DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DIFFERENTIAL GEOMETRICAL CONDITIONS BETWEEN CURVATURE AND OSCULATING STRIP CURVES AND RULED SURFACES

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SUMMARY

In this study, by using the curve of the surface strip invariant magnetudes given at R^3 in the space of lines system of differential equation which is determined ruled surface has been set up.

But general solution of the system could not be obtained. However when the curve of the surface strip could be osculating strip, curvature strip and geodesic strip, the solution of the system and the relationship between the ruled surface's invariant magnetudes and the curve of surface strip's invariant magnetudes have been examined.

ÖZET

Bu çalışmada, IR^3 de verilen bir yüzey şerit eğrisinin invariant büyüklüklerinden yararlanarak, doğrular uzayında regle yüzeyi belirleyen diferansiyel denklem sistemi kurulmuş fakat sistemin genel çözümü elde edilememiştir. Ancak sistemin, yüzey şerit eğrisinin oskülatör şerit, eğrilik şeriti ve jeodezik şerit olması durumlarında çözümü ve regle yüzeyin invariant büyüklükleri ile yüzey şerit eğrisinin invariant büyüklükleri arasındaki bağıntılar incelenmiştir.

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Chapter 1

1. INTRODUCTION

I. 1. Surface Strip

A geometric figure, which is formed by a $\vec{x}$ point and ε plane passing from this point, is called "Surface element." There are two opposite unit vectors which belong to the ε plane and orthogonal to the surface element. These are the unit vectors of the normals of the surface element. If we choose one of these vectors, we will establish the positive direction of the surface element namely the direction it shows. In this way, it is said to be directed to the surface element. By giving the point $\vec{x}$ and normal vector $\vec{\xi}$ at this point, the directed surface element is completely determined. The point belongs to the surface and a surface element which is formed by a tangent plane passing from this point is equivalent to the every ordinary point of a surface. The plane which has all the tangents of the curves passing from this point is called the tangent plane of a point which belongs to a surface. Generally, $\vec{x}$ and $\vec{\xi}$ are given as a function of the parameter t. In this case, a family of surface elements is obtained. But they are generally the points of a curve which is drawn on a surface and they are not joined to each other regularly as they are in the surface elements of a stripped surface which is formed by tangent planes belonging to a surface.

For forming a strip with the surface elements as $\{\vec{x}(t), \vec{\xi}(t)\}$, the plane of the surface element must always pass through the tangent of $\vec{x}(t)$ curve, namely the tangent vector of $\vec{x}'$ of the curve must be orthagonal to the $\vec{\xi}$ vector.

Namely, it is;

$$\vec{x}'.\vec{\xi} = 0 \tag{1}$$

Now, choose the parameter t of the strip of $\left\{\vec{x}(s),\ \vec{\xi}(s)\right\}$ as its own s string (t = s) of the $\vec{x}(s)$ curve. In this case, apart from (1) following relations will be current

$$\vec{x}'^{2} = \vec{x}'.\vec{x}' = 1 \quad \text{and} \quad \vec{\xi}^{2} = 1 \tag{2}$$

Here, we will not examine;

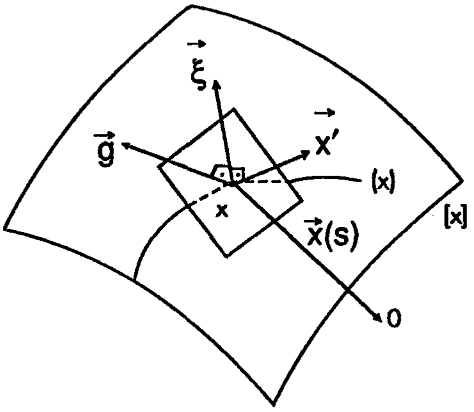
$$\vec{x}'.\vec{x}' = 0 \quad \text{or} \quad \vec{x}' = 0$$

Because in this case a surface element which passes from constant points is being discussed. We add the unit vector;

$$\vec{g} = \vec{\xi} \wedge \vec{x}' \tag{3}$$

to $\vec{x}'$ and $\vec{\xi}$

This vector is orthagonal to the tangent of the $\vec{x}'$ curve that is on the $\vec{x}$ points, and it is existed in the surface element of the strip. so;



the orthanormal base $(\vec{x}', \vec{g}, \vec{\xi})$ is obtained. Namely, the following relations are obtained.

$$\vec{x}'^2 = \vec{\xi}^2 = \vec{g}^2 = 1$$
$$\vec{x}' . \vec{\xi} = \vec{\xi} . \vec{g} = \vec{g} . \vec{x}' = 0 \qquad (4)$$

So, according to the relation (3) we can obtain

$$\vec{g} \wedge \vec{\xi} = \vec{x}'$$
$$\vec{x}' \wedge \vec{g} = \vec{\xi} \qquad (5)$$

Also other than (3) and (4) we can see

$$\begin{aligned}(\vec{x}', \vec{g}, \vec{\xi}) = \vec{x}' (\vec{g} \wedge \vec{\xi}) &= \vec{x}' \left[(\vec{\xi} \wedge \vec{x}') \wedge \vec{\xi}\right] \\ &= \vec{x}' \left[\vec{\xi}^2 . \vec{x}' - (\vec{x}' \vec{\xi})\, \vec{\xi}\right] \qquad (6) \\ &= \vec{x}'^2 \\ &= 1\end{aligned}$$

Now, let us write the derivatives of the orthanormal base vectors again in the form of themselves.

Let us say $\vec{x}' = \vec{x}_1$

We can write the derivative formulas like this;

$$\begin{aligned}\vec{x}'_1 &= \lambda_{11}\vec{x}_1 + \lambda_{12}\vec{g} + \lambda_{13}\vec{\xi} \\ \vec{g}' &= \lambda_{12}\vec{x}_1 + \lambda_{22}\vec{g} + \lambda_{23}\vec{\xi} \qquad (7) \\ \vec{\xi}' &= \lambda_{31}\vec{x}_1 + \lambda_{32}\vec{g} + \lambda_{33}\vec{\xi}\end{aligned}$$

From the equalities (4) we can obtain

$$\lambda_{11} = \lambda_{22} = \lambda_{33} = 0$$

$$\lambda_{21} = -\lambda_{12}\,,\ \lambda_{31} = -\lambda_{13}\,,\ \lambda_{32} = -\lambda_{23}$$

Here, if we take;

$$\lambda_{12} = c\,,\ \lambda_{13} = -b\,,\ \lambda_{32} = -a$$

If we take the derivatives of the equations (7) we obtain the formulas

$$\begin{aligned} \vec{x}'_1 &= * + c\vec{g} - b\,\vec{\xi} \\ \vec{g}' &= -c\vec{x}_1 + * + a\vec{\xi} \\ \vec{\xi} &= b\vec{x}_1 - a\vec{g} + * \end{aligned} \tag{8}$$

From the equations (8) we obtain these equalities

$$\begin{aligned} a &= -\vec{g}\vec{\xi}' = -(\vec{\xi} \wedge \vec{x}_1)\,\vec{\xi}' = (\vec{x}_1\,\vec{\xi}\,\vec{\xi}') = \tau_g \\ b &= \vec{x}_1\,\vec{\xi}' = -\rho_n \\ c &= \vec{g}\,\vec{x}'_1 = (\vec{\xi}' \wedge \vec{x}_1)\,\vec{x}'_1 = (\vec{\xi}\,\vec{x}_1\,\vec{x}'_1) = \rho_g \end{aligned} \tag{9}$$

τ_g, ρ_g and ρ_n in these equalities are invariant to the surface strip. By giving these as a function of the arc length s, it is known by the divergence of the position of the surface strip. In reality, the magnitude of τ_g is invariant by a sign divergence. Because parameter s is established by a sign divergence. Contrary to $-\rho_n$ and ρ_n, the magnitude τ_g is changed its sign with $s = -s^*$ relation. If a direction is choosen for the arc length which is considered to be positive on the curve of the strip, τ_g can be accepted as an absolute invariant. τ_g and $-\rho_n$ are related to the first – degree derivatives of the functions of the strip $\{\vec{x}\,(t)\,,\,\vec{\xi}\,(t)\}$ which is given according to a parameter and ρ_g is related to the second degree derivatives.

By giving τ_g, ρ_n and ρ_g as a function of s only one $\{\vec{x}(s), \vec{\xi}(s)\}$ strip can be determined. Suppose a second $\{\vec{x}^*(s), \vec{\xi}^*(s)\}$ strip is determined

In this case, the following formulas are current

$$\frac{d}{ds}(\vec{x}_1, \vec{x}_1^*) = (\rho_g \vec{g} + \rho_n \vec{\xi})\vec{x}_1^* + \vec{x}_1(\rho_g \vec{g}^* + \rho_n \vec{\xi}^*)$$

$$\frac{d}{ds}(\vec{g}, \vec{g}^*) = (-\rho_g \vec{x}_1 + \tau_g \vec{\xi})\vec{g}^* + \vec{g}(-\rho_g \vec{x}_1^* + \tau_g \vec{\xi}^*) \qquad (10)$$

$$\frac{d}{ds}(\vec{\xi}, \vec{\xi}^*) = (-\rho_n \vec{x}_1 - \tau_g \vec{g})\vec{\xi}^* + \vec{\xi}(-\rho_n \vec{x}_1^* + \tau_g \vec{g}^*)$$

$$\frac{d}{ds}(\vec{x}_1 \vec{x}_1^* + \vec{g}\vec{g}^* + \vec{\xi}\vec{\xi}^*) = 0 \qquad (11)$$

$$\vec{x}_1 \vec{x}_1^* + \vec{g}\vec{g}^* + \vec{\xi}\vec{\xi}^* = \text{Const} = k$$

The formulas (10) and (11) will be current for every s so it will be current for s = 0 too. And for s = 0 we can coincide trihedrons of the two strips

In this case

$$\vec{x}_1 \vec{x}_1^* = \vec{g}\vec{g}^* = \vec{\xi}\vec{\xi}^* = 1$$

So ; $\quad k = 3$

Since each one is a unit vector in order to have k = 3 we must have the following equalities.

$$\vec{x}_1 = \vec{x}_1^*, \; \vec{g} = \vec{g}^*, \; \vec{\xi} = \vec{\xi}^*$$

From this, we can see

$$\frac{d\vec{x}}{ds} = \frac{d\vec{x}^*}{ds} \qquad \vec{x} = \vec{x}^* + \alpha \qquad \alpha \in R$$

Since s = 0 is the common points of the curves (strips), we obtain $\alpha = 0$

So it shows that the strips are coincided, namely $\vec{x} = \vec{x}^*$ [1], [3].

1.2 OSCULATİNG STRİP

Strips given by the equality of $\rho_n = \vec{x}'_1 \cdot \vec{\xi} = 0$ are called osculating strips.

Let $(\vec{x}_1 = \vec{t}, \vec{n}, \vec{b})$ be the Frenet trihedron of the osculating strip. Since the following equations

$$\rho_n = \vec{x}'_1 \vec{\xi} = \rho \vec{n} \vec{\xi} = \rho \operatorname{Cos} \theta = 0$$

$$\theta = \frac{\pi}{2} \quad \text{are current}$$

$\vec{\xi}$, the normal of the surface of the strip is orthagonal to the prime normal of the curve of the strips (At the same time it is $\vec{\xi} \perp \vec{x}_1$). Hence, $\vec{\xi}$ is orthagonal to the osculating plane.

In this way the strips, that are $\rho_n = 0$, the tangent plane of the strip coincide with the osculating plane of the curve of the strip [1], [3].

1.3. Curvature Strip

The strips that proves the following equality

$$\tau_g = (\vec{x}_1, \vec{\xi}, \vec{\xi}') = 0$$

are called the curvature strips. Along these strips, the distribution parameter

$$\vec{y} = \vec{x}(s) + \rho\vec{\xi}(s)$$

of the ruled surface, which is formed by the normals, is zero. Because of

$\tau_g = 0$, this ruled surface can be a developable surface.

The strips that are $\tau_g = 0$ can be characterized by forming a developable ruled surface of the normals [1], [3].

1.4. Geodesic Strips

The strips that satisfy $\rho_g = (\vec{\xi}, \vec{x}_1, \vec{x}'_1) = 0$ are called the geodesic strips.

Since,

$$(\vec{\xi}, \vec{x}_1, \vec{x}'_1) = (\vec{\xi} \wedge \vec{x}_1), \vec{x}'_1 = \vec{g}\,\vec{x}'_1$$

$$\rho\vec{g}\,.\,\vec{n}\,\mathrm{Cos}\,(\frac{\pi}{2} - \theta) = 0$$

$$\rho\,.\,\vec{n}\,\vec{g}\,\mathrm{Sin}\,\theta = 0$$

$$\rho\,\mathrm{Sin}\,\theta = 0$$

$$\theta = 0$$

$\vec{x}_1$ coincide with $\vec{\xi}$ and the osculating plane of the curve of the strip passes from the surface normal.

Result I : If the $\vec{x}(s)$ curve of the strip is a line, $\vec{x}'_1$ must be zero. In this case, since

$$\vec{x}'_1 = \rho_g\vec{x} + \rho_n\,\vec{\xi}$$

We obtain

$$\rho_g = \rho_n = 0$$

As a result, if there is a line on the surface, strip surface along this line is both an osculating strip and a geodesic strip.

Result II : In osculating strips $\rho_n = 0$. From this equation, we can find

$\rho_g = g$, $\tau_g = -\tau_g$. And this shows that the geodesic curvature and the geodesic torsion in osculating strip are equal to the ordinary curvature and the ordinary torsion [1] , [3].

Namely,

$$\rho = \rho_g \, , \tau_g = \tau$$

1.5. Ruled Surface

An oriented straight line in the three - dimensional Euclidean space R^3 may be represented by a dual unit vector in the space R^3 of the triplets of dual numbers as $\vec{A} = \vec{a} + \varepsilon \vec{\bar{a}} \, (\vec{a} \, . \, \vec{a} = 1 \, , \, \vec{a} \, . \, \vec{\bar{a}} = 0)$ where a is a dual unit vector along the oriented line, $\bar{a}$ is the moment of a about the origin 0 and ε is Clifford's operation with the property $\varepsilon^2 = 0$. The set of all oriented lines in R^3 is in one - toone corrospondance with the set of the dual unit sphere $\vec{A}^2 = 1$ in D^3

A ruled surface is represented by a curve in the dual unit sphere as

$$\vec{A}_1(t) = \vec{a}_1(t) + \varepsilon \vec{\bar{a}}_1(t) \tag{12}$$

where t is an arbitrary parameter.

$\{\vec{A}_1(t) \, , \, \vec{A}_2(t) \, , \, \vec{A}_3(t)\}$ which belongs to the generator $\vec{A}_1(t)$ is Blaschke trihedron of the ruled surface and the following derivative formulas exist among Blaschke base vectors.

$$\begin{aligned} \frac{d\vec{A}_1}{dt} &= P\vec{A}_1 \\ \frac{d\vec{A}_2}{dt} &= -P\vec{A}_1 + Q\vec{A}_3 \\ \frac{d\vec{A}_3}{dt} &= -Q\vec{A}_2 \end{aligned} \tag{13}$$

In these formulas, $P = p + \varepsilon\bar{p}$ and $Q = q + \varepsilon\bar{q}$ are the dual invariants the ruled surface which are defined as

$$P = \left\| \left(\frac{d\vec{A}_1}{dt} \right)^2 \right\| \qquad Q = \frac{\left(\frac{d\vec{A}_1}{dt} , \frac{d^2\vec{A}_1}{dt^2} , \frac{d^3\vec{A}_1}{dt^3} \right)}{\left(\frac{d\vec{A}_1}{dt} \right)^2} \tag{14}$$

Also, the dual arc – length of the ruled surface from $t = t_0$ $t = t$ is defined as

$$S = s + \varepsilon\bar{s} = \int_{t_0}^{t} \left\| \left(\frac{d\vec{A}_1}{dt} \right)^2 \right\| dt = \int_{t_0}^{t} P dt \tag{15}$$

and the distribution parameter of the ruled surface is

$$\delta = \frac{\vec{a}'_1 \cdot \vec{\bar{a}}'_1}{\vec{a}'^2_1} = \frac{\bar{p}}{p} \tag{16}$$

If $\delta = 0$ $(\bar{p} = 0)$, the ruled surface is called a developable surface [2], [4].

Chapter 2

2.1. DIFFERANTIAL – GEOMETRICAL CONDITIONS BETWEEN STRIP CURVES and RULED SURFACES

Let $\{\vec{x}_1(s), \vec{g}(s), \vec{\xi}(s)\}$ be Darbaux trihedron of the surface curve $\vec{x} = \vec{x}(s)$ at the point $\vec{x} = \vec{x}(s)$ if the moments of the Darboux vectors are taken with respect to the origin in the (0, x, y, z) coordinate system, the dual unit vectors are defined as

$$\begin{aligned}
&\vec{A}_1(s) = \vec{x}_1 + \varepsilon\,(\vec{x} \wedge \vec{x}_1)\,;\ \overset{\rightrightarrows}{x}_1 = \vec{x} \wedge \vec{x}_1 \\
&\vec{A}_2(s) = \vec{g} + \varepsilon\,(\vec{x} \wedge \vec{g})\,;\ \overset{\rightrightarrows}{g} = \vec{x} \wedge \vec{g} \\
&\vec{A}_3(s) = \vec{\xi} + \varepsilon\,(\vec{x} \wedge \vec{\xi})\,;\ \overset{\rightrightarrows}{\xi} = \vec{x} \wedge \vec{\xi}
\end{aligned} \tag{17}$$

Here

$$\vec{A}_i\vec{A}_j = \begin{cases} 0 & , i \neq j \\ 1 & , i = j \qquad (i, j = 1, 2, 3) \end{cases}$$

the vector $\vec{x} = \vec{x}(s)$ is written in terms of Darboux vectors of the surface curve as

$$\vec{x}(s) = m(s)\,\vec{x}_1 + n(s)\vec{g} + k(s)\vec{\xi} \tag{18}$$

where $m = m(s)$, $n = n(s)$, $k = k(s)$. From the formulas (17) and (18) by using the properties of the cross product we get the following equalities.

$$\begin{aligned}
&\vec{A}_1(s) = \vec{x}_1 + \varepsilon\,(k\vec{g} - n\vec{\xi}) \\
&\vec{A}_2(s) = \vec{g} + \varepsilon\,(-k\vec{x}_1 + m\vec{\xi}) \\
&\vec{A}_3(s) = \vec{\xi} + \varepsilon\,(n\vec{x}_1 - m\vec{g})
\end{aligned} \tag{19}$$

If we determine the coefficients $m = m(s)$, $n = n(s)$ and $k = k(s)$ in terms of the invariants of the surface curve $\vec{x} = \vec{x}(s)$, then $\vec{A}_1 = \vec{A}_1(s)$ ruled surface is determined. By taking the derivative of the equation (18) with respect to s and from the $\frac{d\vec{x}_1}{ds}$; $\frac{d\vec{g}}{ds}$, $\frac{d\vec{\xi}}{ds}$

we obtain

$$\frac{d\vec{x}}{ds} = \vec{x}_1 = (m' - \rho_g n - \rho_n k)\,\vec{x}_1 + (n' + m\rho_g - \tau_g k)\,\vec{g} +$$

$$(k' + m\rho_n + n\tau_g)\,\vec{\xi} \qquad (20)$$

From (20) the following system of differantial equation can be written as

$$\begin{aligned} &m' - \rho_g\, n - \rho_n\, k = 1 \\ &n' + m\,\rho_g - \tau_g\, k = 0 \\ &k' + m\,\rho_n + \tau_g\, n = 0 \end{aligned} \qquad (21)$$

Now, let us solve the (21) system of equations

$$\begin{aligned} &n = \frac{1}{\rho_g} m' - \frac{\rho_n}{\rho_g} k - \frac{1}{\rho_g} \\ &n' = -\rho_g\, m + \tau_g\, k \\ &k' = -\rho_n\, m - \tau_g\, n \end{aligned} \qquad (22)$$

$$m' - \rho_g\, n - \rho_n\, k = 1$$

$$m'' - \rho_g'\, n - \rho_g\, n' - \rho_n'\, k - \rho_n\, k' = 0$$

$$m'' = \rho_g'\, n + \rho_g\, n' + \rho_n'\, k + \rho_n\, k \qquad (23)$$

From the (22) equations, If we write n, n' and k' values to their place in (23) we obtain

$$m'' = \rho_g'\left(\frac{1}{\rho_g}m' - \frac{\rho_n}{\rho_g}k - \frac{1}{\rho_g}\right) + \rho_g(-\rho_g m + \tau_g k) + \rho_n'k + \rho_n(-\rho_n m - \tau_g n)$$

$$m''= \frac{\rho'_g}{\rho_g} m' - \frac{\rho'_g\rho_n}{\rho_g} k - \frac{\rho'_g}{\rho_g} - \rho_g^2 m + \rho_g\tau_g k + \rho'_n k - \rho_g^2 m - \frac{\rho_n\tau_g}{\rho_g} m' + \frac{\rho_n^2\tau_g}{\rho_g} k + \frac{\rho_n\tau_g}{\rho_g}$$

$$\rho_g m''= \rho'_g m' - \rho'_g\rho_n k - \rho'_g - \rho_g^3 m + \rho_g^2\tau_g k + \rho_g\rho'_n k - \rho_n^2\rho_g m - \rho_n\tau_g m' + \rho_n^2\tau_g k + \rho_n\tau_g \qquad (24)$$

$$\rho_g m'' + (\rho_n\tau_g - \rho'_g)m' + (\rho_g^3 + \rho_g \rho_n^2)m + \rho'_n - \rho_n \tau_g = (\rho_g^2 \tau_g + \rho_g \rho'_n - \rho'_g \rho_n + \rho_n^2 \tau_g)k$$

$$A = \rho_g^2 \tau_g + \rho_g\rho_n' - \rho_g' \rho_n + \rho_g^2 \tau_g$$

$$B = \rho_n \tau_g - \rho_g' \qquad (25)$$

$$C = \rho_g^3 + \rho_g\rho_n^2$$

If the equations in (25) are written in (24) we obtain the following

$$\rho_g m'' + Bm' + Cm - B = AK \qquad (26)$$

$$\frac{\rho_g}{A} m'' + \frac{B}{A} m' + \frac{C}{A} m - \frac{B}{A} = K \qquad (27)$$

If we take the derivative of the (26)

$$\rho_g' m'' + \rho_g m''' + B'm' + Bm'' + C'm + Cm' - B' = A'K + AK'$$

$$\rho_g m''' + (\rho_g' + B)\, m'' + (B' + C)\, m' + C'm - B' = A'K + A\,(-\rho_n m - \tau_g n)$$

$$\rho_g m'' + (\rho'_g + B)m'' + (B' + C)m' + C'm - B' = A'K - \rho_n A\, m - \tau_g A\left(\frac{1}{\rho_g}m' - \frac{\rho_n}{\rho_g}K - \frac{1}{\rho_g}\right) \qquad (28)$$

$$\rho_g^2 m''' + (\rho_g\rho'_g + \rho_g B)m'' + (\rho_g B' + \rho_g C)m' + \rho_g C'm - \rho_g B' = \rho_g A'K - \rho_g\rho_n A\, m - \tau_g A m' + \rho_n\tau_g AK + \tau_g A$$

$$\rho_g^2 m''' + (\rho_g\rho'_g + \rho_g B)m'' + (\rho_g B' + \rho_g C + \tau_g A)m' + (\rho_g C' + \rho_g\rho_n A)m - \rho_g B' - \tau_g A = (\rho_g A' + \rho_n\tau_g A)K$$

If the value of k in (27) is written in (28)

$$\rho_g^2 m''' + (\rho_g \rho'_g + \rho_g B)m'' + (\rho_g B' + \rho_g C + \tau_g A)m' + (\rho_g C' + \rho_g \rho_n A)m - \rho_g B' - \tau_g A =$$

$$(\rho_g A' + \rho_n \tau_g A)\left(\frac{\rho_g}{A} m'' + \frac{B}{A} m' + \frac{C}{A} m - \frac{B}{A}\right) \qquad (29)$$

$$\rho_g^2 A\, m''' + (\rho_g \rho'_g A + \rho_g AB - \rho_g^2 A' - \rho_g \rho_n \tau_g A)m'' + (\rho_g AB' + \rho_g AC + \tau_g A^2 - \rho_g A'B - \rho_n \tau_g AB)m' +$$

$$+ (\rho_g AC' + \rho_g \rho_n A^2 - \rho_g A'C - \rho_n \tau_g AC)m - \rho_g AB' - \tau_g A^2 + \rho_g A'B + \rho_n \tau_g AB = 0$$

$$\begin{aligned}
P_4 &= \rho_g^2\, A \\
P_3 &= \rho_g \rho_g{}' A + \rho_g\, AB - \rho_g^2\, A' - \rho_g\, \rho_n\, \tau_g A \\
P_2 &= \rho_g\, AB' + \rho_g\, AC + \tau_g\, A^2 - \rho_g\, A'B - \rho_n\, \tau_g\, AB \\
P_1 &= \rho_g\, AC' + \rho_g\, \rho_n\, A^2 - \rho_g\, A'C - \rho_n\, \tau_g\, AC \qquad (30) \\
P_0 &= \rho_n \tau_g\, AB + \rho_g\, A'B - \tau_g\, A^2 - \rho_g\, AB'
\end{aligned}$$

And if the value of (30) is written in (29) we obtain

$$P_4(t)\frac{d^3m}{dt^3} + P_3(t)\frac{d^2m}{dt^2} + P_2(t)\frac{dm}{dt} + P_1(t)\, m + P_0 = 0 \qquad (31)$$

So, we can get a third - degree differential equation with an arbitrary coefficient. The solution of this equation was not found but in special pasitions we can find the solution of the differential equation systems (20).

2.2. First we can examine the pozition of $\rho_n = \tau_g = \rho_g = cst$ in the equation of (20). Then let us solve it by using the operation method.

$$\begin{aligned}
m' - \rho_g\, n - \rho_n\, k &= 1 \\
n' + \rho_g\, m - \tau_g\, k &= 0 \\
k' + \rho_n\, m + \tau_g\, n &= 0
\end{aligned}$$

In the equation system

$$F(D) = \begin{vmatrix} D & -\rho_g & -\rho_n \\ \rho_g & D & -\tau_g \\ \rho_n & \tau_g & D \end{vmatrix} = D^3 + (\rho_g^2 + \rho_n^2 + \tau_g^2)\,D\,, \quad \text{and } \rho_g^2 + \tau_g^2 + \rho_n^2 = d^2$$

$$= D\,(D^2 + d^2) \neq 0$$

$$f_1(s) = 1\,,\ \ f_2(s) = 0\,,\ \ f_3(s) = 0$$

we obtain

$$\Delta_m(t) = \begin{vmatrix} 1 & -\rho_g & -\rho_n \\ 0 & D & -\tau_g \\ 0 & \tau_g & D \end{vmatrix} = \tau_g^2$$

$$\Delta_n(t) = \begin{vmatrix} D & 1 & -\rho_n \\ \rho_g & 0 & -\tau_g \\ \rho_n & 0 & D \end{vmatrix} = -\rho_n\tau_g$$

$$\Delta_k(t) = \begin{vmatrix} D & -\rho_n & 1 \\ \rho_g & D & 0 \\ \rho_n & \tau_g & 0 \end{vmatrix} = \rho_g\tau_g$$

and then

$$F(D)_m = \Delta_m(t)$$

$$F(D)_n = \Delta_n(t)$$

$$F(D)_k = \Delta_k(t)$$

$$(D^3 + D)_m = \tau_g^2 \qquad (32)$$

$$(D^3 + D)_n = -\rho_n tg$$

$$(D^3 + D)_k = \rho_g\,\tau_g$$

When we use the solutions of the (32) we can obtain the following system

$$
\begin{aligned}
m &= A_1 \cos dt + A_2 \sin dt + A_3 + \tau_g^2 t \\
n &= B_1 \cos dt + B_2 \sin dt + B_3 - \rho_n \tau_g t \\
k &= C_1 . \cos dt + C_2 \sin dt + C_3 + \rho_g \tau_g t
\end{aligned}
\tag{33}
$$

Here A_i, B_i and C_i $(i = 1, 2, 3)$ are constants and

$$m' - \rho_g n - \rho_n k = 1 \tag{34}$$

If we substitute the equation (33) in (34) we can get a new system as follows

$$-d A_1 \sin dt + dA_2 \cos dt + \tau_g^2 - \rho_g B_1 \cos dt - \rho_g B_2 + \sin dt - \rho_g B_3$$

$$+ \rho_g \rho_n \tau_g t - \rho_n C_1 \cos dt - \rho_n C_2 \sin dt - \rho_n C_3 - \rho_g \rho_n \tau_g t = 0$$

$$
\begin{aligned}
-d A_1 - \rho_g B_2 - \rho_n C_2 &= 0 \\
dA_2 - \rho_g B_1 - \rho_n C_1 &= 0 \\
\tau_g^2 - \rho_g B_3 - \rho_n C_3 &= 0
\end{aligned}
\tag{35}
$$

If (33) is written, in its own place at the equation

$$n' + \rho_g m - \tau_g k = 0$$

we obtain

$$-d B_1 \sin dt + d B_2 \cos dt - \rho_n \tau_g + \rho_g A_1 \cos dt + \rho_n A_2 \sin dt + \rho_g A_3$$

$$+ \rho_g \tau_g^2 t - \tau_g C_1 \cos dt - \tau_g C_2 \sin dt - \tau_g C_3 - \rho_g \tau_g^2 t = 0$$

and therefore

$$-d\,B_1 + \rho_g A_2 - \tau_g C_2 = 0$$

$$dB_2 + \rho_g A_1 - \tau_g C_1 = 0 \qquad (36)$$

$$-\rho_n \tau_g + \rho_g A_3 - \tau_g C_3 = 0$$

If (33) is written in its own place at the equation

$$k' + \rho_n m + \tau_g n = 0$$

we obtain

$$-C_1\, d \sin dt + d\, C_2 \cos dt + \rho_g \tau_g + \rho_n A_1 \cos dt + \rho_n A_2 \sin dt +$$

$$+ \rho_n A_3 + \rho_n \tau_g^2\, t + \tau_g B \cos dt + \tau_g B_1 \sin dt + \tau_g B_3 - \rho_n \tau_g^2\, t = 0$$

and so,

$$-dC_1 + \rho_n A_2 + \tau_g B_2 = 0$$

$$d\,C_2 + \rho_n A_1 + \tau_g B_1 = 0 \qquad (37)$$

$$\rho_g \tau_g + \rho_n A_3 + \tau_g B_3 = 0$$

From the (35) , (36) and (37) equations, we can obtain

$$d\,A_1 + \rho_g B_2 + \rho_n C_2 = 0$$

$$d\,A_2 - \rho_g B_1 - \rho_n C_1 = 0$$

$$d\,B_1 + \rho_g A_2 - \tau_g C_2 = 0$$

$$
\begin{aligned}
& d\,B_2 + \rho_g A_1 - \tau_g C_1 = 0 \qquad\qquad (38)\\
& -d\,C_1 + \rho_g A_2 - \tau_g B_2 = 0\\
& d\,C_2 + \rho_n A_1 + \tau_g B_1 = 0\\
& \tau_g^2 - \rho_g B_3 - \rho_n C_3 = 0\\
& -\rho_n \tau_g + \rho_g A_3 - \tau_g C_3 = 0\\
& \rho_g \tau_g + \rho_n A_3 + \tau_g B_3 = 0
\end{aligned}
$$

From the $(38)_8$ we can obtain

$$C_3 = -\rho_n + \frac{\rho_g}{\tau_g} A_3 \qquad (39)$$

From the $(38)_9$ we can obtain

$$B_3 = -\rho_g - \frac{\rho_n}{\tau_g} A_3 \qquad (40)$$

The values of (39) and (40) are written in $(38)_7$

$$\tau_g^2 + \rho_g^2 + \frac{\rho_g \rho_n}{\tau_g} A_3 + \rho_n^2 - \frac{\rho_g \rho_n}{\tau_g} A_3 = 0$$

We obtain

$$\rho_g^2 + \tau_g^2 + \rho_n^2 = 0$$

τ_g , ρ_g , ρ_n are not constants at the same time. $\rho_g^2 + \rho_n^2 + \tau_g^2 = 0$ is possible in the case of $\rho_n = 0$, $\tau_g = 0$ and $\rho_g = 0$. Under this condition

The Darboux trihedron $\{\vec{x}_1(s), \vec{g}(s), \vec{\xi}(s)\}$ of the vector $\vec{x}(s)$ determines a constant point

2.3. Let the curve be a geodesic curve ($\rho_g = 0$) The System (18) takes the following form

$$\begin{aligned} m' - \rho_n\, k &= 1 \\ n' - \tau_g\, k &= 0 \\ k' + \rho_n\, m + \tau_g\, n &= 0 \end{aligned} \tag{41}$$

Now we solve the seystem of differential equation (41) for the following special cases.

2.3.1. If $m = 0$ from the system (41) we have

$$\begin{aligned} -\rho_n\, k &= 1 \\ n^1 - \tau_g\, k &= 0 \\ k^1 + n\,\tau_g &= 0 \end{aligned} \tag{42}$$

From the equations $(42)_2$ and $(42)_3$ we get the following differential equation

$$n'' - \frac{\tau'_g}{\tau_g}\, n' + \tau_g^2 \,.\, n = 0 \tag{43}$$

If we use $t = \int_{t_0}^{t} \tau_g \, ds$ in (43) we obtain

$$\frac{d^2n}{dt^2} + n = 0 \tag{44}$$

Then,

$$n = C_1 \cos \int_0^s \tau_g \, ds + C_2 \sin \int_0^s \tau_g \, ds$$

where C_1 and C_2 are constants. From $(42)_1$ it is clear that

$$k = -\frac{1}{\rho_n}$$

In this case with the aid of the equation (18)

$\vec{x} = \vec{x}\,(s)$ can be written as

$$\vec{x} = (\, C_1 \cos \int_0^s \tau_g \, ds + C_2 \sin \int_0^s \tau_g \, ds \, \vec{g} - \frac{1}{\rho_n} \vec{\xi} \tag{45}$$

Blaschke vectors of the ruled surfaces $\vec{A}_1\,(s)$ are now

$$\begin{aligned} \vec{A}_1 &= \vec{x}_1 + \varepsilon \left\{ -\vec{\xi}\,(C_1 \cos \int_0^s \tau_g \, ds + C_2 \sin \int_0^s \tau_g \, ds) - \frac{1}{\rho_n} \vec{g} \right\} \\ \vec{A}_2 &= \vec{g} + \varepsilon \left(\frac{1}{\rho_n} \vec{x}_1 \right) \\ \vec{A}_3 &= \vec{\xi} + \varepsilon \left\{ C_1 \cos \int_0^s \tau_g \, ds + C_2 \sin \int_0^s \tau_g \, ds \right\} \vec{x}_1 \end{aligned} \tag{46}$$

If $C_1 = C_2 = 0$, then $n = 0$ $\quad k = -\frac{1}{\rho_n}$

$$\begin{aligned} \vec{A}_1 &= \vec{x}_1 + \varepsilon \left(-\frac{1}{\rho_n} \vec{g} \right) \\ \vec{A}_2 &= \vec{g} + \varepsilon \left(\frac{1}{\rho_n} \vec{x}_1 \right) \\ \vec{A}_3 &= \vec{\xi} \end{aligned} \tag{47}$$

Differentiating the equations $(47)_1$ and $(47)_3$ with respect to s and using (1) and (3), the real and dual parts of P and Q are obtained as

$$p = \rho_n \quad , \quad \bar{p} = -\frac{\tau_g}{\rho_n}$$

$$q = \sqrt{\rho_n^2 + \tau_g^2} \quad , \quad \bar{q} = 0 \tag{48}$$

2.3.2. If n = 0, from the system of equation (41), we have

$$\begin{aligned} m' - \rho_n k &= 1 \\ -\tau_g k &= 0 \\ k' + m\,\rho_n &= 0 \end{aligned} \tag{49}$$

Using $(49)_2$ we get k= 0. From $(49)_1$ and $(49)_3$, the following differential equation is obtained.

$$m'' - \frac{\rho'_n}{\rho_n}\left(m' - 1\right) + m\rho_n^2 = 0 \tag{50}$$

With the aid of the parameter change $t = \int_0^s \rho_n \, ds$ in (50)

we get the differential equation

$$\frac{d^2m}{dt^2} + m = \frac{-\rho_n}{\rho_n^2} \tag{51}$$

with constant coeffiecients.

The solution of (51) is

$$m = C_1 \cos \int_0^s \rho_n \, ds + C_2 .\sin \int_0^s \rho_n \, ds + \frac{1}{D^2 + 1}\left(\frac{-\rho_n}{\rho_n^2}\right) \tag{52}$$

Here C_1 and C_2 are constants and,

$\rho_n = \frac{d\rho'_n}{dt}$ $D = \frac{d}{dt}$ denotes the derivative operator. If we substitute

(52), $k = 0$, $n = 0$ in the equation (18), the surface curve $\vec{x} = \vec{x}(s)$ is obtained as

$$\vec{x} = \left\{ C_1 \cos \int_0^s \rho_n \, ds + C_2 \sin \int_0^s \rho_n \, ds + \frac{1}{D^2 + 1} \left(\frac{-\rho_n}{\rho_n^2} \right) \right\} \vec{x}_1$$

Blaschke vectors of the ruled surface $\vec{A}_1(s)$ now take the form

$$\vec{A}_1 = \vec{x}_1$$

$$\vec{A}_2 = \vec{g} + \varepsilon \left\{ C_1 \cos \int_0^s \rho_n \, ds + C_2 \sin \int_0^s \rho_n \, ds \right\} \vec{\xi}$$

$$\vec{A}_3 = \vec{\xi} + \varepsilon \left\{ - C_1 \cos \int_0^s \rho_n \, ds - C_2 \sin \int_0^s \rho_n \, ds \right\} \vec{g}$$

The invariants of the ruled surface $\vec{A}_1(s)$ are

$$p = \rho_n \qquad \bar{p} = 0$$

$$q = \sqrt{\rho_n^2 + \tau_g^2} \quad \bar{q} = - \frac{\tau_g \, \rho_n}{\sqrt{\rho_n^2 + \tau_g^2}} \left\{ C_1 \sin \int_0^s \rho_n \, ds - C_2 \cos \int_0^s \rho_n \, ds \right\},$$

2.3.3. If $k = 0$, from the system of equation (41) we have

$$m' = 1$$

$$n' = 0$$

$$\rho_n m + \tau_g n = 0 \tag{53}$$

From (53) we obtain

$$m = s + C_1, \quad n = C_2$$

In this case $\vec{x} = \vec{x}(s)$ can be written as,

$$\vec{x} = (s + C_1)\,\vec{x}_1 + C_2\,\vec{g}$$

In this case blaschke vectors of the ruled surface $\vec{A}_1(s)$ become

$$\vec{A}_1 = \vec{x}_1 + \varepsilon\left(-C_2\,\vec{\xi}\right)$$

$$\vec{A}_2 = \vec{g} + \varepsilon\,(s + C_1)\,\vec{\xi}$$

$$\vec{A}_3 = \vec{\xi} + \varepsilon\left[-(s + C_1)\,\vec{g} + C_2\,\vec{x}_1\right]$$

The invariants of the ruled surface $\vec{A}_1(s)$ are found as

$$p = \rho_n\,, \qquad \bar{p} = 0$$

$$q = \sqrt{\rho_n^2 + \tau_g^2}\,, \quad \bar{q} = \frac{\tau_g}{\sqrt{\rho_n^2 + \tau_g^2}}$$ [5].

2.4. Let the curve be a normal curve $(\rho_n = 0)$, The system (18) takes the fallowing form.

$$\begin{aligned} &m' - \rho_g n = 1 \\ &n' + \rho_g m - \tau_g k = 0 \\ &k' + \tau_g n = 0 \end{aligned} \tag{54}$$

Now we solve the system of differantial equation (54) for the fallowing special cases.

2.4.I. If $m = 0$ from the system (54) we have

$$\begin{aligned} &-\rho_g n = 1 \\ &n' - \tau_g k = 0 \\ &k' + \tau_g n = 0 \end{aligned} \tag{55}$$

From the equation $(55)_2$ and $(55)_3$ we get the fallowing differential equation

$$k'' - \frac{\tau_g'}{\tau_g} k' + \tau_g^2 k = 0 \tag{56}$$

If we use $t = \int_{to}^{t} \tau_g \, ds$ in (56) we obtain

$$\frac{d^2k}{dt^2} + k = 0 \tag{57}$$

$$k = C_1 \cos \int_0^s \tau_g \, ds + C_2 \sin \int_0^s \tau_g \, ds$$

where C_1 and C_2 are constants. From $(55)_1$ it is clear that

$$n = -\frac{1}{\rho_g} \quad \text{and} \quad \frac{1}{\rho_g} = -C_1 \sin \int_0^s \tau_g \, ds + C_2 \cos \int_0^s \tau_g \, ds$$

In this case with the aid of Eq (18) $\vec{x} = \vec{x}(s)$ can be written as

$$\vec{x} = -\frac{1}{\rho_g}\vec{g} + \left(C_1 \cos \int_0^s \tau_g \, ds + C_2 \sin \int_0^s \tau_g \, ds\right)\vec{\xi} \tag{58}$$

Blaschke vectors of the ruled surfaces $\vec{A}_1(s)$ are now

$$\begin{aligned}
\vec{A}_1 &= \vec{x}_1 + \varepsilon\left[\left(C_1 \cos \int_0^s \tau_g \, ds + C_2 \sin \int_0^s \tau_g \, ds\right)\vec{g} + \frac{1}{\rho_g}\vec{\xi}\right] \\
\vec{A}_2 &= \vec{g} + \varepsilon\left[\left(-C_1 \cos \int_0^s \tau_g \, ds - C_2 \sin \int_0^s \tau_g \, ds\right)\vec{x}_1\right] \\
\vec{A}_3 &= \vec{\xi} + \varepsilon\left(-\frac{1}{\rho_g}\vec{x}_1\right)
\end{aligned} \tag{59}$$

If $C_1 = C_2 = 0$, then $k = 0$ $n = -\frac{1}{\rho_g}$

$$\begin{aligned}
\vec{A}_1 &= \vec{x}_1 + \varepsilon\left(\frac{1}{\rho_g}\vec{\xi}\right) \\
\vec{A}_2 &= \vec{g} \\
\vec{A}_3 &= \vec{\xi} + \varepsilon\left(-\frac{1}{\rho_g}\vec{x}_1\right)
\end{aligned} \tag{60}$$

If the equations $(60)_1$ and $(60)_3$ are diferentialed with respect to s and using (1) and (3), the real and dual parts of P and Q are obtained as

$$p = \rho_g \qquad \bar{p} = -\frac{\tau_g}{\rho_g}$$

2.4.2. If $n = 0$, from the system of equation (54) we have

$$\begin{aligned} &m' = 1 \\ &\rho_g m - \tau_g k = 0 \\ &k' = 0 \end{aligned} \tag{61}$$

From (61) we obtain

$$\begin{aligned} &m = s + C_1 \\ &k = C_2 \end{aligned}$$

In this case, from the $(61)_2$ we obtain $\frac{\rho_g}{\tau_g} = \frac{C_2}{s + C_1}$ and $\vec{x} = \vec{x}\,(s)$ can be written as

$$\vec{x} = (s + C_1)\,\vec{x}_1 + C_2\,\vec{\xi}$$

In this case, blaschke vectors of the ruled surface $\vec{A}_1\,(s)$ become

$$\vec{A}_1 = \vec{x}_1 + \varepsilon\,(C_2\,\vec{g})$$

$$\vec{A}_2 = \vec{g} + \varepsilon\left[-C_2\,\vec{x}_1 + (\boldsymbol{s} + C_1)\,\vec{\xi}\right]$$

$$\vec{A}_3 = \vec{\xi} + \varepsilon\left[-(\boldsymbol{s} + C_1)\,\vec{g}\right]$$

The invariants of the ruled surface $\vec{A}_1\,(s)$ are

$$p = \rho_g \qquad \bar{p} = 0$$

2.4.3. If $k = 0$, from the system of equation (54) we have

$$\begin{aligned} m' - \rho_g n &= 1 \\ n' + \rho_g m &= 0 \\ \tau_g n &= 0 \end{aligned} \tag{62}$$

Using $(62)_3$ we get $n = 0$. From $(62)_1$ and $(62)_2$, the fallowing differential equation is obtained

$$\begin{aligned} & m'' - \rho'_g\, n - \rho_g\, n' = 0 \\ & m'' - \frac{\rho'_g}{\rho_g}\left(m' - 1\right) + \rho_g^2 m = 0 \end{aligned} \tag{63}$$

With the aid of the parameter change $t = \int_0^s \rho_g\, ds$ in (63)

we get the differential equation with constant coefficients.

$$\frac{d^2 m}{dt^2} + m = \frac{-\rho'_g}{\rho_g^2} \tag{64}$$

The solution of (64) is

$$m = C_1 \cos \int_0^s \rho_g \, ds + C_2 \sin \int_0^s \rho_g \, ds + \frac{1}{D^2+1} \cdot \left(\frac{-\dot{\rho}_g}{\rho_g^2} \right) \tag{65}$$

Here C_1 and C_2 are constants and

$\rho_g' = \frac{d\rho_g}{dt}$ $D = \frac{d}{dt}$ denotes the derivative operator.

If we substitute (65), $n = 0$, $k = 0$ in the equation (18)

the surface curve $\vec{x} = \vec{x}\,(s)$ is obtained as

$$\vec{x} = \left\{ C_1 \cos \int_0^s \rho_g \, ds + C_2 \sin \int_0^s \rho_g \, ds + \frac{1}{D^2+1} \cdot \left(\frac{-\dot{\rho}_g}{\rho_g^2} \right) \right\} \vec{x}_1$$

Blaschke vectors of the ruled surface $\vec{A}_1\,(s)$ now take the form

$$A_1 = \vec{x}_1$$

$$\vec{A}_2 = \vec{g} + \varepsilon \left\{ C_1 \cos \int_0^s \rho_g ds + C_2 \sin \int_0^s \rho_g \, ds \right\} \vec{\xi}$$

$$\vec{A}_3 = \vec{g} + \varepsilon \left\{ -C_1 \cos \int_0^s \rho_g ds - C_2 \sin \int_0^s \rho_g \, ds \right\} \vec{g}$$

The invariants of the ruled surface $A_1(s)$ are

$$p = \rho_g \qquad \bar{p} = 0$$

$$q = \sqrt{\rho_g^2 + \tau_g^2}\,, \quad \bar{q} = -\frac{\tau_g \, \rho_g}{\sqrt{\rho_g^2 + \tau_g^2}} \left\{ C_1 \sin \int_0^s \rho_g \, ds - C_2 \cos \int_0^s \rho_g \, ds \right\}$$

2.5. Let the curve be a geodesic torsion ($\tau_g = 0$) The system (18) takes the following form.

$$\begin{aligned} m' - \rho_g n - \rho_g k &= 1 \\ n' + \rho_g m &= 0 \\ k' + \rho_n m &= 0 \end{aligned} \tag{66}$$

Now we solve the system of differential equation (66) for the following special cases.

2.5.1. If $m = 0$ from the system (43) we have

$$\begin{aligned} -\rho_g n - \rho_g k &= 1 \\ n' &= 0 \\ k' &= 0 \end{aligned} \tag{67}$$

from the equations (67) we obtain

$$n = C_1$$

$$k = C_2$$

In this case blaschke vectors of the ruled surface $\vec{A}_1\,(s)$ become

$$\begin{aligned} \vec{A}_1 &= \vec{x}_1 + \varepsilon\left(C_2\,\vec{g} - C_1\,\vec{\xi}\right) \\ \vec{A}_2 &= \vec{g} + \varepsilon\left(-C_2\,\vec{x}_1\right) \\ \vec{A}_3 &= \vec{\xi} + \varepsilon\left(C_1\,\vec{x}_1\right) \end{aligned} \tag{68}$$

The invariants of the ruled surface $\vec{A}_1(s)$ are

$$p=\sqrt{\rho_g^2+\rho_n^2} \qquad \bar{p}=0$$

2.5.2. If $k=0$ from the system (66) we have

$$\begin{aligned} m'-\rho_g n &= 1 \\ n'+\rho_g m &= 0 \\ \rho_n m &= 0 \end{aligned} \tag{69}$$

Using $(69)_3$ we get $m=0$. From $(69)_1$, and $(69)_2$, the following differential equation is obtained

$$m''-\frac{\rho'_g}{\rho_g}\left(m'-1\right)+m\,\rho_g^2=0 \tag{70}$$

With the aid of the parameter change $t=\int_0^s \rho_g\,ds$ in (70)

we get the differential equation with constant coefficients,

$$\frac{d^2m}{dt^2}+m=\frac{-\rho_g}{\rho_g^2} \tag{71}$$

The solution of (71) is

$$m = C_1 \cos \int_0^s \rho_g \, ds + C_2 \sin \int_0^s \rho_g \, ds + \frac{1}{D^2+1}\left(\frac{-\dot{\rho}_g}{\rho_g^2}\right) \tag{72}$$

Here C_1 and C_2 are constants and,

$\rho'_g = \frac{d\rho_g}{dt}$, $D = \frac{d}{dt}$ denotes the derivative operator.

If we substitute (72), $k = 0$, $n = 0$ in the equation (18), the surface curve $\vec{x} = \vec{x}\,(s)$ is obtained as

$$\vec{x} = \left\{ C_1 \cos \int_0^s \rho_g \, ds + C_2 \sin \int_0^s \rho_g \, ds + \frac{1}{D^2+1}\left(\frac{-\dot{\rho}_g}{\rho_g^2}\right)\right\} \vec{x}_1$$

Blaschke vectors of the ruled surface $\vec{A}_1\,(s)$ now take the form

$$\vec{A}_1 = \vec{x}_1$$

$$\vec{A}_2 = \vec{g} + \varepsilon\left(C_1 \cos \int_0^s \rho_g \, ds + C_2 \sin \int_0^s \rho_g \, ds \right)\vec{\xi}$$

$$\vec{A}_3 = \vec{\xi} + \varepsilon\left(-C_1 \cos \int_0^s \rho_g \, ds - C_2 \sin \int_0^s \rho_g \, ds \right)\vec{g}$$

The invariants of the ruled surface $\vec{A}_1\,(s)$ are

$$p = \rho_g \qquad \bar{p} = 0$$

2.5.3. If $n = 0$ from the system (66) we have

$$\begin{aligned} m' - \rho_g k &= 1 \\ \rho_g m &= 0 \\ k' + \rho_g m &= 0 \end{aligned} \qquad (73)$$

Using $(73)_2$ we get $m = 0$. From $(73)_1$ and $(73)_3$, the following differential equation is obtained

$$k'' - \frac{\rho'_g}{\rho_g} k' - \rho_g^2 k + \rho_g = 0 \qquad (74)$$

With the aid of the parameter change $t = \int_0^s \rho_g \, ds$ in (74)

we get the differential equation with constant coefficients.

$$\frac{d^2k}{dt^2} + k = \frac{-\dot{\rho}_g}{\rho_g^2} \qquad (75)$$

The solution of (75) is

$$k = C_1 \cos \int_0^s \rho_g \, ds + C_2 \sin \int_0^s \rho_g \, ds + \frac{1}{D^2 + 1} \left(\frac{-\dot{\rho}_g}{\rho_g^2} \right) \qquad (76)$$

Here C_1 ve C_2 are constants and,

$\dot{\rho}_g = \frac{d\rho_g}{dt}$, $D = \frac{d}{dt}$ donetes the derivative oparator.

If we substitute (76), $m = 0$, $n = 0$ in the equation (18) the surface curve $\vec{x} = \vec{x}\,(s)$ is obtained as

$$\vec{x} = \left\{ C_1 \cos \int_0^s \rho_g \, ds + C_2 \sin \int_0^s \rho_g \, ds + \frac{1}{D^2 + 1} \left(\frac{-\dot{\rho}_g}{\rho_g^2} \right) \right\} \vec{x}_1$$

Blaschke vektors of the ruled surface $\vec{A}_1\,(s)$ now take the form

$$\vec{A}_1 = \vec{x}_1 + \varepsilon \left[C_1 \cos \int_0^s \rho_g \, ds + C_2 \sin \int_0^s \rho_g \, ds + \frac{1}{D^2 + 1} \left(\frac{-\dot{\rho}_g}{\rho_g^2} \right) \right] g$$

$$\vec{A}_2 = \vec{g} + \varepsilon \left[-C_1 \cos \int_0^s \rho_g \, ds - C_2 \sin \int_0^s \rho_g \, ds + \frac{1}{D^2 + 1} \left(- \frac{\dot{\rho}_g}{\rho_g^2} \right) \right] \vec{x}_1$$

$$\vec{A}_3 = \vec{\xi}$$

The invariant of the ruled surface $\vec{A}_1\,(s)$ are

$$p = \rho_g \qquad \bar{p} = 0$$

REFERENCES

1. BLASCHKE, V., (1949) : "Differential Geometry."
İstanbul Üniversitesi yayınları No: 433

2. Hacısalioğlu, H., (1983) : "Differential Geometry"
İnönü Üniversitesi
Fen - Ed. Fak. yayınları No: 2

3. O' Neill, B., (1966) : "Elementary Differential Geometry"
Academic Press. Newyork

4. Guggenheimer, H.W., : "Differential Geometry"
Mc Graw - Hill Book. Company
1963 Newyork

5. Pekmen, Ü., (1994) : "Differential - Geometrical Conditions Between Curvature Strip Curve and Ruled Surfaces"
E.Ü. Journal of the faculty of science seri A