GENERALIZATION OF THE CLASSICAL POLYNOMIAL INEQUALITIES FOR SEVERAL INTERVALS CASE

by

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APPROVAL PAGE

This is to certify that I have read this thesis written by Mehmet Ali AKTÜRK and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy in Mathematics.

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ABSTRACT

In this dissertation we focus on generalization of the classical polynomial inequalities and investigation of the derivative of extremal rational functions.

In chapter 1 we give some basic information about Remez-type, Bernstein-type, Markov-type, Rusak inequalities.

In chapter 2 we obtain new Remez-type inequalities and weighted versions of Bernstein-type inequalities. Remez-type inequality for trigonometric polynomials on a part of the period which is proved in thesis is a new step in the story of the proof of an open hypothesis (Erdélyi's Conjecture).

In chapter 3 we proved Markov-type inequalities for algebraic polynomials and rational functions, as well as Rusak-type inequalities for algebraic fractions. Markov-type inequality for rational functions on several intervals is the first example of a sharp Markov-type inequality for rational functions on several intervals.

In chapter 4 some auxiliary statements are proved which are important to obtain sharp Markov-type inequalities for rational functions on several intervals.

Keywords: Remez Inequality, Bernstein Inequality, Weighted Polynomial Inequalities, Inequalities for Derivatives of Rational Functions, Markov Inequality

KLASİK POLİNOM EŞİTSİZLİKLERİNİN BİRDEN ÇOK ARALIK İÇİN GENELLEŞTİRİLMESİ

Mehmet Ali AKTÜRK

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ÖZ

Bu doktora tezinde genel olarak klasik polinom eşitsizliklerinin genelleştirilmesi ve uç değerli rasyonel fonksiyonların türevleri üzerine yoğunlaştık.

Birinci bölümde Remez-tip, Bernstein-tip, Markov-tip, Rusak eşitsizlikleri hakkında temel bazı bilgiler verdik.

İkinci bölümde yeni Remez-tip eşitsilikler ve Bernstein-tip eşitsizliklerin ağırlıklı versiyonlarını elde ettik. Tezde bölümün bir kısmında ispatlanan trigonometrik polinomlar için Remez-tip eşitsizlik, açık bir hipotezin (Erdélyi'nin varsayımı) ispatının belli bölümünde geçen yeni bir adımdır.

Üçüncü bölümde cebirsel polinomlar ve rasyonel fonksiyonlar için Markov-tip eşitsizlikler ayrıca cebirsel fonksiyonlar için Rusak-tip eşitsizlikler de ispatlandı. Birden çok aralıklarda rasyonel fonksiyonlar için Markov-tip eşitsizlikleri birden çok aralıklarda rasyonel fonksiyonlar için net Markov-tip eşitsizliğinin ilk örneğidir.

Dördüncü bölümde bazı yardımcı ifadelerle ispatı önemli olan birden çok aralıklarda rasyonel fonksiyonlar için net Markov-tip eşitsizlikleri elde edildi.

Anahtar Kelimeler: Remez Eşitsizliği, Bernstein Eşitsizliği, Ağırlıklı Polinom Eşitsizlikleri, Rasyonel Fonksiyonların Türevleri için Eşitsizlikler, Markov Eşitsizliği

To my family

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CHAPTER 1

INTRODUCTION AND BASIC NOTATION

1.1 INTRODUCTION

Inequalities for polynomials have been a classical object of studies for more than one century. Modern expositions can be found in books and surveys (Borwein and Erdélyi, 1995), (Dubinin, 2012), (Milovanović et al., 1994), and (Rahman and Schmeisser, 2002). Recently weighted analogues of classical polynomial inequalities were considered (see, for instance, (Andrievskii, 2012), (Erdélyi, 2003), and (Mastroianni and Totik, 2000)). Other ways of generalizations are in replacing the domain of polynomials by more complicated (disconnected) sets and (or) in considering polynomials in more general Chebyshev systems. Throughout the thesis, we use the notations

$$P_n^{(\mathbb{C})} = \left\{ p : p(x) = \sum_{k=0}^{n} a_k x^k, \ a_k \in \mathbb{R}(\mathbb{C}) \right\}, \tag{1.1}$$

for the set of algebraic polynomials and

$$T_n^{(\mathbb{C})} = \left\{ t : t(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \ a_k, b_k \in \mathbb{R}(\mathbb{C}) \right\}. \tag{1.2}$$

for the set of trigonometric polynomials with real (complex) coefficients; as a weight w, we consider an arbitrary continuous positive function on a suitable set, $\|\cdot\|$ is the uniform norm on this set.

Let E be the set which consists of several intervals $E = \bigcup\limits_{i=1}^{l} [a_{2i-1}, a_{2i}], \quad a_1 < a_2 < a_3 < \ldots < a_{2l}, \quad l > 1$. Let a_j be an endpoint of E, and let E_j be that part of E that lies closer to a_j than to any other endpoint, i.e.

$$E_j = \{ x \in E : |x - a_j| \leq |x - a_i| \, , \ i \neq j \} \, .$$

It is well known (see, for instance, (Totik, 2001)) that the equilibrium measure $\omega(\infty, \alpha, \mathbb{C}\backslash E), \alpha \subset E$ is given by

$$\omega(\infty, \alpha, \mathbb{C}\backslash E) = \frac{1}{\pi} \int\limits_{\alpha} \frac{\prod\limits_{j=1}^{l-1} (x - \tau_j)}{\sqrt{\prod\limits_{j=1}^{2l} |(x - a_j)|}} dx,$$

where $\tau_j \in [a_{2j}, a_{2j+1}], j = 1, \ldots, l-1$ are uniquely determine by

$$\int\limits_{a_{2k}}^{a_{2k+1}} \frac{\prod\limits_{j=1}^{l-1} (x - \tau_j)}{\pi \sqrt{\prod\limits_{j=1}^{2l} |(x - a_j)|}} dx = 0, \;\; k = 1, \ldots, l-1.$$

Theorem 1.1.1. *(Totik, 2001) With the above notations and with*

$$M_j = 2 \frac{\prod\limits_{i=1}^{l-1} (a_j - \tau_i)^2}{\prod\limits_{i \neq j} |a_j - a_i|}$$

we have for each $1 \leq j \leq 2l$

$$\|p_n'\|_{E_j} \leq (1 + o(1)) M_j n^2 \|p_n\|_E\,, \tag{1.3}$$

and this is asymptotically the best possible, for there is a sequence $\{p_n\}$ *of polynomials of corresponding degree at most* $n = 1, 2, \ldots$ *such that*

$$|p_n'(a_j)| \geq (1 + o(1)) M_j n^2 \|p_n\|_E\,.$$

Corollary 1.1.2. *For every* $\varepsilon_1 > 0$ *there exists* n_0 *depending on* ε_1*, and* E *such that*

$$\|p_n'\|_{E_j} \leq (1 + \varepsilon_1) M_j n^2 \|p_n\|_E \tag{1.4}$$

for every polynomial $p_n \in P_n, \;\; n \geq n_0$.

Let us set

$$C\left(n,A,\beta\right)=\sup_{B\subset A,\ |B|\leq\beta}\ \sup_{Q\in T_{n,A,B}}\|Q\|_{C(A)}\,, \tag{1.5}$$

where $A\subset[0,2\pi]$ is a given set, $T_{n,A,B}$ is the set of real trigonometric polynomials Q of order at most n such that

$$\|Q\|_{C(A\setminus B)}\leq 1,\quad \|f\|_{C(E)}=\sup_{t\in E}|f\left(t\right)|$$

and the first supremum in (1.5) is taken over all measurable sets $B\subset A$ of the Lebesque measure $|B|\leq\beta$. In other words, $C\left(n,A,\beta\right)$ is the best constant in the inequality $\|Q_n\|_{C(A)}\leq C\,\|Q_n\|_{C(A\setminus B)}\,,\ |B|\leq\beta,\ Q_n\in T_n$. T. Erdélyi (Erdélyi, 1993) gave the following conjecture.

Conjecture 1.1.1. (T. Erdélyi, 1993)

For every $\beta\in(0,2\pi)$ and $n\geq 1$

$$C\left(n,\left[0,2\pi\right],\beta\right)=v_{n,\pi-\beta/2}\left(\pi\right)$$

where

$$v_n\left(\theta\right)=v_{n,\alpha}\left(\theta\right)=\cos\left(n\arccos\left(\frac{\sin\left(\theta/2\right)}{\sin\left(\alpha/2\right)}\right)\right). \tag{1.6}$$

Erdélyi's conjecture is still open, inspite of many interesting results in this direction ((Andrievskii, 2002), (Ganzburg, 2001), (Ganzburg, 2012), (Erdélyi, 1992), (Erdélyi, 1993), (Erdélyi, 2010), (Nazarov, 1994) and (Nursultanov and Tikhonov, 2013)).

Polynomial (1.6) is a trigonometric polynomial of (half)order $n/2$ and it firstly appeared in V.S. Videnskii's papers in 1960 (for integer order) (Videnskii, 1960) and 1964 (for half-integer order) (Videnskii, 1964). We call it Videnskii polynomial. For the following we need also a description of trigonometric polynomial deviating least from zero on a subset of $[0,2\pi]$ which has the maximal possible number of deviation points ((Lukashov, 2004) compare also (Peherstorfer and Steinbauer, 1996)).

Lemma 1.1.3 ((Lukashov, 2004), (Peherstorfer and Steinbauer, 1996))**.** *: The following assertions are equivalent.*

1. *The trigonometric polynomial* τ_N *deviates least from zero on* $\mathcal{E} = [\theta_1, \theta_2] \cup \ldots \cup [\theta_{2l-1}, \theta_{2l}], \theta_1 < \theta_2 < \ldots < \theta_{2l}$ *with respect to the sup-norm among all trigonometric polynomials of degree* $N/2$ *with leading coefficients* $\cos\psi$ *and* $\sin\psi$*, i.e.,*

$$\max_{\theta\in\mathcal{E}} |\tau_N(\theta)| = \inf_{c_j,d_j\in\mathbb{R}} \max_{\theta\in\mathcal{E}} \Big|\cos\psi\cos\tfrac{N}{2}\theta + \sin\psi\sin\tfrac{N}{2}\theta + \sum_{j=1}^{\lfloor N/2\rfloor} c_j\cos\tfrac{N-2j}{2}\theta + d_j\sin\tfrac{N-2j}{2}\theta\Big|. \tag{1.7}$$

has the maximal possible number of extremum points on $\mathcal{E}$.

2. *For every* $j = 1, \ldots, l$*, the equilibrium measures of the arcs*

$$\Gamma_j = \left\{e^{i\theta} : \theta \in [\theta_{2j-1}, \theta_{2j}]\right\}$$

are positive rational numbers. More precisely,

$$N\omega(\infty, [\theta_{2j-1}, \theta_{2j}], \mathbb{C}\backslash\Gamma_{\mathcal{E}}) = q_{j-1}^{(N)}, q_{j-1}^{(N)} \in \mathbb{N}, j = 2, \ldots, l. \tag{1.8}$$

3. *There is a real trigonometric polynomial* $\sigma_{N-\frac{l}{2}}$ *of order* $N - \frac{l}{2}$ *such that for a constant* $A_N > 0$,

$$\tau_N^2(\theta) - S(\theta)\sigma_{N-\frac{l}{2}}^2(\theta) = A_N^2, \tag{1.9}$$

where $S(\theta)$ *is given by (2.22).*

If any of those assertions is valid, then

i. *the numbers* $q_j^{(N)}$ *are equal to the number of zeros of* $\tau_N(\theta)$ *on* $\mathcal{E}_j = [\theta_{2j-1}, \theta_{2j}]$, $j = 1, \ldots, l$.

ii. *the polynomial* τ_N *may also be written in terms of* $\varpi_{\mathcal{E}}(z, x)$ *as*

$$\tau_N(\theta) = A_N\epsilon\cos\left(2\pi\int_{\mathcal{E}\cap[\theta_1,\theta]} N\varpi_{\mathcal{E}}(\infty, \zeta)\, d\zeta\right), \theta \in \mathcal{E} \tag{1.10}$$

where $\epsilon \in \{-1, 1\}, \varpi(z, x) = \frac{\partial}{\partial x}\omega\left(z, \Gamma_E \cap \{e^{i\varphi} : b \leq \varphi \leq x\}, \mathbb{C}\backslash\Gamma_E\right)$; $\omega(z, G, \mathbb{C}\backslash\Gamma_E)$ *is the harmonic measure of a set* $G \subset \Gamma_E$ *at a point* $z \in \mathbb{C}\backslash\Gamma_E$ *relative to the domain* $\mathbb{C}\backslash\Gamma_E, \Gamma_E = \{e^{i\varphi} : \varphi \in E\}$ *and* $\omega_j(\infty) = \omega(\infty, \Gamma_j, \mathbb{C}\backslash\Gamma_E)$.

Constant $C(n, A, \beta)$ is unknown, while the corresponding constant for algebraic polynomials was found by Remez (Remes, 1936).

Theorem 1.1.4. *(E.Ya. Remez, 1936) :*

The transformed Chebyshev polynomials

$$p(x) = \pm T_n\left(\frac{\pm 2x + s}{2 - s}\right),$$

and only they are extremal for the problem

$$\sup_{p \in P_{n,s}} \|p\|_{C[-1,1]},$$

where

$$P_{n,s} = \{p \in P_n : |\{x \in [-1,1] : |p(x)| \leq 1\}| \geq 2 - s\}.$$

Next is the classical Bernstein inequality for algebraic polynomials.

Theorem 1.1.5. *The inequality*

$$|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|p\|_{[-1,1]}, \qquad -1 < x < 1$$

holds for every $p \in P_n^{(\mathbb{C})}$.

In (Borwein and Erdélyi, 1995) a weighted version of Bernstein inequality was given:

Theorem 1.1.6. *Let* $w \in C[-1,1]$ *be strictly positive on* $[-1,1]$. *For every* $\varepsilon > 0$ *there exists an* n_0 *depending on* ε *and* w *such that*

$$\left\|p'(x)\, w(x) \sqrt{1-x^2}\right\|_{[-1,1]} \leq n(1+\varepsilon) \|pw\|_{[-1,1]}$$

for every $p \in P_n,\ n \geq n_0$.

S.N. Bernstein proved in 1912 (Bernstein, 1958) his famous inequality;

Theorem 1.1.7. *The inequality*

$$\|t^m\|_K \leq n^m \|t\|_K$$

holds for every $t \in T_n$.

If t_n is a trigonometric polynomial degree of $n,\ t_n \in T_n$ then

$$\|t_n'\|_E \leq n\,\|t_n\|_E \tag{1.11}$$

where $\|\cdot\|_E$ denotes the supremum norm on $E,\ E=[0,2\pi]$. Actually, Bernstein had $2n$ instead of n, but a very simple argument based on his result gives also n in (1.11) and (1.11) was published in Russian translation of (Bernstein, 1958) in Bernstein's collected works (Bernstein, 1912) (firstly it was published by M. Riesz (Riesz, 1914)). In 1960 V.S. Videnskii (Videnskii, 1960) proved a generalization of (1.11) for an interval shorter than period (a problem which goes back to I. I. Privalov (Priwaloff, 1925). Namely, Videnskii proved that

$$|t_n'(\theta)| \leq n\left(1-\frac{\cos^2(w/2)}{\cos^2(\theta/2)}\right)^{-1/2}\|t_n\|_E$$

for $\theta \in (-w,w), E=[-w,w], w\in(0,\pi)$,and this is sharp.

In the following we use several auxiliary results.

Corollary 1.1.8. *The inequality of Theorem 1.1.7 remains true for all* $t \in T_n^{\mathbb{C}}$.

Proof. Choose $\alpha \in \mathbb{R}$ such that $e^{i\alpha}t^{(m)}(\theta)$ attains the value $\left\|t^{(m)}\right\|_K$, say, at $\theta=\tau$. Now $\widetilde{t}(\theta)=Re\left(e^{i\alpha}t^{(m)}(\theta)\right)\in T_n$ and $\left\|\widetilde{t}\right\|_K \leq \|t\|_K$. On applying Theorem 1.1.7 to $\widetilde{t}\in T_n$, we obtain

$$\left\|t^{(m)}\right\|_K = e^{i\alpha}t^{(m)}(\tau)=\widetilde{t}^{(m)}(\tau)\leq n^m\left\|\widetilde{t}\right\|_K \leq n^m\|t\|_K.$$

□

Lemma 1.1.9. *(Lukashov, 2004) The following inequality holds for any trigonometric polynomial* $t_n \in \mathrm{T}_n$ *and* $\theta \in int(\mathcal{E}), \mathcal{E}$ *is a real compact subset of* $[0,2\pi]$.

$$\left(\frac{t_n'(\theta)}{2n\pi\varpi_{\mathcal{E}}(\infty,\theta)}\right)^2 + t_n^2(\theta) \leq \|t_n\|_{\mathcal{E}}^2. \tag{1.12}$$

Here,

$$\varpi_{\mathcal{E}}(z,x)=\frac{\partial}{\partial x}\omega\left(z,\Gamma_{\mathcal{E}}\cap\left\{e^{i\theta}:\inf\mathcal{E}\leq\theta\leq x\right\},\mathbb{C}\backslash\Gamma_{\mathcal{E}}\right), \tag{1.13}$$

and $\Gamma_{\mathcal{E}}=\left\{e^{i\theta}:\theta\in\mathcal{E}\right\}$.

A.A. Markov proved in 1889 (Markov, 1889) his famous inequality: if p_n is a real algebraic polynomial of degree n, $p_n \in P_n$ then

$$\|p_n'\|_{[-1,1]} \leq n^2 \|p_n\|_{[-1,1]}, \tag{1.14}$$

where $\|\cdot\|_E$ denotes the supremum norm on E. A weighted version of (1.14) was given in Theorem 1.1.6: let $w \in C[-1,1]$ be a strictly positive on $[-1,1]$. For every $\varepsilon > 0$ there exists an n_0 depending on ε and w such that

$$\|p'w\|_{[-1,1]} \leq n^2(1+\varepsilon) \|pw\|_{[-1,1]}$$

for every $p \in P_n, n \geq n_0$.

CHAPTER 2

REMEZ-TYPE AND BERNSTEIN-TYPE INEQUALITIES

2.1 REMEZ-TYPE INEQUALITIES

The main goal of this section is to give particular assertions about constants $C(n, A, \beta)$ for the case $A = [-\alpha, \alpha]$, $\alpha \in (0, \pi/2)$. To the best of our knowledge they were not studied in the literature up to now.

Theorem 2.1.1. *(Lukashov and Akturk, 2012) If* $0 < \alpha < \frac{\pi}{2}$, $0 < \beta < 2\alpha$ *and for* $n \geq 1$

$$C^{(1)}(n, [-\alpha, \alpha], \beta) = \sup_{\substack{B \subset [-\alpha, \alpha] \\ |B| \leq \beta}} \sup_{Q_n \in T^{(1)}_{n,[-\alpha,\alpha],B}} \|Q_n\|_{C([-\alpha,\alpha])}$$

where

$$T^{(1)}_{n,[-\alpha,\alpha],B} = \left\{ \begin{array}{c} Q_n \in T_{n,[-\alpha,\alpha],B} : \\ \|Q_n\|_{C(conv\ (B))} \leq \|Q_n\|_{C([-\alpha,\alpha]\setminus conv\ (B))} \end{array} \right.$$

then

i. suitable Videnskii polynomials are extremal in the problem $C^{(1)}(n, [-\alpha, \alpha], \beta)$.

ii. extremal polynomials for the problem $C(n, [-\alpha, \alpha], \beta)$ *coincide (up to a constant multiplier) with a trigonometric polynomial deviating least from zero on a subset of* $[0, 2\pi]$ *which has the maximal possible number of deviation points.*

Proof. i) First of all we note that polynomials

$$V_{n,1} = T_{2n}\left(\frac{\sin\frac{2\theta-(\beta-2\alpha)}{4}}{\sin\frac{\beta}{4}}\right), \tag{2.1}$$

and

$$V_{n,2} = T_{2n}\left(\frac{\sin\frac{2\theta-(2\alpha-\beta)}{4}}{\sin\frac{\beta}{4}}\right) \tag{2.2}$$

belong to $T^{(1)}_{n,[-\alpha,\alpha],B}$ with $B = [-\alpha, -\alpha+\beta]$, the other on the interval $B = [\alpha-\beta, \alpha]$.

Let $Q_n \in T^{(1)}_{n,[-\alpha,\alpha],B}$ different from (2.1) and (2.2) with $B \subset [-\alpha, \alpha]$, $|B| \le \beta$.

The set B is evidently composed of a certain number $\nu \le 2n$ of closed intervals some of which can be one point.

Let

$$\sigma_1 = [a_1, b_1], \sigma_2 = [a_2, b_2], \ldots, \sigma_m = [a_m, b_m]$$

be those of them $(m \le \nu)$ of positive length arranged in increasing order. Let $\xi \in [-\alpha, \alpha]$ be a point such that $|Q_n|$ attains its maximum value on the interval $[-\alpha, \alpha]$:

$$|Q_n(\xi)| = \max_{\theta\in[-\alpha,\alpha]} |Q_n(\theta)|.$$

We will show that $|Q_n(\xi)| < C^{(1)}(n, [-\alpha, \alpha], \beta)$, we distinguish between two cases depending on:

$$\xi \in [-\alpha, a_1), and\ \xi \in (b_m, \alpha].$$

We start by first case. Let

$$\theta_1 = -\alpha < \theta_2 < \ldots < \theta_{2n+1} = -\alpha + \beta$$

be the points on the interval $[-\alpha, -\alpha+\beta]$, where the polynomial (2.1) attains, with alternating sign, the values ± 1.

Let, in addition, $\overline{\theta}_1, \overline{\theta}_2, \ldots, \overline{\theta}_{2n+1}$ be the $2n+1$ points that we take in the set B satisfying the following conditions: firstly $\overline{\theta}_1 = a_1$; then for $l = 2, 3, \ldots, 2n+1$ let $\overline{\theta}_l$ be the first of the points of B (traversing this set of points from left to right) for which

$$mes\left(\left[\overline{\theta}_l, \overline{\theta}_1\right] \cap B\right) = \theta_l - \theta_1.$$

Applying the Lagrange interpolation formula, one time with the trigonometric polynomial (2.1) and another time with the trigonometric polynomial Q_n, we can write the following two equalities:

$$V_{n,1}(\alpha) = \sum_{i=1}^{2n+1} V_{n,1}(\theta_i) \prod_{\substack{m=1\\ m\neq i}}^{2n+1} \frac{\sin\frac{1}{2}(\alpha-\theta_m)}{\sin\frac{1}{2}(\theta_i-\theta_m)}, \tag{2.3}$$

$$Q_n(\xi) = \sum_{i=1}^{2n+1} Q_n\left(\overline{\theta}_i\right) \prod_{\substack{m=1\\ m\neq i}}^{2n+1} \frac{\sin\frac{1}{2}\left(\xi - \overline{\theta}_m\right)}{\sin\frac{1}{2}\left(\overline{\theta}_i - \overline{\theta}_m\right)}. \tag{2.4}$$

On comparing their right-hand parts term by term, one notes the following relations:

firstly we note that because of definition of $\overline{\theta}_j$'s inequalities for $j = 1, \ldots, 2n+1$

$$\overline{\theta}_j \geq \theta_j \tag{2.5}$$

and

$$\xi - \overline{\theta}_j \leq \xi - \theta_j \leq \alpha - \theta_j \tag{2.6}$$

holds. Next from

$$mes\left(\left[\overline{\theta}_i, \overline{\theta}_j\right] \cap B\right) = \theta_j - \theta_i$$

we get,

$$|\theta_i - \theta_j| \leq \left|\overline{\theta}_i - \overline{\theta}_j\right|.$$

Now taking into account $l = \alpha < \frac{\pi}{2}$ we deduce (2.5),(2.6) at two inequalities

$$\begin{aligned} \sin\frac{1}{2}\left(\xi - \theta_j\right) &\geq \sin\frac{1}{2}\left(\xi - \overline{\theta}_j\right), \quad j = 1, \ldots, 2n+1 \\ \left|\sin\frac{1}{2}\left(\overline{\theta}_i - \overline{\theta}_m\right)\right| &\geq \left|\sin\frac{1}{2}\left(\theta_i - \theta_m\right)\right|, \quad i \neq m, \text{ and } i, m = 1, \ldots, 2n+1. \end{aligned}$$

On comparing right-hand side of (2.3) and (2.4) term by term we note also the relation

$$\begin{aligned} sign\ V_{n,1}(\theta_i) &= (-1)^{i+1}, \\ sign \prod_{\substack{m=1\\ m\neq i}}^{2n+1} \sin\frac{1}{2}\left(\theta_i - \theta_m\right) &= (-1)^{2n+1-i} = (-1)^{i+1}, \\ sign \prod_{\substack{m=1\\ m\neq i}}^{2n+1} \sin\frac{1}{2}\left(\alpha - \theta_m\right) &= +1, \end{aligned}$$

and

$$|V_{n,1}(\theta_i)| = 1, \quad \left|Q_n\left(\overline{\theta}_i\right)\right| \leq 1.$$

Moreover, one also sees that the $2n+1$ terms on the last part of (2.3) are all the same sign (being +), which need not hold in (2.4). Thus one also has

$$|Q_n(\xi)| < V_{n,1}(\alpha)$$

at least that Q_n is not identical to $\pm V_{n,1}$.

When $\xi \in [b_m, \alpha)$, the reasoning is totally analogous, on replacing the (2.1) by (2.2).

ii) Proof follows from the following lemmas (Similar assertions for $\alpha = \pi$ were given in (Erdélyi, 1992)). □

Lemma 2.1.2. *(Lukashov and Akturk, 2012) P^* has simple real zeros only.*

Proof. Suppose there is a complex zero $z_j = x_j + iy_j$. Then consider the polynomial

$$P_n(t) = (1+h)\, P^*(t) \left(1 - \varepsilon \frac{\sin^2\left(\frac{t-t_0}{2}\right)}{\sin\left(\frac{t-z_j}{2}\right)\sin\left(\frac{t-\overline{z}_j}{2}\right)}\right),$$

where t_0 is such that $|P_n^*(t_0)| = \|P_n^*\|_{C[-\alpha,\alpha]}$. Now $|P_n(t_0)| > |P_n^*(t_0)|$ for sufficiently small $h > 0, \varepsilon > 0$, and $\{t : |P_n(t)| \le 1\} \supset \{t : |P_n^*(t)| \le 1\}$ for sufficiently small $h > 0$, what contradicts the extremality of P_n^*.

Next let us suppose that $P_n^*(t)$ has a multiple zero t_j. Then the polynomial

$$P_n(t) = (1+h)\, P^*(t) \left(1 - \varepsilon \frac{\sin^2 \frac{t-t_0}{2}}{\sin^2 \frac{t-t_j}{2}}\right)$$

leads to the contradiction as well. □

Lemma 2.1.3. *(Lukashov and Akturk, 2012) $P^*(t) - \gamma$ has only real zeros for all $\gamma,\ -1 \le \gamma \le 1$.*

Proof. For $\gamma = 0$ the assertion follows from Lemma 2.1.2. Suppose there are two complex zeros, $z_1 = t_1 + iy_1$ and $\overline{z}_1 = t_1 - iy_1$. Then among preimages $P^{*^{-1}}([0,\gamma])$ there is a curve which connects $z_1, \overline{z}_1$ and intersects $\mathbb{R}$ at a point $\widetilde{t}$. So P^* can not be locally one-to-one in a neighbourhood of $\widetilde{t}$ and hence $P^{*\prime}\left(\widetilde{t}\right) = 0$ what contradicts Lemma 2.1.2. □

2.2 WEIGHTED BERNSTEIN-TYPE INEQUALITIES

We start with properties of the related equilibrium measure.

Lemma 2.2.1. *(Peherstorfer and Steinbauer, 2000) : The density of the equilibrium measure from (1.13),* $\mathcal{E} = [\theta_1, \theta_2] \cup \ldots \cup [\theta_{2l-1}, \theta_{2l}]$ *,* $\theta_1 < \theta_2 < \ldots < \theta_{2l}$ *is given by*

$$\varpi_{\mathcal{E}}(\infty, \theta) = \frac{1}{2\pi} \frac{|Q(\theta)|}{\sqrt{|S(\theta)|}}, \tag{2.7}$$

where $Q(\theta) = \prod_{j=1}^{l} \sin\left(\frac{\theta - \xi_j}{2}\right)$*, and* $\xi_j \in [\theta_{2j}, \theta_{2j+1}]$ *,* $j = 1, \ldots, l$ *are uniquely determined by*

$$\int_{\theta_{2j}}^{\theta_{2j+1}} \frac{Q(\theta)}{\sqrt{|S(\theta)|}} d\zeta = 0, j = 1, \ldots, l. \tag{2.8}$$

Proof. First proof F.Peherstorfer and R.Steinbauer proved that

$$\varpi_{\mathcal{E}}(\infty, \theta) = \frac{1}{2\pi} \frac{S_l\left(e^{i\theta}\right)}{\sqrt{R^0(e^{i\theta})}},$$

where

$$S_l(z) = \sigma \prod_{j=1}^{l} \left(z - e^{i\xi_j}\right).$$

If we put $z = e^{i\varphi}$,

$$\begin{aligned} S_l(e^{i\varphi}) &= \sigma \prod_{j=1}^{l} \left(e^{i\varphi} - e^{i\xi_j}\right) \\ &= \sigma (2i)^l e^{i\frac{l\varphi}{2}} \prod_{j=1}^{l} e^{i\frac{\xi_j}{2}} \prod_{j=1}^{l} \sin\frac{\varphi - \xi_j}{2}. \end{aligned}$$

We know that

$$\begin{aligned} \sin\frac{\varphi - \xi_j}{2} &= \frac{e^{i\frac{\varphi - \xi_j}{2}} - e^{-i\frac{\varphi - \xi_j}{2}}}{2i} \\ &= \frac{e^{i\frac{\varphi}{2}} - e^{i\frac{\xi_j}{2}}}{2ie^{i\frac{\varphi}{2}} e^{i\frac{\xi_j}{2}}}, \end{aligned}$$

and we have

$$R^0(z) = \rho \prod_{j=1}^{2l} \left(z - e^{i\theta_j}\right).$$

If we rearrange same way

$$\begin{aligned} R^0\left(e^{i\varphi}\right) &= \rho \prod_{j=1}^{2l} \left(e^{i\varphi} - e^{i\theta_j}\right) \\ &= \rho (2i)^{2l} e^{i\varphi l} \prod_{j=1}^{2l} \left(e^{i\theta_j}\right) \prod_{j=1}^{2l} \sin\frac{\varphi - \theta_j}{2}. \end{aligned}$$

Then there exists a uniquely determined polynomial $S_l = S_l^*$ of degree l, normalized by $iS_l(0) = \sqrt{R^0(0)}$, which satisfies

$$\int_{\theta_{2j}}^{\theta_{2j+1}} \frac{S_l\left(e^{i\varphi}\right)}{\sqrt{R^0\left(e^{i\varphi}\right)}} d\varphi = 0, j = 1, \ldots, l-1.$$

Lemma 2.2.2. *According to given above*

$$\frac{S_l\left(e^{i\varphi}\right)}{\sqrt{R^0\left(e^{i\varphi}\right)}} = c\frac{\prod_{j=1}^{l} \sin\left(\frac{\varphi-\xi_j}{2}\right)}{\sqrt{\prod_{j=1}^{2l} \sin\frac{\varphi-\theta_j}{2}}}.$$

Proof. We know that

$$R^0(z) = \rho\prod_{j=1}^{2l}\left(z - e^{i\theta_j}\right), \qquad \rho = (-1)^l \exp\left\{-\frac{i}{2}\sum_{j=1}^{2l}\theta_j\right\},$$

and

$$S_l(z) = \sigma\prod_{j=1}^{l}\left(z - e^{i\xi_j}\right).$$

If we use property, we get

$$S_l = S_l^* \Longrightarrow \overline{\sigma} = \sigma\prod_{j=1}^{l} e^{i\xi_j}(-1)^l.$$

Let us rearrange all determined

$$iS_l(0) = i\sigma(-1)^l\prod_{j=1}^{l} e^{i\xi_j} = \sqrt{\rho\prod_{j=1}^{2l} e^{i\theta_j}},$$

and

$$-\sigma^2\prod_{j=1}^{l} e^{2i\xi_j} = \rho\prod_{j=1}^{2l} e^{i\theta_j}.$$

If we substitute to given

$$\sigma^2 = (-1)^{l+1}\prod_{j=1}^{2l} e^{\frac{i\theta_j}{2}}\prod_{j=1}^{l} e^{-2i\xi_j},$$

$$\sigma = \pm i^{l+1}\prod_{j=1}^{2l} e^{\frac{i\theta_j}{4}}\prod_{j=1}^{l} e^{-i\xi_j},$$

$$\overline{\sigma} = \pm(-1)^{l+1} i^{l+1}\prod_{j=1}^{2l} e^{\frac{-i\theta_j}{4}}\prod_{j=1}^{l} e^{i\xi_j}.$$

Now we compare to the equations

$$(-1)^{l+1} i^{l+1} \prod_{j=1}^{2l} e^{\frac{-i\theta_j}{4}} \prod_{j=1}^{l} e^{i\xi_j} = i^{l+1}(-1)^l \prod_{j=1}^{2l} e^{\frac{i\theta_j}{4}} \prod_{j=1}^{l} e^{i\xi_j} \prod_{j=1}^{l} e^{-i\xi_j},$$

and

$$(-1) = \prod_{j=1}^{2l} e^{\frac{i\theta_j}{2}} \prod_{j=1}^{l} e^{-i\xi_j},$$

with them

$$\frac{S_l\left(e^{i\varphi}\right)}{\sqrt{R^0\left(e^{i\varphi}\right)}} = \frac{\sigma 2^l i^l e^{i\frac{\varphi l}{2}} \prod_{j=1}^{l} e^{-i\xi_j} \prod_{j=1}^{l} \sin\left(\frac{\varphi-\xi_j}{2}\right)}{\sqrt{\rho}(2i)^l e^{i\frac{\varphi l}{2}} \sqrt{\rho \prod_{j=1}^{2l} e^{i\frac{\theta_j}{2}}} \sqrt{\prod_{j=1}^{2l} \sin\frac{\varphi-\theta_j}{2}}}.$$

So we have

$$\frac{S_l\left(e^{i\varphi}\right)}{\sqrt{R^0\left(e^{i\varphi}\right)}} = c\frac{\prod_{j=1}^{l} \sin\left(\frac{\varphi-\xi_j}{2}\right)}{\sqrt{\prod_{j=1}^{2l} \sin\frac{\varphi-\theta_j}{2}}}$$

thus proof of Lemma 2.2.2 is completed. □

We want to present here a different partial proof of the lemma which uses the representations of extremal polynomials in (1.10).

1. Suppose firstly $\omega\left(\infty, \Gamma_j, \mathbb{C}\backslash\Gamma_{\mathcal{E}}\right) = \frac{p_j}{2N}, p_j \in \mathbb{N}, j = 1, \ldots, l$. Then by Lemma 2.2.1 the function

$$\tau_N(\theta) = \cos\left(\pi \int_{\mathcal{E}\cap[\theta_1,\theta]} 2N\varpi_{\mathcal{E}}(\infty,\zeta)\, d\zeta\right), \theta \in \mathcal{E} \tag{2.9}$$

is a real trigonometric polynomial of order N. If we take a derivative, we get

$$\tau_N'(\theta) = NQ(\theta) \prod_{j=1}^{2N-l} \sin\frac{\theta-\beta_j}{2}, \tag{2.10}$$

where $\beta_j,\ j = 1, \ldots, 2N-l$, are zeros of $\sigma_{N-\frac{l}{2}}(\theta)$ and there is a real trigonometric polynomial $\sigma_{N-\frac{l}{2}}$ of order $N - \frac{l}{2}$ such that

$$\tau_N^2(\theta) - S(\theta)\, \sigma_{N-\frac{l}{2}}^2(\theta) = 1. \tag{2.11}$$

Hence

$$\sigma_{N-\frac{l}{2}}(\theta)=c\prod_{j=1}^{2N-l}\sin\frac{\theta-\beta_j}{2}=\frac{\sin\left(\pi\int\limits_{\mathcal{E}\cap[\theta_1,\theta]}2N\varpi_{\mathcal{E}}(\infty,\zeta)\,d\zeta\right)}{\sqrt{|S(\theta)|}}. \tag{2.12}$$

Moreover, $\tau_N(\theta)$ has a maximal number of deviation points, and inner zeros of its derivative coincide with zeros of $\sigma_{N-\frac{l}{2}}(\theta)$, and τ_N has one zero ξ_j at each gap $(\theta_{2j},\theta_{2j+1})$, $j=1,\ldots,l$. Hence

$$\begin{aligned}\tau_N'(\theta) &= \pm N\frac{\sin\left(\pi\int\limits_{\mathcal{E}\cap[\theta_1,\theta]}2N\varpi_{\mathcal{E}}(\infty,\zeta)\,d\zeta\right)}{\sqrt{|S(\theta)|}}Q(\theta) \qquad (2.13)\\ &= -\sin\left(\pi\int\limits_{\mathcal{E}\cap[\theta_1,\theta]}2N\varpi_{\mathcal{E}}(\infty,\zeta)\,d\zeta\right)\pi 2N\varpi_{\mathcal{E}}(\infty,\theta),\end{aligned}$$

so we have

$$\varpi_{\mathcal{E}}(\infty,\theta)=\frac{1}{2\pi}\frac{|Q(\theta)|}{\sqrt{|S(\theta)|}}. \tag{2.14}$$

Now equality (2.8) follows from the representation (1.10). Uniqueness of ξ_j's follows from the uniqueness of extremal trigonometric polynomials in Lemma 2.2.1.

2. Using density of the systems of l arcs satisfying $\omega(\infty,\Gamma_j,\mathbb{C}\backslash\Gamma_{\mathcal{E}})\in\mathbb{Q}, j=1,\ldots,l$, among all systems of l arcs (see,for instance, (Khrushchev, 2009), (Totik, 2011)and references therein), we obtain the lemma.

□

Remark 2.2.1. Lemma 2.2.1 was recently proved by V. Totik (manuscript, for symmetric with respect to 0 intervals see (Totik, 2012)).

The main goal of this section is to give simple proofs of weighted analogues of Bernstein-type inequalities on several intervals. We start with the Bernstein inequality.

Theorem 2.2.3. *(Akturk and Lukashov, 2013) Let E be a set consisting of a finite number $l\geq 2$ of disjoint intervals, $E=\bigcup_{j=1}^{l}[a_j,b_j]\subset[0,1]$, $a_1<b_1<a_2<\ldots<b_l$,*

then there exists n_0 depending on w and E such that

$$\left| p_n'(x)w(x)\sqrt{\prod_{j=1}^{l} \left|(x-a_j)(x-b_j)\right|}\right| \le n\left\|p_n w\right\|_E, \ x \in E \tag{2.15}$$

for every polynomial $p_n \in P_n^{\mathbb{C}}, n \ge n_0$.

Proof. By the Weierstrass approximation theorem, for any $\eta > 0$ there is a $q_k \in P_k, k \ge l$ such that

$$w(x) \le \frac{q_k(x)}{\prod_{j=1}^{l-1} |x-\tau_j|} \le (1+\eta)\, w(x),\ x \in E. \tag{2.16}$$

Hence,

$$\begin{aligned}
\left| p_n'(x)w(x)\prod_{j=1}^{l} \left|(x-a_j)(x-b_j)\right|^{1/2}\right| &\le \left| p_n'(x)q_k(x)\frac{\prod_{j=1}^{l} \left|(x-a_j)(x-b_j)\right|^{1/2}}{\prod_{j=1}^{l-1} |x-\tau_j|}\right| \\
&\le \left| (p_n q_k)'(x)\frac{\prod_{j=1}^{l} \left|(x-a_j)(x-b_j)\right|^{1/2}}{\prod_{j=1}^{l-1} |x-\tau_j|}\right| \\
&\quad + \left| p_n(x)q_k'(x)\frac{\prod_{j=1}^{l} \left|(x-a_j)(x-b_j)\right|^{1/2}}{\prod_{j=1}^{l-1} |x-\tau_j|}\right|.
\end{aligned}$$

Denote $m := \min\{w(x) : x \in E\}$ then we obtain taking into account inequality

(2.16) and using Theorem 1.1.1

$$\begin{aligned}
\left|p_n'(x)w(x)\prod_{j=1}^{l}\left|(x-a_j)(x-b_j)\right|^{1/2}\right| &\leq (n+k)\left\|p_nq_k\right\|_E+\left\|p_n\right\|_E k\left\|q_k\right\|_E \\
&\leq (n+k)(1+\eta)\left\|p_nw\right\|_E\left\|\prod_{j=1}^{l-1}|x-\tau_j|\right\|_E + \\
&\quad +\frac{k}{m}(1+\eta)\left\|p_nw\right\|_E\left\|w\right\|_E\left\|\prod_{j=1}^{l-1}|x-\tau_j|\right\|_E \\
&\leq n(1+\eta)\left\|p_nw\right\|_E\left\|\prod_{j=1}^{l-1}|x-\tau_j|\right\|_E + \\
&\quad +k(1+\eta)\left\|p_nw\right\|_E\left\|\prod_{j=1}^{l-1}|x-\tau_j|\right\|_E + \\
&\quad +\frac{k}{m}(1+\eta)\left\|p_nw\right\|_E\left\|w\right\|_E\left\|\prod_{j=1}^{l-1}|x-\tau_j|\right\|_E \\
&\leq n\left\|p_nw\right\|_E\left[1+\eta+\frac{(1+\eta)}{n}\left(k+\frac{k}{m}\left\|w\right\|_E\right)\right] \\
&\quad \times\left\|\prod_{j=1}^{l-1}|x-\tau_j|\right\|_E
\end{aligned}$$

for every $p_n \in P_n$, provided $\eta > 0$ and $\varepsilon > 0$ are sufficiently small, $n \geq n_0$ such that $\varepsilon \geq \eta + \frac{(1+\eta)}{n}\left(k+\frac{k}{m}\left\|w\right\|_E\right)$, we get

$$\begin{aligned}
\left|p_n'(x)w(x)\prod_{j=1}^{l}\left|(x-a_j)(x-b_j)\right|^{1/2}\right| &\leq n(1+\varepsilon)\left\|\prod_{j=1}^{l-1}|x-\tau_j|\right\|_E\left\|p_nw\right\|_E \\
&\leq n\left\|p_nw\right\|_E . \qquad (2.17)
\end{aligned}$$

Theorem 2.2.4. *Let E be a set consisting of a finite number $l \geq 2$ of disjoint intervals, $E = \bigcup_{j=1}^{l}[a_j, b_j], a_1 < b_1 < a_2 < \ldots < b_l$, then there exists n_0 depending on w and E such that*

$$\left|p_n'(x)w(x)\sqrt{\prod_{j=1}^{l}\left|(x-a_j)(x-b_j)\right|}\right| \leq n(1+\varepsilon)\left\|\prod_{j=1}^{l-1}|x-\tau_j|\right\|_E\left\|p_nw\right\|_E, \; x \in E \qquad (2.18)$$

for every polynomial $p_n \in P_n^{\mathbb{C}}, n \geq n_0$.

Proof. By the Weierstrass approximation theorem, for any $\eta > 0$ there is a $q_k \in P_k, k \geq l$ such that

$$w(x) \leq \frac{q_k(x)}{\prod_{j=1}^{l-1} |x-\tau_j|} \leq (1+\eta)\, w(x), \;\; x \in E. \tag{2.19}$$

Hence,

$$\begin{aligned}
\left| p_n'(x) w(x) \prod_{j=1}^{l} \left| (x-a_j)(x-b_j) \right|^{1/2} \right| &\leq \left| p_n'(x) q_k(x) \frac{\prod_{j=1}^{l} \left| (x-a_j)(x-b_j) \right|^{1/2}}{\prod_{j=1}^{l-1} |x-\tau_j|} \right| \\
&\leq \left| (p_n q_k)'(x) \frac{\prod_{j=1}^{l} \left| (x-a_j)(x-b_j) \right|^{1/2}}{\prod_{j=1}^{l-1} |x-\tau_j|} \right| \\
&\quad + \left| p_n(x) q_k'(x) \frac{\prod_{j=1}^{l} \left| (x-a_j)(x-b_j) \right|^{1/2}}{\prod_{j=1}^{l-1} |x-\tau_j|} \right|.
\end{aligned}$$

Denote $m := \min\{w(x) : x \in E\}$ then we obtain taking into account inequality (2.19) and using Theorem 1.1.1, we get

$$\begin{aligned}
\left| p_n'(x) w(x) \prod_{j=1}^{l} \left| (x-a_j)(x-b_j) \right|^{1/2} \right| &\leq (n+k) \left\| p_n q_k \right\|_E + \left\| p_n \right\|_E k \left\| q_k \right\|_E \\
&\leq (n+k)(1+\eta) \left\| p_n w \right\|_E \left\| \prod_{j=1}^{l-1} |x-\tau_j| \right\|_E + \\
&\quad + \frac{k}{m} (1+\eta) \left\| p_n w \right\|_E \left\| w \right\|_E \left\| \prod_{j=1}^{l-1} |x-\tau_j| \right\|_E \\
&\leq n(1+\eta) \left\| p_n w \right\|_E \left\| \prod_{j=1}^{l-1} |x-\tau_j| \right\|_E + \\
&\quad + k(1+\eta) \left\| p_n w \right\|_E \left\| \prod_{j=1}^{l-1} |x-\tau_j| \right\|_E + \\
&\quad + \frac{k}{m} (1+\eta) \left\| p_n w \right\|_E \left\| w \right\|_E \left\| \prod_{j=1}^{l-1} |x-\tau_j| \right\|_E \\
&\leq n \left\| p_n w \right\|_E \left[1 + \eta + \frac{(1+\eta)}{n} \left(k + \frac{k}{m} \left\| w \right\|_E \right) \right] \\
&\quad \times \left\| \prod_{j=1}^{l-1} |x-\tau_j| \right\|_E
\end{aligned}$$

for every $p_n \in P_n$. Now choose $\eta > 0$ and n_0 such that for all $n \geq n_0$, $\varepsilon \geq \eta + \frac{(1+\eta)}{n}\left(k + \frac{k}{m}\|w\|_E\right)$, we get

$$\left|p_n'(x)w(x)\prod_{j=1}^{l}\left|(x-a_j)(x-b_j)\right|^{1/2}\right| \leq n(1+\varepsilon)\left\|\prod_{j=1}^{l-1}|x-\tau_j|\right\|_E \|p_n w\|_E. \quad (2.20)$$

Next result is a weighted version of the Bernstein-type inequality for trigonometric polynomials on several intervals.

Theorem 2.2.5. *(Akturk and Lukashov, 2013) Let w be any function which is continuous and positive on*

$$\mathcal{E} = \bigcup_{j=1}^{l}[\theta_{2j-1}, \theta_{2j}] \subsetneq [0, 2\pi], \theta_1 < \theta_2 < \ldots < \theta_{2l} < \theta_{2l+1} = \theta_1 + 2\pi, \quad (2.21)$$

and

$$S(\theta) = \prod_{j=1}^{2l}\sin\left(\frac{\theta-\theta_j}{2}\right). \quad (2.22)$$

Then there exists n_0 depending on w and $\mathcal{E}$ such that

$$\left|t_n'(\theta)w(\theta)\sqrt{|S(\theta)|}\right| \leq n\|t_n w\|_{\mathcal{E}}, \theta \in \mathcal{E} \quad (2.23)$$

for every polynomial $t_n \in \mathrm{T}_n^{\mathbb{C}}, n \geq n_0$. Inequality (2.23) is sharp in the sense that it is not possible to replace n in (2.23) by $n(1-\varepsilon)$ for arbitrary $\varepsilon > 0$.

Proof. First consider $t_n \in T_n$. By the Weierstrass approximation theorem, for any $\eta > 0$, there is $q_k \in T_k$ such that

$$w(\theta) \leq \frac{q_k(\theta)}{\prod_{j=1}^{l}\left|\sin\left(\frac{\theta-\xi_j}{2}\right)\right|} \leq (1+\eta)w(\theta),\ \theta \in \mathcal{E}, \quad (2.24)$$

where ξ_j are given by (2.8) in Lemma 2.2.1. Hence

$$\begin{aligned}\left|t_n'(\theta)w(\theta)|S(\theta)|^{1/2}\right| &\leq \left|t_n'(\theta)q_k(\theta)\frac{|S(\theta)|^{1/2}}{\prod_{j=1}^{l}\left|\sin\left(\frac{\theta-\xi_j}{2}\right)\right|}\right| \\ &\leq \left|(t_nq_k)'(\theta)\frac{|S(\theta)|^{1/2}}{\prod_{j=1}^{l}\left|\sin\left(\frac{\theta-\xi_j}{2}\right)\right|}\right| + \\ &+ \left|t_n(\theta)q_k'(\theta)\frac{|S(\theta)|^{1/2}}{\prod_{j=1}^{l}\left|\sin\left(\frac{\theta-\xi_j}{2}\right)\right|}\right|, \end{aligned} \quad (2.25)$$

and, using Lemmas 1.1.9 and 2.2.1, we have

$$\begin{aligned}
\left|t_n'(\theta)w(\theta)\left|S(\theta)\right|^{1/2}\right| &\leq (n+k)\left\|t_n q_k\right\|_{\mathcal{E}}+\left\|t_n\right\|_{\mathcal{E}} k\left\|q_k\right\|_{\mathcal{E}} \\
&\leq (n+k)(1+\eta)\left\|t_n w\right\|_{\mathcal{E}}\left\|\prod_{j=1}^{l}\sin\left(\frac{\theta-\xi_j}{2}\right)\right\|_{\mathcal{E}}+ \\
&+ \frac{1}{m}(1+\eta)\left\|t_n w\right\|_{\mathcal{E}}\left\|w\right\|_{\mathcal{E}}\left\|\prod_{j=1}^{l}\sin\left(\frac{\theta-\xi_j}{2}\right)\right\|_{\mathcal{E}} \\
&\leq n\left\|t_n w\right\|_{\mathcal{E}}\left[1+\eta+\frac{k}{n}(1+\eta)+\frac{1}{mn}(1+\eta)\left\|w\right\|_{\mathcal{E}}\right]\times \\
&\times \left\|\prod_{j=1}^{l}\sin\left(\frac{\theta-\xi_j}{2}\right)\right\|_{\mathcal{E}}, \qquad (2.26)
\end{aligned}$$

where $m:=\min\{w(\theta):\theta\in\mathcal{E}\}$. Now, for every $t_n\in T_n$ and $\varepsilon>0$, provided $\eta>0$ is sufficiently small, $n\geq n_0$ such that $\varepsilon\geq\eta+\frac{(1+\eta)}{n}\left(k+\frac{1}{m}\left\|w\right\|_{\mathcal{E}}\right)$, we get

$$\left|t_n'(\theta)w(\theta)\left|S(\theta)\right|^{1/2}\right|\leq n(1+\varepsilon)\left\|t_n w\right\|_{\mathcal{E}}\left\|\prod_{j=1}^{l}\sin\left(\frac{\theta-\xi_j}{2}\right)\right\|_{\mathcal{E}}, \qquad (2.27)$$

and because of $\left\|\prod_{j=1}^{l}\sin\left(\frac{\theta-\xi_j}{2}\right)\right\|_{\mathcal{E}}<1$, we obtain, for sufficiently small $\varepsilon>0$,

$$\left|t_n'(\theta)w(\theta)\sqrt{\left|S(\theta)\right|}\right|\leq n\left\|t_n w\right\|_{\mathcal{E}}. \qquad (2.28)$$

The case of $t_n\in T_n^{\mathbb{C}}$ is proved then similarly to the proof of Corollary 1.1.8. The theorem is sharp even for the case $w\equiv 1$. Namely, we cannot replace the multiplier n by $n(1-\varepsilon)$ with any $\varepsilon>0$ in the right hand side of (2.23).

Take $\mathcal{E}=[-\alpha,\alpha], 0<\alpha<\pi$. Then we have $S(\theta)=\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta+\alpha}{2}\right), \xi_1=\pi$ and

$$\left|\sin\left(\frac{\theta-\xi_j}{2}\right)\right|=\left|\cos\left(\frac{\theta}{2}\right)\right|. \qquad (2.29)$$

Consider

$$t_n(\theta)=\cos\left(2n\arccos\left(\frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}\right)\right). \qquad (2.30)$$

Take $\theta=\theta_n=2\arcsin\left(\sin\frac{\alpha}{2}\sin\frac{\pi}{4n}\right)$, then

$$\left|t_n'(\theta_n)\right|=n\frac{\cos\frac{\theta_n}{2}}{\sqrt{\left|S(\theta_n)\right|}}, \qquad (2.31)$$

and

$$\left|t_n'(\theta_n)\sqrt{|S(\theta_n)|}\right| = n\cos\frac{\theta_n}{2} > n(1-\varepsilon), \tag{2.32}$$

for sufficiently large n such that

$$\sin^2\frac{\pi}{4n} < \frac{\varepsilon}{\sin^2\frac{\alpha}{2}}. \tag{2.33}$$

□

Now let $0<\alpha<\pi$, and let

$$K_\alpha = \left\{ e^{i\theta} \middle|\, \theta \in [-\alpha,\alpha] \right\} \tag{2.34}$$

be the circular arc on the unit circle of central angle 2α and with a midpoint at 1.

Theorem 2.2.6. *(Nagy and Totik, 2012) If P_n is a polynomial of degree at most n, then*

$$\left|P_n'\left(e^{i\theta}\right)\right| \le \frac{n}{2}\left(1+\frac{\sqrt{2}\cos\left(\theta/2\right)}{\sqrt{\cos\theta-\cos\alpha}}\right)\|P_n\|_{K_\alpha}, \quad \theta\in(-\alpha,\alpha).$$

Next result is a weighted version of the inequality from Theorem 2.2.6.

Theorem 2.2.7. *(Akturk and Lukashov, 2013) With the above notations and for any continuous positive function w, there exists n_0 depending on w and α such that*

$$\left|p_n'\left(e^{i\theta}\right)w\left(e^{i\theta}\right)\right|\sqrt{\left|\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta+\alpha}{2}\right)\right|} \le n\,\|p_n w\|_{K_\alpha}, \theta\in[-\alpha,\alpha] \tag{2.35}$$

for every polynomial $p_n \in P_n^{\mathbb{C}}, n \ge n_0$.

Proof. By the Weirstrass approximation theorem, for any $\eta>0$ there is a $q_k\in P_k$ such that

$$w\left(e^{i\theta}\right) \le \frac{q_k\left(e^{i\theta}\right)}{\left|\sqrt{\cos\theta-\cos\alpha}+\sqrt{2}\cos\frac{\theta}{2}\right|} \le (1+\eta)\,w\left(e^{i\theta}\right), \quad \theta\in K_\alpha. \tag{2.36}$$

Hence,

$$\begin{aligned}
\left|p_n'\left(e^{i\theta}\right)w\left(e^{i\theta}\right)\sqrt{\cos\theta-\cos\alpha}\right| &\le \left|p_n'\left(e^{i\theta}\right)q_k\left(e^{i\theta}\right)\frac{\sqrt{\cos\theta-\cos\alpha}}{\sqrt{\cos\theta-\cos\alpha}+\sqrt{2}\cos\left(\frac{\theta}{2}\right)}\right| \\
&\le \left|\left(p_nq_k\right)'\left(e^{i\theta}\right)\frac{\sqrt{\cos\theta-\cos\alpha}}{\sqrt{\cos\theta-\cos\alpha}+\sqrt{2}\cos\left(\frac{\theta}{2}\right)}\right| + \\
&\quad + \left|p_n\left(e^{i\theta}\right)q_k'\left(e^{i\theta}\right)\frac{\sqrt{\cos\theta-\cos\alpha}}{\sqrt{\cos\theta-\cos\alpha}+\sqrt{2}\cos\left(\frac{\theta}{2}\right)}\right|.
\end{aligned}$$

Denote $m := \left\{w\left(e^{i\theta}\right) : \theta \in K_\alpha\right\}$ then we obtain taking into account inequality (Nagy and Totik, 2012) and (2.36) the estimate

$$\begin{aligned}
\left|p_n'\left(e^{i\theta}\right) w\left(e^{i\theta}\right) |\cos\theta - \cos\alpha|^{\frac{1}{2}}\right| &\leq \left[\frac{(n+k)}{2}\left\|p_n q_k\right\|_{K_\alpha} + \frac{k}{2}\left\|p_n\right\|_{K_\alpha}\left\|q_k\right\|_{K_\alpha}\right] \\
&\leq \frac{(n+k)}{2}(1+\eta)\left\|p_n w\right\|_{K_\alpha} + \\
&\quad + \frac{1}{m}(1+\eta)\left\|p_n w\right\|_{K_\alpha} \times \\
&\quad \times \|w\|_{K_\alpha}\left\|\sqrt{\cos\theta - \cos\alpha} + \sqrt{2}\cos\frac{\theta}{2}\right\|_{K_\alpha} \\
&\leq \frac{n}{2}\left\|p_n w\right\|_{K_\alpha}\left[1 + \eta + \frac{ik}{n}(1+\eta) + \frac{2}{mn}(1+\eta)\|w\|_{K_\alpha}\right] \times \\
&\quad \times \left\|\sqrt{\cos\theta - \cos\alpha} + \sqrt{2}\cos\frac{\theta}{2}\right\|_{K_\alpha}
\end{aligned}$$

for every $p_n \in P_n$, provided $\eta > 0$ is sufficiently small, $n \geq n_0$ such that $\varepsilon \geq \eta + (1+\eta)\left(\frac{ik}{n} + \frac{2}{mn}\|w\|_{K_\alpha}\right)$,

$$\left|p_n'\left(e^{i\theta}\right) w\left(e^{i\theta}\right) |\cos\theta - \cos\alpha|^{\frac{1}{2}}\right| \leq \frac{n}{2}(1+\varepsilon)\left\|p_n w\right\|_{K_\alpha}\left\|\sqrt{2}\left(1 + \sin\frac{\alpha}{2}\right)\right\|_{K_\alpha}.$$

□

CHAPTER 3

MARKOV-TYPE INEQUALITIES

3.1 MARKOV-TYPE INEQUALITIES FOR ALGEBRAIC POLYNOMIALS

The main goal of this part is to give a weighted version of Totik's generalization in Theorem 1.1.1 of (1.14). Let E be the set which consists of several intervals $E = \bigcup_{i=1}^{l} [a_{2i-1}, a_{2i}], \quad a_1 < a_2 < a_3 < \ldots < a_{2l}, \quad l > 1$. Let a_j, M_j be defined as on page 2.

Theorem 3.1.1. *(Akturk and Lukashov, 2012) With the above notations for every $\varepsilon > 0, w \in C(E), w(x) > 0$, for all $x \in E$, there exists an n_0 depending on ε, w and E such that*

$$\|p_n' w\|_{E_j} \leq (1+\varepsilon) M_j n^2 \|p_n w\|_E \, , \quad x \in E_j$$

for every polynomial $p_n \in P_n, \ n \geq n_0$.

Proof. By the Weierstrass approximation theorem, for any $\eta > 0$ there is a $q_k \in P_k, \ k \geq l$ such that

$$w(x) \leq q_k(x) \leq (1+\eta)\, w(x) \ , \ x \in E. \tag{3.1}$$

Hence,

$$\begin{aligned} |p_n'(x) w(x)| &\leq |p_n'(x) q_k(x)| \\ &\leq \left|(p_n q_k)'(x)\right| + |p_n(x) q_k'(x)| \, . \end{aligned}$$

Denote $m := \min\{w(x) : x \in E\}$ then we obtain taking into account inequality (1.4) and (3.1) the estimate

$$\begin{aligned}
|p_n'(x)w(x)| &\leq (1+\varepsilon_1)(n+k)^2 M_j \|p_n q_k\|_E + (1+\varepsilon_1) M_j \|p_n\|_E k^2 \|q_k\|_E \\
&\leq (1+\varepsilon_1)(1+\eta)(n+k)^2 M_j \|p_n w\|_E + \frac{1}{m}(1+\varepsilon_1)(1+\eta)\|p_n w\|_E \|w\|_E \\
&\leq n^2 M_j \|p_n w\|_E [1+\varepsilon_1+\eta(1+\varepsilon_1)+ \\
&\quad + \frac{2k}{n}(1+\varepsilon_1)(1+\eta) + \frac{k^2}{n^2}(1+\eta)\left(1+\varepsilon_1+\frac{1}{mM_j}\|w\|_E\right)\Big] \\
&\leq (1+\varepsilon) M_j n^2 \|p_n w\|_E
\end{aligned}$$

for every $p_n \in P_n$, provided $\eta > 0$ and $\varepsilon_1 > 0$ are sufficiently small, $n \geq n_0$ such that

$$\varepsilon \geq \varepsilon_1 + \eta(1+\varepsilon_1) + \frac{2k}{n}(1+\varepsilon_1)(1+\eta) + \frac{k^2}{n^2}(1+\eta)\left(1+\varepsilon_1+\frac{1}{mM_j}\|w\|_E\right).$$

□

Remark 3.1.1. Theorem is asymptotically sharp. It follows from asymptotic sharpness of Totik's inequality (1.3).

3.2 PROPERTIES OF CHEBYSHEV- MARKOV ALGEBRAIC FRACTIONS

The content of this section is translation of a part of the book (Rusak, 1979) which is not available in English.

We will consider here the problem on boundaries for derivatives of algebraic fractions

$$r_n(x) = \frac{p_n(x)}{\sqrt{t_{2n}(x)}},$$

$$t_{2n}(x) = \prod_{k=1}^{2n}(1+a_k x), \quad -1 \leq x \leq 1, \tag{3.2}$$

where $p_n(x)$- is an algebraic polynomial of degree at most n with complex (real) coefficients, numbers a_k are either real $|a_k| < 1$, or pairwise complex conjugate. In particular, if $t_{2n}(x) = t_n^2(x)$, algebraic fractions are transformed to rational functions.

Firstly the problem about construction of rational functions with given denominator deviated least from zero on $[-1, 1]$, was solved by Chebyshev. A. A. Markov generalized that problem and constructed algebraic fractions of the form (3.2) which deviate least from 0 on $[-1,1]$. These authors and also Bernstein discovered deep analogue of these fractions with Chebyshev polynomials.

We will use systematically algebraic fractions deviated least from 0 on $[-1,1]$ in estimating derivatives of the functions of the form (3.2).

To constructed Chebyshev-Markov algebraic fraction it is useful to passed to $(-\infty,\infty)$ and to obtain then as a particular case of Bernstein algebraic fractions. This fact seems to be note observed before. In any case nowhere in Bernstein works the connection between the algebraic fraction deviated least from 0 on the real axis (which were constructed by him) and Chebyshev-Markov algebraic fraction was mentioned.

Put (3.2), $x=\frac{1-y^2}{1+y^2}$, then

$$\begin{aligned} R_{2n}(y) &= r_n\left(\frac{1-y^2}{1+y^2}\right)=\frac{P_{2n}(y)}{\sqrt{h_{2n}(y)}}, \\ h_{2n}(y) &= \prod_{k=1}^{2n}\left[y^2+\frac{1+a_k}{1-a_k}\right], \qquad -\infty<y<\infty, \end{aligned}$$

where $P_{2n}(y)$- is an even algebraic polynomial of degree at most $2n$. The polynomial $h_{2n}(y)$ is positive on the real axis and its zeros are pairwise complex conjugate. Denote by $z_j=\alpha_j+i\beta_j$, $j=1,\dots,2n$, the zeros of $h_{2n}(y)$, with $\beta_j>0$, and let

$$\Phi_{2n}(y)=\sum_{j=1}^{2n}\arg(\alpha_j+i\beta_j-y) \tag{3.3}$$

will introduced Bernstein cosine fraction by

$$M_{2n}(y)=\cos\Phi_{2n}(y)=\frac{1}{2}\left\{\prod_{j=1}^{2n}\frac{z_j-y}{|z_j-y|}+\prod_{j=1}^{2n}\frac{\overline{z_j}-y}{|\overline{z_j}-y|}\right\}\equiv\frac{\widetilde{P}_{2n}(y)}{\sqrt{h_{2n}(y)}} \tag{3.4}$$

Numbers $\{z_j\}_{j=1}^{2n}$ are roots of equations $y^2+\frac{1+a_k}{1-a_k}=0$, $k=1,\dots,2n$, and they obviously symmetric with respect to the imaginary axis. Then the polynomial $\widetilde{P}_{2n}(y)$ in (3.4) is an even of algebraic polynomial of degree $2n$.

Go back to the variable x in (3.4), and we obtain the algebraic Chebyshev-Markov cosine fraction for the interval $[-1,1]$

$$m_n(x)\overset{def}{=}\cos\varphi_{2n}(x)\equiv\cos\Phi_{2n}(y). \tag{3.5}$$

Note that

$$\begin{aligned}
\varphi_{2n}'(x) &= \Phi_{2n}'(y)\frac{dy}{dx} = \frac{\left(y^2+1\right)^2}{-4y}\sum_{j=1}^{2n}\frac{\beta_j}{\left(\alpha_j-y\right)^2+\beta_j^2}, \quad 0<y<\infty, \qquad (3.6)\\
\varphi_{2n}(-1) &= \Phi_{2n}(\infty) = 2n\pi,\\
\varphi_{2n}(1) &= \Phi_{2n}(0) = n\pi.
\end{aligned}$$

The last equality follows from (3.3) since the points $\{\alpha_j+i\beta_j\}$ are pairwise symmetric with respect to the imaginary axis.

So, the argument $\varphi_{2n}(x)$ is a decreasing function on the interval $[-1,1]$, the function $m_n(x)$- is an algebraic fraction of the form (3.2), which has maximum modules 1 with alternate signs at $n+1$ points. In particular,

$$m_n(-1)=1, \quad m_n(1)=(-1)^n. \qquad (3.7)$$

Hence, $m_n(x)$ deviates least from 0 among real algebraic fractions of the form (3.2) with fixed leading coefficient of the polynomial $p_n(x)$. It follows (3.6) so that the zeros $\{x_k\}_{k=1}^n$ of the cosine fraction $m_n(x)$ are determined by equation

$$\varphi_{2n}(x) = \left[2n-\left(k-\frac{1}{2}\right)\right]\pi, \ k=1,\ldots,n, \qquad (3.8)$$

and situated $(-1,1)$ in such a way that

$$-1<x_1<x_2<\ldots<x_n<1 \qquad (3.9)$$

together with form will define Chebyshev- Markov sine fraction by

$$\begin{aligned}
v_n(x) \stackrel{def}{=} \sin\varphi_{2n}(x) &\equiv \sin\Phi_{2n}(y) =\\
&= \frac{1}{2i}\left\{\prod_{j=1}^{2n}\frac{z_j-y}{|z_j-y|}+\prod_{j=1}^{2n}\frac{\overline{z_j}-y}{|\overline{z_j}-y|}\right\} \qquad (3.10)\\
&\equiv \frac{y\widetilde{P}_{2n-2}(y)}{\sqrt{h_{2n}(y)}}
\end{aligned}$$

Here $\widetilde{P}_{2n-2}(y)$ is an even polynomial of degree $2n-2$. Go back to the variable x, and we obtain

$$v_n(x) = \frac{\sqrt{1-x^2}\widetilde{p}_{n-1}(x)}{\sqrt{t_{2n}(x)}}$$

it is clear that $\sin\varphi_{2n}(x)$ has $n+1$ distinct zeros on the interval $[-1,1]$. Denote them by $\{y_j\}_{j=0}^n$, so

$$\begin{aligned}
-1 &= y_0<y_1<\ldots<y_n=1, \qquad (3.11)\\
\varphi_{2n}(y_j) &= (2n-j)\pi,
\end{aligned}$$

the point $\{y_j\}$ and $\{x_k\}$ interlace and

$$\begin{aligned} m_n(y_j) &= (-1)^j, \ j = 0, 1, \ldots, n; \\ v_n(x_k) &= (-1)^k, \ k = 1, 2, \ldots, n. \end{aligned} \tag{3.12}$$

It is necessary also to find an explicit expression for the function $\varphi'_{2n}(x)$ in terms of the variable x and numbers a_k. Take into account the connections between $\{a_k\}$ and $\{z_j\}$, suppose that all square roots have positive real parts and we have

$$\begin{aligned} \varphi'_{2n}(x) &= \frac{(y^2+1)^2}{-4y} \sum_{j=1}^{2n} \left(\frac{i}{\alpha_j - y + i\beta_j} - \frac{i}{\alpha_j - y - i\beta_j} \right) \\ &= \frac{-1}{2\sqrt{1-x^2}} \sum_{k=1}^{2n} \frac{\sqrt{1-a_k^2}}{1+a_k x} \stackrel{def}{=} \frac{-1}{\sqrt{1-x^2}} \lambda_n(x). \end{aligned} \tag{3.13}$$

Now we will prove 2 lemmas. Which are necessary to obtain Markov-type inequalities for derivatives of algebraic fractions.

Lemma 3.2.1. *If zeros of the polynomial $t_{2n}(x)$ are complex conjugate or a real of even multiplicity then the inequality*

$$\left| \frac{\sin \varphi_{2n}(x)}{\sqrt{1-x^2}} \right| \le \lambda_n(x), \tag{3.14}$$

is valid for $-1 \le x \le 1$ where $\lambda_n(x)$ is defined by formula (3.13). The equality sine in (3.14) is attained at points $x = \pm 1$ only.

Proof. Using the substitution $x = \frac{1-y^2}{1+y^2}$, we transformed inequality (3.14) to the form (see, (3.6) and (3.13))

$$\left| \frac{\sin \Phi_{2n}(y)}{y} \right| \le \sum_{j=1}^{2n} \frac{\beta_j}{(\alpha_j - y)^2 + \beta_j^2}, \quad 0 \le y \le \infty.$$

Under supposition Lemma 3.2.1 the argument $\Phi_{2n}(y)$ can be represented as

$$\Phi_{2n}(y) = \sum_{k=1}^{n} \Psi_k(y),$$

where $\Psi_k(y)$- is the argument which correspondence to a pair of complex conjugate of zeros of the polynomial $t_{2n}(x)$ or to a double zero of that polynomial. Corresponding four zeros of the polynomial $h_{2n}(y)$ are symmetric with respect to the both coordinate axes, and we obtain

$$\begin{aligned} \sin \Psi_k(y) &= \frac{1}{2i} \left(e^{i\Psi_k(y)} - e^{-i\Psi_k(y)} \right) \\ &= \frac{-2\beta_k y}{|\alpha_k - y + i\beta_k| \, |\alpha_k + y + i\beta_k|} \end{aligned}$$

it follows,

$$\left|\frac{\sin\Psi_k(y)}{y}\right| = \frac{2\beta_k}{|\alpha_k+i\beta_k-y|\,|\alpha_k+i\beta_k+y|} \le \frac{\beta_k}{|\alpha_k+i\beta_k-y|^2}+\frac{\beta_k}{|\alpha_k+i\beta_k+y|^2}, \tag{3.15}$$

where the equality is attained at $y=0$ and $y=\infty$ only.

Let inequality (3.14) be valid with $n=p$, i.e.

$$\left|\frac{\sin\Phi_{2p}(y)}{y}\right| \le \sum_{j=1}^{2p}\frac{\beta_j}{(\alpha_j-y)^2+\beta_j^2} \tag{3.16}$$

is full feel. We have from (3.15) and (3.16) that

$$\begin{aligned}\left|\frac{\sin\Phi_{2p+2}(y)}{y}\right| &= \frac{1}{y}\left|\sin\Phi_{2p}(y)\cos\Psi_{p+1}(y)+\sin\Psi_{p+1}(y)\cos\Phi_{2p}(y)\right| \\ &\le \sum_{j=1}^{2p}\frac{\beta_j}{(\alpha_j-y)^2+\beta_j^2}+\frac{\beta_{p+1}}{|\alpha_{p+1}-y+i\beta_{p+1}|^2}+\frac{\beta_{p+1}}{|\alpha_{p+1}+y+i\beta_{p+1}|^2} \\ &= \sum_{j=1}^{2p+2}\frac{\beta_j}{(\alpha_j-y)^2+\beta_j^2}.\end{aligned}$$

□

Lemma 3.2.2. *If zeros of the polynomial $t_{2n}(x)$ are real of the same sign or symmetric with respect to the point $x=0$, then the inequality*

$$\frac{\sin\varphi_{2n}(x)}{\sqrt{1-x^2}} \le \max\{\lambda_n(1),\lambda_n(-1)\} \tag{3.17}$$

is valid for $-1\le x\le 1$. Equality sign in (3.17) is attained at one or at both end points of the interval $[-1,1]$.

Proof. As in the proof of Lemma 3.2.1 we searched a majorant for the function $\sin\Phi_{2n}(y)\frac{1+y^2}{2y}$, $0\le y\le\infty$. At this time we have

$$\left|\frac{1+y^2}{2y}\sin\Phi_{2n}(y)\right| \le \frac{1}{2}\sum_{k=1}^{n}\frac{(\beta_k+\beta_{n+k})(1+y^2)}{\sqrt{\beta_k^2+y^2}\sqrt{\beta_{n+k}^2+y^2}}, \tag{3.18}$$

and equality is attained at points $y=0$ and $y=\infty$.

Taking for definedness $\beta_k=\sqrt{\frac{1+a_k}{1-a_k}}>1,\ k=1,\dots 2n$, it is not difficult to find that the maximum of the majorant in (3.18) is equal to

$$\sum_{k=1}^{2n}\beta_k=\sum_{k=1}^{2n}\sqrt{\frac{1+a_k}{1-a_k}}=\lambda_n(-1).$$

In the case of symmetric $\beta_k\beta_{n+k} = 1,\ \beta_k > 1$ and the maximum of majorant of (3.18) is equal to

$$\sum_{k=1}^{n}\left(\beta_k + \frac{1}{\beta_{n+k}}\right) = \lambda_n(1) = \lambda_n(-1).$$

Note that as a particular case it follows from (3.17) the inequality

$$\left|\frac{\sin n \arccos x}{\sqrt{1-x^2}}\right| \leq n, \qquad -1 \leq x \leq 1.$$

□

Theorem 3.2.3. *If a real algebraic fraction of the form (3.2) satisfies inequality*

$$|r_n(x)| \leq 1, \qquad -1 \leq x \leq 1, \tag{3.19}$$

then the estimate

$$|r'_n(x)| \leq \sqrt{1 - r_n^2(x)}\frac{\lambda_n(x)}{\sqrt{1-x^2}}, \qquad -1 < x < 1 \tag{3.20}$$

is valid for its derivatives. Equality in (3.20) is valid only at the points where $|r_n(x)| = 1$, *and for the function* $r_n(x) = \varepsilon \cos\varphi_{2n}(x),\ \varepsilon = \pm 1$, *at all points.*

Proof. After substitute $x = \frac{1-y^2}{1+y^2}$, which transforms $-\infty < y < \infty$ into double traversed interval $(-1, 1)$, the function $r_n(x)$ will be certain real function of the form

$$\begin{aligned} R_{2n}(y) &= r_n\left(\frac{1-y^2}{1+y^2}\right) = \frac{P_{2n}(y)}{\sqrt{h_{2n}(y)}}, \\ h_{2n}(y) &= \prod_{k=1}^{2n}\left[(\alpha_k - y)^2 + \beta_k^2\right]. \end{aligned}$$

It follows from (3.19) that, $|R_{2n}(y)| \leq 1, \quad -\infty < y < \infty$. Hence the estimate

$$|R'_{2n}(y)| \leq \sqrt{1 - R_{2n}^2(y)}\Phi'_{2n}(y) \tag{3.21}$$

is valid for its derivative (see, for instance, (Rusak, 1973) and references therein), where equality sign is attained at those points only where $|R_{2n}(y)| = 1$, and for Bernstein cosine fractions $M_{2n,\varphi}(y) = \cos\left[\Phi_{2n}(y) + \varphi\right]$ everywhere. Now observe that

$$r'_n(x) = R'_{2n}(y)\frac{(1+y^2)^2}{-4y}. \tag{3.22}$$

It follows from relations from (3.21) and (3.22) (see also (3.6), (3.13))

$$\begin{aligned} |r_n'(x)| &= \left| R_{2n}'(y) \frac{(1+y^2)^2}{4y} \right| \\ &\leq \sqrt{1-R_{2n}^2(y)}\Phi_{2n}'(y) \frac{(1+y^2)^2}{4y} \\ &= \left| \sqrt{1-r_n^2(x)}\varphi_{2n}'(x) \right| = \sqrt{1-r_n^2(x)} \frac{\lambda_n(x)}{\sqrt{1-x^2}}, \end{aligned}$$

where equality sign is attained only for those algebraic fractions $r_n(x) = R_{2n}(y)$, which realized equality sign in estimate (3.21). It is necessary to distinguish only two cosine fractions: $M_{2n,0}(y)$ and $M_{2n,\pi}(y)$ from the set of Bernstein cosine fractions $M_{2n,\varphi}(y)$ which are extremal in inequality (3.21), i.e. the extremal in estimate (3.20) are the functions

$$r_n(x) = R_{2n}(y) \equiv \pm \cos \Phi_{2n}(y) = \pm \cos \varphi_{2n}(x).$$

The reason is that for other values of the parameter φ, Bernstein cosine fractions after transformed to variable x are not algebraic fractions of the form (3.2) as it is clear from (3.4) and (3.10). □

Remark 3.2.1. Under suppositions of Theorem 3.2.3 the estimate

$$|r_n'(x)| \leq \frac{\lambda_n(x)}{\sqrt{1-x^2}}, \qquad -1 < x < 1 \tag{3.23}$$

is valid obviously. Chebyshev-Markov cosine fractions $\pm \cos \varphi_{2n}(x)$ is still extremal in (3.23) but the equality sign is attained now only at those points where $\cos \varphi_{2n}(x) = 0$. Unlike (3.20) inequality (3.23) is valid also for algebraic fractions of the form (3.2), which have a polynomial with complex coefficients as a numerator $p_n(x)$. Extremal algebraic fractions are the functions

$$r_n(x) \equiv \varepsilon \cos \varphi_{2n}(x), \qquad |\varepsilon| = 1$$

only. It can be deduces directly from the validity of (3.23) for real algebraic fractions by known methods. Since we will refer to this method in the following we presented here.

Let $r_n(x)$- be complex algebraic fraction of the form (3.2), $|r_n(x)| \leq 1$ Then it can be represented

$$r_n(x) = u_n(x) + iv_n(x),$$

where $u_n(x)$ and $v_n(x)$- are real algebraic fractions of the form (3.2). It follows from inequality $|r_n(x)| \le 1, \quad -1 \le x \le 1$ that for arbitrary real α

$$|u_n(x)\cos\alpha - v_n(x)\sin\alpha| \le 1, \quad -1 \le x \le 1,$$

and because of validity of the estimate (3.23) for real algebraic fraction we have

$$|u_n'(x)\cos\alpha - v_n'(x)\sin\alpha| \le \frac{\lambda_n(x)}{\sqrt{1-x^2}}, \qquad -1 < x < 1 \tag{3.24}$$

since α is arbitrary it follows from (3.24) that

$$|r_n'(x)| = \sqrt{u_n'^2(x) + v_n'^2(x)} \le \frac{\lambda_n(x)}{\sqrt{1-x^2}}, \qquad -1 < x < 1$$

i.e. inequality (3.23) is proved for complex valued algebraic fractions. If the equality sign is realized in the last inequality then it is realized for a certain α_0 in (3.24). Then we have

$$u_n(x)\cos\alpha_0 - v_n(x)\sin\alpha_0 \equiv \pm\cos\varphi_{2n}(x), \tag{3.25}$$

or what is the same

$$Re\left\{r_n(x)e^{i\alpha_0}\right\} = \pm\cos\varphi_{2n}(x)$$

from the other side,

$$Im\left\{r_n(x)e^{i\alpha_0}\right\} = u_n(x)\cos\alpha_0 - v_n(x)\sin\alpha_0$$

and from inequality $|r_n(x)e^{i\alpha_0}| \le 1$ it follows that

$$|u_n(x)\cos\alpha_0 + v_n(x)\sin\alpha_0| \le |\sin\varphi_{2n}(x)|\,.$$

In particular, $u_n(x)\cos\alpha_0 + v_n(x)\sin\alpha_0$ vanishes at $n+1$ zeros of the sign fraction $\sin\varphi_{2n}(x)$. It follows from here

$$u_n(x)\cos\alpha_0 + v_n(x)\sin\alpha_0 \equiv 0. \tag{3.26}$$

It is clear from (3.25) and (3.26) that

$$u_n(x) = \pm\cos\alpha_0\cos\varphi_{2n}(x), \quad v_n(x) = \pm\sin\alpha_0\cos\varphi_{2n}(x)$$

and, hence the function $r_n(x)$, which realized inequality (3.23) has necessarily the form

$$\begin{aligned} r_n(x) &= \pm\cos\alpha_0\cos\varphi_{2n}(x) \pm \sin\alpha_0\cos\varphi_{2n}(x) \\ &= \pm\cos\varphi_{2n}(x)\left(\cos\alpha_0 - i\sin\alpha_0\right) \\ &= \varepsilon\cos\varphi_{2n}(x), \quad |\varepsilon| = 1. \end{aligned}$$

Theorem 3.2.4. *Let complex valued algebraic fraction $r_n(x)$ satisfies condition (3.19) and $\gamma(x)$- is a differentiable function on $[-1,1]$. Then the estimate*

$$\left|(r_n(x)\gamma(x))'\right| \leq \sqrt{\left[\varphi_{2n}'(x)\right]^2 \gamma^2(x) + \gamma'^2(x)}, \qquad -1 < x < 1 \tag{3.27}$$

is valid. If the point x is not a multiple root of the function $\gamma(x)$, then the equality sign in (3.27) is valid only for algebraic fractions

$$r_n(x) \equiv \varepsilon \cos \varphi_{2n}(x), \qquad |\varepsilon| = 1$$

at those points where $(\gamma(x) \sin \varphi_{2n}(x))' = 0$.

Proof. It is sufficient to prove (3.27) only for the case when numerator $p_n(x)$ has real coefficients. The validity of the estimate for complex valued algebraic fractions is proved in the same way as in Remark 3.2.1.

Put $r_n(x) = \cos\omega = \cos\left[\arccos r_n(x)\right]$, and we obtain

$$\begin{aligned}
\left|(r_n(x)\gamma(x))'\right| &= \left|\frac{\sin\omega r_n'(x)\gamma(x)}{\sqrt{1-r_n^2(x)}} + \cos\omega\gamma'(x)\right| \\
&\leq \sqrt{\sin^2\omega + \cos^2\omega}\left[\frac{\left(r_n'(x)\right)^2\gamma^2(x)}{1-r_n^2(x)} + \gamma'^2(x)\right]^{\frac{1}{2}} \\
&\leq \sqrt{\left[\varphi_{2n}'(x)\right]^2\gamma^2(x) + \gamma'^2(x)}.
\end{aligned}$$

Equality sign in the last inequality because of Remark 3.2.1 and Cauchy- Schwart $\left(\sum (a_k b_k) \leq (\sum a_k^2)^{\frac{1}{2}} (\sum b_k^2)^{\frac{1}{2}}\right)$ inequality is valid only for the function at those points where

$$\frac{-\sin\varphi_{2n}(x)}{\varphi_{2n}'(x)\gamma(x)} = \frac{\cos\varphi_{2n}(x)}{\gamma'(x)}.$$

The last equality is equivalent to $(\gamma(x)\sin\varphi_{2n}(x))' = 0$. □

Theorem 3.2.5 (Bocher- Walsch)**.** *If disks $|z| \leq r_1$ and $|z| \geq r_2 (> r_1)$ contain respectively all zeros and poles of rational function $f(z)$ of order n, then those regions contain all zeros of the derivative $f'(z)$, and the first region contains precisely $n-1$ zeros of derivative.*

Theorem 3.2.6. *If algebraic fraction of the from (3.2) satisfies condition (3.19) and $|a_k| < 1,\ k = 1, \ldots, 2n$, then the estimate*

$$|r_n'(x)| \leq \begin{cases} \frac{\lambda_n(x)}{\sqrt{1-x^2}}, & x_1 \leq x \leq x_n, \\ |m_n'(x)|, & -1 \leq x \leq x_1,\ x_n \leq x \leq 1, \end{cases} \tag{3.28}$$

is valid. Where $\{x_k\}$, $-1 < x_1 < \ldots < x_n < 1$- *are zeros of the cosine fraction* $m_n(x)$. *Equality sign in (3.28) is attained only for functions* $r_n(x) \equiv \varepsilon m_n(x), \ |\varepsilon| = 1$.

Proof. Obviously, estimate (3.28) is necessarily to prove only for intervals $[-1, x_1]$ and $[x_n, 1]$. Since inequality (3.28) coincides with the estimate (3.23) which was proved before on the interval $[x_1, x_n]$. Furthermore it is sufficient to prove it only for the interval $[-1, x_1]$; for the interval $[x_1, 1]$ it will follow from the symmetricity of the problem under consideration. Finally taking into account Remark 3.2.1 we will consider only the case of real algebraic fractions.

Let $\{y_j\}_{j=0}^n$- be zeros of the sine Chebyshev- Markov fraction $v_n(x)$ (see (3.11)). If $r_n(x) \not\equiv \pm m_n(x)$ for $0 < \lambda < 1$ because of (3.12) the difference $m_n(x) - \lambda r_n(x)$ takes opposite signs at points $\{y_j\}_{j=0}^n$. Hence it has simple zeros on interval $(-1, 1)$ and can be written as

$$m_n(x) - \lambda r_n(x) = C \prod_{k=1}^{n} \frac{(x - z_k)}{\sqrt{t_{2n}(x)}}, \quad |z_k| < 1.$$

Next, the derivative of this difference has form

$$m_n'(x) - \lambda r_n'(x) = \frac{C}{2} \left(m_n(x) - \lambda r_n(x)\right) \times \left[2 \sum_{k=1}^{n} \frac{1}{x - z_k} - \sum_{k=1}^{2n} \frac{1}{x - \frac{1}{a_k}} \right]. \quad (3.29)$$

Zeros of the difference $m_n'(x) - \lambda r_n'(x)$ coincide with zeros of square brackets in (3.29), and they are zeros of the logarithmic derivative of the rational function $R_{2n}(x) = \prod\limits_{k=1}^{n} \frac{(x-z_k)^2}{t_{2n}(x)}$. It follows from condition $|a_k| < 1, \ k = 1, \ldots, 2n$, that poles of the rational functions $R_{2n}(z)$ are contained in region $|z| \geq r > 1$, zeros of that function $\{z_k\}$ lie on the interval $(-1, 1)$. It follows from Boher- Walsch theorem that $R_{2n}'(z)$ has $2n - 1$ zeros in $|z| \leq 1$ and other zeros are in $|z| \geq r > 1$. Among the zeros of the function $R_{2n}'(z)$ which lie in $|z| < 1$, there are obviously points $\{z_k\}_{k=1}^n$, and if we will exclude them then the logarithmic derivative of the function $R_{2n}(z)$ has precisely $n - 1$ zeros in $|z| \leq 1$ and other zeros are in $|z| \geq r > 1$. In the same way because of (3.29) the zeros of the difference $m_n'(x) - \lambda r_n'(x)$ are located. What is more important for us is that inside the disk $|z| \leq 1$ and in particular on the interval $[-1, 1]$, the function $m_n'(x) - \lambda r_n'(x), \ 0 < \lambda < 1$ has no more than $n - 1$ zeros.

In this consideration the multiply λ was introduced for the convenient . Essentially the same consideration can be given also for the difference $m_n(x) - r_n(x)$ but the zeros $\{z_k\}$ can be of different multiplicity. As a result we will obtain the function $m_n'(x) - r_n'(x)$ has at most $n-1$ zeros on the interval $[-1,1]$.

Since the derivative of the cosine fraction $m_n'(x) = -\sin\varphi_{2n}(x)\varphi_{2n}'(x)$ has alternate signs at points $\{x_k\}_{k=1}^{2n}$ (see (3.12)) and $|r_n'(x_k)| < |m_n'(x_k)|$ because of (3.23), we have sign $\{m_n'(x_k) - r_n'(x_k)\} = (-1)^k$, $k = 1, \dots, n$. Hence $n-1$ zeros of the difference $m_n'(x) - r_n'(x)$ are on the interval (x_1, x_n) and this difference preserves sign on intervals $[-1, x_1]$ and $[x_n, 1]$. In particular it follows from $m_n'(x_1) - r_n'(x_1) < 0$ that $m_n'(x) - r_n'(x) < 0$ for $-1 \le x \le x_1$. Analogues consideration is valid for the algebraic fraction- $r_n(x)$. Hence we obtain inequality $m_n'(x) + r_n'(x) < 0$ for $-1 \le x \le x_1$, and it gives the estimate $|r_n'(x)| < |m_n'(x)|$, $-1 \le x \le x_1$.

In the case $r_n(x) = \varepsilon m_n(x)$, $|\varepsilon| = 1$, the equality sign in (3.28) is attained obviously. □

Theorem 3.2.7. *If the conditions of Theorem 3.2.6 are satisfied and polynomial $t_{2n}(x)$ has no real zeros of odd multiplicity then the inequality*

$$r_n'(x) \le \begin{cases} \frac{\lambda_n(x)}{\sqrt{1-x^2}}, & x_1 \le x \le x_n, \\ \lambda_n^2(x), & -1 \le x \le x_1, \ x_n \le x \le 1, \end{cases} \tag{3.30}$$

is valid. Equality sign in (3.30) is attained at points $\{x_k\}$ only and $x = \pm 1$ for the function $r_n(x) = \varepsilon m_n(x)$, $|\varepsilon| = 1$.

Proof. Estimate (3.30) follows easily from (3.28) it is sufficient to note that (see (3.13))

$$m_n'(x) = (\cos\varphi_{2n}(x))' = \frac{-\sin\varphi_{2n}(x)}{\sqrt{1-x^2}}\lambda_n(x),$$

and to use Lemma 3.2.1. □

Theorem 3.2.8. *If the conditions of theorem 3.2.6 are satisfied zeros of polynomial $t_{2n}(x)$ are real and of the same sign or symmetric with respect to point $x = 0$, then the inequality*

$$|r_n'(x)| \le \max\left\{\lambda_n^2(1), \lambda_n^2(-1)\right\} \tag{3.31}$$

is valid. Equality sign in (3.31) is attained at one or at both end points of the interval $[-1,1]$ for the functions only $r_n(x) = \varepsilon m_n(x)$, $|\varepsilon| = 1$.

Proof. For the definiteness let $\max\limits_{-1\le x\le 1}\lambda_n(x)=\lambda_n(1)$. Proceed in the same way as in the proof of the Theorem 3.2.7, but applying Lemma 3.2.2 instead of Lemma 3.2.1 to obtain inequality

$$|r_n'(x)|\le\begin{cases}\frac{\lambda_n(x)}{\sqrt{1-x^2}}, & x_1\le x\le x_n,\\ \lambda_n^2(1), & -1\le x\le x_1,\ x_n\le x\le 1\end{cases}\tag{3.32}$$

Now let us estimates lengths of intervals $[-1,x_1]$ and $[x_1,1]$. Supposition $x=\cos\theta,\ x_k=\cos\theta_k$ we will obtain because of (3.8) and (3.11)

$$\varphi_{2n}(\cos\pi)-\varphi_{2n}(\cos\theta_1)=\frac{\pi}{2},\qquad(\theta_n-0)\,\lambda_n(\cos\theta'')=\frac{\pi}{2}.$$

Apply in those equalities mean value theorem with respect to the variable θ we obtain

$$(\pi-\theta_1)\,\lambda_n(\cos\theta')=\frac{\pi}{2},\qquad(\theta_n-0)\,\lambda_n(\cos\theta'')=\frac{\pi}{2},$$

has it follows

$$\pi-\theta_1\ge\frac{\pi}{2\lambda_n(1)},\qquad\theta_n\ge\frac{\pi}{2\lambda_n(1)}$$

then for $x_1\le x\le x_n$ the inequality

$$\frac{1}{\sqrt{1-x^2}}=\frac{1}{\sin\theta}\le\max\left\{\frac{1}{\sin\theta_1},\frac{1}{\sin\theta_n}\right\}\le\lambda_n(1)$$

is valid. Estimate (3.31) follows from (3.32), equality case in it can be deduce from the extremality from the (3.28) and Lemma 3.2.2. □

3.3 RUSAK-TYPE INEQUALITY FOR ALGEBRAIC FRACTIONS

We use the following theorems:

Theorem 3.3.1. *(Lukashov, 2004) Let $\rho_\nu(x)$ be a real algebraic polynomial of degree ν which is positive on a system $E=\bigcup\limits_{j=1}^{l}[a_{2j-1},a_{2j}]\subset\mathbb{R}$ of disjoint intervals. Write*

$$\rho_\nu(x)=\prod_{j=1}^{\nu^*}(x-x_j)^{\nu_j},\qquad l\ge 2.\tag{3.33}$$

Then the following assertions are equivalent.

1. *The rational function with denominator $\sqrt{\rho_\nu(x)}$ and least deviation from zero among all such functions has the maximal number of deviation points on E, that is*

$$\max_{x\in E}\left|\frac{P_n(x)}{\sqrt{\rho_\nu(x)}}\right| = \min_{c_i\in\mathbb{R}}\max_{x\in E}\left|\frac{x^n + c_1x^{n-1}+\ldots+c_n}{\sqrt{\rho_\nu(x)}}\right| = M_n > 0 \tag{3.34}$$

and

$$\frac{P_n}{\sqrt{\rho_\nu}}(a_{2j}) = \frac{P_n}{\sqrt{\rho_\nu}}(a_{2j+1}), \quad j=1,\ldots,l-1, \quad 2n\geq\nu. \tag{3.35}$$

2. *For every $j=1,\ldots,l$, the sum of the harmonic measures of the interval $[a_{2j-1},a_{2j}]$ with respect to zeros of ρ_ν is a positive integer. More precisely,*

$$\left(n-\frac{\nu}{2}\right)\omega_j(\infty) + \frac{1}{2}\sum_{k=1}^{\nu^*}\nu_k\omega_j(x_k) = q_j, \quad q_j\in\mathbb{N}, \quad j=1,\ldots,l.$$

3. *There is a real algebraic polynomial Q_{n-l} of degree $n-l$ such that*

$$P_n^2(x) - H(x)\,Q_{n-l}^2(x) = M_n^2\rho_\nu(x) \tag{3.36}$$

for some constant $M_n>0$, and branch of the root may be chosen in such a way that

$$P_n(x_k) = \left(\sqrt{H}Q_{n-l}\right)(x_k), \quad k=1,\ldots,\nu^* \tag{3.37}$$

and $H(x) = \prod_{j=1}^{2l}(x-a_j)$.

In this case, the fraction

$$\frac{P_n(x)}{\sqrt{\rho_\nu(x)}}$$

can be written in terms of the densities $\varpi_a(z,x) = \frac{\partial}{\partial x}\omega(z, E\cup[a_1,x], \mathbb{C}\setminus E)$ *of the harmonic measure as*

$$\frac{P_n(x)}{\sqrt{\rho_\nu(x)}} = \varepsilon M_n\cos\left(\pi\int_{a_1}^{x}\left(\left(n-\frac{\nu}{2}\right)\varpi_a(\infty,x) + \frac{1}{2}\sum_{j=1}^{\nu^*}\nu_j\varpi_a(x_j,x)\right)dx\right) \tag{3.38}$$

where $\varepsilon\in\{-1,1\}$.

Theorem 3.3.2. *(Lukashov, 2004) Consider any algebraic fraction* $r(x) = \frac{p_n(x)}{\sqrt{\rho_\nu(x)}}$, *where* $p_n(x)$ *is algebraic polynomial and* $\rho_\nu(x) = \prod_{j=1}^{\nu^*}(x - x_j)^{\nu_j}$ *is a real polynomial of degree* ν *which is positive on* $E = [a_1, a_2] \cup \ldots \cup [a_{2l-1}, a_{2l}] \subset \mathbb{R},\ a_1 < a_2 < \ldots < a_{2l}$. *Then*

$$\left(\frac{r'(x)}{\frac{\pi}{2}\left((2n-\nu)_+ \varpi(\infty, x) + \sum_{j=1}^{\nu^*} \nu_j \varpi(x_j, x)\right)}\right)^2 + r^2(x) \le \|r\|^2_{C(E)}, \tag{3.39}$$

here $\varpi(z, x) = \frac{\partial}{\partial x}\left(w\left(z, [a_1, x] \cap E, \mathbb{C}\backslash E\right)\right)$ *is the harmonic measure.*

Theorem 3.3.3. *(Akturk and Lukashov, 2013) Let complex valued algebraic fraction* $r_n(x) = \frac{p_n(x)}{\sqrt{\rho_\nu(x)}}$ *be satisfying condition*

$$|r_n(x)| \le 1, \quad x \in E, \tag{3.40}$$

and $\gamma(x)$*- is a differentiable function on* E*. Then the estimate*

$$\left\|(r_n\gamma)'\right\|_E \le \sqrt{(\varphi_n'(x))^2 \gamma^2(x) + \gamma'^2(x)}, \quad x \in int\ E$$

is valid. Where

$$\varphi_n(x) = \frac{\pi}{2}\left((2n-\nu) + \varpi_E(\infty, x) + \sum_{j=1}^{\nu} \nu_j \varpi_E(x_j', x)\right).$$

Here, $\varpi_E(z, x) = \frac{d}{dx}\left(\omega\left(z, [\inf(E), x] \cap E, E\right)\right),$ *and* x_j' *are those* x_j *which are poles of the function* r_n^2*. Then the equality sign in (3.40) is valid only for algebraic fractions*

$$r_n(x) \equiv \varepsilon \cos \varphi_n(x), \quad |\varepsilon| = 1$$

in the case when, $\sum_{j=1}^{\nu} \nu_j \omega_k(x_j) = \frac{q_k}{n}$.

Proof. It is sufficient to prove (3.40) only for the case when numerator $p_n(x)$ has real coefficients. The validity of the estimate for complex valued algebraic fractions is proved in the same way

Put $r_n(x) = \cos\omega = \cos(\arccos r_n(x))$, and we obtain

$$\begin{aligned} \left|(r_n(x)\gamma(x))'\right| &= \left|\frac{\sin\omega r_n'(x)\gamma(x)}{\sqrt{1 - r_n^2(x)}} + \cos\omega\gamma'(x)\right| \\ &\le \sqrt{\sin^2\omega + \cos^2\omega}\left[\frac{(r_n'(x))^2\gamma^2(x)}{1 - r_n^2(x)} + \gamma'^2(x)\right]^{\frac{1}{2}} \\ &\le \sqrt{(\varphi_n'(x))^2\gamma^2(x) + \gamma'^2(x)}. \end{aligned}$$

□

3.4 MARKOV-TYPE INEQUALITIES FOR RATIONAL FUNCTIONS ON SEVERAL INTERVALS

V.N. Rusak (Rusak, 1979) proved if a real algebraic fraction of the form (3.2) satisfies inequality (3.19) then the estimate (3.20) is valid for its derivatives where

$$\lambda_n(x) = \frac{1}{2}\sum_{k=1}^{2n} \frac{\sqrt{1-a_k^2}}{1+a_k x},\ \ |a_k|<1,\ \ k=1,\ldots,2n.$$

Equality in (3.20) is valid only at the points where $|r_n(x)| = 1$, and for the function $m_n(x) = \varepsilon \cos \varphi_{2n}(x)$,

$$\varphi_{2n}(x) = \int \frac{-1}{\sqrt{1-x^2}} \lambda_n(x) dx, \varepsilon = \pm 1,$$

at all points.

Theorem 3.4.1. *Suppose that* $\sum\limits_{j=1}^{2n} \omega_k(\xi_j) = 2q_k,\ q_k \in \mathbb{N},\ k=1,\ldots,l$ *and* r_n *is a rational function*

$$r_n(x) = \frac{b_0 x^n + b_1 x^{n-1} + \ldots + b_n}{\sqrt{\rho_\nu(x)}},$$

where $|r_n(x)| \le 1,\ b_0,\ldots,b_n \in \mathbb{C}$ *and* $\rho_\nu(x) = \prod\limits_{j=1}^{2n} (x-\xi_j)^{\nu_j}$ *is a real polynomial of degree* ν *which is positive on* $E = \bigcup\limits_{j=1}^{l} [a_{2j-1}, a_{2j}]\,,\ \ -1 = a_1 < a_2 < \ldots < a_{2l} = 1,\ \ |\xi_j| > 1,\ j = 1,\ldots,2n.$ *Then*

$$|r_n'(x)| \le M_n(z_1,\ldots,z_n;E)\,, \tag{3.41}$$

where

$$M_n(z_1,\ldots,z_n;E) = \max\left(\frac{\pi}{2}\max_{x\in\widetilde{E}_n}\left|\sum_{j=1}^{2n} \varpi_E(x,\xi_j)\right|,\ \max_{x\in E\setminus\widetilde{E}_n} |m_n'(x)|\,,\right)$$

and

$$\begin{aligned}
\widetilde{E}_n &= [x_1, x_{q_1}] \cup [x_{q_1}, x_{q_1+q_2}] \cup \ldots \cup \left[x_{q_1+\ldots+q_{l-1}}, x_n\right],\\
m_n(x) &= \cos\left(\frac{\pi}{2}\int_{a_1}^{x} \sum_{j=1}^{2n} \varpi_E(x,\xi_j)\,dx\right),\\
\varphi_{2n}'(x) &= \frac{\pi}{2}\sum_{j=1}^{2n} \varpi_E(x,\xi_j)\,,
\end{aligned}$$

where $\{x_k\}$ are zeros of the cosine fraction $m_n(x)$. Here

$$\varpi_E\left(z,x\right)=\frac{\partial}{\partial x}w\left(z,E\cup\left[a_1,x\right],\mathbb{C}\backslash E\right);\ w\left(z,G,\mathbb{C}\backslash E\right)$$

is the harmonic measure of a set $G\subset E$ at a point $z\in\mathbb{C}\backslash E$.

The equality sign (3.41) is attained $r_n(x)\equiv\varepsilon m_n(x),\ \ |\varepsilon|=1$.

Proof. (3.41) is necessarily to prove only for intervals $[a_1,x_1],\ldots,[x_n,a_{2l}]$ since the first inequality in (3.41) coincides with the estimate (3.39).

Let $\{y_j\}_{j=0}^{n+l}$- be zeros of the sine Chebyshev-Markov fraction $\frac{\sqrt{-H(x)}v_{n-1}(x)}{\sqrt{\rho_\nu(x)}}$, where $v_{n-1}(x)$ is determined from

$$m_n^2(x)-H(x)v_{n-1}^2(x)=\rho_\nu(x),$$

and $H(x)=\prod_{j=1}^{2l}(x-a_j)$ (see (Lukashov, 2004)).

$$\begin{aligned} m_n(y_k) &= (-1)^{k+j+1}, \quad k=\sum_{i=1}^{j-1}q_{i+j-1};\ j=1,\ldots,l, \\ m_n(x_j) &= (-1)^j, \quad j=1,\ldots,n, \end{aligned} \tag{3.42}$$

if $r_n(x)\neq m_n(x)$ then for $0<\lambda<1$ the difference $m_n(x)-\lambda\, r_n(x)$ takes the same signs as m_n at points $\{y_j\}_{j=0}^{n+l}$. Hence it has simple zeros on the set E and can be written as

$$m_n(x)-\lambda r_n(x)=C\prod_{k=1}^{n}\frac{(x-z_k)}{\sqrt{\prod_{j=1}^{2n}(x-\xi_j)}},\qquad |z_k|<1.$$

Next, the derivative of this difference has form

$$m_n'(x)-\lambda r_n'(x)=\frac{C}{2}\left(m_n(x)-\lambda r_n(x)\right)\left[2\sum_{k=1}^{n}\frac{1}{x-z_k}-\sum_{j=1}^{2n}\frac{1}{x-\xi_j}\right]. \tag{3.43}$$

Zeros of the difference $m_n'(x)-\lambda r_n'(x)$ coincide with zeros of square brackets in (3.43), and they are zeros of the logarithmic derivative of the rational function $R_{2n}(x)=\frac{\prod_{k=1}^{n}(x-z_k)^2}{\rho_\nu(x)}$.

It follows from condition $|\xi_j|>1,\ j=1,\ldots,2n$ that poles of the rational function $R_{2n}(z)$ are contained in region $|z|\geq r>1$, zeros of that function $\{z_k\}$ lie in the interval $(-1,1)$.

It follows Bocher-Walsch Theorem 3.2.5 that $R_{2n}'(z)$ has $n-1$ zeros in $|z|\leq 1$ and other zeros are in $|z|\geq r>1$. In the same way because of (3.43) the zeros of the difference $m_n'(x)-\lambda r_n'(x)$ are located. So the function $m_n'(x)-\lambda r_n'(x),\ 0<\lambda<1$ has no more than $n-1$ zeros in $(-1,1)$ and other zeros are in $|z|\geq r>1$.

By the continuity as λ tends to 1 we obtain the function $m_n'(x)-r_n'(x)$ has at most $n-1$ zeros on $(-1,1)$.

Since the derivative of cosine fraction

$$\begin{aligned} m_n'(x) &= -\sin\varphi_{2n}(x).\varphi_{2n}'(x) \\ \varphi_{2n}(-1) &= 0, \\ \varphi_{2n}(1) &= n\pi, \end{aligned}$$

has alternate sign at points $\{x_k\}_{k=1}^n$ and from the identity

$$\left(\frac{m_n'(x)}{\varphi_{2n}'(x)}\right)^2+m_n^2(x)=1,$$

and the inequality (with $\leq$sign it was given in (Lukashov, 2004), for $m_n\neq\pm r_n$ strict inequality was obtained in (Kalmykov, 2009))

$$\frac{(r_n'(x))^2}{\varphi_{2n}'(x)}+r_n^2(x)<1,$$

we get

$$|r_n'(x_k)|<m_n'(x_k).$$

Hence $sign\left\{m_n'(x_k)-r_n'(x_k)\right\}=(-1)^k,\ k=1,\ldots,n$. On the interval (x_1,x_{q_1}) there are q_1-1 zeros y_j's $(j=1,\ldots,q_1-1)$, on $(x_{q_1+1},x_{q_1+q_2})$ there are q_2-1 zeros y_j's $(j=q_1+2,\ldots,q_1+q_2-1)$, and so on $\left(x_{q_1+\ldots+q_{l-1}},x_n\right)$ there are $n-q_1-\ldots-q_{l-1}$ zeros y_j's $(j=q_1+\ldots+q_{l-1}-1,\ldots,n+1)$, hence $n-l+1$ zeros of difference $m_n'(x)-r_n'(x)$ are on the set $\widetilde{E}_n$ and because of

$$sign\left\{m_n'(a_{2k})-r_n'(a_{2k})\right\}=sign\left\{m_n'(a_{2k+1})-r_n'(a_{2k+1})\right\},\ k=1,\ldots,l-1$$

that difference preserves sign on intervals from $E\backslash\widetilde{E}_n$.

In particular it follows from $m_n'(a_1)-r_n'(a_1)<0$ that $m_n'(x)-r_n'(x)<0$ for $a_1\leq x\leq x_1$. Analogous consideration is valid for the algebraic fraction $-r_n(x)$. Therefore we obtain inequality $m_n'(x)+r_n'(x)<0$ for $a_1\leq x\leq x_1$, and it gives the estimate $|r_n'(x)|<|m_n'(x)|,\ a_1\leq x\leq x_1$. Analogously other intervals are considered.

In the case $r_n(x)\equiv\varepsilon m_n(x),\ \ |\varepsilon|=1$, the equality sign in (3.41) is attained obviously. $\square$

CHAPTER 4

INVESTIGATION OF THE DERIVATIVE OF EXTREMAL RATIONAL FUNCTIONS

Theorem 3.4.1 shows that it is necessary to investigate $m_n'(x)$ in the case of several intervals.

4.1 INVESTIGATION OF THE DERIVATIVE OF EXTREMAL RATIONAL FUNCTIONS ON SEVERAL INTERVALS AT THE END POINTS

Lemma 4.1.1. *Under suppositions of Theorem 3.3.1 i.e.*

$$m_n^2(x) - H(x)\, v_{n-1}^2(x) = \rho_\nu^2(x) \tag{4.1}$$

where m_n is the cosine fraction

$$m_n(x) = \cos\left(\int_{-1}^{x}\left(\sum_{r=1}^{\nu^*} \nu_r \omega_E(\xi_r, x)\right) dx\right),$$

and $\rho_\nu(x) = \prod_{r=1}^{\nu^*}(x-\xi_r)^{\nu_r}$ is a polynomial of degree ν, $\xi_j \in \mathbb{R}\setminus[a_1, a_{2l}]$, the following formula holds

$$|m_n'(a_m)| = \frac{2}{|-H'(a_m)|}\left(\sum_{r=1}^{\nu^*}\nu_r \frac{\prod_{k=1}^{l-1}(a_m - c_{k,r})\sqrt{H(\xi_r)}}{(a_m-\xi_r)\prod_{k=1}^{l-1}(\xi_r - c_{k,r})}\right)^2$$

where $H(x) = \prod_{j=1}^{2l} |x - a_j|$, $c_k \in (a_{2k}, a_{2k+1})$, $k = 1, \ldots, l-1$ *are uniquely determined by*

$$\int_{a_{2k}}^{a_{2k+1}} \omega_E(\infty, x)\, dx = 0,$$

and $c_{k,j} \in (a_{2k}, a_{2k+1})$, $k = 1, \ldots, l-1$ *are uniquely determined by*

$$\int_{a_{2k}}^{a_{2k+1}} \frac{\prod_{k=1}^{l-1} (x - c_{k,j})}{(x - \xi_j)\sqrt{H(x)}} dx = 0. \tag{4.2}$$

Proof. Recall that (see (Widom, 1969))

$$\omega_E(\infty, x) = \frac{\prod_{k=1}^{l-1} |x - c_k|}{\sqrt{|H(x)|}}, \tag{4.3}$$

and

$$\omega_E(\xi_j, x) = \frac{\prod_{k=1}^{l-1} |x - c_{k,j}| \sqrt{|H(\xi_j)|}}{|x - \xi_j| \sqrt{|H(x)|} \prod_{k=1}^{l-1} |\xi_j - c_{k,j}|}. \tag{4.4}$$

On differentiating $m_n(x)$, we get

$$m_n'(x) = -\sin\left(\int_{-1}^{x} \left(\sum_{r=1}^{\nu^*} \nu_r \omega_E(\xi_r, x)\right) dx\right) \sum_{r=1}^{\nu^*} \nu_r \omega_E(\xi_r, x)$$

so setting $x = a_m$ we obtain

$$m_n'(a_m) = -\lim_{x \to a_m} \left(\frac{\sin\left(\int_{-1}^{x} \left(\sum_{r=1}^{\nu^*} \nu_r \omega_E(\xi_r, x)\right) dx\right)}{\sqrt{-H(x)}}\right) \sum_{r=1}^{\nu^*} \nu_r \frac{\prod_{k=1}^{l-1} (a_m - c_{k,r}) \sqrt{H(\xi_r)}}{(a_m - \xi_r) \prod_{k=1}^{l-1} (\xi_r - c_{k,r})}. \tag{4.5}$$

Let $I = \lim\limits_{x\to a_m}\left(\frac{\sin\left(\int\limits_{-1}^{x}\left(\sum\limits_{r=1}^{\nu^*}\nu_r\omega_E(\xi_r,x)\right)dx\right)}{\sqrt{-H(x)}}\right)$. By L'Hospital rule,

$$\begin{aligned}
I &= \lim_{x\to a_m}\frac{\cos\left(\int\limits_{-1}^{x}\left(\sum\limits_{r=1}^{\nu^*}\nu_r\omega_E\left(\xi_r,x\right)\right)dx\right)\sum\limits_{r=1}^{\nu^*}\nu_r\omega_E\left(\xi_r,x\right)}{\frac{1}{2\sqrt{-H(x)}}\left(-H'(x)\right)}\\
&= \lim_{x\to a_m}\frac{\cos\left(\int\limits_{-1}^{x}\left(\sum\limits_{r=1}^{\nu^*}\nu_r\omega_E\left(\xi_r,x\right)\right)dx\right)\sum\limits_{r=1}^{\nu^*}\nu_r\frac{\prod\limits_{k=1}^{l-1}\left(x-c_{k,r}\right)\sqrt{H(\xi_r)}}{(x-\xi_r)\sqrt{-H(x)}\prod\limits_{k=1}^{l-1}\left(\xi_r-c_{k,r}\right)}}{\frac{1}{2\sqrt{-H(x)}}\left(-H'(x)\right)}\\
&= \lim_{x\to a_m}\frac{2\cos\left(\int\limits_{-1}^{x}\left(\sum\limits_{r=1}^{\nu^*}\nu_r\omega_E\left(\xi_r,x\right)\right)dx\right)}{\left(-H'(x)\right)}\sum_{r=1}^{\nu^*}\nu_r\frac{\prod\limits_{k=1}^{l-1}\left(x-c_{k,r}\right)\sqrt{H(\xi_r)}}{\left(x-\xi_r\right)\prod\limits_{k=1}^{l-1}\left(\xi_r-c_{k,r}\right)}\\
&= \frac{2m_n\left(a_m\right)}{\left(-H'(x)\right)}\sum_{r=1}^{\nu^*}\nu_r\frac{\prod\limits_{k=1}^{l-1}\left(a_m-c_{k,r}\right)\sqrt{H(\xi_r)}}{\left(a_m-\xi_r\right)\prod\limits_{k=1}^{l-1}\left(\xi_r-c_{k,r}\right)}.
\end{aligned}$$

If we take this at (4.5) and making use of the fact that $m_n\left(a_m\right)=\pm 1$ we obtain

$$\begin{aligned}
\left|m_n'\left(a_m\right)\right| &= \frac{2}{\left|-H'(a_m)\right|}\sum_{r=1}^{\nu^*}\nu_r\frac{\prod\limits_{k=1}^{l-1}\left(a_m-c_{k,r}\right)\sqrt{H(\xi_r)}}{\left(a_m-\xi_r\right)\prod\limits_{k=1}^{l-1}\left(\xi_r-c_{k,r}\right)}\sum_{r=1}^{\nu^*}\nu_r\frac{\prod\limits_{k=1}^{l-1}\left(a_m-c_{k,r}\right)\sqrt{H(\xi_r)}}{\left(a_m-\xi_r\right)\prod\limits_{k=1}^{l-1}\left(\xi_r-c_{k,r}\right)}\\
&= \frac{2}{\left|-H'(a_m)\right|}\left(\sum_{r=1}^{\nu^*}\nu_r\frac{\prod\limits_{k=1}^{l-1}\left(a_m-c_{k,r}\right)\sqrt{H(\xi_r)}}{\left(a_m-\xi_r\right)\prod\limits_{k=1}^{l-1}\left(\xi_r-c_{k,r}\right)}\right)^2.
\end{aligned}$$

□

Next result concerns the investigation of m_n for the case of two symmetric intervals $E=[-1,-a]\cup[a,1]$, n is even, and denominator is an even polynomial. In this case the cosine fraction maybe written as

$$m_k(x)=\cos\left(\pi\int\limits_{-1}^{x}\left(\left(2k-\nu\right)\frac{x}{\sqrt{-H(x)}}+\frac{2}{\pi}\sum_{j=1}^{\frac{\nu^*}{2}}\nu_j\sqrt{H(\xi_j)}\left(\frac{x}{\left(x^2-\xi_j^2\right)\sqrt{-H(x)}}\right)\right)dx\right).$$

Lemma 4.1.2. *Under supposition from above*

$$|m'_k(a)| = \frac{2}{|-H'(a)|}\left((2k-\nu)\pi + \sum_{j=1}^{\frac{\nu^*}{2}} \nu_j \frac{2\sqrt{H(\xi_j)}}{(a^2-\xi_j^2)}\right)^2,$$

and

$$|m'_k(1)| = \frac{2}{|-H'(1)|}\left((2k-\nu)\pi + \sum_{j=1}^{\frac{\nu^*}{2}} \nu_j \frac{\sqrt{H(\xi_j)}}{(1-\xi_j^2)}\right)^2.$$

Proof. On differentiating $m_k(x)$, we get

$$\begin{aligned} m'_k(x) &= -\sin\left(\pi\int_{-1}^{x}\left((2k-\nu)\,\omega_E(\infty,x) + \sum_{j=1}^{\frac{\nu^*}{2}} \nu_j\left(\omega_E(\xi_j,x)+\omega_E(-\xi_j,x)\right)\right)dx\right)\times \\ &\quad \times\pi\left((2k-\nu)\,\omega_E(\infty,x) + \sum_{j=1}^{\frac{\nu^*}{2}} \nu_j\left(\omega_E(\xi_j,x)+\omega_E(-\xi_j,x)\right)\right) \end{aligned}$$

setting $x=a$

$$\begin{aligned} m'_k(a) &= \lim_{x\to a}\left(\frac{\sin\left(\pi\int_{-1}^{x}\left((2k-\nu)\,\omega_E(\infty,x) + \sum_{j=1}^{\frac{\nu^*}{2}} \nu_j\left(\omega_E(\xi_j,x)+\omega_E(-\xi_j,x)\right)\right)dx\right)}{\sqrt{-H(x)}}\right)\times \\ &\quad \times(2k-\nu)\,\pi a + \sum_{j=1}^{\frac{\nu^*}{2}} \nu_j \frac{2a\sqrt{H(\xi_j)}}{\left(a^2-\xi_j^2\right)} \end{aligned} \tag{4.6}$$

Let $I = \lim_{x\to a}\left(\frac{\sin\left(\pi\int_{-1}^{x}\left((2k-\nu)\omega_E(\infty,x)+\sum_{j=1}^{\frac{\nu^*}{2}}\nu_j(\omega_E(\xi_j,x)+\omega_E(-\xi_j,x))\right)dx\right)}{\sqrt{-H(x)}}\right)$. By the L'Hospital

Rule

$$I = \lim_{x\to a} \frac{\cos\left(\pi\int\limits_{-1}^{x}\left(\left(2k-\nu\right)\omega_E\left(\infty,x\right)+\sum\limits_{j=1}^{\frac{\nu^*}{2}}\nu_j\left(\omega_E\left(\xi_j,x\right)+\omega_E\left(-\xi_j,x\right)\right)\right)dx\right)}{\frac{1}{2\sqrt{-H(x)}}\left(-H'(x)\right)}\times$$

$$\times\left(\left(2k-\nu\right)\pi\frac{x}{\sqrt{-H(x)}}+\sum_{j=1}^{\frac{\nu^*}{2}}\nu_j\frac{2x\sqrt{H(\xi_j)}}{\left(x^2-\xi_j^2\right)\sqrt{-H(x)}}\right)$$

$$= \lim_{x\to a} \frac{2\cos\left(\pi\int\limits_{-1}^{x}\left(\left(2k-\nu\right)\omega_E\left(\infty,x\right)+\sum\limits_{j=1}^{\frac{\nu^*}{2}}\nu_j\left(\omega_E\left(\xi_j,x\right)+\omega_E\left(-\xi_j,x\right)\right)\right)dx\right)}{\left(-H'(x)\right)}\times$$

$$\times\left(\left(2k-\nu\right)\pi x+\sum_{j=1}^{\frac{\nu^*}{2}}\nu_j\frac{2x\sqrt{H(\xi_j)}}{\left(x^2-\xi_j^2\right)}\right)$$

$$= \frac{2m_k(a)}{\left(-H'(a)\right)}\left(\left(2k-\nu\right)\pi a+\sum_{j=1}^{\frac{\nu^*}{2}}\nu_j\frac{2a\sqrt{H(\xi_j)}}{\left(a^2-\xi_j^2\right)}\right).$$

If we take this at (4.6) and making use of the fact that $m_k(a)$ we get

$$|m_k'(a)| = \frac{2}{|-H'(a)|}\left(\left(2k-\nu\right)\pi a+\sum_{j=1}^{\frac{\nu^*}{2}}\nu_j\frac{2a\sqrt{H(\xi_j)}}{\left(a^2-\xi_j^2\right)}\right)\times$$

$$\times\left(\left(2k-\nu\right)\pi a+\sum_{j=1}^{\frac{\nu^*}{2}}\nu_j\frac{2a\sqrt{H(\xi_j)}}{\left(a^2-\xi_j^2\right)}\right)$$

$$= \frac{2}{|-H'(a)|}\left(\left(2k-\nu\right)\pi+\sum_{j=1}^{\frac{\nu^*}{2}}\nu_j\frac{2\sqrt{H(\xi_j)}}{\left(a^2-\xi_j^2\right)}\right)^2,$$

and similar to the above method, setting $x=1$

$$|m_k'(1)| = \frac{2}{|-H'(1)|}\left(\left(2k-\nu\right)\pi+\sum_{j=1}^{\frac{\nu^*}{2}}\nu_j\frac{\sqrt{H(\xi_j)}}{\left(1-\xi_j^2\right)}\right)^2.$$

□

4.2 MONOTONICITY PROPERTIES OF HARMONIC MEASURES

This section is devoted to monotonicity properties of densities harmonic measures from (4.3), (4.4).

Lemma 4.2.1. *There exist $R \geq 1$ and $\delta > 0$ such that for all $\xi_j \in \mathbb{R},\ |\xi_j| \geq R > 1$, any of $\omega_E(\xi_j, x),\ j = 1, \ldots, \nu^*$ is monotonic on each interval*

$$[a_{2k+1}, a_{2k+1} + \delta], [a_{2k} - \delta, a_{2k}], k = 1, \ldots, l.$$

Proof. $c_{k,j}$ continuously depend on ξ_j and $c_{k,j} \in (a_{2k}, a_{2k+1})$,

$$\lim_{\xi_j \to \infty} c_{k,j} = c_k.$$

Namely,

$$\begin{aligned} \prod_{k=1}^{l-1} (x - c_{k,j}) &= x^{l-1} + \gamma_{l-2,j} x^{l-2} + \ldots + \gamma_{1,j} x + \gamma_{0,j}, \\ \gamma_{k-1,j} &= (-1)^{k-1} s_{k-1}(c_{1,j}, \ldots, c_{l-1,j}),\ k = 1, \ldots, l-1 \end{aligned}$$

where s_{k-1} is the $k-1$ th symmetric polynomial in $l-1$ variables.

If we write in (4.2) we get

$$\int_{a_{2k}}^{a_{2k+1}} \frac{x^{l-1} + \gamma_{l-2,j} x^{l-2} + \ldots + \gamma_{0,j}}{(x - \xi_j)\sqrt{H(x)}} dx = 0,\ k = 1, \ldots, l-1.$$

We can rewrite

$$\begin{aligned} \int_{a_{2k}}^{a_{2k+1}} \frac{\gamma_m x^m}{\left(x - \xi_{\widehat{j}}\right)\sqrt{H(x)}} dx &= 0,\ k = 1, \ldots, l-1,\ m = 0, \ldots, l-2, \\ \int_{a_{2k}}^{a_{2k+1}} \frac{\sum_{m=0}^{l-2} \gamma_m x^m}{\left(x - \xi_{\widehat{j}}\right)\sqrt{H(x)}} dx &= 0. \end{aligned}$$

Here $\left(x - \xi_{\widehat{j}}\right)\sqrt{H(x)}$ sign is constant on $[a_{2k}, a_{2k+1}]$ and let $\overset{\#}{p}(x) = \sum_{m=0}^{l-2} \gamma_m x^m$. There exists a zero of $\overset{\#}{p}(x)$ on $[a_{2k}, a_{2k+1}]$, $k = 1, \ldots, l-1$.

Consider $U_k = \{|z - c_k| < \varepsilon\}$ is an open subset with smooth boundary ∂U_k,

$$N_{U_k}(P_j) = \frac{1}{2\pi i} \int_{\partial U_k} \frac{P'_{\widehat{j}}(z)}{P_{\widehat{j}}(z)} dz,$$

where $P_{\widehat{j}}(x) = x^{l-1} + \gamma_{l-2,\widehat{j}} x^{l-2} + \ldots + \gamma_{0,\widehat{j}}$. When it is defined, $N_{U_k}(P_{\widehat{j}})$ is the number of zeros $P_{\widehat{j}}$ in U_k counted with multiplicity. Also here

$$\frac{P'_{\widehat{j}}(z)}{P_{\widehat{j}}(z)} = \frac{(l-1)z^{l-2} + \gamma_{l-2,\widehat{j}}(l-2)z^{l-3} + \ldots + \gamma_{2,\widehat{j}}}{z^{l-1} + \gamma_{l-2,\widehat{j}} z^{l-2} + \ldots + \gamma_{1,\widehat{j}} z + \gamma_{0,\widehat{j}}}$$

as $\xi_j \to \infty$,

$$N_{U_k}(P_j) \longrightarrow \frac{1}{2\pi i} \int\limits_{\partial U_k} \frac{P'_\infty(z)}{P_\infty(z)} dz = 1.$$

It means that P_j has precise one zero at each of U_k for and we have sufficiently large ξ_j's. So

$$\lim_{\xi_j \to \infty} c_{k,j} = c_k.$$

Hence

$$\begin{aligned}
\lim_{\xi_j \to \infty} \omega_E(\xi_j, x) &= \lim_{\xi_j \to \infty} \frac{\prod\limits_{k=1}^{l-1} |x - c_{k,j}| \sqrt{|H(\xi_j)|}}{|x - \xi_j| \sqrt{|H(x)|} \prod\limits_{k=1}^{l-1} |\xi_j - c_{k,j}|} \\
&= \lim_{\xi_j \to \infty} \frac{\prod\limits_{k=1}^{l-1} |x - c_{k,j}| \sqrt{\left|\frac{1}{\xi_j^{2l}} H(\xi_j)\right|}}{\left|\frac{x}{\xi_j} - 1\right| \sqrt{|H(x)|} \prod\limits_{k=1}^{l-1} \left|1 - \frac{c_{k,j}}{\xi_j}\right|} \\
&= \frac{\prod\limits_{k=1}^{l-1} |x - c_k|}{\sqrt{|H(x)|}} \\
&= \omega_E(\infty, x).
\end{aligned}$$

Thus

$$\omega_E(\xi_j, x) \longrightarrow \omega_E(\infty, x), \text{ as } \xi_j \to \infty$$

uniformly with respect to x on compacts the interior of E.

Monotonicity properties of $\omega_E(\infty, x)$ and of $\omega_E^2(\infty, x)$ coincide. Put $R(x) = \frac{\prod_{k=1}^{l-1} (x - c_k)^2}{H(x)}$. Then because of $|R(x)| = \omega_E^2(\infty, x)$, $x \in E$ monotonicity properties of $\omega_E^2(\infty, x)$ and $R(x)$ coincide up to the sense. Hence there exists a $\delta > 0$ such that $\omega_E(\infty, x)$ is monotonic on each interval $[a_{2k+1}, a_{2k+1} + \delta]$, $[a_{2k} - \delta, a_{2k}]$, $k = 1, \ldots, l$.

Consider $\left(\omega_E^2\left(\xi_j,x\right)\right)'$. It equals to

$$\left(\omega_E^2\left(\xi_j,x\right)\right)' = H\left(\xi_j\right)\left[\frac{\left(2\sum_{k=1}^{l-1}\prod_{\substack{i=1\\k\neq i}}^{l-1}\left(x-c_i\left(\xi_j\right)\right)-\frac{\prod_{k=1}^{l-1}\left(x-c_k\left(\xi_j\right)\right)^2}{\left(x-\xi_j\right)}\right)\left(-H\left(x\right)\right)}{\left(x-\xi_j\right)^2\left(-H\left(x\right)\right)^2\prod_{k=1}^{l-1}\left(\xi_j-c_k\left(\xi_j\right)\right)^2}+\right.$$
$$\left.+\frac{\left[H'\left(x\right)\prod_{k=1}^{l-1}\left(x-c_k\left(\xi_j\right)\right)^2\right]}{\left(x-\xi_j\right)^2\left(-H\left(x\right)\right)^2\prod_{k=1}^{l-1}\left(\xi_j-c_k\left(\xi_j\right)\right)^2}\right].$$

Now $\left(-H\left(x\right)\right)^2\left(\omega_E^2\left(\xi_j,x\right)\right)'$ tends to $\left(-H\left(x\right)\right)^2\left(\omega_E^2\left(\infty,x\right)\right)'$ uniformly on E as $\xi_j\to\infty$. Since $\left(\omega_E^2\left(\infty,x\right)\right)'$ has no zeros on $\left[a_{2k+1},a_{2k+1}+\delta\right]$, $\left[a_{2k}-\delta,a_{2k}\right]$ then $\left(\omega_E^2\left(\xi_j,x\right)\right)'$ has no zeros on $\left[a_{2k+1},a_{2k+1}+\delta\right]$, $\left[a_{2k}-\delta,a_{2k}\right]$ as well for $\xi_j>R$.

□

Corollary 4.2.2. *$\varphi_n'\left(x\right)$ is monotonic on $\left[a_k,x_p\right]$ where δ_1,δ_2 are given*

$$\delta_1 = \min\left(\min_{1\leq j\leq\frac{\nu^*}{2}}\left(\sqrt{t_{2,j}},\sqrt{a}\right)\right),$$
$$\delta_2 = \max\left(\max_{1\leq j\leq\frac{\nu^*}{2}}\left(\sqrt{t_{2,j}},\sqrt{a}\right)\right),$$

and

$$t_{2,j} = \frac{\left(2a^2+\xi_j^2+2\right)}{9}-\frac{\left(\sqrt[3]{R_j+\sqrt{Q_j^3+R_j^2}}+\sqrt[3]{R_j-\sqrt{Q_j^3+R_j^2}}\right)}{2}-$$
$$-i\frac{\sqrt{3}}{2}\left(\sqrt[3]{R_j+\sqrt{Q_j^3+R_j^2}}-\sqrt[3]{R_j-\sqrt{Q_j^3+R_j^2}}\right),$$
$$\varphi_n\left(x\right) = \int_{-1}^{x}\left(\sum_{r=1}^{\nu^*}\nu_r\omega_E\left(\xi_r,x\right)\right)dx.$$

Lemma 4.2.3. *If n is sufficiently large and $|\xi_j|\geq R>1$, then*

$$|x_p-a_{2k+1}|<\delta.$$

where x_p is one of zeros of the cosine fraction from Theorem 3.4.1.

Proof. Suppose $a_{2k+1} = y_m$, then $\varphi_n(a_{2k+1}) = \pi(m-1)$, and let x_p be the closest to y_m on E, then

$$\begin{aligned}\varphi_n(x_p) &= \pi(m-1) + \frac{\pi}{2},\\ \varphi_n(x_p) - \varphi_n(a_{2k+1}) &= \frac{\pi}{2}\\ &= \varphi_n'(\xi)(x_p - a_{2k+1}),\end{aligned}$$

$$\begin{aligned}|x_p - a_{2k+1}| &= \frac{\pi}{2\varphi_n'(\xi)} \le \frac{\pi}{2\varphi_n'(x_p)}\\ &= \frac{\pi}{2\sum \omega_E(\xi_j, x_p)}\\ &= \frac{\pi\sqrt{H(x_p)}}{2\sum \nu_j P(x_p, \xi_j)}.\end{aligned}$$

Here

$$P(x_p, \xi_j) = \frac{\prod_{k=1}^{l-1} |x_p - c_k(\xi_j)| \sqrt{|H(\xi_j)|}}{|x_p - \xi_j| \prod_{k=1}^{l-1} |\xi_j - c_k(\xi_j)|},$$

and we have

$$\begin{aligned}\lim_{\xi_j\to\infty} P(x_p, \xi_j) &= \prod_{k=1}^{l-1} |x_p - c_k| = P(x_p, \infty)\\ \lim_{\xi_j\to\infty} \prod_{k=1}^{l-1} (x - c_k(\xi_j)) &= \prod_{k=1}^{l-1} (x - c_k)\end{aligned}$$

uniformly on E. Hence for every ε and every $x \in E$ if $|\xi_j| \ge R > 1$, then

$$|P(x, \xi_j) - P(x, \infty)| < \varepsilon,$$

and

$$\begin{aligned}|x_p - a_{2k+1}| &\le \frac{\pi\sqrt{H(x_p)}}{2(nP(x_p, \infty) - \sum \nu_j \varepsilon)}\\ &= \frac{\pi\sqrt{H(x_p)}}{2n(P(x_p, \infty) - \varepsilon)}.\end{aligned}$$

If we let $\varepsilon = \min\limits_{x\in E} \frac{P(x_p,\infty)}{2}$, we get

$$|x_p - a_{2k+1}| \le \frac{\pi\sqrt{\|H\|_E} \left\| \frac{1}{P(\cdot,\infty)} \right\|_E}{n},$$

and $\delta > 0,\ n \ge n(\delta, E) = \frac{\pi\sqrt{\|H\|_E} \left\| \frac{1}{P(\cdot,\infty)} \right\|_E}{\delta}$ such that

$$|x_p - a_{2k+1}| < \delta.$$

□

Next we note that the densities of the harmonic measures of the domain $\mathbb{C}\backslash E$, $E = [a_1, a_2] \cup [a_3, a_4]\,, -1 = a_1 < a_2 = -a < a_3 = a < a_4 = 1$, with respect to poles $\xi_j \in \bar{\mathbb{R}}\backslash[a_1, a_4]$, are given by

$$\begin{aligned} \omega_E(\infty, x) &= \frac{x}{\sqrt{-H(x)}}, \qquad (4.7) \\ \omega_E(\xi_j, x) &= \frac{1}{\pi}\frac{(x-c_j)}{(x-\xi_j)\sqrt{-H(x)}}\frac{\sqrt{-H(\xi_j)}}{(\xi_j-c_j)}, \\ \omega_k(z) &= \int\limits_{a_{2k-1}}^{a_{2k}} \omega_E(z,x)dx, \quad k=1,2, \end{aligned}$$

where $H(x) = \prod\limits_{j=1}^{4}(x-a_j)$, $c_j \in (a_2, a_3)$ are uniquely determined by

$$\int\limits_{a_2}^{a_3} \frac{(x-c_j)}{(x-\xi_j)\sqrt{-H(x)}}dx = 0.$$

Lemma 4.2.4. *For all* $\xi_j > 1$, *any of* $(\omega_E(\xi_j, x) + \omega_E(-\xi_j, x))\,,\ j = 1, \dots, \nu^*$ *is monotonic on each interval*

$$\left[-1, -\sqrt{t_{2,j}}\right], \left[-\sqrt{t_{2,j}}, -a\right], \left[a, \sqrt{t_{2,j}}\right], \left[\sqrt{t_{2,j}}, 1\right].$$

Here

$$\begin{aligned} t_{2,j} = \; & \frac{(2a^2+\xi_j^2+2)}{9} - \frac{\left(\sqrt[3]{R_j+\sqrt{Q_j^3+R_j^2}}+\sqrt[3]{R_j-\sqrt{Q_j^3+R_j^2}}\right)}{2} - \\ & -i\frac{\sqrt{3}}{2}\left(\sqrt[3]{R_j+\sqrt{Q_j^3+R_j^2}}-\sqrt[3]{R_j-\sqrt{Q_j^3+R_j^2}}\right) \end{aligned}$$

where

$$\begin{aligned} Q_j &= \frac{\left(2a^2+\xi_j^2+2\right)^2-9a^2}{3^4}, \\ R_j &= \frac{1}{2.3^6}\left(-16a^6-24a^4\xi_j^2+6a^4-12a^2\xi_j^4+222a^2\xi_j^2+6a^2-2\xi_j^6-12\xi_j^4-24\xi_j^2-16\right). \end{aligned}$$

Proof. First, we have function

$$\begin{aligned}\left(\omega_E\left(\xi_j,x\right)+\omega_E\left(-\xi_j,x\right)\right) &= \left(\frac{1}{\pi}\frac{\sqrt{H(\xi_j)}}{\left(\xi_j-c_j\right)\sqrt{-H(x)}}\times\right. \\ &\quad \left.\times\left(\frac{\left(x-c_j\right)\left(x+\xi_j\right)-\left(x+c_j\right)\left(x-\xi_j\right)}{x^2-\xi_j^2}\right)\right) \\ &= \frac{2\sqrt{H(\xi_j)}}{\pi}\frac{x}{\left(x^2-\xi_j^2\right)\sqrt{-H(x)}}.\end{aligned} \tag{4.8}$$

Here

$$\begin{aligned}\omega_E\left(\xi_j,x\right) &= \frac{1}{\pi}\frac{x-c_j}{\left(x-\xi_j\right)\sqrt{-H(x)}}\frac{\sqrt{H(\xi_j)}}{\left(\xi_j-c_j\right)}, \\ \omega_E\left(-\xi_j,x\right) &= -\frac{1}{\pi}\frac{x+c_j}{\left(x+\xi_j\right)\sqrt{-H(x)}}\frac{\sqrt{H(\xi_j)}}{\left(\xi_j-c_j\right)}.\end{aligned}$$

We want to show monotonicity of (4.8). The following figure illustrates Figure 4.1.

Consider $\left(\left(\omega_E\left(\xi_j,x\right)+\omega_E\left(-\xi_j,x\right)\right)^2\right)'$.

$$\left(\omega_E\left(\xi_j,x\right)+\omega_E\left(-\xi_j,x\right)\right)^2=\frac{4H(\xi_j)}{\pi^2}\frac{x^2}{\left(x^2-\xi_j^2\right)^2H(x)}.$$

Put

$$f_j\left(t\right)=\frac{t}{\left(t-\xi_j^2\right)^2(t-1)(t-a^2)}.$$

The graph of function $f_j\left(t\right)$ is shown below in Figure 4.2.

If we take the derivative of $f_j\left(t\right)$

$$\begin{aligned}f_j'\left(t\right) &= \frac{\left(t-\xi_j^2\right)^2(t-1)(t-a^2)-2t\left(t-\xi_j^2\right)(t-1)(t-a^2)-t\left(t-\xi_j^2\right)^2(t-a^2)}{\left(t-\xi_j^2\right)^4(t-1)^2(t-a^2)^2}- \\ &\quad -\frac{t\left(t-\xi_j^2\right)^2(t-1)}{\left(t-\xi_j^2\right)^4(t-1)^2(t-a^2)^2}.\end{aligned}$$

Let us determine all zeros of function

$$\left(t-\xi_j^2\right)\left(\left(t-\xi_j^2\right)(t-1)(t-a^2)-2t(t-1)(t-a^2)-t\left(t-\xi_j^2\right)(t-a^2)-t\left(t-\xi_j^2\right)(t-1)\right)=0.$$

If we arrange

$$\begin{aligned}-3t^3+\left(2a^2+\xi_j^2+2\right)t^2-a^2t-a^2\xi_j^2 &= 0, \\ t^3-\frac{\left(2a^2+\xi_j^2+2\right)}{3}t^2+\frac{a^2t}{3}+\frac{a^2\xi_j^2}{3} &= 0.\end{aligned}$$

This form is general cubic equation which is $\alpha t^3 + bt^2 + ct + d = 0,\ \alpha, b, c, d \in \mathbb{R}$. Substituting

$$t = y - \frac{b}{3\alpha} = y + \frac{\left(2a^2 + \xi_j^2 + 2\right)}{9},$$

we get

$$y^3 + \left(\tfrac{9a^2 - \left(2a^2+\xi_j^2+2\right)^2}{27}\right) y + \tfrac{1}{3^6}\left(-16a^6 - 24a^4\xi_j^2 + 6a^4 - 12a^2\xi_j^4 + 222a^2\xi_j^2 + \right.$$
$$\left. + 6a^2 - 2\xi_j^6 - 12\xi_j^4 - 24\xi_j^2 - 16\right) = 0.$$

Now, we have the equation

$$y^3 + py + q = 0 \tag{4.9}$$

where

$$p = \frac{9a^2 - \left(2a^2 + \xi_j^2 + 2\right)^2}{27} = -3Q_j$$

and

$$2R_j = q = \frac{1}{3^6}\left(-16a^6 - 24a^4\xi_j^2 + 6a^4 - 12a^2\xi_j^4 + 222a^2\xi_j^2 + 6a^2 - 2\xi_j^6 - 12\xi_j^4 - 24\xi_j^2 - 16\right).$$

We introduce two variables u and v linked by the condition

$$u + v = y,\ \text{where } uv = -Q_j$$

and substitute this in the depressed cubic (4.9), giving

$$\begin{aligned} (u+v)^3 + 3Q_j(u+v) - 2R_j &= 0, \\ u^3 + 3u^2v + 3uv^2 + v^3 + 3Q_j(u+v) - 2R_j &= 0, \\ u^3 + v^3 - 2R_j &= 0 \text{ as } uv = -Q_j, \\ u^3 - \frac{Q_j^3}{u^3} - 2R_j &= 0, \end{aligned}$$

and we get

$$\left(u^3\right)^2 - 2R_ju^3 - Q_j^3 = 0.$$

This is quadratic in u^3 :

$$\begin{aligned} u^3 &= \frac{2R_j \pm \sqrt{4Q_j^3 + 4R_j^2}}{2} \\ &= R_j \pm \sqrt{Q_j^3 + R_j^2}. \end{aligned}$$

We have from above $uv = -Q_j$ and hence $v^3 = -\frac{Q_j^3}{v^3}$. Let us take the principal branch of the root which is $u^3 = R_j + \sqrt{Q_j^3 + R_j^2}$. Then

$$\begin{aligned} v^3 &= -\frac{Q_j^3}{R_j + \sqrt{Q_j^3 + R_j^2}} \\ &= \frac{-Q_j^3\left(R_j - \sqrt{Q_j^3 + R_j^2}\right)}{R_j^2 - (Q_j^3 + R_j^2)} \\ &= R_j - \sqrt{Q_j^3 + R_j^2}. \end{aligned}$$

Let $S_j = \sqrt[3]{R_j + \sqrt{Q_j^3 + R_j^2}}$, $T_j = \sqrt[3]{R_j - \sqrt{Q_j^3 + R_j^2}}$, where principal branch of cubic roots is considered. Then , we have the three cube roots of u^3 and v^3 :

$$u = \begin{cases} S_j, \\ \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) S_j, \\ \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) S_j, \end{cases}$$

and

$$v = \begin{cases} T_j, \\ \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) T_j, \\ \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) T_j. \end{cases}$$

Next, we must find the cube roots of these equations to solve for u and v

* if $Q_j^3 + R_j^2 < 0$, the roots will be real numbers,

* if $Q_j^3 + R_j^2 > 0$, then one root is real and two are complex conjugates,

* is $Q_j^3 + R_j^2 = 0$, $t_{2,j} = t_{3,j}$, and $t_{1,j}$ are real numbers.

Obviously (compare picture) $Q_j^3 + R_j^2 < 0$. Then $\sqrt{Q_j^3 + R_j^2} = i\left|Q_j^3 + R_j^2\right| = iE_j$, where $E_j > 0$. Thus $S_j^3 = R_j + iE_j$, $T_j^3 = R_j - iE_j$.

Let $\sqrt[3]{R_j + iE_j} = m + in$, and so $\sqrt[3]{R_j - iE_j} = m - in$. Hence

$$S_j + T_j = 2m,\ S_j - T_j = 2in.$$

So

$$\begin{aligned} y_{1,j} &= -\frac{(S_j + T_j)}{2} + i\frac{\sqrt{3}}{2}(S_j - T_j) = -m - \sqrt{3}n, \\ y_{2,j} &= -\frac{(S_j + T_j)}{2} - i\frac{\sqrt{3}}{2}(S_j - T_j) = -m + \sqrt{3}n, \\ y_{3,j} &= S_j + T_j = 2m. \end{aligned}$$

If we put these root in $t = y - \frac{b}{3\alpha}$ then we get

$$
\begin{aligned}
t_{1,j} &= -m - \sqrt{3}n - \frac{b}{3\alpha} \\
&= -\frac{(S_j + T_j)}{2} + i\frac{\sqrt{3}}{2}(S_j - T_j) + \frac{\left(2a^2 + \xi_j^2 + 2\right)}{9} \\
&= -\frac{\left(\sqrt[3]{R_j + \sqrt{Q_j^3 + R_j^2}} + \sqrt[3]{R_j - \sqrt{Q_j^3 + R_j^2}}\right)}{2} + \\
&\quad + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{R_j + \sqrt{Q_j^3 + R_j^2}} - \sqrt[3]{R_j - \sqrt{Q_j^3 + R_j^2}}\right) + \frac{\left(2a^2 + \xi_j^2 + 2\right)}{9},
\end{aligned}
$$

$$
\begin{aligned}
t_{2,j} &= -m + \sqrt{3}n - \frac{b}{3\alpha} \\
&= -\frac{(S_j + T_j)}{2} - i\frac{\sqrt{3}}{2}(S_j - T_j) + \frac{\left(2a^2 + \xi_j^2 + 2\right)}{9} \\
&= -\frac{\left(\sqrt[3]{R_j + \sqrt{Q_j^3 + R_j^2}} + \sqrt[3]{R_j - \sqrt{Q_j^3 + R_j^2}}\right)}{2} - \\
&\quad - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{R_j + \sqrt{Q_j^3 + R_j^2}} - \sqrt[3]{R_j - \sqrt{Q_j^3 + R_j^2}}\right) + \frac{\left(2a^2 + \xi_j^2 + 2\right)}{9},
\end{aligned}
$$

$$
\begin{aligned}
t_{3,j} &= 2m - \frac{b}{3\alpha} \\
&= S + T + \frac{\left(2a^2 + \xi_j^2 + 2\right)}{9} \\
&= \sqrt[3]{R_j + \sqrt{Q_j^3 + R_j^2}} + \sqrt[3]{R_j - \sqrt{Q_j^3 + R_j^2}} + \frac{\left(2a^2 + \xi_j^2 + 2\right)}{9}.
\end{aligned}
$$

Here

$$t_{3,j} > 1,$$

$$
\begin{aligned}
t_{1,j} &= \frac{\left(2a^2 + \xi_j^2 + 2\right)}{9} - \frac{\left(R_j + \sqrt{Q_j^3 + R_j^2}\right)^{\frac{1}{3}} + \left(R_j - \sqrt{Q_j^3 + R_j^2}\right)^{\frac{1}{3}}}{2} + \\
&\quad + i\frac{\sqrt{3}}{2}\left(\left(R_j + \sqrt{Q_j^3 + R_j^2}\right)^{\frac{1}{3}} - \left(R_j - \sqrt{Q_j^3 + R_j^2}\right)^{\frac{1}{3}}\right) < 0,
\end{aligned}
$$

and

$$
\begin{aligned}
t_{2,j} &= -\frac{\left(R_j + \sqrt{Q_j^3 + R_j^2}\right)^{\frac{1}{3}} + \left(R_j - \sqrt{Q_j^3 + R_j^2}\right)^{\frac{1}{3}}}{2} - \\
&\quad - i\frac{\sqrt{3}}{2}\left(\left(R_j + \sqrt{Q_j^3 + R_j^2}\right)^{\frac{1}{3}} - \left(R_j - \sqrt{Q_j^3 + R_j^2}\right)^{\frac{1}{3}}\right) \\
&\quad < t_{3,j}.
\end{aligned}
$$

Thus argument of $R + iE$ belongs to $[0, \pi]$, hence $\arg(m + in) \in \left[0, \frac{\pi}{3}\right]$

$$\frac{n}{m} < \sqrt{3}$$

if we edit

$$\begin{aligned}\sqrt{3}n &< 3m,\\ -m + \sqrt{3}n &< 2m\end{aligned}$$

it is obviously

$$t_{1,j} < t_{2,j} < t_{3,j}.$$

$t_{2,j}$ is unique point of $f_j(t)$ on $[-1, -a] \cup [a, 1]$. In this expression, shape we can show in Figure 4.3. □

Corollary 4.2.5. *$\varphi_n'(x)$ is monotonic on $[a, \delta_1]$, $[\delta_2, 1]$,*

$$\varphi_n(x) = \pi \int_{-1}^{x} \left((2k - \nu)\, \omega_E(\infty, x) + \sum_{j=1}^{\frac{\nu^*}{2}} \nu_j \left(\omega_E(\xi_j, x) + \omega_E(-\xi_j, x) \right) \right) dx.$$

Proof. Firstly, we have

$$\omega_E^2(\infty, x) = \frac{x^2}{-H(x)\pi^2}.$$

If we take the derivative of this function

$$\begin{aligned}\left(\omega_E^2(\infty, x)\right)' &= \frac{2x\,(-H(x)) + x^2 H'(x)}{\pi^2 H^2(x)}\\ &= \frac{-2x\,(x^2-1)\,(x^2-a^2) + x^2\,(2x\,(x^2-a^2) + 2x\,(x^2-1))}{\pi^2 H^2(x)}\\ &= \frac{2x}{\pi^2 H^2(x)} \left(x^2\left(x^2-a^2\right) + x^2\left(x^2-1\right) - \left(x^2-1\right)\left(x^2-a^2\right)\right)\\ &= \frac{2x}{\pi^2 H^2(x)} \left(x^4 - a^2\right).\end{aligned}$$

Let us determine all zeros of function

$$2x\left(x^4 - a^2\right) = 0.$$

Then we get

$$x_{1,2} = \pm\sqrt{a},\ \ x_{3,4} = \pm i\sqrt{a}.$$

Hence using Lemma 4.2.4 we finish the proof.

□

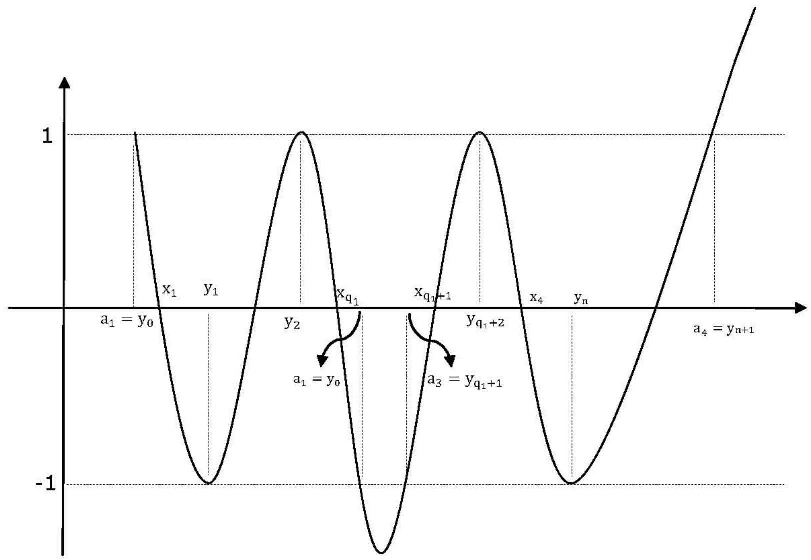

Figure 4.1

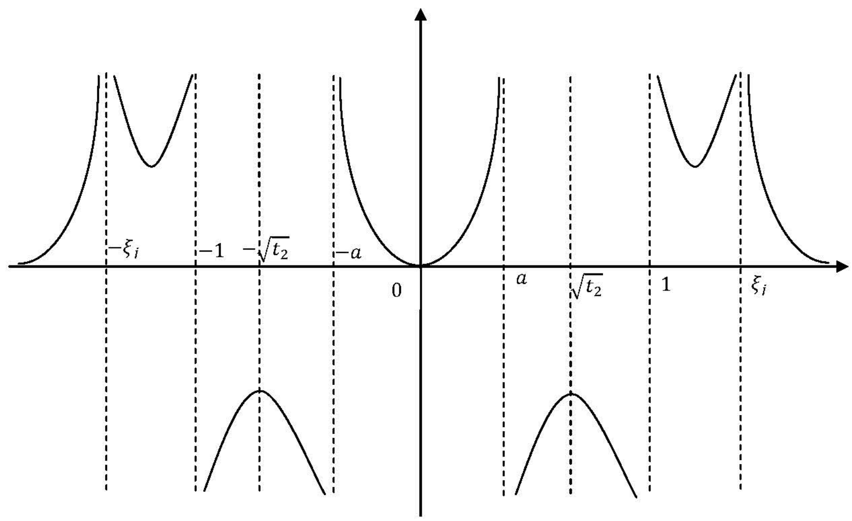

Figure 4.2

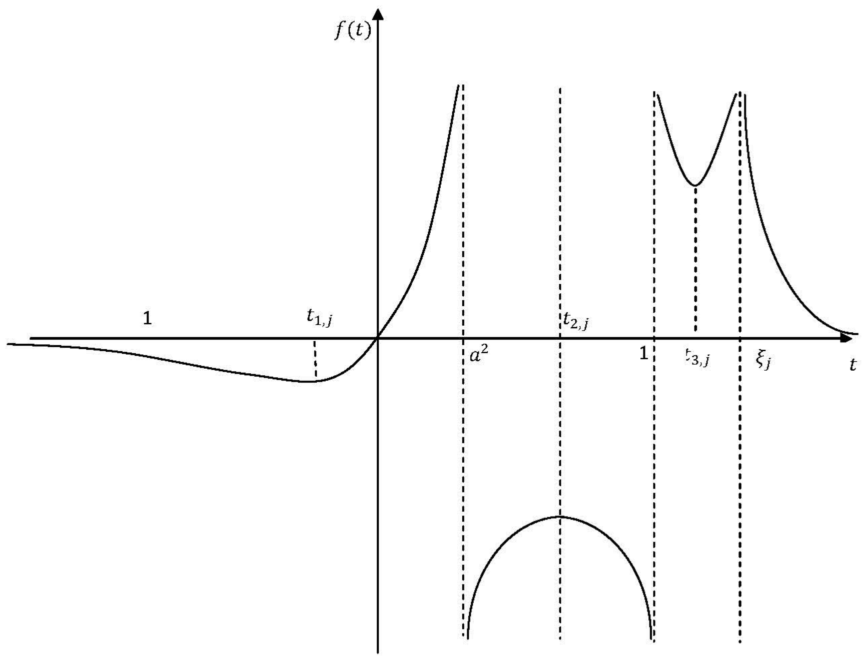

Figure 4.3

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APPENDIX A

DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS, FURTHER STUDIES AND PUBLICATIONS FROM THESIS WORK

A.1 DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS

I hereby declare that this thesis comprises my original work. No material in this thesis has been previously published and written by another person, except where due reference is made in the text of the thesis. I further declare that this thesis contains no material which has been submitted for a degree or diploma or other qualifications at any other university.

Signature:

Date: February 15, 2014

A.2 FURTHER STUDIES

A.3 PUBLICATIONS FROM THESIS WORK

1. Mehmet Ali Aktürk, and Alexey Lukashov, "Weighted Markov Inequality on Several Intervals", AIP Conference Proceedings, ICNAAM 2012, Vol. 1479(1), pp. 568-569, Sep. 2012.

2. Alexey Lukashov, and Mehmet Ali Aktürk, "Remez Type Inequality for Trigonometric Polynomials on An Interval", AIP Conference Proceedings, ICNAAM 2012, Vol. 1470, pp. 42, Octo. 2012.

3. Mehmet Ali Aktürk, and Alexey Lukashov, "Weighted analogues of Bernstein-Type Inequalities on Several Intervals" Journal of Inequalities and Applications, 2013, 487, 2013.

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- Mehmet Ali Aktürk and Alexey Lukashov, ”O neravenstve Remeza dlia trigonometricheskikh polinomov (On Remez Inequality for Trigonometric Polynomials)”, Modern Problems of Function Theory and Their Applications (Materials of 16th Saratov Winter School), Saratov, Russia, 27 Jan.- 3 Feb., 2012.
- Mehmet Ali Aktürk and Alexey Lukashov, ”Weighted Bernstein Inequality On Several Intervals”, International Conference on Applied Analysis and Algebra (ICAAA 2012), Istanbul, Turkey, 20-24 June, 2012.
- Mehmet Ali Aktürk and Alexey Lukashov, ”Weighted Markov Inequality On Several Intervals”, International Conference of Numerical Analysis and Aplied Mathematics (ICNAAM 2012), Kos, Greece, 19-25 Sep., 2012.
- Alexey Lukashov and Mehmet Ali Aktürk, ”Remez Type Inequality for Trigonometric Polynomials On An Intervals”, the First International Conference on Analysis and Applied Mathematics (ICAAM 2012), Gumushane, Turkey, 18-21 Oct., 2012.
- Mehmet Ali Aktürk and Alexey Lukashov, ”Inequality for Derivatives of Rational Functions on Several Intervals”, Algerian Turkish International Days on Mathematics (ATIM 2013), Istanbul, Turkey, 12-14 Sep., 2013.