\ &= \int \lim_{N\to\infty} \prod_{i=1}^{N} \frac{da_i}{\sqrt{2\pi\hbar}} e^{-\sum_{n,m}\delta_{nm}a_n a_m \lambda_m},\\ &= \int \lim_{N\to\infty} \prod_{i=1}^{N} \frac{da_i}{\sqrt{2\pi\hbar}} e^{-\sum_n a_n^2\lambda_n},\end{aligned} \tag{2.15}$$

For the last line of (2.15), we see that the result involves product of infinite number of Gaussian integrals with each integral being proportional to $\frac{1}{\sqrt{\lambda_n}}$, we find

$$\begin{aligned}\langle\Omega|\Omega\rangle_\Phi &= \lim_{N\to\infty} \frac{\tilde{C}}{\sqrt{\lambda_1\lambda_2 ... \lambda_N}},\\ &= \frac{\tilde{C}}{\sqrt{Det\,\mathcal{O}}},\end{aligned} \tag{2.16}$$

where $\tilde{C}$ is the constant we get from the integration. Going back to (2.10) and squaring the result in 2.16 we have

$$\langle\Omega|\Omega\rangle = \frac{C}{Det\,\mathcal{O}}. \tag{2.17}$$

We could write the $1/Det(\mathcal{O})$ term as

$$\begin{aligned}\frac{1}{Det\,\mathcal{O}} &= exp(-log\,(Det\,\mathcal{O})),\\ &= exp(-log\,(\prod_n \lambda_n)),\\ &= exp(-\sum_n log\,(\lambda_n)),\\ &= exp(-Tr\,log(D^2+m^2)),\end{aligned} \tag{2.18}$$

and therefore the vacuum to vacuum transition amplitude can be cast in the form

$$\langle\Omega|\Omega\rangle = C \times e^{-Tr\,log(D^2+m^2)}. \tag{2.19}$$

Here, the exponent of the exponential is the one-loop effective action, and in terms of the Feynman diagrams it is the sum of all one-loop diagrams with the even number of external photon legs [18].

2.2 Pair Production in Scalar Fields

In this section, we will calculate the pair production for scalar fields, given that there is a constant, uniform electric field. If we assume that the electric field is on the z-direction, we can write $\vec{E} = E\hat{z}$. The vector potential for such field is easily written as $A_0 = Ez$, in the Landau gauge, with other components of A_μ being zero. This shows that the operator $\mathcal{O}$ has the form $\mathcal{O} = (\partial_t - iEz)^2 + \vec{\nabla}_\perp^2 + m^2$. We have to solve the eigenvalue equation satisfied by this operator, since the vacuum to vacuum transition amplitude depends on the determinant of $\mathcal{O}$. It is possible to reorder the operator $\mathcal{O}$ and write it as,

$$\mathcal{O} = p_z^2 + E^2 Z^2 + p_\perp^2 + m^2. \tag{2.20}$$

Here we have defined $Z \equiv (-\partial_t/E - iz)$. This is a valid definition since we could write the commutation relation as $[Z, p_z] = 1$. Also it is important to realize that the definitions we have used for the momentum are just for convenience, meaning $p_z^2 = \partial_z^2$ and $\partial_\perp^2 = \partial_x^2 + \partial_y^2$. That being said, the term $p_z^2 + E^2Z^2$ is just the harmonic oscillator term. Spectrum of $\mathcal{O}$ is therefore given as

$$Spec(\mathcal{O}) = p_\perp^2 + m^2 + E(2n+1),\ n = 0, 1, 2, \dots, \tag{2.21}$$

where $p_\perp$ stands for the continuous eigenvalues of p_x and p_y, $p_\perp = \sqrt{p_x^2 + p_y^2}$, and the corresponding eigenvalues are $e^{i\vec{p}_\perp\cdot\vec{x}}\varphi_n(Z)$. We now focus on the vacuum to vacuum transition amplitude given by (2.19), from this amplitude we can calculate the probability of finding the state in a vacuum state, given enough time by just taking the modulus square of the amplitude. This gives us,

$$Prob. = |\,\langle\Omega|\Omega\rangle\,|^2 = \tilde{C}\, e^{-2Re(Tr\,log\,(\mathcal{O}))}, \tag{2.22}$$

which means that the probability of pair production (p.p.p) must be obtained by subtracting the probability in (2.22) from 1. From which we get,

$$p.p.p = 1 - \tilde{C}\, e^{-2\Gamma}. \tag{2.23}$$

Here we have defined $\Gamma \equiv Re(Tr\,log\,\mathcal{O})$. Therefore, we can simply focus on calculating Γ. We start with the calculation of $Tr\,log\,\mathcal{O}$,

$$Tr\,log\,\mathcal{O} = Tr\int \frac{ds}{s} e^{-s\,\mathcal{O}}. \tag{2.24}$$

Here the trace indicates the summation of the eigenvalues for the given operator. As we have already indicated the operator $\mathcal{O}$ has a harmonic oscillator part with frequency $\omega = E$, and a degeneracy coming from the fact that the solution for the x and y directions are freely propagating plane waves. This means that we should write the trace as a combination of integrals over the transverse direction and a summation over the discrete harmonic oscillator eigenvalues. In a spatial cube of side length L, and time interval T, we write this as,

$$\begin{aligned} Tr\,log\,\mathcal{O} &= \int \frac{d^2p_\perp}{(\frac{2\pi}{L})^2}\int \frac{dp_0}{(\frac{2\pi}{T})}\int dp_z \int \frac{ds}{s}\,\langle p|\, e^{-s\,\mathcal{O}}\,|p\rangle \\ &= L^2T\int_{-\infty}^{\infty} \frac{d^2p_\perp}{(2\pi)^2}\int_0^{EL} \frac{dp_0}{2\pi}\int dp_z \int \frac{ds}{s}\, e^{-s\,(p_z^2+E^2Z^2+p_\perp^2+m^2)} \\ &= L^3T\frac{E}{8\pi^2}\int dp_z \int \frac{ds}{s^2}\, e^{-s\,(p_z^2+E^2Z^2+m^2)}. \end{aligned} \tag{2.25}$$

In (2.25), when going from first to second line, we have used the degeneracy per unit space-time volume which is $\int \frac{d^2p_\perp}{(2\pi)^2}$ factor. Also the integration over the p_z simply converts to the summation over the eigenvalues. Therefore, we find

$$\begin{aligned} Tr\,log\,\mathcal{O} &= L^3T\frac{E}{8\pi^2}\int \frac{ds}{s^2}\sum_{n=0}^{\infty} e^{-s\,(E(2n+1)+m^2)} \\ &= L^3T\frac{E}{16\pi^2}\int \frac{ds}{s^2}\,\frac{e^{-sm^2}}{sinh(sE)}. \end{aligned} \tag{2.26}$$

In order to take the integration over s, we have to perform Wick rotation by $x^0 \to ix^0$, and $E \to iE$, then we can write (2.26) as,

$$Tr\,log\,\mathcal{O} = L^3T\frac{iE}{16\pi^3}I. \tag{2.27}$$

Here we have defined I to be,

$$I = \int_0^{\infty} \frac{ds}{s^2}\,\frac{e^{-sm^2}}{sin(sE)}, \tag{2.28}$$

for convenience. Notice that the integrand has poles at $s = s_n = n\pi/E$. To get around that problem, we define the integration contour as small semicircular deviations from the real axis, say $z = \epsilon e^{i\theta}$, centered at each s_n. This means we could write $s - s_n = z$. Using this in (2.28), we get

$$
\begin{aligned}
I &= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\int_{C_n}\frac{ds}{s^2}\,\frac{e^{-sm^2}}{sin(n\pi+Ez)},\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\int_{C_n}\frac{ds}{s^2}\,\frac{e^{-sm^2}}{sin(n\pi)cos(Ez)+cos(n\pi)sin(Ez)},\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\int_{C_n}\frac{ds}{s^2}\,\frac{(-1)^n\,e^{-sm^2}}{sin(Ez)},\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\int_{C_n}\frac{ds}{s^2}\,\frac{(-1)^n\,e^{-sm^2}}{Ez},\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\int_{C_n}\frac{dz}{s_n^2}\,\frac{(-1)^n\,e^{-sm^2}}{Ez},\\
&= \lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\int_{C_n}\frac{dz}{s_n^2}\,\frac{(-1)^n\,e^{-s_nm^2}}{Ez},\\
&= i\sum_{n=1}^{\infty}\frac{(-1)^n\,e^{-s_nm^2}}{Es_n^2}\int_{\pi}^{0}d\theta,\\
&= iE\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,e^{-n\pi m^2/E}}{n^2\pi}.
\end{aligned}
\tag{2.29}
$$

Which means we could write Γ with the help of the result we have found in 2.29 in conjunction with 2.27, and we could write

$$
\Gamma = L^3T\frac{E^2}{16\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-n\pi m^2/E}, \tag{2.30}
$$

dividing by the volume, L^3T, gives the Schwinger result for the pair production per unit space-time volume under constant electric field.

2.3 Transition Amplitude for Spinor Fields

Obtaining the vacuum to vacuum transition amplitude for spinor fields is similar in vein to the complex scalar case. However, we have to use the Grassmann variables to perform the path integrals. Therefore, it will be appropriate to start with a concise review of the Grassmann variables.

2.3.1 Grassmann Variables

Grassmann variables are the elements of the exterior (or, Grassmann) algebra and they anti-commute [19]. This means for a Grassmann variable, θ, the following is true

$$\{\theta, \theta\} = 0. \tag{2.31}$$

Thus, for a Grassmann number, we have $\theta^2 = 0$. We can define the derivation, which is the operator when applied to a Grassmann number, gives 1. For this purpose, consider the equation

$$\begin{aligned} \left\{ \frac{d}{d\theta}, \theta \right\} \theta' &= \frac{d}{d\theta}(\theta\theta') + \theta \frac{d}{d\theta}\theta', \\ &= \theta', \end{aligned} \tag{2.32}$$

where θ and θ' are both Grassmann numbers. Therefore we see that the differentiation is defined by the operator $d/d\theta$ such that,

$$\left\{ \frac{d}{d\theta}, \theta \right\} = 1. \tag{2.33}$$

Note that the most general function $f(\theta)$ of a single Grassmann variable is $f(\theta) = a + b\theta$, where a and b are constants. This is because $\theta^2 = 0$.

The integration for Grassmann variables is defined as the operator which gives the result of zero for the integral of a total derivative, and it is a linear operation. We have,

$$\int d\theta \, \frac{\partial f(\theta)}{\partial \theta} = 0, \tag{2.34}$$

which gives

$$\int d\theta \, a = 0, \qquad \int d\theta \, \theta = 1. \tag{2.35}$$

We can generalize the definition for the derivation to N Grassmann variables, with $(i, j, k = 1, 2, ..., N)$ as

$$\begin{aligned} \left\{ \frac{\partial}{\partial\theta_i}, \theta_j \right\} \theta_k &= \frac{\partial}{\partial\theta_i}(\theta_j\theta_k) + \theta_j \frac{\partial}{\partial\theta_i}\theta_k \\ &= \frac{\partial\theta_j}{\partial\theta_i}\theta_k - \frac{\partial\theta_k}{\partial\theta_i}\theta_j \\ &= \delta_{ij}\theta_k \end{aligned} \tag{2.36}$$

On the second term of the second line the minus sign comes from the fact that the derivation is only applied to the variable right next to it. So, after changing the places of the variables, we get a minus sign. With this, we can write,

$$\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta_{ij}. \tag{2.37}$$

The integration, for N Grassmann numbers is defined as

$$\int d\theta_i \theta_j = \delta_{ij}, \qquad \int d\theta_i = 0. \tag{2.38}$$

We can take multiple integrations using the fact that the Grassmann variable anti-commute

$$\int\int d\theta_1 d\theta_2\, \theta_1\theta_2 = -\int d\theta_1 \left[\int d\theta_2\, \theta_2\right]\theta_1 = -1. \tag{2.39}$$

Let us look to the expansion of particular exponential function involving Grassmann variables, we have

$$e^{\theta^T \mathcal{O}\theta} = 1 + (\theta^T\mathcal{O}\theta) + \frac{1}{2!}(\theta^T\mathcal{O}\theta)^2 + \cdots \tag{2.40}$$

Here $\mathcal{O}$ is an anti-symmetric $N \times N$ matrix, with the elements o_{ij} satisfying $o_{ij} = -o_{ji}$ and $\theta = (\theta_1, \theta_2, ..., \theta_N)^T$, $\theta^T = (\theta_1, \theta_2, ..., \theta_N)$ are column and row matrices with N Grassmann variables . It is also important to realize that if N is even this series truncates in the $(N/2)$th term, since there are only N Grassmann variables, and if N is an odd number, the terms after the $[(N+1)/2]$th term are zero because of (2.31). If we want to integrate this expression we can write

$$\int d\theta_1 \int d\theta_2 \cdots \int d\theta_N \left[1 + \Big(\sum_{i,j}\theta_i o_{ij}\theta_j\Big) + \frac{1}{2!}\Big(\sum_{i,j}\theta_i o_{ij}\theta_j\Big)^2 + \cdots \right.$$
$$\left. \cdots + \frac{1}{N!}\Big(\sum_{i,j}\theta_i o_{ij}\theta_j\Big)^N\right]. \tag{2.41}$$

It is easy to see that the only term that survives this integral is the $(N/2)$nd term if N is even, and the integral vanishes if N is odd. Hence N can only be even. So we write the only surviving term more explicitly as

$$\int d\theta_1 \int d\theta_2 \cdots \int d\theta_N \left[\frac{1}{(N/2)!}\sum_{i_1,\cdots,i_N} o_{i_1,i_2} o_{i_3,i_4} \cdots o_{i_{N-1},i_N} \right.$$
$$\left. \times \theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\cdots\theta_{i_{N-1}}\theta_{i_N}\right]. \tag{2.42}$$

We can see from this expression that the indices $i_1, i_2, ..., i_N$ cannot contain any two, or more, numbers that are the same. Possible combinations of these indices form the elements of the symmetric group, S_N. Also, in order to take the integrals we have to change the position of the Grassmann numbers, and any change between two adjacent θ's yields a factor of -1. Since N must be an even number, the factor ± 1 can be found from the $sgn(\sigma)$ where $\sigma \in S_N$ is a permutation of $\{i_1, i_2, ..., i_N\}$. Hence, we can write

$$\frac{1}{(N/2)!} \sum_{\sigma \in S_N} \left[\prod_{i=1}^{N} sgn(\sigma)\, o_{\sigma(i),\sigma(i+1)} \right]. \tag{2.43}$$

Here, since we are summing over all possible permutations of the group elements, the term $o_{\sigma(i),\sigma(i+1)}$ contains all of the matrix elements of O. This is the definition for the Pfaffian, of a $N \times N$, anti-symmetric matrix up to a multiplicative constant $2^{N/2}$ [20]. Pfaffian of O equal to square root of the determinant [21]. Hence we could finally write

$$\int d\theta_1 \int d\theta_2 \cdots \int d\theta_N e^{\theta^T \mathcal{O} \theta} = 2^{N/2} \sqrt{Det\, \mathcal{O}}. \tag{2.44}$$

2.3.2 Transition amplitude for spinor fields

Now we proceed to discuss the calculation of vacuum to vacuum transition amplitude of a Dirac field. We start with the Lagrangian [15],

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi, \tag{2.45}$$

where $\not{D} = \gamma^\mu D_\mu$ is the gauged Dirac operator, with $D_\mu = \partial_\mu - iA_\mu$ and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ is a 4-component Dirac spinor. Here γ^μ are the Dirac gamma matrices, satisfying $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

Let us use the auxiliary notation $\mathcal{O} \equiv (i\not{D} - m)$, and write the vacuum to vacuum transition amplitude as

$$\langle \Omega | \Omega \rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{\int d^4x\, \bar{\psi}\mathcal{O}\psi}. \tag{2.46}$$

Expanding the Dirac spinors as $\psi = \sum b_n \varphi_n$, where b_n are Grassmann variables, and integrating the b_n out, with the help of the result in (2.44), allows us to write (2.46) as,

$$
\begin{aligned}
\langle\Omega|\Omega\rangle &= \int \prod_{n\neq m} d\bar{b}_n db_m \, e^{\int d^4x \, \bar{b}_n (\phi_n \mathcal{O} \phi_m) b_m}, \\
&= \int \prod_{n\neq m} d\bar{b}_n db_m \, e^{\int d^4x \, \bar{b}_n \mathcal{O}_{mn} b_m}, \\
&= \int \mathcal{D}\bar{b} \, \mathcal{D}b \, e^{\int d^4x \, \bar{b}^T \mathcal{O} b}, \\
&= C \times Det \, \mathcal{O}, \qquad (2.47)
\end{aligned}
$$

where C is the constant of integration and the multiplicative factors from (2.44). This expression can also be written as the determinant of $\mathcal{O}$ and determinant of its conjugate $\mathcal{O}^\dagger$ as

$$
\langle\Omega|\Omega\rangle = C \times \sqrt{Det \, (\mathcal{O}\mathcal{O}^\dagger)}, \qquad (2.48)
$$

and finally we may cast it into the form

$$
\begin{aligned}
\langle\Omega|\Omega\rangle &= C \times exp\big[log \, \sqrt{Det \, (\mathcal{O}\mathcal{O}^\dagger)}\big], \qquad (2.49) \\
&= C \times e^{\frac{1}{2} Tr \, log \, (\mathcal{O}\mathcal{O}^\dagger)}.
\end{aligned}
$$

2.4 Pair Production in Spinor Fields

With vacuum to vacuum amplitude formally expressed for the spinor fields in (2.49), we now set out to derive the Schwinger's result for pair production of spin-$1/2$ charged particle under constant electric field. Calculations will follow similarly to the scalar case, the main difference being the eigenvalues and their degeneracies of the operator $\mathcal{O}$. We will use the pair to pair transition amplitude we have found in (2.49) to calculate the pair creation probability.

In the same way we have encountered in Section 2.2, $1 - |\langle\Omega|\Omega\rangle|^2$ will give us the pair creation probability. Therefore, it is meaningful to calculate the $Tr \, log \, (\mathcal{O}\mathcal{O}^\dagger)$.

Let us remember that the operator $\mathcal{O}\mathcal{O}^\dagger$ is given by

$$\begin{aligned}
\mathcal{O}\mathcal{O}^\dagger &= \not{D}^2 + m^2, \\
&= (\gamma^\mu D_\mu)(\gamma^\nu D_\nu) + m^2, \\
&= \gamma^\mu \gamma^\nu D_\mu D_\nu + m^2, \\
&= (\eta^{\mu\nu} - i\sigma^{\mu\nu}) D_\mu D_\nu, \\
&= D^2 - i\sigma^{\mu\nu}(\partial_\mu - iA_\mu)(\partial_\nu - iA_\nu), \\
&= D^2 - i\sigma^{\mu\nu}(\partial_\mu\partial_\nu - i\partial_\mu A_\nu - iA_\nu\partial_\mu - iA_\mu\partial_\nu - A_\mu A_\nu), \\
&= D^2 - \sigma_{\mu\nu}(\partial_\mu A_\nu), \\
&= D^2 - \frac{1}{2}\sigma^{\mu\nu}(\partial_\mu A_\nu + \partial_\mu A_\nu), \\
&= D^2 - \frac{1}{2}\sigma^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu), \\
&= D^2 - \frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu}.
\end{aligned} \tag{2.50}$$

From the last line of (2.50) we see that the operator $\mathcal{O}\mathcal{O}^\dagger$ consists of two parts. The first term, D^2, is the gauged Laplacian we have already examined in Section 2.2. The second term can be called the Zeeman term as it encodes the interaction of the spin with the external electromagnetic field. In (2.50) we have used the definition,

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \tag{2.51}$$

a partial explicit representation is provided by

$$\sigma^{0i} = \begin{pmatrix} 0 & i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \sigma^{ij} = \epsilon_{ijk} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}. \tag{2.52}$$

Proceeding with the same gauge choice we have used in Section 2.2, $A_0 = Ez$, we are able to rewrite (2.50) using (2.52) as,

$$\begin{aligned}\mathcal{O}\mathcal{O}^\dagger &= D^2 - \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}, \\ &= D^2 - \frac{1}{2}(\sigma^{0\nu}F_{0\nu} + \sigma^{\mu 0}F_{\mu 0}), \\ &= D^2 - \frac{1}{2}(\sigma^{0j}F_{0j} + \sigma^{i0}F_{i0}), \\ &= D^2 - \frac{1}{2}(\sigma^{0z}F_{0z} + \sigma^{z0}F_{z0}), \\ &= D^2 - \frac{1}{2}(-\sigma^{0z}\partial_z A_0 + \sigma^{z0}\partial_z A_0), \\ &= D^2 + \sigma^{0z}\partial_z A_0, \\ &= \begin{pmatrix} D^2\mathbb{1}_2 & iE\sigma_z \\ iE\sigma_z & D^2\mathbb{1}_2 \end{pmatrix}.\end{aligned} \tag{2.53}$$

By Wick rotating the result to Euclidean space, we will get

$$\mathcal{O}\mathcal{O}^\dagger = \begin{pmatrix} D^2 & E\sigma_z \\ E\sigma_z & D^2 \end{pmatrix} \tag{2.54}$$

Which means the eigenvalues for such operator is given by

$$\lambda_n^\pm = p_\perp^2 + (2n+1)E \pm E + m^2 \tag{2.55}$$

Here the $\pm$ indicates the spin for the system, we can also shift the indices to write this result in a more suggestive form just like we have done in Appendix D.2,

$$\lambda_n = p_\perp^2 + 2nE + m^2 \tag{2.56}$$

with the same degeneracy factors as (D.9). Now we will calculate the trace of $log\,\mathcal{O}\mathcal{O}^\dagger$ since the pair to pair transition amplitude is proportional to that, as it can be seen from

Eq. (2.49).

$$
\begin{aligned}
Tr\,log\,\mathcal{O}\mathcal{O}^\dagger &= \int d^4p \int \frac{ds}{s}\ \langle p|\, e^{-s\,\mathcal{O}\mathcal{O}^\dagger}\, |p\rangle\,, \\
&= \int_{-\infty}^{\infty} \frac{d^2p_\perp}{(\frac{2\pi}{L})^2} \int_0^{EL} \frac{dp_0}{(\frac{2\pi}{T})} \int \frac{ds}{s} \sum_n e^{-s\lambda_n}, \\
&= L^3T\frac{E}{8\pi^2}\int \frac{ds}{s^2}\sum_n e^{-s\lambda_n}, \\
&= L^3T\frac{E}{8\pi^2}\int \frac{ds}{s^2}\,(e^{-sm^2}+2\sum_{n=1}^{\infty} e^{-s\lambda_n}), \\
&= L^3T\frac{E}{8\pi^2}\int \frac{ds}{s^2}\,(1+2\sum_{n=1}^{\infty} e^{-s(2nE)})e^{-sm^2}, \\
&= L^3T\frac{E}{8\pi^2}\int \frac{ds}{s^2}e^{-sm^2}\,coth\,sE,
\end{aligned}
\tag{2.57}
$$

we will this expression back to the Minkowski space via Wick rotation, then obtain

$$
Re(iS_{eff}) = -L^3T\frac{iE}{8\pi^2}\int \frac{ds}{s^2}e^{-sm^2}\,cot\,sE, \tag{2.58}
$$

again just like the scalar case, we will have singularities at $s = s_n \equiv n\pi/E$ because of the $coth\,sE$ term, and we will move around them with the small semicircles centered around each s_n. We also use $cot\,sE \approx 1/Ez$, and write (2.58) as

$$
\begin{aligned}
Re(iS_{eff}) &= -L^3T\frac{iE}{8\pi^2}\sum_{n=1}^{\infty}\int_{C_n} \frac{dz}{zE}\frac{E^2}{n^2\pi^2}e^{-(n\pi/E)m^2}, \\
&= -L^3T\frac{iE}{8\pi^2}\sum_{n=1}^{\infty}\int_{\pi}^{0} id\theta\frac{E}{n^2\pi^2}e^{-(n\pi/E)m^2}, \\
&= -L^3T\frac{E^2}{8\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-(n\pi/E)m^2}.
\end{aligned}
\tag{2.59}
$$

Dividing the last line of (2.59) we find the Schwinger result for the pair production of charged spin-$1/2$ particles per unit volume of the Minkowski space.

2.5 Magnetic Fields Effects on Pair Production

We would like to proceed with exploring the effects of additional constant magnetic field B applied to the system in a direction parallel to the constant electric field, E.

Consider that we have $F_{03} \equiv E$ and $F_{12} \equiv B_1$ with E & B_1 constants in the Minkowski space $\mathbb{R}^{3,1}$. In order to facilitate the evaluation of the necessary path integrals, we may work in the Euclidean space $\mathbb{R}^4$ with $F_{12} \equiv B_1$, and $F_{34} \equiv B_2$[1], then at the appropriate stage of our calculation we will rotate $\mathbb{R}^4$ to the Minkowski space $\mathbb{R}^{3,1}$ and B_2 to $-iE$ ($F_{34} \to iF_{30}$), i.e. to the electric field.

2.5.1 Scalar Fields on $\mathbb{R}^{3,1}$

Let us start with writing the vacuum to vacuum transition amplitude in (2.17) with the notation

$$\langle \Omega | \Omega \rangle = C \times e^{iS_{eff}}, \tag{2.60}$$

where, we introduced

$$iS_{eff} \equiv Tr\, log(D^2 + m^2), \tag{2.61}$$

we are interested in obtaining the pair production probability (or, p.p.p.). This can be computed from the vacuum to vacuum transition probability, by subtracting it from the total probability 1. Therefore, we have

$$\begin{aligned} p.p.p. &= 1 - |\langle \Omega | \Omega \rangle|^2 \\ &= C \times (1 - e^{-2Re(iS_{eff})}) \\ &= C \times (1 - e^{-\Gamma}) \end{aligned} \tag{2.62}$$

In (2.62) $Re(iS_{eff})$ is present instead of iS_{eff} because in order to calculate the probability, we compute the modulus square of (2.60), and in the last line used definition of $\Gamma = Re(Tr\, log\,(\mathcal{O}))$ given after (2.22), corresponding to the real part of the iS_{eff}, with S_{eff} being the one-loop effective action.

We can compute Γ by writing the trace explicitly, and expressing the logarithm as an integral. In the Euclidean signature we have

$$\begin{aligned} \Gamma_E &= -Tr \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{ds}{s} e^{-s(-D^2+m^2)}, \\ &= -\int d^4x \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{ds}{s} \langle x | e^{-s(-D^2+m^2)} | x \rangle \,. \end{aligned} \tag{2.63}$$

As mentioned earlier we will compute this integral in the Euclidean space and then we will Wick rotate back to Minkowski space.

[1] The subscripts in B_1 and B_2 are not tensor indices, they are just an auxiliary notation to label the magnetic fields perpendicular to the (x_1, x_2) and (x_3, x_4) planes, respectively.

In this case we have B_1 as the magnetic field that is perpendicular to (x_1, x_2) plane, and the magnetic field B_2 that is perpendicular to the (x_3, x_4) plane. For which we can write the spectrum for the D^2 operator as,

$$D^2\psi_{n_1,n_2,\alpha}(x) = [(2n_1+1)B_1 + (2n_2+1)B_2]\psi_{n_1,n_2,\alpha}(x) \begin{cases} n_1 = 0,1,2,... \\ n_2 = 0,1,2,... \end{cases} \tag{2.64}$$

This is essentially obtained from superposing the solution of two distinct Landau problems on two orthogonal planes, a detailed calculation of which is given in Appendix D.1.

Expanding the position ket $|x\rangle$ in terms of the energy eigenbasis kets, labeled as $|n_1, n_2\rangle$, i.e. $|x\rangle = \sum_{n_1,n_2} |n_1, n_2\rangle \langle n_1, n_2|x\rangle$, we may write

$$\begin{aligned} \Gamma_E &= \int d^4x \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s} \sum_{n_1,n_2,\alpha} \langle x| e^{-s(-D^2+m^2)} |n_1,n_2\rangle_\alpha \langle n_1,n_2|x\rangle_\alpha\,, \\ &= \int d^4x \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s} \sum_{n_1,n_2,\alpha} \langle x|n_1,n_2\rangle_\alpha e^{-s[B_1(2n_1+1)+B_2(2n_2+1)+m^2]} \langle n_1,n_2|x\rangle_\alpha\,, \\ &= \int d^4x \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s} \sum_{n_1,n_2,\alpha} \psi^*_{n_1,n_2,\alpha}(x)\psi_{n_1,n_2,\alpha}(x) e^{-s[B_1(2n_1+1)+B_2(2n_2+1)+m^2]}. \end{aligned} \tag{2.65}$$

Here $\langle n_1, n_2|x\rangle \equiv \psi_{n_1,n_2,\alpha}(x)$ with the α index labeling the degeneracy in each Landau level. We may use the orthogonality of the wavefunctions and the degeneracy of the Landau levels to manipulate the last line of (2.65). Note that we have the normalization condition,

$$\int d^4x\, \psi_{n_1,n_2,\alpha}(x)\psi_{n_1,n_2,\alpha}(x) = 1, \tag{2.66}$$

and since the density of states for Landau levels are $\frac{B}{2\pi}$ [22] and we have two distinct Landau levels, sum over the degeneracy index α is given by

$$\int dx_1 dx_2 \frac{B_1}{2\pi} \times \int dx_3 dx_4 \frac{B_1}{2\pi} = \int d^4x \frac{B_1 B_2}{4\pi^2}. \tag{2.67}$$

Putting these facts together, we have

$$\Gamma_E = \lim_{\epsilon\to 0} \frac{B_1 B_2}{4\pi^2} \int d^4x \int_\epsilon^\infty \frac{ds}{s} \sum_{n_1,n_2} e^{-s[B_1(2n_1+1)+B_2(2n_2+1)+m^2]}. \tag{2.68}$$

Here we can perform the summation on n_1 and n_2 explicitly, using geometrical series. In order to put this expression in a form which is better suited for physical interpretation we only perform the summation over n_2. This yields

$$\Gamma_E = -\frac{1}{8\pi^2}\int d^4x\, B_1 \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s^2}\frac{sB_2}{sinh\, sB_2}\sum_{n_1} e^{-s[B_1(2n_1+1)+m^2]}, \tag{2.69}$$

and Wick rotating the result by $x_4 \to ix_0$ and $B_2 \to -iE$ and identifying the continuation of Γ_E with iS_{eff}. We get

$$iS_{eff} = \frac{i}{8\pi^2}\int d^4x\, B_1 \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s^2}\frac{sE}{sin\, sE}\sum_{n_1} e^{-s[B_1(2n_1+1)+m^2]}. \tag{2.70}$$

Now, we are in a position to perform the s-integral. First we have to realize that the integral has singularities at $s = n\pi/E$, $n = 1, 2, \cdots$. It can be seen from (2.62) that we are interested with the real part of the iS_{eff}. This contribution comes from the integrals around the small semicircles, C_n, centered around the singularities. Hence our integration variable can be written as $s = n\pi/E + z$, with $|z|$ being small. We can also write $sin\, sE \approx (-1)^n zE$. Hence the real part of the iS_{eff} becomes

$$\begin{aligned} Re(iS_{eff}) &= \frac{i}{8\pi^2}\int d^4x\, B_1 \sum_n \int_{C_n} \frac{dz}{(n\pi/E)^2}\frac{(-1)^n n\pi}{zE}\sum_{n_1} e^{-(n\pi/E)[B_1(2n_1+1)+m^2]}, \\ &= \frac{iEB_1}{8\pi^2}\int d^4x \sum_n \int_{C_n} \frac{dz}{z}\frac{1}{(n\pi)}(-1)^n \sum_{n_1} e^{-(n\pi/E)[B_1(2n_1+1)+m^2]}, \\ &= \frac{iEB_1}{8\pi^2}\int d^4x \sum_n \int_\pi^0 id\theta\frac{1}{(n\pi)}(-1)^n \sum_{n_1} e^{-(n\pi/E)[B_1(2n_1+1)+m^2]}, \\ &= \frac{EB_1}{8\pi^2}\int d^4x \sum_{n=1}^\infty \frac{(-1)^n}{n}\sum_{n_1} e^{-(n\pi/E)[B_1(2n_1+1)+m^2]}. \end{aligned} \tag{2.71}$$

It is possible to take the $B_1 \to 0$ limit of the last line of (2.71), but to do so we must first carry out the summation over n_1. Carrying out the summation gives us

$$\begin{aligned} Re(iS_{eff}) &= \lim_{B_1\to 0}\frac{E}{8\pi^2}\int d^4x \sum_{n=1}^\infty e^{-m^2(n\pi/E)}\frac{(-1)^n}{n}\frac{B_1}{2\, sinh\,(n\pi B_1/E)}, \\ &= \lim_{B_1\to 0}\frac{E}{8\pi^2}\int d^4x \sum_{n=1}^\infty e^{-m^2(n\pi/E)}\frac{(-1)^n}{n}\frac{E/n\pi}{2\, cosh\,(n\pi B_1/E)}, \\ &= \frac{E^2}{16\pi^3}\int d^4x \sum_{n=1}^\infty e^{-m^2(n\pi/E)}\frac{(-1)^n}{n^2}, \end{aligned} \tag{2.72}$$

which is the Schwinger's result, as we have shown in Section 2.2, (2.30).

2.5.2 Spinor Fields on $\mathbb{R}^{3,1}$

We now compute the pair production rates for spin-$1/2$ particles in the presence of parallel E & B fields. To do so we again start with the Euclidean problem with B_1 on the (x_1, x_2) & B_2 on the (x_3, x_4) plane. We have [2]

$$\begin{aligned}
\not{D}_2^2 + m^2 &= (\tau_x D_x + \tau_y D_y)(\tau_x D_x + \tau_y D_y) \\
&= \tau_x^2 D_x^2 + \tau_y^2 D_y^2 + \tau_x \tau_y (D_x D_y - D_y D_x) \\
&= D_2^2 + i\tau_z[(\partial_x + iA_x)(\partial_y + iA_y) - (\partial_y + iA_y)(\partial_x + iA_x)] \\
&= D_2^2 - B_1 \tau_z
\end{aligned} \tag{2.73}$$

Here the τ_x and τ_y operators indicate the usual Pauli spin operators. It is readily seen from (2.73) that the spectrum depends on the spin of the particle, and it can be written as

$$\begin{aligned}
Spec(\not{D}_2^2) &= D_2^2 \mathbb{1} - B_1 \tau_z = \begin{pmatrix} D_2^2 - B_1 & 0 \\ 0 & D_2^2 + B_1 \end{pmatrix} \\
&= \begin{cases} (2n_1 + 1)B_1 + B_1 = 2(n_1 + 1)B_1, & n_1 = 0, 1, 2, ... \\ (2n_1 + 1)B_1 - B_1 = 2n_1 B_1, & n_1 = 0, 1, 2, ... \end{cases}
\end{aligned} \tag{2.74}$$

Adding two copies of (2.74) with the quantum numbers n_1 and n_2 and magnetic fields B_1 and B_2 we find the spectrum $\not{D}^2$ on $\mathbb{R}^4$ as

$$Spec(\not{D}^2) = (2n_1 + 1)B_1 + (2n_2 + 1)B_2 \pm B_1 \pm B_2. \tag{2.75}$$

Compared to the spectrum of the operator, $D^2 + m^2$, for the scalar case, we see that there are the Zeeman terms $\pm B_1$ $\pm B_2$ with all possible $(++, +-, -+, --)$ sign combinations. This causes an important change in the spectrum. For instance; for the spin down state corresponding two lower signs, the ground level which is given by $n_1 = n_2 = 0$ is a zero energy state i.e. it is a zero mode. This will have consequences for pair production as it is a state that can be filled by produced pairs without any energy cost.

The degeneracy factor is the same as in the scalar case for each spin branch, and given

[2] Subscript 2 in $\not{D}_2^2$ indicates that the operator is in two dimensions.

by

$$\int d^4x \frac{B_1 B_2}{4\pi^2}. \tag{2.76}$$

Putting these factors together, we can write the one loop effective action as

$$\Gamma_E = -\frac{B_1\,B_2}{8\pi^2}\int d^4x \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s}\sum_{n_1,n_2}\Big[e^{-s[B_1(2n_1+2)+m^2]} + e^{-s[B_1 2n_1+m^2]} + e^{-s[B_2(2n_2+2)+m^2]} + e^{-s[B_2 2n_2+m^2]}\Big] \tag{2.77}$$

We can perform the summation on n_1 in the (2.77), and write it as

$$\begin{aligned}\Gamma_E &= -\frac{B_1\,B_2}{8\pi^2}\int d^4x \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s}\sum_{n_1,n_2}\Big[e^{-s[B_1 2n_1+m^2]}(1+e^{-s[2B_1+m^2]}) \\ &+ e^{-s[B_2(2n_2+2)+m^2]} + e^{-s[B_2 2n_2+m^2]}\Big] \\ &= -\frac{B_1\,B_2}{8\pi^2}\int d^4x \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s} coth\, sB_1 \sum_{n_2}\Big[e^{-s[B_1(2n_1+2)+m^2]} + e^{-s[B_2 2n_2+m^2]}\Big]\end{aligned} \tag{2.78}$$

Now we perform Wick rotation into the Minkowski space with $B_2 \to -iE$, and identify continuation of Γ_E with iS_{eff}. This allows us to write

$$\begin{aligned}Re(iS_{eff}) &= -\frac{iE\,B_1}{8\pi^2}\int d^4x \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s} cot\, sE \sum_{n_2}\Big[e^{-s[B_1(2n_2+2)+m^2]} + e^{-s[B_2 2n_2+m^2]}\Big] \\ &= -\frac{E\,B_1}{8\pi^2}\int d^4x \sum_{n=1}^{\infty}\frac{1}{n}\sum_{n_1}\Big[e^{-(n\pi/E)[B_1(2n_1+2)+m^2]} + e^{-(n\pi/E)[B_1 2n_1+m^2]}\Big]\end{aligned} \tag{2.79}$$

2.5.3 Spherical Scalar Case

We will now consider a situation in which not only the effects of magnetic field B and an electric field E is taken into account, but also spatial curvature effect is also examined. To do so, we will consider the pair production effects on the product manifold $S^2 \times \mathbb{R}^{1,1}$ with $\mathbb{R}^{1,1}$ is spanned by the coordinates x_0 & x_3. We will consider a uniform electric field in the x_3-direction, while the uniform magnetic field on S^2 is provided by a magnetic monopole.

In order to compute the one-loop effective action, we follow the same approach as before, and first consider this problem on the Euclidean space $S^2 \times \mathbb{R}^2$ with a magnetic field B_1 perpendicular to S^2 provided by a magnetic monopole, and B_2 being perpendicular to the plane $\mathbb{R}^2$. We will Wick rotate B_2 to an electric field E at an appropriate stage of this calculation, as in the flat case before.

Since, we need the spectrum of $-D^2+m^2$ to evaluate the one-loop effective action, we may write $-D^2 = -D^2_{S^2} + D^2_{\mathbb{R}^2}$ where the spectrum of $D^2_{\mathbb{R}^2}$ is given by $(2n_2+1)B_2$ as this is the standard Landau problem solution we have used in Section 2.5.1 and detailed out in Appendix D.1

Spectrum of $D^2_{S^2}$ can be computed using group theory. The uniform magnetic field on S^2 is provided by a Dirac monopole placed at the center of this sphere. Due to Dirac quantization condition [23] the magnetic field is quantized as $B_1 = \frac{N}{2a^2}$ where N is an integer, and a is the radius of the sphere. The solution of this problem is provided in Appendix E.1 and given as

$$Spec(-D^2+m^2) = \frac{1}{a^2}\left(n_1(n_1+1) + Nn_1 + \frac{N}{2}\right) + (2n_2+1)B_2 + m^2, \quad (2.80)$$

one-loop effective action takes the form

$$\begin{aligned}\Gamma_E &= -\int_{S^2} d\Omega \int dx_3 dx_4 \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s} \sum_{n_1,n_2,\alpha} \psi^*_{n_1,n_2,\alpha}(x) e^{-s(-D^2+m^2)} \psi_{n_1,n_2,\alpha}(x) \\ &= -\int_{S^2} d\Omega \int dx_3 dx_4 \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s} \sum_{n_1,n_2,\alpha} \psi^*_{n_1,n_2,\alpha}(x) \psi_{n_1,n_2,\alpha}(x) \qquad (2.81) \\ &\times e^{-s[(n_1(n_1+1)+Nn_1+N/2)/a^2 + B_2(2n_2+1)+m^2]}\end{aligned}$$

Here $d\Omega$ denotes surface element of the 2-sphere, and the wavefunctions $\psi_{n_1,n_2,\alpha}(x)$ are on the $\mathbb{R}^2 \times S^2$ space. In contrast to the flat case the degeneracy in each Landau level is finite and given by the dimension of the spin $J = n_1 + \frac{N}{2}$ irreducible representation of $SU(2) \simeq SO(3)$, which is $2n_1 + 1 + N$. Therefore the density of states on S^2 is $\rho_{S^2} = \frac{2k+1+N}{4\pi a^2}$.

Using ρ_{S^2}, the orthogonality of the wavefunctions, and the degeneracy of the Landau levels, which as we recall is $B_2\, dx_3\, dx_4/(2\pi)$ for the flat case. Summing over the degeneracy index, α we obtain

$$\sum_\alpha \int_{S^2} d\Omega \int dx_3 dx_4 \psi^*_{n_1,n_2,\alpha}(x)\psi_{n_1,n_2,\alpha}(x) = \int_{S^2} d\Omega \int dx_3 dx_4 \, \frac{B_2}{2\pi}\frac{2k+1+N}{4\pi a^2}. \tag{2.82}$$

Then the one-loop effective action takes the form

$$\Gamma_E = \frac{1}{16\pi^2 a^2}\int_{S^2} d\Omega \int dx_3 dx_4 \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s^2}\frac{sB_2}{sin\, sB_2} \times \sum_{n_1=0}^{\infty}(2n_1+1+N)e^{-s[(n_1(n_1+1)+Nn_1+N/2)/a^2+m^2]}. \tag{2.83}$$

As we did previously, we again continue this expression into the $\mathbb{R}^{1,1}\times S^2$ space by means of the Wick rotation, $x_4 \to ix_0$ and $B_2 \to -iE$. Taking the s-integral using the residue theorem due to the singularities at $sE = n\pi$, $n = 1, 2, 3, ..$ we may write the real part of iS_{eff} as (recall that analytic continuation of $-\Gamma_E$ to $\mathbb{R}^{1,1}\times S^2$ is identified as iS_{eff}.)

$$Re(iS_{eff}) = \frac{E}{16\pi^2 a^2}\int d\Omega_2 \int dx_3 dx_0 \sum_{n=0}^{\infty}\frac{(-1)^n}{n} \times \sum_{n_1=0}^{\infty}(2n_1+1+N)\, e^{-(n\pi/E)[(n_1(n_1+1)+Nn_1+N/2)/a^2+m^2]}. \tag{2.84}$$

Now we define the dimensionless variable $\omega = \pi/Ea^2$, and express (2.84) as

$$Re(iS_{eff}) = -\frac{E^2}{16\pi^3}\int_{S^2} d\Omega \int dx_3 dx_0 \beta_0(\omega), \tag{2.85}$$

where we have also introduced,

$$\begin{aligned}\beta_0(\omega) &:= \omega \sum_{n=0}^{\infty}\frac{(-1)^n}{n}\sum_{n_1=0}^{\infty}(2n_1+1+N)\, e^{-(n\omega)[B_1(n_1(n_1+1)+Nn_1+N/2)+m^2a^2]}, \\ &= \omega \sum_{n_1=0}^{\infty}(2n_1+1+N)\, log\,(1+e^{-\omega[n_1(n_1+1)+Nn_1+N/2+m^2a^2]})\end{aligned} \tag{2.86}$$

In passing to the second line of (2.86) we have performed the sum over [3] n.

It is possible to take the flat limit of $\beta_0(\omega)$, by letting $N \to \infty$ and $a^2 \to \infty$ while keeping N/a^2 constant. This gives

$$\beta_0^{flat}(\omega) = \omega N \sum_{n_1=0}^{\infty} log\,(1+e^{-\omega[Nn_1+N/2+m^2a^2]}). \tag{2.87}$$

[3] We recall that $log\,(1+\Lambda) = \sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\Lambda^n$.

In order to appreciate the effects of the curvature and magnetic field on the pair production rates, we may consider the profile of the function $\gamma_0(\omega) := \beta_0(\omega)/\beta_0^{flat}(\omega)$. Explicitly, we have

$$\gamma_0(\omega) = \begin{cases} \frac{\sum_{n_1=0}^{\infty}(2n_1+1+N)\, log\,(1+e^{-\omega[n_1(n_1+1)+Nn_1+N/2+m^2a^2]})}{N\sum_{n_1=0}^{\infty} log\,(1+e^{-\omega[Nn_1+N/2+m^2a^2]})} & N \neq 0, \\ \frac{12}{\pi^2}\,\omega\sum_{n_1=0}^{\infty}(2n_1+1)\, log\,(1+e^{-\omega n_1(n_1+1)}) & N = 0. \end{cases} \tag{2.88}$$

In figures 2.1a and 2.1b we plot the profile of γ_0 with respect to ω, at several values of N. Since $\gamma_0(\omega) > 1$, we see that the pair production rates are larger than that is found for flat space. This is mainly due to the fact that in the present case the degeneracy is $\frac{B_1}{2\pi} + \frac{2n_1+1+N}{4\pi a^2}$ compared to just $\frac{B_1}{2\pi}$ of the flat case. The term proportional to $\frac{1}{a^2}$ is the contribution of the curvature of S^2 to the degeneracy, and acts to increase the pair production rates. For $N = 0$, there is also an increase caused by the zero mode states at $n_1 = 0$, which are absent in the flat case, and they can be filled by produced pairs without any energy cost. As N increases, we see that the $\gamma_0(\omega)$ tends back towards the value 1, meaning that as the magnetic field is increased it counter acts the effect of positive curvature and tends to lower the pair production. In fact at large ω, $\gamma_0(\omega) \approx \frac{N+1}{N} = \frac{1}{N} + 1$.

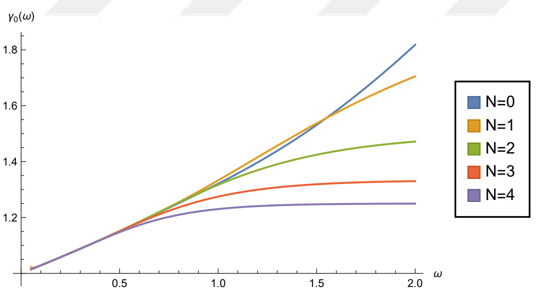

Figure 2.1a: $\gamma_0(\omega)$

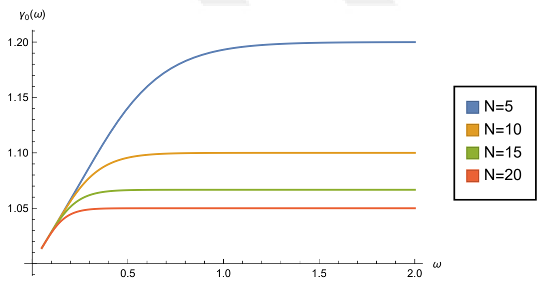

Figure 2.1b: $\gamma_0(\omega)$

2.5.4 Spherical Spinor Case

We now proceed to consider the pair production rates for spin-1/2 particles on $\mathbb{R}^{1,1} \times S^2$ following the results in [12]. In this case we may start with

$$\Gamma_E = -Tr\, log\,(i\gamma^\mu D_\mu + m) = \frac{1}{2}\int \frac{ds}{s} Tr\, e^{-s(m^2 - \not{D}^2)} \tag{2.89}$$

In order to proceed, we need the spectrum of $\not{D}^2$. This is worked out in the Appendix E.2. Here we give the results in Table 2.1. As will be explained in detail in Appendix E.2 in Table 2.1, R_3 denotes the eigenvalues of right acting $U(1)$ generator of the $SU(2) \sim SO(3)$ symmetry combined with the spin of the particle. Effective

Eigenvalues	Degeneracy	R_3 Eigenvalues
$2n_2B_2 + ((n_1+1)^2 + N(n_1+1))/a^2$	$\frac{2n_1+2+N}{4\pi a^2}\frac{B_2}{2\pi}$	$\frac{-1-N}{2}$
$(2n_2+2)B_2 + (n_1^2 + Nn_1)/a^2$	$\frac{2n_1+N}{4\pi a^2}\frac{B_2}{2\pi}$	$\frac{1-N}{2}$
$(2n_2+2)B_2 + ((n_1+1)^2 + N(n_1+1))/a^2$	$\frac{2n_1+2+N}{4\pi a^2}\frac{B_2}{2\pi}$	$\frac{-1-N}{2}$
$2n_2B_2 + (n_1^2 + Nn_1)/a^2$	$\frac{2n_1+N}{4\pi a^2}\frac{B_2}{2\pi}$	$\frac{1-N}{2}$

Table 2.1: Eigenvalues of $\not{D}^2$, corresponding degeneracies and R_3 eigenvalues

action can be cast in the form

$$\Gamma_E = \frac{1}{16\pi^2 a^2}\int d\Omega_2 dx_3 dx_4 \int \frac{ds}{s} coth\, sB_2 \tag{2.90}$$
$$\times \sum_{n_1}\left[(2n_1+N)e^{-s[(n_1^2+Nn_1)/a^2+m^2]} + (2n_1+2+N)e^{-s[((n_1+1)^2+N(n_1+1))/a^2+m^2]}\right].$$

Analytically continuing this expression to $\mathbb{R}^{1,1}\times S^2$ via the Wick rotation, and also performing the s-integral in a manner similar to the scalar case, we can write

$$Re(iS_{eff}) = -\int d\Omega_2 dx_3 dx_0 \frac{E^2}{8\pi^3}\beta_{1/2}(\omega), \tag{2.91}$$

where we have introduced

$$\beta_{1/2}(\omega) = \omega\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{N}{2}e^{-n\omega m^2a^2} + \sum_{n_1=1}^{\infty}(2n_1+N)e^{-n\omega(n_1^2+Nn_1+m^2a^2)}\right], \tag{2.92}$$
$$= -\omega\left(\frac{N}{2}log\,(1-e^{-\omega m^2a^2}) + \sum_{n_1=1}^{\infty}(2n_1+N)log\,(1-e^{-\omega(n_1^2+Nn_1+m^2a^2)})\right).$$

In the second line of (2.92) we have carried out the summation over n. Notice also that the contribution of the zero modes are written out explicitly. Clearly there is zero mode contribution for only the $N \neq 0$ case.

Just like in the scalar case as $N\to\infty$, $a\to\infty$ with N/a^2 held fixed, we can write the flat limit i.e. $S^2\to\mathbb{R}^2$ of $\beta_{1/2}(\omega)$ as

$$\beta_{1/2}^{flat}(\omega) = -\omega\left(\frac{N}{2}log\,(1-e^{-\omega m^2a^2}) + \sum_{n_1=1}^{\infty}N log\,(1-e^{-\omega(Nn_1+m^2a^2)})\right). \tag{2.93}$$

Finally, we define the ratio $\gamma_{1/2}(\omega) \equiv \beta_{1/2}(\omega)/\beta^{flat}_{1/2}(\omega)$,

$$\gamma_{1/2}(\omega) = \begin{cases} \frac{N\,log\,(1-e^{-\omega m^2 a^2})+2\sum_{n_1=1}^{\infty}(2n_1+N)\,log\,(1-e^{-\omega(n_1^2+Nn_1+m^2a^2)})}{N\,log\,(1-e^{-\omega m^2 a^2})+2\sum_{n_1=1}^{\infty}N\,log\,(1-e^{-\omega(Nn_1+m^2a^2)})}, & N \neq 0, \\ \frac{12}{\pi^2}\,\omega\sum_{n_1=0}^{\infty} n_1\,log\,(1-e^{-\omega n_1^2}), & N = 0. \end{cases} \tag{2.94}$$

Profile of $\gamma_{1/2}(\omega)$ is provided in Figure 2.2

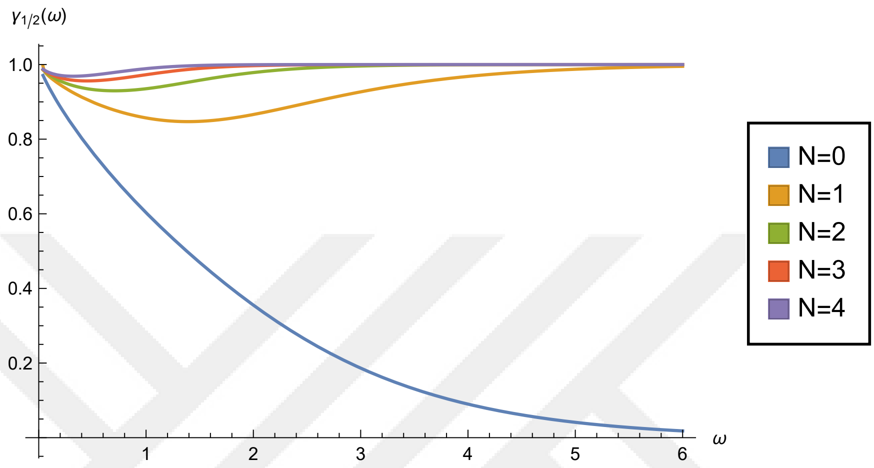

Figure 2.2: $\gamma_{1/2}(\omega)$.

By looking at the figure 2.2, we observe that, since $\gamma_{1/2} < 1$ pair production effects are diminished compared to the flat case for spin-$1/2$ particles. Note that the sign of the exponential term inside the logarithms is minus and this ensures that the numerator of $\gamma_{1/2}(\omega)$ remains smaller then its denominator. In the absence of any transverse magnetic field the pair production effect is significantly reduced, while at $N \neq 0$ contribution of the zero modes counter acts the diminishing effect of the curvature. Note also that the overall ratio of pair production is increasing with increasing N.

CHAPTER 3

PAIR PRODUCTION IN THE PRESENCE OF NON-ABELIAN GAUGE FIELDS

We now shift our focus to the treatment of the pair production in the presence of electric fields as well as both abelian and non-abelian magnetic fields. The latter renders the calculations to be treatable in several cases of interest as we will see in the ensuing sections and the next chapter.

Presence of non-abelian gauge fields adds an isospin component to the already existing degrees of freedom. This will complicate calculations somewhat, but the main essence will remain the same, while various novel physical effects will be noted and discussed in detail. This chapter is devoted to the treatment of the problem of pair production for spin-0 and spin-$1/2$ fields in on $\mathbb{R}^{3,1}$.

3.1 Scalar Fields

3.1.1 D^2 operator and its spectrum

We start our discussion by considering the Landau problem for potentials subject to a uniform magnetic field B_z, which is due to both an abelian component and a non-abelian one. This magnetic field can be written in the form [13]

$$B_z = B_1 \mathbb{1}_2 + 2\beta^2 \sigma_z. \tag{3.1}$$

Here, we may think of the first term is an abelian magnetic field component while the second term represents a non-abelian uniform magnetic field due to an $SU(2)$ pure gauge field [24]. Considering the abelian component of B_z as being generated by a

$U(1)$ subgroup of $SU(2)$, we may write the gauge field associated to B_z as

$$\vec{A} = \frac{B_1}{2}(-y\hat{x} + x\hat{y})\mathbb{1}_2 + \beta(-\sigma_y\hat{x} + \sigma_x\hat{y}) \tag{3.2}$$

We can see that the (3.1) follows from our gauge choice by making the calculation

$$\begin{aligned} B_z = F_{xy} &= \partial_x A_y - \partial_y A_x + i[A_x, A_y], \\ &= \left[\partial_x\left(\frac{B_1\,x}{2}\right) - \partial_y\left(-\frac{B_1\,y}{2}\right)\right]\mathbb{1}_2 - i\beta^2[\sigma_y, \sigma_x], \\ &= B_1\mathbb{1}_2 + 2\beta^2\sigma_z. \end{aligned} \tag{3.3}$$

Now we consider the covariant derivative operator subject to this external gauge field. We have $D_\mu = \partial_\mu + iA_\mu$, and we want to compute the spectrum of the gauged Laplacian operator $D^2 = D_\mu D^\mu$ similarly to the previous chapters.

$$\begin{aligned} D^2 &= D_\mu D^\mu, \\ &= (\partial_x + iA_x)(\partial_x - iA_x) + (\partial_y + iA_y)(\partial_y - iA_y), \\ &= (\partial_x^2 - 2iA_x\partial_x + A_x^2) + (\partial_y^2 - 2iA_y\partial_y + A_y^2), \\ &= (\partial_x^2 + \partial_y^2) - i(-B_1y\partial_x + B_1x\partial_y - 2\beta\sigma_y\partial_x + 2\beta\sigma_x\partial_y) + \frac{B_1^2}{4}(x^2 + y^2) \\ &+ B_1\beta(x\sigma_x + y\sigma_y) + 2\beta^2, \\ &= 4\partial\bar{\partial} + B_1(\bar{z}\bar{\partial} - z\partial) + 4\beta(\bar{\partial}\sigma_- - \partial\sigma_+) + \frac{B_1^2}{4}\bar{z}z + B_1\beta(z\sigma_- + \bar{z}\sigma_+) + 2\beta^2, \\ &= 2B_1\Big(\frac{2\partial\bar{\partial}}{B_1} + \frac{1}{2}(\bar{z}\bar{\partial} - z\partial) + \frac{2\beta}{B_1}(\bar{\partial}\sigma_- - \partial\sigma_+) - \frac{B_1}{8}\bar{z}z - \frac{\beta}{2}(z\sigma_- + \bar{z}\sigma_+) + \frac{\beta^2}{B_1}\Big), \\ &= 2B_1\Big(a^\dagger a + \sqrt{2}\frac{\beta}{\sqrt{B}}(a^\dagger\sigma_+ + a\sigma_-) + \frac{1}{2} + \frac{\beta^2}{B_1}\Big), \\ &= 2B_1\Big(a^\dagger a + \sqrt{2}\beta'(a^\dagger\sigma_+ + a\sigma_-) + \frac{1}{2}(1 + 2\beta'^2)\Big). \end{aligned} \tag{3.4}$$

In passing to the last line of (3.4), we have defined the dimensionless quantity $\beta' \equiv \beta/\sqrt{B_1}$. Passing from line four to line five in (3.4), we have used the definitions,

$$\begin{aligned} z &= x + iy, & \bar{z} &= x - iy \\ \partial &= \frac{\partial_x - i\partial_y}{2}, & \bar{\partial} &= \frac{\partial_x + i\partial_y}{2} \\ \sigma_+ &= \frac{\sigma_x + i\sigma_y}{2}, & \sigma_- &= \frac{\sigma_x - i\sigma_y}{2} \end{aligned} \tag{3.5}$$

and introduced the creation and annihilation operators in the penultimate line via

$$a = \frac{1}{\sqrt{2B_1}}\Big(\frac{B}{2}z + 2\bar{\partial}\Big),\; a^\dagger = \frac{1}{\sqrt{2B_1}}\Big(\frac{B}{2}\bar{z} - 2\partial\Big). \tag{3.6}$$

It can be readily checked that $[a, a^\dagger] = 1$. Spectrum of D^2 can be obtained manner similar to one applied to the Jaynes-Cummings Hamiltonian [14]. This means that it can be diagonalized by using the subspace of states, $|n+1, +\rangle$ and $|n, -\rangle$, where n is the eigenvalue of the number operator $N = a^\dagger a$ and $\pm$ denotes the isospin up and isospin down respectively. This problem is already solved in [13], we are reproducing the results for completeness. We can write out the operator D^2 in the matric form

$$D^2 = 2B_1 \begin{pmatrix} a^\dagger a + \frac{1}{2}(1+2\beta'^2) & \sqrt{2}\beta' a^\dagger \\ \sqrt{2}\beta' a & a^\dagger a + \frac{1}{2}(1+2\beta'^2) \end{pmatrix}, \tag{3.7}$$

in the aforementioned subspace, we can write the matrix elements of D_n^2 by writing

$$D^2 = 2B_1 \begin{pmatrix} n + \frac{1}{2}(1+2\beta'^2) & \sqrt{2(n+1)}\beta' \\ \sqrt{2(n+1)}\beta' & n + \frac{1}{2}(1+2\beta'^2) \end{pmatrix}, \tag{3.8}$$

Finally we can write the eigenvalues of D^2 operator by simply computing the eigenvalues of D_n^2 as

$$\Lambda_{n_1}^{\pm} = \begin{cases} 2B_1\Big(n_1^2 + \sqrt{2\beta'^2 n_1 + 1/4} + \beta'^2\Big), & n_1 = 0, 1, 2, ... \\ 2B_1\Big(n_1^2 - \sqrt{2\beta'^2 n_1 + 1/4} + \beta'^2\Big), & n_1 = 1, 2, 3, ... \end{cases} \tag{3.9}$$

Note in particular that the ground state is given by $\Lambda_0^+ = 2B_1(\beta'^2 + 1/2)$, using the upper sign in (3.9).

3.1.2 Pair production

Now we will calculate the pair production rates in a manner similar to computation presented in Section 2.4. We will start with the spectrum for the $D^2 + m^2$ operator in the $\mathbb{R}^2 \times \mathbb{R}^2 \equiv \mathbb{R}^4$ space that we have described earlier and rotate the second of the $\mathbb{R}^2$ to $\mathbb{R}^{1,1}$ to obtain the electric field, while the first $\mathbb{R}^2$ is subject to the magnetic field made up both abelian and non-abelian components. We can write down the spectrum of the whole system as

$$Spec(D^2 + m^2) = \begin{cases} 2B_1(n_1^2 + \sqrt{2\beta'^2 n_1 + 1/4} + \beta'^2) + B_2(2n_2+1), \\ 2B_1(n_1^2 - \sqrt{2\beta'^2 n_1 + 1/4} + \beta'^2) + B_2(2n_2+1), \end{cases} \tag{3.10}$$

In (3.10) first spectrum is for the isospin up case and the quantum numbers $n_1 = 0, 1, 2, ...$, $n_2 = 0, 1, 2, ..$, and the second is for the isospin down case and the quantum numbers $n_1 = 1, 2, ..., n_2 = 0, 1, 2, ...$ The degeneracy in each of the $\mathbb{R}^2$ is as before. The density of states on $\mathbb{R}^2 \times \mathbb{R}^2$ is given by

$$\frac{B_1 B_2}{(2\pi)^2}. \tag{3.11}$$

Euclidean effective action, Γ_E, on $\mathbb{R}^2 \times \mathbb{R}^2$ takes the form.

$$\Gamma_E = -\int d^4x \int_\epsilon^\infty \frac{ds}{s} \int d^4x \frac{B_1 B_2}{(2\pi)^2} \sum_{n_1, n_2} e^{-s(2B_1(n_1^2 + \sqrt{2\beta'^2 n_1 + 1/4} + \beta'^2) + B_2(2n_2+1) + m^2)} \times e^{-s(2B_1(n_1^2 + \sqrt{2\beta'^2 n_1 + 1/4} + \beta'^2))}. \tag{3.12}$$

As before, all we need for pair production is to focus on to evaluate $Re(iS_{eff})$ this is done by by Wick rotating Γ_E and then calculating the residue integral, and then evaluating the sum over the index n, labeling the poles, along the lines given in (2.71) in Chapter 2. This gives

$$Re(iS_{eff}) = -\int d^4x \frac{EB_1}{8\pi^2} \left[\sum_{n_1=1}^{\infty} log\,(1 + e^{-2\pi B_1/E(n_1^2 - \sqrt{2\beta'^2 n_1 + 1/4} + \beta')}) + \sum_{n_1=0}^{\infty} log\,(1 + e^{-2\pi B_1/E(n_1^2 + \sqrt{2\beta'^2 n_1 + 1/4} + \beta')}) \right] \tag{3.13}$$

Notice that, for the isospin up branch of the spectrum, sum over n_1 starts from 0 and for the isospin down part it starts from $n_1 = 1$.

Let us define $y \equiv B_1/E$ for convenience, then by taking the limit $m^2 \to 0$ we are able to write

$$Re(iS_{eff}) = -\int d^4x \frac{E^2}{96\pi^2} f_0(y, \beta'), \tag{3.14}$$

where

$$f_0(y, \beta') = \frac{12y}{\pi} \left[\sum_{n_1=1}^{\infty} log\,(1 + e^{-2\pi y(n_1 - \sqrt{2\beta'^2 n_1 + 1/4} + \beta'^2)}) + \sum_{n_1=0}^{\infty} log\,(1 + e^{-2\pi y(n_1 + \sqrt{2\beta'^2 n_1 + 1/4} + \beta'^2)}) \right]. \tag{3.15}$$

In order to understand the pair production rates, we will need to take a look at the plots of $f_0(y, \beta')$, for different β' values

3.1.2.1 Pair production only for non-abelian case

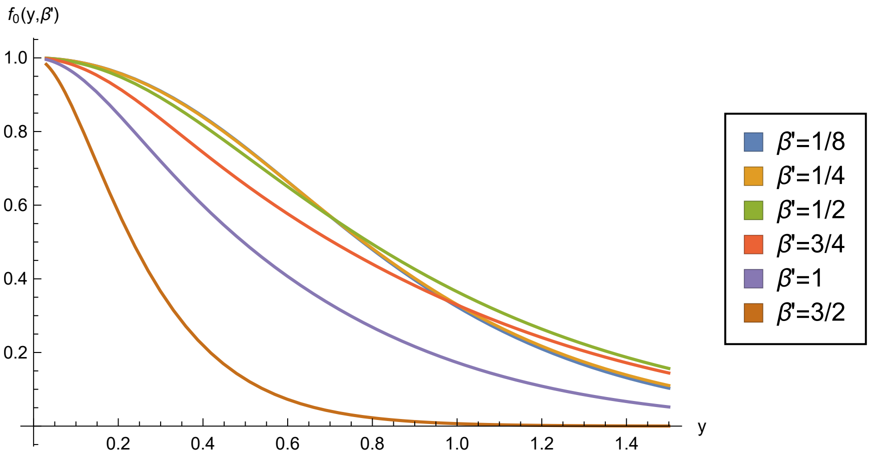

Figure 3.1: $f_0(y, \beta')$

In Figure 3.1, we are observing the profile of the function $f_0(y, \beta')$. We see that $f_0(y, \beta')$ decreases with increasing y. This means that with the increasing abelian field strength, B_1, we see a decrease in the pair production. This effect is already encountered in the abelian field case of Figure 2.1a. It is also possible to see further decrease in the pair production when β', non-abelian field strength, is increased. However, this effect becomes significant for $\beta' \gtrsim 0.61$ at sufficiently large y, and when $\beta' \lesssim 0.61$ the hierarchy between pair productions are reversed compared to the $\beta' \gtrsim 0.61$ case. But it is also important to note that when y is large this means that the corresponding electric field must be small, this means we cannot observe pair production for these values anyways.

3.1.2.2 Pair production comparison of non-abelian with abelian

Here we define the function $F_0(y, \beta') \equiv f_0(y, \beta')/f_0(y, 0)$, which is good for comparison of the pair production rates or spin-0 particles subject to a combination of an abelian and non-abelian magnetic field to that with purely an abelian magnetic field. In other words, we will be able to observe how the variation of β' affects the pair production rates.

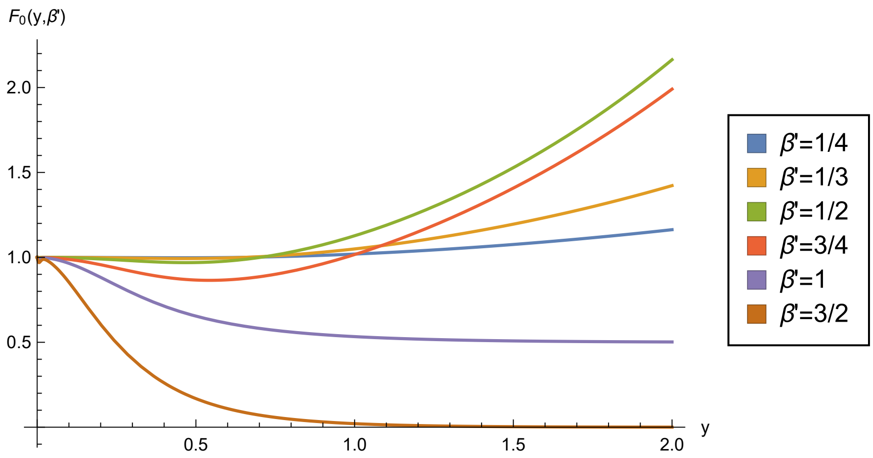

Figure 3.2: $F_0(y, \beta')$

Inspecting the graph of $F_0(y, \beta')$ in Figure 3.2, we see an interesting behavior is emerging, mainly there is a critical value for β' called, $\beta'_c = 1$, and when $\beta' = \beta'_c$ we see that in the limit of large y, $F_0(y, \beta')$ approaches to $1/2$. In another words when the non-abelian field strength has the value of β'_c, pair production rate quickly converges to half of what it was for the purely abelian case. This β'_c value can be found by comparing the energy values between abelian and non-abelian cases. More concretely, the lowest values of the energies [1] have the same value of B_1 [2] but since there are essentially double the amount of states for the abelian case with the same energy [3] the graph approaches to the value $1/2$.

For the cases with $\beta' > 1$ the situation is clear, the pair production decreases with the increasing non-abelian magnetic charge. This is because the ground state energy for the non-abelian case is always larger then the ground state energy for the purely abelian case, hence the states become harder and harder to fill compared to the abelian case, resulting in $F_0(y, \beta')$ to approach zero faster. For $0 < \beta' < 1$ case $F_0(y, \beta')$ has a small dip below one, then we see an increase for the larger values of y. However, for the large y values, we see two cases emerging with one for $\beta' \gtrsim 0.61$ and another for $\beta' \lesssim 0.61$ comparing the hierarchy between these different non-abelian

[1] isospin down part of Eq (3.9) with $n_1 = 1$ and $\beta' = 1$ and both isospin up, and down parts with $\beta' = 0$.

[2] Here, it is sufficient to focus on the lowest energy eigenvalues since the lowest energy eigenvalues are the most easily filled states. This fact can also be seen, numerically, from the form of $F_0(y, \beta')$.

[3] Since the value B_1 both comes from the isospin up and isospin down part for the abelian part.

magnetic charges, first case we will look at is $\beta' \lesssim 0.61$. Here the ground state energy eigenvalue for the non-abelian configuration is always less than of the abelian, and, decreases further with the increasing β' hence the function $F_0(y, \beta')$ for large y decrease with the increasing β'. For the $b < \beta' < 1$ case, the situation is somewhat reversed, i.e. here we see again, that the ground state eigenvalues are always lower for the non-abelian case, however the difference keeps getting smaller and smaller, hence we see that the hierarchy is increasing with the increasing β'.

3.2 Spinor Fields

We will now consider the pair production for spin-$1/2$ particles under the influence of both the abelian and non-abelian magnetic field introduced in the previous section. For this purpose, we will need the spectrum of the appropriate gauged Dirac operator. This problem is not solved before, therefore we first proceed to handle this task, outcome of which is relevant and interesting in its own right, and may be of relevance in the context of condensed matter physics.

3.2.1 Gauged Dirac operator and its spectrum

We may launch the discussion by writing out the gauged Dirac operator on the 2D flat space, $\mathbb{R}^2$. This is given as

$$\not{D} = \gamma_i (p_i - A_i) \tag{3.16}$$

γ_i are the 2×2 matrices spanning the Clifford algebra on $\mathbb{R}^2$, we may take them as $\gamma_1 = \tau_1$ and $\gamma_2 = \tau_2$, where τ_1, τ_2 are the 2×2 Pauli matrices. Components of the gauge fields are given as before (3.2).

$$A_x = -\frac{B}{2} y - \beta \sigma_y, \quad A_y = \frac{B}{2} x + \beta \sigma_x. \tag{3.17}$$

Note that since A_i has dimensions $[length]^{-1}$ we have non-abelian field strength β given dimensions of $[length]^{-1}$ as well, while B_1 has dimensions $[length]^{-2}$.

Now, with all this in mind, we can express $\not{D}$ as the following 2×2 block matrix

$$\not{D} = \begin{pmatrix} 0 & (p_x + \frac{B}{2} y + \beta \sigma_y) - i(p_y - \frac{B}{2} x - \beta \sigma_x) \\ (p_x + \frac{B}{2} y + \beta \sigma_y) + i(p_y - \frac{B}{2} x - \beta \sigma_x) & 0 \end{pmatrix}, \tag{3.18}$$

which, after some rearrangement of terms can be written as

$$\not{D} = -i \begin{pmatrix} 0 & (\partial_x - i\partial_y) + \frac{B}{2}(-x+iy) + \beta(-\sigma_x + i\sigma_y) \\ (\partial_x + i\partial_y) + \frac{B}{2}(x+iy) + \beta(\sigma_x + i\sigma_y) & 0 \end{pmatrix}. \quad (3.19)$$

Using the definitions (3.5) in $\not{D}$, we can write it in terms of the complex coordinates

$$\not{D} = -i \begin{pmatrix} 0 & 2\partial - \frac{B}{2}\bar{z} - 2\beta\sigma_- \\ 2\bar{\partial} - \frac{B}{2}z - 2\beta\sigma_+ & 0 \end{pmatrix}. \quad (3.20)$$

Using the operators a & $a^\dagger$ defined in (3.6), we can rewrite (3.20) as,

$$\not{D} = -i \begin{pmatrix} 0 & -(\sqrt{2B}a^\dagger + 2\beta\sigma_-) \\ (\sqrt{2B}a + 2\beta\sigma_+) & 0 \end{pmatrix}, \quad (3.21)$$

Let us introduce the magnetic length $l_B = \frac{1}{\sqrt{B}}$, and the dimensionless non-abelian field strength $\beta' = l_B\beta$. We may then, write the Dirac operator as,

$$\not{D} = \frac{i\sqrt{2}}{l_B} \begin{pmatrix} 0 & a^\dagger + \sqrt{2}\beta'\sigma_- \\ -a - \sqrt{2}\beta'\sigma_+ & 0 \end{pmatrix}, \quad (3.22)$$

We may consider the operator $\not{D}^2$. This is evaluated as

$$\begin{aligned} \not{D}^2 &= \omega_c \begin{pmatrix} 0 & a^\dagger + \sqrt{2}\beta'\sigma_- \\ -a - \sqrt{2}\beta'\sigma_+ & 0 \end{pmatrix} \begin{pmatrix} 0 & a^\dagger + \sqrt{2}\beta'\sigma_- \\ -a - \sqrt{2}\beta'\sigma_+ & 0 \end{pmatrix}, \\ &= \omega_c \begin{pmatrix} a^\dagger a + \sqrt{2}\beta'(a\sigma_- + a^\dagger\sigma_+) + 2\beta'^2 p_- & 0 \\ 0 & aa^\dagger + \sqrt{2}\beta'(a\sigma_- + a^\dagger\sigma_+) + 2\beta'^2 p_+ \end{pmatrix}, \end{aligned} \quad (3.23)$$

where $\omega_c = \frac{2}{l_B^2}$ and $p_\pm = \sigma_\pm\sigma_\mp$. Now we can expand the 2×2 blocks in (3.23) by writing the Pauli matrices explicitly. We have

$$\not{D}^2 = \omega_c \begin{pmatrix} a^\dagger a & \sqrt{2}\beta' a^\dagger & 0 & 0 \\ \sqrt{2}\beta' a & a^\dagger a + 2\beta'^2 & 0 & 0 \\ 0 & 0 & aa^\dagger + 2\beta'^2 & \sqrt{2}\beta' a^\dagger \\ 0 & 0 & \sqrt{2}\beta' a & aa^\dagger \end{pmatrix}. \quad (3.24)$$

In order to obtain spectrum of $\not{D}^2$, we write the eigenvalue equation for (3.24) as

$$\omega_c \begin{pmatrix} a^\dagger a & \sqrt{2}\beta' a^\dagger & 0 & 0 \\ \sqrt{2}\beta' a & a^\dagger a + 2\beta'^2 & 0 & 0 \\ 0 & 0 & aa^\dagger + 2\beta'^2 & \sqrt{2}\beta' a^\dagger \\ 0 & 0 & \sqrt{2}\beta' a & aa^\dagger \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}. \quad (3.25)$$

This leads to the set of operator equations

$$\omega_c(a^\dagger a\phi_1 + \sqrt{2}\beta' a^\dagger \phi_2) = \lambda\phi_1, \tag{3.26a}$$

$$\omega_c(\sqrt{2}\beta' a\phi_1 + a^\dagger a\phi_2 + 2\beta'^2\phi_2) = \lambda\phi_2, \tag{3.26b}$$

$$\omega_c(aa^\dagger\phi_3 + 2\beta'^2\phi_3 + \sqrt{2}\beta' a^\dagger\phi_4) = \lambda\phi_3, \tag{3.26c}$$

$$\omega_c(\sqrt{2}\beta' a\phi_3 + aa^\dagger\phi_4) = \lambda\phi_4. \tag{3.26d}$$

From (3.25) and (3.26), the eigenket space for the operator $\not{D}^2$ is observed to be the tensor product space $\mathcal{H} = \mathbb{C}^4 \otimes \mathcal{F}$. Here the Fock space spanned by the eigenstates of the number operator $N = a^\dagger a$, the basis of $\mathcal{F}$ may be denoted by $\{|n\rangle\}$ with $n \in \mathbb{Z}^+$. In order to solve the (3.26), it is sufficient to consider the subspace $\mathcal{F}$ for which we have the states $|n+1\rangle$, $|n\rangle$, and $|n-1\rangle$, with all possible combinations of the spin and isospin, namely, we have

$$|1\rangle = |n+1,+,+\rangle \doteq \begin{pmatrix} |n+1\rangle \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = |n,+,-\rangle \doteq \begin{pmatrix} 0 \\ |n\rangle \\ 0 \\ 0 \end{pmatrix},$$

$$|3\rangle = |n,-,+\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ |n\rangle \\ 0 \end{pmatrix}, \quad |4\rangle = |n-1,-,-\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ |n-1\rangle \end{pmatrix}. \tag{3.27}$$

Here the first spin component in the states refers to the spin, and the second refers to the isospin[4]. Computing the $\not{D}^2$ within this subspace, i.e determining the matrix elements $\not{D}^2_{ij} = \langle i|\not{D}^2|j\rangle$ with $i, j = 1, 2, 3, 4$, we find

$$\not{D}^2_{(n)} = \omega_c \begin{pmatrix} n+1 & \sqrt{2}\beta'\sqrt{n+1} & 0 & 0 \\ \sqrt{2}\beta'\sqrt{n+1} & n+2\beta'^2 & 0 & 0 \\ 0 & 0 & n+1+2\beta'^2 & \sqrt{2}\beta'\sqrt{n} \\ 0 & 0 & \sqrt{2}\beta'\sqrt{n} & n \end{pmatrix} \tag{3.28}$$

Eigenvalues of $\not{D}^2$ can be readily computed and we find.

$$\lambda_n^\pm = \frac{\omega_c}{2}(1 + 2n + 2\beta'^2 \pm \sqrt{1 + 4\beta'^2(1 + 2n + \beta'^2)}) \tag{3.29}$$

[4] Note that for the upper two component spinor spanned by $|1\rangle$ and $|2\rangle$, $n = 0, 1, 2, 3, \ldots$, while for the lower two component spinor, spanned by $|3\rangle$ and $|4\rangle$, $n = 1, 2, 3, \ldots$.

Subscript $\pm$ in $\lambda_n^{\pm}$ represents the isospin up and isospin down components of the energy spectrum, and each eigenvalue is two fold degenerate, within this subspace.

To prepare for writing down the corresponding eigenvectors let us define the following

$$a_{\pm} = \frac{1 + 2\beta'^2 \pm \sqrt{1 + 4\beta'^2(1 + 2n + \beta'^2)}}{2\sqrt{2n}\beta'}, \tag{3.30a}$$

$$b_{\pm} = \frac{1 - 2\beta'^2 \pm \sqrt{1 + 4\beta'^2(1 + 2n + \beta'^2)}}{2\sqrt{2n + 2\beta'}}. \tag{3.30b}$$

Now, using (3.30), we can write the eigenvectors of (3.28). After orthonormalizing the eigevectors, we may express them in the form

$$\psi_n^{+,1} = a_+ |3\rangle + |4\rangle, \tag{3.31a}$$

$$\psi_n^{+,2} = b_+ |1\rangle + |2\rangle, \tag{3.31b}$$

$$\psi_n^{-,1} = a_- |3\rangle + |4\rangle, \tag{3.31c}$$

$$\psi_n^{-,2} = b_- |1\rangle + |2\rangle, \tag{3.31d}$$

with $\psi_n^{+,(1,2)}$ corresponding to eigenvalues λ_n^+ and $\psi_n^{-,(1,2)}$ corresponding to λ_n^-. It is important to see that the eigenvectors written in (3.31) are only valid when $n \geq 1$. More specifically, eigenvectors $\psi^{+,1}$ and $\psi^{-,1}$ require $n \geq 1$ while the eigenvectors $\psi^{+,2}$ and $\psi^{-,2}$ require $n \geq 0$. This is caused by the definitions of the subspaces in (3.27). So, in order to complete the solution of the eigenvalue equation, we should inspect the solutions that include $|0, +, +\rangle$, $|0, +, -\rangle$, and $|0, -, +\rangle$ more carefully[5].

To elaborate on the last point, we may start by looking at the zero mode solutions taking the $\lambda = 0$ in (3.26). (3.26c) and (3.26d) are satisfied by only the trivial solutions, $\phi_3 = \phi_4 = 0$. Solving (3.26a) and (3.26b) we find the normalized eigenvectors

[5] Observe that these are the states that lies outside the limits we have given for n.

corresponding to the zero modes

$$\psi_0^1 = |0,+,+\rangle = \begin{pmatrix} |0\rangle \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.32a}$$

$$\psi_0^2 = \frac{(-\sqrt{2}\,|1,+,+\rangle + |0,+,-\rangle)}{\sqrt{3}} = \frac{1}{\sqrt{3}} \begin{pmatrix} -\sqrt{2}\,|1\rangle \\ |0\rangle \\ 0 \\ 0 \end{pmatrix}. \tag{3.32b}$$

Here, the first eigenvector corresponds to both spin and isospin up case, while the second is a combination of spin up isospin up, and spin up isospin down states. Another state that is not covered by the subspace in (3.27) can be seen by taking $\phi_3 = |0\rangle$ and $\phi_1 = \phi_2 = \phi_4 = 0$, which gives us $\lambda = \omega_c(1 + 2\beta'^2)$ [6]. Taking $\phi_3 = |0\rangle$, and all others zero means that we have this eigenvalue for spin down, isospin up.

3.2.2 Pair production

Having calculated the eigenvalues of the Dirac operator, we are in a position to calculate the pair production rates. We start by writing the spectrum for the total system on the $\mathbb{R}^2 \times \mathbb{R}^2$

$$Spec(\not{D}^2 + m^2) = \begin{cases} B_1\Big(1 + 2n_1 + 2\beta'^2 \pm \sqrt{1 + 4\beta'^2(1 + 2n_1 + \beta'^2)}\Big) \\ +2B_2(n_2 + 1) + m^2, \\ B_1\Big(1 + 2n_1 + 2\beta'^2 \pm \sqrt{1 + 4\beta'^2(1 + 2n_1 + \beta'^2)}\Big) \\ +2B_2 n_2 + m^2, \end{cases} \tag{3.33}$$

The degeneracies for each case given as before for the abelian case in Section 2.4, we can write the degeneracies simply as $B_1 B_2/(2\pi)^2$.

Now, we are in a position to write the one-loop effective action, for the Euclidean space as

$$\Gamma_E = -\frac{1}{2} Tr\, log\, (\not{D}^\dagger + m)(\not{D} + m). \tag{3.34}$$

[6] Although this eigenvalue can be obtained by taking $n_1 = 0$ in $\lambda^+_{n_1}$, we have to make this calculation to obtain its eigenstate.

By writing the spectrum, and the zero modes explicitly, then performing the summation over the n_2, which are the variables used to describe energy eigenvalues B_2, we get

$$\Gamma_E = \frac{B_1 B_2}{4\pi^2} \int d^4x \lim_{\epsilon\to 0} \frac{ds}{s} coth\, sB_2 \Bigg[\sum_{n_1=0}^{\infty} e^{-s(1+2n_1+2\beta'^2+\sqrt{1+4\beta'^2(1+2n_1+\beta'^2)})+m^2} + \sum_{n_1=1}^{\infty} e^{-s(1+2n_1+2\beta'^2-\sqrt{1+4\beta'^2(1+2n_1+\beta'^2)})+m^2} + e^{-sm^2} \Bigg]. \quad (3.35)$$

Note that, since the eigenvalues are two fold degenerate, the factor $\frac{1}{2}$ in (3.34) is multiplied by a factor of 2 resulting in the $\frac{1}{4\pi^2}$ factor in (3.35) instead of $\frac{1}{8\pi^2}$ in (2.77) in subsection 2.5.2. We also keep in mind that we have also two fold degeneracy in the zero modes.

Performing the s-integration, evaluating the summation due to the residue integration and Wick rotating the Γ_E to Minkowski time allows us to write $Re(iS_{eff})$ as

$$Re(iS_{eff}) = -\int d^4x \frac{E^2}{2\pi^2} y \Bigg[log\,(1 - e^{\pi m^2/E}) + \sum_{n_1=0} log\,(1 - e^{-y\pi\left(1+2n_1+2\beta'^2+\sqrt{1+4\beta'^2(1+2n_1+\beta'^2)}\right)}) + \sum_{n_1=1} log\,(1 - e^{-y\pi\left(1+2n_1+2\beta'^2-\sqrt{1+4\beta'^2(1+2n_1+\beta'^2)}\right)}) \Bigg]. \quad (3.36)$$

Here, we have again used the definition $y \equiv B_1/E$. We are also defining $f_{1/2}(y, \beta')$ by

$$Re(iS_{eff}) = -\int d^4x \frac{E^2}{24\pi} f_{1/2}(y, \beta'), \quad (3.37)$$

with $f_{1/2}(y, \beta')$

$$f_{1/2}(y, \beta') = \frac{12y}{\pi} \Bigg[log\,(1 - e^{\pi m^2/E}) + \sum_{n_1=0} log\,(1 - e^{-y\pi\left(1+2n_1+2\beta'^2+\sqrt{1+4\beta'^2(1+2n_1+\beta'^2)}\right)}) + \sum_{n_1=1} log\,(1 - e^{-y\pi\left(1+2n_1+2\beta'^2-\sqrt{1+4\beta'^2(1+2n_1+\beta'^2)}\right)}) \Bigg]. \quad (3.38)$$

We can also define the ratio $F_{1/2}(y, \beta') \equiv f_{1/2}(y, \beta')/f_{1/2}(y, 0)$, measuring the effect of the non-abelian gauge field. Now we may proceed to explain the explain the behavior of the pair production rates by inspecting the graphs of $f_{1/2}(y, \beta')$ and $F_{1/2}(y, \beta')$, for different values of β'.

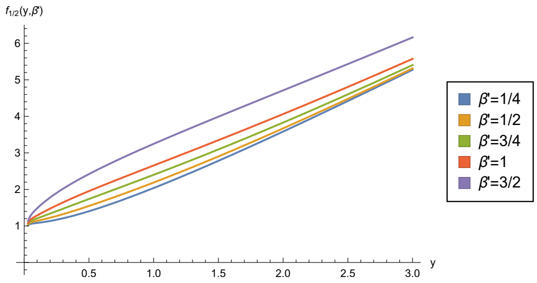

Figure 3.3a: $f_{1/2}(y, \beta')$

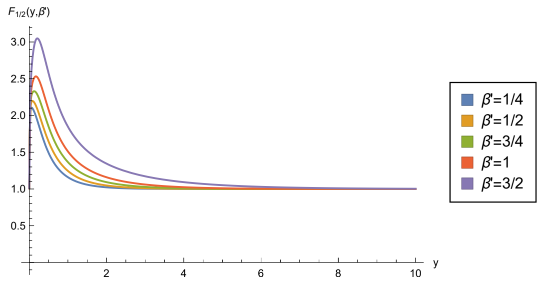

Figure 3.3b: $F_{1/2}(y, \beta')$

Looking at the Figure 3.3a we see a quite simple behavior. The pair production increases with increasing y, this is caused by the existence of the zero modes in the energy spectrum. Since there exists zero modes, the minimum energy is always the same, irrespective of the value of y or β', hence the energy of easiest to fill states do not change. How many states there can be for a given y value changes, in other words the degeneracy changes since it is proportional to B_1. Therefore, if we increase y there are more and more zero energy states to fill, so the pair production increases with increasing y. It is also clear that the pair production increases with the increasing β', for a given y value.

Effect of the increase in the non-abelian magnetic charge is easier to see for the Figure 3.3b, since we are comparing the non-abelian and abelian cases directly. Clearly, the effect we observed, mainly the increase in the non-abelian charge causes an increase in the pair production is also apparent here. Explanation for this behavior goes two-fold, first the increase in the non-abelian magnetic charge, β', is decreasing the energies for the isospin down excited energy values. This shows us that the decrease in the energies of isospin down states "outweigh" the increase in the energies of the isospin up states. Second part for the explanation comes from the fact, for smaller values of y there is a large increase in $F_{1/2}(y, \beta')$, this is followed by a sharp decline, meaning pair production of non-abelian and abelian cases with $F_{1/2}(y, \beta')$ the value 1 for large y values. This is caused by the fact that the increase in y increases the excited energies for both non-abelian and the abelian cases, hence the effect of the zero modes on the pair production becomes much more pronounced, and dominant in both cases, irrespective of this hierarchy of the excited states.

CHAPTER 4

PAIR PRODUCTION ON $S^2 \times \mathbb{R}^{1,1}$ WITH NON-ABELIAN MAGNETIC FIELDS

We will now direct our attention to the pair production effects on the manifold $S^2 \times \mathbb{R}^{1,1}$. In addition to the effects of abelian and non-abelian magnetic fields on S^2, here we will inspect how the inclusion of the curvature changes the pair production rates, for scalar and spinor fields. In particular, we will examine the ratios of pair production rates of spherical to flat cases, both for the scalar and the spinor fields. The new ingredient here compared to the review given in Section 2.4 is the non-abelian magnetic field, and the additional isospin degree of freedom of the charged particles, and these lead to novel physical effects.

4.1 Complex Scalar Field Subject to Uniform Non-abelian Magnetic field

In order to set up the problem, we may first consider the spectrum of the gauged Laplacian on the product Euclidean manifold $S^2 \times \mathbb{R}^2$. Gauge field strength is composed of two magnetic fields, with one abelian magnetic field on the $\mathbb{R}^2$ component and non-abelian magnetic field on S^2. Compared to the discussion in Section 2.4, difference here will be caused by the presence of the non-abelian magnetic field and the isospin degree of freedom of the charged fields mentioned above, and consequently there will be important distinctions compared to the abelian case treated in Section 2.4. Details of the calculation for the spectrum of non-abelian magnetic field on the 2-sphere is given in [13], however we will provide a review here, for completeness.

4.1.1 Laplacian operator and its spectrum

Consider a two-sphere of radius a, with the radial vector denoted as $\vec{r} = a\hat{r}$. Let us introduce an $SU(2)$ gauge field $\vec{A}$ as follows

$$\vec{A} = \vec{A}_{abelian} + \alpha \frac{\vec{r} \times \vec{\sigma}}{a^2} \tag{4.1}$$

In (4.1), $\vec{A}_{abelian}$ stands for the gauge potential of a Dirac monopole with magnetic charge $N/2$, $N \in \mathbb{Z}$, and $\vec{\sigma}$ are the Pauli matrices spanning the $SU(2)$ gauge symmetry, which we call the isospin in this context. Associated magnetic field strength is computed via $\vec{B} = \vec{\nabla} \times \vec{A}$ and yields a radial magnetic field; which takes the form[1]

$$B_r = \frac{N}{2a^2} + 2\alpha(\alpha - 1)\frac{\vec{r} \cdot \vec{\sigma}}{a^3} \tag{4.2}$$

From (4.2) it is manifestly seen that the magnetic field has the symmetry $\alpha \to (\alpha - 1)$, which is caused by the gauge transformation $U := \sigma_r = \vec{\sigma} \cdot \hat{r}$ [2]. This will become important later in our discussion. Gauged Laplace operator on S^2 may be written as

$$D^2 = \frac{\vec{\Lambda}^2}{a^2}, \tag{4.3}$$

where $\vec{\Lambda}$ is given as

$$\vec{\Lambda} \equiv \vec{r} \times (\vec{p} - \vec{A}). \tag{4.4}$$

After several steps of calculation (4.3) can be expressed as

$$D^2 = \frac{1}{a^2}\left[\vec{J}^2 + \frac{1}{4} + 2\alpha(\alpha - 1) + (2\alpha - 1)\left(\vec{J} \cdot \vec{\sigma} - \frac{1}{2} + \frac{N}{2}\sigma_r\right) + \frac{N}{2}\sigma_r\right]. \tag{4.5}$$

In (4.5) the total angular momentum, $\vec{J}$, is defined as

$$\vec{J} = \vec{r} \times (\vec{p} - \vec{A}_{abelian}) + \frac{N}{2}\hat{r} + \frac{\vec{\sigma}}{2} = \vec{\Lambda}_{abelian} + \frac{N}{2}\hat{r} + \frac{\vec{\sigma}}{2}, \tag{4.6}$$

This involves the contribution of the orbital angular momentum of charged particles, angular momentum of the particle-Dirac monopole electromagnetic field and that due to the $SU(2)$ isospin. We find the spectrum of D^2 is given as [13],.

$$\lambda^{\pm}_{n_1} = \frac{1}{a^2}\left[n_1(N + n_1) + 2\alpha(\alpha - 1) \pm \sqrt{(2\alpha - 1)^2(n_1 + N)n_1 + N^2/4}\right] \tag{4.7}$$

$$n_1 = 0, 1, 2, ...$$

[1] Note that choice of gauge for $\vec{A}_{abelian}$ is immaterial for our purposes.
[2] $UB_r(\alpha)U = B_r(\alpha - 1)$

It is straightforward to see that D^2 and $\vec{J}$ commute. Only nontrivial commutators are $[J_i, \vec{J}\cdot\vec{\sigma}]$ and $[J_i, \sigma_r]$, and they yield zero after the explicit calculation which are given below:

$$\begin{aligned}
[J_i, J_j\sigma_j] &= [L_i + \frac{1}{2}\sigma_i, (L_j + \frac{1}{2}\sigma_j)\sigma_j] = [L_i + \frac{1}{2}\sigma_i, L_j\sigma_j], \\
&= [L_i, L_j]\sigma_j + L_j[L_i, \sigma_j] + \frac{1}{2}[\sigma_i, L_j]\sigma_j + \frac{1}{2}L_j[\sigma_i, \sigma_j], \\
&= [L_i, L_j]\sigma_j + \frac{1}{2}L_j[\sigma_i, \sigma_j] = i\epsilon_{ijk}(L_k\sigma_j + L_j\sigma_k) = 0,
\end{aligned} \tag{4.8a}$$

$$\begin{aligned}
[J_i, \sigma_r] &= \Big[J_i, \frac{\sigma_n r_n}{\sqrt{r_m r_m}}\Big] = \Big[\epsilon_{ijk} r_j(p_k - A_k) - \frac{N}{2}\frac{r_i}{\sqrt{r_l r_l}} + \frac{\sigma_i}{2}, \frac{\sigma_n r_n}{\sqrt{r_m r_m}}\Big], \\
&= \epsilon_{ijk}\Big[r_j(p_k - A_k), \frac{\sigma_n r_n}{\sqrt{r_m r_m}}\Big] - \frac{N}{2}\Big[\frac{r_i}{\sqrt{r_l r_l}}, \frac{\sigma_n r_n}{\sqrt{r_m r_m}}\Big] + \frac{r_n}{2r}[\sigma_i, \sigma_n], \\
&= \epsilon_{ijk} r_j\Big[(p_k - A_k), \frac{\sigma_n r_n}{\sqrt{r_m r_m}}\Big] + \epsilon_{ijk}\Big[r_j, \frac{\sigma_n r_n}{\sqrt{r_m r_m}}\Big](p_k - A_k) + \frac{r_n}{2r}[\sigma_i, \sigma_n], \\
&= \epsilon_{ijk} r_j\Big[p_k, \frac{\sigma_n r_n}{\sqrt{r_m r_m}}\Big] + \frac{r_n}{2r}[\sigma_i, \sigma_n] = -i\epsilon_{ijk} r_j \sigma_n \frac{r^2\delta_{nk} - r_n r_k}{r^3} + \frac{r_n}{2r}[\sigma_i, \sigma_n], \\
&= -i\epsilon_{ijk}\frac{r_j}{r}\sigma_n\delta_{nk} + \frac{r_n}{2r}[\sigma_i, \sigma_n] = -i\epsilon_{ijk}\frac{r_j}{r}\sigma_k + \frac{r_n}{2r}2i\epsilon_{ink}\sigma_k = 0.
\end{aligned} \tag{4.8b}$$

Therefore, we conclude that each energy level in (4.7) is $2j+1$ fold degenerate, where $j = n_1 + \frac{N-1}{2}$, with $N \in \mathbb{Z}$ and $N \geq 1$ and $n_1 = 0, 1, 2, ...$

We may express the spectrum of the operator $D^2 + m^2$ operator on the $\mathbb{R}^2 \times S^2$ as

$$\begin{aligned}
Spec(D^2 + m^2) =& \Big(n_1(N + n_1) + 2\alpha(\alpha - 1) \pm \sqrt{(2\alpha - 1)^2(n_1 + N)n_1 + N^2/4)}\Big)/a^2 \\
&+ B_2(2n_2 + 1) + m^2 \begin{cases} n_1 = 0, 1, 2, ... \\ n_2 = 0, 1, 2, ... \end{cases},
\end{aligned} \tag{4.9}$$

where we have added the Landau level energies for the $\mathbb{R}^2$ part subject to a B_2 field in the standard manner.

4.1.2 Pair production for the scalar fields

Now we will calculate the pair production rates for scalar fields under the influence of non-abelian fields on a $\mathbb{R}^{1,1} \times S^2$. To do so, we will start with the Euclidean counterpart of this space, meaning $\mathbb{R}^2 \times S^2$, in this space using the result (4.9), the

Euclidean one-loop effective action takes the form,

$$\Gamma_E = -\frac{1}{16\pi^2 a^2}\int d^2x \int d\Omega_2 \int \frac{ds}{s}\frac{B_2}{sinh\, sB_2}$$
$$\times\left\{\sum_{n_1=0}^{\infty}(2n_1+N)\, e^{-s[n_1(N+n_1)+2\alpha(\alpha-1)+\sqrt{(2\alpha-1)^2(n_1+N)\,n_1+N^2/4}]/a^2}\right.$$
$$\left.+\left\{\sum_{n_1=1}^{\infty}(2n_1+N)\, e^{-s[n_1(N+n_1)+2\alpha(\alpha-1)-\sqrt{(2\alpha-1)^2(n_1+N)\,n_1+N^2/4}]/a^2}\right\}\right. \quad (4.10)$$

After performing the Wick rotation $\mathbb{R}^2 \to \mathbb{R}^{1,1}$, $B_2 \to -iE$ and performing integration over s, we can write the $Re(iS_{eff})$ as

$$Re(iS_{eff}) = -\int d^2x \int d\Omega_2\, \frac{E^2}{16\pi^2}\beta_0(\omega), \quad (4.11)$$

where $\beta_0(\omega)$ given as

$$\beta_0(\omega) = \omega\left\{\sum_{n_1=0}^{\infty}(2n_1+N)\, log\,(1+e^{-\omega[n_1(N+n_1)+2\alpha(\alpha-1)+\sqrt{(2\alpha-1)^2 n_1(n_1+N)+N^2/4}]})\right.$$
$$\left.+\sum_{n_1=1}^{\infty}(2n_1+N)\, log\,(1+e^{-\omega[n_1(N+n_1)+2\alpha(\alpha-1)-\sqrt{(2\alpha-1)^2 n_1(n_1+N)+N^2/4}]})\right\}. \quad (4.12)$$

In (4.12) we have defined the ω to be, $\omega := \pi/Ea^2$. In order to compare the pair production rates on this geometry to the result on $\mathbb{R}^{3,1}$, in Section 3.1, we first of our result for limit $S^2 \to \mathbb{R}^2$ the flat limit of the result we have found above in Eq. (4.12). To compute this limit, we take $a^2 \to \infty$, while keeping both $N/2a^2$ and α/a constant. Since the definition of ω already contains the term $1/a^2$, we can keep ωN or similar combinations as such.

Another important remark is that when comparing the effect of non-abelian fields on the pair production, using the α as the non-abelian field strength presents some difficulties. This is mainly due to the fact that the non-abelian magnetic field strength is symmetric for $\alpha \to (\alpha - 1)$, as can be seen from (4.2), but for the abelian magnetic field strength there is no such symmetry. For ease in comparison we define the variable $\gamma \equiv \alpha(\alpha-1)$ as a convenient parameter to work with. While taking the flat limit, $a \to \infty$, we can keep the product $\omega\gamma$ the same for both flat and spherical cases. For the flat case γ will be proportional to β^2 where $\beta^2 = B_1\beta'^2$ as it is apparent from Eq. (3.1).

Now we express our result in (4.12) in terms of γ as

$$\beta_0(\omega) = \omega \Bigg\{ \sum_{n_1=0}^{\infty} (2n_1 + N)\, log\,(1 + e^{-\omega[n_1(N+n_1)+2\gamma+\sqrt{(4\gamma+1)n_1(n_1+N)+N^2/4}]}) + \sum_{n_1=1}^{\infty} (2n_1 + N)\, log\,(1 + e^{-\omega[n_1(N+n_1)+2\gamma-\sqrt{(4\gamma+1)n_1(n_1+N)+N^2/4}]}) \Bigg\}. \tag{4.13}$$

By taking the flat limit we obtain

$$\beta_0^{flat}(\omega) = \omega N \Bigg\{ \sum_{n_1=0}^{\infty} log\,(1 + e^{-\omega[n_1 N+2\gamma+\sqrt{4\gamma N n_1+N^2/4}]}) + \sum_{n_1=1}^{\infty} log\,(1 + e^{-\omega[n_1 N+2\gamma-\sqrt{4\gamma N n_1+N^2/4}]}) \Bigg\} \tag{4.14}$$

We can see that this matches with what we have found in Subsection 3.1.2 in (3.15), since we have identified $B_1 = \frac{N}{2a^2}$ and $\gamma = \frac{\alpha^2}{a^2}$.

Now we can define the ratio $\gamma_0(\omega) \equiv \beta_0(\omega)/\beta_0^{flat}(\omega)$, and compare the pair production rates by plotting $\gamma_0(\omega)$ for different values of the abelian, and the non-abelian magnetic charges. We first inspect the profile of $\gamma_0(\omega)$ at fixed values of γ. To be precise, we consider $\gamma = 1, 2$ for $N = 1, 2, 3, 4, 5$. Profiles of $\gamma_0(\omega)$ are given in the figures below.

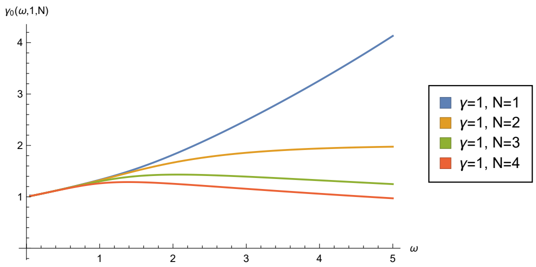

Figure 4.1a: $\gamma = 1$

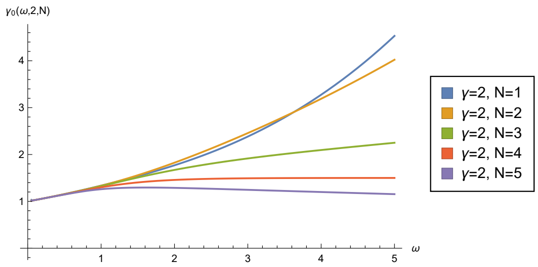

Figure 4.1b: $\gamma = 2$

From Figure 4.1a and Figure 4.1b we see for all values of γ and N there is an increase of the pair production rates at small ω values, since $\gamma_0(\omega) \geq 1$. This increase is caused by the fact that there is an extra term in the density of states factor, $\frac{2j+1}{4\pi a^2}$ in $\beta_0(\omega)$, which arises from the curvature. However, for larger ω values we see a more complicated result compared to the abelian case in Section 2.4. Mainly, there is now a critical value for the non-abelian charge, which we may call γ_c, and $\gamma_c = N/2$. Accordingly the limiting behavior of $\gamma_0(\omega)$ at large ω is described as follows.

$$\gamma_0(\omega, \gamma, N) \xrightarrow{\omega\to\infty} \begin{cases} \infty & \gamma > \gamma_c \\ \frac{N+2}{N} & \gamma = \gamma_c \\ 0 & \gamma < \gamma_c \end{cases} \tag{4.15}$$

Effect in Eq. (4.15) is quite general and can be traced back to the energy spectrum and degeneracies of the gauged Laplacian, as it is discussed in Appendix B. Physical meaning of this effect may be stated as follows: If the ground state energy of the spherical case is larger compared to the flat one, $\gamma_0(\omega)$ will diverge at large ω; if the ground state energy for the flat case is higher than the spherical, $\gamma_0(\omega)$ converges to zero at large ω; and finally if the two ground state energies are the same, then the limiting behavior of $\gamma_0(\omega)$ is given by the ratio of the degeneracies. This is because when the energies are lower the states are easier to fill by the produced pairs, hence they will have a tendency to fill the states with the lower energies, for example if the ground state energy of the spherical case are higher than the flat one, denominator of $\gamma_0(\omega)$ will be larger, hence the limiting value 0. If the energies are the same for both cases, more states that are available to be filled i.e. with larger degeneracy will yield more pair production.

Besides the dependence of γ_c and limiting value of $\gamma_0(\omega)$ on N, we also see that N effects the hierarchy of the pair production rates. For smaller N values we see that the pair production rates are generally higher, this is caused by the fact that the energies increase with increasing abelian charge, hence the states become harder and harder to fill by the produced pairs.

We may also consider the profiles of $\gamma_0(\omega)$ at fixed values of N ($N = 1, 2$), while we take γ to be $\gamma = 1, 2, 3, 4$. These plots are given in Figure 4.2a and Figure 4.2b.

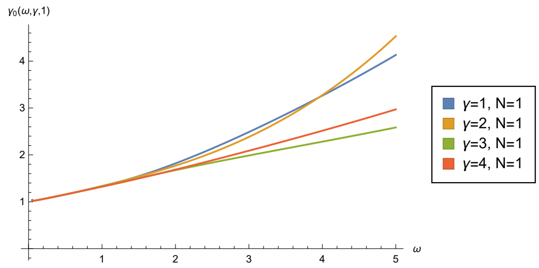

Figure 4.2a: $N = 1$

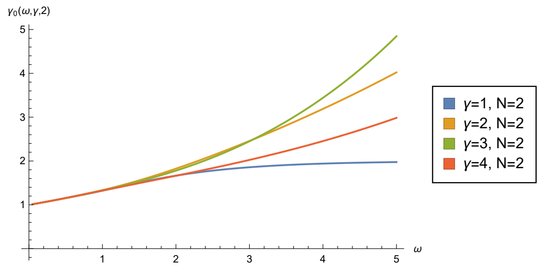

Figure 4.2b: $N = 2$

Here we have, $\gamma_c = 0.5$ and $\gamma_c = 1$ respectively, and the plots are, in agreement with (4.15). For the plot in Figure 4.2a, since all γ values we inspect are larger then γ_c, $\gamma_0(\omega)$ increases monotonically with γ. For the plot in Figure 4.2b the profiles with $\gamma = 2, 3, 4$ increase monotonically, while $\gamma = 1$ profile first increases then settles to the limiting value predicted in (4.15).

Finally, we may compare the pair production rates of $\beta_0(\omega, \gamma, N)$ with that in the absence of the non-abelian magnetic fields. For this purpose we define $R_0(\omega, \gamma, N) \equiv \beta_0(\omega, \gamma, N)/\beta_0(\omega, 0, N)$. This will allow us to further elaborate on the effects of the non-abelian magnetic fields on the pair production.

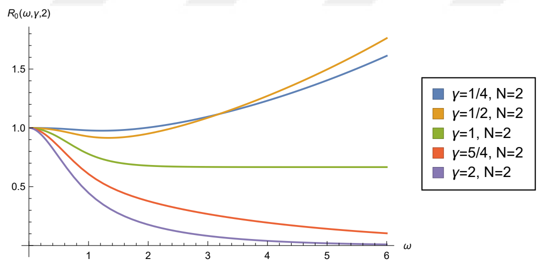

Figure 4.3a: $N = 2$

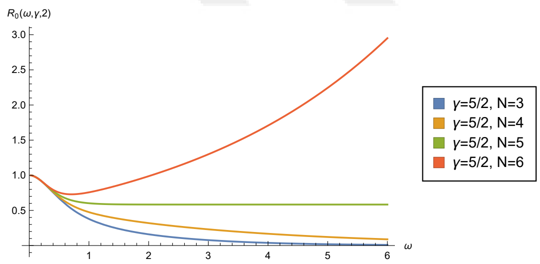

Figure 4.3b: $\gamma = 2.5$

Inspecting the Figure 4.3a and Figure 4.3b we see that there is a decrease in the pair production rates under the influence of both abelian and non-abelian case compared to fields under the influence of purely abelian magnetic field, for smaller values of ω. This can be essentially attributed to the fact that the profile of the function $R_0(\omega, \gamma, N)$ is governed by the leading terms in both $\beta_0(\omega, \gamma, N)$ and $\beta_0(\omega, 0, N)$. These leading terms are the states with the larger larger contribution to $\beta_0(\omega)$, and it can be seen in Figure C.6 in Appendix C, that the spin-up eigenvalues of the energy of the particles under the influence of both abelian and non-abelian magnetic fields are always higher than the particles under the influence of purely abelian magnetic fields. Possibly spin down states are not at a sufficiently low energy to balance or alter this behaviour. More numerical analysis may be useful to further elaborate on this point, but such an analysis is beyond the scope of this thesis.

We also see that our previous conclusion with γ_c holds true indeed. In this case we find that the limiting behavior of $R_0(\omega, \gamma, N)$ is described by

$$R_0(\omega, \gamma, N) \xrightarrow{\omega \to \infty} \begin{cases} \infty & \gamma < \gamma_c \\ \frac{N+2}{2N+2} & \gamma = \gamma_c \\ 0 & \gamma > \gamma_c \end{cases} \tag{4.16}$$

This time for limit to be ∞ we need to have $\gamma < \gamma_c$, opposite of what we have found for the $\gamma_0(\omega)$ case, since compared to the ground state energy of the particles under the influence of both abelian and non-abelian magnetic fields, the ground state energy of the purely abelian magnetic field case becomes smaller when $\gamma < \gamma_c$.

4.2 Dirac fields on $S^2 \times \mathbb{R}^{1,1}$

Now let us investigate the same case for the Dirac fields. Here we again have the total space of the form $\mathbb{R}^2 \times S^2$ on which we have our two magnetic fields, an abelian magnetic field on the $\mathbb{R}^2$ and a non-abelian field on the S^2. We will rotate $\mathbb{R}^2$ to $\mathbb{R}^{1,1}$ as usual to obtain an electric field in the course of the calculation of the pair production rate. We start with discussing of the spectrum of the Dirac operator with the non-abelian uniform magnetic field background on the two-sphere.

4.2.1 Spectrum of the gauged Dirac operator

We consider the Dirac operator in the background of the total magnetic field introduced in (4.1). This Dirac operator can be written as $\not{D} = \vec{\tau} \cdot \vec{\Lambda}$ where $\vec{\tau}$ are the Pauli matrices, spanning the Clifford algebra $\{\tau_\alpha \tau_\beta\} = 2\delta_{\alpha\beta}$, and $\vec{\Lambda}$ is defined as previously in (4.4).

$\vec{\Lambda}$ can be written as

$$\vec{\Lambda} = \vec{r} \times (\vec{p} - \vec{A}_{abelian} - \vec{A}_{non-abelian}),$$
$$= \vec{\Lambda}_{abelian} - \vec{r} \times \vec{A}_{non-abelian}, \tag{4.17}$$

where the $\vec{\Lambda}_{abelian}$ and $\vec{A}_{non-abelian}$ are given as before

$$\vec{\Lambda}_{abelian} = \vec{L} + \frac{N}{2}\hat{r}, \quad \vec{A}_{non-abelian} = \alpha\frac{\vec{r} \times \vec{\sigma}}{r^2}. \tag{4.18}$$

Here $\vec{L}$ is the angular momentum with $[L_\alpha, L_\beta] = i\epsilon_{\alpha\beta\gamma}L_\gamma$, which involves the orbital angular momentum of the charged particles and the angular momentum of the abelian electromagnetic field due to the Dirac monopole. It neither involves the spin nor the isospin, so it is not the total angular momentum. With these we can write $\vec{\Lambda}$ as

$$\vec{\Lambda} = \vec{L} + \frac{N}{2}\hat{r} + \alpha(\vec{\sigma} - \sigma_r\hat{r}) \tag{4.19}$$

To determine the spectrum of $\not{D}$ we may work with its square $\not{D}^2 = (\vec{\tau} \cdot \vec{\Lambda})^2$. This can be expressed as

$$\begin{aligned}
\not{D}^2 &= (\vec{\tau} \cdot \vec{\Lambda})^2, \\
&= \tau_i\tau_j\Lambda_i\Lambda_j = (\delta_{ij} + i\epsilon_{ijk}\tau_k)\Lambda_i\Lambda_j, \\
&= \Lambda^2 + \frac{i}{2}\epsilon_{ijk}[\Lambda_i, \Lambda_j]\tau_k, \\
&= \Lambda^2 + \frac{i}{2}\epsilon_{ijk}[L_i + \frac{N}{2}\hat{r}_i + \alpha(\sigma_i - \sigma_n\hat{r}_n\hat{r}_i), L_j + \frac{N}{2}\hat{r}_j + \alpha(\sigma_j - \sigma_m\hat{r}_m\hat{r}_j)]\tau_k, \\
&= \Lambda^2 + \frac{i}{2}\epsilon_{ijk}\Big([L_i, L_j] + \frac{N}{2}[L_i, \hat{r}_j] + \frac{N}{2}[\hat{r}_i, L_j] - \alpha[L_i, \sigma_m\hat{r}_m\hat{r}_j] - \alpha[\sigma_n\hat{r}_n\hat{r}_i, L_j], \\
&- \alpha^2[\sigma_i, \sigma_m\hat{r}_m\hat{r}_j] - \alpha^2[\sigma_n\hat{r}_n\hat{r}_i, \sigma_j] + \alpha^2[\sigma_i, \sigma_j] + \alpha^2[\sigma_n\hat{r}_n\hat{r}_i, \sigma_m\hat{r}_m\hat{r}_j]\Big)\tau_k, \\
&= \Lambda^2 + \frac{i}{2}\epsilon_{ijk}\Big([L_i, L_j] + \frac{N}{2}[L_i, \hat{r}_j] + \frac{N}{2}[\hat{r}_i, L_j] - \alpha\sigma_m[L_i, \hat{r}_m\hat{r}_j] - \alpha\sigma_n[\hat{r}_n\hat{r}_i, L_j], \\
&- \alpha^2\hat{r}_m\hat{r}_j[\sigma_i, \sigma_m] - \alpha^2\hat{r}_n\hat{r}_i[\sigma_n, \sigma_j] + \alpha^2[\sigma_i, \sigma_j] + \alpha^2\hat{r}_n\hat{r}_i\hat{r}_m\hat{r}_j[\sigma_n, \sigma_m]\Big)\tau_k, \\
&= \Lambda^2 + \vec{\tau} \cdot \vec{\Lambda} - \frac{N}{2}\hat{r} \cdot \vec{\tau} + 1 + 2\alpha(1 - \alpha)(\vec{\tau} \cdot \hat{r})(\vec{\sigma} \cdot \hat{r}).
\end{aligned} \tag{4.20}$$

In order to proceed we may first write an auxiliary step for $\not{D}^2$ in terms of $\vec{J}$ which is defined as $\vec{J} = \vec{L} + \frac{\vec{\sigma}}{2}$.

$$\begin{aligned}
\not{D}^2 =& J^2 + (2\alpha - 1)\vec{J} \cdot \vec{\sigma} + \frac{3}{4} - 3\alpha + 2\alpha^2 + N\alpha\vec{\sigma} \cdot \hat{r} - \frac{N^2}{4} + \vec{\tau} \cdot \vec{J} \\
&+ (\alpha - \frac{1}{2})\vec{\tau} \cdot \vec{\sigma} + \alpha(1 - 2\alpha)(\vec{\sigma} \cdot \hat{r})(\vec{\tau} \cdot \hat{r}) + 1
\end{aligned} \tag{4.21}$$

Introducing the total angular momentum operator $\vec{K}$ and the operator $\vec{\tau} \cdot \vec{J}$

$$\vec{K} = \vec{J} + \frac{\vec{\tau}}{2} = \vec{L} + \frac{\vec{\sigma}}{2} + \frac{\vec{\tau}}{2}, \tag{4.22a}$$

$$\vec{\tau} \cdot \vec{J} = K^2 - J^2 - \frac{\tau^2}{4}, \tag{4.22b}$$

we may express $\not{D}^2$ as

$$\begin{aligned}\not{D}^2 =& K^2 - \Big(\frac{N^2}{4} + 2\alpha(1-\alpha) - \frac{1}{2}\Big)\mathbb{1}_4 \\ &+ [(2\alpha - 1)(\vec{K} \cdot \vec{\sigma} - \frac{1}{2}) + N\alpha\vec{\sigma} \cdot \hat{r} + \alpha(1-2\alpha)(\vec{\sigma} \cdot \hat{r})(\vec{\tau} \cdot \hat{r})].\end{aligned} \tag{4.23}$$

In this expression first two terms are already diagonal, but we have to diagonalize the operator in the square parenthesis. We define this part of $\not{D}^2$ as

$$\chi := (2\alpha - 1)(\vec{K}.\vec{\sigma} - \frac{1}{2}) + N\,\alpha\,\vec{\sigma}.\hat{r} + \alpha(1-2\alpha)(\vec{\sigma}.\hat{r})(\vec{\tau}.\hat{r}), \tag{4.24}$$

and consider its square, which is given as

$$\chi^2 = (2\alpha - 1)^2(K^2 + \frac{1}{4}) + \alpha(1-\alpha)[N^2 - (2\alpha - 1)^2]. \tag{4.25}$$

Since K^2 is the square of the total angular momentum, its eigenvalues are $k(k+1)$ with k taking integer and half integer values, and hence we can write the $\not{D}^2$ operators eigenvalues

$$\begin{aligned}\not{D}_k^2 =& k(k+1) - N^2/4 - 2\alpha(1-\alpha) + 1/2 \\ &\pm \sqrt{(2\alpha-1)^2(k+1/2)^2 + \alpha(1-\alpha)[N^2 - (2\alpha-1)^2]}.\end{aligned} \tag{4.26}$$

Inspecting the total angular momentum we see that its possible values can be obtained by the tensor product $l \otimes 1/2 \otimes 1/2$ where $l = n_1 + N/2$ with $n_1 = 0, 1, 2, ...,$ $N = 1, 2, 3, ...$ and where one of the $1/2$ factor is for spin and the other is for isospin. This means that k can take three different values, namely $k = l, l \pm 1$, where $l = n_1 + N/2$. Let us note that the irreducible representation of $\vec{K}$ for the spin down and isospin down case $k = n_1 + N/2 - 1$, is valid for $N > 1$ and when $N = 1$ we have $n_1 \geq 1$ for this case

$$l \otimes \frac{1}{2} \otimes \frac{1}{2} = (l \otimes 1) \oplus l = (l+1) \oplus \mathbf{2}l \oplus (l-1) = \begin{cases} n_1 + \frac{N}{2} + 1 \\ n_1 + \frac{N}{2} \\ n_1 + \frac{N}{2} - 1 \end{cases} \tag{4.27}$$

With this information we can write the spectrum of $\not{D}^2$ as

$$\not{D}_{l+1}^2 = \xi_{n_1+1} + (2\alpha - 1)^2/4 - \sqrt{(2\alpha-1)^2\xi_{n_1+1} + N^2/4}, \tag{4.28a}$$

$$\not{D}_{l}^2 = \xi_{n_1} + (2\alpha - 1)^2/4 \pm \sqrt{(2\alpha-1)^2\xi_{n_1} + N^2/4}, \tag{4.28b}$$

$$\not{D}_{l-1}^2 = \xi_{n_1-1} + (2\alpha - 1)^2/4 + \sqrt{(2\alpha-1)^2\xi_{n_1-1} + N^2/4}, \tag{4.28c}$$

where ξ_{n_1} is

$$\xi_{n_1} = n_1^2 + n_1 N + N/2 + n_1 + 1/4 + \alpha(\alpha - 1). \tag{4.29}$$

In order to see how we have decided the signs in front of the square roots in (4.28), we should inspect the operator χ at $\alpha = 0$, or χ_0 for convenience

$$\chi_0 = -(\vec{K}.\vec{\sigma} - 1/2). \tag{4.30}$$

Let us recall that $\vec{K} = \vec{L} + \vec{\sigma}/2 + \vec{\tau}/2$, also $\vec{L}$ has the irreducible representations $n_1 + N/2$. Let us consider the lowest lying state with $n_1 = 0$. In this case, $\vec{K}$ has the irreducible representations given by the direct sum obtained as follows

$$\begin{aligned} &\frac{N}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}, \\ =&\Big(\frac{N-1}{2} \oplus \frac{N+1}{2}\Big) \otimes \frac{1}{2}, \\ =&\Big(\frac{N}{2} - 1\Big) \oplus \mathbf{2}\ \frac{N}{2} \oplus \Big(\frac{N}{2} + 1\Big). \end{aligned} \tag{4.31}$$

We can write $\vec{K} \cdot \vec{\sigma} - 1/2 = \vec{K}^2 - \vec{J}^2 + 1/4$, from which we can find the eigenvalues of χ_0. These are given with the corresponding IRRs of $\vec{K}$ and $\vec{J}$ in the table below To get these eigenvalues from square terms in (4.28a), (4.28b), and (4.28c) we chose

χ_0	K	J
$-N/2 - 3/2$	$N/2 + 1/2$	$N/2 + 1/2$
$N/2 + 1/2$	$N/2$	$N/2 + 1/2$
$-N/2 - 1/2$	$N/2$	$N/2 - 1/2$
$N/2 - 1/2$	$N/2 - 1/2$	$N/2 - 1/2$

Table 4.1: IRRs of χ_0 and the corresponding IRRs of K and J

the minus sign in front of the square root in (4.28a), both signs for each IRRs of $\vec{K}$ in (4.28b), and the plus sign for (4.28c). Since the spectrum of $\not{D}^2$ is continuous with respect to α, these signs are correct for all values of α.

4.2.2 Pair Production

We will now compute the pair production rates for Dirac fields on $\mathbb{R}^{1,1} \times S^2$. On the Euclidean space $\mathbb{R}^2 \times S^2$ the spectrum of the square of the gauged Dirac operator and the corresponding density of states are given as where ξ_{n_1} is given in Eq. (4.29).

$Spec(\not{D}^2)$	Density of states
$2n_2B_2 + (\xi_{n_1-1} + (2\alpha-1)^2/4 + \sqrt{(2\alpha-1)^2\xi_{n_1-1} + N^2/4})$	$\frac{B_2}{2\pi}\frac{B_1(2n_1+N-1)}{4\pi a^2}$
$2n_2B_2 + (\xi_{n_1} + (2\alpha-1)^2/4 \pm \sqrt{(2\alpha-1)^2\xi_{n_1} + N^2/4})$	$\frac{B_2}{2\pi}\frac{B_1(2n_1+N+1)}{4\pi a^2}$
$2n_2B_2 + (\xi_{n_1+1} + (2\alpha-1)^2/4 - \sqrt{(2\alpha-1)^2\xi_{n_1+1} + N^2/4})$	$\frac{B_2}{2\pi}\frac{B_1(2n_1+N+3)}{4\pi a^2}$
$2n_2B_2 + 2 + (\xi_{n_1-1} + (2\alpha-1)^2/4 + \sqrt{(2\alpha-1)^2\xi_{n_1-1} + N^2/4})$	$\frac{B_2}{2\pi}\frac{B_1(2n_1+N-1)}{4\pi a^2}$
$2n_2B_2 + 2 + (\xi_{n_1} + (2\alpha-1)^2/4 \pm \sqrt{(2\alpha-1)^2\xi_{n_1} + N^2/4})$	$\frac{B_2}{2\pi}\frac{B_1(2n_1+N+1)}{4\pi a^2}$
$2n_2B_2 + 2 + (\xi_{n_1+1} + (2\alpha-1)^2/4 - \sqrt{(2\alpha-1)^2\xi_{n_1+1} + N^2/4})$	$\frac{B_2}{2\pi}\frac{B_1(2n_1+N+3)}{4\pi a^2}$

Table 4.2: Spectrum of $\not{D}^2$ and the corresponding density of states.

The quantum number $n_1 = 1, 2, 3, ...$ for the eigenvalues that include ξ_{n_1} with the minus sign and eigenvalues that include $\xi n_1 - 1$, for others $n_1 = 0, 1, 2, ...$, and also $n_2 = 0, 1, 2, ...$ for all of them.

Now we are in a position to write the pair production rate on $\mathbb{R}^{1,1} \times S^2$. We will first evaluate the effective action on $\mathbb{R}^2 \times S^2$ and then Wick rotate to $\mathbb{R}^{1,1} \times S^2$ at an appropriate state of the calculation. Γ_E is given as

$$\Gamma_E = \frac{1}{2}\int \frac{ds}{s} Tr\left[e^{-s(\not{D}^2+m^2)}\right] \tag{4.32}$$

Now we can write the real part of iS_{eff} in Minkowski space, by letting $x_4 \to ix_0$,

$B_2 \to -iE$ and taking the integral over s

$$Re(iS_{eff}) = \frac{-1}{16\pi^2 a^2} \int d\mu dx_3 dx_4 \sum_{n=1}^{\infty} e^{-n\pi m^2/E} \frac{E}{n} \Bigg\{ \tag{4.33}$$
$$\sum_{n_1=1} (2n_1 + N - 1) \Big[e^{-n\pi[B_1(\xi_{n_1-1} + (2\alpha-1)^2/4 + \sqrt{(2\alpha-1)^2 \xi_{n_1-1} + N^2/4})]/(Ea^2)} \Big]$$
$$+ \sum_{n_1=0} (2n_1 + N + 1) \Big[e^{-n\pi[B_1(\xi_{n_1} + (2\alpha-1)^2/4 + \sqrt{(2\alpha-1)^2 \xi_{n_1} + N^2/4})]/(Ea^2)} \Big]$$
$$+ \sum_{n_1=1} (2n_1 + N + 1) \Big[e^{-n\pi[B_1(\xi_{n_1} + (2\alpha-1)^2/4 - \sqrt{(2\alpha-1)^2 \xi_{n_1} + N^2/4})]/(Ea^2)} \Big]$$
$$+ \sum_{n_1=0} (2n_1 + N + 3) \Big[e^{-n\pi[B_1(\xi_{n_1+1} + (2\alpha-1)^2/4 - \sqrt{(2\alpha-1)^2 \xi_{n_1+1} + N^2/4})]/(Ea^2)} \Big] \Bigg\}.$$

Using $\omega = \frac{\pi}{Ea^2}$ as before we can also write $Re(iS_{eff})$ in the following form

$$Re(iS_{eff}) = -\int d\mu dx_3 dx_4 \frac{E^2}{8\pi^3} \beta_{1/2}(\omega), \tag{4.34}$$

where

$$\beta_{1/2}(\omega) = -\frac{\omega}{2} \Bigg\{ 2\,log\,(1 - e^{-\omega m^2 a^2}) \tag{4.35}$$
$$+ \sum_{n_1=1}^{\infty} (2n_1 + N - 1) log\,(1 - e^{-\omega(\xi_{n_1-1} + (4\gamma+1)/4 + \sqrt{(2\alpha-1)^2 \xi_{n_1-1} + N^2/4}) + m^2 a^2)})$$
$$+ \sum_{n_1=0}^{\infty} (2n_1 + N + 1) log\,(1 - e^{-\omega(\xi_{n_1} + (4\gamma+1)/4 + \sqrt{(4\gamma+1)\xi_{n_1} + N^2/4}) + m^2 a^2)})$$
$$+ \sum_{n_1=1}^{\infty} (2n_1 + N + 1) log\,(1 - e^{-\omega(\xi_{n_1} + (4\gamma+1)/4 - \sqrt{(4\gamma+1)\xi_{n_1} + N^2/4}) + m^2 a^2)})$$
$$+ \sum_{n_1=0}^{\infty} (2n_1 + N + 3) log\,(1 - e^{-\omega(\xi_{n_1+1} + (4\gamma+1)/4 - \sqrt{(4\gamma+1)\xi_{n_1+1} + N^2/4}) + m^2 a^2)}) \Bigg\}.$$

In (4.35), we have written the contribution of the zero modes explicitly, and also introduced $\gamma = \alpha(\alpha - 1)$ as before.

Let us now calculate the flat limit Eq. (4.35), that is $\mathbb{R}^{1,1} \times S^2 \to \mathbb{R}^{3,1}$. In order to do this we have to take $a^2 \to \infty$, $N \to \infty$, and $\gamma \to \infty$, while keeping N/a^2 and γ/a^2 constant. Since ω is proportional to $1/a^2$ this practically means we can keep $\omega\gamma$ and

ωN as such. While we drop the terms which vanish as $a^2 \to \infty$. We find

$$\beta_{1/2}^{flat}(\omega) = -\frac{\omega N}{2}\Bigg\{ 2\,log\,(1 - e^{-\omega m^2 a^2}) \tag{4.36}$$
$$+ \sum_{n_1=1}^{\infty} log\,(1 - e^{-\omega(-N/2 + n_1 N + 2\beta^2 + \sqrt{4\beta^2(-N/2 + n_1 N + \beta^2) + N^2/4} + m^2 a^2)})$$
$$+ \sum_{n_1=0}^{\infty} log\,(1 - e^{-\omega(N/2 + n_1 N + 2\beta^2 + \sqrt{4\beta^2(N/2 + n_1 N + \beta^2) + N^2/4} + m^2 a^2)})$$
$$+ \sum_{n_1=1}^{\infty} log\,(1 - e^{-\omega(N/2 + n_1 N + 2\beta^2 - \sqrt{4\beta^2(N/2 + n_1 N + \beta^2) + N^2/4} + m^2 a^2)})$$
$$+ \sum_{n_1=0}^{\infty} log\,(1 - e^{-\omega(3N/2 + n_1 N + 2\beta^2 - \sqrt{4\beta^2(3N/2 + n_1 N + \beta^2) + N^2/4} + m^2 a^2)})\Bigg\}.$$

Shifting the index of first sum in (4.36) to zero it is seen that the sum is equal to the second sum in (4.36) and similarly, shifting the index of the last sum to the starting value 1, it is seen to be equivalent to the third sum in (4.36). Thus, we obtain

$$\beta_{1/2}^{flat}(\omega) = -\omega N\Bigg\{ log\,(1 - e^{-\omega m^2 a^2}) \tag{4.37}$$
$$+ \sum_{n_1=0}^{\infty} log\,(1 - e^{-\omega(N/2 + N n_1 + 2\beta^2 + \sqrt{2\beta^2(N + 2N n_1 + 2\beta^2) + N^2/4} + m^2 a^2)})$$
$$+ \sum_{n_1=1}^{\infty} log\,(1 - e^{-\omega(N/2 + N n_1 + 2\beta^2 - \sqrt{2\beta^2(N + 2N n_1 + 2\beta^2) + N^2/4} + m^2 a^2)})\Bigg\}.$$

In order to examine the effects of curvature and the non-abelian magnetic field, we define the ratio

$$\gamma_{1/2}(\omega) := \beta_{1/2}(\omega)/\beta_{1/2}^{flat}(\omega) \tag{4.38}$$

We plot the profile of $\gamma_{1/2}(\omega)$ at a fixed non-abelian charge, γ, at different values of the abelian monopole charge N. Figure 4.4a, shows us that when the non-abelian charge is taken as zero our results exactly match with the results found in [12], which we have reproduced in Section 2.4. Clearly this is because, when $\gamma = 0$, the non-abelian part of the magnetic field becomes zero and we are left with just the abelian part.

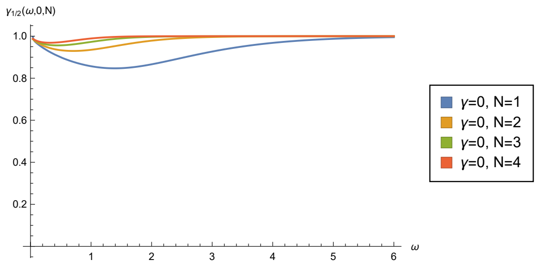

Figure 4.4a: $\gamma = 0$

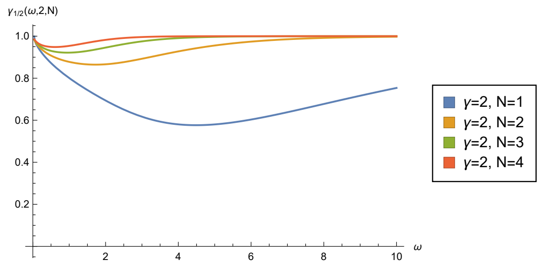

Figure 4.4b: $\gamma = 2$

Figure 4.4b shows us that at $\gamma \neq 0$ the general profile/hierarchy at $\gamma = 0$ is preserved, meaning that we continue to see a decrease in the pair production rate for small values of ω, then grows back and settles at $\gamma_{1/2}(\omega) = 1$, at large values of ω. The initial decrease is caused by the fact that the energies of the quantum states for the spherical case is higher compared tp those of the flat case. Also when we are at low values of the abelian charge this dip becomes much more pronounced. This is essentially because of the fact that energies generally increase when the abelian charge is increased, hence the $\gamma_{1/2}(\omega)$ settles much more quickly to the limiting value 1. The reason for $\gamma_{1/2}(\omega)$ approaching 1 at larger values of N is due to increasing number of zero modes which cost no energy to be filled by the produced pairs. Let us also note that from Figure $4.4b$, we see a relative decrease at pair production rates for $\gamma \neq 0$ compared to $\gamma = 0$; This is due to the further increase in the energies of available quantum the states due to the non-abelian field on $\mathbb{R}^{1,1} \times S^2$ compared to $\mathbb{R}^{3,1}$ as it can be seen from Figures C.9 and C.10 in Appendix C.

Now we will keep the abelian charge constant while increasing the non-abelian charge. This will allow us to comment on the effects of the non-abelian magnetic field further.

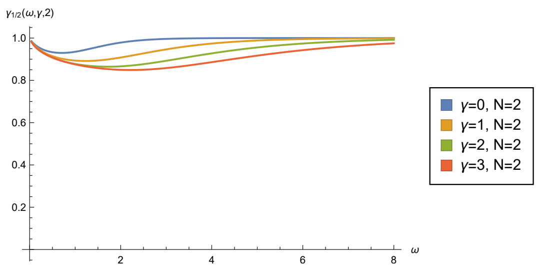

Figure 4.5a: $N = 2$

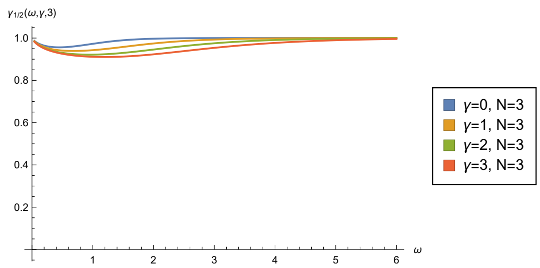

Figure 4.5b: $N = 3$

From the Figure 4.5a and Figure 4.5b, we see at fixed N, increasing γ, results in an initial decrease in the pair production rates which, eventually settles back at 1 at sufficiently large ω. We see that the increasing the non-abelian charge γ causes a decrease in the pair production rates, this is due to the higher energies of the available quantum states.

Another way to inspect the non-abelian charges effect on the pair production is to compare it with the pair production rates at $\gamma = 0$. In other words, we compare $\gamma \neq 0$ case with $\gamma = 0$ case on the same geometry $\mathbb{R}^{1,1} \times S^2$. To do this we define the $R_{1/2}(\omega, N, \gamma)$ function as

$$R_{1/2}(\omega, N, \gamma) \equiv \beta_{1/2}(\omega, N, \gamma)/\beta_{1/2}(\omega, N, 0) \tag{4.39}$$

$R_{1/2}(\omega, N, \gamma)$ function allows us to inspect the non-abelian charge effect directly. We look at two graphs. In Figure 4.6a we take the non-abelian charge $\gamma = 2$ and look at the different abelian charge values. In Figure 4.6b we take the abelian charge $N = 2$ to be constant and vary the non-abelian charge.

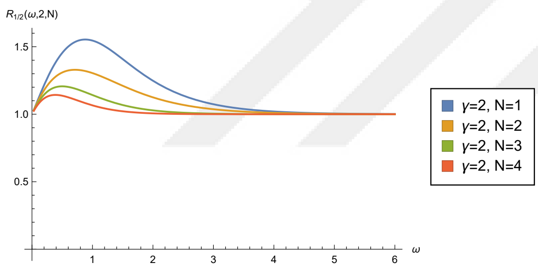

Figure 4.6a: $\gamma = 2$

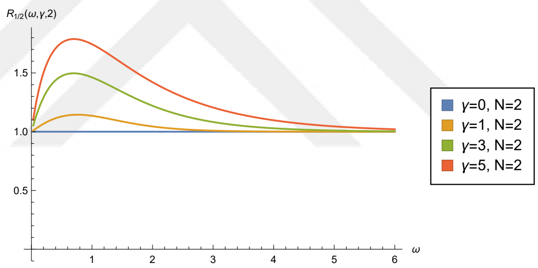

Figure 4.6b: $N = 2$

Inspecting the Figure 4.6a we see that there is an increase in the pair production when the ω values are smaller, then it approaches to 1. Initial increase in the pair production is caused by the fact that energies of the quantum states with $\gamma \neq 0$ are lower than those for $\gamma = 0$ as it can be seen from Figures C.9 and C.10 in Appendix C. Let us also note that with increasing values of the abelian magnetic charge N, energies of the quantum states also increase, and hence we see that increase in $R_0(\omega, \gamma, N)$ is milder, when N is larger, and settles back to the limiting value 1 faster.

Figure 4.6b behaves very similarly to Figure 4.12, mainly the pair production has an increase, and converges to the limiting value of 1, at large enough ω. This is as before due to the dominant contribution of the zero modes at large ω. Since the energies of the available quantum states decrease with the increasing non-abelian charge, we see an increase in the pair production rates with increasing γ at fixed N.

CHAPTER 5

CONCLUSIONS

Purpose of this thesis was to explore the effects of non-abelian magnetic field, as well as the constant curvature on the pair production rate. In Chapter 2 we gave a review for the Schwinger mechanism, and then also examined this effect on $\mathbb{R}^{3,1}$ and $\mathbb{R}^{1,1} \times S^2$ with an additional uniform abelian magnetic field. We have seen that the pair production rate increased for scalar fields, due to the curvature of two-sphere compared to the flat case, this was seen by the extra factor related to the curvature of S^2, the $\frac{1}{a^2}$ factor. Nevertheless with increasing abelian monopole charge, pair production rates tended back to those of the flat case. This can be seen as the abelian magnetic charge counterbalancing the effect of positive curvature on spin-0 fields, as the energies of the quantum states at higher values of abelian magnetic charge are higher and harder to be filled by the produced pairs. For spinor field we saw that pair production on $\mathbb{R}^{1,1} \times S^2$ is relatively smaller than that on $\mathbb{R}^{3,1}$, see Figure 2.2. The reason for this effect is that, the energies of the quantum states on $\mathbb{R}^{1,1} \times S^2$ are higher than those on $\mathbb{R}^{3,1}$ and they cost more energy to be filled. However, with increasing abelian-charge there are increasing number of zero energy modes, which cost no energy to be filled and therefore the relative pair production rate settles back to 1 at sufficiently large ω, and this happens faster with increasing N.

In Chapter 3 we have inspected the pair production rates for spinor and scalar fields under the influence of both abelian and non-abelian magnetic fields, on Minkowski space $\mathbb{R}^{3,1}$. For the scalar fields case we have defined the function $f_0(y, \beta')$ to examine the pair production rates, and also defined $F_0(y, \beta')$ to see the effects of non-abelian field strength on the pair production. Our results show that the pair production decreases with increasing y and in general, decrease with increasing non-abelian field strength β'. For the spinor fields, we have defined the similar functions $f_{1/2}(y, \beta')$ and $F_{1/2}(y, \beta')$. Plotting these showed us that pair production in general, increased with increasing y, in contrast with the scalar field case. Similarly we have also seen that the pair production rate was increase with increasing non-abelian magnetic field strength, β'.

In Chapter 4 we have looked at the pair production rates on $\mathbb{R}^{1,1} \times S^2$ under the influence of non-abelian fields. We have compared two different cases for both spinor fields and scalar fields. First case was for the comparison of pair production on spherical spaces to pair production on flat spaces. This was done by defining the function $\gamma_0(\omega)$ for scalar fields, and $\gamma_{1/2}(\omega)$ for the spinor fields. Secondly we looked at the effects of the non-abelian magnetic fields directly by defining the function $R_0(\omega)$ for the scalar case, and $R_{1/2}(\omega)$ for the spinor case. From these considerations for scalar fields we have seen that the pair production decreases when we increase the non-abelian field strength γ, and for the spinor fields the situation is reversed. However, profiles of these functions are much more complicated for large ω values, compared to the flat case. Mainly there exists a critical γ value for which the limiting values of these functions are determined.

We may consider the pair production effects on manifolds of the type $AdS_2 \times \mathbb{R}^2$, $AdS_2 \times S^2$, with an electric field on AdS_2 and non-abelian magnetic fields on $\mathbb{R}^2$ and S^2 parts. Pair production on AdS_2 areas considered in [9] and on de-Sitter in [10, 11]. We hope to tackle these problems in near future.

Appendix A

PATH INTEGRAL FORMULATION

Path integral formulations were first used in physics in the context of diffusion equations and Brownian motion in the late 1920's by Norbert Wiener. After Wiener, P.A.M. Dirac used this concept in quantum mechanical context in 1933 [23]. The method was rigorously applied to quantum mechanics, by R. Feynman, first in his thesis.

A.1 Formulating the Path Integrals

We start with a simple quantum mechanical system in Schrödinger picture of quantum mechanics, a time dependent position operator $\hat{q}$, and with the state, $|q, t\rangle$ that satisfies the following eigenvalue equation

$$\hat{q}(t)\,|q, t\rangle = q(t)\,|q, t\rangle \tag{A.1}$$

Let us say that this system has initial and final positions, with q_a, t_a and q_b, t_b respectively. For this system we can find the transition amplitude of finding the system at state b, given that it was initially in state a. This transition amplitude is also referred to as the kernel, and defined as

$$K(a, b) = \langle q_b, t_b | q_a, t_a \rangle \tag{A.2}$$

We can expand A.2 in a way that there are different alternatives to go from a to b, thanks to completeness of quantum states. Meaning we can write A.2 as

$$K(a, b) = \int dq' \, \langle q_b, t_b | q', t' \rangle \, \langle q', t' | q_a, t_a \rangle \tag{A.3}$$

Since

$$\int dq' \, |q', t'\rangle \, \langle q', t'| = \mathbb{1} \tag{A.4}$$

Which is the completeness relation. We can summarize what is happening in A.3 in a space time graph of the form

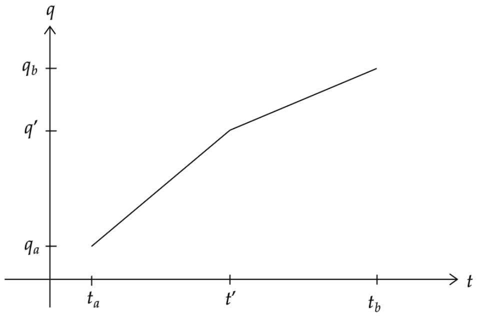

Figure A.1: Particle path.

It is important to realize that the point q', t' is not a fixed point but stands for all intermediate paths that the state can take. We can imagine there is some imaginary "obstacle" at the position q', t', although the analogy is not % 100 accurate since the obstacles we imagine are in the space time graph rather than in the $2D$ space. So we will call these what they are, intermediate points. We can increase the number of intermediate points, let us say we have N number of points in total, including a and b, which are separated in equal time to one another. This time separation is $\Delta t = \epsilon$. In this case the space time graph is Here, the only fixed points are $t_0 = t_a$ and $t_N = t_b$,

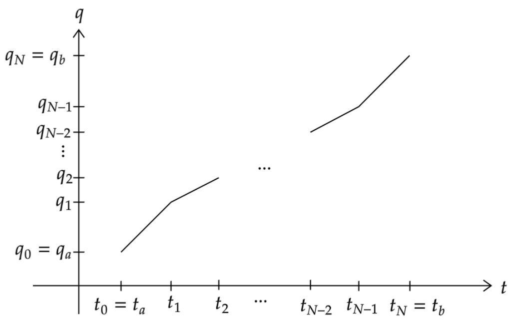

Figure A.2: Divided particle path.

hence we can further expand the Kernel as

$$K(a,b) = \int\int\cdots\int dq_1 dq_2 \cdots dq_{N-1} \langle q_b, t_b | q_{N-1}, t_{N-1}\rangle \langle q_{N-1}, t_{N-1} | q_{N-2}, t_{N-2}\rangle \times \\ \langle q_{N-2}, t_{N-2}| \times \cdots \times |q_1, t_1\rangle \langle q_1, t_1 | q_a, t_a\rangle \tag{A.5}$$

Here the integration is not over q_a and q_b since they are fixed points, as discussed before. Before going any further, we should look at to the Schrödinger picture of quantum mechanics and the time evolution of the states. It is known, according to our current understanding, that the differences between the different pictures of quantum mechanics do not effect the observed reality, and they are purely theoretical.

Let us say that we have a state at some time t_0. We are interested to find out how does this state evolve with time. Our initial state is $|\alpha, t_0\rangle$ and we want to find out the evolved state at time, t_1. For this we define an operator called $U(t_1, t_0)$ which is both a function of the initial and the final time. This operator should give us the state at time t_2 when applied to the state in time t_0, hence it should satisfy

$$|\alpha, t_1\rangle = U(t_1, t_0) |\alpha, t_0\rangle \tag{A.6}$$

This time evolution operator should also have the following properties

- It should preserve normalization of the states, i.e. if a state is normalized, it should stay normalized after the time evolution operator is applied to it.
- If t_1 and t_0 are equal to each other operator in A.5 should be reduced to the identity operator.
- It should follow $U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0)$.

Using the time evolution operator, it is easy to see that every factor in A.5 can be written in following form

$$\begin{aligned} \langle q_i, t_i | q_{i-1}, t_{i-1}\rangle &= \langle q_i| e^{-i\hat{H}t_i/\hbar} e^{i\hat{H}t_{i-1}/\hbar} |q_{i-1}\rangle \\ &= \langle q_i| e^{-i\hat{H}\epsilon/\hbar} |q_{i-1}\rangle \end{aligned} \tag{A.7}$$

Now let $\epsilon \to 0$ while keeping $t_b - t_a$, or $N\epsilon$, constant. Which is very similar to what we do when we are going from Riemann sum to integration. Since ϵ is very small, we can use the following approximation

$$\begin{aligned}\langle q_i, t_i | q_{i-1}, t_{i-1}\rangle &= \langle q_i| e^{-i\hat{H}\epsilon/\hbar} |q_{i-1}\rangle \\ &= \langle q_i| \mathbb{1} - \frac{i\hat{H}}{\hbar}\epsilon + O(\epsilon^2) |q_{i-1}\rangle \end{aligned} \tag{A.8}$$

First term in the right hand side of the equation A.8 is just the Dirac delta function $\delta(q_i - q_{i-1})$, and the second term is the matrix elements of the Hamiltonian, multiplied by some factors. To go further from this point, we should use the general Hamiltonian, written as

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \tag{A.9}$$

We will write the matrix elements of the kinetic energy and the potential energy separately. We start with kinetic energy term. Which is

$$\langle q_i| \frac{\hat{p}^2}{2m} |q_{i-1}\rangle = \int\int dp_{i-1}\, dp_i \, \langle q_i|p_i\rangle \langle p_i| \frac{\hat{p}^2}{2m} |p_{i-1}\rangle \langle p_{i-1}|q_{i-1}\rangle \tag{A.10}$$

Since

$$\int dp_i \, |p_i\rangle \langle p_i| = \mathbb{1} \tag{A.11}$$

We also know that the inner product of the momentum eigenstates, $|p_i\rangle$ with the corresponding position eigenstates, $|q_i\rangle$ is

$$\langle q_i|p_i\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{i(p_i q_i)/\hbar} \tag{A.12}$$

Then the kinetic term becomes

$$\begin{aligned}\langle q_i| \frac{\hat{p}^2}{2m} |q_{i-1}\rangle =& \frac{1}{2\pi\hbar} \int\int dp_{i-1}\, dp_i \, \delta(p_i - p_{i-1}) e^{i(p_i q_i)/\hbar} e^{-i(p_{i-1}q_{i-1})/\hbar} \\ &\times \frac{p_{i-1}^2}{2m} \end{aligned} \tag{A.13}$$

With the help of the Dirac delta function, this result can be further simplified by integrating once, which will give us

$$\langle q_i| \frac{\hat{p}^2}{2m} |q_{i-1}\rangle = \frac{1}{2\pi\hbar} \int dp_i \, e^{ip_i(q_i - q_{i-1})/\hbar} \, \frac{p_i^2}{2m} \tag{A.14}$$

Now we calculate the potential term, however there is one slight caveat, we can either write

$$\langle q_i| V(\hat{q}) |q_{i-1}\rangle = \delta(q_i - q_{i-1}) \, V(q_i) \tag{A.15}$$

Or similarly, we can write

$$\langle q_i| V(\hat{q}) |q_{i-1}\rangle = \delta(q_i - q_{i-1}) V(q_{i-1}) \tag{A.16}$$

Notice that, thanks to the Dirac delta function, the result we get when integrating A.15 with respect to dq_i is the same with integrating A.16 with respect to dq_{i-1}, to get around this issue, we write

$$\langle q_i| V(\hat{q}) |q_{i-1}\rangle = \delta(q_i - q_{i-1}) V\Big(\frac{q_{i-1}+q_i}{2}\Big) \tag{A.17}$$

Just for the notational convenience we define $\bar{q}_i \equiv (q_{i-1}+q_i)/2$, and writing the Dirac delta function as an integral will get us

$$\langle q_i| V(\hat{q}) |q_{i-1}\rangle = \frac{1}{2\pi\hbar}\int dp_i\, e^{ip_i(q_i-q_{i-1})/\hbar}\, V(\bar{q}_i) \tag{A.18}$$

Finally if we combine A.14 with A.18, we can write the matrix element for the Hamiltonian as

$$\langle q_i| \hat{H} |q_{i-1}\rangle = \frac{1}{2\pi\hbar}\int dp_i\, e^{ip_i(q_i-q_{i-1})/\hbar}\Big(\frac{p_i^2}{2m} + V(\bar{q}_i)\Big) \tag{A.19}$$

Now, if we get back to the original problem at hand, A.8 can be written as

$$\langle q_i, t_i|q_{i-1}, t_{i-1}\rangle = \frac{1}{2\pi\hbar}\int dp_i\, e^{ip_i(q_i-q_{i-1})/\hbar}\left[\mathbb{1} - \frac{i\epsilon}{\hbar}H(p_i,\bar{q}_i) + O(\epsilon^2)\right] \tag{A.20}$$

Remember that we used the approximation for the exponential in A.7, and ignored the ϵ^2 terms and higher. Keeping the same level of approximation we can write

$$\langle q_i, t_i|q_{i-1}, t_{i-1}\rangle = \frac{1}{2\pi\hbar}\int dp_i\, e^{ip_i(q_i-q_{i-1})/\hbar} e^{-i\epsilon H(p_i,\bar{q}_i)/\hbar} \tag{A.21}$$

Here it is appropriate to give some definitions before going any further. For a function $\alpha(t)$ we define the following

$$\mathcal{D}\alpha = \lim_{N\to\infty}\prod_{i=1}^{N}\frac{d\alpha_i}{\sqrt{2\pi\hbar}} \tag{A.22}$$

The factor $1/\sqrt{2\pi\hbar}$ is just there to allow us to write A.5 in a more compact way. Which is

$$\begin{aligned} K(a,b) &= \int \mathcal{D}p\mathcal{D}q \prod_{i=1}^{N} e^{i[p_i(q_i-q_{i-1})-\epsilon H(p_i,\bar{q}_i)]/\hbar} \\ &= \int \mathcal{D}p\mathcal{D}q\, e^{i\lim_{N\to\infty}\sum_{i=1}^{N}[p_i(q_i-q_{i-1})-\epsilon H(p_i,\bar{q}_i)]/\hbar} \end{aligned} \tag{A.23}$$

Notice that since we are taking the limits $N \to \infty$ and $\epsilon \to 0$, while keeping $N\epsilon$ constant, we can divide and multiply everything by ϵ to write

$$K(a,b) = \int \mathcal{D}p\mathcal{D}q\, e^{i \lim_{N\to\infty} \sum_{i=1}^{N} \epsilon[p_i(q_i-q_{i-1})/\epsilon - H(p_i,\bar{q}_i)]/\hbar} \tag{A.24}$$

It is easy to see that the $(q_i - q_{i-1})/\epsilon$ is just the time derivative of the momentum, since ϵ is just the time steps we took, and the q_i and q_{i-1} are the momenta separated by those time steps. Also we are multiplying everything with ϵ while summing over all of the time steps, this is just the definition of the integral, so we can written

$$K(a,b) = \int \mathcal{D}p\mathcal{D}q\, e^{\frac{i}{\hbar}\int dt[p\dot{q} - H(p,q)]} \tag{A.25}$$

Since the part inside the integration is just the Lagrangian, we might also written

$$\begin{aligned} K(a,b) &= \int \mathcal{D}p\mathcal{D}q\, e^{\frac{i}{\hbar}\int dt L(q,\dot{q})} \\ &= \int \mathcal{D}p\mathcal{D}q\, e^{\frac{i}{\hbar}S(q,\dot{q})} \end{aligned} \tag{A.26}$$

If the Hamiltonian has the form it has in A.9, we can take the integral over the potential. To do this we will go back to A.24 and look to just one i value, and generalize it after taking the integral for that. So we will start with

$$\begin{aligned} K(a,b) &= \int_{-\infty}^{\infty} dq_i\, e^{-\frac{i\epsilon}{2m\hbar}[p_i^2 - 2mp_i(q_i-q_{i-1})]} \\ &= e^{\frac{im\epsilon}{2\hbar}(q_i-q_{i-1})^2} \int_{-\infty}^{\infty} dq_i\, e^{-\frac{i\epsilon}{2m\hbar}[p_i - m(q_i-q_{i-1})]^2} \\ &= \sqrt{\frac{2m\pi\hbar}{i\epsilon}} e^{\frac{im\epsilon}{2\hbar}(q_i-q_{i-1})^2} \end{aligned} \tag{A.27}$$

Now since we have defined $\mathcal{D}q$ with the factor $1/\sqrt{2\pi\hbar}$, we can redo the steps when going from A.24 to A.26 and write our final result as

$$K(a,b) = \int \mathcal{D}q\, e^{\frac{i}{\hbar}\int dt L(q,\dot{q})} \tag{A.28}$$

Appendix B

LIMITING BEHAVIOR OF THE GRAPHS

When we look to the pair production ratios, we see that their form is the same with all of them, i.e.

$$f(x) = \frac{a_1 \, log\,(1 \pm e^{-x\lambda_1}) + a_2 \, log\,(1 \pm e^{-x\lambda_2}) + ... + a_n \, log\,(1 \pm e^{-x\lambda_n}) + ...}{b_1 \, log\,(1 \pm e^{-x\theta_1}) + b_2 \, log\,(1 \pm e^{-x\theta_2}) + ... + b_m \, log\,(1 \pm e^{-x\theta_m}) + ...} \tag{B.1}$$

If we assume that the θ_n and λ_n are the lower when the subscript is lower, then we can say that the limiting behavior when $x \to \infty$ is written as

$$\lim_{x \to \infty} f(x) = \begin{cases} \infty & \text{if } \theta_1 > \lambda_1 \\ a_1/b_1 & \text{if } \theta_1 = \lambda_1 \\ 0 & \text{if } \theta_1 < \lambda_1 \end{cases} \tag{B.2}$$

Appendix C

ENERGY GRAPHS

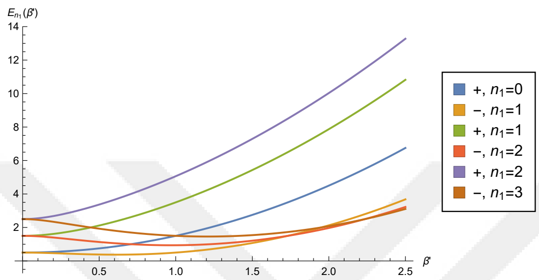

Figure C.1: spin-0 flat energies

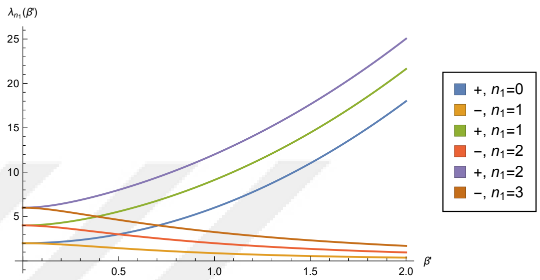

Figure C.2: spin-$1/2$ flat energies

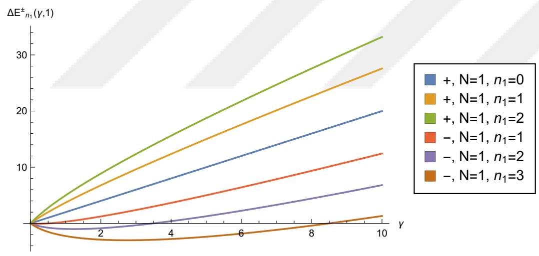

Figure C.3: ΔE of non-abelian spherical spin-0 and abelian spherical spin-0 $N = 1$

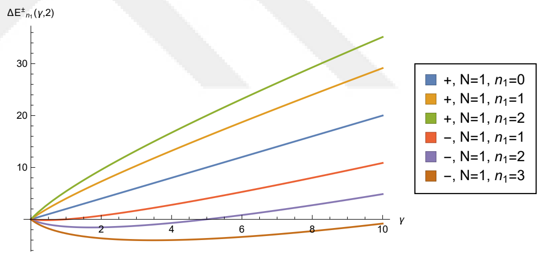

Figure C.4: ΔE of non-abelian spherical spin-0 and abelian spherical spin-0 $N = 2$

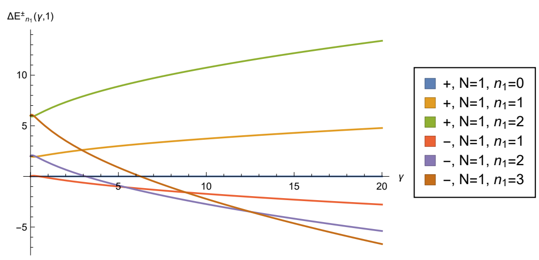

Figure C.5: ΔE of non-abelian spherical spin-0 and flat spin-0 $N = 1$

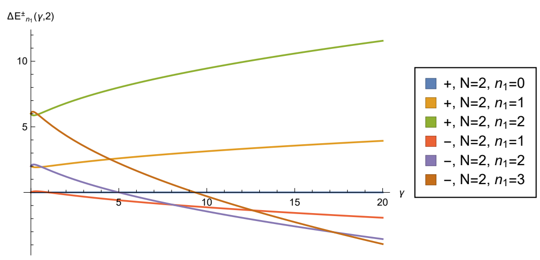

Figure C.6: ΔE of non-abelian spherical spin-0 and flat spin-0 $N = 2$

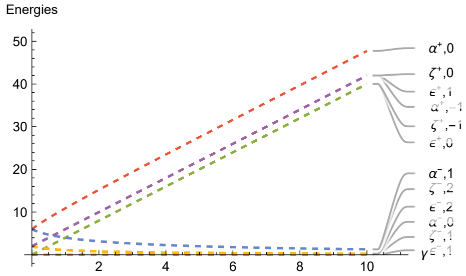

Figure C.7: Spectrum of $\not{D}^2$ for spherical case, $N = 1$

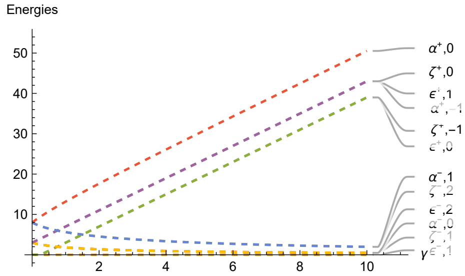

Figure C.8: Spectrum of $\not{D}^2$ for spherical case, $N = 2$

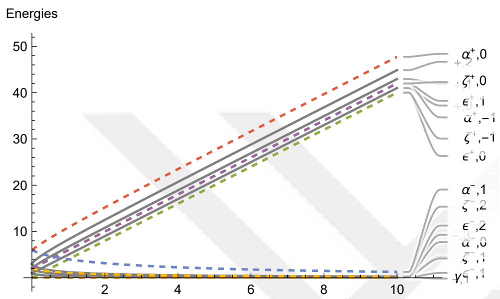

Figure C.9: Spectrum of $\not{D}^2$ for spherical vs flat cases, $N = 1$

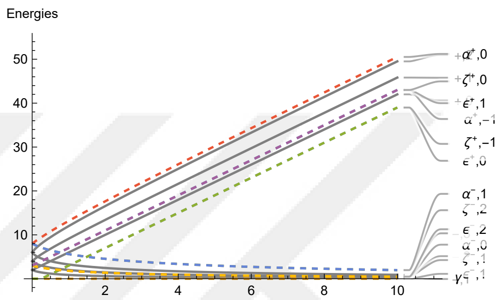

Figure C.10: $\not{D}^2$ for spherical vs flat cases, $N = 2$

Appendix D

LANDAU PROBLEM ON $\mathbb{R}^2$

D.1 For Scalar Fields

Here we will discuss the solution for the Landau problem for scalar fields, that are constraint to move on the flat $\mathbb{R}^2$ space. For starters we have a perpendicular magnetic field that is also constant. Since the particles will only move in the flat surface, we may reduce the problem to two dimensions. We take the plane to be the (x, y) plane, then the magnetic field will have the form

$$\vec{B} = B\hat{z}. \tag{D.1}$$

We can write the vector potential, $\vec{A}$, using the Landau gauge in the following form

$$\vec{A} = Bx\hat{y}. \tag{D.2}$$

Which means we can write the operator D^2 as

$$\begin{aligned} D^2 = D_\mu D^\mu &= (\partial_\mu - iA_\mu)(\partial^\mu - iA^\mu), \\ &= \partial_x^2 + (\partial_y - iBx)^2, \end{aligned} \tag{D.3}$$

now we define the modified position operator as $X = (\partial_y/B - ix)$, so that the commutation relation is written as $[X, \partial_x] = [-ix, \partial_x] = i$, with of course $\hbar = 1$. Then with this definition we are able to write

$$D^2 = \partial_x^2 + B^2 X^2. \tag{D.4}$$

Which is should just give us the energy spectrum for the quantum harmonic oscillator, meaning the spectrum should be

$$\lambda = B(2n + 1) \tag{D.5}$$

D.2 For Spinor Fields

Now we will make the calculation we have done in appendix D.1 with spinor fields. Again in the same manner as before we will just focus on the two dimensional case with an external magnetic field, B, perpendicular to the (x, y) plane, with using the same gauge we have used in appendix D.1. The spectrum of the operator we will be calculating is the Dirac operator, or more precisely its square. $\not{D}^2$ operator can be written for the aforementioned case is

$$\begin{aligned}
\not{D}^2 &= \left[\tau_x(\partial_x - iA_x) + \tau_y(\partial_y - iA_y)\right]^2, \\
&= \tau_x^2(\partial_x - iA_x)^2 + \tau_y^2(\partial_y - iA_y)^2 + \tau_x\tau_y[\partial_x - iA_x, \partial_y - iA_y], \\
&= \partial_x^2 + (\partial_y - iBx)^2 + \tau_x\tau_y[\partial_x, \partial_y - iBx], \\
&= D^2 + i\tau_z(-iB), \\
&= D^2 + B\tau_z.
\end{aligned} \tag{D.6}$$

Which means that the spectrum for the operator $\not{D}^2$ is just

$$\lambda_n^{\pm} = B(2n + 1) \pm B. \tag{D.7}$$

Here $\pm B$ terms obviously comes from the spin of the particle, plus for spin up and minus for spin down. Also notice that by shifting the index of the spin up eigenvalue by one we are able to put it in a more manageable form,

$$\lambda_n = 2nB, \tag{D.8}$$

With the degeneracy for each index n, d_n, given by

$$d_n = \begin{cases} 1 & n = 0 \\ 2 & n \neq 0 \end{cases} \tag{D.9}$$

Appendix E

LANDAU PROBLEM ON S^2

E.1 For Scalar Fields

Now, we will inspect the spectrum for the Landau levels on a sphere. To do this we will need a magnetic field perpendicular to the surface and constant everywhere, we can obtain this configuration by a Dirac monopole on the center of the said sphere. Our magnetic field on the sphere will be given as

$$\vec{B} = \frac{N}{2a^2}\hat{r}. \tag{E.1}$$

With $N \in \mathbb{Z}^+$ for the Dirac quantization condition. Since the Laplacian can be written in the form $D^2 = \frac{\Lambda^2}{a^2}$, where $\vec{\Lambda} = \vec{r} \times (\vec{p} - \vec{A})$. We see that the commutation relation $[\Lambda_i, \Lambda_j]$ is

$$[\Lambda_i, \Lambda_j] = i\epsilon_{ijk}\Big(\Lambda_k - \frac{N}{2}\hat{r}_k\Big), \tag{E.2}$$

here we define the operator $\vec{J} = \vec{\Lambda} - \frac{N}{2}\hat{r}$, with commutation relations being

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{E.3}$$

We see that the gauged Laplacian operator can be written as

$$D^2 = \frac{1}{a^2}\Big(J^2 - \frac{N^2}{4}\Big), \tag{E.4}$$

and since the eigenvalues of J are $j = n - \frac{N}{2}$, so the spectrum for the gauged Lagrangian is found to be

$$\begin{aligned} Spec(D^2) &= \frac{1}{a^2}\Big(j(j+1) - \frac{N^2}{4}\Big), \\ &= \frac{1}{a^2}\Big(n(N+n+1) + \frac{N}{2}\Big). \end{aligned} \tag{E.5}$$

E.2 For Spinor Fields

With the same magnetic field given in Eq (E.1), we can write the operator $\not{D}^2$ as

$$\begin{aligned} \not{D}^2 &= \frac{\tau_i \Lambda_i}{a^2}, \\ &= \Big(\Lambda^2 + \vec{\tau}\cdot\vec{\Lambda} - \frac{N}{2}\hat{r}\cdot\vec{\tau} + 1\Big)/a^2. \end{aligned} \tag{E.6}$$

Here we also define the total angular momentum operator J, with also including a spin component, explicitly, J_i reads,

$$J_i = \Lambda_i - \frac{N}{2}\hat{r}_i + \frac{\tau_i}{2}, \tag{E.7}$$

it is easy to check that the operators J_i can be described with the $SU(2)$ algebra, in other words, they satisfy the commutation relationship $[J_i, J_j] = i\epsilon_{ijk}J_k$. We can also compute the operator $\vec{J}^2$, which gives

$$\begin{aligned} J^2 &= \Big(\Lambda_i - \frac{N}{2}\hat{r}_i + \frac{\tau_i}{2}\Big)\Big(\Lambda_i - \frac{N}{2}\hat{r}_i + \frac{\tau_i}{2}\Big), \\ &= \Big(\Lambda^2 + \vec{\tau}\cdot\vec{\Lambda} - \frac{N}{2}\hat{r}\cdot\vec{\tau} + \frac{N^2}{4} + \frac{3}{4}\Big). \end{aligned} \tag{E.8}$$

This result shows us that the operator $\not{D}^2$ can be simply written in terms of J^2 as,

$$\not{D}^2 = \Big(J^2 - \frac{N^2}{4} + \frac{1}{4}\Big)/a^2. \tag{E.9}$$

Since the eigenvalues of the operator J are $n - \frac{N}{2} \pm \frac{1}{2}$ the spectrum of the Dirac operator on 2-sphere with radius a can be written as

$$Spec(\not{D}^2_{S^2}) = \begin{cases} \frac{1}{a^2}((n+1)^2 + N(n+1)) \\ \frac{1}{a^2}(n^2 + Nn) \end{cases} \tag{E.10}$$

REFERENCES

[1] J. Schwinger, "On gauge invariance and vacuum polarization," *Physical Review*, vol. 82, pp. 664–679, June 1951.

[2] E. Brezin and C. Itzykson, "Pair production in vacuum by an alternating field," *Physical Review D*, vol. 2, pp. 1191–1199, Oct. 1970.

[3] G. Dunne and T. Hall, "QED effective action in time dependent electric backgrounds," *Physical Review D*, vol. 58, Oct. 1998.

[4] F. Hebenstreit, R. Alkofer, and H. Gies, "Pair production beyond the schwinger formula in time-dependent electric fields," *Physical Review D*, vol. 78, Sept. 2008.

[5] R. Schützhold, H. Gies, and G. Dunne, "Dynamically assisted schwinger mechanism," *Physical Review Letters*, vol. 101, Sept. 2008.

[6] G. V. Dunne and C. Schubert, "Worldline instantons and pair production in inhomogenous fields," *Physical Review D*, vol. 72, Nov. 2005.

[7] H. Gies and K. Klingmüller, "Pair production in inhomogeneous fields," *Physical Review D*, vol. 72, Sept. 2005.

[8] I. K. Affleck and N. S. Manton, "Monopole pair production in a magnetic field," *Nuclear Physics B*, vol. 194, pp. 38–64, Jan. 1982.

[9] B. Pioline and J. Troost, "Schwinger pair production in AdS2," *Journal of High Energy Physics*, vol. 2005, pp. 043–043, Mar. 2005.

[10] E. Mottola, "Particle creation in de sitter space," *Physical Review D*, vol. 31, pp. 754–766, Feb. 1985.

[11] S. P. Kim and D. N. Page, "Schwinger pair production indS2andads2," *Physical Review D*, vol. 78, Nov. 2008.

[12] D. Karabali, S. Kürkçüoğlu, and V. Nair, "Magnetic field and curvature effects on pair production. i. scalars and spinors," *Physical Review D*, vol. 100, Sep 2019.

[13] B. Estienne, S. M. Haaker, and K. Schoutens, "Particles in non-abelian gauge potentials: Landau problem and insertion of non-abelian flux," *New Journal of Physics*, vol. 13, p. 045012, Apr 2011.

[14] E. Jaynes and F. Cummings, "Comparison of quantum and semiclassical radiation theories with application to the beam maser," *Proceedings of the IEEE*, vol. 51, no. 1, pp. 89–109, 1963.

[15] B. R. Holstein, "Strong field pair production," *American Journal of Physics*, vol. 67, pp. 499–507, June 1999.

[16] E. Fradkin, *Quantum Field Theory: An Integrated Approach*. Princeton University Press, kindle edition ed., 3 2021.

[17] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals*. International series in pure and applied physics, New York, NY: McGraw-Hill, 1965.

[18] G. V. DUNNE, "HEISENBERG–EULER EFFECTIVE LAGRANGIANS: BASICS AND EXTENSIONS," in *From Fields to Strings: Circumnavigating Theoretical Physics*, pp. 445–522, WORLD SCIENTIFIC, Feb. 2005.

[19] U. Mosel, *Path Integrals in Field Theory*. Springer Berlin Heidelberg, 2004.

[20] H. Haber, "Notes on antisymmetric matrices and the pfaffian," 2009.

[21] A. Dress and W. Wenzel, "A simple proof of an identity concerning pfaffians of skew symmetric matrices," *Advances in Mathematics*, vol. 112, pp. 120–134, Apr. 1995.

[22] Z. F. Ezawa, *Quantum Hall Effects*. WORLD SCIENTIFIC, May 2013.

[23] "Quantised singularities in the electromagnetic field,," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 133, pp. 60–72, Sept. 1931.

[24] L. S. Brown and W. L. Weisberger, “Vacuum polarization in uniform non-abelian gauge fields,” *Nuclear Physics B*, vol. 157, pp. 285–326, Sept. 1979.