ON A WELFARE ECONOMICS PROBLEM INVOLVING PUBLIC INTEREST GOODS

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ABSTRACT

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Governments worldwide devise policies to assist the public in adapting to a new technology or good that is known to be beneficial to society or the environment. A well-known example of such a good is electric vehicles. The government uses subsidy policies aimed at both customers and manufacturers to incentivize the consumption and production of such so-called public-interest goods. This thesis studies how the government aims to increase social welfare and can set its subsidy policy to consumers and manufacturers to promote a specific public-interest good. Part of the government's subsidy policy is to increase the supply of such a good by giving subsidies to the manufacturer, and another part is to give rebates or tax reductions to the consumers to increase the demand. In this thesis, we analyze this problem in both centralized and decentralized economies. In a centralized economy, the government, next to determining its own subsidy policy, has complete control over the market's supply side and dictates the manufacturer the price and production quantity of the public interest good. This problem can be formulated as a nonlinear optimization problem consisting of the decision variables production quantity and price demanded by the manufacturer, subsidy for each produced product, and rebate to be offered to each buying customer. In a decentralized economy, the government has only control over its subsidy policy and cannot dictate to the manufacturer the supply and price of the product. As such, the government only knows that for any announced subsidy policy, the manufacturer taking into account this subsidy policy

will determine the price and production to increase its profit. In this thesis, this problem is formulated as a so-called bilevel optimization problem with the same decision variables as used before. Such a formulation is called a Stackelberg game, where the manufacturer is the follower and decides over price and quantity, and the government is the leader and decides over rebate and subsidy. To keep the analysis tractable we only consider in this thesis the deterministic demand case where the demand is given by some demand function. The more complicated case of random demand is not considered in this thesis. The purpose of this research is to provide, under the most general conditions on the demand side, an almost analytical solution to the above problem in both a centralized and decentralized economy. In our first formulation of the problem in both a centralized and decentralized economy, the objective consists of adding the consumer surplus and profit of the manufacturer and externality effects for the economy and subtracting the total expenditures of the government spend on its subsidy policy. In our second formulation, we consider in both economy settings the objective from which we delete from the previous one the expenditures of the government but now include these expenditures as a budget restriction. In the first formulation, we analyze the problem only assuming that the demand function is continuous. In the second formulation, including the budget constraint, we analyze this problem for a very general class of demand functions covering a subset of the class of log-concave demand functions. Finally, these results are refined for some well-known demand functions like to power demand function, the linear demand function, the log-linear demand function, and the logit demand function.

ÖZET

TOPLUMA YARARLI ÜRÜNLERE DAİR BİR REFAH EKONOMİSİ PROBLEMİ ÜZERİNE

SINA SHAHRI MAJARSHIN

ENDÜSTRİ MÜHENDİSLİĞİ YÜKSEK LİSANS TEZİ, ARALIK 2022

Tez Danışmanı: Prof. Hans Frenk

Tez Eş Danışmanı: Doç. Semih Onur Sezer

Anahtar Kelimeler: İki seviyeli optimizasyon, doğrusal olmayan optimizasyon, devlet sübvansiyonu

Tüm dünyada devletler, topluma ve çevreye faydalı olan yeni bir teknoloji veya ürünün benimsenmesine yardımcı olmak için politikaları bir araç olarak kullanır. Elektrikli araçlar böyle bir ürün için iyi bir örnektir. Toplumsal faydaya sahip ürünlerin tüketimini ve üretimini teşvik etmek için devlet, üretici ve tüketicilere yönelik sübvansiyon politikaları kullanır. Bu tez, sosyal refahı arttırmaya çalışan bir devletin, tüketici ve üreticiye yönelik sübvansiyon politikalarını, toplumsal faydaya sahip belirli bir ürünü öne çıkarmak hedefiyle nasıl oluşturabileceği üzerine bir çalışmadır. Sübvansiyon politikalarının bir yönü, üreticilere yönelik sübvansiyon sağlayarak böyle bir ürünün arzını arttırmaktır, diğer bir yönü ise vergilerde kesinti veya geri ödeme yöntemi ile tüketicilerin talebini arttırmaktır. Bu tezde, merkeziyetçi ve merkeziyetçi olmayan ekonomiler için bu problem analiz edilmektedir. Merkeziyetçi bir ekonomide yönetim, sübvansiyon politikasını belirlemenin yanında pazarın arz tarafında mutlak bir kontrole sahiptir ve toplumsal faydaya sahip bir ürünün üretim miktarını ve fiyatını da belirler. Bu durum, üretim miktarı, ürünün fiyatı, üretilen her ürün için verilen sübvansiyon miktarı ve satılan her ürün için tüketiciye verilen geri ödeme miktarı karar değişkenlerinden oluşan doğrusal olmayan bir optimizasyon problemi ile formüle edilebilir. Merkeziyetçi olmayan bir ekonomide yönetim sadece sübvansiyon politikaları üzerinde kontrol sahibidir ve ürünün miktarını veya fiyatını belirlemez. Böyle bir yönetim, ilan edilmiş her-

hangi bir sübvansiyon politikasına göre hareket eden bir üreticinin ürün miktarını ve fiyatını kendi kârını arttırma hedefiyle belirleyeceğini bilir. Bu tezde bu durum, belirtilen karar değişkenleri ile oluşturulmuş iki seviyeli bir optimizasyon problemi ile formüle edilmektedir. Bu problem, üreticinin takipçi olduğu, fiyat ve miktar üzerine karar verdiği, yönetimin ise lider olduğu, geri ödeme ve sübvansiyon üzerine karar verdiği bir Stackelberg oyunu olarak gösterilmektedir. Bu çalışmada, analizi çözülebilir tutmak için, sadece talebin deterministik bir talep fonksiyonu ile verildiği durumu inceliyoruz. Bu çalışmada, rassal talebin daha karmaşık durumları incelenmemiştir. Bu çalışmanın amacı, talep tarafındaki en kapsayıcı durumlar altında, yukarıda belirtilen problem üzerine merkeziyetçi ve merkeziyetçi olmayan ekonomiler için tam analitik olmaya yakın bir çözüm sağlamaktır. Hem merkeziyetçi hem de merkeziyetçi olmayan ekonomilerdeki probleme dair ilk formülasyonumuzdaki amaç fonksiyonunda, tüketici refahı, üretici kârı, dışsal etkiler eklenmiş ve yönetimin sübvansiyon politikası üzerindeki toplam harcaması çıkarılmıştır. İkinci formülasyonumuzda, her iki ekonomi koşulunda da, bir önceki amaç fonksiyonundaki yönetimin toplam harcamasını dahil etmiyor ancak, bu defa, bu harcamaları bütçe kısıtı altında düşünüyoruz. İlk gösterimde problemi talep fonksiyonunun sadece sürekli olduğunu varsayarak analiz ediyoruz. Bütçe kısıtını içeren ikinci formülasyonumuzda ise problemi talep fonksiyonlarının logaritmik içbükey talep fonksiyonlarını alt küme olarak içeren genel bir sınıfı için analiz ediyoruz. Son olarak bu sonuçlar, kuvvet talep fonksiyonu, doğrusal talep fonksiyonu, logaritmik doğrusal talep fonksiyonu ve lojistik regresyon talep fonksiyonu gibi bazı bilinen talep fonksiyonları için yeniden değerlendirilmektedir.

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Dedication page

I dedicate this work to my future self, I believe the ideas are going to come in handy

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1. Introduction and Literature Review

In this thesis, we study the intervention of government or a central authority on public-interest goods with the goal of maximizing the overall social welfare. Public-interest goods are products that benefit both the consumer and society. Health-related products like vaccines, energy efficient appliances, electronic vehicles, and developments in green technology are some noteworthy examples. In order to stimulate the demand for this public interest good, the government has two intervention tools. first, giving rebate or tax reduction on the price to each customer buying this product and second, giving subsidy to the manufacturer for each produced item. Depending on the type of economy the government or central authority has the possibility to intervene in two different settings.

- Decentralized economy: The government only decides on rebate and subsidy values. Knowing the rebate and subsidy values the manufacturer determines the price and the production quantity using the manufacturer's objective of profit maximization. This means that the government is the leader and the manufacturer is the follower in this bi-level optimization problem.
- Centralized economy: The government next to determining the tax reduction and subsidy dictates the manufacturer how to set its price and produced quantity.

The main goal of this research is to formulate and analyze a mathematical model which determines the rebate and subsidy values that maximize the objective function in both a centralized and decentralized economy. The concept of optimization of social welfare and coordination of a supply chain with incentives such as subsidies and rebates has been thoroughly studied by researchers in both welfare economics and operational management (for a review on welfare economics see Jones (2005)). However, the main focus of most of the research is on social welfare in a decentralized economy. In the first part of this section, we discuss the different related specific public interest goods discussed in the literature giving rise to product related modeling issues and also sometimes different objectives and optimization problems.

In the second part, we consider some of the literature which is closely related to the objective function and optimization problems discussed in this thesis.

The first main stream of research in the literature deals with papers describing the availability of vaccines against flu in a vaccine supply chain and show how the government should intervene in this market to improve this availability. For this specific example one deals with related modelling problems as the one we discuss in this thesis for public interest goods. Although this literature is well-developed due to the recent pandemic we only refer to Arifoglu, Deo & Iravani (2012), Demirci & Erkip (2020) and Bo, Haoyang & Shuxia (2022) since these are the ones which are most relevant for our problem. In all of these papers depending on the generality of the problem formulation different models are used to study how the government can act in this market to increase the welfare of the society. In Arifoglu et al. (2012) the government either subsidizes the manufacturer or the consumer and not both and conclude that the government should intervene on both the supply side and the demand side to coordinate the system. However, their objective also includes the infection rate of the disease by means of a epidemiology model describing the progress of the infections and is completely unrelated to our used objective. In Demirci & Erkip (2020) government intervention is not formulated in terms of subsidy for each produced product and tax reduction for each buying customer. In this paper it is assumed that the government reserves funds to improve the demand for the vaccine and this demand is represented by some lognormal distributed random variable of which the mean and variance is a function of the amount of the reserved funds. Also it is assumed that not all vaccines after production are suitable for use against the disease and so the government subsidy giving to the manufacturer is used to reduce the total random number of unusable vaccines in the number of vaccines produced. Since vaccines which cannot be used against the disease cannot be sold by the manufacturer, the manufacturer also incurs a loss due to this. This random yield effect is included in the profit maximizing behaviour of the manufacturer. The optimization problem is now formulated as a bilevel optimization problem which optimizes the allocation of the total available budget over the amount of money spend on increasing the demand and the subsidy given to the manufacture to decrease the number of fall outs of the produced quantity of vaccines. Finally in Bo et al. (2022) the impact of different government subsidies in a vaccine supply chain with one manufacturer and multiple retailers is studied.

Another main stream of research within the literature are those papers discussing how the government should coordinate the market for green products. We mention therefore some of the papers studying the green products and how to invest in greener technology with the goal of carbon emission reduction. Again the complexity of this market and the specific characteristic of each problem formulation leads to different models. In Krass, Nedorezov & Ovchinnikov (2013) and Ruidas, Seikh & Nayak (2022) it is possible that the manufacturer can choose between different technologies and so the main question is which technology should be chosen to increase profit. In Krass et al. (2013) the government imposes an environmental tax and as a conclusion the authors show that higher taxation does not necessarily lead to greener technology. Ruidas et al. (2022) focuses on a manufacturer that receives a subsidy from the government and invests in green technology and emission reduction innovations. In Yu, Tang & Shen (2018) a system is studied with two competing manufacturers producing green products and a government that needs to decide whether to subsidize the manufacturers or the consumers. In Chemama, Cohen, Lobel & Perakis (2019) the government's two key strategies given by the so-called commitment strategy vs the flexible strategy are compared. The authors minimize government expenditure while having a target adoption constraint. In Bai, Gong, Jin & Xu (2019) a supply chain with one manufacturer and two retailers is analyzed. They conclude that for optimal coordination the government should subsidize the retailers instead of the manufacturer. In Babich, Lobel & Yücel (2020) the government aims to motivate households to adopt investing in renewable energy resources. In this paper the advantages of the tax rebate policy versus the feed-in tariff policy are compared. In Bai, Hu, Gui, So & Ma (2021) the government's subsidization programs that aim to optimally motivate consumers to trade-in their products with greener ones are studied. Junsong & Xiaolong (2022) examine the advantages of imposing an environmental tax over giving subsidy to the manufacturer in terms of emission reduction cost and environmental damage.

Another main stream of research in the literature are those papers discussing how the government should intervene in the market for electric vehicles. In Arar (2010), Carlsson & Johansson-Stenman (2013), Yan (2018), it is discussed whether it is sensible to give subsidies to electric cars and will the stimulation of the sales of electric vehicles by means of subsidies contribute a lot to the reduction of carbon dioxide emissions. In Carlsson & Johansson-Stenman (2013) it is also concluded that the government subsidies for electric vehicles could result in high budget expenditures and so one should be careful in applying government subsidizing policies to this market. In Arar (2010) it is shown that the adoption of electric vehicles in the US will certainly result in a reduction of carbon emissions and hence is environmentally sensible to do. In Yan (2018), the role of government incentives like subsidies and taxation which increase the sales of electric vehicles and their environmental benefits are studied. The models studied in Cohen, Perakis & Thraves (2015), Raz & Ovchinnikov (2015) and Cohen, Lobel & Perakis (2016) can also be applied to the electric vehicle market. These authors study models where the government aims to coordinate the supply by giving rebates or subsidies. Raz & Ovchinnikov (2015) also products with negative externality are considered. In Cohen et al. (2015) there are multiple suppliers and the focus of the research is on studying the impact of competition between suppliers. In the papers of Cohen et al. (2016) and Cohen et al. (2015) also the social welfare is maximized under a target adoption constraint. They suggest that the subsidy given to the end consumer is sufficient for coordinating the supply chain system. Demirci & Erkip (2017) present a model where the government simultaneously invests in demand-increasing strategies and pays consumer subsidy. Their objective function summarizes the welfare in externality. Up to now we only discussed literature related to a particular public interest good. Due to the specific good some of these papers use different objectives than used in this thesis and in the remainder of this section we will focus on papers discussing objective functions and models closely related to the objective functions and models used in this thesis.

The objective function in our model consists of consumer surplus, manufacturer's profit, positive externality effect, and government expenditure. The use of these separate components seems to be a standard approach in welfare economics. We explore in detail two different optimization problems, both aiming to maximize social welfare using a deterministic demand function. The first optimization problem subtracts in the objective function the government expenditure from the sum of the consumer surplus, manufacturer profit, and externalities and has only restrictions on subsidy and rebate. The second optimization problem has a budget constraint and deletes the government expenditures from the objective function used in the first optimization problem. Both problems are solved in a centralized and decentralized economy with decision variables given by rebate, subsidy, production quantity, and price. From a modeling perspective, our optimization problems are closely related to the ones used in Cohen et al. (2016) and Raz & Ovchinnikov (2015). Both of these papers have objective functions and decision variables similar to ours but their demand is modelled by a very special demand functions within a stochastic environment. In our thesis we consider general demand functions without any random error term. In Cohen et al. (2016) the price is not a decision variable and there is no simultaneous optimization over rebate and subsidy. Also, in that paper, an easier alternative target level instead of a budget constraint is used. Our first optimization problem is similar to the one used in Raz & Ovchinnikov (2015) but Raz & Ovchinnikov (2015) models the demand using the stochastic additive demand model with a linear demand function and only analyzes this model in a decentralized economy. In Krass et al. (2013), Cohen et al. (2015), and Chemama et al. (2019), one can also find some similarities with our work but as already mentioned the papers have again a stochastic demand setting. Krass et al. (2013) analyzes a related model in a centralized and decentralized economy with the goal of maximizing the total welfare but the authors do not introduce the price demanded by the manufacturer as a decision variable. Instead of this, technology type is used as a decision variable. Chemama et al. (2019) extends the results of Cohen et al. (2015) and Cohen et al. (2016) and deals with a different and more complicated problem in a decentralized economy. As in Cohen et al. (2015) and Cohen et al. (2016) an easier target adoption level constraint is included instead of the more difficult budget constraint. Demirci & Erkip (2017), Yu et al. (2018), Bai et al. (2021) have rather different objective functions. In Demirci & Erkip (2017) discuss other demand-increasing strategies, and they optimize over budget for demand-increasing strategies, rebate, and production quantity in a decentralized economy. Yu et al. (2018) optimize consumer welfare instead of social welfare, and this objective function has different components. The objective function in Bai et al. (2021) is similar to ours and they also have a budget constraint but the problem they address is in fact about trade-in programs and hence deals with different modeling components.

To express our contribution to the literature, this thesis formulates and analyzes, under very weak assumptions on the deterministic demand functions the welfare economics optimization problems dealing simultaneously with price, production quantity, subsidy, and rebate. This holds for a centralized and decentralized economy. As a result, we are able to identify for almost all cases the optimal solution of both optimization problems by means of easy analytical formulas for both centralized and decentralized economies. A future research topic would be to analyze in detail the same models using a stochastic demand model. The main problem under this more general stochastic demand framework is that the production quantity will be more difficult to determine and due to this the overall objective function although easy to compute will become more difficult to analyze. In this case, it is expected that no easy analytical expressions for the optimal decision variables will be available but instead, they can be determined numerically by some algorithms.

2. On the Objective Function in Welfare Economics

To start the introduction of the objective function used for public interest goods we first give the used symbols.

$$
\begin{aligned}
q &= \text{total number of items of a particular product produced by a manufacturer}\\
p &= \text{price per item}\\
c &= \text{cost of production per item}\\
s &= \text{subsidy per item produced}\\
r &= \text{rebate value per item sold}\\
b_s &= \text{salvage value per leftover item}\\
B &= \text{total government budget}
\end{aligned}
$$

The deterministic demand for the product is given by a demand function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathbb{R}_+ = [0, \infty)$. Since by definition the value $\lambda(p)$ denotes the demand of customers willing to pay the price p for an item of this particular product the demand is a decreasing function of the price p and so the function λ is decreasing and non-negative. To identify the class of all decreasing non-negative functions that can serve as demand functions, we note that if the price is set at zero then every customer in the economy consisting of a finite number of customers will obtain the product for free. This means

$$\lambda(0) := \lim_{p \downarrow 0} \lambda(p) < \infty. \tag{2.1}$$

Hence the value $\lambda(0)$ denotes the size of the market. To keep the analysis simple, we assume that the demand function λ is continuous on $[0, \infty)$. It might also happen that for a given price or higher there occurs no demand and so we introduce the value

$$p_{\max} = \sup\{p \geq 0 : \lambda(p) > 0\} \leq \infty. \tag{2.2}$$

If the function λ has no compact support (this means $\lambda(p) > 0$ for every $0 \leq p < \infty$) it follows that $p_{\max} = \infty$ while for the function λ having a compact support (this means there exists some $0 \leq p < \infty$ satisfying $\lambda(p) = 0$ we obtain $p_{\max} < \infty$. By the continuity of the demand function, it follows $\lambda(p_{\max}) = 0$. Since there should be a demand if the price is set to the production cost c of the product we need to assume that $p_{\max} > c$. This means $\lambda(c)$ is finite and positive and so we incur a positive demand if the product is sold at the cost price c. To avoid pathological cases, we also impose that

$$\lim_{p \uparrow \infty} p\lambda(p) = 0. \tag{2.3}$$

The most well known examples of demand functions $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ fitting the above framework are given by the linear demand function (Mills (1959))

$$\lambda(p) = (a - bp)^+, \ \ a > 0, b > 0, \tag{2.4}$$

with $x^+ := \max\{x, 0\}$, the loglinear demand function (Jeuland & Shugan (1988))

$$\lambda(p) = ae^{-bp}, \ \ a > 0, b > 0, \tag{2.5}$$

the power demand function (cf.Jeuland & Shugan (1988))

$$\lambda(p) = (a + bp)^{-\gamma}, \ \ a > 0, b > 0, \gamma > 1$$

and the logit demand function (cf.Chen & Simchi-Levi (2012), Jeuland & Shugan (1988))

$$\lambda(p) = \frac{ae^{-bp}}{1 + e^{-bp}}, a > 0, b > 0.$$

In all the above examples of demand functions except the power demand function and the logit demand function $\lambda(0) = a$ and so a denotes for these demand functions the size of the market. For an overview on specific demand functions used in the economics and operations management literature, the reader is referred to the survey paper by (Jiang Huang & Parlar (2013)) and Talluri & Van Ryzin (2004). A rebate $0 < r < c$ will now be given by the government to every customer buying this product and this means that each customer needs to pay $p - r \geq 0$ to the manufacturer with p the price of the product. Hence the rebate value can be seen as a tax reduction. Since for $p - r > p_{max}$ there is no demand it is clear for p_{max} is finite that $0 \leq p - c \leq p - r \leq p_{max}$, while for $p_{max} = \infty$, we have $0 \leq p - c \leq p - r < \infty$. Also the government gives a subsidy $0 \leq s \leq c - b_s$ to the manufacturer for each produced product. The total number of produced items of this particular product is given by

q and each item is offered at price p. The total sales $S(p)$ of this product equals

$$S(p) = \min\{q, \lambda(p-r)\}. \tag{2.6}$$

Given the subsidy s and rebate r the amount of money the government spends on stimulating the sales of this particular product is given by the budget function

$$b(s,r,p,q) := r\min\{q, \lambda(p-r)\} + sq. \tag{2.7}$$

If B denotes the available budget of the government and the budget is not a soft constraint the budget constraint

$$b(s,r,p,q) = r\min\{q, \lambda(p-r)\}) + sq \leq B \tag{2.8}$$

needs to be included in our optimization problem. Another more refined option is to divide the budget into a budget B_r for rebates and a budget B_s for subsidies and to include the constraints

$$r\min\{q, \lambda(p-r)\}) \leq B_r, \;\; sq \leq B_s.$$

This approach will not be followed in this thesis. To determine the objective function of our optimization problem we first introduce the different components of this objective function for rebate r, subsidy s, price p and offered quantity q satisfying the constraints $0 \leq s \leq c - b_s, 0 \leq r \leq c$ and $r \leq p \leq r + p_{max}$. To simplify the notation in the remaining part of the thesis we introduce the nonempty sets

$$F := \{(r,p) : 0 \leq r \leq c, r \leq p \leq p_{\max} + r\} \tag{2.9}$$

and

$$G := \{(s,r,p) : 0 \leq s \leq c - b_s, 0 \leq r \leq c, r \leq p \leq p_{\max} + r\}. \tag{2.10}$$

The components of the objective function consist of the so-called consumer surplus discussed in Appendix, the external effects for the economy, and the profit of the manufacturer.

1.1 The total consumer surplus $cs : F \times \mathbb{R}_+ \to \mathbb{R}_+$ is given by (see Appendix)

$$cs(r,p,q) = \beta(p-r)\frac{\min\{q, \lambda(p-r)\}}{\lambda(p-r)} \tag{2.11}$$

with the function $\beta:[0,p_{max})\to\mathbb{R}_+$ defined by

$$\beta(u) := \int_u^{p_{\max}} \lambda(x)dx. \tag{2.12}$$

Since we include the consumer surplus in the objective function we need to restrict the set of demand functions $\lambda:\mathbb{R}_+\to\mathbb{R}_+$ and impose that

$$\int_0^{p_{\max}} \lambda(x)dx < \infty, \tag{2.13}$$

For $p_{max}=\infty$ it follows for any $u>0$ that

$$\int_{\frac{u}{2}}^{p_{max}} \lambda(x)dx \geq \int_{\frac{u}{2}}^{u} \lambda(x)dx \geq \frac{u}{2}\lambda(u).$$

and this shows for $\int_0^{p_{\max}} \lambda(x)dx$ finite that relation (2.3) is satisfied. To simplify the statements in the next sections, let $C((0,\infty))$ denote the set of all continuous functions on $(0,\infty)$ and introduce the set $\mathcal{D}$ of the so-called feasible demand functions given by

$$\mathcal{D} = \left\{\lambda \in C((0,\infty)) : \lambda \text{ decreasing}, \lambda(0)<\infty, \int_0^{p_{max}} \lambda(x)dx < \infty\right\}. \tag{2.14}$$

1.2 The external positive effects $e:F\times\mathbb{R}_+\to\mathbb{R}_+$ for the economy are given

$$e(r,p,q) = \alpha \min\{q,\lambda(p-r)\} \tag{2.15}$$

for some $\alpha \geq 0$. The nonnegative parameter α represents the additional monetary value of each item sold to a customer in the market and is determined outside the model. Hence the external effects measure the additional benefit to society for each item sold. The main problem is to determine the value of α. In Demirci & Erkip (2017), the value of α is determined solving a subproblem using Lagrangian multipliers, while in Raz & Ovchinnikov (2015), Cohen et al. (2015) and Cohen et al. (2016), the value of α is determined outside the model. In particular for the electric vehicle market, estimates of α based on cost considerations are given in Arar (2010).

1.3 The total profit $m:G\times\mathbb{R}_+\to\mathbb{R}$ of the manufacturer is given by

$$m(s,r,p,q) = p\min\{q,\lambda(p-r)\} + b_s(q-\lambda(p-r))^+ - (c-s)q \tag{2.16}$$

with b_s denoting the salvage value of any item left over in stock at the end of the period and $x^+ := \max\{x,0\}$. This salvage value might be negative or positive, and since the salvage value should be less than the production cost, we assume $b_s < c$. Since

$$\min\{q, \lambda(p-r)\} + (q - \lambda(p-r))^+ = q,$$

it follows that the profit of the manufacturer offering the item at price p and producing q items is given by

$$m(s,r,p,q) = (p - b_s)\min\{q, \lambda(p-r)\} - (c - b_s - s)q. \tag{2.17}$$

Since we need to optimize the function m in (2.17) over $q > 0$ it is natural to assume that the subsidy s satisfies $s < c - b_s$. This means that the subsidy per produced item given by the government to the producer should not be above the production cost minus the salvage value.

1.4 The total expenditure of the government $b : G \times \mathbb{R}_+ \to \mathbb{R}$ given by

$$b(s,r,p,q) = r\min\{q, \lambda(p-r)\}) + sq. \tag{2.18}$$

The objective function of our optimization problem can now be defined in two different ways. We either include the expenditure of the government in this objective function and do not include a budget constraint or exclude the expenditure in the objective function and add a budget constraint. In the first case the objective function $f : G \times \mathbb{R}_+ \to \mathbb{R}$ reduces to

$$\begin{aligned} f(s,r,p,q) &:= cs(r,p,q) + e(r,p,q) + m(s,r,p,q) - b(s,r,p,q) \\ &= \min\{q, \lambda(p-r)\}\left(p - r - b_s + \alpha + \tfrac{\beta(p-r)}{\lambda(p-r)}\right) - (c - b_s)q. \end{aligned} \tag{2.19}$$

This objective function is used in Raz & Ovchinnikov (2015) for the stochastic version of the above model with stochastic demand $\mathbf{D}(p)$ given by

$$\mathbf{D}(p) = a - b + \mathbf{Y}$$

with $\mathbf{Y}$ a bounded random variable satisfying $\mathbb{E}(\mathbf{Y}) = 0$. If $\mathbf{Y}$ is a degenerate random variable equal to 0, we obtain the above objective function for the linear demand function $\lambda(p) = a - p$.

In the last case the expenditure of the government is not included in the objective function and so we introduce the function $g : G \times \mathbb{R}_+ \to \mathbb{R}$ given by

$$
\begin{aligned}
g(s,r,p,q) &:= cs(r,p,q) + e(r,p,q) + m(s,r,p,q) \\
&= \min\{q, \lambda(p-r)\}\left(p - b_s + \alpha + \tfrac{\beta(p-r)}{\lambda(p-r)}\right) - (c - b_s - s)q.
\end{aligned} \tag{2.20}
$$

In Raz & Ovchinnikov (2015), Cohen et al. (2015) and Chemama et al. (2019), the function f is used in a special stochastic additive demand function setting, while in Krass et al. (2013) the function g is used. We will consider in the next two subsections the centralized and decentralized versions of the above decision problems.

3. Formulation of the Model in a Centralized Economy

In the centralized economy the government dictates the manufacturer the price $p \geq r$ of the product and the produced quantity q. The manufacturer is not allowed to set his own price and quantity to maximize his profit. If we do not include the budget constraint

$$b(s,r,p,q) = r\min\{q,\lambda(p-r)\}) + sq \leq B, \tag{3.1}$$

our optimization problem reduces to

$$\sup\{f(s,r,p,q) : (r,s,p,q) \in X\} \tag{P}$$

with the feasible set X given by

$$X = \{(s,r,p,q) : 0 \leq s \leq c - b_s, 0 \leq r \leq c, r \leq p \leq p_{max} + r, q \geq 0\}. \tag{3.2}$$

If the budget constraint

$$r\min\{q,\lambda(p-r)\}) + sq \leq B$$

is included we obtain the optimization problem

$$\sup\{g(s,r,p,q) : (r,s,p,q) \in H\} \tag{P_1}$$

with the nonempty set H given by

$$H := \{(s,r,p,q) \in X : b(s,r,p,q) \leq B\} \subseteq X. \tag{3.3}$$

The functions g and f are listed in (2.19) and (2.20), and the function b is given in (3.1). If the government selects any (s,r,p,q) belonging to the feasible set H or X then in optimization problems (P) the objective value is given by $f(s,r,p,q)$ while in optimization problem (P_1) the objective value equals $g(s,r,p,q)$. To restrict the feasible region of both optimization problems we observe the following. For every

(s,r,p,q) belonging to H or X satisfying $q > \lambda(p-r)$ it follows that $(s,r,p,\lambda(p-r))$ also belongs to H or X and by relation (2.19) and (2.20) we obtain using $c > b$ that

$$f(s,r,p,q) \le f(s,r,p,\lambda(p-r)) \tag{3.4}$$

and using $s < c-b$ that

$$g(s,r,p,q) \le g(s,r,p,\lambda(p-r)). \tag{3.5}$$

for any $q \ge \lambda(p-r)$. This means it is suboptimal for the goverment to demand from the manufacturer that $q > \lambda(p-r)$ items need to be produced. Hence by relation (3.4) the optimization problem (P) with no budget constraint reduces to

$$(P) \quad \begin{aligned} \upsilon(P) &= \sup\{f(s,r,p,q) : (s,r,p,q) \in X\} \\ &= \sup\{f_0(s,r,p,q) : q \le \lambda(p-r), (s,r,p,q) \in X\} \end{aligned}$$

with objective function

$$f_0(s,r,p,q) = q\left(p-r-c+\alpha+\frac{\beta(p-r)}{\lambda(p-r)}\right). \tag{3.6}$$

By the same argument using relation (3.5) the centralized optimization problem (P_1) with a budget, constraint is given by

$$\begin{aligned} \upsilon(P_1) &= \sup\{g(s,r,p,q) : (s,r,p,q) \in H\} \\ &= \sup\{g_0(s,r,p,q) : q \le \lambda(p-r), (s,r,p,q) \in H\} \\ &= \sup\{g_0(s,r,p,q) : q \le \lambda(p-r), (s,r,p,q) \in X, (s+r)q \le B\} \\ &= \sup\{g_0(s,r,p,q) : (s,r,p) \in G, 0 \le q \le \min\{\lambda(p-r), \tfrac{B}{r+s}\}\} \end{aligned} \tag{3.7}$$

with objective function

$$g_0(s,r,p,q) := q\left(p+s-c+\alpha+\frac{\beta(p-r)}{\lambda(p-r)}\right) \tag{3.8}$$

and the set G defined in relation (2.10). In the next subsections we will analyze in detail these two different optimization problems and identify their optimal solutions.

3.1 Analysis of Optimization Problem (P).

In this subsection, we analyze any demand function belonging to the set $\mathcal{D}$ introduced in relation (2.14) the centralized version of optimization problem (P) with no budget constraint. Before showing that optimization problem (P) has an optimal solution and determine this optimal solution we introduce for any $\gamma \in \mathbb{R}$ the function $f_\gamma : \mathbb{R}_+ \to \mathbb{R}$ given by

$$f_\gamma(u) = \lambda(u)(u+\gamma) + \int_u^{p_{\max}} \lambda(x)dx, u > 0 \tag{3.9}$$

and $f_\gamma(0) := \lim_{u \downarrow 0} f_\gamma(u) = \gamma\lambda(0) + \int_0^{p_{\max}} \lambda(x)dx$. For this functions $f_\gamma, \gamma \in \mathbb{R}$ one can show the following monotonicity result.

Lemma 3.1.1. *If the function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is (strictly) decreasing on $(0, p_{max})$ then $f_\gamma(p_{max}) = 0$ for every γ with f_γ listed in relation (3.9). If $\gamma \geq 0$ the function f_γ is (strictly) decreasing on $(0, p_{max})$ and for $-p_{\max} < \gamma < 0$ the function f_γ is (strictly) increasing on $(0, -\gamma)$ and (strictly) decreasing on $(-\gamma, p_{\max})$.*

Proof. We only show the result for a demand function strictly decreasing on $(0, p_{max})$. A similar proof applies for a decreasing demand function on $(0, p_{max})$. For any $0 \leq u < u+h < p_{max}$ and $\gamma \geq 0$ it follows that

$$f_\gamma(u) - f_\gamma(u+h) = (u+\gamma)(\lambda(u) - \lambda(u+h)) - h\lambda(u+h) + \int_u^{u+h} \lambda(x)dx. \tag{3.10}$$

Since λ is strictly decreasing on $(0, p_{max})$ we obtain both $\lambda(u) - \lambda(u+h) > 0$ and $h\lambda(u+h) < \int_u^{u+h} \lambda(x)dx$. This shows by relation (3.10) and $\gamma \geq 0$ the first result. The result for $-p_{\max} < \gamma < 0$ can be verified in a similar way. □

Using Lemma 3.1.1 it is easy to verify for every $\alpha \geq 0$ and any demand function λ belonging to the set $\mathcal{D}$ that the optimization problem

$$\sup_{(r,p) \in F} \{f_{\alpha - c}(p - r)\} \tag{3.11}$$

with the set F listed in relation (2.9) has an optimal solution. We will now relate in the next lemma the above optimization problem to the centralized version of optimization problem (P).

Lemma 3.1.2. *It follows for every demand function λ belonging to the set $\mathcal{D}$ and*

$\alpha \geq 0$ *that*

$$v(P) = \sup_{(r,p)\in F}\{f_{\alpha-c}(p-r)\} > 0 \tag{3.12}$$

with the set F defined in relation (2.9) and the function $f_{\alpha-c}$ in relation (3.9). It is optimal to choose in optimization problem (P) the subsidy equal to 0 *and the production quantity equal to $\lambda(p-r)$ with (p,r) an optimal solution of optimization problem* $\sup_{(r,p)\in F}\{f_{\alpha-c}(p-r)\}$.

Proof. It follows using relation (3.6) that

$$\begin{aligned} f_1(r,p) &:= \sup_{0\leq q\leq \lambda(p-r),0\leq s\leq c-b}\{f_0(s,r,p,q)\} \\ &= \sup_{0\leq q\leq \lambda(p-r),0\leq s\leq c-b} q\left(p-r-c+\alpha+\tfrac{\beta(p-r)}{\lambda(p-r)}\right) \\ &= \max\{0, f_{\alpha-c}(p-r)\} \end{aligned} \tag{3.13}$$

and so it is optimal to choose the subsidy equal to 0. Hence by relation (3.13) we obtain

$$v(P) = \sup_{0\leq r\leq c, c\leq p\leq p_{max}+r}\{f_1(r,p)\} = \sup_{(r,p)\in F}\{\max\{0, f_{\alpha-c}(p-r)\}.$$

By Lemma A.0.1 it follows that

$$\sup_{(r,p)\in F}\{\max\{0, f_{\alpha-c}(p-r)\}\} = \max\{0, \sup_{(r,p)\in F}\{f_{\alpha-c}(p-r)\}\}$$

and this shows

$$v(P) = \max\{0, \sup_{(r,p)\in F}\{f_{\alpha-c}(p-r)\}\}. \tag{3.14}$$

Since $c < p_{\max}$ we obtain for every $0 \leq r \leq c$ that the vector $(r, r+(c-\alpha)^+)$ belongs to set F listed in relation (2.9) and this implies

$$\sup_{(r,p)\in F}\{f_{\alpha-c}(p-r)\} \geq f_{\alpha-c}((c-\alpha)^+) > 0.$$

Applying relation (3.14) the desired result in relation (3.12) follows. The selection of an optimal production quantity follows easily from relation (3.13). □

Applying Lemma 3.1.1 and 3.1.2 the next result identifying optimal solutions of the centralized version of optimization problem (P) for every $\alpha \geq 0$ follows easily.

Theorem 3.1.1. *If follows for every demand function λ belonging to $\mathcal{D}$ and $\alpha \geq 0$ it is optimal to select in problem (P) rebate equal to $0 \leq r \leq c$, subsidy equal to 0,* *price equal to* $r+(c-\alpha)^+$ *and production quantity equal to* $\lambda((c-\alpha)^+)$*. This choice yields optimal objective value*

$$(3.15) \qquad \upsilon(P) = f_{\alpha-c}((c-\alpha)^+) = \lambda((c-\alpha)^+)((c-\alpha)^+ + \alpha - c) + \beta((c-\alpha)^+)$$

The contribution to the optimal objective value of the profit of the manufacturer is $(r+(c-\alpha)^+ - c)\lambda((c-\alpha)^+)$*, the contribution of the consumer surplus is* $\beta((c-\alpha)^+)$ *and the contribution of the externalities is* $\alpha\lambda((c-\alpha)^+)$*,while the government spends* $r\lambda((c-\alpha)^+$*.*

Proof. It is easy to see that the vector $(s,r,p,q) = (0,r,r+(c-\alpha)^+, \lambda((c-\alpha)^+)$ belongs to the set X listed in (3.2) and so this vector is a feasible solution of optimization problem (P). This shows by (2.19) that

$$\begin{aligned} \upsilon(P) &\geq f(0,r,r+(c-\alpha)^+, \lambda((c-\alpha)^+)) \\ &= \lambda((c-\alpha)^+)((c-\alpha)^+ + \alpha - c) + \beta((c-\alpha)^+) \\ &= f_{\alpha-c}((c-\alpha)^+) \end{aligned}$$

Applying Lemma 3.1.1 and 3.1.2 it follows that $\upsilon(P) \leq f_{\alpha-c}((c-\alpha)^+)$ and we obtain that the vector $(s,r,p,q) = (0,r,r+(c-\alpha)^+, \lambda((c-\alpha)^+)$ is an optimal solution. Substituting this optimal solution into (2.11),(2.17) and (2.18) gives the total spending of the government and the contribution of the profit of the manufacturer, the consumer surplus and the externalities to the optimal objective value in a centralized economy. □

To give an interpretation of the above set of optimal solutions we observe the following. By Theorem 3.1.1 it follows if the externality factor α satisfies $\alpha \geq c$ and so the public interest good has a high impact on the social welfare that the government tries to minimize in the centralized version of problem (P) the price $p-r$ to be paid by the customer. This means since $p-r \geq 0$ that by Theorem 3.1.1, any $(r,p(r))$ with $p(r) = r$, and $0 \leq r \leq c$, is an optimal solution. Hence in this case the consumer will get the public interest good for free, and the government can force the industry to incur a loss of $(c-r)\lambda(0)$ if $0 < r < c$ is chosen. However, if the government also decides beforehand that the price demanded from the manufacturer should be such that the manufacturer will not incur any losses then the only feasible optimal solution is given by $r = c$ and $p = c$. For $\alpha < c$ and so the public interest good has a lower impact on the welfare of the society the customer does not obtain the good for free. By Theorem 3.1.1 it follows that any optimal solution is given by $(r,p(r))$ with $p(r) = r+c-\alpha$, for any $0 \leq r \leq c$. This means that the consumer always has to pay the positive amount $c-\alpha$ to obtain the product and the manufacturer needs to produce $\lambda(c-\alpha)$ and does not produce for the whole market contrary to the case $\alpha \geq c$. At the same time the manufacturer earns $(r-\alpha)\lambda(c-\alpha)$ and the overall welfare objective value equals

$$f_{\alpha-c}(c-\alpha) = \beta(c-\alpha)$$

denoting the consumer surplus of the customers paying the price $c-\alpha$. Hence if the government decides in this scenario that the manufacturer also should benefit then the only feasible optimal solutions are given by $(r, r+c-\alpha)$ with $r-\alpha \geq 0$. Since the government might also like to minimize their own expenditures then under the additional condition that the industry will not incur any losses the only feasible optimal solution is given by $r=\alpha$ and $p=c$. We next summarize the algorithm solving optimization problem (P) in case the manufacturer produces at cost price and the government minimizes its expenses.

Numerical Solution Procedure 3.1.1. *Solving centralized version of optimization problem (P) for any nonnegative* α.

- *STEP 1. The optimal rebate value equals* $\min\{\alpha, c\}$, *the optimal subsidy value equals* 0, *the optimal price demanded by the manufacturer equals* c, *the price paid by the customer equals* $(c-\alpha)^+$, *the optimal production quantity equals* $\lambda((c-\alpha)^+)$ *and the optimal objective value equals* $f_{\alpha-c}((c-\alpha)^+)$.
- *STEP 2: output* $r_{opt} = \min\{\alpha, c\}$, $s_{opt} = 0$, $p_{opt} = c$, $q_{opt} = \lambda((c-\alpha)^+)$ *and* $\upsilon(P) = f_{\alpha-c}((c-\alpha)^+)$.

We now determine the optimal objective value of optimization problem (P) for $\alpha \geq 0$ for some important demand functions belonging to the set $\mathcal{D}$.

Example 3.1.1. *If we use the linear demand function given by*

$$\lambda(p) = (a-bp)^+, a > 0, b > 0$$

then $p_{\max} = \frac{a}{b}$ *and for every* $u \leq p_{\max}$

$$\int_u^{p_{\max}} (a-bx)dx = \frac{1}{2b}(a-bu)^2.$$

This shows for every $u \leq p_{\max}$ *that*

$$f_{\alpha-c}(u) = (a-bu)(u+\alpha-c) + \frac{1}{2b}(a-bu)^2. \tag{3.16}$$

Hence it follows by Theorem 3.1.1 and relation (3.16) that for $\alpha \geq c$ *the optimal* *objective value of optimization problem* (P) *is given by*

$$\upsilon(P) = f_{\alpha-c}(0) = \alpha(a-c) + \frac{a^2}{2b}$$

and for $0 \leq \alpha < c$ *we obtain*

$$\upsilon(P) = f_{\alpha-c}(c-\alpha) = \frac{(a+b\alpha-bc)^2}{2b}.$$

In the next example we consider the loglinear demand function.

Example 3.1.2. *If we use the log linear demand function* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *given by*

$$\lambda(p) = ae^{-bp},\ a > 0, b > 0$$

we obtain $p_{\max} = \infty$ *and for every* $u < p_{\max}$

$$\int_u^{p_{\max}} ae^{-bx}dx = \frac{ae^{-bu}}{b}.$$

This shows for every $u < p_{\max}$ *that*

$$f_{\alpha-c}(u) = ae^{-bu}\left(u+\alpha-c+\frac{1}{b}\right). \tag{3.17}$$

Hence it follows by Theorem 3.1.1 and relation (3.17) that for $\alpha \geq c$ *the optimal objective value of optimization problem* (P) *is given by*

$$\upsilon(P) = f_{\alpha-c}(0) = a(\alpha-c) + \frac{a}{b}$$

and for $0 \leq \alpha < c$ *we obtain*

$$\upsilon(P) = f_{\alpha-c}(c-\alpha) = \frac{ae^{-b(\alpha-c)}}{b}.$$

We next consider the power demand function.

Example 3.1.3. *If we use the power demand function* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *given by*

$$\lambda(p) = (a+bp)^{-\gamma},\ \ a > 0, b > 0, \gamma > 1$$

we obtain $p_{\max} = \infty$ *and for every* $u < p_{\max}$ *we obtain*

$$\int_u^{p_{\max}} (a+bp)^{-\gamma}dx = \frac{(a+bu)^{-\gamma+1}}{b(\gamma-1)}.$$

This shows for every $u < p_{\max}$ *that*

$$f_{\alpha-c}(u) = (a+bu)^{-\gamma}\left(\frac{\gamma u}{\gamma-1} + \alpha - c + \frac{a}{b(\gamma-1)}\right) \tag{3.18}$$

Hence it follows by Theorem 3.1.1 and relation (3.18) that for $\alpha \geq c$ *the optimal objective value of optimization problem* (P) *is given by*

$$\upsilon(P) = f_{\alpha-c}(0) = a^{-\gamma}\left(\alpha - c + \frac{a}{b(\gamma-1)}\right)$$

and for $0 \leq \alpha < c$ *we obtain*

$$\upsilon(P) = f_{\alpha-c}(c-\alpha) = \frac{(a+bc-b\alpha)^{-\gamma+1}}{b(\gamma-1)}.$$

Finally we consider the logit demand function.

Example 3.1.4. *If we use the logit demand function* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *given by*

$$\lambda(p) = \frac{ae^{-bp}}{1+e^{-bp}}, a > 0, b > 0$$

we obtain $p_{\max} = \infty$*. Since the derivative of the function* $f : \mathbb{R}_+ \to \mathbb{R}$ *given by*

$$f(p) = \ln(1+e^{-p})$$

is given by $\frac{-e^{-p}}{1+e^{-p}}$ *we obtain*

$$\int_u^{p_{\max}} \frac{e^{-v}}{1+e^{-v}} dv = \ln(1+e^{-u}).$$

and so for every $u < p_{\max}$ *we obtain*

$$\int_u^{p_{max}} \frac{ae^{-bx}}{1+e^{-bx}} dx = \frac{a}{b}\int_{bu}^{p_{\max}} \frac{e^{-v}}{1+e^{-v}} dv = \frac{a}{b}\ln(1+e^{-bu}).$$

This shows for every $u < p_{\max}$ *that*

$$f_{\alpha-c}(u) = \frac{ae^{-bu}}{1+e^{-bu}}(u+\alpha-c) + \frac{a}{b}\ln(1+e^{-bu}) \tag{3.19}$$

Hence it follows by Theorem 3.1.1 and relation (3.19) that for $\alpha \geq c$ *the optimal objective value of optimization problem* (P) *is given by*

$$\upsilon(P) = f_{\alpha-c}(0) = a(\alpha-c) + \frac{a\ln(2)}{b}$$

and for $0 \leq \alpha < c$ *we obtain*

$$\upsilon(P) = f_{\alpha-c}(c-\alpha) = \frac{a}{b}\ln(1+e^{-b(c-\alpha)}).$$

The table below displays the optimal solutions to the problem (P). It also shows the contribution of each of the components, consumer surplus, manufacturer profit, and externality effect on the overall welfare. Observe by Theorem 3.1.1 the rebate value r in this table is any value between 0 and c. In all these cases the consumer (obtaining the rebate r from the government!) has to pay $(c-\alpha)^+$ for the considered public interest good. Also, observe there are multiple optimal solutions since the government can force the manufacturer to pay part of the total cost for producing the public interest good.

Table 3.1 displays a summary of results for optimization problem (P) for any $\alpha \geq 0$ and any continuous demand function.

Table 3.1 Results for problem (P) in centralized economy.

Results	
optimal subsidy	0
optimal rebate	$0 \leq r \leq c$
optimal price - optimal rebate	$(c-\alpha)^+$
optimal production	$\lambda((c-\alpha)^+)$
optimal consumer surplus	$\beta((c-\alpha)^+)$
optimal externality	$\alpha\lambda((c-\alpha)^+)$
optimal profit manufacturer	$(r+(c-\alpha)^+-c)\lambda((c-\alpha)^+)$
optimal expenditure government	$r\lambda((c-\alpha)^+)$

In the next section, we analyze the optimization problem (P_1).

3.2 Analysis of Optimization Problem (P_1).

In this subsection, we analyze the optimization problem (P_1). We first show the following intermediate result for the optimization problem (P_1) eliminating the decision variable q from this optimization problem.

Lemma 3.2.1. *It follows for every demand function λ belonging to the set $\mathcal{D}$ and $\alpha \geq 0$ that*

$$(P_1) \qquad v(P_1) = \sup_{(s,r,p)\in G}\{f(s,r,p)\} > 0$$

with the set G defined in relation (2.10) and

$$f(s,r,p) := \min\left\{\lambda(p-r), \frac{B}{r+s}\right\}\left(p+s-c+\alpha+\frac{\beta(p-r)}{\lambda(p-r)}\right). \tag{3.20}$$

Proof. Applying relation (3.7) and Lemma A.0.1 it follows that

$$\begin{aligned} v(P_1) &= \sup\{q\left(p+s-c+\alpha+\tfrac{\beta(p-r)}{\lambda(p-r)}\right) : 0 \leq q \leq \min\{\lambda(p-r), \tfrac{B}{r+s}\}, (s,r,p) \in G\} \\ &= \sup_{(s,r,p)\in G}\{\max\{0, f(s,r,p)\}\} \\ &= \max\{0, \sup_{(s,r,p)\in G}\{f(s,r,p)\}\}. \end{aligned} \tag{3.21}$$

Since by relation (2.10) and $c < p_{\max}$ the vector $(0, r, r+(c-\alpha)^+)$ belongs to G we obtain

$$\begin{aligned} \sup_{(s,r,p)\in G}\{f(s,r,p)\} &\geq f(0, r, r+(c-\alpha)^+) \\ &= \min\{\lambda((c-\alpha)^+), \tfrac{B}{r}\}\left(r+(c-\alpha)^+ + \alpha - c + \tfrac{\beta((c-\alpha)^+)}{\lambda((c-\alpha)^+)}\right) \\ &\geq \min\left\{\lambda((c-\alpha)^+), \tfrac{B}{r}\right\}\left(r + \tfrac{\beta((c-\alpha)^+)}{\lambda((c-\alpha)^+)}\right) \\ &> 0 \end{aligned}$$

and by relation (3.21) the desired result follows. □

Before discussing a dominance result we list the following intermediate result.

Lemma 3.2.2. *If the demand function λ is (strictly) increasing on $(0, p_{\max})$ then for every γ the function $\rho_\gamma : (0, p_{\max}) \to \mathbb{R}_+$ given by*

$$\rho_\gamma(u) := \frac{f_\gamma(u)}{\lambda(u)} \tag{3.22}$$

and f_γ listed in relation (3.9) is (strictly) increasing on $(0, p_{\max})$.

Proof. We only give a proof for λ strictly decreasing on $(0, p_{\max})$. For λ decreasing the proof is similar. For any $0 < u < u+h < p_{\max}$. we obtain

$$
\begin{aligned}
\rho_\gamma(u+h)-\rho_\gamma(u) &= h+\frac{\int_{u+h}^{p_{\max}}\lambda(x)dx}{\lambda(u+h)}-\frac{\int_{u}^{p_{\max}}\lambda(x)dx}{\lambda(u)} \\
&= h-\frac{1}{\lambda(u)}\int_u^{u+h}\lambda(x)dx+\int_{u+h}^{p_{\max}}\lambda(x)dx\left(\frac{1}{\lambda(u+h)}-\frac{1}{\lambda(u)}\right)
\end{aligned} \tag{3.23}
$$

Since λ is strictly decreasing on $(0,p_{\max})$ we obtain $\int_u^{u+h}\lambda(x)dx < h\lambda(u)$ and $\frac{1}{\lambda(u+h)}-\frac{1}{\lambda(u)} > 0$. Applying now relation (3.23) yields the desired result. □

To show some dominance result for optimization problem (P_1) we introduce the sets

$$G_2 = \{(s,r,p) \in G : (r+s)\lambda(p-r) \geq B\} \tag{3.24}$$

and

$$G_1 = \{(s,r,p) \in G : (r+s)\lambda(p-r) \leq B\}. \tag{3.25}$$

with the nonempty set G listed in relation (2.10). Clearly the set G_1 is always nonempty while the set G_2 might be empty.

Lemma 3.2.3. *If the vector (s,r,p) belongs to G_2 then there exists some $p_* \geq p$ satisfying (s,r,p_*) belongs to G_1 and $f(s,r,p_*) \geq f(s,r,p)$ with the function f listed in relation (3.20).*

Proof. It follows for $(s,r,p) \in G_2$ that by relation (3.20)

$$
\begin{aligned}
f(s,r,p) &= \frac{B}{r+s}\left(p+s-c+\alpha+\frac{\beta(p-r)}{\lambda(p-r)}\right) \\
&= B+B\left(p-r-c+\alpha+\frac{\beta(p-r)}{(r+s)\lambda(p-r)}\right) \\
&= B+\frac{Bf_{\alpha-c}(p-r)}{(r+s)\lambda(p-r)} \\
&= B+\frac{B\rho_{\alpha-c}(p-r)}{r+s}
\end{aligned} \tag{3.26}
$$

If $(r,s,p) \in G_2$ we obtain since the function $p \to (r+s)\lambda(p-r)$ is continuous and decreasing on (r,∞) and $\lim_{p\uparrow\infty}(r+s)\lambda(p-r) = 0$ for every fixed $0 \leq r \leq c$ and $0 \leq s \leq c-b$ that there exists some $r+p_{\max} \geq p_* \geq p$ satisfying $(r+s)\lambda(p_*-r) = B$. For this vector (s,r,p_*) it follows that (s,r,p_*) belongs to G_1 and by Lemma 3.2.2 and $p_* \geq p$ we obtain by relation (3.26) that

$$f(s,r,p_*) = B+\frac{B\rho_{\alpha-c}(p_*-r)}{r+s} \geq B+\frac{B\rho_{\alpha-c}(p-r)}{r+s} = f(s,r,p).$$

This shows the desired result. □

Using Weierstrass theorem (Rudin (1982)) it is easy to verify by standard techniques for every $\alpha \geq 0$ and any demand function λ belonging to $\mathcal{D}$ that the optimization problem

$$\sup_{(s,r,p)\in G_1}\{\lambda(p-r)(p+s-c+\alpha)+\beta(p-r)\}$$

has an optimal solution. We will now relate in the next lemma the above optimization problem to the centralized version of optimization problem (P_1).

Lemma 3.2.4. *It follows for every demand function λ belonging to the set $\mathcal{D}$ and $\alpha \geq 0$ that*

$$\begin{aligned} v(P_1) &= \sup_{(s,r,p)\in G_1}\{\lambda(p-r)(p+s-c+\alpha)+\beta(p-r)\} \\ &= \sup_{(s,r,p)\in G_1}\{f_{\alpha-c}(p-r)+(r+s)\lambda(p-r)\} \end{aligned} \tag{3.27}$$

with the function $f_{\alpha-c}$ defined in relation (3.9). Moreover, it is optimal to choose in optimization problem (P_1) the subsidy equal to s_{opt}, the rebate equal to r_{opt}, the price equal to p_{opt} and the production quantity equal to $\lambda(p_{opt}-r_{opt})$ with $(s_{opt}, p_{opt}, r_{opt})$ an optimal solution of optimization problem

$$\sup_{(s,r,p)\in G_1}\{\lambda(p-r)(p+s-c+\alpha)+\beta(p-r)\}.$$

Proof. By Lemma 3.2.1 and 3.2.3 it follows that

$$\begin{aligned} v(P_1) &= \sup_{(s,r,p)\in G}\{f(s,r,p)\} \\ &= \sup_{(s,r,p)\in G_1}\{f(s,r,p)\} \\ &= \sup_{(s,r,p)\in G_1}\{\lambda(p-r)(p+s-c+\alpha)+\beta(p-r)\} \end{aligned}$$

and this shows relation (3.27). Since the above optimization problem has an optimal solution the remaining part of this lemma follows easily. □

By Lemma 3.2.4 we need to analyze the optimization problem

$$v(P_1)=\sup_{(s,r,p)\in G_1}\{\lambda(p-r)(p+s-c+\alpha)+\beta(p-r)\}. \tag{3.28}$$

To analyze the above optimization problem we consider two mutually exclusive subcases. We first analyze the case that the budget constraint is redundant for every (s,r) satisfying $0 \leq s \leq c-b_s, 0 \leq r \leq c$. This means for every feasible rebate and subsidy value that the available budget B is sufficient. In the second subcase we consider the complementary case that the available budget is not sufficient for some feasible rebate and subsidy value. To start with the case that the budget constraint is always redundant we give in the next result a necessary and sufficient condition on the demand function λ such that the available budget is sufficient for every feasible rebate and subsidy value.

Lemma 3.2.5. *It follows for any demand function λ belonging to $\mathcal{D}$ that the budget constraint is redundant if and only if $(2c - b_s)\lambda(0) \leq B$. This means that $G = G_1$ with the set G_1 listed in relation (3.25).*

Proof. If $G = G_1$ and hence $(r+s)\lambda(p-r) \leq B$ for every $(r,s,p) \in G$ it is obvious for any demand function λ belonging to $\mathcal{D}$ for $p = r$ and letting $s \uparrow c - b_s$ and $r \uparrow c$ that $(2c - b_s)\lambda(0) \leq B$. Moreover, if $(2c - b_s)\lambda(0) \leq B$ and $(r,s,p) \in G$ then for any demand function λ belonging to $\mathcal{D}$ we obtain by the monotonicity of the function λ that

$$(r+s)\lambda(p-r) \leq (r+s)\lambda(0) \leq (2c - b_s)\lambda(0) \leq B$$

and we have verified the result. □

We now analyze optimization problem (P_1) in case the budget B is sufficient for all feasible rebate and subsidy values given by the set

$$M := \{(s,r) : 0 \leq s \leq c - b_s, 0 \leq r \leq c\}. \tag{3.29}$$

Theorem 3.2.1. *If $\alpha \geq 0$ and the selected demand function λ belonging to $\mathcal{D}$ satisfies $(2c - b_s)\lambda(0) \leq B$ it is optimal to select in optimization problem (P_1) the rebate value equal to c, the subsidy value equal to $c - b_s$, the price equal to c and the production quantity equal to $\lambda(0)$. This choice has optimal objective value*

$$\upsilon(P_1) = \lambda(0)(\alpha + c - b_s) + \beta(0). \tag{3.30}$$

The contribution to the optimal objective value of the profit of the manufacturer is $(c - b_s)\lambda(0) > 0$, the contribution of the consumer surplus is $\beta(0)$ and the contribution of the externalities is $\alpha\lambda(0)$, while the government spends $(2c - b_s)\lambda(0) \leq B$.

Proof. By Lemma 3.2.5 and relation (3.27) we obtain with the set F listed in relation (2.9) and the function $\gamma : (0,r) \to \mathbb{R}$ given by

$$\begin{aligned} \gamma(r) &:= \sup_{r \leq p \leq r + p_{\max}} \{\lambda(p-r)(p + \alpha - b_s) + \beta(p-r)\} \\ &= \sup_{0 \leq u \leq p_{\max}} \{\lambda(u)(u + r + \alpha - b_s) + \beta(u)\}. \end{aligned} \tag{3.31}$$

that

$$
\begin{aligned}
v(P_1) &= \sup_{(s,r,p)\in G_1}\{\lambda(p-r)(p+s+\alpha-c)+\beta(p-r)\} \\
&= \sup_{(s,r,p)\in G}\{\lambda(p-r)(p+s+\alpha-c)+\beta(p-r)\} \\
&= \sup_{(r,p)\in F}\{\lambda(p-r)(p+\alpha-b_s)+\beta(p-r)\} \\
&= \sup_{0\le r\le c}\gamma(r).
\end{aligned}
\tag{3.32}
$$

By relation (3.32) it is obvious that the optimal subsidy equals $c-b_s$. Since $\lambda(u)\geq 0$ for every $0\leq u\leq p_{\max}$ it follows for every $0\leq u\leq p_{\max}$ that the function

$$r\to\lambda(u)(u+r+\alpha-b_s)+\beta(u)$$

is increasing. This shows that the function γ in (3.31) is increasing on $(0,c)$ and by relation (3.32) we obtain that the optimal rebate value equals c and

$$
\begin{aligned}
v(P_1) &= \gamma(c) \\
&= \sup_{0\le u\le p_{\max}}\{\lambda(u)(u+c+\alpha-b_s)+\beta(u)\}.
\end{aligned}
$$

Since $\alpha\geq 0$ and $c>b_s$ and hence $c+\alpha-b_s\geq 0$ it follows by Lemma 3.1.1 that

$$v(P_1)=\sup_{0\le u\le p_{\max}}\{\lambda(u)(u+c+\alpha-b_s)+\beta(u)\}=\lambda(0)(c+\alpha-b_s)+\beta(0)\}$$

and so the optimal price equals $r=c$. Substituting the optimal price $p=c$, the optimal rebate $r=c$ and the optimal subsidy $s=c-b_s$ into relations (2.11),(2.17) and (2.18) gives the total spending of the government and the contribution of the profit of the manufacturer, the consumer surplus and the externalities to the optimal objective value in a centralized economy. □

We now consider any demand function λ for which the available budget B is not sufficient to cover some feasible rebate and subsidy values belonging to the set M. To cover this case we consider two mutually exclusive subcases. First we consider the subcase (for $0\leq\alpha<c$) that

$$B\leq(2c-b_s)\lambda(c-\alpha)<(2c-b_s)\lambda(0) \tag{3.33}$$

If we have selected a demand function satisfying relation (3.33) we will identify in the next result optimal solutions of optimization problem (P_1).

Theorem 3.2.2. *If $\alpha\geq 0$ and the selected demand function λ belonging to $\mathcal{D}$ satisfies*

$(2c-b_s)\lambda((c-\alpha)^+) \geq B$ it is optimal to select in optimization problem (P_1) the rebate value equal to $0 \leq r \leq c$, the subsidy value equal to $0 \leq s \leq c-b_s$ such that $(r+s)\lambda((c-\alpha)^+) = B$. Moreover, the price equals $r+(c-\alpha)^+$ and production quantity equals $\lambda((c-\alpha))^+)$ and this choice has optimal objective value

$$v(P_1) = \lambda((c-\alpha)^+)((c-\alpha)^+ + \alpha - c) + \beta((c-\alpha)^+) + B. \tag{3.34}$$

The contribution to the optimal objective value of the profit of the manufacturer is $B+((c-\alpha)^+ - c)\lambda((c-\alpha)^+)$, the contribution of the consumer surplus is $\beta((c-\alpha)^+)$ and the contribution of the externalities is $\alpha\lambda((c-\alpha)^+)$, while the government spends the budget B.

Proof. Since $(2c-b_s)\lambda((c-\alpha)^+) \geq B$ it is easy to find some $0 \leq s_* \leq c-b_s$ and $0 \leq r_* \leq c$ satisfying

$$(r_* + s_*)\lambda((c-\alpha)^+) = B$$

It is obvious that the vector $(s_*, r_*, r_* + (c-\alpha)^+)$ belongs to the set G_1 and we will now show that this vector is an optimal solution of optimization problem (P_1) . For any chosen $(r,s,p) \in G_1$ it follows using Lemma 3.1.1 and $(r+s)\lambda(p-r) \leq B$ that

$$\begin{aligned} f(s,r,p) &= f_{\alpha-c}(p-r) + (r+s)\lambda(p-r) \\ &\leq f_{\alpha-c}((c-\alpha)^+) + B \\ &= f_{\alpha-c}(p_* - r_*) + (r_* + s_*)\lambda((c-\alpha)^+) \\ &= f(s_*, r_*, p_*) \end{aligned}$$

Hence by Lemma 3.2.4 we obtain that $p = r + (c-\alpha)^+$ and (s,r) satisfying $(s + r)\lambda((c-\alpha)^+) = B$ are the optimal solutions. Substituting these optimal solutions into relations (2.11),(2.17) and (2.18) yields the last part of the result. □

It is not clear under the conditions of the above theorem whether the manufacturer always gains a profit. We only can conclude that his profit is always bounded above by $(c-b_s+(c-\alpha)^+)\lambda(c-\alpha)^+)$. In the next theorem we consider the second case $0 \leq \alpha < c$ and we select a demand function λ belonging to $\mathcal{D}$ satisfying

$$(2c-b_s)\lambda(0) > B > (2c-b_s)\lambda(c-\alpha). \tag{3.35}$$

Theorem 3.2.3. *If $0 \leq \alpha < c$ and the selected demand function λ belonging to $\mathcal{D}$ satisfies $(2c-b_s)\lambda(0) > B > (2c-b_s)\lambda(c-\alpha)$ it is optimal to select in optimization* *problem* (P_1) *the rebate value equal to* c*, the subsidy value equal to* $c - b_s$*, the price equal to* $c + u_*$ *with* $0 < u_* < c - \alpha$ *satisfying:*

$$\lambda(u_*)(2c - b_s) = B \tag{3.36}$$

and the production quantity equal to $\lambda(u_*)$*. The optimal objective value is given by*

$$\upsilon(P_1) = \lambda(u_*)(u_* + \alpha - c) + \beta(u_*) + B = \lambda(u_*)(u_* + c + \alpha - b_s) + \beta(u_*),$$

The contribution to the optimal objective value of the profit of the manufacturer is $(u_* + c - b_s)\lambda(u_*) > 0$*, the contribution of the consumer surplus is* $\beta(u_*)$ *and the contribution of the externalities is* $\alpha\lambda(u_*)$*, while the government spends the budget* B*.*

Proof. By relation (3.28) it follows using the definition of the set M in relation (3.29) that the objective value $\upsilon(P_1)$ can be rewritten as

(3.37)

$$\begin{aligned} &\upsilon(P_1) \\ = \ & \sup_{(s,r,p)\in G_1}\{f_{\alpha-c}(p-r) + (r+s)\lambda(p-r)\} \\ = \ & \sup_{(s,r)\in M}\{\sup_{r\le p\le r+p_{\max},(s+r)\lambda(p-r)\le B}\{f_{\alpha-c}(p-r) + (r+s)\lambda(p-r)\}\} \\ = \ & \sup_{(s,r)\in M}\{\sup_{0\le u\le p_{\max},(s+r)\lambda(u)\le B}\{f_{\alpha-c}(u) + (r+s)\lambda(u)\}\} \\ = \ & \sup_{0\le u\le p_{\max},(s+r)\lambda(u)\le B,(s,r)\in M}\{f_{\alpha-c}(u) + (r+s)\lambda(u)\} \\ = \ & \sup_{0\le u\le p_{\max}}\{\gamma(u)\} \end{aligned}$$

with

$$\gamma(u) := \sup_{(s,r)\in M,(r+s)\lambda(u)\le B}\{f_{\alpha-c}(u) + (r+s)\lambda(u)\}$$

To simplify the expression for $\gamma(u)$ we observe for every $0 \le u \le p_{\max}$ and λ is a positive function on $(0, p_{\max})$ that

$$\begin{aligned} \gamma(u) &= f_{\alpha-c}(u) + \lambda(u)\sup_{0\le s\le c-b_s,0\le r\le c,(r+s)\lambda(u)\le B}\{r+s\} \\ &= f_{\alpha-c}(u) + \lambda(u)\sup_{0\le s\le c-b_s,0\le r\le c,(r+s)\le \min\{2c-b_s,\frac{B}{\lambda(u)}\}}\{r+s\} \\ &= f_{\alpha-c}(u) + \lambda(u)\min\{2c-b_s, \tfrac{B}{\lambda(u)}\} \\ &= \min\{f_{\alpha-c}(u) + (2c-b_s)\lambda(u), f_{\alpha-c}(u) + B\} \end{aligned} \tag{3.38}$$

Introduce now the function $f:(0,p_{\max})\to\mathbb{R}$ given by

$$(3.39)\qquad \begin{aligned} f(u) &= \min\{f_{\alpha-c}(u)+(2c-b_s)\lambda(u), f_{\alpha-c}(u)+B\} \\ &= \min\{f_{c+\alpha-b_s}(u), f_{\alpha-c}(u)+B\} \end{aligned}$$

Since $0\le\alpha\le c$ and $c>b_s$ we obtain $c+\alpha-b_s\ge 0$ and by Lemma 3.1.1 the function $f_{c+\alpha-b_s}$ is decreasing on $(0,p_{\max})$ and the function $f_{\alpha-c}$ is increasing on $(0,c-\alpha)$ and decreasing on $(c-\alpha,p_{\max})$. This shows that the function f in (3.39) is decreasing on $(c-\alpha,p_{\max})$ and by relations (3.37) and (3.38) we obtain

$$v(P_1)=\sup_{0\le u\le c-\alpha}\{f(u)\}$$

Since $(2c-b_s)\lambda(0)>B>(2c-b_s)\lambda(c-\alpha)$ it follows

$$(3.40)\qquad f(0)=\min\{f_{c+\alpha-b_s}(0), f_{\alpha-c}(0)+B\}=f_{\alpha-c}(0)+B<f_{c+\alpha-b_s}(0)$$

and

$$(3.41)\qquad f(c-\alpha)=f_{c+\alpha-b_s}(c-\alpha)<f_{\alpha-c}(c-\alpha)+B$$

By the continuity of both functions in the definition of the function f and relations (3.40) and (3.41) this implies the existence of some $0<u_*<c-\alpha$ satisfying $f_{c+\alpha-b_s}(u_*)=f_{\alpha-c}(u_*)+B$ or equivalently

$$(3.42)\qquad \lambda(u_*)(2c-b_s)=B.$$

Since by Lemma 3.1.1 the function $f_{c+\alpha-b_s}$ is decreasing on $(0,p_{\max})$ and the function $f_{\alpha-c}$ increasing on $(0,c-\alpha)$ we obtain that the function f is increasing on $(0,u_*)$ and decreasing on $(u_*,c-\alpha)$ and this shows that the optimal price to be paid by the customer equals $r+u_*$ with $(s,r)\in M$ satisfying $\lambda(u_*)(2c-b_s)=B$. Using relation (3.38) we need $r+s=2c-b_s$ for $(s,r)\in M$ and so the only possible candidates are $r=c$ and $s=c-b_s$ showing that an optimal solution is given by $s=c-b_s$, $r=c$ and $p=c+u_*$. Substituting these optimal solutions into relations (2.11),(2.17) and (2.18) yields the last part of the result. □

Observe in Theorem 3.2.3 it follows for $0<\alpha<c$ that in a centralized economy the price paid by the consumer is less than in Theorem 3.2.2. This is caused by the lower demand at price $c-\alpha$ for the public interest good in Theorem 3.2.3 and so the government can increase rebates and subsidies without exceeding the available budget. This reduces the price to be paid by the customer. If in a centralized

economy the public interest good is more important to society and so α increases the price a customer has to pay decreases. Eventually for $\alpha \geq c$ the customer obtains the public interest good for free and the manufacturer is forced to produce for the whole market. Finally, if in a centralized economy the available budget is not sufficient to cover every feasible rebate and subsidy policy the government will spend all its budget on the selected rebate and subsidy. At the same time the government forces the manufacturer to pay for the remaining cost of producing the imposed production quantity.

In the next examples we give for the linear, loglinear and power demand function an analytical solution for the price u_* to be paid by the customer (after reduction rebate!) defined in Theorem 3.2.3.

Example 3.2.1. *Let the demand function* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *be given by the linear demand function*

$$\lambda(p) = (a - bp)^+ \tag{3.43}$$

and assume the conditions of Theorem 3.2.3 hold. This means in this particular case that the parameters a, b, c, b_s *and* α *satisfy* $0 \leq \alpha < c < \frac{a}{b}$ *and* $(2c - b_s)a > B > (2c - b_s)(a + b(c - \alpha))$. *By (3.36) we observe* $a - bu_* = \frac{B}{2c - b_s}$ *and so the price the customer has to pay (after reduction rebate value!) is given by*

$$u_* = \frac{1}{b}(a - \frac{B}{2c - b_s}) \tag{3.44}$$

Since the optimal rebate value equals c the price the manufacturer demands is given by $\frac{1}{b}(a - \frac{B}{2c - b_s}) + c$.

In the next example we consider the loglinear demand function

Example 3.2.2. *Let the demand function* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *be given by the loglinear demand function*

$$\lambda(p) = ae^{-bp}, a > 0, b > 0 \tag{3.45}$$

and assume the conditions of Theorem 3.2.3 hold. This means for this demand function that the parameters a, b, c, b_s *and* α *satisfy* $0 \leq \alpha < c$ *and* $a(2c - b_s) > B > a(2c - b_s)e^{-b(c-\alpha)}$. *By (3.36 we know* $ae^{-bu_*} = \frac{B}{2c - b_s}$. *This shows that the price the customer (after reduction of optimal rebate c) is given by*

$$u_* = -\frac{1}{b}\ln(\frac{B}{a(2c - b_s)}) \tag{3.46}$$

The optimal price for the loglinear demand would be $p_{opt} = c - \frac{1}{b}\ln(\frac{B}{a(2c-b_s)})$.

In the next example we consider the power demand function.

Example 3.2.3. *Let the demand function* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *be given by the power demand function*

$$\lambda(p) = (a+bp)^{-\gamma}, a>0, b>0, \gamma>1 \tag{3.47}$$

and assume the conditions of Theorem 3.2.3 hold. This means for this demand function that the parameters a, b, c, b_s*,* γ *and* α *satisfy* $(2c-b_s)a^{-\gamma} > B > (2c-b_s)(a+b(c-\alpha)^{-\gamma})$*. By (3.36) we know that* $(a+bu_*)^{-\gamma} = \frac{B}{2c-b_s}$ *and this shows that the price paid by the customer is given by*

$$u_* = \frac{1}{b}(\frac{B}{2c-b_s})^{-\gamma^{-1}} - \frac{a}{b} \tag{3.48}$$

Since the optimal rebate equals c it follows that price demanded by the manufacturer is given by $c - \frac{a}{b} + \frac{1}{b}(\frac{B}{2c-b_s})^{-\gamma^{-1}}$.

In the final example we consider the logit demand function.

Example 3.2.4. *Let the demand function* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *be given by the logit demand function*

$$\lambda(p) = \frac{ae^{-bp}}{1+e^{-bp}}, a>0, b>0 \tag{3.49}$$

and assume the conditions of Theorem 3.2.3 hold. This means for this demand function that the parameters a, b, c, b_s *and* α *satisfy* $(2c-b_s)\frac{a}{2} > B > a(2c-b_s)\frac{e^{-b(c-\alpha)}}{1+e^{-b(c-\alpha)}}$*. By (3.36) we know that* $\frac{ae^{-bu_*}}{1+e^{-bu_*}} = \frac{B}{2c-b_s}$ *and this shows that the price paid by the customer is given by*

$$u_* = \frac{-1}{b}\ln(\frac{a(2c-b_s)}{B} - 1). \tag{3.50}$$

The next procedure summarizes the solution steps for the optimization problem (P_1).

Numerical Solution Procedure 3.2.1. *Solution procedure centralized version of optimization problem* (P_1) *for any* $\alpha \geq 0$ *and any continuous demand function.*

- *STEP 1. Compute* $(2c-b_s)\lambda(0)$.
- *STEP 2. If* $(2c-b_s)\lambda(0) < B$*, go to step 3. If not compute* $(2c-b)\lambda((c-\alpha)^+)$*. If* $(2c-b)\lambda((c-\alpha)^+) \geq B$ *go to step 4. Otherwise go to step 5.*

- *STEP 3. Optimal rebate equals c, optimal subsidy equals* $c-b_s$*, optimal price equals c and optimal production quantity equals* $\lambda(0)$ *and the expenditure of the government equals* $(2c-b_s)\lambda(0) \leq B$*.*

- *STEP 4. Compute the value*

$$r_* := \sup\{0 \leq r \leq c : r\lambda((c-\alpha)^+) \leq B\}.$$

An optimal rebate equals r_**, an optimal subsidy equals* $\frac{B}{\lambda((c-\alpha)^+)} - r_*$*, the optimal price equals* $r_* + (c-\alpha)^+$ *and the optimal production quantity equals* $\lambda((c-\alpha)^+)$ *and the expenditure of the government equals its available budget B.*

- *STEP 5. The optimal rebate equals c, the optimal subsidy equals* $c-b_s$*, the optimal price equals* $u_* + c$ *with* u_* *satisfying* $\lambda(u_*)(2c-b_s) = B$*, the optimal quantity* q_{opt} *equals* $\lambda(u_*)$ *and the expenditure of the government equals its available budget B.*

The table below summarizes the optimal solution and the contribution of the consumer surplus, profit manufacturer, and externalities to the optimal objective value of the centralized version of problem (P_1) for every budget B. In the last two columns, the government spends all its available budget B while in the first column the government spends $(2c-b_s)\lambda(0) \leq B$.

Table 3.2 shows a summary of results for optimization problem (P_1) considering the budget classification. The value for $r_* \leq c$ and $0 < u_* \leq c-\alpha$ are defined in Solution procedure 3.2.1 and Theorem 3.2.3.

Table 3.2 Results for problem (P_1) in centralized economy.

Budget Value Condition	$(2c-b_s)\lambda(0) \leq B$	$(2c-b_s)\lambda(0) > B > (2c-b_s)\lambda((c-\alpha)^+)$	$(2c-b_s)\lambda((c-\alpha)^+) \geq B$
optimal subsidy	$c-b_s$	$c-b_s$	$\frac{B}{\lambda((c-\alpha)^+)} - r_* \leq c-b_s$
optimal rebate	c	c	$r_* \leq c$
optimal price-optimal rebate	0	$0 \leq u_* \leq (c-\alpha)^+$	$(c-\alpha)^+$
optimal production quantity	$\lambda(0)$	$\lambda(u_*) = \frac{B}{2c-b_s} < \lambda(0)$	$\lambda((c-\alpha)^+)$
optimal consumer surplus	$\beta(0)$	$\beta(u_*)$	$\beta((c-\alpha)^+)$
optimal externality	$\alpha\lambda(0)$	$\alpha\lambda(u_*)$	$\alpha\lambda((c-\alpha)^+)$
optimal profit manufacturer	$(c-b_s)\lambda(0)$	$(u_*+c-b_s)\lambda(u_*)$	$(r_*+s_*+(c-\alpha)^+-c)\lambda((c-\alpha)^+)$

4. Formulation of the Model in a Decentralized Economy

In the decentralized version of our problem the government lets the manufacturer decide about the price and the offered quantity. Given the rebate value $0 \leq r \leq c$ and subsidy $0 \leq s \leq c - b_s$ the manufacturer optimizes his profit. This means that the manufacturer solves for every λ belonging to $\mathcal{D}$ satisfying $\lambda(c) > 0$ and every $(r,s) \in M$ with the set M defined in (3.29) the optimization problem

$$(M(s,r)) \qquad v(s,r) := \sup_{r \leq p \leq p_{\max}+r, q \geq 0} m(s,r,p,q)$$

and the function m listed in relation (2.17) given by

$$m(s,r,p,q) = (p - b_s)\min\{q, \lambda(p-r)\} - (c - s - b_s)q.$$

In the next subsection, we analyze the optimization problem for the manufacturer facing a rebate value $0 \leq r \leq c$ and a subsidy value $0 \leq s \leq c - b_s$.

4.1 Analysis of the Optimization Problem for the Manufacturer

We first show by standard arguments that the optimization problem $(M(s,r))$ of the manufacturer has an optimal solution. Before discussing this result we introduce for every $(s,r) \in M$ the set $B(s,r) \subseteq \mathbb{R}_+$ given by

$$(4.1) \qquad B(s,r) := [r + (c - s - r)^+, r + p_{\max}]$$

with $x^+ := \max\{0,x\}$. Since $\lambda(c) > 0$ and hence $c < p_{\max}$ this implies using $r,s \geq 0$ that $(c-s-r)^+ < p_{\max}$ and so the set $B(r,s)$ is nonempty for every $(s,r) \in M$. In particular it follows using $c > b_s$ that

$$B(0,0) = [c, p_{\max}], B(c-b_s, c) = [c, c+p_{\max}]. \tag{4.2}$$

Lemma 4.1.1. *If λ belongs to $\mathcal{D}$ and $\lambda(c) > 0$ then for every (s,r) belonging to M the optimization problem $(M(s,r))$ to be solved by the manufacturer reduces to*

$$(M(s,r)) \qquad v(s,r) = \sup_{r \le p \le p_{\max}+r}\{(p-c+s)\lambda(p-r)\} > 0.$$

The optimization problem $(M(s,r))$ has an optimal solution and for (s,r) belonging to M and $s < c - b_s$ every optimal solution $(p_{opt}(s,r), q_{opt}(s,r))$ satisfies $p_{opt}(s,r) \ge r + (c-s-r)^+$ and

$$q_{opt}(s,r) = \lambda(p_{opt}(s,r) - r).$$

For (s,r) belonging to M and $s = c - b_s$ the optimization problem $(M(s,r))$ has multiple optimal production quantities and we alway select again $q_{opt}(s,r) = \lambda(p_{opt}(s,r) - r)$.

Proof. Clearly the optimization problem to be solved by the manufacturer facing a rebate $0 \le r \le c$ and subsidy $0 \le s \le c - b_s$ is given by

$$v(s,r) := \sup_{r \le p \le p_{max}+r, q \ge 0} m(s,r,p,q) = \sup_{r \le p \le p_{max}+r} v_0(s,r,p)$$

with

$$(M_0) \qquad v_0(s,r,p) := \sup_{q \ge 0}\{(p-b_s)\min\{q, \lambda(p-r)\} - (c-s-b_s)q\}.$$

It is easy to see using $0 \le s \le c - b_s$ that an optimal solution of the above parametrized optimization problem (M_0) for each price $p \ge r$ equals $q_{opt}(p) = \lambda(p-r)$ and this optimal production quantity is unique if $s < c - b_s$. Hence the optimization problem $(M(s,r))$ reduces to the optimization problem

$$v(s,r) = \sup_{r \le p \le p_{max}+r}\{m_0(s,r,p)\}$$

with the function $m_0 : M \times (r, r+p_{max}) \to \mathbb{R}$ given by

$$m_0(s,r,p) := m(r,s,p,\lambda(p-r)) = (p-c+s)\lambda(p-r). \tag{4.3}$$

Since $\lambda(c) > 0$ or $c < p_{\max}$ there exists some $r \le c \le p_* < p_{\max} + r$ satisfying $\lambda(p_* -$

$r) > 0$ and hence for every (s,r) belonging to M we obtain

$$m_0(s,r,p_*) = (p_* - c + s)\lambda(p_* - r) > 0. \tag{4.4}$$

This shows $\sup_{r \le p \le p_{max}+r}\{m_0(s,r,p)\} > 0$ and by the continuity of the function m_0 and for $p_{\max}$ finite the result follows using Weierstrass theorem (cf.Rudin (1982)). If $p_{\max} = \infty$ we obtain using $\lambda \in \mathcal{D}$ that for every $0 \le r \le c$ and $0 \le s \le c - b_s$

$$\lim_{p \uparrow \infty} m_0(r,s,p) = 0.$$

Since by (4.4) we know that $\gamma := \sup_{r \le p \le p_{max}+r}\{m_0(s,r,p)\} > 0$ we may restrict the unbounded feasible region (r,∞) to the bounded and closed feasible region $\{p \ge r : m_0(s,r,p) \ge \frac{1}{2}\gamma\}$. Applying again Weierstrass theorem yields that an optimal solution $p_{opt}(s,r)$ exists. Hence

$$0 < \gamma = m_0(s,r,p_{opt}(s,r)) = (p_{opt}(s,r) - c + s)\lambda(p_{opt}(s,r) - r)$$

and so $p_{opt}(s,r) > c - s$ Since $r + p_{\max} \ge p_{opt}(s,r) \ge r$ this implies

$$r + p_{\max} \ge p_{opt}(s,r) \ge \max\{r, c - s\} = r + (c - s - r)^+$$

and we have verified the results. □

We will first analyze the optimization problem $(M(s,r))$ for some often used demand functions (cf.Talluri & Van Ryzin (2004)) before analyzing this problem in more detail for a general class of demand functions.

Example 4.1.1. *If the demand function λ is given by the linear demand function*

$$\lambda(p) = (a - bp)^+, a > 0, b > 0$$

then $\lambda(c) > 0$ implies that the parameters a and b should satisfy

$$a > bc \tag{4.5}$$

For this particular choice $p_{max} = ab^{-1} < \infty$ and so the linear demand function belongs to the set $\mathcal{D}$ listed in relation (2.14). Also by reation (4.3) we obtain that

$$m_0(s,r,p) = (p - c + s)(a - b(p - r))^+$$

and so for every $(s,r) \in M$ the optimization problem $(M(s,r))$ for the linear demand, *function reduces to*

$$(4.6) \qquad v(s,r) = \sup_{r \le p \le ab^{-1}+r}\{(p-c+s)(a-b(p-r))\}.$$

By Lemma 4.1.1 this optimization problem has for every $(s,r) \in M$ *an optimal solution* $r+(c-s-r)^+ \le p_{opt}(s,r) < ab^{-1}+r$. *We observe that for every* $(s,r) \in M$ *the objective function*

$$p \to m_0(s,r,p) = (p-c+s))(a-b(p-r))$$

is logconcave on $\left(r+(c-s-r)^+, ab^{-1}+r\right)$ *and hence the logarithmic transformation of the optimization problem* $(M(s,r))$ *is a concave maximization problem. The partial derivative of the objective function* $m_0 : M \times (r, r+p_{max}) \to \mathbb{R}$ *with respect to* p *equals*

$$(4.7) \qquad \frac{\partial m_0}{\partial p}(s,r,p) = a + b(r+c-s) - 2bp.$$

Since for $p = r$ *we obtain using* $a > bc$ *that*

$$\frac{\partial m_0}{\partial p}(s,r,r) = a + b(c-s-r) > bc + b(c-s-r) = b(2c-s-r) > 0$$

it follows by relation (4.7) and the logarithmic transformation of optimization problem $(M(s,r))$ *is a concave maximization problem that*

$$(4.8) \qquad p_{opt}(s,r) = \frac{1}{2}\left(ab^{-1}+r+c-s\right).$$

Its optimal objective is then given by

$$(4.9) \qquad v(s,r) = (p_{opt}(s,r) - c + s)(a - bp_{opt}(s,r)).$$

In the next example we consider the loglinear function.

Example 4.1.2. *If the demand function* λ *is given by the loglinear demand function*

$$\lambda(p) = ae^{-bp}, a > 0, b > 0$$

then we obtain $\lambda(c) > 0$ *for any* $a, b > 0$ *and* $p_{\max} = \infty$. *It is easy to verify that the loglinear demand function belongs to the set* $\mathcal{D}$. *Also by reation (4.3) we obtain that*

$$m_0(s,r,p) = a(p-c+s)e^{-b(p-r)}$$

and so for every (s,r) *belonging to* M *the optimization problem* $(M(s,r))$ *for the loglinear demand function reduces to*

$$v(s,r) = ae^{br}\sup_{r \leq p < \infty}\{(p-c+s)e^{-bp}\}. \tag{4.10}$$

By Lemma 4.1.1 this optimization problem has for every (s,r) *belonging to* M *an optimal solution* $p_{opt}(r,s)$ *satisfying* $r+(c-s-r)^+ \leq p_{opt}(s,r) < \infty$*. We now observe for every* $(s,r) \in M$ *that the objective function*

$$p \to m_0(s,r,p) = ae^{br}(p-c+s)e^{-bp}$$

is log-concave on $(r+(c-s-r)^+,\infty)$ *and the logarithmic transformation of the optimization problem* $(M(s,r))$ *is a concave maximization problem. The partial derivative of the objective function* $m_0 : M \times (r, r+p_{max}) \to \mathbb{R}$ *with respect to* p *is given by*

$$\frac{\partial m_0}{\partial p}(s,r,p) = ae^{-b(p-r)}(1-b(p+s-c)). \tag{4.11}$$

Hence for $p=r$ *we obtain* $\frac{\partial m_0}{\partial p}(s,r,r) = a(1-b(r+s-c))$ *and this implies for* $r+s \geq c+b^{-1}$ *or* $\frac{\partial m_0}{\partial p}(s,r,r) \leq 0$ *that* $p_{opt}(s,r)$ *equals*

$$p_{opt}(s,r) = r. \tag{4.12}$$

For $0 \leq r+s < c+b^{-1}$ *it follows by relation (4.11) that* $p_{opt}(s,r)$ *equals*

$$p_{opt}(s,r) = b^{-1}+c-s. \tag{4.13}$$

Combining both cases discussed in (4.12) and (4.13) a compact notation for $p_{opt}(s,r)$ *is given by*

$$p_{opt}(s,r) = \max\{r, b^{-1}+c-s\}. \tag{4.14}$$

Its optimal objective value is then given by

$$v(s,r) = ae^{br}(p_{opt}(s,r)-c+s)e^{-bp_{opt}(s,r)}. \tag{4.15}$$

In the next example we list the power demand function.

Example 4.1.3. *If the demand function* λ *is given by the power demand function*

$$\lambda(p) = (a+bp)^{-\gamma}, a>0, b>0, \gamma>1$$

then we obtain $\lambda(c) > 0$ *for any* $a, b > 0$ *and* $\gamma > 1$ *and* $p_{max} = \infty$. *It is easy to verify that the power demand function belongs to the set* $\mathcal{D}$. *Also by relation (4.3) it follows*

$$m_0(s,r,p) = (p-c+s)\Big(a+b(p-r)\Big)^{-\gamma}$$

and so for every (s,r) *belonging to* M *the optimization problem* $M(s,r)$ *for the power demand function reduces to*

$$v(s,r) = \sup_{r \leq p \leq \infty} \{(p-c+s)(a+b(p-r))^{-\gamma}\}. \tag{4.16}$$

We now observe for every $(s,r) \in M$ *that the partial derivative of the objective function* $m_0 : M \times (r, r+p_{max}) \to \mathbb{R}$ *with respect to* p *is given by*

$$\frac{\partial m_0}{\partial p}(s,r,p) = (a+b(p-r))^{-\gamma}\Big(1 - \frac{\gamma b(p-c+s)}{a+b(p-r)}\Big) \tag{4.17}$$

Hence for $p = r$ *we obtain* $\frac{\partial m_0}{\partial p}(s,r,r) = a^{-\gamma}\Big(1 - \frac{\gamma b(r+s-c)}{a}\Big)$ *and this implies for* $r+s-c \geq \frac{a}{\gamma b}$ *or* $\frac{\partial m_0}{\partial p}(s,r,r) \leq 0$ *that* $p_{opt(s,r)}$ *equals*

$$p_{opt}(s,r) = r. \tag{4.18}$$

Also for $r+s-c \leq \frac{a}{\gamma b}$ *we obtain by relation (4.17) that* $p_{opt(s,r)}$ *equals*

$$p_{opt}(s,r) = \frac{a - br + (c-s)\gamma b}{(\gamma-1)b}. \tag{4.19}$$

Combining both cases discussed in (4.18) and (4.19) a compact notation for $p_{opt}(s,r)$ *is given by*

$$p_{opt}(s,r) = \max\{r, \frac{a-br+(c-s)\gamma b}{(\gamma-1)b}\}. \tag{4.20}$$

The optimal objective value equals

$$v(s,r) = (p_{opt}(s,r) - c + s)\Big(a + b(p_{opt}(s,r) - r)\Big)^{-\gamma} \tag{4.21}$$

Observe for the power demand function the function $ln(\lambda)$ *is convex but with* λ' *denoting the derivative of the function* λ *the function*

$$p \mapsto (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)} = -\gamma + \frac{\gamma(b(c-s)+a)}{a+bp} \tag{4.22}$$

is strictly decreasing. Hence we can apply the results of this thesis also to the power demand function.

In the next example we consider the logit demand function.

Example 4.1.4. *If the demand function* λ *is given by the logit demand function*

$$\lambda(p) = \frac{ae^{-bp}}{1+e^{-bp}},\ a>0, b>0$$

we obtain $\lambda(c) > 0$ *for any* $a, b > 0$ *and* $p_{\max} = \infty$*. It is easy to verify that the logit demand function belongs to the set* $\mathcal{D}$*. Also by relation (4.3) it follows*

$$m_0(s,r,p) = (p-c+s)\frac{ae^{-b(p-r)}}{1+e^{-b(p-r)}}$$

and so for every (s,r) *belonging to* M *the optimization problem* $(M(s,r))$ *reduces to*

$$v(s,r) = ae^{br}\sup_{r \le p < \infty}\left\{\frac{(p-c+s)e^{-bp}}{1+e^{-b(p-r)}}\right\}.$$

Since

$$\ln(\lambda(p)) = \ln(a) - bp - \ln(1+e^{-bp})$$

and the function $p \to \ln(1+e^{-p})$ *is convex on* $(0,\infty)$ *the logit demand function is log-concave. Hence the logarithmic transformation of optimization problem* $(M(s,r))$ *is a concave maximization problem. It follows after some analysis that for* $\frac{b(r-c+s)}{1-e^{(-b(0))}} \le 1$ *the optimal unique price* $p_{opt}(s,r)$ *is the solution of the nonlinear equation.*

$$1-e^{-b(p-r)} = b(p-c+s)$$

and for $\frac{b(r-c+s)}{1-e^{(-b(0^+))}} \ge 1$ *we obtain* $p_{opt}(s,r) = r$*.*

To determine under which conditions on the demand function the optimization problem $(M(s,r))$ can be easily solved and its optimal solution is unique we list the following result. Remember λ' denotes the derivative of the function λ.

Lemma 4.1.2. *If the function* λ *is differentiable and for every* $(s,r) \in M$ *the function*

$$p \to (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing on $B(s,r)$ *the optimal solution of optimization problem* $(M(s,r))$ *is unique and it is given by*

$$p_{opt}(s,r) = \inf\left\{p \in B(s,r) : (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)} \le -1\right\} \tag{4.23}$$

with $B(s,r)$ *listed in relation (4.1).*

Proof. For every (s,r) belonging to M the partial derivative of the function $m_0 : M\times(r,r+p_{max})\to\mathbb{R}$ listed in relation (4.3) with respect to p is given by

$$\begin{aligned}\frac{\partial m_0}{\partial p}(s,r,p) &= (p-c+s)\lambda'(p-r)+\lambda(p-r)\\ &= \lambda(p-r)\left(1+(p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}\right).\end{aligned}\tag{4.24}$$

By assumption the function $p\to(p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$ is strictly decreasing on $B(r,s)=(r+(c-s-r)^+,p_{\max}+r)$ and since $p_{\max}+r$ cannot be an optimal solution and by Lemma 4.1.1 there always exists an optimal solution the result follows by relation (4.24) using $\lambda(p-r)>0$ for every $r\le p<r+p_{\max}$. □

Although the power demand function is logconvex and not logconcave (see Lemma 4.1.3) it satisfies the condition of Lemma 4.1.2. If no easy analytical expression for $p_{opt}(s,r)$ exists it is easy to use bisection on the function

$$p\to(p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

to determine the optimal price $p_{opt}(s,r)$ demanded by the manufacturer. For the function

$$p\to(p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing and continuous on $B(r,s)$ we can slightly improve the above result. If $r+s\le c$ it follows for $p=r+(c-s-r)^+$ being the smallest value of the set $B(s,r)$ that $p=c-s$ and this implies

$$(c-s-c+s)\frac{\lambda'((c-s-r)^+)}{\lambda((c-s-r)^+)}=0>-1.$$

Hence for $r+s\le c$ we obtain by (4.23) that $p_{opt}(s,r)$ satisfies

$$(p_{opt}(s,r)-c+s)\frac{\lambda'(p_{opt}(s,r)-r)}{\lambda(p_{opt}(s,r)-r)}=-1\tag{4.25}$$

If $r+s>c$ it follows for $p=r+(c-s-r)^+$ being the smallest value of the set $B(s,r)$ that $p=r$. This shows in case $(r-c+s)\frac{\lambda'(0)}{\lambda(0)}\le-1$ that by (4.23)

$$p_{opt}(r,s)=\max\{r,c-s\}=r.\tag{4.26}$$

For $(r-c+s)\frac{\lambda^{(1)}(0)}{\lambda(0)} > -1$ we obtain by the same argument that $p_{opt}(s,r)$ satisfies

$$(4.27) \qquad (p_{opt}(s,r)-c+s)\frac{\lambda'(p_{opt}(s,r)-r)}{\lambda(p_{opt}(s,r)-r)} = -1.$$

Looking at the extreme cases of no subsidy and no rebate and subsidy $c-b_s$ and rebate c we obtain by Lemma 4.1.2 and relation (4.2) that

$$(4.28) \qquad p_{opt}(0,0) = \inf\left\{c \le p \le p_{max} : (p-c)\frac{\lambda'(p)}{\lambda(p)} \le -1\right\}$$

and

$$(4.29) \qquad p_{opt}(c-b_s,c) = \inf\left\{c \le p \le c+p_{\max} : (p-b_s)\frac{\lambda'(p-c)}{\lambda(p-c)} \le -1\right\}.$$

At the same time we obtain by Lemma 4.1.2 for every $(s,r) \in M$

$$(4.30) \qquad p_{opt}(s,r)-r = \inf\{u \in B(s,r)-r : (u-c+r+s)\frac{\lambda'(u)}{\lambda(u)} \le -1\}.$$

We now list a sufficient condition on the demand function λ for which the condition in Lemma 4.1.2 holds.

Lemma 4.1.3. *If the function $\lambda_1 : (0,p_{\max}) \to \mathbb{R}$ given by*

$$\lambda_1(p) = \ln(\lambda(p))$$

is concave and $\lambda'(u) < 0$ for every $0 < u < p_{\max}$ then it follows for every (s,r) belonging to M that the function

$$p \to (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing and negative on the set $B(s,r)$.

Proof. Since by assumption λ_1 is logconcave and $\lambda'(u) < 0$ for every $0 < u < p_{\max}$ we obtain for every (s,r) belonging to M that the function $p \to \frac{\lambda'(p-r)}{\lambda(p-r)}$ is decreasing and negative on $(r, r+p_{\max})$. This shows that the function

$$p \to (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing and negative on $B(s,r) = (r+(c-s-r)^+, p_{\max}+r)$ and we have shown the result. □

Observe the linear demand function, the loglinear demand function and the logit demand function satisfy the above sufficient condition. In the next result we show how the optimal price to be paid by the customer depends on the rebate r and the subsidy s. In particular, we show the intuitively clear result that the profit optimizing price demanded by the manufacturer minus the rebate value (so the actual price a customer has to pay) is decreasing when either the subsidy given by the government is increasing or the rebate given by the government is increasing.

Lemma 4.1.4. *If the function*

$$p \to (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing on $B(r,s)$ for every $(s,r) \in M$ then for every $0 \leq r \leq c$ the function $s \to p_{opt}(s,r)$ is decreasing on $(0,c-b_s]$ and for every $0 \leq s \leq c-b_s$ the function $r \to p_{opt}(s,r)-r$ is decreasing on $(0,c)$.

Proof. By the definition of $B(s,r)$ in relation (4.1) it is easy to see that for $0 \leq s_2 < s_1 < c-b_s$ we obtain

$$B(s_2,r) \subseteq B(s_1,r).$$

This implies by relation (4.23) that

$$\begin{aligned} p_{opt}(r,s_1) &= \inf\{p \in B(s_1,r) : (p-c+s_1)\tfrac{\lambda'(p-r)}{\lambda(p-r)} \leq -1\} \\ &\leq \inf\{p \in B(s_2,r) : (p-c+s_1)\tfrac{\lambda'(p-r)}{\lambda(p-r)} \leq -1\}. \end{aligned} \tag{4.31}$$

Since the function λ' is a nonpositive function and $s_1 > s_2$, we obtain for every $p \in B(s_2,r)$ that

$$(p-c+s_1)\frac{\lambda'(p-r)}{\lambda(p-r)} \leq (p-c+s_2)\frac{\lambda'(p-r)}{\lambda(p-r)},$$

and this shows by relation (4.31) that

$$p_{opt}(s_1,r) \leq \inf\{p \in B(s_2,r) : g_r(s_2,p) \leq -1\} = p_{opt}(s_2,r).$$

Hence the function $s \to p_{opt}(r,s)$ is decreasing on $(0,c-b_s)$. To show that the function $r \to p_{opt}(r,s)-r$ is decreasing we first observe that for $c \geq r_1 > r_2 \geq 0$ that by relation (4.1)

$$B(s,r_2)-r_2 \subseteq B(s,r_1)-r_1.$$

This implies by relation (4.30) that for $c \geq r_1 > r_2 \geq 0$

$$\begin{aligned} p_{opt}(s,r_1) &= \inf\{u \in B(s,r_1) - r_1 : (u - c + r_1 + s)\tfrac{\lambda'(u)}{\lambda(u)} \leq -1\} \\ &\leq \inf\{u \in B(s,r_2) - r_2 : (u - c + r_1 + s)\tfrac{\lambda'(u)}{\lambda(u)} \leq -1\}. \end{aligned}$$

Since λ' is a nonpositive function and $r_1 > r_2$ we obtain for every $u \in B(s,r_2) - r_2$ that

$$(u - c + r_1 + s)\frac{\lambda'(u)}{\lambda(u)} \leq (u - c + r_2 + s)\frac{\lambda'(u)}{\lambda(u)}$$

and this yields

$$\begin{aligned} p_{opt}(s,r_1) &\leq \inf\{u \in B(s,r_2) - r_2 : (u - c + r_1 + s)\tfrac{\lambda'(u)}{\lambda(u)} \leq -1\} \\ &\leq \inf\{u \in B(s,r_2) - r_2 : (u - c + r_2 + s)\tfrac{\lambda'(u)}{\lambda(u)} \leq -1\} \\ &= p_{opt}(s,r_2) - r_2 \end{aligned}$$

showing the desired result. □

If $\lambda'(u) < 0$ for every $0 < u < p_{\max}$ then it is easy to see from the above proof that the functions $s \to p_{opt}(s,r)$ and $r \to p_{opt}(s,r) - r$ are strictly decreasing on $(0, c - b_s)$, respectively $(0, c)$. An important implication of Lemma 4.1.4 is given by the following result.

Lemma 4.1.5. *If the function*

$$p \to (p - c + s)\frac{\lambda'(p-r)}{\lambda(p-r)} \tag{4.32}$$

is strictly decreasing on $B(s,r)$ *for every* $(s,r) \in M$ *then for every* (s,r) *belonging to* M

$$0 \leq p_{opt}(c - b_s, c) - c \leq p_{opt}(s,r) - r \tag{4.33}$$

If additionally the function in relation (4.32) is continuous then for any $\alpha < c$ *it follows*

$$p_{opt}(c - b_s, c) - c > c - \alpha \Leftrightarrow (2c - \alpha - b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} > -1. \tag{4.34}$$

Proof. By Lemma 4.1.4 and relation (4.29) we obtain for every (s,r) belonging to M that

$$p_{opt}(s,r) - r \geq p_{opt}(c - b_s, r) - r \geq p_{opt}(c - b_s, c) - c \geq 0$$

Since by relation (4.29)

$$p_{opt}(c-b_s,c) = \inf\left\{c \leq p \leq c+p_{\max:} : (p-b_s)\frac{\lambda'(p-c)}{\lambda(p-c)} \leq -1\right\}$$

and by assumption the function $p \to (p-b)\frac{\lambda'(p-c)}{\lambda(p-c)}$ is strictly decreasing on $[c, c+p_{\max}]$ and continuous it is easy to see for any $0 \leq \alpha < c$ that

$$p_{opt}(c-b_s,c) > 2c-\alpha \Leftrightarrow (2c-\alpha-b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} > -1$$

and so the last part follows. □

Taking into account the profit optimizing behaviour of the manufacturer facing some subsidy and rebate, we introduce in the next section the two different decentralized optimization problems.

4.2 A More Detailed Formulation of the Model

To analyze the optimization problem for the government in a decentralized economy we introduce for every $\lambda \in \mathcal{D}$ the nonempty set $M_\lambda \subseteq M$ with M defined in relation (3.29) given by

(4.35) $$M_\lambda := \{(s,r) \in M : (s+r)\lambda(p_{opt}(s,r)-r) \leq B\}$$

It is always assumed in this section that the optimization problem $(M(s,r))$ has a unique optimal solution $p_{opt}(s,r)$ for any $(s,r) \in M$. The next result is now easy to verify.

Lemma 4.2.1. *If $\alpha \geq 0$ and for every (s,r) belonging to M the manufacturers optimization problem $(M(s,r))$ has a unique optimal price $p_{opt}(s,r)$, then*

(P) $$v(P) = \sup_{(s,r)\in M}\{f_{\alpha-c}(p_{opt}(s,r)-r)\}$$

and

(P_1) $$v(P_1) = \sup_{(s,r)\in M_\lambda}\{f_{\alpha-c}(p_{opt}(s,r)-r)+(r+s)\lambda(p_{opt}(s,r)-r)\}$$

with the function $f_{\alpha-c}$ listed in relation (3.9).

Proof. By Lemma 4.1.1 and relations (2.19) and (3.9) and $q_{opt}(s,r) = \lambda(p_{opt}(s,r) - r)$ the representation of the first optimization problem follows. Again by Lemma 4.1.1 and relations (2.20) and (3.9) using again $q_{opt}(s,r) = \lambda(p_{opt}(s,r) - r)$ we obtain the second optimization problem. □

In the next section we give a more detailed analysis of optimization problem (P) and identify its optimal solution.

4.3 Analysis of Optimization Problem (P)

In this section, we will give an analysis of the decentralized version of optimization problem (P). By Lemma 4.2.1, we know

$$(P) \qquad v(P) = \sup_{(s,r)\in M}\{f_{\alpha-c}(p_{opt}(s,r) - r)\}$$

with f_γ defined in relation (3.9). We will first identify optimal solutions of optimization problem (P) for $\alpha \geq c$.

Theorem 4.3.1. *If $\alpha \geq c$ and for every (s,r) belonging to M the function*

$$p \to (p - c + s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing on the set $B(s,r)$ it follows that in optimization problem (P) it is optimal to select the rebate value equal to c, the subsidy equal to $c - b_s$, the price equal to $p_{opt}(c - b_s, c)$, and the production quantity equal to $\lambda(p_{opt}(c - b_s, c) - c)$. The optimal objective value of optimization problem (P) is given by

$$v(P) = f_{\alpha-c}(p_{opt}(c - b_s, c) - c). \tag{4.36}$$

Proof. Since by Lemma 3.1.1 the function $f_{\alpha-c}$ is decreasing on $(0,\infty)$ for $\alpha \geq c$ it follows using relation (4.33) that for every $(s,r) \in M$

$$f_{\alpha-c}(p_{opt}(c - b_s, c) - c) \geq f_{\alpha-c}(p_{opt}(s,r) - r)$$

and this shows the desired result. □

By the above theorem using $f_{\alpha-c}$ is decreasing on $(0,\infty)$ for $\alpha \geq c$ it is equivalent to solve the optimization problem

$$(Q) \qquad \inf_{(s,r)\in M}\{p_{opt}(s,r)-r\}.$$

Using Lemma 4.1.4 this shows that the optimal subsidy is equal to $c-b_s$ and optimal rebate value equal to c. Using this alternative approach the government selects the optimal subsidy and rebate in such a way that given the optimization approach of the manufacturer the price the customer has to pay is minimal. We will now identify optimal solutions of optimization problem (P) for $0\leq\alpha<c$.

Theorem 4.3.2. *If $0\leq\alpha<c$ and for every (s,r) belonging to M the function*

$$p\to(p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing on the set $B(s,r)$, then the next results hold.

2.1 If $(2c-\alpha-b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)}>-1$ it is optimal to select in optimization problem (P) the rebate value equal to c, the subsidy equal to $c-b_s$, the price equal to $p_{opt}(c-b_s,c)$ and the optimal production quantity $\lambda(p_{opt}(c-b_s,c)-c)$. The optimal objective value of optimization problem (P) is given by

$$\upsilon(P)=f_{\alpha-c}(p_{opt}(c-b_s,c)-c).$$

2.2 If $(2c-\alpha-b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)}\leq-1$ it is optimal to select in optimization problem (P) the rebate value equal to γ_*c, the subsidy equal to $\gamma_*(c-b_s)$, the price equal to $p_{opt}(\gamma_*(c-b_s),\gamma_*c)$ with $0<\gamma_*\leq1$ satisfying

$$(4.37) \qquad p_{opt}(\gamma_*(c-b_s),\gamma_*c)-\gamma_*c=c-\alpha$$

and the optimal production quantity equal to $\lambda(p_{opt}(\gamma_*(c-b_s),\gamma_*c)-\gamma_*c)$. The optimal objective value of optimization problem (P) is given by

$$\upsilon(P)=f_{\alpha-c}(c-\alpha).$$

Proof. By Lemma 4.1.5 we know for $\alpha<c$ that

$$(4.38) \qquad c-\alpha<p_{opt}(c-b_s,c)-c\leq p_{opt}(s,r)-r$$

Since for $0\leq\alpha<c$ we obtain by Lemma 3.1.1 that the function $f_{\alpha-c}$ is increasing on $(0,c-\alpha)$ and decreasing on $(c-\alpha,r+p_{\max})$ this implies using relation (4.38) that

for every $(s,r) \in M$

$$f_{\alpha-c}(p_{opt}(c-b_s,c)) \geq f_{\alpha-c}(p_{opt}(s,r)-r)$$

Hence we have verified part 1 of this lemma. For $(2c-\alpha-b_s)\frac{\lambda^{(1)}(c-\alpha)}{\lambda(c-\alpha)} \leq -1$, it follows by Lemma 4.1.5 that

$$p_{opt}(c-b_s,c)-c \leq c-\alpha < c \leq p_{opt}(0,0) \tag{4.39}$$

Consider now the continuous function $g:[0,1] \to \mathbb{R}$ given by

$$g(\gamma) = p_{opt}(\gamma(c-b_s),\gamma c)-\gamma c$$

By relation (4.39) there exists some $0 < \gamma_* \leq 1$ satisfying

$$c-\alpha = g(\gamma_*) = p_{opt}(\gamma_*(c-b_s),\lambda_* c)-\gamma_* c$$

Since by Lemma 3.1.1 the function $f_{\alpha-c}$ has a maximum at $c-\alpha$ this yields for every $(s,r) \in M$ that

$$f_{\alpha-c}(p_{opt}(\gamma_*(c-b_s),\gamma_* c)-\gamma_* c) = f_{\alpha-c}(c-\alpha) \leq f_{\alpha-c}(p_{opt}(s,r)-r)$$

and this shows the desired result. □

Since by Lemma 4.1.4 the function

$$\gamma \to p_{opt}(\gamma(c-b_s),\lambda c)-\lambda c$$

is decreasing on $[0,1]$ it is easy to determine by bisection the value γ_* in part 2 of Theorem 4.3.2. We start our bisection algorithm at $\gamma = \frac{1}{2}$ and continue by each time selecting the interval which contains γ_* and evaluating the midpoint of that selected interval until a certain stopping rule is satisfied. At $\gamma = 0$ we know $p_{opt}(0,0) > c \geq (c-\alpha)$ and so we start at $\gamma = \frac{1}{2}$. Remember the condition of Theorem 4.3.1 and 4.3.2 is satisfied for λ logconcave on $(0,p_{\max})$. As already observed this holds for the linear demand function, the loglinear demand function and the logit demand function. In the next example, we give an analytical formula for γ_* for the linear demand function.

Example 4.3.1. *If the demand function λ is given by the linear demand function*

$$\lambda(p) = (a-bp)^+, a > 0, b > 0$$

and the conditions of the last part of Theorem 4.3.2 hold then it is possible to give an analytical formula for γ_*. *Since by (4.8) we know that* $p_{opt}(s,r) = \frac{1}{2}\left(ab^{-1} + r + c - s\right)$ *it follows by equation (4.37) that* γ_* *satisfies*

$$\frac{1}{2}(ab^{-1} + \gamma_* c + c - \gamma_*(c - b_s)) - \gamma_* c = c - \alpha. \tag{4.40}$$

This implies

$$ab^{-1} + 2\alpha - c = \gamma_*(2c - b_s) \tag{4.41}$$

and so

$$\gamma_* = \frac{ab^{-1} + 2\alpha - c}{2c - b_s}. \tag{4.42}$$

Hence for the linear demand function the optimal subsidy is $\gamma_*(c - b_s)$ *and optimal rebate* $\gamma_* c$.

Example 4.3.2. *If the demand function* λ *is given by the loglinear demand function*

$$\lambda(p) = ae^{-bp}, a > 0, b > 0$$

and the conditions of the last part of Theorem 4.3.2 hold then it is possible to give an analytical formula for γ_*. *Since by (4.14) we know that* $p_{opt}(s,r) = \max\{r, b^{-1} + c - s\}$ *it follows using equation (4.37) for* γ_* *that*

$$\max\{0, b^{-1} + c - \gamma_*(2c - b_s)\} = \max\{\gamma_* c, b^{-1} + c - \gamma_*(c - b_s)\} - \gamma_* c = c - \alpha \tag{4.43}$$

This implies

$$(2c - b_s)\max\{-\frac{c + b^{-1}}{2c - b_s}, -\gamma_*\} = -\alpha - b^{-1} \tag{4.44}$$

and hence γ_* *must satisfy*

$$\min\{\frac{c + b^{-1}}{2c - b_s}, \gamma_*\} = \frac{\alpha + b^{-1}}{2c - b_s} \tag{4.45}$$

Since $\alpha \leq c$, *this yields*

$$\gamma_* = \frac{\alpha + b^{-1}}{2c - b_s}. \tag{4.46}$$

Hence for the loglinear demand function the optimal subsidy is $\gamma_*(c - b_s)$ *and optimal rebate* $\gamma_* c$.

In the next example we consider the power demand function.

Example 4.3.3. *If the demand function λ is given by the power demand function*

$$\lambda(p) = (a+bp)^{-\gamma}, a>0, b>0, \gamma>1$$

and the conditions of the last part of Theorem 4.3.2 hold then it is possible to give an analytical formula for γ_*. By (4.20) we know that $p_{opt}(s,r) = \max\{r, \frac{a-br+(c-s)\gamma b}{(\gamma-1)b}\}$ and this implies by equation (4.37) that γ_* satisfies

$$\begin{aligned}
\max\Big\{\gamma_* c, \frac{a-\gamma_* bc+(c-\gamma_*(c-b_s))\gamma b}{(\gamma-1)b}\Big\} - \gamma_* c &= c-\alpha \\
\max\Big\{0, \frac{a+\gamma bc-\gamma\gamma_* b(2c-b_s)}{(\gamma-1)b}\Big\} &= c-\alpha \\
\max\Big\{-a-\gamma bc, -\gamma_*\gamma b(2c-b_s)\Big\} &= (c-\alpha)(\gamma-1)b-a-\gamma bs \\
\min\Big\{\frac{a+\gamma bc}{\gamma b(2c-b_s)}, \gamma_*\Big\} &= \frac{a+b(c+\alpha(\gamma-1))}{\gamma b(2c-b_s)}
\end{aligned}$$

Since $\alpha \le c$, then we can easily show that $a+\gamma bc \ge a+b(c+\alpha(\gamma-1))$. This implies that

$$\frac{a+b(c+\alpha(\gamma-1))}{\gamma b(2c-b_s)} \le \frac{a+\gamma bc}{\gamma b(2c-b_s)}$$

and we may conclude

$$\gamma_* = \frac{a+b(c+\alpha(\gamma-1))}{\gamma b(2c-b_s)}. \tag{4.47}$$

Hence for the power demand function the optimal subsidy is $\gamma_*(c-b_s)$ and optimal rebate $\gamma_* c$.

We now summarize the solution procedure for the decentralized version of optimization problem (P).

Numerical Solution Procedure 4.3.1. *Solution procedure for (P)*

- *STEP 1. Check $\alpha > c$, if true then move to step 2. Otherwise move to step 3.*
- *STEP 2. The optimal rebate is equal to c, optimal subsidy is equal to $c-b_s$ and the optimal price is equal to $p_{opt}(c-b_s, c)$. The optimal objective value is given by*

  $$\upsilon(P) = f_{\alpha-c}(p_{opt}(c-b_s,c)-c)$$

- *STEP 3. Check $(2c-\alpha-b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} > -1$, if true move to step 2. If not, move to step 4.*

- *STEP 4. Compute* $0 \le \gamma_* \le 1$ *satisfying*

$$p_{opt}(\gamma_*(c-b_s), \lambda_* c) - \gamma_* c = c - \alpha$$

The optimal rebate is equal to $\gamma_* c$*, optimal subsidy is equal to* $\gamma_*(c-b_s)$*, and the optimal price is equal to* $p_{opt}(\gamma_*(c-b_s), \gamma_* c)$*. The optimal objective value is given by*

$$\upsilon(P) = f_{\alpha-c}(c-\alpha).$$

Now we summarize in the tables below the optimal solutions and related values of decentralized problem (P). In Table 4.1 we consider optimization problem (P) for either $\alpha > c$ or $\alpha \le c$ and

$$(2c - \alpha - b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} > -1 \tag{4.48}$$

Table 4.1 Results for optimization problem (P) if $c \le \alpha$ or $\alpha > c$ and (4.48) holds.

Results	
optimal subsidy	$c - b_s$
optimal rebate	c
optimal price - optimal rebate	$p_{opt}(c-b_s, c) - c$
optimal production	$\lambda(p_{opt}(c-b_s, c) - c)$
optimal consumer surplus	$\beta(p_{opt}(c-b_s, c) - c)$
optimal externality	$\alpha\lambda(p_{opt}(c-b_s, c) - c)$
optimal profit manufacturer	$\lambda(p_{opt}(c-b_s, c) - c)(p_{opt}(c-b_s, c) - b_s)$
optimal government expenditure	$(2c - b_s)\lambda(p_{opt}(c-b_s, c) - c)$

In Table 4.2 a summary is given of optimization problem (P) for $\alpha \le c$ and $(2c - \alpha - b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} \le -1$. In this case we need γ_* defined in (4.37).

Table 4.2 Results for optimzation problem (P) if $0 \leq \alpha < c$ and (4.48) does not hold.

Results	
optimal subsidy	$\gamma_*(c-b_s)$
optimal rebate	$\gamma_* c$
optimal price - optimal rebate	$p_{opt}(\gamma_*(c-b_s),\gamma_* c)-\gamma_* c$
optimal production	$\lambda(p_{opt}(\gamma_*(c-b_s),\gamma_* c)-\gamma_* c)$
optimal consumer surplus	$\beta(p_{opt}(\gamma_*(c-b_s),\gamma_* c)-\gamma_* c)$
optimal externality	$\alpha\lambda(p_{opt}(\gamma_*(c-b_s),\gamma_* c)-\gamma_* c)$
optimal profit manufacturer	$\lambda(p_{opt}(\gamma_*(c-b_s),\gamma_* c)-\gamma_* c)(p_{opt}(\gamma_*(c-b_s),\gamma_* c)-c+\gamma_*(c-b_s))$
optimal government expenditure	$\gamma_*(2c-b_s)\lambda(p_{opt}(\gamma_*(c-b_s),\gamma_* c)-\gamma_* c)$

In the next subsection, we analyze in detail the decentralized version of optimization problem (P_1).

4.4 Analysis of Optimization Problem (P_1)

In this section, we will analyze the optimization problem (P_1). The problem has the form

$$(P_1) \qquad v(P_1) = \sup_{(s,r)\in M_\lambda}\{f_{\alpha-c}(p_{opt}(s,r)-r)+(r+s)\lambda(p_{opt}(s,r)-r)\}$$

with f_γ defined in (3.9). To analyze the above problem we consider the two mutually exclusive subcases that the budget constraint is redundant or binding. As in Lemma 3.2.5 one can show the following result.

Lemma 4.4.1. *If for every (s,r) belonging to M the function*

$$p \to (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing on $B(r,s)$ then the value $p_{opt}(c-b_s,c)-c$ satisfies the budget constraint if and only if $M = M_\lambda$ with the set M_λ given in (4.35).

Proof. Since the demand function λ is decreasing this implies using Lemma 4.1.5

that for every (s,r) belonging to M

$$\begin{aligned}(r+s)\lambda(p_{opt}(r,s)-r) &\leq (r+s)\lambda(p_{opt}(c,c-b_s)-c)\\ &\leq (2c-b_s)\lambda(p_{opt}(c,c-b_s)-c)\\ &\leq B\end{aligned} \tag{4.49}$$

and so the budget restriction holds for any (s,r) belonging to M. This shows the result. Moreover, if $(r+s)\lambda(p_{opt}(r,s)-r) \leq B$ for every (s,r) belonging to M then by letting $s \uparrow c-b_s$ and $r \uparrow c$ and using $(s,r) \to p_{opt}(s,r)$ is continuous and λ is continuous, we obtain $(2c-b_s)\lambda(p_{opt}(c-b_s,c)-c) \leq B$. □

To check in Lemma 4.4.1 that $p_{opt}(c-b_s,c)-c$ satisfies the budget constraint given by

$$(2c-b)\lambda(p_{opt}(c-b_s,c)-c) \leq B,$$

we need to solve the manufacturers optimization problem $(M(c-b_s,c))$ and determine its optimal solution $p_{opt}(c-b_s,c)$ and optimal objective value. After solving this we compute

$$(2c-b_s)\lambda(p_{opt}(c-b_s,c)-c) = \frac{\upsilon(c-b_s,c)(2c-b_s)}{p_{opt}(c-b_s,c)-c}$$

with $\upsilon(c-b_s,c)$ the optimal objective value of the optimization problem $(M(c-b_s,c))$ In the next example we check under which conditions the budget constraint is redundant for the linear loglinear and power demand functions.

Example 4.4.1. *In this example, we write the condition under which the budget constraint is redundant when the demand function is linear. Since the linear demand function is log-concave by Lemma 4.4.1 we only need to check if* $(2c-b_s)\lambda(p_{opt}(c-b_s,c)-c) \leq B$ *holds. From example 4.1.1 we have*

$$\lambda(p) = (a-bp)^+, a>0, b>0$$

and it follows that

$$p_{opt}(s,r)-r = \frac{1}{2}(ab^{-1}+c-r-s).$$

Hence we only need to check if the next inequality holds.

$$(2c-b_s)(a-\frac{b}{2}(\frac{a}{b}-c+b_s)) \leq B. \tag{4.50}$$

This reduces to

$$a - b(b_s - c) \leq \frac{2B}{2c - b_s}. \tag{4.51}$$

Example 4.4.2. *We write in this example the budget constraint redundancy condition for the loglinear demand function. Since for the loglinear demand function* $\frac{\lambda'(p)}{\lambda(p)}$ *is a constant negative value, we only need to check by Lemma 4.4.1 that* $(2c - b_s)\lambda(p_{opt}(c - b_s, c) - c) \leq B$. *From example 4.1.2 we know that*

$$p_{opt}(s,r) - r = (\frac{1}{b} + c - r - s)^+.$$

Hence we only need to check if the inequality

$$a(2c - b_s)e^{-b(\frac{1}{b} - c + b_s)^+\}} \leq B, \tag{4.52}$$

holds and this is the same as

$$\frac{a}{B} \leq \frac{\exp(\max\{0, 1 - b(c - b_s)\})}{2c - b_s}, \tag{4.53}$$

Example 4.4.3. *We write the budget constraint redundancy condition for power demand in this example. From example 4.1.3 we have*

$$\lambda(p) = (a + bp)^{-\gamma}, a > 0, b > 0, \gamma > 1$$

Since the function

$$p \mapsto (p - c + s)\frac{-\gamma b}{a + b(p - r)}$$

is strictly decreasing, by Lemma 4.4.1 we only need to check if $(2c - b_s)\lambda(p_{opt}(c - b_s, c) - c) \leq B$ *holds. Again using the results of example 4.1.3 we can write*

$$p_{opt}(s,r) - r = \max\{0, \frac{a + \gamma b(c - r - s)}{(\gamma - 1)b}\}. \tag{4.54}$$

Hence we only need to check if the inequality below holds.

$$(2c - b_s)(\max\{a, \frac{\gamma(a + b(b_s - c))}{\gamma - 1}\})^{-\gamma} \leq B,$$

Which could be simplified to

$$(\frac{2c - b_s}{B})^{\gamma^{-1}} \leq \max\{a, \frac{\gamma(a + b(b_s - c))}{\gamma - 1}\}.$$

We now list the optimal solutions to problem (P_1) for $\alpha \geq c$ and the value $p_{opt}(c - b_s, c) - c$ satisfies the budget constraint.

Theorem 4.4.1. *If $\alpha \geq c$ and for every (s,r) belonging to M the function*

$$p \to (p - c + s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing on $B(r,s)$ then for $p_{opt}(c-b_s,c)-c$ satisfying the budget constraint it is optimal to select in optimization problem (P_1) the rebate value equal to c, the subsidy value equal to $c - b_s$, the price equal to $p_{opt}(c-b_s,c)$ and the production quantity equal to $\lambda(p_{opt}(c-b_s,c)-c)$. The optimal objective value of optimization problem (P_1) is given by

$$\upsilon(P_1) = f_{\alpha-c}(p_{opt}(c-b_s,c)-c) + (2c-b_s)\lambda(p_{opt}(c-b_s,c)-c).$$

Proof. Since $p_{opt}(c-b_s,c)-c$ satisfies the budget constraint it follows by Lemma 4.2.1 that

$$\begin{aligned} \upsilon(P_1) &= \sup_{(s,r)\in M_\lambda}\{f_{\alpha-c}(p_{opt}(s,r)-r) + (r+s)\lambda(p_{opt}(s,r)-r)\} \\ &\geq f_{\alpha-c}(p_{opt}(c-b_s,c)-c) + (2c-b_s)\lambda(p_{opt}(c-b_s,c)-c). \end{aligned}$$

Since by Lemma 4.1.4 and relation (4.29) we know for every (s,r) belonging to M that

$$0 \leq p_{opt}(c-b_s,c)-c \leq p_{opt}(s,r)-r$$

and by Lemma 3.1.1 the function $f_{\alpha-c}$ is decreasing on $(0,\infty)$ and λ is decreasing it follows again by Lemma 4.4.1 that

$$\upsilon(P_1) \leq f_{\alpha-c}(p_{opt}(c-b_s,c)-c) + (2c-b_s)\lambda(p_{opt}(c-b_s,c)-c)$$

and this shows the desired result. □

We will now consider the case $0 \leq \alpha < c$ and $p_{opt}(c-b_s,c)-c$ satisfies the budget constraint.

Theorem 4.4.2. *If $0 \leq \alpha < c$ and for every (s,r) belonging to M the function*

$$p \to (p - c + s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing and continuous on $B(r,s)$ and it satisfies $(2c-\alpha-b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} > -1$ then for $p_{opt}(c-b_s,c)-c$ satisfying the budget constraint it is optimal to select *in optimization problem (P_1) the rebate value equal to c, the subsidy value equal to $c - b_s$ and the optimal price equal to $p_{opt}(c - b_s, c)$.The optimal objective value of optimization problem (P_1) is given by*

$$\upsilon(P_1) = f_{\alpha-c}(p_{opt}(c - b_s, c) - c) + (2c - b_s)\lambda(p_{opt}(c - b_s, c) - c).$$

Proof. Since $(2c - \alpha - b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} > -1$ we know by Lemma 4.1.5 that

$$c - \alpha \leq p_{opt}(c - b_s, c) - c \leq p_{opt}(s, r) - r$$

and by assumption $p_{opt}(c - b_s, c) - c$ satisfies the budget constraint. This implies using $f_{\alpha-c}$ is decreasing on $(c-\alpha, p_{\max})$ and λ decreasing that for every (s, r) belonging to M

$$\begin{aligned} & f_{\alpha-c}(p_{opt}(s,r) - r) + (r+s)\lambda(p_{opt}(s,r) - r) \\ \leq \; & f_{\alpha-c}(p_{opt}(c - b_s, c) - c) + (r+s)\lambda(p_{opt}(c - b_s, c) - c) \\ \leq \; & f_{\alpha-c}(p_{opt}(c - b_s, c) - c) + (2c - b_s)\lambda(p_{opt}(c - b_s, c) - c) \end{aligned}$$

Hence we have shown the desired result. □

To give a more insightful economic interpretation of the results for optimization problem (P_1) we observe the following.

Remark 4.4.1. *It follows under the conditions of Theorem 4.4.1 and 4.4.2 that by the same theorems it is equivalent to solve the optimization problem*

$$\inf_{(s,r)\in M_\lambda}\{p_{opt}(s,r) - r\} = \inf_{(s,r)\in M}\{p_{opt}(s,r) - r\}.$$

Observe by Lemma 4.1.4 the above optimization problem has the same set of optimal solutions. Hence in these particular cases the optimal choice of the decision variables for the objective function used in optimization problem (P_1) is to select taking into account the behavior of the manufacturer the rebate and subsidy in such a way that the price paid by the customer is minimized.

For the last remaining case $(2c - \alpha - b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} \leq -1$ and $p_{opt}(c - b_s, c) - c$ satisfies the budget constraint, one can conclude by Lemma 4.1.5 and relation (4.28) that

$$p_{opt}(c - b_s, c) - c \leq c - \alpha < p_{opt}(0, 0)$$

In this case it is still an open problem for $0 \leq \alpha < c$ and $p_{opt}(c - b_s, c) - c \leq c - \alpha$ to identify optimal solution for the optimization problem

$$\begin{aligned} v(P_1) &= \sup_{(s,r)\in M_\lambda}\{f_{\alpha-c}(p_{opt}(s,r)-r)+(r+s)\lambda(p_{opt}(s,r)-r)\} \\ &= \sup_{0\leq s\leq c-b_s, 0\leq r\leq c}\{f_{\alpha-c}(p_{opt}(s,r)-r)+(r+s)\lambda(p_{opt}(s,r)-r)\}. \end{aligned}$$

We now consider the case that the subsidy value $c-b_s$ and rebate value c violate the budget restriction and so

$$(2c-b_s)\lambda(p_{opt}(c-b_s,c)-c) > B$$

To analyze optimization problem (P_1) we need the following dominance result for optimization problem (P_1) and show that under certain conditions the budget constraint is binding. Before discussing this result we introduce the possible empty set $\partial M_\lambda \subseteq M_\lambda$ given by

$$\partial M_\lambda := \{(s,r)\in M : (s+r)\lambda(p_{opt}(s,r)-r) = B\} \tag{4.55}$$

and for ∂M_λ nonempty the optimization problem

$$v(Q_1) := \inf_{(s,r)\in\partial M_\lambda}\{p_{opt}(s,r)-r\}. \tag{Q_1}$$

Theorem 4.4.3. *If for every (s,r) belonging to M the function*

$$p \to (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing on $B(s,r)$ and $p_{opt}(c-b_s,c)-c$ violates the budget constraint then the next results hold

3.1 If $\alpha \geq c$, then

$$v(P_1) = B + f_{\alpha-c}(v(Q_1)).$$

3.2 If $0 \leq \alpha < c$ and $(2c-\alpha-b)\frac{\lambda^{(1)}(c-\alpha)}{\lambda(c-\alpha)} > -1$, then

$$v(P_1) = B + f_{\alpha-c}(v(Q_1)).$$

Proof. To start our proof we first show that for any (s,r) belonging to $M_\lambda\backslash\partial M_\lambda$ there exists some (s_*,r_*) belonging to ∂M_λ having a larger objective value. Let $(s,r)\in M_\lambda\backslash\partial M_\lambda$ and introduce the function $s:[0,1]\to\mathbb{R}$ and $r:[0,1]\to\mathbb{R}$ given by

$$s(\gamma) := (1-\gamma)s+\gamma(c-b), r(\gamma) := (1-\gamma)r+\gamma c$$

Since (s,r) belongs to $M_\lambda \backslash \partial M_\lambda$ we know

$$(s(0)+r(0))\lambda(p_{opt}(s(0),r(0))-r(0)) = (s+r)\lambda(p_{opt}(s,r)-r) < B \tag{4.56}$$

and because $p_{opt}(c-b,c)-c$ violates the budget constraint we conclude

$$(s(1)+r(1))\lambda(p_{opt}(s(1),r(1))-r(1)) = (2c-b)\lambda(p_{opt}(c-b,c)-c) > B. \tag{4.57}$$

Since the function

$$\gamma \to (s(\gamma)+r(\gamma))\lambda(p_{opt}(s(\gamma),r(\gamma))-r(\gamma))$$

is continuous on $(0,1)$, this implies by relations (4.56) and (4.57) that there exists some $0 < \gamma_* < 1$ satisfying

$$(s(\gamma_*)+r(\gamma_*))\lambda(p_{opt}(s(\gamma_*),r(\gamma_*))-r(\gamma_*)) = B. \tag{4.58}$$

This shows the set ∂M_λ is nonempty and since $0 \leq s \leq c-b$ and $0 \leq r \leq c$ it follows that

$$s_* := s(\gamma_*) \geq s, \ r_* = r(\gamma_*) \geq r.$$

By Lemma 4.1.4 and 4.1.5 this implies for $\alpha \geq c$ or $0 \leq \alpha < c$ and $(2c-\alpha-b)\frac{\lambda^{(1)}(c-\alpha)}{\lambda(c-\alpha)} > -1$ that

$$(c-\alpha)^+ \leq p_{opt}(s_*,r_*)-r_* \leq p_{opt}(s,r)-r. \tag{4.59}$$

Since the function $f_{\alpha-c}$ is decreasing on $((c-\alpha)^+,\infty)$ we obtain by relations (4.59) and and (s,r) belongs to $M_\lambda \backslash \partial M_\lambda$ that

$$\begin{aligned} & f_{\alpha-c}(p_{opt}(s,r)-r)+(r+s)\lambda(p_{opt}(s,r)-r) \\ < \ & f_{\alpha-c}(p_{opt}(s_*,r_*)-r_*)+B \\ = \ & f_{\alpha-c}(p_{opt}(s_*,r_*)-r_*)+(s_*+r_*)\lambda(p_{opt}(s_*,r_*)-r_*) \end{aligned} \tag{4.60}$$

By relations (4.58) and (4.60), it now follows that

$$\begin{aligned} v(P_1) &= \sup_{(s,r)\in M_\lambda}\{f_{\alpha-c}(p_{opt}(s,r)-r)+(r+s)\lambda(p_{opt}(s,r)-r)\} \\ &= \sup_{(s,r)\in \partial M_\lambda}\{f_{\alpha-c}(p_{opt}(s,r)-r)+(r+s)\lambda(p_{opt}(s,r)-r)\} \\ &= B+\sup_{(s,r)\in \partial M_\lambda}\{f_{\alpha-c}(p_{opt}(s,r)-r)\} \end{aligned} \tag{4.61}$$

Since by Lemma 4.1.5 and 4.1.4 we know for every (s,r) belonging to M that

$$p_{opt}(s,r)-r \geq p_{opt}(c-b_s,c)-c \geq (c-\alpha)^+$$

and the function $f_{\alpha-c}$ is decreasing on $((c-\alpha)^+,\infty)$ the desired result follows using relation (4.61) □

If $(p_{opt}(c-b_s,c)-c)$ violates the budget constraint it follows under the conditions of Theorem 4.4.3 and using λ belongs to $\mathcal{D}$ and hence λ decreasing that

$$\begin{aligned}
\sup_{(s,r)\in M_\lambda}\{(r+s)\} &= \sup_{(r,s)\in\partial M_\lambda}\left\{\frac{B}{\lambda(p_{opt}(s,r)-r)}\right\} \\
&= \frac{B}{\inf_{(s,r)}\{\lambda(p_{opt}(s,r)-r)\}} \\
&= \frac{B}{\lambda(\sup_{(s,r)\in\partial M_\lambda}\{p_{opt}(s,r)-r)\})}.
\end{aligned}$$

This shows that the optimization problem $\sup_{(s,r)\in M_\lambda}\{(r+s)\}$ has the same set of optimal solutions as the optimization problem $\sup_{(s,r)\in\partial M_\lambda}\{p_{opt}(s,r)-r)\}$ and by Theorem 4.4.3 it has the same set of optimal solutions as optimization problem (P_1).

It is still an open question how to analyze the optimization problem (P_1) for

$$0 \leq \alpha \leq c \text{ and } (2c-\alpha-b)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} \leq -1.$$

By Theorem 4.4.3 it is obvious that solving optimization problem (P_1) is the same as solving optimization problem (Q_1). In the remainder of this section, we assume that the conditions of Theorem 4.4.3 hold. To analyze the problem (Q_1) we do the following. If the condition

$$(2c-b_s)\lambda(p_{opt}(c-b_s,c)-c) > B$$

holds and so $p_{opt}(c-b_s,c)$ violates the budget constraint we introduce for every $0 \leq s \leq c-b$ the possibly empty set

$$M(s) = \{0 \leq r \leq c : (r+s)\lambda(p_{opt}(s,r)-r) \leq B\}. \tag{4.62}$$

By Theorem 4.4.3 the optimization problem (Q_1) can now be written as

$$\inf_{(s,r)\in\partial M_\lambda}\{p_{opt}(r,s)-r\} = \upsilon(Q_1) = \inf_{0\leq s\leq c-b_s}\inf\{p_{opt}(s,r)-r : r \in M(s)\} \tag{4.63}$$

By Lemma 4.1.4 and λ decreasing, the function $s \to \lambda(p_{opt}(s,0))$ is increasing and since 0 belongs to the set $\{0 \leq s \leq c - b_s : s\lambda(p_{opt}(s,0)) \leq B\}$ the expression

$$s_* := \sup\{0 \leq s \leq c - b_s : s\lambda(p_{opt}(s,0)) \leq B\}. \tag{4.64}$$

is well defined. For $s_* < c - b_s$ we obtain by the continuity of the increasing function $s \to s\lambda(p_{opt}(s,0))$ that

$$s_*\lambda(p_{opt}(s_*,0)) = B. \tag{4.65}$$

Clearly if $s_* < c - b_s$ it follows that s_* represents the largest subsidy value for which it is not possible to give a positive rebate to each customer without violating the budget constraint. At the same time we introduce for every $s \leq c - b_s$ the function $r_* : [0, s_*] \to [0, c]$ defined by

$$r_*(s) := \sup\{0 \leq r \leq c : (r+s)\lambda(p_{opt}(s,r) - r) \leq B\}. \tag{4.66}$$

and for each given $s \leq s_*$ the value $r_*(s)$ represents the largest value of the rebate $r \leq c$ satisfying the budget constraint in case the subsidy value equals s. Clearly the nonnegative function $r_* : [0, s_*] \to [0, c]$ is bounded above by c and decreasing. This means that we can define

$$s_{**} = \begin{cases} \sup\{0 \leq s \leq c - b_s : r_*(s) = c\} & \text{if } r_*(0) = c \\ 0 & \text{otherwise.} \end{cases} \tag{4.67}$$

This implies for $r_*(0) = c$ that $r_*(s_{**}) = c$. Also it follows using $r_*(s)$ is non-negative on $[0, s_*]$ that

$$\begin{aligned} s_{**}\lambda(p_{opt}(s_{**},0)) &\leq s_{**}\lambda\Big(p_{opt}(s_{**}, r_*(s_{**})) - r_*(s_{**})\Big) \\ &\leq (s_{**} + r_*(s_{**}))\lambda(p_{opt}(s_{**}, r_*(s_{**})) - r_*(s_{**})) \\ &\leq B \end{aligned}$$

This shows using $s \to \lambda(p_{opt}(s,0))$ is increasing that

$$s_{**} \leq s_* \tag{4.68}$$

In the next intermediate result, we relate the set $M(s)$ defined in (4.62) and $r_*(s)$ for $s \leq s_*$ and show that the set $M(s)$ is empty for $s > s_*$.

Lemma 4.4.2. *It follows for every s satisfying $s_* < s \leq c - b_s$ that the set $M(s)$ is* *empty and for every* $0 \leq s \leq s_*$

$$M(s) = [0, r_*(s)].$$

Moreover, if $s_{**} < s_*$ *and* $s_{**} < s < s_*$*, the value* $r_*(s)$ *is the unique solution of the nonlinear equation*

$$(v+s)\lambda(p_{opt}(s,v)-v) = B,\ 0 < v < c$$

and the function $s \to s + r_*(s)$ *is continuous on* $[s_{**}, s_*]$.

Proof. To show that the set $M(s)$ is empty for every s satisfying $s_* < s \leq c - b_s$ we observe by relation (4.65) that for $s_* < c - b_s$

$$s_*\lambda(p_{opt}(s_*,0)) = B \tag{4.69}$$

This implies using Lemma 4.1.4 that for every s satisfying $0 < s_* < s \leq c - b_s$ and $0 \leq r \leq c$

$$\begin{aligned}(r+s)\lambda(p_{opt}(s,r)-r) &> (r+s_*)\lambda(p_{opt}(s_*,r)-r)\\ &\geq s_*\lambda(p_{opt}(s_*,r)-r)\\ &\geq s_*\lambda(p_{opt}(s_*,0)\\ &= B\end{aligned}$$

This shows that the set $M(s)$ is empty for every $0 < s_* < s \leq c - b_s$. Also if $0 \leq s \leq s_*$ the value 0 belongs to the set $\{0 \leq r \leq c : (r+s)\lambda(p_{opt}(s,r)-r) \leq B\}$ and since the function $r \to p_{opt}(s,r) - r$ is increasing on $(0,c)$ the value

$$r_*(s) := \sup\{0 \leq r \leq c : (r+s)\lambda(p_{opt}(s,r)-r) \leq B\} \tag{4.70}$$

is well defined and we may conclude for $0 \leq s \leq s_\star$ that $M(s) = [0, r_*(s)]$. Finally if $s_{**} < s \leq s_*$ it follows by the definition of s_{**} that $r_*(s) < c$ and using $v \to (v+s)\lambda(p_{opt}(s,v)-v)$ is strictly increasing and continuous the value $r_*(s)$ is the unique solution of the nonlinear equation

$$(v+s)\lambda(p_{opt}(s,v)-v) = B,\ 0 < v < c \tag{4.71}$$

This shows the second result. Also by relation (4.71) it is easy to verify using $\lambda \in \mathcal{D}$ that the function $s \to s + r_*(s)$ is continuous on $[s_{**}, s_*]$. $\square$

For each $s \leq s_*$ and $r_*(s) < c$ we can identify $r_*(s)$ by a Newton-Raphson type of root-finding procedure (Cheney & Kincaid. (1999)). Another way to identify $r_*(s)$ is to use a standard bisection method. One can now show the following result.

Lemma 4.4.3. *If for every* (s,r) *belonging to* M *the function*

$$p \to (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing on $B(s,r)$ *and* $(p_{opt}(c-b_s,c)-c)$ *violates the budget constraint then*

$$\upsilon(Q_1) = \inf_{s_{**}\leq s\leq s_*}\{p_{opt}(s,r_*(s))-r_*(s)\}$$

with s_* *defined in relation (4.64) and* s_{**} *in relation (4.67).*

Proof. We know by relation (4.63) and Lemma 4.4.2 that

$$\upsilon(Q_1) = \inf_{0\leq s\leq s_*}\gamma(s)$$

with

$$\gamma(s) := \inf\{p_{opt}(s,r)-r : r \in M(s)\}.$$

Applying Lemma 4.4.2 and 4.1.4 it follows for every $s \leq s_*$ that

$$\gamma(s) = \inf\{p_{opt}(s,r)-r : r \in [0,r_*(s)]\} = p_{opt}(s,r_*(s))-r_*(s)$$

and using $r_*(s) = c$ for every $s \leq s_{**}$ yields the desired result. □

Since $r_*(s)$ is a decreasing function it is unclear how the function $s \to s+r_*(s)$ behaves on $[s_{**},s_*]$. This is discussed in the following result.

Lemma 4.4.4. *If* λ *belongs to* $\mathcal{D}$ *and for every* (s,r) *belonging to* M *the function*

$$p \to (p-c+s)\frac{\lambda'(p-r)}{\lambda(p-r)}$$

is strictly decreasing on $B(s,r)$ *and* $p_{opt}(c-b_s,c)-c$ *violates the budget constraint, then the function*

$$s \to p_{opt}(s,r_*(s))-r_*(s)$$

is constant on $[s_{**},s_*]$.

Proof. Introduce the possibly empty set

$$I = \left\{s_{**} \leq s \leq s_* : s+r_*(s) > c, (r_\star(s)-c+s)\frac{\lambda^{(1)}(0)}{\lambda(0)} \leq -1\right\}$$

For every $s \in I$ it follows by relation (4.26) that

$$f(s) = p_{opt}(s, r_*(s)) - r_*(s) = 0$$

For every $s \in [s_{**}, s_*] \backslash I$ we obtain by relations (4.25) and (4.27) that $f(s)$ is the unique solution of the nonlinear equation

$$(u - c + r_*(s) + s)\frac{\lambda^{(1)}(u)}{\lambda(u)} = -1. \tag{4.72}$$

We also know by Lemma 4.4.2 that for every $s_{**} \leq s \leq s_*$

$$B = (s + r_*(s))\lambda(f(s)). \tag{4.73}$$

We will now show for every s_1, s_2 belonging to $[s_{**}, s_*] \backslash I$ that $s_1 + r_*(s_1) = r_2 + s_*(r_2)$. Suppose by contradiction that

$$s_1 + r_*(s_1) > s_2 + r_*(s_2).$$

This implies using $\lambda^{(1)}$ is negative that by relation (4.72)

$$\begin{aligned} (f(s_2) - c + s_1 + r_*(s_1))\tfrac{\lambda^{(1)}(f(s_2))}{\lambda(f(s_2)} &< (f(s_2) - c + s_2 + r_*(s_2))\tfrac{\lambda^{(1)}(f(s_2))}{\lambda(f(s_2)} \\ &= -1. \end{aligned} \tag{4.74}$$

Since the function $u \to (u - c + s_1 + r_*(s_1))\frac{\lambda^{(1)}(u)}{\lambda(u)}$ is strictly decreasing on $B(s_1, r_\star(s_1))$ we obtain by relation (4.74) that

$$f(s_1) < f(s_2). \tag{4.75}$$

By relation (4.75) and λ is a decreasing function it follows that

$$\begin{aligned} B &= (s_1 + r_*(s_1))\lambda(f(s_1)) \\ &\geq (s_1 + r_*(s_1))\lambda(f(s_2)) \\ &> (s_2 + r_*(s_2))\lambda(f(s_2)) \\ &= B. \end{aligned} \tag{4.76}$$

Hence we obtain a contradiction and since s_1, s_2 are arbitrarily chosen from the set $[s_{**}, s_*] \backslash I$ it must follows that $s_2 + r_*(s_2) = s_1 + r_*(s_1)$. This shows by relation (4.76) that

$$f(s_1) = f(s_2).$$

Since the function f is continuous on $[s_{**}, s_*]$ it must therefore follow that $f(s) = 0$ if I is nonempty and $f(s)$ is a fixed constant if I is empty. This shows the desired result. □

Finally we summarize the results in a solution procedure. As shown in the previous theorems the optimal solutions are identified for $\alpha \geq c$, while for $\alpha < c$ and $(2c-\alpha-b)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} \leq -1$ it is not clear what are the optimal solutions. Hence we only list the solution procedure for $\alpha \geq c$.

Numerical Solution Procedure 4.4.1. *Solving the decentralized version of the optimization problem (P_1) for $\alpha \geq c$.*

- *STEP 1. Solve optimization problem $(M(c-b,c))$ and compute its optimal solution $p_{opt}(c-b,c)$. Check $(2c-b)\lambda(p_{opt}(c-b,c)-c) \leq B$, if true then move to step 2. otherwise go to step 3.*

- *STEP 2. Optimal subsidy is $c-b$, optimal rebate value is c, optimal price is $p_{opt}(c-b,c)$, optimal production quantity is $q_{opt} = \lambda(p_{opt}(c-b,c)-c)$, optimal objective value is*

$$f_{\alpha-c}(p_{opt}(c-b,c)-c) + (2c-b)\lambda(p_{opt}(c-b,c)-c)$$

  and the budget restriction is redundant.

- *STEP 3. Compute by bisection s_{**} and $r_*(s_{**})$. If $s_{**} > 0$ go to step 4 otherwise to step 5.*

- *STEP 4. Optimal rebate value is c, optimal subsidy value is s_{**}, optimal price is $p_{opt}(s_{**},c)$, optimal production quantity is $q_{opt} = \lambda(p_{opt}(s_{**},c)-c)$, optimal objective value is*

$$f_{\alpha-c}(p_{opt}(s_{**},c)-c) + B$$

  and the budget constraint is binding.

- *STEP 5. Optimal subsidy is 0, optimal rebate value is $r_*(0)$, optimal price is $p_{opt}(0,r_*(0))$ and optimal production quantity is $q_{opt} = \lambda(p_{opt}(0,r_*(0)-r_*(0))$, optimal objective value is*

$$f_{\alpha-c}(p_{opt}(0,r_*(0)) - r_*(0)) + B$$

  and the budget restriction is binding.

We present the results for either the budget constraint is redundant or binding in two separate tables. Table 4.3 displays a summary of the results for problem (P_1) when the budget constraint is redundant and we have either $\alpha > c$ or $\alpha \leq c$ and $(2c-\alpha-b_s)\frac{\lambda'(c-\alpha)}{\lambda(c-\alpha)} > -1$.

Table 4.3 Results for problem (P_1) with redundant budget constraint.

Results	
optimal subsidy	$c-b_s$
optimal rebate	c
optimal price - optimal rebate	$p_{opt}(c-b_s,c)-c$
optimal production	$\lambda(p_{opt}(c-b_s,c)-c)$
optimal consumer surplus	$\beta(p_{opt}(c-b_s,c)-c)$
optimal externality	$\alpha\lambda(p_{opt}(c-b_s,c)-c)$
optimal profit manufacturer	$\lambda(p_{opt}(c-b_s,c)-c)(p_{opt}(c-b_s,c)-b_s)$

Table 4.4 summarizes the results of problem (P_1) in case the budget constraint is binding and either $\alpha > c$ or $\alpha \leq c$ and $(2c-\alpha-b_s)\frac{\lambda^{(1)}(c-\alpha)}{\lambda(c-\alpha)} > -1$. Observe s_*, $r_*(s)$ and s_{**} are defined in (4.64), (4.66) and (4.67).

Table 4.4 Results for problem (P_1) with binding budget constraint.

Results	
optimal subsidy	$s \in [s_{**}, s_*]$
optimal rebate	$r_*(s)$
optimal price - optimal rebate	$p_{opt}(s,r_*(s))-r_*(s)$
optimal production	$\lambda(p_{opt}(s,r_*(s))-r_*(s))$
optimal consumer surplus	$\beta(p_{opt}(s,r_*(s))-r_*(s))$
optimal externality	$\alpha\lambda(p_{opt}(s,r_*(s))-r_*(s))$
optimal profit manufacturer	$\lambda(p_{opt}(s,r_*(s))-r_*(s))(p_{opt}(s,r_*(s))-c+s)$

5. Conclusion and Comparison

In this chapter, we will compare the results we attained from optimization problems (P) and (P_1) in the centralized and decentralized economy settings. Since the government also optimizes over price and quantity instead of just rebate and subsidy, the value of the optimal welfare both in problem (P) and (P_1) is larger in the centralized economy compared to its value in the decentralized economy. But it is of interest to know how the government's expenditure changes or how the manufacturer's profit and externality are affected.

5.1 A Comparison for Problem (P)

We start by comparing the results of the centralized economy with the decentralized economy for optimization problem (P). For simplicity, we assume that $\alpha \geq c$ in the tables below and so the public interest good is highly valuable for the welfare of the society.

Table 5.1 A comparison between centralized and decentralized results for problem (P)

Results	Centralized Economy	Decentralized Economy
optimal subsidy	0	$c - b_s$
optimal rebate	$0 \leq r \leq c$	c
optimal price - optimal rebate	0	$p_{opt}(c-b_s,c)-c$
optimal production	$\lambda(0)$	$\lambda(p_{opt}(c-b_s,c)-c)$
optimal consumer surplus	$\beta(0)$	$\beta(p_{opt}(c-b_s,c)-c)$
optimal externality	$\alpha\lambda(0)$	$\alpha\lambda(p_{opt}(c-b_s,c)-c)$
optimal profit manufacturer	$(r-c)\lambda(0)$	$\lambda(p_{opt}(c-b_s,c)-c)(p_{opt}(c-b_s,c)-b_s)$
optimal government expenditure	$r\lambda(0)$	$(2c-b_s)\lambda(p_{opt}(c-b_s,c)-c)$

The optimal production quantity, optimal consumer surplus, and optimal externality are highest in a centralized economy. If in such an economy the government decides to give a rebate strictly smaller than the production cost, the manufacturer will suffer a loss of $(r-c)\lambda(0)$. In the extreme case the government can force the manufacturer to cover all the production costs by paying 0 rebate.

In a decentralized economy, optimal production quantity, optimal consumer surplus, and optimal externality are smaller. Since $p_{opt}(c-b_s, c) \geq c$ the manufacturer will not suffer a loss and gains a profit.

5.2 A Comparison for Problem (P1)

We compare the results of the centralized economy with the results of the decentralized economy for problem (P_1). First, we assume that the budget constraint is redundant for both problem settings, meaning the value of budget B is so large that it covers all possible rebate and subsidy values. This means $(2c-b_s)\lambda(0) \leq B$. Again for simplicity, we assume $\alpha \geq c$ in the tables below.

Table 5.2 A comparison between centralized and decentralized results for problem (P_1) if the budget covers all possible rebate and subsidy settings.

Results	Centralized Economy	Decentralized Economy
optimal subsidy	$c-b_s$	$c-b_s$
optimal rebate	c	c
optimal price - optimal rebate	0	$p_{opt}(c-b_s,c)-c$
optimal production	$\lambda(0)$	$\lambda(p_{opt}(c-b_s,c)-c)$
optimal consumer surplus	$\beta(0)$	$\beta(p_{opt}(c-b_s,c)-c)$
optimal externality	$\alpha\lambda(0)$	$\alpha\lambda(p_{opt}(c-b_s,c)-c)$
optimal profit manufacturer	$(c-b_s)\lambda(0)$	$\lambda(p_{opt}(c-b_s,c)-c)(p_{opt}(c-b_s,c)-b_s)$

From table 5.2, it is clear that the government sets optimal rebate and subsidy values to c and $c-b_s$ in both centralized and decentralized economies. Again the production, consumer surplus, and externality are larger in the centralized economy, but we cannot determine whether the manufacturer will gain more profit in a centralized or decentralized economy.

Next we assume that the budget constraint is binding in both economies and this means $(2c-b_s)\lambda(p_{opt}(c-b_s,c)-c) \geq B$. The notations s_*, $r_*(s)$ and s_{**} have been introduced in (4.64), (4.66) and (4.67). In order to provide a clear table of results, we define

$$r_* = \sup\{r \in [0,c] : r\lambda(0) \leq B\}. \tag{5.1}$$

Table 5.3 A comparison between centralized and decentralized results for problem (P_1) if the budget B does not cover all possible rebate and subsidy settings.

Results	Centralized Economy	Decentralized Economy
optimal subsidy	$\frac{B}{\lambda(0)} - r_*$	$s_{**} \leq s \leq s_*$
optimal rebate	r_*	$r_*(s)$
optimal price - optimal rebate	0	$p_{opt}(s,r_*(s)) - r_*(s)$
optimal production	$\lambda(0)$	$\lambda(p_{opt}(s,r_*(s)) - r_*(s))$
optimal consumer surplus	$\beta(0)$	$\beta(p_{opt}(s,r_*(s)) - r_*(s))$
optimal externality	$\alpha\lambda(0)$	$\alpha\lambda(p_{opt}(s,r_*(s)) - r_*(s))$
optimal profit manufacturer	$B - c\lambda(0)$	$\lambda(p_{opt}(s,r_*(s)) - r_*(s))(p_{opt}(s,r_*(s)) - c + s)$

Since $p_{opt}(s,r_*(s)) - r_*(s) \geq 0$ we conclude from Table 5.3 that in a central economy optimal production, optimal consumer surplus, and optimal externality are still higher then in a decentralised economy.

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APPENDIX A

In this appendix, we list in the first subsection the following auxiliary result, while in the second subsection we discuss the consumer surplus.

An Auxiliary Result.

In the first part of this appendix, we prove the following result.

Lemma A.0.1. *For any functions $f_i : (0,r) \to [-\infty,\infty]$ it follows that*

$$\sup_{0\le r\le c}\max\{f_0(r), f_1(r)\} = \max\{\sup_{0\le r\le c} f_0(r), \sup_{0\le r\le c} f_1(r)\}$$

Proof. Since $\max\{f_0(r), f_1(r)\} \ge f_i(r), i=0,1$ we obtain

$$\sup_{0\le r\le c}\max\{f_0(r), f_1(r)\} \ge \max\{\sup_{0\le r\le c} f_0(r), \sup_{0\le r\le c} f_1(r)\}$$

Suppose now by contradiction that

$$\sup_{0\le r\le c}\max\{f_0(r), f_1(r)\} > \max\{\sup_{0\le r\le c} f_0(r), \sup_{0\le r\le c} f_1(r)\}$$

Hence by the definition of supremum we can find some $0 \le r_0 \le c$ satisfying

$$\max\{f_0(r_0), f_1(r_0)\} > \max\{\sup_{0\le r\le c} f_0(r), \sup_{0\le r\le c} f_1(r)\} \tag{A.1}$$

Since $0 \le r_0 \le c$ we obtain for every $i = 0,1$ that

$$f_i(r_0) \le \sup_{0\le r\le c} f_i(r)$$

and so

$$\max\{f_0(r_0), f_1(r_0)\} \le \max\{\sup_{0\le r\le c} f_0(r), \sup_{0\le r\le c} f_1(r)\}$$

contradicting relation (A.1). □

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On the Consumer Surplus

In this part of the appendix we introduce the consumer surplus for two different demand generating models. We first consider a demand function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ with $p_{\max} \leq \infty$ the maximum price customers are willing to pay for a given product.By definition this demand function is decreasing and satisfies $\lambda(p_{\max}) = 0$. The value $\lambda(p)$ represents the demand of customers within a population willing to buy the product at price p. The consumer surplus of an individual customer is now given by the difference this individual customer is willing to pay and the actual price this customer needs to pay. If the price of the product is given by $p \geq c$ and we introduce for every $h > 0$ the sequence $p_k = p + kh$ for $k = 0, ...M < \infty$ for $p_M = p_{\max} < \infty$, then according to this demand function the total number of customer willing to pay between p_k and p_{k+1} is given by

$$\Delta\lambda_k = \lambda(p_k) - \lambda(p_{k+1}), k = 0, ..., M$$

This shows that the total consumer surplus of customers willing to pay between p_k and p_{k+1} is bounded above by $(p_{k+1} - p)\Delta\lambda_k = (k+1)h\Delta D_k$ and below by $(p_k - p)\Delta\lambda_k = kh\Delta\lambda_k$. The total consumer surplus of the demand function at price p is then bounded above by

$$\begin{aligned}
\textstyle\sum_{k=0}^{M}(p_{k+1} - p)\Delta\lambda_k &= h\textstyle\sum_{k=0}^{M}(k+1)\Delta\lambda_k \\
&= h\textstyle\sum_{k=0}^{M}\sum_{j=0}^{k}\Delta\lambda_k \\
&= h\textstyle\sum_{j=0}^{M}\lambda(p_j) - \lambda(p_{\max}) \\
&= h\textstyle\sum_{j=0}^{M}\lambda(p_j)
\end{aligned}$$

and bounded below by

$$\begin{aligned}
\textstyle\sum_{k=0}^{M}(p_k - p)\Delta\lambda_k &= \textstyle\sum_{k=0}^{M}(p_{k+1} - p)\Delta\lambda_k + \sum_{k=0}^{M}(p_k - p_{k+1})\Delta\lambda_k \\
&= h\textstyle\sum_{j=0}^{M}\lambda(p_j) - h\lambda(p).
\end{aligned}$$

Letting now $h \downarrow 0$ we obtain that

$$C_s(p) = \int_p^{p_{\max}} \lambda(u)du. \tag{A.2}$$

If the demand function is given by the linear demand function $\lambda(p) = a - p$ with $a > 0$ (Raz & Ovchinnikov (2015)) then $p_{\max} = a$ and we obtain that the consumer surplus equals

$$C_s(p) = \int_p^a (a-p)dp = \frac{1}{2}(a-p)^2 = \frac{\lambda^2(p)}{2}$$

By a similar analys we obtain for $p_{\max} = \infty$ that

$$C_s(p) = \int_p^\infty \lambda(u)du. \tag{A.3}$$

In the latter case, we assume that this integral is finite. If the offered quantity of the product is given by q and $\lambda(p)$ is the demand for the product at price p, then the total sales are given by $\min\{q, D(p)\}$. As in Cohen et al. (2016) and Raz & Ovchinnikov (2015), we define the consumer surplus $C_s(p,q)$ by the product of the fraction of customer being able to buy the product and the consumer surplus of all the customers willing to pay the price p. This means that

$$C_s(p,q) = \frac{\min\{q, \lambda(p)\}}{\lambda(p)} \int_p^{p_{\max}} \lambda(u)du \tag{A.4}$$

Observe for the special demand function in Raz & Ovchinnikov (2015) it follows for $\lambda(p) > q$ (see formula (2) of Raz & Ovchinnikov (2015)) that

$$C_s(p,q) = \frac{\min\{q, \lambda(p)\}}{\lambda(p)} \int_p^{p_{\max}} \lambda(u)du = \frac{q}{\lambda(p)} \frac{\lambda^2(p)}{2} = \frac{q\lambda(p)}{2}$$

For $\lambda(p) < q$ we obtain (see formula (2) of Raz & Ovchinnikov (2015)) that

$$C_s(p,q) = \frac{\lambda^2(p)}{2}.$$