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THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
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EXTREMAL GRAPH THEORY AND APPLYING THE REGULARITY LEMMA

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FOR
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IN
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BY

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August 2022

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

EXTREMAL GRAPH THEORY AND APPLYING THE REGULARITY LEMMA

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Advisor: Asst. Prof. Dr. Celalettin KAYA
August 2022

The main source used in the preparation of this study is Diestel's (2017) "Graph Theory" book. This thesis mainly consists of studying the seventh chapter of the aforementioned book to understand and explain the subject of “extremal graph theory”. But of course, no part of the mentioned book has been quoted exactly, a study has been put forward with our own words and our own sentences; and on the one hand, some very difficult or technical proofs were skipped, on the other hand, some parts of the book that were left to the reader were explained and the subject was presented more understandably. In addition to these, the articles listed in the references were also consulted. To summarize in outline: In the first chapter, basic definitions and theorems of graph theory, which are the prerequisites for understanding the main subject, are covered by using the first chapter of the aforementioned book. In the first section of the second chapter, the question of what edge density is required to force a graph to contain a copy of a given graph is sought; and in the second section, the same problem is discussed in terms of inclusion as a minor. In the third section, Hadwiger's (1943) conjecture and related results; and in the fourth section, the Szemeredi lemma of regularity and its proof are studied. In the fifth and the last section of the second chapter, a general method of applying the Szemeredi lemma is introduced and as an example of this method, the theorem of Erdös and Stone (1946) is proved. In the third chapter of the thesis, before the conclusions and recommendation chapter, a brief literature review on the subject of the thesis is presented by using the notes section of the seventh chapter of the mentioned book.

2022, 63 pages

Keywords: Subgraphs, Minors, Regularity Lemma

ÖZET

EKSTREM GRAF TEORİ VE REGÜLERLİK LEMMASININ UYGULANMASI

Ali Farooq Sadeq SADEQ
Matematik, Yüksek Lisans
Tez Danışmanı: Dr. Öğr. Üyesi Celalettin KAYA
Ağustos 2022

Bu çalışmanın hazırlanmasında kullanılan başlıca kaynak, Diestel'in (2017) "Graph Theory" kitabıdır. Bu tez esas itibariyle, "ekstrem graf teori" konusunu anlamak ve anlatmak için, söz konusu kitabın yedinci bölümünün çalışılmasından ibarettir. Fakat tabi ki mevzubahis kitabın herhangi bir kısmı aynen alıntılanmamış, kendi sözcüklerimiz ve kendi cümlelerimizle bir çalışma ortaya konulmuş ve bir yandan bazı çok zorlu veya teknik ispatlar atlanırken, diğer yandan kitabın okuyucuya bırakılan bazı kısımları şerh edilerek konu daha anlaşılır bir şekilde sunulmuştur. Bunlara ek olarak, kaynaklar kısmında listelenmiş olan makalelere de başvurulmuştur. Ana hatlarıyla özetlemek gerekirse: Birinci bölümde, mevzubahis kitabın birinci bölümü kullanılarak, esas konunun anlaşılabilmesi için ön şart durumunda olan graf teorinin temel tanım ve teoremleri işlenmiştir. İkinci bölümün ilk alt bölümünde, bir grafı, verilen bir grafın kopyasını içermeye zorlamak için hangi kenar yoğunluğunun gerektiği sorusuna cevap aranmış ve ikinci alt bölümünde, aynı problem minör olarak içermek açısından ele alınmıştır. Üçüncü alt bölümünde, Hadwiger (1943) kestirimi ve ilgili sonuçlar ve dördüncü alt bölümünde, Szemeredi regülerlik lemması ve ispatı çalışılmıştır. Son beşinci ve son alt bölümünde ise, Szemeredi lemmasının uygulanması ile ilgili genel bir metot tanıtılmış ve bu metodun bir örneği olarak, Erdös and Stone (1946) teoremi ispat edilmiştir. Tezin üçüncü, sonuç ve öneriler bölümünden önceki son bölümünde ise, mevzubahis kitabın yedinci bölümünün notlar kısmı kullanılarak, tezin konusu ile ilgili kısa bir literatür taraması sunulmuştur.

2022, 63 sayfa

Anahtar Kelimeler: Altgraflar, Minörler, Regülerlik lemması

PREFACE AND ACKNOWLEDGEMENTS

This thesis is a review study, and the main source used in its preparation is Diestel (2017): "Graph Theory (Graduate Texts in Mathematics, 173)". In order for this book to be the main source for the thesis, our advisor has communicated with the author Reinhard Diestel via e-mail, and he has received the following reply: "It's completely ok to have MSc theses consist of re-writes, or summaries, of chapters of my book. As you say, you must state clearly that whatever material is used originates from the book, but then that's fine. The theses should not include long passages verbatim; if they're needed, they can refer to the book instead. But summaries in the candidates' own words are completely ok if you state clearly what you're summarising (namely, passages from the book)".

After the permission of the author, the mentioned book, which is one of the most excellent books prepared at the graduate level in graph theory, has been adopted as the main source of the thesis. Therefore, almost all the figures, definitions, and theorems in the thesis have been taken from Diestel (2017), and instead of citing each one separately, it was found more convenient to state this situation here. Also, proofs of almost all theorems that have been stated but not proven in the thesis can be found in the aforementioned book.

In addition, the basic definitions and theorems required by anyone who wants to study any subject related to graph theory are naturally the same. For this reason, the "basics of graph theory" chapter of this thesis was prepared jointly by Rusul Hadi Mahdi AL-NASER, Hiba Munneer Mohammed AL-RUBAYE, Bashar Buraa Khalaf KHALAF, and Hajir Shakir Mahmood MAHMOOD with the work done by our advisor. Therefore, the mentioned chapter of the theses of each of the aforementioned graduate students is almost the same.

After stating these, at the outset, we thank God Almighty for His generosity towards us in completing the educational journey and for granting us success. I thank my family who have been my support and strength at this stage. I thank my father and mother, who have the great credit and great effort for reaching this stage. I would like to thank my professors at Çankırı Karatekin University who gave us all the support and information. I hope that God Almighty will help us and benefit us with what we know.

I dedicate this message to: Hajir Sadeq, Wedad Majed, Shakir Mahmood, Hajir Shakir, Halema Abdulah.

Ali Farooq Sadeq SADEQ

Çankırı - 2022

CONTENTS

LIST OF SYMBOLS

$\Gamma = (V, E)$	A graph with a vertex set V and an edge set E
$\mathcal{T}$	A tree
$\overline{\Gamma}$	Complement of a graph Γ
$\mathbb{K}_{s,t}$	Complete bipartite graph with partite sets of sizes s and t
$\mathbb{K}^n$	Complete graph of order n
$\mathbb{C}^n$	Cycle of order n
$E(\Gamma) = E$	Edge set of a graph Γ
$I\Gamma$	Inflated minor of a graph Γ
$\simeq$	Isomorphism between two graphs
$\mathfrak{L}(\Gamma)$	Line graph of a graph Γ
$\preccurlyeq$ or $\preceq$	Minor relation between two graphs
$\mathbb{P}^n$	Path of order n
$\mathfrak{N}_\Gamma(\alpha)$	Set of neighbors of a vertex α in a graph Γ
$\Gamma[D]$	Subgraph of Γ induced by a set D
$d(\Gamma)$	The average degree of a graph Γ
$deg_\Gamma(\alpha)$	The degree of a vertex α in a graph Γ
$diam(\Gamma)$	The diameter of a graph Γ
$\lambda(\Gamma)$	The edge-connectivity of a graph Γ
$\Gamma = (\emptyset, \emptyset)$ or $\emptyset$	The empty graph
$\varrho(\Gamma)$	The girth of a graph Γ
$\ell(\mathbb{C})$	The length of a cycle $\mathbb{C}$
$\ell(\mathbb{P})$	The length of a path $\mathbb{P}$
$\Delta(\Gamma)$	The maximum degree of a graph Γ
$\delta(\Gamma)$	The minimum degree of a graph Γ
$\|\Gamma\|$	The order of a graph Γ
$rad(\Gamma)$	The radius of a graph Γ
$\|\|\Gamma\|\|$	The size of a graph Γ
$\kappa(\Gamma)$	The (vertex-)connectivity of a graph Γ
$T\Gamma$	Topological minor of a graph Γ
$\leq_{\mathcal{T}}$ or $\leqslant_{\mathcal{T}}$	Tree order induced by a rooted tree $\mathbb{T}$
$V(\Gamma) = V$	Vertex set of a graph Γ

LIST OF ABBREVIATIONS

a.d.	Average degree
c.n.	Chromatic number
e.s.	Exceptional set
g.p.	Graph property
iff	If and only if
i.s.	Induced subgraph
r.p.	Regular partition
s.s.	Spanning subgraph
s.t.	Spanning tree
t.m.	Topological minor
t.o.	Tree order
TFAE	The following are equivalent
wlog	Without loss of generality
wrt	With respect to

LIST OF FIGURES

1. BASICS OF GRAPH THEORY

In this chapter, we will introduce basic notions and fundamental theorems in graph theory.

1.1 Basic Definitions

Definition 1.1 A pair (V, E), where $E \subseteq [V]^2$, that is, E is a subset of the set of 2-element subsets of V, is called a *graph*. An element of V (respectively, E) is called a *vertex* or *node* (respectively, *edge* or *line*).

From now on, $\Gamma = (V, E)$ always denotes a graph.

A graph is illustrated by using dots and lines in such a way that we draw a dot for each vertex and we draw a line or a curve between two dots corresponding $a, b \in V$, if $\{a, b\} \in E$. How the dots and lines are pictured is not important, the only important matter is to preserve adjacency and non-adjacency relation between vertices.

Example 1.1 Let $V = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ and
$E = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_1, a_4\}, \{a_5, a_6\}, \{a_6, a_7\}\}$. Then it is possible to draw this graph as in Figure 1.1 below.

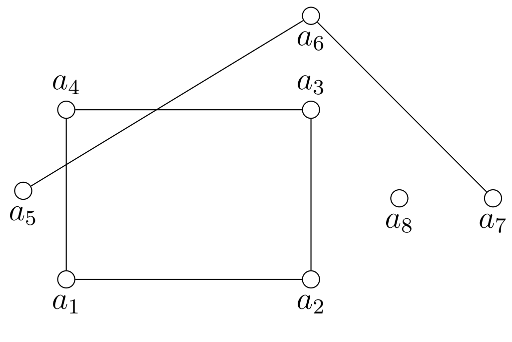


Figure 1.1 The graph $\Gamma = (V, E)$

The vertex set (respectively, the edge set) of the graph Γ is denoted by $V(\Gamma)$ (respectively, $E(\Gamma)$). And instead of writing $a \in V(\Gamma)$ and $e \in E(\Gamma)$, we simply write $a \in \Gamma$ or $e \in \Gamma$, respectively.

Definition 1.2 The *order* (respectively, the *size*) of a graph Γ, denoted by $|\Gamma|$ (respectively, $\|\Gamma\|$), is the number of vertices (respectively, the number of edges) in Γ.
A graph is called a *finite graph*, an *infinite graph*, a *countable graph*, etc. wrt its order.

Definition 1.3 The graph $\Gamma = (\emptyset, \emptyset)$ is called the *empty graph*.
If the order of a graph is 0 or 1, then it is called a *trivial graph*.

Definition 1.4 Let $a \in V$ and $e \in E$ in a graph $\Gamma = (V, E)$.
· If $a \in e$, the a is said to be *incident with* e.
· If $e = \{a, b\}$, then a and b are called *end-vertices* or *ends* of e, and it is said that e *joins* a and b.
· An edge $e = \{a, b\}$ can be written as ab or ba.

Definition 1.5 Let $A, B \subseteq V$ in a graph $\Gamma = (V, E)$. If $e = ab \in E$ such that $a \in A$ and $b \in B$, then $e = ab$ is called an $A - B$ edge.

Notation 1.1 Let $K \subseteq E$, and $a, b \in V$.
· $K(A, B)$ denotes the set of all $a - b$ edges in K such that $a \in A$ and $b \in B$.
· The sets $K(\{a\}, B)$ and $K(A, \{b\})$ is simply denoted by $K(a, B)$ and $K(A, b)$, respectively.
· The set of all edges in K incident with a is denoted by $K(a)$.

Definition 1.6 Let $a, b \in V(\Gamma)$, $f \neq g \in E(\Gamma)$.
· If $ab \in E(\Gamma)$, then a, b are called *adjacent* or *neighbours*.
· If f and g have a common endpoint, then they are called *adjacent*.
· If any distinct pair of vertices of Γ are adjacent, then Γ is called a *complete graph*.
· The complete graph on n vertices is denoted by $\mathbb{K}^n$, and $\mathbb{K}^3$ is a *triangle*.

Definition 1.7 Let $\Gamma = (V, E)$ and $\Pi = (W, F)$ be two graphs, and let $\psi : V \longrightarrow W$ be a map. If the map ψ preserves the adjacency relation between vertices, that is:
· If "$ab \in E \Longrightarrow \psi(a)\psi(b) \in F$", then ψ is called a *homomorphism* from Γ to Π.

Proposition 1.1 *Let* $\Gamma = (V, E)$ *and* $\Pi = (W, F)$ *be two graphs, and let* $\psi : V \longrightarrow W$ *be a homomorphism. If* $y \in Im(\psi)$*, then* $\psi^{-1}(y) = \{a \in V : \psi(a) = y\} \subseteq V$ *is independent.*

Proof. Suppose to the contrary that $\psi^{-1}(y)$ is not independent. Then there are two adjacent vertices, a_0 and $a_1 \in \psi^{-1}(y)$ such that $\{a_0, a_1\} \in E$, and thus $\{\psi(a_0), \psi(a_1)\} \in F$. But since $\psi(a_0) = y = \psi(a_1)$, then $\{\psi(a_0), \psi(a_1)\} = \{y\}$, which is not a 2-element subset of W. ■

Definition 1.8 · If the map ψ given in the previous definition is a bijection and its inverse map ψ^{-1} is also a homomorphism, then ψ is an *isomorphism*.
· If $\psi : \Gamma \to \Pi$ is an isomorphism, then Γ and Π are called *isomorphic* graphs, and the notation $\Gamma \simeq \Pi$ is used.
· If ψ is an isomorphism and $\Pi = \Gamma$, then ψ is called an *automorphism* of Γ.

Proposition 1.2 *The map* $\psi : V(\Gamma) \longrightarrow V(\Pi)$ *is an isomorphism iff* ψ *is a bijection, and:*

$$ab \in E(\Gamma) \Longleftrightarrow \psi(a)\psi(b) \in E(\Pi).$$

Proof.
($\Rightarrow$): ψ is trivially a bijection, and $ab \in E(\Gamma) \Longleftrightarrow \psi(a)\psi(b) \in E(\Pi)$:
· $\Rightarrow$: Since ψ is a homomorphism, this direction is true by definition.

$$\begin{aligned} \cdot\Leftarrow : a'b' \in E(\Pi) &\implies \text{There are } a, b \in V(\Gamma) \text{ such that } \psi(a) = a' \text{ and } \psi(b) = b' \\ &\implies \psi(a)\psi(b) \in E(\Pi) \\ &\implies \psi^{-1}(\psi(a))\psi^{-1}(\psi(b)) \in E(\Gamma) \;\; (\psi^{-1} \text{ is a homomorphism}) \\ &\implies ab \in E(\Gamma). \end{aligned}$$

($\Leftarrow$): $\Rightarrow$ shows that ψ is a homamorphism, and $\Leftarrow$ shows that ψ^{-1} is a homamorphism. Therefore, ψ is an isomorphism. ■

Definition 1.9 Let Γ and Π be two graphs.

· The *union* of Γ and Π is the graph $\Gamma \cup \Pi := (V(\Gamma) \cup V(\Pi), E(\Gamma) \cup E(\Pi))$.

· The *intersection* of Γ and Π is the graph $\Gamma \cap \Pi := (V(\Gamma) \cap V(\Pi), E(\Gamma) \cap E(\Pi))$.

· If $\Gamma \cap \Pi$ is the empty graph, then the graphs Γ and Π are said to be *disjoint*.

Definition 1.10 Let $\Gamma = (V, E)$ and $\Pi = (W, F)$ be two graphs.

· If $W \subseteq V$ and $F \subseteq E$, then the graph Π is said to be a *subgraph* of Γ, and it is denoted by $\Pi \subseteq \Gamma$.

· If $\Pi \subseteq \Gamma$ and $\Gamma \neq \Pi$, then Π is said to be a *proper subgraph* of Γ.

· If $\Pi \subseteq \Gamma$ and if Π contains every edge in Γ with two of its ends in $V(\Pi) = W$ (that is, if $\{a, b\} \in E$ and $a, b \in W$, then $\{a, b\} \in F$), then Π is called an *induced subgraph (i.s.)* of Γ, denoted by $\Pi := \Gamma[W]$, and we say that W *induces / spans* Π in Γ.

· If $\Pi \subseteq \Gamma$, induced or not, then instead of $\Gamma[V(\Pi)]$, the notation $\Gamma[\Pi]$ is used.

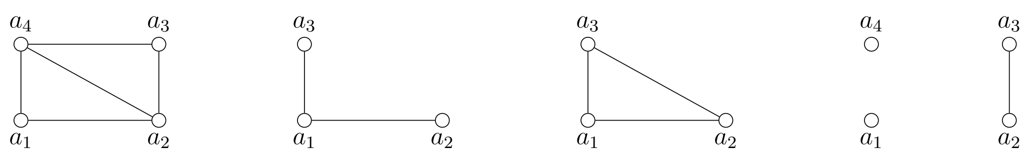
· Π is a *spanning subgraph (s.s.)* of Γ if it is a subgraph of Γ and $W = V$.

Example 1.2 Consider the graphs Γ, Γ_1, Γ_2 and Γ_3 in Figure 1.2.

The graphs Γ_1, Γ_2 and Γ_3 are all subgraphs of Γ.

· Γ_1 is not an i.s. of Γ, but $\Gamma_2 = \Gamma[\{a_1, a_2, a_3\}]$ is induced.

· $V(\Gamma_2) \neq V(\Gamma)$, thus Γ_2 is not a s.s. of Γ, but Γ_3 is a s.s. of Γ.



(a) Γ **(b)** Γ_1 **(c)** Γ_2 **(d)** Γ_3

Figure 1.2 The graphs Γ, Γ_1, Γ_2 and Γ_3

Definition 1.11 Let W be a vertex set, $W \subseteq V$ or not, and $L \subseteq [V]^2$ be an edge set.

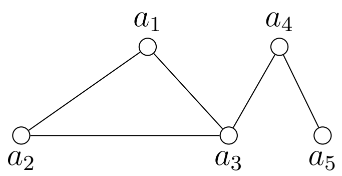
· The graph $\Gamma - W := \Gamma[V \diagdown W]$, (i.e., $\Gamma - W$ is the subgraph of Γ such that $\Gamma - W = (V \diagdown W, E \diagdown \{h = ab : a \in W \cap V$ or $b \in W \cap V\})$.

· The graphs $\Gamma - L$ and $\Gamma + L$ are defined respectively as follows:

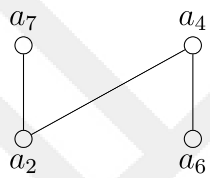
$$\Gamma - L := (V, E \diagdown L) \text{ and } \Gamma + F := (V, E \cup L).$$

· If $W = \{a\}$, then instead of $\Gamma - \{a\}$, we simply write $\Gamma - a$. Also, $\Gamma - V(\Pi)$, where Π is any graph, can be denoted by $\Gamma - \Pi$, and is called the *difference* of the graphs Γ and Π.
· If $L = \{l\}$, instead of $\Gamma - \{l\}$ and $\Gamma + \{l\}$, we simply write $\Gamma - l$ and $\Gamma + l$.

Example 1.3 Let Γ and Π be the graphs given in Figure 1.3.



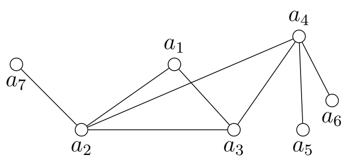
(a) Γ



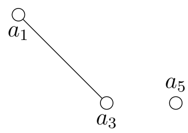
(b) Π

Figure 1.3 The graphs Γ and Π

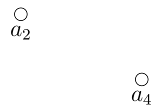
Then the union, intersection and difference of Γ and Π are given in Figure 1.4.



(a) $\Gamma \cup \Pi$



(b) $\Gamma - \Pi$



(c) $\Gamma \cap \Pi$

Figure 1.4 The union, difference and intersection of Γ and Π

Definition 1.12 If a graph Γ satisfies a g.p., but there exists no graph $\Pi = (V, L)$ with $E(\Gamma) \subsetneqq L$ satisfying the same g.p., then Γ is called *edge-maximal* wrt this property.

In general, a graph is called *maximal* or *minimal* with a g.p., if it is maximal or minimal wrt the subgraph relation, unless a particular ordering relation is specified.

Definition 1.13 Let Γ and Π be disjoint graphs. Then, the *join* of Γ and Π, denoted by $\Gamma * \Pi$, is defined as follows:

$\Gamma * \Pi = (V(\Gamma) \cup V(\Pi), E(\Gamma) \cup E(\Pi) \cup \{h = ab : a \in V(\Gamma), b \in V(\Pi)\})$.

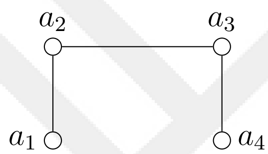
Definition 1.14 · The *complement graph* of Γ, denoted by $\overline{\Gamma}$, is the following graph:

$\overline{\Gamma} := (V(\Gamma), [V]^2 - E(\Gamma))$.

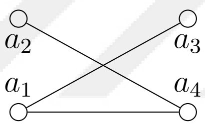
· The *line graph* of Γ, denoted by $\mathfrak{L}(\Gamma)$, is the following graph:

$\mathfrak{L}(\Gamma) = (V(\mathfrak{L}(\Gamma)) = E, E(\mathfrak{L}(\Gamma)) = \{ab : a, b \in E$ and a, b are adjacent edges in $\Gamma\})$.

Example 1.4 Consider the graph Γ and $\overline{\Gamma}$ in Figure 1.5. Note that $\Gamma \simeq \overline{\Gamma}$.

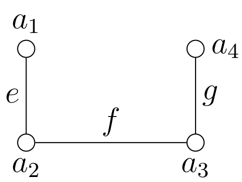


(a) Γ

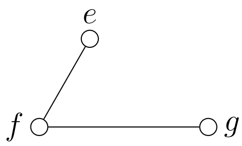


(b) $\overline{\Gamma}$

Figure 1.5 A graph Γ and its complement $\overline{\Gamma}$



(a) Γ



(b) $\overline{\Gamma}$

Figure 1.6 A graph Γ and its line graph $\mathfrak{L}(\Gamma)$

Example 1.5 Consider the graph Γ given in Figure 1.6. Its line graph $\mathfrak{L}(\Gamma)$ is also given in the same figure.

Definition 1.15 Let Γ be a nonempty graph, and $L \subseteq V = V(\Gamma)$.

· Let $a \in V(\Gamma)$. The *neighbor* of a, $\mathfrak{N}_\Gamma(a)$ or simply $\mathfrak{N}(a)$, is the collection of all adjacent vertices of a.

· In general, the set $\mathfrak{N}(L) := \{a \in V \setminus L : ab \in E$ for some $b \in L\}$ is called the *neighbor* of L.

Definition 1.16 The *degree* of a vertex a, denoted by $d_\Gamma(a)$ or simply $d(a)$, is the number of edges at a. That is: $d_\Gamma(a) = d(a) = |E(a)| = |\mathfrak{N}(a)|$.

· A degree zero vertex is called an *isolated vertex*.

· $\delta(\Gamma)$ =min$\{d(a) : a \in V\}$ is called the *minimum degree* of Γ.

· $\Delta(\Gamma)$ =max$\{d(a) : a \in V\}$ is called the maximum degree of Γ.

· If for some $j \in \mathbb{N}$, $d(a) = j$ for all $a \in V$, then Γ is called a *j-regular* graph.

· A 3-regular graph is called a *cubic* graph.

Definition 1.17 The *average degree (a.d.)* of Γ is the number:

$$d(a) = \frac{1}{|V|} \sum_{a \in V} d(a).$$

Another parameter is given as a function of the number of edges and the number of vertices in a graph is defined as follows:

$$\sigma(\Gamma) = \frac{|E|}{|V|}.$$

Proposition 1.3 *Let Γ be a graph. Then, the following inequalities hold:*

$$\delta(\Gamma) \leq d(\Gamma) \leq \Delta(\Gamma).$$

Proof. Let $|V| = n$. Assume that $a \in V$ is a vertex in V. Then:

$$\begin{aligned}\delta(\Gamma) \leq d(a) \leq \Delta(\Gamma) &\implies \sum_{a \in V} \delta(\Gamma) \leq \sum_{a \in V} d(a) \leq \sum_{a \in V} \Delta(\Gamma) \\ &\implies n.\delta(\Gamma) \leq \sum_{a \in V} d(a) \leq n.\Delta(\Gamma) \\ &\implies \delta(\Gamma) \leq \frac{1}{n} \sum_{a \in V} d(a) \leq \Delta(\Gamma) \\ &\implies \delta(\Gamma) \leq d(\Gamma) \leq \Delta(\Gamma).\end{aligned}$$

■

Proposition 1.4 *Let Γ be a graph. Then:*

$$\sigma(\Gamma) = \frac{1}{2} d(\Gamma).$$

Proof. Let $|E|$ be the size of Γ.

$$\begin{aligned}\sum_{a \in V} d(a) = 2|E| &\implies |E| = \frac{1}{2} \sum_{a \in V} d(a) = \frac{1}{2}|V| \frac{1}{|V|} \sum_{a \in V} d(a) \\ &\implies |E| = \frac{1}{2}|V| d(\Gamma) \\ &\implies \frac{|E|}{|V|} = \frac{1}{2} d(\Gamma) \\ &\implies \sigma(\Gamma) = \frac{1}{2} d(\Gamma).\end{aligned}$$

■

Proposition 1.5 *In a graph, there exists an even number of odd degree vertices.*

Proof. Since $\sum_{a \in V} d(a) = 2|E|$ is an even integer, the number of odd degree vertices is even. ■

Definition 1.18 A nonempty graph $\mathbb{P} = (V, E)$, where $V = \{a_0, a_1, \ldots, a_k\}$ and $E = \{a_0 a_1, a_1 a_2, \ldots, a_{k-1} a_k\}$, is called a *path* from a_0 to a_k (or between a_0 and a_k).
Note that all the vertices are distinct for a path.

· We say that the end-vertices a_0 and a_k are *connected* or *linked* by $\mathbb{P}$.

· The *inner* vertices of $\mathbb{P}$ are $a_1, a_2, \ldots, a_{k-1}$.

· The *length* of $\mathbb{P}$ is the number of edges of $\mathbb{P}$, and denoted by $\ell(\mathbb{P})$.

· A *j-path* is a path of length j, and it is denoted by $\mathbb{P}^j$. Note that $j = 0$ is possible, and $\mathbb{P}^0 = \mathbb{K}^1$.

Notation 1.2 Let $\mathbb{P} = (\{a_0, a_1, \ldots, a_k\}, \{a_0a_1, a_1a_2, \ldots, a_{k-1}a_k\})$ be a path.

· $\mathbb{P}$ is denoted by $\mathbb{P} = a_0a_1 \ldots a_k$, and $\mathring{\mathbb{P}} = a_1a_2 \ldots a_{k-1}$.

· For $0 \leq i \leq j \leq k$, we use the following notations:

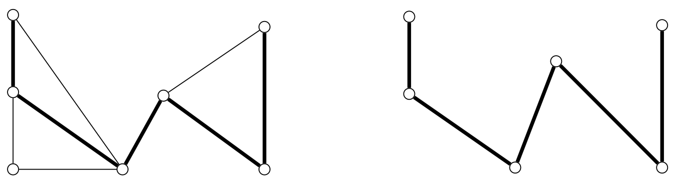
· $\mathbb{P}a_i = a_0 \ldots a_i$, · $a_i\mathbb{P} = a_i \ldots a_k$;

· $\mathbb{P}\mathring{a}_i = a_0 \ldots a_{i-1}$, · $\mathring{a}_i\mathbb{P} = a_{i+1} \ldots a_k$;

· $a_i\mathbb{P}a_j = a_i \ldots a_j$, · $\mathring{a}_i\mathbb{P}\mathring{a}_j = a_{i+1} \ldots a_{j-1}$.

· For the cancatenation of paths, a similar intuitive notation is used. For instance, suppose that the union of the paths $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$ is again a path, say $\mathbb{P}a \cup a\mathbb{Q}b \cup b\mathbb{R}$. Then this path is simply denoted by $\mathbb{P}a\mathbb{Q}b\mathbb{R}$.

Example 1.6 For the graph Γ given in Figure 1.7, a path $\mathbb{R}^5$ in Γ is given on the right of Γ.



(a) Graph Γ **(b)** A path $\mathbb{R}^5$

Figure 1.7 A path $\mathbb{R}^5$ in the graph Γ

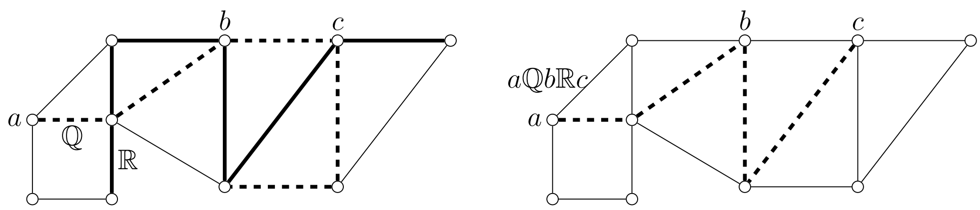


Figure 1.8 Paths $\mathbb{Q}$, $\mathbb{R}$ and $a\mathbb{Q}b\mathbb{R}c$

Example 1.7 For the graph Γ in Figure 1.8, paths $\mathbb{Q}$ and $\mathbb{R}$ indicated by dashed and bold line segments, respectively. On the right side, the path $a\mathbb{Q}b\mathbb{R}c$ is drawn by using dashed line segments.

Definition 1.19 Let $\Gamma = (V, E)$ and Π be graphs. Let $A, B \subseteq V$, and $\mathbb{Q} = a_0 \dots a_k$.
· If $V(\mathbb{Q}) \cap A = \{a_0\}$ and $V(\mathbb{Q}) \cap B = \{a_k\}$, then $\mathbb{Q}$ is called an $A - B$ *path*.
· If the set of inner vertices of two or more paths are pairwise disjoint, then these paths are called *independent*.
· If $\mathbb{Q}$ intersects Π exactly at its end vertices, $\mathbb{Q}$ is called an Π-*path*.

Definition 1.20 Let $j \geq 3$ and $\mathbb{P} = a_0 \dots a_{j-1}$ be a path of length $j - 1$. The graph $\mathbb{C} = \mathbb{P} + a_{j-1}a_0$ is called a *cycle*. The *length* of a cycle is the number of edges in the cycle, denoted by $\ell(\mathbb{C})$, and a k-*cycle* is a cycle of length k. A k- cycle is denoted by $\mathbb{C}^k$.

Notation 1.3 A cycle can be denoted by listing its vertices cyclically. For example, the cycle $\mathbb{C} = \mathbb{P} + a_{j-1}a_0$ can be denoted by $\mathbb{C} = a_0a_1 \dots a_{j-1}a_0$.

Definition 1.21 Let Γ be a graph containing a cycle.
· The *girth* of Γ, denoted by $\varrho(\Gamma)$, is the minimum length of a cycle in Γ.
· The *circumference* of Γ, denoted by $\varsigma(\Gamma)$, is the maximum length of a cycle in Γ.

Note 1.1 If there is no cycle in Γ, then the girth of Γ is defined as ∞, and the circumference of Γ is defined as 0.

Definition 1.22 Let $\mathbb{C}$ be a cycle in Γ. An edge ab of Γ is called a *chord* of $\mathbb{C}$ if $a, b \in V(\mathbb{C})$, but $ab \notin E(\mathbb{C})$.

Note 1.2 Assume that $\mathbb{C}$ is an induced cycle of the graph Γ. Then, since $\mathbb{C}$ is an i.s. of Γ and since $\mathbb{C}$ is a cycle, $\mathbb{C}$ does not have a chord.

Example 1.8 A cycle $\mathbb{C}^7$ and induced cycles $\mathbb{C}^3$ and $\mathbb{C}^6$ are given in Figure 1.9.
Note that since $\mathbb{C}^7$ has chord ab, it is not an induced cycle.

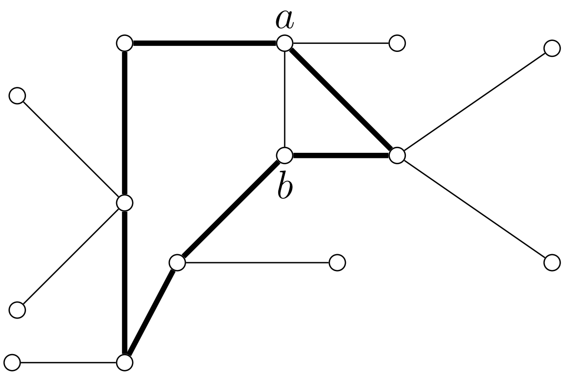


Figure 1.9 A cycle $\mathbb{C}^7$ with chord ab and induced cycles $\mathbb{C}^3$ and $\mathbb{C}^6$

Proposition 1.6 *Suppose that* Γ *is a graph such that* $\delta(\Gamma) \geq 2$*. Then there is a path of length* $\delta(\mathbb{H})$ *and a cycle of length at least* $\delta(\Gamma) + 1$ *in* Γ.

Proof. Let $\mathbb{Q} = a_0 \dots a_k$ be a path of maximum length in Γ. Then $\mathfrak{N}_\Gamma(a_k) \subseteq V(\mathbb{Q})$ by the maximality of $\mathbb{Q}$.

Therefore, "the length of $\mathbb{Q}$" $= k \geq d_\Gamma(a_k) \geq \delta(\Gamma)$.

Now, let $i < k$ be the minimum index such that $a_i a_k \in E(\Gamma)$. Then $a_i a_{i+1} \dots a_k a_i$ is a cycle of length at least $\delta(\Gamma) + 1$, because "the length of the cycle $\mathbb{C} = a_i \dots a_k a_i$" = $\ell(\mathbb{C}) \geq d(a_k) + 1 \geq \delta(\Gamma) + 1$.

The first inequality is true because $\mathfrak{N}_\Gamma(a_k) \subseteq V(\mathbb{Q})$, and $i < k$ is minimal with $a_i \in \mathfrak{N}_\Gamma(a_k) \subseteq V(\mathbb{Q})$. Also 1 is added for the vertex a_k.

The second inequality is also true, because $d(a_k) \geq \delta(\Gamma)$. ■

Definition 1.23 Let $s, t \in V(\Gamma)$.

· The minimum of the lengths of $s-t$ paths in Γ is called the *distance* between the vertices s and t, and it is denoted by $d_\Gamma(s,t)$ or simply $d(s,t)$.

If there is no path joining the vertices s and t, then $d(s,t)$ is defined as ∞.

· The *diameter* of Γ, denoted by $diam(\Gamma)$, is the maximum of the distances in Γ, that is:

$$diam(\Gamma) := \max\{d(s,t) : s, t \in V(\Gamma)\}.$$

The notions girth and diameter are related.

Proposition 1.7 *Let* Γ *be a graph containing a cycle. Then:*

$$\varrho(\Gamma) \leq 2diam(\Gamma) + 1.$$

Proof. Let $\mathbb{C}$ and $\mathbb{Q}$ be a cycle and a path in Γ, respectively.
Suppose to the contrary that $\varrho(\Gamma) > 2diam(\Gamma) + 1$. Then $\varrho(\Gamma) \geq 2diam(\Gamma) + 2$. Now, assume that the equality $\ell(\mathbb{C}) = \varrho(\Gamma)$ holds. Then $\mathbb{C}$ must have two vertices, say c and d such that $d_{\mathbb{C}}(c, d) \geq diam(\Gamma) + 1$. ($d_{\mathbb{C}}(c, d)$ is the distance in the subgraph $\mathbb{C}$.)
But in Γ, $d_\Gamma(c, d) \leq diam(\Gamma)$. Therefore, any path $\mathbb{Q}$ of length $d_\Gamma(c, d)$ between c and d can not be a subgraph of the cycle $\mathbb{C}$. Therefore, there is a $\mathbb{C}$-path $a\mathbb{Q}b$.
Now, there are two $a - b$ paths in $\mathbb{C}$. Let $\mathbb{R}$ be the shorter of them. Then the cycle $a\mathbb{Q}b\mathbb{R}a$ has length strictly less that $\ell(\mathbb{C})$:
$\ell(a\mathbb{Q}b\mathbb{R}a) = \ell(\text{shorter of the two } a - b \text{ paths in } \mathbb{C}) + \ell(a\mathbb{Q}b)$
$\leq \lfloor\frac{\ell(\mathbb{C})}{2}\rfloor + \ell(\mathbb{Q}) \leq \lfloor\frac{\ell(\mathbb{C})}{2}\rfloor + \text{diam}(\Gamma) \leq \lfloor\frac{\ell(\mathbb{C})}{2}\rfloor + \frac{\ell(\mathbb{C})}{2} - 1 \leq \frac{\ell(\mathbb{C})}{2} + \frac{\ell(\mathbb{C})}{2} - 1$
$= \ell(\mathbb{C}) - 1.$
This is a contradicition, which shows that $\varrho(\Gamma) \leq 2\text{diam}(\Gamma) + 1$. ■

Definition 1.24 The natural number rad(Γ) $= min_{a\in V(\Gamma)} max_{b\in V(\Gamma)} d_\Gamma(a, b)$ is called the *radius* of Γ. A vertex a satisfying rad(Γ) $= max_{b\in V(\Gamma)} d_\Gamma(a, b)$ is called a *central* vertex, that is, a has the maximum distance from the other vertices of Γ.

Proposition 1.8 *For a graph* Γ*, we have:*

$$rad(\Gamma) \leq diam(\Gamma) \leq 2rad(\Gamma)$$

Proof. · rad(Γ) $\leq$ diam(Γ): This is true from the definitions of radius and diameter.
· diam(Γ) $\leq$ 2rad(Γ): Let a, b be two vertices such that $d(a, b) =$ diam(Γ), and let a be a central vertex. Then: diam(Γ)$= d(a, b) \leq d(a, s) + d(s, d) \leq rad(\Gamma) + rad(\Gamma)$. ■

Definition 1.25 · An alternating sequence $a_0 f_0 a_1 f_1 \ldots f_{k-1} a_k$ of vertices and edges in a graph Γ, where $f_i = \{a_i, a_{i+1}\}$ for all $i < k$, is called a *walk*.

· The *length* of a walk is the number of edges in the walk.

· If $a_0 = a_k$, $\mathbb{W}$ is called a *closed* walk.

Note 1.3 In a walk $\mathbb{W}$, if all the vertices are distinct, then $\mathbb{W}$ is also a path.

Proposition 1.9 *Let Γ be a graph, and let $\mathbb{W}$ be a walk between the vertices s and t. Then, there is a path $\mathbb{P} \subseteq \mathbb{W}$ joining the vertices s and t.*

Proof. Let $\mathbb{P}$ be a minimum length $s - t$ walk in $\mathbb{W}$, that is, $\mathbb{P} \subseteq \mathbb{W}$ and $\mathbb{P}$ is a walk between the vertices s and t, and also its length is minimum in such $s - t$ walks.

We claim that $\mathbb{P}$ is an $s - t$ path. Say $\mathbb{P} := s = a_1 a_2 \ldots a_n = t$. And, suppose to the contrary that $\mathbb{P}$ is not a path. Then there exists a repeated vertex a_i $(1 \leq i \leq n)$ in $\mathbb{P}$.

Therefore, $\mathbb{P} := s = a_1 a_2 \ldots a_j = a_i \ldots a_n = t$ for some $j > i$.

But then $\mathbb{Q} = s = a_1 a_2 \ldots a_i a_{j+1} \ldots a_n = t$ is also an $s - t$ walk in $\mathbb{W}$ and the length of $\mathbb{Q}$ is strictly less than the length of $\mathbb{P}$. This gives a contradiction.

$\therefore$ There is no repeated vertex in $\mathbb{P}$. $\therefore$ $\mathbb{P}$ is an $s - t$ path in $\mathbb{W}$. ■

1.2 Connectivity

Definition 1.26 Let Γ be a nonempty graph.

· Γ is called *connected* if there exist a path between any two of its vertices.

· Let $Y \subseteq V(\Gamma)$. If $\Gamma[Y]$ is connected, then we say that Y is connected in Γ.

· If a graph is not connected, it is called *disconnected*.

Proposition 1.10 *Let Γ be a connected graph, and let $|V(\Gamma)| = n$. The vertices of Γ can be labeled as $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that for every $i = 1, \ldots n$, $\Gamma_i = \Gamma[\sigma_1, \sigma_2, \ldots, \sigma_i]$ is connected.*

Proof. Choose a vertex of Γ, and label it as σ_1. Then $\Gamma_1 = \Gamma[\sigma_1]$ is trivially connected. Assume that there is such a labeling for $1 \leq j < |\Gamma|$.

Now, for $j+1$, take a vertex $y \in \Gamma - \Gamma_j$. Since Γ is connected, there is a $y - \sigma_1$ path $\mathbb{R} = y = c_1 \ldots c_l = \sigma_1$.

Let $1 \leq k \leq l$ be the index such that $c_k \notin \Gamma_j$ but $c_{k+1} \in \Gamma_j$. Take c_k as σ_{j+1}. Then σ_{j+1} has a neighbor, namely c_{k+1} in Γ_j, and thus $\Gamma_{j+1} = \Gamma_j[\sigma_{i+1}]$ is connected.

Therefore, by induction on n, Γ_j is connected for every j. ■

Definition 1.27 Let Π be a subgraph of Γ.

If Π is connected, and it there is no connected subgraph of Γ properly containing Π, then Π is called a *component* of Γ.

Note 1.4 A component of a graph Γ is an i.s. Because a component is a maximal connected subgraph of a graph Γ by definition. Therefore, if it contains two vertices, say $\sigma, \Gamma \in V(\Gamma)$, and if $\sigma\Gamma \in E(\Gamma)$, then it must also contain the edge $\sigma\Gamma$.

Example 1.9 There is a graph Γ with four components is Figure 1.10. A minimal spanning connected subgraph of each component is drawn by bold edges. Note also that the sets of vertices of components partition $V(\Gamma)$.

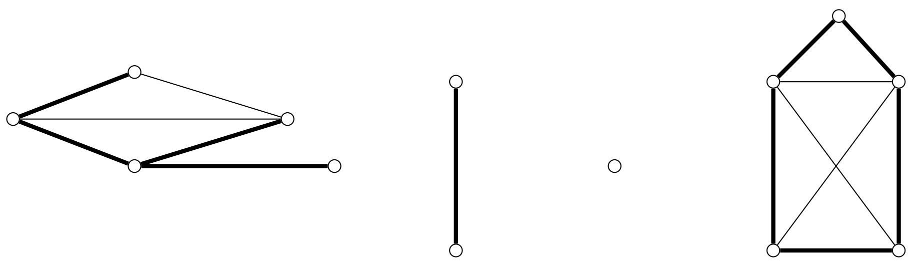


Figure 1.10 A graph Γ with 4 components

Definition 1.28 Let Γ be a graph, $\alpha, \beta \in V(\Gamma)$, $\mathbb{A}, \mathbb{B} \subseteq V(\Gamma)$, $\mathbb{S} \subseteq V(\Gamma) \cup E(\Gamma)$.

· If the intersection of $\mathbb{S}$ with each $\mathbb{A} - \mathbb{B}$ path in Γ is nonempty, then it is said that $\mathbb{S}$ *separates* $\mathbb{A}$ and $\mathbb{B}$ in Γ.

· If $\alpha, \beta \notin \mathbb{S}$ and if $\mathbb{S}$ separates the sets $\{\alpha\}$, $\{\beta\}$, then it is said that $\mathbb{S}$ *separates* the vertices α and β.

· If $\mathbb{S}$ separates two vertices of Γ, then it is said that $\mathbb{S}$ *separates* Γ.

· If $\mathbb{S} \subseteq V(\Gamma)$, and if $\mathbb{S}$ separates Γ, then $\mathbb{S}$ is called a *separator*.

Note 1.5 If $\mathbb{S}$ separates $\mathbb{A}$ and $\mathbb{B}$ in Γ, then $\mathbb{A} \cap \mathbb{B} \subseteq \mathbb{S}$:

Let $\gamma \in \mathbb{A} \cap \mathbb{B}$. Then there is no trivial $\mathbb{A} - \mathbb{B}$ path with an endpoint γ.

Suppose not. Then there is a nontrivial $\mathbb{A} - \mathbb{B}$ path with an endpoint γ. Say $\mathbb{Q} = \gamma c_1 \dots c_k$ (or $\mathbb{Q} = c_1 \dots c_k \gamma$) is such a path. Then $V(\mathbb{Q}) \cap \mathbb{A} = \{\gamma\}$ and $V(\mathbb{Q}) \cap \mathbb{B} = \{\gamma, c_k\}$ (or $V(\mathbb{Q}) \cap \mathbb{A} = \{c_1, \gamma\}$ and $V(\mathbb{Q}) \cap \mathbb{B} = \{\gamma\}$). This is a contradiction, because $V(\mathbb{Q}) \cap \mathbb{B} \neq \{c_k\}$ (or $V(\mathbb{Q}) \cap \mathbb{A} \neq \{c_1\}$). Thus, the only $\mathbb{A} - \mathbb{B}$ path with an endpoint γ is the trivial path $\mathbb{Q} = \gamma$. Therefore, since $\mathbb{S}$ separates $\mathbb{A}$ and $\mathbb{B}$ in Γ, $\gamma \in \mathbb{S}$. As a result, $\mathbb{A} \cap \mathbb{B} \subseteq \mathbb{S}$.

Definition 1.29 If a vertex α separates two vertices in its component, then α is called a *cutvertex*. If an edge f separates its end-vertices, then it is called a *bridge*.

Note 1.6 An edge f is a bridge $\Longleftrightarrow$ There is no cycle $\mathbb{C}$ containing f.

Equivalently, an edge f is not a bridge $\Longleftrightarrow$ There is a cycle $\mathbb{C}$ containing f:

$f = \alpha\beta$ is not a bridge $\Longleftrightarrow$ There is a path $\mathbb{R}$ between α and β not containing the edge f $\Longleftrightarrow \mathbb{C} = \mathbb{R} + f$ is a cycle.

Example 1.10 For the graph Γ given in Figure 1.11, a, b, c, d, e are cutvertices and $h = bc$ is a bridge.

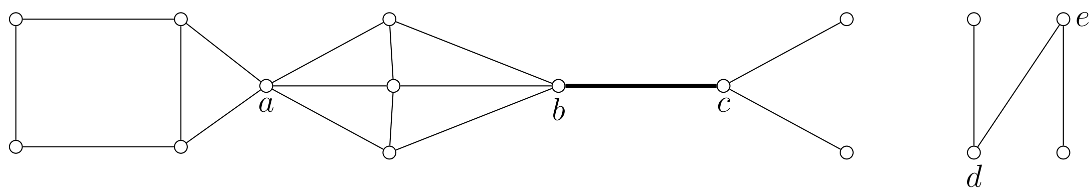


Figure 1.11 A graph Γ with cutvertices a, b, c, d, e and bridge $h = bc$

Definition 1.30 Let $\mathbb{A}, \mathbb{B} \subseteq V(\Gamma)$.

· If the union of the vertex sets $\mathbb{A}$ and $\mathbb{B}$ is V, and if the edge set $E(\mathbb{A} \setminus \mathbb{B}, \mathbb{B} \setminus \mathbb{A})$ is empty (recall that $E(S, T)$ denotes the collection of $S - T$ edges in $E(\Gamma)$), then $\{\mathbb{A}, \mathbb{B}\}$ is called a *separation* of Γ.

· If both $\mathbb{A} \setminus \mathbb{B} \neq \emptyset$ and $\mathbb{B} \setminus \mathbb{A} \neq \emptyset$ hold, then $\{\mathbb{A}, \mathbb{B}\}$ is called a *proper separation*.

· The *order* of $\{\mathbb{A}, \mathbb{B}\}$ is the number $|\mathbb{A} \cap \mathbb{B}|$, and $\mathbb{A}$, $\mathbb{B}$ are called the *sides* of $\{\mathbb{A}, \mathbb{B}\}$.

Note 1.7 $E(\mathbb{A} \setminus \mathbb{B}, \mathbb{B} \setminus \mathbb{A})$ is empty $\Longleftrightarrow$ $\mathbb{A} \cap \mathbb{B}$ separates $\mathbb{A}$ and $\mathbb{B}$:
$E(\mathbb{A} \setminus \mathbb{B}, \mathbb{B} \setminus \mathbb{A})$ is empty $\Longleftrightarrow$ Any $\mathbb{A} - \mathbb{B}$ path in Γ has a vertex in the intersection $\mathbb{A} \cap \mathbb{B}$ $\Longleftrightarrow$ $\mathbb{A} \cap \mathbb{B}$ separates $\mathbb{A}$ and $\mathbb{B}$.

Definition 1.31 If $|\Gamma| > d$ and $\Gamma - \mathbb{S}$ is connected for every vertex set $\mathbb{S}$ of order $< d$, then Γ is said to be *d-connected*. And the maximum d for which Γ is d-connected is called the *vertex connectivity* of Γ, and this maximum number is denoted by $\kappa(\Gamma)$.

Note 1.8 · Let Γ be a nonempty graph. Then Γ is 0-connected, because there is no set $\mathbb{S} \subseteq V$ with $|\mathbb{S}| < 0$.
· Let Γ be a nontrivial graph. Then: Γ is 1-conencted $\Longleftrightarrow$ Γ is connected, because $|\mathbb{S}| < 1$ implies $\mathbb{S}$ is empty.
· Moreover, $\kappa(\Gamma) = 0 \Longleftrightarrow \Gamma$ is disconnected or $\Gamma = \mathbb{K}^1$.
· Also, $\kappa(\Gamma) = n - 1 \Longleftrightarrow \Gamma = \mathbb{K}^n$ for every $n \geq 1$.

Definition 1.32 Let $|\Gamma| > 1$. If $\Gamma - \mathbb{F}$ is connected for each edge set $\mathbb{F}$ of size $< d$, then Γ is said to be *d-edge-connected*. and the maximum d for which Γ is d-edge-connected is called the *edge connectivity* of Γ, and this maximum number is denoted by $\lambda(\Gamma)$.

Note 1.9 Let Γ be a nontrivial graph. Then:
· $\lambda(\Gamma) = 0 \Longleftrightarrow \Gamma$ is disconnected.
· And, $\lambda(\Gamma) \geq 1 \Longleftrightarrow \Gamma$ is connected.

Example 1.11 For the graph given in Figure 1.12, $\kappa(\Gamma) = 1$, $\lambda(\Gamma) = 2$ and $\delta(\Gamma) = 3$.

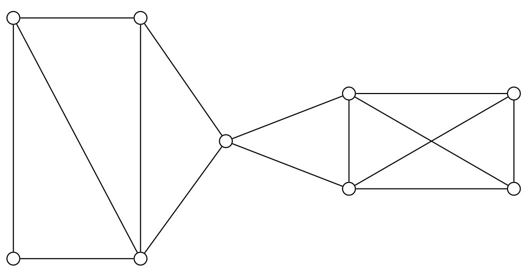


Figure 1.12 A graph Γ with $\kappa(\Gamma) = 1$, $\lambda(\Gamma) = 2$ and $\delta(\Gamma) = 3$

Proposition 1.11 *Let Γ be a nontrivial graph. Then,*

$$\kappa(\Gamma) \leq \lambda(\Gamma) \leq \delta(\Gamma).$$

Proof. · $\lambda(\Gamma) \leq \delta(\Gamma)$: For every $\alpha \in V$, $\lambda(\Gamma) \leq d_\Gamma(\alpha)$, because all the edges incident with α separates Γ. Therefore, $\lambda(\Gamma) \leq \min\{d_\Gamma(\alpha) : \alpha \in V\} = \delta(\Gamma)$.
· $\kappa(\Gamma) \leq \lambda(\Gamma)$: Let $F \subseteq E$ such that $|F| = \lambda(\Gamma)$ and $\Gamma - F$ is disconnected. We know that there is such a separating set by definition of the edge connectivity.
Now, we prove that $\kappa(\Gamma) \leq |F|$.
Case 1: Suppose that there is a vertex $\alpha \in V$ which is not an end-vertex of any edge in F. Let us denote the component containing α in $\Gamma - F$ by $\mathbb{C}$.
Let $Z = \{\beta \in V(\mathbb{C}) : \beta$ is an end-vertex of an edge in $F\}$. Then Z separates α and $\Gamma - \mathbb{C}$.
On the other hand, for any edge $e \in F$, at most one end of e is in $V(\mathbb{C})$: Suppose to the contrary that there exists $f = \beta\gamma$ in F with both $\beta, \gamma \in V(\mathbb{C})$. Then $F - f$ is a set with $\lambda(\Gamma) - 1$ edges such that $\Gamma - (F - f)$ is disconnected. But this contradicts with the minimality of F. Therefore, $|Z| \leq |F|$. And so, $\kappa(\Gamma) \leq |F|$.
Case 2: Suppose that each vertex is an end-vertex of some edge in F. Let $\alpha \in V(\Gamma)$. Let us denote the component containing α in $\Gamma - F$ by $\mathbb{C}$.
Let $Z = \{z \in \mathfrak{N}_\Gamma(\alpha) : z\alpha \notin F\}$. Then $Z \subseteq V(\mathbb{C})$, because $\mathbb{C}$ is a component and $\alpha \in \mathbb{C}$. And again, since for an edge f in F, two vertices in Z can not be ends of the same edge in F. Then, $d_\Gamma(\alpha) \leq |F|$. Therefore, since $\mathfrak{N}_\Gamma(\alpha)$ separates α from $\Gamma - \mathfrak{N}_\Gamma(\alpha)$, $\kappa(\Gamma) \leq |F|$, if of course, $\Gamma - \mathfrak{N}_\Gamma(\alpha) \neq \{\alpha\}$.
On the other hand, if $\Gamma - \mathfrak{N}_\Gamma(\alpha) = \{\alpha\}$, i.e , if $\{\alpha\} \cap \mathfrak{N}_\Gamma(\alpha) = V(\Gamma)$, since α is an arbitrary vertex, we can suppose that $\Gamma = \mathbb{K}^n$, where $n = |\Gamma|$. But in this case, $\kappa(\Gamma) = \lambda(\Gamma) = |\Gamma| - 1$. That is, the inequalities are again satisfied. ■

Note that by the previous proposition, if $\kappa(G)$ (or $\Gamma(G)$) is great, then $\delta(G)$ must be great. But the converse of this statement need not be true:

Example 1.12 In Figure 1.13, two copies of $\mathbb{K}^4$ are connected by an edge f with two end-vertices in two copies of $\mathbb{K}^4$ respectively. Note that $\delta(\Gamma) = 3$, but $\kappa(\Gamma) = \lambda(\Gamma) = 1$.

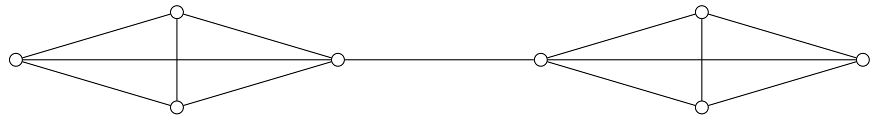


Figure 1.13 A graph Γ obtained by an edge and two copies of $\mathbb{K}^4$

Note 1.10 In general, if we connect two vertices of two copies of $\mathbb{K}^n$ by an edge f, then we obtain a graph Γ such that $\delta(\Gamma) = n - 1$, but $\kappa(\Gamma) = \lambda(\Gamma) = 1$.

1.3 Trees and Forests

Definition 1.33 An *acyclic* graph is a graph containing no cycles. A *forest* is an acyclic graph. A *tree* is a connected forest. A vertex of degree 1 of a tree is called *leaf*. An *inner vertex* of a tree is a vertex which is not a leaf.

Note 1.11 · If $\mathcal{T}$ is a nontrivial tree, then $\mathcal{T}$ has a leaf:
Let $\mathbb{R}$ be a longest path in $\mathcal{T}$. Then the both ends of $\mathbb{R}$ must have degree 1. Otherwise, we can extend $\mathbb{R}$ and this contradicts with the maximality of $\mathbb{R}$. Therefore, $\mathcal{T}$ has at least two leaves.
· If t is a leaf of $\mathcal{T}$, then $\mathcal{T}' = \mathcal{T} - t$ is also a tree:
First, $\mathcal{T}$ does not contain any cycles, so does $\mathcal{T}'$. And since t is a leaf of $\mathcal{T}$, it has only one neighbor, and thus $\mathcal{T}'$ is connected. Therefore, $\mathcal{T}'$ is a tree.

Theorem 1.1 *Let $\mathcal{T}$ be a graph. Then, TFAE:*
(1) *$\mathcal{T}$ is a tree.*
(2) *There is a unique path in $\mathcal{T}$ between any two vertices a and b of $\mathcal{T}$.*
(3) *$\mathcal{T}$ is connected, but for any edge $t \in E(\mathcal{T})$, $\mathcal{T} - t$ is disconnected.*
(4) *$\mathcal{T}$ does not contain any cycles, but for any two nonadjacent vertices $a, b \in V(\mathcal{T})$, $\mathcal{T} + ab$ does contain a cycle.*

Proof.
· $(1 \Rightarrow 2)$: First of all, since $\mathcal{T}$ is a tree, it is connected. Therefore, there is a path $\mathbb{Q}$ between any two vertices t and s of $\mathcal{T}$. And such a path is unique:

Suppose not. Then there are two vertices, say t_0 and t_1 in $\mathcal{T}$ such that there are two distinct paths, say $\mathbb{R}_1 = t_0 = z_1 \dots z_n = t_1$ and $\mathbb{R}_2 = t_0 = y_1 \dots y_n = t_1$ between t_0 and t_1. Let $S = V(\mathbb{R}_1) \cap V(\mathbb{R}_2)$. Then, since t_0 and $t_1 \in S$, $S \neq \emptyset$; and since $\mathbb{R}_1 \neq \mathbb{R}_2$, $S \subsetneq V(\mathbb{R}_1)$ or $S \subsetneq V(\mathbb{R}_2)$. Say $S \subsetneq V(\mathbb{R}_1)$.

Let z_{s+1} be the first vertex (wrt the increasing of indices) in $V(\mathbb{R}_1) - V(S)$, and z_t be the first vertex after z_s in S. Then z_s, $z_t \in S$, and $\mathbb{C} = z_s\mathbb{R}_1 z_t \mathbb{R}_2 z_s$ is a cycle in $\mathcal{T}$. But this contradicts with the fact that $\mathcal{T}$ is acyclic. Therefore, such a path must be unique.

· $(2 \Rightarrow 3)$: Since there is a path joining any two vertices of $\mathcal{T}$, $\mathcal{T}$ is connected. Let $f = wz$ be an edge in $\mathcal{T}$. Then $\mathcal{T}' = \mathcal{T} - f$ is disconnected:

Suppose not. Then there is a path $\mathbb{Q}$ in $\mathcal{T}'$ between the vertices w and z. Then, since $\mathcal{T}' \subseteq \mathcal{T}$, $\mathbb{Q} \subseteq \mathcal{T}$. But $\mathbb{R} = wz$ is another path between w and z in $\mathcal{T}$. This contradicts with the uniqueness of such path. Therefore, $\mathcal{T}'$ must be disconnected.

· $(3 \Rightarrow 4)$: First, suppose to the contrary that $\mathcal{T}$ contains a cycle $C = w_1 w_2 \dots w_n w_1$. Let $f = w_1 w_2$. Then $\mathcal{T}' = \mathcal{T} - f$ is connected:

Let $\alpha, \beta \in V(\mathcal{T}')$. Then, since $\mathcal{T}$ is connected, there is a path $\mathbb{Q} = \alpha = y_1 y_2 \dots y_m = \beta$ between α and β in $\mathcal{T}$. If $f \notin E(\mathbb{Q})$, then $\mathbb{Q} \subseteq \mathcal{T}'$. If $f \in E(\mathbb{Q})$, then $f = y_i y_{i+1}$ for some $1 \leq i < m$. Then, $\mathbb{R} = \mathbb{Q} y_i = w_1 w_n w_{n-1} \dots w_2 = y_{i+1}\mathbb{Q}$ is a walk between α and β in $\mathcal{T}'$. Therefore, there is a path between α and β in $\mathcal{T}'$, because every $\alpha - \beta$ walk contains a $\alpha - \beta$ path. And so, since $\mathcal{T}' = \mathcal{T} - f$ is connected, this contradicts with the assumption. Therefore, $\mathcal{T}$ contains no cycles.

Now, let α and β be two non-adjacent vertices in $\mathcal{T}$. Then $\mathcal{T}'' = \mathcal{T} + \alpha\beta$ contains a cycle: Let $\mathbb{P}$ be a path between α and β in $\mathcal{T}$. Then $\mathbb{Q} = \mathbb{P} + \alpha\beta$ is a cycle in $\mathcal{T}''$.

· $(4 \Rightarrow 1)$: First of all, by hypothesis, $\mathcal{T}$ is acyclic. Therefore, we only need to demonstrate that $\mathcal{T}$ is connected: Let $\alpha, \beta \in V(\mathcal{T})$. If $f = \alpha\beta \in E(\mathcal{T})$, then $\mathbb{P} = \alpha\beta$ is a path between α and β in $\mathcal{T}$. If $f = \alpha\beta \notin E(\mathcal{T})$, then, by hypothesis, $\mathcal{T} + \alpha\beta$ contains a cycle C. But since $\mathcal{T}$ does not contain a cycle, $f = \alpha\beta \in E(C)$. Then $\mathbb{P} = C - f$ is a path in $\mathcal{T}$ between the vertices α and β. Therefore, $\mathcal{T}$ is connected. ■

Notation 1.4 For a tree $\mathcal{T}$, let $\alpha, \beta \in V(\mathcal{T})$. Then $\alpha\mathcal{T}\beta$ denotes the unique path in $\mathcal{T}$ between α and β.

Note 1.12 Any connected graph Γ has a s.t.:

Let $\mathcal{T}$ be a minimally connected s.s. of Γ. Then $\mathcal{T}$ is a tree by part (3) of Theorem 1.1. Therefore, $\mathcal{T}$ is a s.t.

OR: Let $\mathcal{T}$ be a maximally acyclic subgraph of Γ. Then $\mathcal{T}$ is a tree by part (4) of Theorem 1.1. Also, $\mathcal{T}$ is a s.s.: Suppose not. Then, there is a vertex α of Γ such that $\alpha \notin \mathcal{T}$. Now, since $\mathcal{T} \subseteq \Gamma$ and Γ is connected, there is a vertex γ of $\mathcal{T}$ such that $\alpha\gamma \in E(\Gamma)$. Then $\mathcal{T} + \alpha\gamma$ is still acyclic, because $\mathcal{T}$ is acyclic and $\alpha \notin \mathcal{T}$. But then, since $\mathcal{T} \subsetneqq \mathcal{T} + \alpha\gamma$, this contradicts with the maximality of $\mathcal{T}$.

Example 1.13 The graph given in Figure 1.14 is an example of a tree.

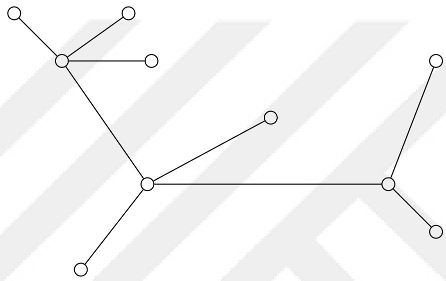


Figure 1.14 An example of a tree

Example 1.14 Figure 1.15 shows three s.trees with bold edges of the graph $\mathbb{H} = \mathbb{K}^5$

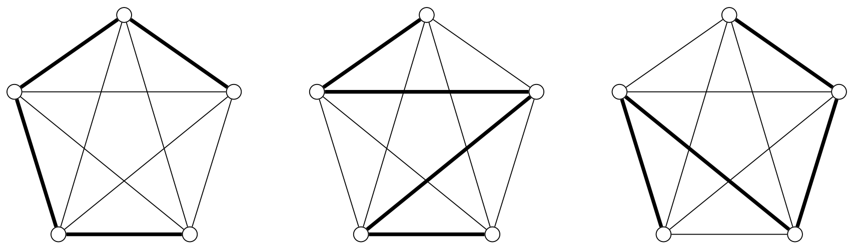


Figure 1.15 $\mathbb{K}^5$ with three spanning trees

Definition 1.34 Let Γ be a graph, and $\mathcal{T}$ be a s.t. of Γ. An edge $f \in E(\Gamma) \setminus E(\mathcal{T})$ is called a *chord* of $\mathcal{T}$ in Γ.

Corollary 1.1 *Let $\mathcal{T}$ be a tree with order n. Then the vertices of $\mathcal{T}$ can be labeled $w_1, \ldots, w_n$ such that for every $i \geq 2$, there is a unique neighbor of w_i in $\{w_1, \ldots, w_{i-1}\}$.*

Proof. We know that we can label the vertices of $\mathcal{T}$ as $w_1, \ldots, w_n$ such that for every $i \geq 2$, $\mathcal{T}_i = \mathcal{T}[w_1, \ldots, w_i]$ is connected. Therefore, there is a neighbor of w_i in $\{w_1, \ldots, w_{i-1}\}$. And since $\mathcal{T}_i$ is a tree, this neighbor is unique: Suppose not. Then, w_i has two neighbors, say w_k and w_l with $1 \leq k < l \leq i-1$. Now, since $\mathcal{T}_{i-1}$ is a tree, it is connected, and thus there is a (unique) path $\mathbb{P}$ connecting w_k and w_l. But then $w_i w_k + \mathbb{P} + w_l w_i$ is a cycle in $\mathcal{T}_i$, which is a contradiction (since $\mathcal{T}_i$ is a tree, it is acyclic). ■

Corollary 1.2 *Let Γ be a connected graph, and $|V(\Gamma)| = n$. Then:*

$$\Gamma \textit{ is a tree} \iff |E(\Gamma)| = n - 1.$$

Proof.

· $(\Rightarrow)$: Proof can be done by induction on the order i.

For $i = 1$: Since Γ is a graph, there is no edge with the same endpoint. Therefore, $|E(\Gamma)| = 1 - 1 = 0$.

Suppose the assertion is true for $i < n$.

Now for $l = i + 1$, we can use the previous corollary. Γ_i has $i - 1$ edges by the induction assumption. Therefore, since w_{i+1} ha s a unique neighbor in Γ_i, Γ_{i+1} has $i - 1 + 1 = i$ edges.

Therefore, by induction, $\Gamma = \Gamma_n$ has $n - 1$ edges.

· $(\Leftarrow)$: Let $|E(\Gamma)| = n-1$. And let $\mathcal{T}$ be a s.t. in Γ. Then, by the first part, $|E(\mathcal{T})| = n-1$. Therefore, $\Gamma = \mathcal{T}$, i.e., Γ is a tree. ■

Corollary 1.3 *Let $\mathcal{T}$ be a tree, and Γ be a graph such that $\delta(\Gamma) \geq |\mathcal{T}| - 1$. Then $\mathcal{T} \subseteq \Gamma$, that is, there is a subgraph $\mathcal{J}$ of Γ such that $\mathcal{J} \simeq \mathcal{T}$.*

Proof. We can find a copy of $\mathcal{T}$ in Γ by using the vertex enumeration $w_1, w_2, \ldots, w_n$ of $\mathcal{T}$ (given in next to last corollary) by using the induction on the number of vertices n of $\mathcal{T}$: Assume inductively that $w_1, \ldots, w_i \in \Gamma$ is chosen for some $1 \leq i < |\mathcal{T}| = n$. Now, choose $\alpha \in \Gamma - \{w_1, .., w_i\}$ as w_{i+1} such that $w_k \in E(\Gamma)$, where w_k is the unique neighbor of $\alpha = w_{i+1}$ in $\mathcal{T}$. Such a vertex α exists, because $d(w_k) \geq \delta(\Gamma) \geq |\mathcal{T}| - 1$. Therefore, by induction, we find an isomorphic copy of $\mathcal{T}$ in Γ. ■

Definition 1.35 If we consider one vertex of a tree as a distinguished vertex, then this vertex is called the *root* of the tree. A tree with a chosen root is called a *rooted tree*.

Note 1.13 Define a relation on the set of vertices of a rooted tree $\mathcal{T}$ as follows:
Let β be a root of $\mathcal{T}$, α, γ be two vertices of $\mathcal{T}$. Then:

$$\alpha \leq \gamma \Longleftrightarrow \alpha \in \beta\mathcal{T}\gamma.$$

In this case, $V(\mathcal{T})$ is a partially ordered set wrt the realtion $\leq$. That is, the realtion $\leq$ is a partial order on $V(\mathcal{T})$:
· $\leq$ is reflexive: Let $\alpha \in V(\mathcal{T})$. Then $\alpha \leq \alpha$, because $\alpha \in \beta\mathcal{T}\alpha$.
· $\leq$ is anti-symmetric: Let $\alpha, \gamma \in V(\mathcal{T})$. Suppose that $\alpha \leq \gamma$ and $\gamma \leq \alpha$. Then $\alpha \in \beta\mathcal{T}\gamma$ and $\gamma \in \beta\mathcal{T}\alpha$. Thus, $\beta\mathcal{T}\alpha \subseteq \beta\mathcal{T}\gamma$ and $\beta\mathcal{T}\gamma \subseteq \beta\mathcal{T}\alpha$, because there is a unique path between any two vertices of a tree. (For example, if $\beta\mathcal{T}\alpha \nsubseteq \beta\mathcal{T}\gamma$, then there are at least two different paths between β and α). Therefore, since $\beta\mathcal{T}\alpha \subseteq \beta\mathcal{T}\gamma \subseteq \beta\mathcal{T}\alpha$, $\beta\mathcal{T}\alpha = \beta\mathcal{T}\gamma$. And so, $\alpha = \gamma$.
· $\leq$ is transitive: Let $\alpha, \gamma, \delta \in V(\mathcal{T})$. Suppose that $\alpha \leq \beta$ and $\beta \leq \gamma$. Then, $\alpha \in \beta\mathcal{T}\gamma$ and $\gamma \in \beta\mathcal{T}\delta$. Then, since $\gamma \in \beta\mathcal{T}\delta$, $\beta\mathcal{T}\gamma \subseteq \beta\mathcal{T}\delta$. Therefore, since $\alpha \in \beta\mathcal{T}\gamma$, $\alpha \in \beta\mathcal{T}\delta$. And so, $\alpha \leq \delta$.

Definition 1.36 The partial order defined in the previous note is called the *tree-order* wrt $\mathcal{T}$ and β.
If $\alpha < \gamma$, i.e., $\alpha \neq \gamma$, wrt the tree-order, then α is *below* γ in $\mathcal{T}$.
The sets $\lceil \delta \rceil = \{\theta \mid \theta < \delta\}$ and $\lfloor \delta \rfloor = \{\theta \mid \theta \geq \delta\}$ are called the *down-closure* and *up-closure* of δ, respectively.
Let $Y \subseteq V(\mathcal{T})$. If Y is equal to its down-closure (respectively, up-closure), that is, $Y = \lceil Y \rceil = \cup_{y \in Y} \lceil y \rceil$ (respectively, $Y = \lfloor Y \rfloor = \cup_{y \in Y} \lfloor y \rfloor$), then Y is said to be *closed downwards* (respectively, *closed upwards*), or an *down-set* (respectively, *up-set*).

Note 1.14 $\cdot$ Wrt the t.o., the root of $\mathcal{T}$ is the minimum element of the collection of vertices, and maximal elements are the leaves of $\mathcal{T}$.

$\cdot$ In $\mathcal{T}$, the end-vertices of an edge are comparable.

$\cdot$ Let $\alpha \in \mathcal{T}$. Then the down-closure of α is a chain: Let $\alpha, \gamma \in \lceil \beta \rceil$. Then $\alpha \leq \beta$ and $\gamma \leq \beta$. Thus $\alpha \in \theta \mathcal{T} \beta$ and $\gamma \in \theta \mathcal{T} \alpha$. Therefore, $\alpha \leq \gamma$ or $\gamma \leq \alpha$.

Definition 1.37 The *height* of a vertex α is the length of the path $\theta \mathcal{T} \alpha$.
The set of all height l vertices is called the *l-th level* of $\mathcal{T}$.

Definition 1.38 Let Γ be a graph and $\mathcal{T} \subseteq \Gamma$ be a rooted tree. If for any $\mathcal{T}$-path $\mathbb{P}$ in Γ with end-vertices α and β, $\alpha \leq \beta$ or $\beta \leq \alpha$ holds wrt t.o. of $\mathcal{T}$, then $\mathcal{T}$ is called *normal* in Γ.

Note 1.15 Let $\mathcal{T}$ be a s.t. in Γ. Then:

$$\mathcal{T} \text{ is normal in } \Gamma \Longleftrightarrow [\alpha\beta \in E(\Gamma) \Longrightarrow \alpha \leq \beta \text{ or } \beta \leq \alpha \text{ in } \mathcal{T}].$$

First, $\mathcal{T}$ is a normal tree iff the end-vertices of any $\mathcal{T}$-path in Γ are comparable wrt t.o. of $\mathcal{T}$.

Second, $\mathcal{T}$ is a s.s. of Γ iff $V(\mathcal{T}) = V(\Gamma)$. Now:

$\cdot$ ($\Rightarrow$): Suppose that $\mathcal{T}$ is normal in Γ. And let $\alpha\beta \in E(\Gamma)$.

$\cdot$ **Case 1:** If $\alpha\beta \in E(\mathcal{T})$, then $\alpha \leq \beta$ or $\beta \leq \alpha$

$\cdot$ **Case 2:** If $\alpha\beta \notin E(\mathcal{T})$, then $\alpha\beta$ is a $\mathcal{T}$-path in Γ. Therefore,since $\mathcal{T}$ is normal in Γ, $\alpha \leq \beta$ or $\beta \leq \alpha$.

$\cdot$ ($\Leftarrow$): Suppose that if $\alpha\beta \in E(\Gamma)$, then $\alpha \leq \beta$ or $\beta \leq \alpha$ in $\mathcal{T}$. Since $\mathcal{T}$ spans Γ, any $\mathcal{T}$-path in Γ is an edge $f \in E(\Gamma) - E(\mathcal{T})$. Therefore, to show that $\mathcal{T}$ is normal, we only need to show that for any edge $f = \alpha\beta \in E(\Gamma) - E(\mathcal{T})$, $\alpha \leq \beta$ or $\beta \leq \alpha$. Now, assume that $f \in E(\Gamma) - E(\mathcal{T})$, Then, since $f = \alpha\beta \in E(\Gamma)$, $\alpha \leq \beta$ or $\beta \leq \alpha$ in $\mathcal{T}$ by hypothesis. Therefore, $\mathcal{T}$ is normal in Γ.

Since the separation properties of $\mathcal{T}$ is reflected by Γ, we can understand the structure of Γ by using a normal tree in Γ.

Lemma 1.1 *Let Γ be a graph, and $\mathcal{T} \subseteq \Gamma$ be a tree. Suppose that $\mathcal{T}$ is normal in Γ, and α, β are two arbitrary vertices of $\mathcal{T}$. Then:*

· The set $\lceil \alpha \rceil \cap \lceil \beta \rceil$ separates α and β in Γ.

· Let $R \subseteq V(\mathcal{T})$. Assume that $\mathcal{T}$ is also a s.t., and R is down-closed. Let $W = \{\theta \mid \theta$ is a minimal elements in $\mathcal{T} - R\}$. Then the sets $\lfloor \theta \rfloor$, where $\theta \in W$, span the components of $\Gamma - R$.

We finish this section with a proposition which guarantees the existence of a normal s.t. of a connected graph Γ.

Proposition 1.12 *Let Γ be a connected graph, and let β be a vertex of Γ. Then, Γ has normal s.t. $\mathcal{T}$ with root β.*

1.4 Bipartite Graphs

Definition 1.39 Let $r \geq 2$ be an integer, and let $\Gamma = (V, E)$ be a graph. If there is a partition of V containing r classes such that there is no edge between any two vertices in the same class, that is, the end vertices of an edge belongs to the different classes; then Γ is called an *r-partite graph*. And every class in the partition is called a *partite* set.
A 2-partite graph is called *bipartite*.

Definition 1.40 Let Γ be an r-partite graph. If there is an edge between any two vertices of different partite sets, then Γ is called a *complete r-partite graph*, without mentionning r, Γ is called a *complete multi-partite graph*.

Notation 1.5 $\mathbb{K}_{n_1,n_2,\ldots,n_r}$ denotes the complete r-partite graph $\overline{\mathbb{K}^{n_1}} * \ldots * \overline{\mathbb{K}^{n_r}}$.
If $n_1 = \ldots = n_r = t$, then $\mathbb{K}_{n_1,n_2,\ldots,n_r}$ is denoted by $\mathbb{K}_t^r$.
Note that if every vertex of a $\mathbb{K}^r$ is replaced by an independent set, that is, a set of vertices containing no edges, of size t, then we obtain a $\mathbb{K}_t^r$.

Example 1.15 In Figure 1.16, there are three 3-partite graphs. Note that the last two of these graphs are two different drawings of the octahedron $\mathbb{K}_2^3$.

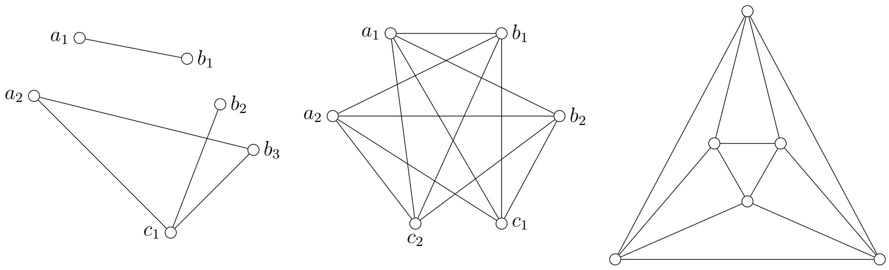


Figure 1.16 Three 3-partite graphs

Definition 1.41 A *star* is a bipartite graph $\mathbb{K}_{1,n}$ for some $n \in \mathbb{N}$. The *star's center* is the vertex in the partite set of order one.

Now, we give a proposition which characterize a bipartite graph completely:

Proposition 1.13 *Let* $\Gamma = (V, E)$ *be a graph with at least two vertices. Then:*

Γ *is bipartite* $\Longleftrightarrow$ Γ *contains no odd cycles, that is, a cycle with odd length.*

Proof.
· ($\Rightarrow$): Suppose that Γ is bipartite. Then Γ does not contain any odd cycles, because the adjacent vertices of a cycle must be in different partite sets. Therefore, the length of a cycle must be even.
· ($\Leftarrow$): Suppose that Γ does not contain any odd odd cycles. First of all, we can assume wlog that Γ is connected, because Γ is bipartite iff all the components of Γ are bipartite or trivial.
Let $\mathcal{T}$ be a rooted s.t. of Γ, and r be the root of $\mathcal{T}$. Denote the corresponding t.o. on $V = V(\Gamma)$ by $\leq_{\mathcal{T}}$.
Now, for each $\alpha \in V$, there is a unique path $r\mathcal{T}\alpha$, and this path has either odd or even length.
Let $Q = \{\alpha \in V \mid$ the length of $r\mathcal{T}\alpha$ is odd $\}$ and $R = \{\alpha \in V \mid$ the length of $r\mathcal{T}\alpha$ is even $\}$. Then Q and R are nonempty: If α is adjacent to r, then $\alpha \in Q$; and $r \in R$. Also, obviously $Q \cap R = \emptyset$. We pove that Γ is bipartite with partite sets Q and R.
let $f = \alpha\beta \in E(\Gamma)$.

· **Case 1:** Suppose that $f \in \mathcal{T}$, and sat $\alpha <_{\mathcal{T}} \beta$. Then, $r\mathcal{T}\beta = r\mathcal{T}\alpha\beta$. Therefore, α and β are in different partite sets, because $r\mathcal{T}\alpha$ has odd (respectively, even) length $\Leftrightarrow$ $r\mathcal{T}\alpha\beta = r\mathcal{T}\beta$ has even (respectively, odd) length.

· **Case 2:** Suppose that $f \notin \mathcal{T}$. Then $C_f = \alpha\mathcal{T}\beta + f$ is a cycle (of even length by hypothesis).

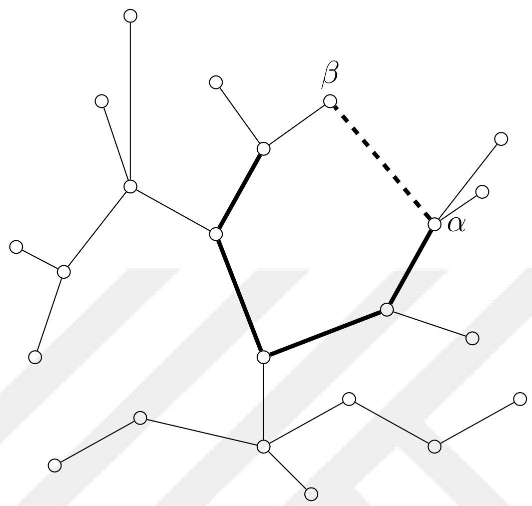


Figure 1.17 The graph $\mathcal{T} + f$ and the cycle C_f in it

Now, by Case 1, the vertices in the path $\alpha\mathcal{T}\beta$ alternate between the partite sets Q and R. Therefore, since C_f is even by hypothesis, $\alpha\mathcal{T}\beta = C_f - f$ has odd length, that is , $\alpha\mathcal{T}\beta$ has even number of vertices Therefore, α and β belong to different partite sets. ■

1.5 Contraction and Minor

Let Γ be a given fixed graph.

Definition 1.42 · A *subdivision* of the graph Ω is a graph gotten from Ω by *subdividing* some edges by using new vertices. More precisely, if some edges of the graph Ω are replaced with new paths between the end vertices of the corresponding edges such that these paths do not contain any inner vertex in $V(\Omega)$ or any vertex of some other new path, then the obtained graph is called a *subdivision* of Ω. If a graph is a subdivision of Ω , then it is also said that Ω is a $T\Omega$ (where T corresponds to the word topological).

· The vertices of the graph Ω are called the *branch vertices* of $T\Omega$; and the new added vertices are named as *subdividing vertices* of $T\Omega$.

Note 1.16 The degree of all subdividing vertices is 2; on the other hand, $deg_{\Gamma}(\mu) = deg_{T\Gamma}(\mu)$ for every $\mu \in \Gamma$.

Definition 1.43 Let Π be a graph. If $T\Gamma$ is a subgraph of Π, then Γ is said to be a *topological minor (t.m.)* of Π.

Definition 1.44 · If some of the vertices $\mu \in V(\Gamma)$ are replaced with disjoint and connected graphs Θ_μ, and the corresponding edges $h = \mu\nu \in E(\Gamma)$ are replaced with nonempty sets of $\Theta_\mu - \Theta_\nu$, then the obtained graph is called an $I\Gamma$ (where I corresponds to the word inflated). More precisely, if the vertex set of a graph Θ has a partition $\{V_\mu \mid \mu \in V(\Gamma)\}$ into connected subsets V_μ in such a way that for any two distinct vertices $\mu, \nu \in V(\Gamma)$, $\mu\nu \in E(\Gamma)$ iff Θ has a $V_\mu - V_\nu$ edge, then the graph Θ is called an $I\Gamma$.
· The sets V_μ are called the *branch sets* of $I\Gamma$.
· If Θ is an $I\Gamma$, then we say that Γ is obtained from Θ by *contracting* the (disjoint and connected) subgraphs $\Theta_\mu \subseteq \Theta$, and Γ is called as a *contraction minor* of Θ.

Definition 1.45 Let Π be a graph. If $I\Gamma$ is a subgraph of Π, then Γ is said to be a *minor* of Π. In this case, $I\Gamma$ is called a *model* of the graph Γ in Π, and it is denoted by $\Gamma \preceq \Pi$.

Note 1.17 Since branch sets can consist of a single vertex, each subgraph of a graph is also a minor of the graph.

Theorem 1.2 *The t.m. and the minor relations are partial order relations on the set of finite graphs.*

Notation 1.6 · If the graph Π is an $I\Gamma$, then by definition $\mathcal{L} = \{V_\mu \mid \mu \in \Gamma\}$ is a partition of $V(\Pi)$, and the contraction minor Γ of Π is denoted by $\Pi/\mathcal{L}$.
· If $Y = V_\mu$ is the only branch set containing more than one vertex, then Γ is denoted by Π/Y; in this case, the vertex $\mu \in \Gamma$ to which the set Y contracts is denoted

by μ_Y, and the remaining part of the graph Π is considered as an i.s. of the graph Π.
· The smallest possible case for Y is that Y has precisely two vertices which are adjacent to each other; that is, $Y = \{h\}$, a set containing a single edge. In this case, $\Gamma = \Pi/h$ is said to be obtained from the graph Π by *contracting the edge* h.

We know that the minor relation $\preceq$ is a partial order relation. Therefore it is transitive. And so, each sequence of single edge deletions or contractions or single vertex deletions produces a minor. Conversely, each minor of a finite graph can be gotten with this process:

Corollary 1.4 *Let* Γ *and* Π *be graphs which are finite. The graph* Γ *is a minor of the graph* Π *iff there is a sequence of graphs* $\Pi_0, \Pi_1 \ldots, \Pi_n$ *such that* $\Pi = \Pi_0$ *and* $\Gamma = \Pi_n$*, and every* Π_{i+1} *is obtained from* Π_i *by deletion of an edge, by contraction of an edge, or by deletion of a vertex.*

Finally, the following relations exist between t.minors and minors:

Theorem 1.3 (1) *If a graph* Π *is a* $T\Gamma$*, then it is also an* $I\Gamma$*. Therefore, each t.m. of a graph is also a minor of this graph.*
(2) *If* $\Delta(\Gamma) \leq 3$*, then each* $I\Gamma$ *has a* $T\Gamma$ *in it. Therefore, each minor of maximum degree less than or equal to* 3 *of a graph is also a t.m. of this graph.*

2. EXTREMAL GRAPH THEORY

In this main chapter of the thesis, we consider effects of global parameters, like the edge density, a.d. or c.n., to local substructures of a graph. For example, for a graph on n vertices, how many edges do we need to have to guarantee that the graph contains a complete graph $\mathbb{K}^s$ (for some fixed s) as a subgraph or as a minor? Can the existence of such substructures be guaranteed by sufficiently large c.n. or a.d.? These types of questions constitute *extremal graph theory*, and this the topic of what we study in this chapter.

Before starting the first section, we give a basic definition:

Definition 2.1 Let Γ be a graph.
· If the number of edges $\|\Gamma\|$ of Γ is about linear wrt its number of vertices $|\Gamma|$, then Γ is called *sparse*.
· If the number of edges $\|\Gamma\|$ of Γ is about quadratic wrt its number of vertices $|\Gamma|$, then Γ is called *dense*.
· The number $\dfrac{|\Gamma|}{\binom{|\Gamma|}{2}}$ is called *edge-density* of Γ.

Note that *dense* and *sparse* notions do not make sense for individual graphs, these notions become meaningful for sets of graphs whose number of vertices goes to infinity.

2.1 Subgraphs

Let Π be a given graph, $n \geq |\Pi|$, and Γ be any graph of order n. How many edges will be enough to guarantee that Γ has a subgraph isomorphic to Π? Or, in other words: What is the highest possible edge number that Γ can contain without having a subgraph isomorphic to Π?

Definition 2.2 A graph Γ having n vertices and not containing (an isomorphic copy of) Π with the greatest possible edge number is called *extremal* for Π and n; and in this case, the edge number of Γ is denoted by $ex(n, \Pi)$.

Note 2.1 Obviously, if a graph Γ is extremal for some given Π and n, then Γ is also edge-maximal with the property that $\Pi \nsubseteq \Gamma$. However, the converse statement is not true: There is an edge-maximal graph Γ with the property that $\Pi \nsubseteq \Gamma$, but such that Γ contains fewer edges than $ex(n, \Pi)$. (In Figure 2.1, there are two graphs on 4 vertices such that the left one is edge-maximal with the g.p. that $P^3 \nsubseteq \Gamma$, but the right one is not extremal wrt this g.p.)

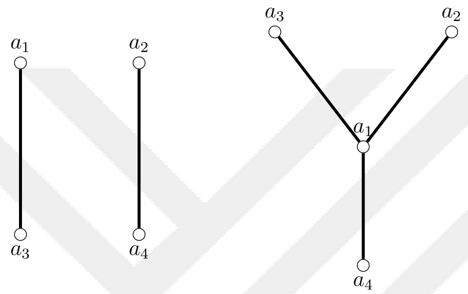


Figure 2.1 The left graph is edge-maximal, but the right one is not with $P^3 \nsubseteq \Gamma$

Consider the problem for the complete graph $\Pi = \mathbb{K}^s$, where $s > 1$. For extremality, trivial candidates are $(s-1)$-partite complete graphs which are obviously edge-maximal in regard to not containing the complete graph $\mathbb{K}^s$. The question is the following: Which of these graphs contain the largest number of edges? We have the restricition that $n_1 + \cdots + n_s = n$, where n_i is the number of vertices in the i-th partite set, and we want to maximize the product $n_1 n_2 \ldots n_s$. Therefore, we must choose n_i values as equal as possible, that is, for $i \neq j$, $n_i - n_j$ must be less than or equal to 1: If $n_i - n_j \geq 2$ for some $1 \leq i, j \leq r$, the number of edges can be increased by shifting a vertex from the corresponding partite set with n_i vertices to the corresponding partite set with n_j vertices.

Definition 2.3 For each $n \geq s-1$, the $(s-1)$-partite complete graph on n vertices in which the difference between any two partite sets is at most 1 is called *Turan graph*, denoted by $\Theta^{s-1}(n)$, and its number of edges is denoted by $\theta_{r-1}(n)$ in this thesis.

Example 2.1 In Figure 2.2, the Turan graph $\Theta^3(8)$ is pictured.

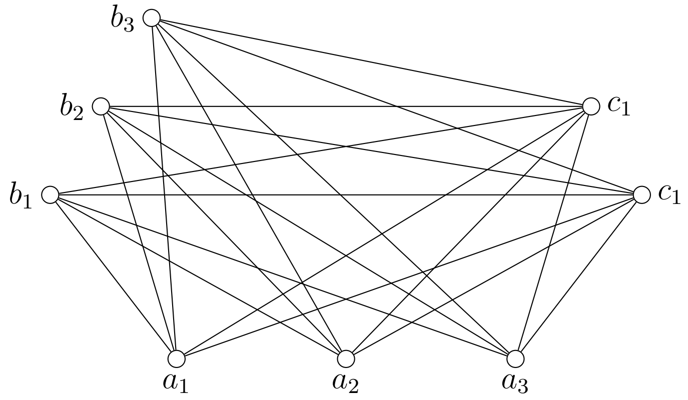


Figure 2.2 The 3-partite Turan graph on 8 vertices.

Note 2.2 For $n < s-1$, we formally use this definition, but now the partite sets may be empty. Then for each $n \leq s-1$, $\Theta^{s-1}(n) = \mathbb{K}^n$: We simply distribute n vertices to the n partite sets one by one, and the remaining $(s-1)-n$ partite sets are empty.

$\Theta^{s-1}(n)$ is really extremal for n and the complete graph $\mathbb{K}^s$, and in this respect it is unique; particullary, $ex(n, \mathbb{K}^s) = \theta_{s-1}(n)$:

Theorem 2.1 (Turan 1941) *Let s and n be integers with $s > 1$. If Γ is a graph on n vertices and $ex(n, \mathbb{K}^s)$ edges such that $\Gamma \not\supseteq \mathbb{K}^s$, then $\Gamma = \Theta^{s-1}(n)$.*

First proof of Turan's Theorem. Proof is by induction on the number of vertices n. For $n \leq s-1$, $\Gamma = \mathbb{K}^n = \Theta^{s-1}$ as stated in the theorem. Now, let $n \geq s$.
First, Γ contains $\mathbb{K} = \mathbb{K}^{s-1}$, because it is edge-maximal without containing the complete graph $\mathbb{K}^s$ as a subgraph. The number of edges of $\Gamma - \mathbb{K}$ is at most $\theta_{s-1}(n-(s-1))$ by the induction hypothesis, and the number of neighbors in $\mathbb{K}$ of each vertex in $\Gamma - \mathbb{K}$ is less than or equal to $s-2$ (if there exists a vertex $\alpha \in \Gamma - \mathbb{K}$ with $s-1$ neighbors in $\mathbb{K}$, then Γ contains a $\mathbb{K}^s$). Therefore, we have: (To understand the rightmost equality in $(*)$, study Figure 2.3.)

$$\|\Gamma\| \leq \theta_{s-1}(n-s+1) + (n-s+1)(s-2) + \binom{s-1}{2} = \theta_{s-1}(n). \qquad (*)$$

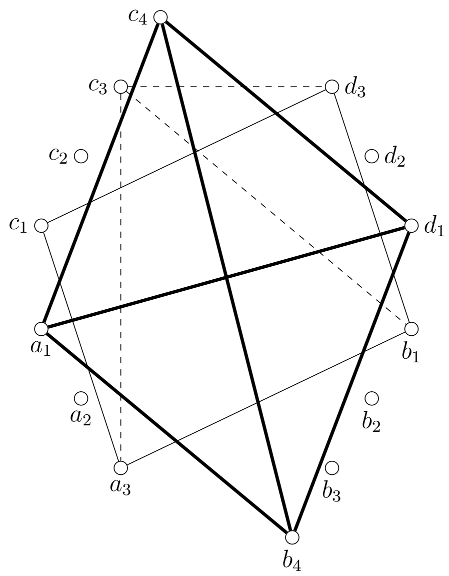


Figure 2.3 The equality in $(*)$ for $n = 14$ and $s = 5$

Now, Γ is extremal without having a $\mathbb{K}^s$ in it and $\mathbb{K}^s \nsubseteq \Theta^{s-1}(n)$, where $\|\Theta^{s-1}(n)\| = \theta_{s-1}(n)$. Therefore, instead of $\leq$, equality occurs in the equation $(*)$. Thus, the number of neighbors in $\mathbb{K}$ of each vertex in $\Gamma - \mathbb{K}$ is axactly $s - 2$, just as the vertices, say $\alpha_1, \ldots, \alpha_{s-1}$, of $\mathbb{K}$. For each $i = 1, \ldots, s-1$, define:

$$V_i = \{\alpha \in V(\Gamma) \mid \alpha\alpha_i \notin E(\Gamma)\}.$$

Note that V_i is the set of vertices such that their $s - 2$ neighbors in $\mathbb{K}$ are the vertices $\alpha_j \neq \alpha_i$. For each $i = 1, \ldots, s-1$, V_i is independent, because $\mathbb{K}^s \nsubseteq \Gamma$. (If $\alpha, \beta \in V(\Gamma)$ and $\alpha\beta \in E(\Gamma)$, then since α and β are both adjacent to the same $s - 2$ vertices of $\mathbb{K}$, these two vertices and $\mathbb{K}$ give us a $\mathbb{K}^s$.) Also, the vertex sets V_i partition $V(\Gamma)$. Therefore, Γ is $(s - 1)$-partite graph on n vertices with the maximum number of edges. Therefore, by the extremality of Γ, i.e., by $\|\Gamma\| = \theta_{s-1}(n)$, $\Gamma = \Theta^{s-1}(n)$. ■

In the second proof of Turan's theorem, a new operation (*vertex dublication*) is used:

Definition 2.4 Let $\alpha \in \Gamma$ be a vertex. *Dublication* of α means that adding a new vertex, say α', to Γ and joining this new vertex α' to all the neighbors of α (but not to the α).

Second Proof of Turan's Theorem. We know that the Turan graphs $\Theta^l(n)$ have the most possible edges in all the l-partite complete graphs on n vertices. And from the vertex degrees, it is obvious that $\Theta^{s-1}(n)$ contains more edges than $\Theta^l(n)$ for every $l < s-1$. Therefore, it is enough to show that Γ is a complete multi-partite graph.

Suppose to the contrary that Γ is not a complete multi-partite graph. Then non-adjacency relation on $V(\Gamma)$ is not an equivalence relation. Therefore, there are vertices $\beta_1, \alpha, \beta_2 \in V(\Gamma)$ such that β_1 and α, α and β_2 are not adjacent, but β_1 and β_2 are adjacent.

Now, $d(\beta_2) \leq d(\alpha)$ and $d(\beta_1) \leq d(\alpha)$: Suppose to the contrary that $d(\beta_2) > d(\alpha)$. Then delete α and dublicate β_1. After this operation, we obtain another graph without having $\mathbb{K}^s$ as a subgraph, but with having more edges than the graph Γ. This contradicts with the extremality of Γ. Therefore $d(\beta_1) \leq d(\alpha)$. With exactly the same manner, $d(\beta_2) \leq d(\alpha)$. But then delete β_1 and dublicate α, and delete β_2 and dublicate α again. After these two operations, we get another graph without having $\mathbb{K}^s$ as a subgraph, but with having more edges than the graph Γ. Again, this gives us a contradiction.

As a result, Γ must be a complete multipartite graph. ■

It can be shown easily that for all n and s , the following inequality is true:

$$\theta_{s-1}(n) \leq \frac{1}{2}n^2\frac{s-2}{s-1}.$$

and equality occurs if $s-1$ divides n. Therefore, it is striking that, for any given $\sigma > 0$ and large enough n, σn^2 more edges gives not only $\mathbb{K}^s$ as a subgraph (as stated in Turan's Theorem), but also $\mathbb{K}_t^s$ as a subgraph for any given $0 < t \in \mathbb{N}$ as stated in Erdös and Stone Theorem:

Theorem 2.2 (Erdös and Stone 1946) *Let $s \geq 2$ and $t \geq 1$ be integers, and let $\sigma > 0$ be a positive real number. Then there is a natual number n_0 such that every graph Γ with the number of vertices $n \geq n_0$ and the number of edges*

$$\|\Gamma\| \geq \theta_{s-1}(n) + \sigma n^2$$

contains a copy of $\mathbb{K}_t^s = \mathbb{K}_{t,t,\ldots,t} = \overline{\mathbb{K}_t} * \overline{\mathbb{K}_t} * \ldots * \overline{\mathbb{K}_t}$ (s copies of $\overline{\mathbb{K}_t}$).

The Erdös-Stone theorem is not only a striking result, but it also has a note-worthy corollary. Before stating this corollary, for a graph Π and a natural number n, define:

$$\tau_n = ex(n, \Pi)/\binom{n}{2}.$$

Note that τ_n is the greatest edge density of an n-vertex graph not having a subgraph isomorphic to Π.

Corollary 2.1 *Let Π be a graph with size ≥ 1. Then:*

$$\lim_{n\to\infty} ex(n, \Pi)\binom{n}{2}^{-1} = \frac{\chi(\Pi) - 2}{\chi(\Pi) - 1},$$

where χ denotes the well-knwon invariant of a graph, namely, the c.n. of a graph.

First, we state a lemma which will be used in the proof of the previous corollary.

Lemma 2.1 $\lim_{n\longrightarrow\infty} \theta_{s-1}(n)/\binom{n}{2} = \frac{s-2}{s-1}.$

Proof of the Previous Corollary. Let $s = \chi(\Pi)$. Since the c.n. of a graph is the minimum number of colors needed to color the vertices of a graph properly (properly means that no two adjacent vertices have the same color), it is not possible to color the vertices of Π with $s - 1$ colors. Therefore, $\Pi \nsubseteq \Theta^{s-1}(n)$ for every $n \in \mathbb{N}$. Thus, since $ex(n, \Pi)$ denotes the maximum number of edges that a graph can have without containing Π as a subgraph, we have:

$$\theta_{s-1}(n) \leq ex(n, \Pi).$$

On the other side, since $s = \chi(\Pi)$, for all sufficiently large t values, $\Pi \subseteq \mathbb{K}_t^s$. Therefore, for all such t values, we have:

$$ex(n, \Pi) \leq ex(n, \mathbb{K}_t^s).$$

Now, let us take a sufficiently large t, and fix it. Then for every $\sigma > 0$, and for sufficiently large n, we have:

$$ex(n, \mathbb{K}_t^s) < \theta_{s-1}(n) + \sigma n^2$$

by Erdös-Stone Theorem. Therefore, for large enough n values, we have:

$$\begin{aligned}\theta_{s-1}(n)/\binom{n}{2} &\leq ex(n,\Pi)/\binom{n}{2} \\ &\leq ex(n,\mathbb{K}_t^s)/\binom{n}{2} \\ &< \theta_{s-1}/\binom{n}{2} + \sigma n^2/\binom{n}{2} \\ &= \theta_{s-1}/\binom{n}{2} + 2\sigma/(1-\frac{1}{n}) \\ &\leq \theta_{r-1}/\binom{n}{2} + 4\sigma\end{aligned}$$

As a result, since $\lim_{n\longrightarrow\infty} \theta_{s-1}(n)/\binom{n}{2} = \frac{s-2}{s-1}$ by the previous lemma, we get the require limit value by the squeezing theorem: $\lim_{n\longrightarrow\infty} \theta_{s-1}(n)/\binom{n}{2} = \frac{s-2}{s-1} = \frac{\chi(\Pi)-2}{\chi(\Pi)-1}$. ■

2.2 Minors

In this section, we try to understand under which conditions a graph contains some other given graph as a minor or as a t.m.

As a first question, let's consider the analog of Turan's theorem: How many edges in a graph Γ on n vertices guarantee to have a $\mathbb{K}^s$ as a minor or as a t.m. of Γ?

Theorem 2.3 *Let Γ be a graph of a.d. $d(\Gamma) \geq 2^{s-2}$. Then, for every $s \in \mathbb{N}$, Γ contains a $\mathbb{K}^s$ as a minor.*

Proof. Proof is by induction on s.
For $s \leq 2$, the statement is trivially true. Now, let $s \geq 3$. Then since we assume that $d(\Gamma) \geq 2^{s-2}$, $\sigma(\Gamma)(= \frac{1}{2}d(\Gamma)) \geq 2^{s-3}$.
Let Π be a minor of Γ such that it is minimal with $\sigma(\Pi) \geq 2^{s-3}$.
Choose a vertex $\alpha \in V(\Pi)$. The vertex α can not be an isolated vertex, because Π is minimal. Also, if β is a neighbor of α, then α and β have at least 2^{s-3} common neighbors: Suppose to the contrary that there is a neighbor β of α such that the number of common

neighbors of α and β is less than 2^{s-3}. Contract the edge $\alpha\beta$. By this contraction, we get a smaller minor Π' such that $|\Pi'| = |\Pi| - 1$ and $\|\Pi'\| > \|\Pi\| - (2^{s-3} + 1)$, i.e., $\|\Pi'\| \geq \|\Pi\| - 2^{s-3}$. But then:

$$\sigma(\Pi') = \frac{\|\Pi'\|}{|\Pi|} \geq \frac{\|\Pi'\| - 2^{s-3}}{|\Pi| - 1} = \frac{\sigma(\Pi)|\Pi| - 2^{s-3}}{|\Pi| - 1} \geq \frac{2^{s-3}|\Pi| - 2^{s-3}}{|\Pi| - 1} = 2^{s-3}.$$

That is, $\sigma(\Pi') \geq 2^{s-3}$. This contradicts with the minimality of Π. Therefore, α and β have at least 2^{s-3} common neighbors. As a result, the minimum degree of the subgraph of Π induced by the neighbors of α is at least 2^{s-3}, and thus, the a.d. of this subgraph is at least 2^{s-3}. Therefore, by the induction hypothesis, this subgraph contains $\mathbb{K}^{s-1}$ as a minor. This minor and the vertex α produce the sought $\mathbb{K}^s$ minor of the original graph Γ.

■

To guarantee the existence of $\mathbb{K}^s$ as a t.m. is a little harder. The branch vertices of $\mathbb{K}^s$ are fixed beforehand, and the subdivided edges of $\mathbb{K}^s$ are constructed inductively; this forces to begin with an a.d. of at least $2^{\binom{s}{2}}$. Except for this difference, the main idea of the proof is the same.

Theorem 2.4 *Let Γ be a graph of a.d. $d(\Gamma) \geq 2^{\binom{s}{2}}$. Then, for every $2 \leq s \in \mathbb{N}$, Γ contains $\mathbb{K}^s$ as a t.m.*

Proof. For $s = 2$, the statement is trivially true. Now, let $s \geq 3$.

Proof is by induction on $m = s, s+1, \ldots, \binom{s}{2}$. Let Γ be a graph with a.d. $d(\Gamma) \geq 2^m$. We prove that Γ contain a t.m. Γ with $|\Gamma| = s$ and $\|\Gamma\| = m$.

Let $m = s$. Then we know that Γ contains a subgraph Π with

$$\delta(\Pi) > \sigma(\Pi) \geq \sigma(\Gamma) = \frac{1}{2}d(\Gamma) \geq 2^{s-1};$$

and thus, Π has a cycle of length

$$\ell \geq \delta(\Pi) + 1 > \sigma(\Pi) + 1 \geq \sigma(\Gamma) + 1 \geq 2^{s-1} + 1.$$

Therefore, Γ has a cycle of length

$$\ell \geq 2^{s-1} \geq s + 1.$$

As a result, since $\Gamma = \mathbb{C}^s$ satisfies the requirement, the statement is true.

Now, let $s < m \leq \binom{s}{2}$, and suppose that the theorem is true for smaller m values. Let Γ be a graph with a.d. $d(\Gamma) \geq 2^m$. Therefore, $\sigma(\Gamma) = \frac{d(\Gamma)}{2} \geq 2^{m-1}$.

First of all, since there is a component $\mathcal{C}$ of Γ with $\sigma(\mathcal{C}) \geq \sigma(\Gamma)$, wlog, we may suppose that Γ is itself connected. Let $W \subseteq V(\Gamma)$ be a maximal set such that it is connected in Γ and $\sigma(\Gamma/W) \geq 2^{m-1}$. (There is such a set W, because for $W \subseteq V(\Gamma)$ with $|W| = 1$, Γ is itself of the form Γ/W.) By the connectivity of Γ, $\mathfrak{N}(W) \neq \emptyset$.

Let $\Pi = \Gamma[\mathfrak{N}(W)]$. Then every vertex α of Π has degree $d_\Pi(\alpha) \geq 2^{m-1}$ in Π: Suppose not. Then there is a vertex $\alpha \in \Pi$ with $d_\Pi(\alpha) < 2^{m-1}$. If the edge $\alpha\alpha_W$ in Γ/W is contracted, then one vertex and $d_\Pi + 1 \leq 2^{m-1}$ edges are lost. Therefore, if α is added to W, then the obtained graph has still $\sigma \geq 2^{m-1}$. But this contradicts with the maximality of the vertex set W. As a result, $d(\Pi) \geq \delta(\Pi) \geq 2^{m-1}$. By the induction hypothesis, the graph Π has $T\Gamma$ such that $|\Gamma| = s$ and $\|\Gamma\| = m - 1$. Choose two branch vertices, say β and γ, of this $T\Gamma$ such that $\beta\gamma$ is not an edge in Γ. Since $\beta, \gamma \in W$, and since W is connected in Γ, there is an $\beta - \gamma$ path in Γ whose inner vertices belong to W. Finally, add this path to the $T\Gamma$. Then we get the required $T\Gamma$. ■

We can reduce the bound in the previous theorem from exponential to quadratic by using the following involved Thomas-Wollan theorem. To state that theorem, we need the following definition:

Definition 2.5 Let $\Gamma = (V, E)$ be a graph, and $Z \subseteq V(\Gamma)$.

· If for every distinct vertices $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l$ in Z, there are disjoint paths, say $Q_1, \ldots, Q_l$ in Γ such that for each $i = 1, \ldots, j$, Q_i links the vertex α_i to the vertex β_i and Q_i does not have any inner vertex in Z, then Z is called *linked* in Γ.

· If $|\Gamma| \geq 2l$ and every $Z \subseteq V(\Gamma)$ with $|Z| \leq 2l$ is linked in Γ, then Γ is called *l-linked.*

Theorem 2.5 (Thomas and Wollan 2005) *Let Γ be a graph, and let $l \in \mathbb{N}$. Suppose that Γ is $2l$-connected and $\sigma(\Gamma) \geq 8l$. Then Γ is l-linked.*

And, in the proof of the theorem reducing the mentioned bound from exponential to quadratic, we also need the following theorem of Mader:

Theorem 2.6 (Mader 1972) *Let $0 \neq s \in \mathbb{N}$, and let Γ be a graph with $d(\Gamma) \geq 4s$. Then Γ contains a $(k+1)$-connected subgraph Γ satisfying the inequality $\sigma(\Gamma) \geq \sigma(\Gamma) - k$.*

Now, we can state and prove the theorem reducing the bound given in Theorem 2.4:

Theorem 2.7 *There is a constant $\gamma \in \mathbb{R}$ satisfying the following property: For each natural number s, every graph Γ with $d(\Gamma) \geq \gamma s^2$ has $\mathbb{K}^s$ in it as a t.m.*

Proof. We prove the claim by taking $\gamma = 10$.
Let Γ be a given graph with $d(\Gamma) \geq 10s^2$. Let $l = s^2$. Then, Mader theorem, there is a subgraph Π of Γ such that $\kappa(\Pi) \geq s^2$ and $\sigma(\Pi) > \sigma(\Gamma) - s^2 \geq \dfrac{10s^2}{2} - s^2 = 4s^2$.
For a t.m. $\mathbb{K}^s$ in the subgraph Π, choose a set $Z \subseteq V(\Pi)$ with $|Z| = s$, as branch vertices; and by picking $s-1$ vertices for each vertex in the set Z, choose a set $A \subseteq V(\Pi)$ with $|A| = s(s-1)$ consisting of neighbors of Z, as initial subdividing vertices. Note that the chosen number of vertices is s^2, and since $\delta(\Pi) \geq \kappa(\Pi) \geq s^2$, we can choose these vertices distinctly.
To finish the proof, the only remaining part is to join the vertices of the set A in pairs, by disjoint paths in $\Pi' = \Pi - Z$, and of course these paths correspond to the edges of the complete graph $\mathbb{K}^s$. Such connections is possible if A is linked in Π'. More generally, in fact, Π' is $\dfrac{1}{2}s(s-1)$-linked, because Π' satisfies the hypothesis of Thomas-Wollan theorem for $l = \dfrac{1}{2}s(s-1)$: $\kappa(\Pi') \geq \kappa(\Pi) - s \geq s^2 - s = s(s-1) = 2l$, thus the first hypothesis is satisfied. And since Π' was gotten from Π by deleting at most $s|\Pi|$ edges (also some vertices), simply becasue $\Pi' = \Pi - Z$ and $|Z| = s$; the second hypothesis is also satisfied:

$$\sigma(\Pi') \geq \sigma(\Pi) - s \geq 4s^2 - s = 4s(s-1) = 8l.$$

As a result, by Thomas-Wollan theorem, Γ is l-linked. ■

The a.d. required to gaurantee the existence of an arbitrary $\mathbb{K}^s$ minor is less than that for an arbitrary $T\mathbb{K}^s$ t.m., and this a.d. is known exactly:

Theorem 2.8 (Kostochka 1982) *There is a constant $\gamma \in \mathbb{R}$ satisfying the following property: For each natural numebr s, every graph Γ with $d(\Gamma) \geq \gamma s\sqrt{\log s}$ has $\mathbb{K}^s$ in it as a minor. Moreover, as a function of s, this lower bound is best possible up to the value of the constant γ. ($\log s = \log_2 s$.)*

In the rest of the section, we study another remarkable condition which again guarantees the existence of minors: By increasing the girth of a graph (but not solely subdividing edges), the existence of a $\mathbb{K}^s$ minor can be guaranteed: If the girth, say ϱ, of a graph Γ is big, then the ball around a vertex α with radius $\lfloor \varrho/2 \rfloor$, namely, $S = \{\beta \mid d(\beta, \alpha) < \lfloor \varrho/2 \rfloor\}$ induces a tree (since for every $\beta, \delta \in S$, $d(\beta, \alpha) \leq (\beta, \alpha) + d(\alpha, \delta) < \lfloor \varrho/2 \rfloor + \lfloor \varrho/2 \rfloor = \varrho$, S does not contain any cycles; and thus, the graph induced by S is a tree) and this tree has many leaves. In Γ, only one of the incident edges of such a leave belongs to the tree, the other edges are away from the tree. Therefore, by contracting enough numbers of disjoint trees, we can expect to get a minor with a large a.d., which successively give a complete minor with a large a.d. This idea is realised by the following lemma:

Lemma 2.2 *Let j and l be natural numbers with $j \geq 3$. Let Γ be a graph with the minimum degree $\delta(\Gamma) \geq j$ and the girth $\varrho(\Gamma) \geq 8l + 3$. Then there is a minor Γ of Γ with $\delta(\Gamma) \geq j(j-1)^l$.*

This lemma and the previous theorem give the following result:

Theorem 2.9 (Thomassen 1983) *There is a function $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $s \in \mathbb{N}$ the following is satisfied: If a graph Γ has the minimum degree $\delta(\Gamma) \geq 3$ and the grith $\varrho(\Gamma) \geq \psi(s)$, then Γ contains a $\mathbb{K}^s$ minor.*

Proof. We show that $\psi(s) = 8\log(s) + 4\log(\log(s)) + \gamma$, for some real constant γ, is one of the the desired functions, that is, this function ψ satisfies the required condition:

Let $l = l(s)$ be a minimal natural number satisfying the inequality $32^l \geq \gamma' s\sqrt{\log(s)}$, where $\gamma' \in \mathbb{R}$ is the constant mentioned in the previous theorem. (At this stage, we just choose a minimal l satisfying $3\,2^l \geq \gamma' s\sqrt{\log(s)}$.) Then since l is minimal, we have:

$$\begin{aligned} 3\,2^l \geq \gamma' s\sqrt{\log(s)} &\implies \log(3) + (s-1) < \log(\gamma') + \log(s) + \frac{1}{2}\log(\log(s)). \\ &\implies \text{Since } \log 3 - 1 > 0, l < \log(\gamma') + \log(\gamma) + \frac{1}{2}\log(\log(s)). \\ &\implies 8l + 3 < 8\log(s) + 4\log(\log(s)) + (8\log(\gamma') + 3). \\ &\implies 8l + 3 < 8\log(s) + 4\log(\log(s)) + \gamma, \gamma = 8\log(\gamma') + 3. \end{aligned}$$

Now, since $\varrho(\Gamma) \geq \psi(s) \geq 8l + 3$, and since $\delta(\Gamma) \geq 3$, the graph Γ contains a minor Π with $\delta(\Pi) \geq 3(3-1)^l = 3\,2^l$, by the previous lemma.
Then, since $d(\Pi) \geq \delta(\Pi) \geq 3\,2^l \geq \gamma' s\sqrt{\log(s)}$, Π contains a minor $\mathbb{K}^s$, by the previous theorem. As a result, since $\mathbb{K}^s$ is a minor of Π and Π is a minor of Γ, $\mathbb{K}^s$ is a minor of Γ.

■

Big girth value can also be employed to guarantee a t.m. $\mathbb{K}^s$. For this, as branch vertices, some vertices of degree $\geq s-1$ are needed. By assuming a minimum degree $\geq s-$, the existence of such vertices can be ensured. But even in this case, a girth bound not depending on s is needed to guarantee a t.m. $\mathbb{K}^s$.

Theorem 2.10 (Kühn and Osthus 2002) *There is some constant ξ such that every graph Γ with $\delta(\Gamma) \geq s-1$ and $\varrho(\Gamma) \geq \xi$ contains $T\mathbb{K}^s$ for all natural numbers s.*

2.3 Hadwiger's Conjecture

In the previous section, we proved that a.d. $\geq \gamma s \log(s)$ is enough to guarantee a graph to contain a minor $\mathbb{K}^s$, and a.d. $\geq \gamma s^2$ is enough to guarantee a graph to contain a t.m. $\mathbb{K}^s$. If *c.n.* puts in place of *a.d.*, then, with nearly the same constants, these two statements hold true, because a graph Γ with $\chi(\Gamma) = l$ contains a subgraph Π with $d(\Pi) \geq l-1$:

Lemma 2.3 *Let Γ be a graph with $\chi(\Gamma) = l$. Then Γ has a subgraph Π with $d(\Pi) \geq l-1$.*

Proof. Let $\Pi \subseteq \Gamma$ be a minimal subgraph such that $\chi(\Pi) = l$. Then $\delta(\Pi) \geq l - 1$: Suppose not. Then there is a vertex $\alpha \in V$ with $d_{\Pi}(\alpha) \leq l - 2$. But then, since $\Pi - \alpha$ is $(l-1)$-colorable, by extending such a coloring, we obtain a $(l-1)$-coloring of Π which contradicts by the choice of Π. ■

With the hypothesis on *a.d.*, given on the related theorems, both functions, $\gamma s\sqrt{\log(s)}$ and γs^2, (up to a constant) are best possible. Are they still best possible if *c.n.* is used instead of *a.d.*? Do some slower-growing functions guarantee the same conclusions? An answer of this question is not known for general or t.minors. However, for general minors, Hadwiger proposes an affirmative answer with the following conjecture:

Conjecture 2.1 (Hadwiger Conjecture 1943) *For every graph* Γ *and every natural number* $s \geq 1$*, the following statement holds:*

$$\chi(\Gamma) \geq s \Longrightarrow \Gamma \succcurlyeq \mathbb{K}^s.$$

Hadwiger's conjecture is obviously true for $s \geq 2$, it is easy to see that it also holds for $s = 3$ and $s = 4$. For $s = 5$ and $s = 6$, the four color theorem and Hadwiger's conjecture are equivalent. For $s \geq 7$, the conjecture holds for line graphs and for graphs with large girth value, but it is still open for arbitrary graphs. Generally, if the conjecture is true for $s + 1$, then it is true for s.

For any fixed $s \in \mathbb{N}$, the Hadwiger's conjecture is equivalent to the statement that if a graph does not contain a $\mathbb{K}^s$ minor, then it is $(s-1)$-colorable. With this reformulation, it is reasonable to ask the following question: Can we characterize the graphs not containing a $\mathbb{K}^s$ minor?

For example, for $s = 3$, the graphs not containing a $\mathbb{K}^3$ minor are exactly the forests, because such graph does not contain any cycles. Thus, since a forest is a bipartite graph, such graphs are really 2-colorable.

For $s = 4$, there exists as well a modest characterization for the graphs not containing $\mathbb{K}^4$ minor. First, we need a definition:

Definition 2.6 Let Γ be a graph, and let Γ_1, Γ_2 and T be i.subgraphs of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $T = \Gamma_1 \cap \Gamma_2$. In this case, it is said that Γ results from Γ_1 and Γ_2 by *pasting* them along T.

Theorem 2.11 *Let Γ be a graph with $|\Gamma| \geq 3$. Then: Γ is edge-maximal not containing a $\mathbb{K}^4$ minor $\Longleftrightarrow$ It can be obtained from triangles by pasting along $\mathbb{K}^2$ s recursively.*

One of the corollaries of the previous theorem is the following:

Corollary 2.2 *Let Γ be an edge-maximal graph not containing a $\mathbb{K}^4$ minor. Then the number of edges of Γ is $2|\Gamma| - 3$. Therefore, every such graph Γ has the same order, and thus they are extremal.*

Proof. Proof is by induction on the number of vertices of Γ. ■

Corollary 2.3 *For $s = 4$, Hadwiger's conjecture is true.*

Proof. We prove the contrapositive of Hadwiger's conjecture, that is: If Γ does not have a $\mathbb{K}^4$ minor, then $\chi(\Gamma) < 4$, i.e., Γ is 3-colorable.
Suppose first that Γ results from Γ_1 and Γ_2 by pasting along $\mathbb{K}^n$ for some $n \in \mathbb{N}$. Then $\chi(\Gamma) = max\{\chi(\Gamma_1), \chi(\Gamma_2\}$:
$\Gamma_1 \subseteq \Gamma, \Gamma_2 \subseteq \Gamma \implies \chi(\Gamma_1) \leq \chi(\Gamma), \chi(\Gamma_2) \leq \chi(\Gamma) \implies \max\{\chi(\Gamma_1), \chi(\Gamma_2)\} \leq \chi(\Gamma):$
Let $\gamma_i : V(\Gamma_i) \longrightarrow \{1, 2, \ldots, \chi(\Gamma_i)\}$ be a coloring of Γ_i for $i = 1, 2$. Then, these two colorings can be united to obtain a coloring of Γ with $max\{\chi(\Gamma_1), \chi(\Gamma_2\}$ colors. (If it is needed, the colors of one of Γ_1 or Γ_2 can be permuted.) Therefore, $\chi(\Gamma) \leq \max\{\chi(\Gamma_1), \chi(\Gamma_2\}$. As a result, $\chi(\Gamma) = \max\{\chi(\Gamma_1), \chi(\Gamma_2)\}$.
Now, let Γ be any edge-maximal graph not containing a $\mathbb{K}^4$ minor. Then by the previous theorem, Γ can be obtained recursively from $\mathbb{K}^3$ s by pasting along $\mathbb{K}^2$ s. Therefore, by induction on the number of vertices of Γ, since $\chi(\Gamma) = \max\{\chi(\Gamma_1), \chi(\Gamma_2)\} \leq \{3, 3\}$, $\chi(\Gamma) \leq 3$. As a result, $\chi(\Gamma) < 4$. (Since this is true for edge-maximal graphs, it is true for all graphs.) ■

To prove Hadwiger's conjecture for $s = 5$, we need the following theorem for graphs not containing a $\mathbb{K}^5$ minor:

Theorem 2.12 (Wagner 1937) *Let Γ be a graph such that it is edge-maximal and it does not contain a $\mathbb{K}^5$ minor. Suppose that $|\Gamma| \geq 4$. Then Γ can be built recursively, by pasting along $\mathbb{K}^3$ s and $\mathbb{K}^2$ s, from the copies of the Wager graph W (see Figure 2.4) and from the plane triangulations.*

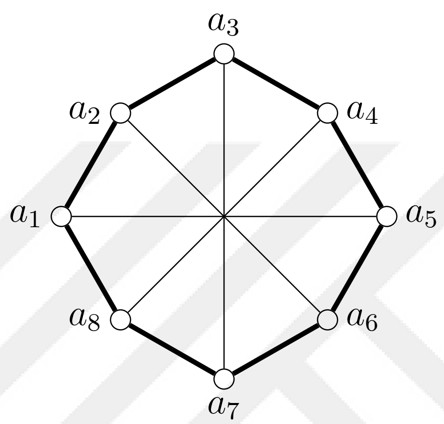


Figure 2.4 A representation of he Wagner graph W

The numbers of edges of extremal graphs not containing a $\mathbb{K}^5$ minor, built as in the previous theorem, is less than or equal to those of the maximal planar graphs:

Corollary 2.4 *Let Γ be a graph not containing a $\mathbb{K}^5$ minor with n vertices. Then the number of edges of Γ is less than or equal to $3n - 6$.*

Now, since $\chi(W) = 3$, by the previous theorem and the 4-color theorem (namely, for a planar graph Γ, $\chi(\Gamma) \leq 4$), we get Hadwiger's conjecture for the next s value:

Corollary 2.5 *Hadwiger's conjecture is true for $s = 5$.*

For $s = 6$, the proof of Hadwiger's conjecture is considerably difficult than the previous s values, and the proof again relies upon the 4-color theorem:

Theorem 2.13 (Robertson *et al.* 1993) *Hadwiger's conjecture is true for* $s = 6$.

In the last theorem of the previous section, we saw that with the additional assumption of large girth, the hypothesis $\delta(\Gamma) \geq s - 1$ guarantees the existence of a t.m. $T\mathbb{K}^s$. In fact, a weaker additional supposition is sufficient:
"For any given natural number t and for all sufficiently big j values which depend only on t; the graphs Γ not containing the complete bipartite graph $\mathbb{K}_{t,t}$ with a.d. $d(\Gamma) \geq j$ contains $\mathbb{K}^s$ minors for s values which are substantially bigger than j."
For Hadwiger's conjecture, as a result of this statement, we obtain:

Theorem 2.14 (Kühn and Osthus 2005) *For each* $t \in \mathbb{N}$*, there exists* $s_t \in \mathbb{N}$ *satisfying the following property: For each graph* Γ *not containing the complete bipartite graph* $\mathbb{K}_{t,t}$*, and for all* $s \geq s_t$*, the Hadwiger's conjecture is true.*

The strengthening version of Hadwiger's conjecture is the following conjecture:

Conjecture 2.2 (Hajos Conjecture 1961) *For every graph* Γ *and every natural number* $s \geq 1$*, the following statement holds:*

$$\chi(\Gamma) \geq s \Longrightarrow \Gamma \text{ contains } \mathbb{K}^s \text{ as a t.m.}$$

Generally, Hajos's conjecture is not true, however from the last theorem of the previous section, it is true for graphs with large girth values:

Corollary 2.6 *There exist some constant* ξ *such that for all graphs* Γ *with* $\varrho(\Gamma) \geq \xi$*, the following implication is true:*

$$\chi(\Gamma) \geq l \Longrightarrow \Gamma \supseteq T\mathbb{K}^s \text{ for all natural numbers } s.$$

Proof. By the last theorem of the previous section, there is some constant ξ such that for any graph Γ with $\delta(\Gamma) \geq s-1$ and $\varrho(\Gamma) \geq \xi$, Γ contains $\mathbb{K}^s$ as a topological minor. Now, consider the same constant ξ. Since $\chi(\Gamma) \geq s$ by hypothesis, we know that there is a subgraph Π of Γ such that $\delta(\Pi) \geq s - 1$.

On the other hand , since $\varrho(\Pi) \geq \varrho(\Gamma) \geq \xi$, again the last theorem of the previous section implies that $\Pi \supseteq T\mathbb{K}^s$. Therefore, since $\Gamma \supseteq \Pi$, $\Gamma \supseteq T\mathbb{K}^s$. ■

2.4 Szemeredi's Regularity Lemma

Let $\Gamma = (V, E)$ be a graph.

Definition 2.7 Let $C, D \subseteq V$ be two disjoint vertex sets. $\|C, D\|$ denotes the total number of $C - D$ edges in Γ, and

$$d(C, D) = \frac{\|C, D\|}{|C||D|}$$

is called the *density* of (C, D). (Note that $d(C, D) \in [0, 1]$.)

Definition 2.8 Let $X, Z \subseteq V$ be two disjoint vertex sets, and let $\sigma > 0$ be a given fixed positive real number. (X, Z) is called *σ-regular pair* if for all $C \subseteq X$ and $D \subseteq Z$ satisfying

$$|C| \geq \sigma|X| \qquad \text{and} \qquad |D| \geq \sigma|Z|,$$

the following inequality holds:

$$|d(C, D) - d(X, Z)| \leq \sigma.$$

Definition 2.9 Let $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_\iota\}$ be a partition of V. From this partition, discriminate one set Γ_0, and call it as an *exceptional set (e.s.)*. (The e.s. need not be non-empty.) If this partition satisfies the following conditions:
(i) $|\Gamma_0| \leq \sigma|V|$;
(ii) $|\Gamma_1| = \ldots = |\Gamma_\iota|$;
(iii) For $1 \leq i < j \leq l$, excpet for at most σl^2 pairs (Γ_i, Γ_j), almost all such pairs (Γ_i, Γ_j) are σ-regular;
then it is called *σ-regular partition (σ-r.p.)* of the graph Γ.

Note that the e.s. Γ_0 enables to satisfy that the orders of the remaining partition sets are equal. Also, since condition (iii) is related to $\Gamma_1, \ldots, \Gamma_\iota$, the vertices of Γ_0 are ignored when the regularity of some partition is measured; but by condition (i), there are only a small number of vertices in the e.s. Γ_0.

Theorem 2.15 (Regularity Lemma) *Let $\sigma > 0$ be a given positive real number and $\upsilon \geq 1$ be a given natural number. Then there is a natural number Υ such that if Γ is a graph of order $\geq \upsilon$, then Γ accepts an σ-r.p. $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_\iota\}$ satisfying $\upsilon \leq l \leq \Upsilon$.*

Therefore, the Regularity Lemma states that, for any given $\sigma > 0$, each graph has got an σ-r.p. such that the number of sets in this partition is bounded. The upper bound Υ guarantees that for large graphs, the sets in the partition is also large. Note that if the partition sets contain a single vertex, then σ-regularity is trivially satisfied; but if partition sets are large, then σ-regularity is a strong property. The Regularity Lemma also gives us a lower bound υ for the number of sets in the partition. The lower bound υ can be used to raise the ratio of edges between different sets in the partition (that is, the edges controlled by the Regularity Lemma) and edges within partition sets (about these edges we do not know anything).

The rest of this section is dedicated to the proof of the Regularity Lemma.
First, we need the following inequality: Let $\nu_1, \ldots, \nu_l > 0$ and $f_1, \ldots, f_l \geq 0$ be real numbers. Then:

$$\sum \frac{f_i^2}{\nu_i} \geq \frac{(\sum f_i)^2}{\sum \nu_i}.$$

The proof of this inequality can be obtained by using the Cauchy-Schwarz inequality: $\sum \alpha_i^2 \beta_i^2 \geq (\sum \alpha_i \beta_i)^2$ by substituting $\alpha_i = \sqrt{\nu_i}$ and $\beta_i = f_i/\sqrt{\nu_i}$.

Now, let $\Gamma = (V, E)$ be a graph with the number of vertices n, and let $P, R \subseteq V$ be disjoint vertex sets. Define:

$$\varsigma(P, R) = \frac{|P||R|}{n^2}\, d^2(P, R) = \frac{\|P, R\|^2}{|P||R|n^2}.$$

Let $\mathcal{P}$ (respectively, $\mathcal{R}$) be a partition of P (respectively, R), and let $\mathcal{O} = \{\nu_1, \ldots, \nu_l\}$ be a partition of V. Define:

$$\varsigma(\mathcal{P}, \mathcal{R}) = \sum_{P' \in \mathcal{P}, R' \in \mathcal{R}} \varsigma(P', R'), \qquad \varsigma(\mathcal{O}) = \sum_{i<j} \varsigma(\nu_i, \nu_j).$$

On the other hand, if $\mathcal{O} = \{\nu_0, \nu_1, \ldots, \nu_l\}$, where ν_0 is an e.s., is a partition of the vertex set V of Γ, then we behave ν_0 as a family of sets containing single vertices, and define:

$$\varsigma(\mathcal{O}) = \varsigma(\widetilde{\mathcal{O}}),$$

where $\widetilde{\mathcal{O}} = \{\nu_1, \ldots, \nu_l\} \cup \{\{\alpha\} \mid \alpha \in \nu_0\}$.

The function $\varsigma(\mathcal{O})$ not only plays a role in the Regularity Lemma's proof, but also $\varsigma(\mathcal{O})$ assesses the regularity of the partition $\mathcal{O}$: If there are lots of pairs (P, R) which are irregular, then the subsets S and T of the pairs (S, T), which violates the regularity of (P, R), maybe taken and made partition sets. As we will see, this process refines the partition $\mathcal{O}$ into a new partition $\mathcal{Q}$ such that $\varsigma(\mathcal{Q})$ is 'considerably' greater than $\varsigma(\mathcal{O})$ ('considerably' means that the increment of $\varsigma(\mathcal{O})$ is bounded from below by a constant which depends only on σ). On the other hand, we have:

$$\varsigma(\mathcal{O}) = \sum_{i<j} \varsigma(\nu_i, \nu_j) = \sum_{i<j} \frac{|\nu_i||\nu_j|}{n^2}\, d^2(\nu_i, \nu_j) \leq \frac{1}{n^2} \sum_{i<j} |\nu_i||\nu_j| \leq 1.$$

Therefore, since the increment of $\varsigma(\mathcal{O})$ is bounded (from below) by a constant, the number of refinements of $\mathcal{O}$ is also bounded (from above) by a constant, that is, after some bounded number of (i.e., finitely many) refinements, the partition will become σ-regular. Thus, to finish the Regularity Lemma's proof, we only need to determine the upper bound Υ stated in the theorem; in other words, if one starts with a partition containing υ elements, then we have to decide the number of partition sets in the last partition.

All these will be made explicit. Let us begin by proving that if a partition is refined, then the ς value does not decrease:

Lemma 2.4 (1) *Let P, R be disjoint vertex sets. Let $\mathcal{P}$ (respectively $\mathcal{R}$) be a partition of P (respectively, R). Then, $\varsigma(\mathcal{P}, \mathcal{R}) \geq \varsigma(P, R)$.*
(2) *Let $\mathcal{O}'$ be a refinement of $\mathcal{O}$, where $\mathcal{O}$ and $\mathcal{O}'$ are both partitions of the vertex set of Γ. Then $\varsigma(\mathcal{O}') \geq \varsigma(\mathcal{O})$.*

Proof. (1) Let $\mathcal{P} = \{\nu_1, \dots, \nu_\iota\}$ and $\mathcal{R} = \{\eta_1, \dots, \eta_l\}$. Then, we have:

$$\begin{aligned}\varsigma(\mathcal{P}, \mathcal{R}) = \sum_{i,j} \varsigma(\nu_i, \eta_j) &= \frac{1}{n^2} \sum_{i,j} \frac{\|\nu_i, \eta_j\|^2}{|\nu_i||\eta_j|} \\ &\geq \frac{1}{n^2} \frac{(\sum_{i,j} \|\nu_i, \eta_j\|)^2}{\sum_{i,j} |\nu_i||\eta_j|} \\ &= \frac{1}{n^2} \frac{\|P, R\|^2}{(\sum_i |\nu_i|)(\sum_j |\eta_j|)} = \varsigma(P, R).\end{aligned}$$

(2) Let $\mathcal{O} = \{\nu_1, \dots, \nu_l\}$ be a partition of V; and for each $i = 1, \dots, l$, let $\mathcal{P}_i$ be the partition of ν_i consisiting of partition sets in $\mathcal{O}'$, which are subsets of ν_i. Then, we have:

$$\varsigma(\mathcal{O}) = \sum_{i<j} \varsigma(\nu_i, \nu_j) \leq \sum_{i<j} \varsigma(\mathcal{P}_i, \mathcal{P}_j) \leq \varsigma(\mathcal{O}').$$

The last inequality is true due to the fact that $\varsigma(\mathcal{O}') = \sum_i q(\mathcal{P}_i) + \sum_{i<j} q(\mathcal{P}_i, \mathcal{P}_j)$. ■

Next, we prove that of one refines a partition by just subpartitionning a single irregulaar pair of sets in the partition, then the increase in the value of ς is very small, and less than any given constant.

Lemma 2.5 *Let $\sigma > 0$ be given, and let P, R be disjoint vertex sets of Γ. Suppose that the pair (P, R) is not σ-regular. Then there exists partition $\mathcal{P} = \{\nu_1, \nu_2\}$ (respectively, $\mathcal{R} = \{\eta_1, \eta_2\}$) of P (respectively, R) such that*

$$\varsigma(\mathcal{P}, \mathcal{R}) \geq \varsigma(P, R) + \sigma^4 \frac{|P||R|}{n^2}.$$

Proof. First, since (P, R) is not σ-regular, there exists sets $P_1 \subseteq P$ and $R_1 \subseteq R$ such that: $|P_1| \geq \sigma|P|$ and $|R_1| \geq \sigma|R|$ satisfying the following strict inequality for $\xi = d(P_1, R_1) - d(P, R)$:

$$|\xi| > \sigma.$$

Let $P_2 = P \setminus P_1$ (respectively, $R_2 = R \setminus R_1$), and $\mathcal{P} = \{P_1, P_2\}$ (respectively, $\mathcal{R} = \{R_1, R_2\}$).

Now, we prove that $\mathcal{P}$ and $\mathcal{R}$ fulfill the requirement of the lemma.

Let $p_i = |P_i|$, $r_i = |R_i|$, $f_{i,j} = \|P_i, R_i\|$, and $p = |P|$, $r = |R|$, $f = \|P, R\|$. Then like in the proof of the previous lemma, we have:

$$\varsigma(\mathcal{P}, \mathcal{R}) = \frac{1}{n^2} \sum_{i,j} \frac{f_{ij}^2}{p_i r_j} = \frac{1}{n^2}(\frac{f_{11}^2}{p_1 r_1} + \sum_{i+j>2} \frac{f_{ij}^2}{p_i r_j})$$
$$\geq \frac{1}{n^2}(\frac{f_{11}^2}{p_1 r_1} + \frac{(f - f_{11})^2}{pr - p_1 r_1}).$$

Furthermore, by the definitions of ξ, $f_{11} = \frac{c_1 d_1 f}{cd} + \eta c_1 d_1$, therefore we have:

$$n^2 \varsigma(\mathcal{P}, \mathcal{R}) \geq \frac{1}{p_1 r_1}(\frac{p_1 r_1 f}{pr} + \xi p_1 r_1)^2 + \frac{1}{pr - p_1 r_1}(\frac{pr - p_1 r_1}{pr} f - \xi p_1 r_1)^2$$
$$= \frac{p_1 r_1 f^2}{p^2 r^2} + \frac{2 f \xi p_1 r_1}{pr} + \xi^2 p_1 r_1 + \frac{pr - p_1 r_1}{p^2 r^2} f^2 - \frac{2 f \xi p_1 r_1}{pr} + \frac{\xi^2 p_1^2 r_1^2}{pr - p_1 r_1}$$
$$\geq \frac{f^2}{pr} + \xi^2 p_1 r_1 \geq \frac{f^2}{pr} + \sigma^4 pr.$$

The last inequality is true due to the choice of P_1 and R_1: $p_1 \geq \sigma o$ and $r_1 \geq \sigma r$. ■

Finally, we prove that if there are enough number of irregular pairs of sets in a partition, then subpartitioning all these irregular pairs leads to an increment of ς by some constant:

Lemma 2.6 *Let $\sigma \in (0, 1/4]$. Let $\mathcal{O} = \{P_0, P_1, \ldots, P_\iota\}$ be a partition of the vertex set V, where P_0 is the e.s. with $|P_0| \leq \sigma n$ ($n = |V|$) and $|P_1| = \ldots = |P_\iota| = p$. Suppose that $\mathcal{O}$ is not σ-regular. Then there exists a partition $\mathcal{O}' = \{P'_0, P'_1, \ldots, P'_l\}$ of the vertex set V, where P'_0 is the e.s. with $|P'_0| \leq |P_0| + n/2^\iota$ and $|P'_1| = \ldots = |P'_l|$, also where $\iota \leq l \leq \iota 4^{\iota+1}$ such that either $\mathcal{O}'$ is σ-regular or $\varsigma(\mathcal{O}') \geq \varsigma(\mathcal{O}) + \sigma^5/2$.*

Proof. For each i, j satisfying $1 \leq i < j \leq \iota$, define a partition $\mathcal{P}_{ij}$ (respectively, $\mathcal{P}_{ji}$) of P_i (respectively, P_j) as the following:

· If (P_i, P_j) is σ-regular, then $\mathcal{P}_{ij} = \{P_i\}$ and $\mathcal{P}_{ji} = \{P_j\}$.

· If (P_i, P_j) is not σ-regular, then, by the previous lemma, there exist partitions $\mathcal{P}_{ij}$ of P_i and $\mathcal{P}_{ji}$ of P_j such that $|\mathcal{P}_{ij}| = |\mathcal{P}_{ji}| = 2$, and

$$\varsigma(\mathcal{P}_{ij}) \geq \varsigma(P_i, P_j) + \sigma^4 \frac{|P_i||P_j|}{n^2} = \varsigma(P_i, P_j) + \frac{\sigma^4 p^2}{n^2}.$$

Now, for every $i = 1, \ldots, \iota$, define an equivalence $\sim$ on P_i as follows: Let $\alpha, \beta \in P_i$. Then $\alpha \sim \beta$ iff they belong to the same set of the partition $\mathcal{P}_{ij}$ for every $i \neq j$. Let $\mathcal{P}_i$

be the set of equivalence classes of this equivalence relation on P_i. Then $\mathcal{P}_i$ is the unique partition of the set P_i such that it is minimal in the sense that $\mathcal{P}_i$ refines every partition $\mathcal{P}_{ij}$ for $j \neq i$. Therefore, $|\mathcal{P}_i| \leq 2^{\iota-1}$ (For each $1 \leq i \leq \iota - 1$, the number of j s with $1 \leq i < j \leq \iota$ is $\leq \iota - 1$. Thus, since every $\mathcal{P}_{ij}$ contains exactly two elements, $|\mathcal{P}_i| \leq 2^{\iota-1}$). Now, let

$$\mathcal{P} = \{P_0\} \cup \bigcup_{i=1}^{\iota} \mathcal{P}_i$$

be the partition of the vertex set V, where P_0 is an e.s. Then $\mathcal{P}$ refines the partition $\mathcal{O}$ and $|\mathcal{P} \setminus \{P_0\}| \leq \iota 2^{\iota-1}$. Therefore:

$$\iota \leq |\mathcal{P}| \leq \iota 2^{\iota}.$$

(For each $i = 1, \dots, \iota$, $\mathcal{P}_i$ is nonempty, i.e., it contains at least one element. Thus, $|\mathcal{P}| \geq \iota$. On the other hand, since $|P_0| \leq \sigma n \leq \frac{1}{4}n$, $|\bigcup_{i=1}^{\iota} \mathcal{P}_i| = \sum_{i=1}^{\iota} |P_i| \geq \frac{3}{4}n$. Therefore, since $\sum_{i=1}^{\iota} |P_i| \leq \iota 2^{\iota-1}$, $\iota 2^{\iota-1} \geq \frac{3}{4}n \Longleftrightarrow n \leq \frac{4}{3}\iota 2^{\iota-1} \leq \iota 2^{\iota}$. Thus, $|\mathcal{P}| = n \leq \iota 2^{\iota}$.)

Let $\mathcal{P}_0 = \{\{\alpha\}\} \mid \alpha \in P_0\}$. Now, if the partition $\mathcal{P} = \{P_0, P_1, \dots, P_l\}$ is not σ-regular, since $|P_0| \leq \sigma n$ and $|P_1| = \dots = |P_\iota|$, the third condition of being σ-regular must be violated. That is, more than $\sigma\iota^2$ pairs (P_i, P_j) with $1 \leq i < j \leq \iota$ are not σ-regular, i.e., by the definition of $\mathcal{P}_{ij}$, more than $\sigma\iota^2$ partitions $\mathcal{P}_{ij}$ are non-trivial. (Let Z be a non-empty set. Then both partitions $\mathcal{P} = \{Z\}$ and $\mathcal{P} = \{\{\beta\}\} \mid \beta \in Z\}$, that is, the partition with only one partition set and the partition in which all the partition sets are singletons, are called *trivial partitions*.) Therefore, since the definition of ς for $\mathcal{P} = \{P_0, P_1, \dots, P_l\}$ containing an e.s. P_0 is $\varsigma(\mathcal{P}) = \varsigma(\widetilde{\mathcal{P}})$, where $\widetilde{\mathcal{P}} = \{P_1, \dots, P_l\} \cup \{\{\alpha\}\} \mid \alpha \in P_0\}$, by part (1) of the next to last lemma, we have:

$$\begin{aligned}
\varsigma(\mathcal{P}) &= \sum_{1 \leq i < j} \varsigma(\mathcal{P}_i, \mathcal{P}_j) + \sum_{1 \leq i} \varsigma(\mathcal{P}_0, \mathcal{P}_i) + \sum_{0 \leq i} \varsigma(\mathcal{P}_i) \\
&\geq \sum_{1 \leq i < j} \varsigma(\mathcal{P}_{ij}, \mathcal{P}_{ji}) + \sum_{1 \leq i} \varsigma(\mathcal{P}_0, \{P_i\}) + \varsigma(\mathcal{P}_0) \\
&\geq \sum_{1 \leq i < j} \varsigma(P_i, P_j) + \sigma\iota^2 \frac{\sigma^4 p^2}{n^2} + \sum_{1 \leq i} \varsigma(\mathcal{P}_0, \{P_i\}) + \varsigma(\mathcal{P}_0) \\
&= \varsigma(\mathcal{P}) + \sigma^5 (\frac{\iota p}{n})^2 \geq \varsigma(\mathcal{P}) + \sigma^5/2.
\end{aligned}$$

(The last inequality is true, because $|P_0| \leq \sigma n \leq \frac{1}{4}n$ implies that $\iota p \geq \frac{3}{4}n$.)

To transform $\mathcal{P}$ into the desired partition $\mathcal{O}'$, the only remaining thing is to cut up the partition sets of $\mathcal{P}$ into sets of same fixed common size; and this fixed common size should be small enough so that the residual vertices can be put into the e.s. without making it too big.

Case 1. If $p < 4^\iota$, then let $P_0' = P_0$ and for each $\alpha \in V \setminus P_0$, the singletons $\{\alpha_i\}$ be P_i' for $i = 1, \dots, p$. Then, $l = \iota p$, and $\iota \leq l = \iota p < \iota 4^\iota$. Thus, $\iota \leq l \leq \iota 4^\iota$ as required.

Case 2. Now, assume that $p \geq 4^\iota$. Let $P_1', P_2', \dots, P_l'$ be a maximal family of disjoint sets with size $d = \lfloor p/4^\iota \rfloor \geq 1$ such that P_i' is subset of some $P \in \mathcal{P} \setminus \{P_0\}$ for each $i = 1, 2, \dots, l$, and let $P_0' = V \setminus \bigcup_{i=1}^{l} P_i'$. Then, $\mathcal{O}' = \{P_0', P_1', \dots, P_l'\}$ is a partition of V. Furthermore, $\widetilde{\mathcal{O}'}$ refines $\widetilde{\mathcal{P}}$. Therefore, by part (2) of the next to last lemma, we have:

$$\varsigma(\mathcal{O}') \geq \varsigma(\mathcal{P}) \geq \varsigma(\mathcal{O}) + \sigma^5/2.$$

Moreover, for each $i = 1, 2, \dots, l$, $P_i' \neq P_0'$ is subset of one of the sets $P_1, \dots, P_\iota$; but since for each i and j, $|P_i'| = d$ and $|P_j| = p$, the number of sets P_i' contained in the same P_j is less than or equal to p/d, and by the choice of d, $p/d \leq 4^{\iota+1}$:

$$\frac{p}{d} = \frac{p}{\lfloor p/4^\iota \rfloor} \leq \frac{p}{p/4^{\iota+1}} = 4^{\iota+1}.$$

(It can be shown easily by induction on p that $\frac{p}{4^\iota} - \frac{p}{4^{\iota+1}} \geq 1$ for all $p \geq 4^\iota$. Therefore, $\lfloor \frac{p}{4^\iota} \rfloor \geq \frac{p}{4^{\iota+1}}$.) Thus, $\iota \leq l \leq \iota 4^{\iota+1}$ as desired. (Each P_j contains at most $\frac{p}{d} \leq 4^{\iota+1}$ P_i' for each $j = 1, 2, \dots, \iota$. Therefore, $l \leq \iota \frac{p}{d} \leq \iota 4^{\iota+1}$.)

Finally, the partition sets $P_1', P_2', \dots, P_l'$ do not use at most d (in fact, at most $d - 1$, because $P_1', P_2', \dots, P_l'$ is a maximal family of disjoint sets with size d such that every P_i' is a subset of some $P \in \mathcal{P} \setminus P_0$) vertices from every partition set $P \neq P_0$ of $\mathcal{P}$. Therefore, we have:

$$|P_0'| \leq |P_0| + d|\mathcal{P}| \leq |P_0| + \frac{p}{4^\iota}\iota 2^\iota = |P_0| + \frac{p\iota}{2^\iota} \leq |P_0| + \frac{n}{2^\iota}.$$

■

Now, the proof of the Regularity Lemma can done by applying the previous lemma recurrently.

Proof of the Regularity Lemma. Let $\sigma > 0$ be a given positive real number, and $\upsilon \geq 1$ be a given natural number. Wlog, we can assume that $\sigma \leq 1/4$. Let $t = 2/\sigma^5$. Note that t is an upper bound for the iteration number of the previous lemma, that is, a partition turns into a σ-r.p. after applying the previous lemma at most t-times. (Note that for any partition $\mathcal{P}$, $\varsigma(\mathcal{P}) \leq 1$.)

To apply the previous lemma to a partition $\mathcal{P} = \{P_0, P_1, \ldots, P_l\}$ with $|P_1| = \ldots = |P_l|$, $|P_0|$ must be less than or equal to σn. However, with each application of the lemma, this size can increase by up to the value $n/2^\iota$. (More explicitly, $|P_0|$ can increase by up to the value $n/2^\iota$, where ι is the number of sets other than the e.s. in the present partition. But we know from the previous lemma that $l \geq \iota$. Therefore, $n/2^\iota$ is an upper bound for the increment.) Therefore, we want the starting value of ι to be large enough so that even if the lemma is applied t times, the total increment of $|P_0|$ is at most $\frac{1}{2}\sigma n$; and we also want the number of vertices n to be large enough to guarantee that for any starting value of $|P_0| < \iota$, $|P_0| \leq \frac{1}{2}\sigma n$. (If the starting partition $\mathcal{P}$ consists of ι non-exceptional partition sets $P_1, P_2, \ldots, P_\iota$, then to satisfy the condition $|P_1| = \ldots = |P_\iota|$, the initial size of P_0 should be allowed up to the value ι.)

Therefore, let $\iota \geq \upsilon$ be such that $2^{\iota-1} \geq \frac{t}{\sigma}$. Then $\frac{t}{2^\iota} \leq \frac{\sigma}{2}$, and so, if $\frac{\iota}{n} \leq \frac{\sigma}{2}$, i.e., if $n \geq \frac{2\iota}{\sigma}$, we have:

$$\iota + \frac{t}{2^\iota}n \leq \iota + \frac{\sigma}{2}n \leq \frac{\sigma}{2}n + \frac{\sigma}{2}n = \sigma n. \qquad (*)$$

Now, let us choose Υ. We know that Υ is an upper bound for the number of non-exceptional partition sets in a σ-r.p. obtained at most t applications of the previous lemma. In each application, the number of non-exceptional partition sets can increase from its present value s up to $s4^{s+1}$. Therefore, let $\psi : \mathbb{N} \longrightarrow \mathbb{N}$ be the function defined by $y \longrightarrow y4^{y+1}$, and choose $\Upsilon = \max\{\psi^t(\iota), \frac{2\iota}{\sigma}\}$. (Note that the second term in the maximum guarantees that any $n \geq \Upsilon$ is sufficiently large to satisfy inequality $(*)$.)

Finally, we must prove that every graph $\Gamma = (V, E)$ of order greater than or equal to υ has a partition $\{P_0, P_1, \ldots, P_l\}$ such that it is σ-regular and $\upsilon \leq l \leq \Upsilon$. Thus, let Γ be a given graph, and let n be the order of Γ, i.e., $n = |\Gamma|$.

Case 1. If $n \leq \Upsilon$, choose $P_0 = \emptyset$ and partition the graph Γ into $l = n$ sets $P_1, P_2, \ldots, P_l$ such that $|P_1| = \ldots = |P_l| = 1$. This partition is trivially σ-regular.

Case 2. Now assume that $n > \Upsilon$. Let $P_0 \subseteq V$ be a minimal vertex set such that the former ι divides $|V \setminus P_0|$, and take $l = \iota$. Then take any partition $\{P_1, P_2, \ldots, P_l\}$ of the vertex set $V \setminus P_0$ into sets such that $|P_1| = \ldots = |P_l|$. Then $|P_0| < l$ (since P_0 is minimal wrt the property that $l \mid |P \setminus P_0|$, $0 \leq |P_0| < l$), and so $|P_0| \leq \sigma n$ by the inequality $(*)$.) Now, start with $P_0, P_1, \ldots, P_l$, and apply the previous lemma repeatedly, until to obtain a σ-r.p. of the graph Γ; this will take place after at most t applications of the previous lemma. Note that in each step of the applications, the number of elements in the current e.s. is less than or equal to σn by the inequality $(*)$. Therefore, the lemma is certainly be applied up to t times which we know is the maximum value of the number of times the previous lemma should apply to obtain an σ-r.p. of the graph Γ. ■

2.5 Applying the Regularity Lemma

The aim of this section is to demonstrate the usage of the Regularity Lemma in the extremal graph theory.

Let Γ and Π be given graphs, and suppose that there exists a σ-r.p. of the graph Γ. Now, assume that we want to show that a specific edge density of Γ is enough to guarantee the existence of an isomorphic copy of Π in Γ. First of all, for almost all the partition sets pairs (P_i, P_j), the edges between the pairs are distributed quite uniformly; however, the density of these edges may depend upon the pair. But this density can not be so small for lots of pairs, that is, sizable ratio of the pairs have at least a specific positive density, because Γ contains many edges. Furthermore, if Γ is large, then the pairs are also large, because the number of partite sets is bounded, and each partite set have the same number of vertices. But any sufficiently large bipartite graph with partite sets of equal size, with a specific positive edge density, and with a uniform edge distribution contains an isomorphic copy of any given bipartite graph; below, we will explain this precisely. Expressing Π as a union of its bipartite subgraphs, for example, consider a proper vertex coloring of Π, and take the corresponding bipartite subgraph induced by pairs of color classes, then we obtain an isomorphic copy of Π in Γ as required.

This approach is formalized by the lemma after the next lemma. Then, by using that lemma and the Regularity Lemma, we give a proof of the Erdös-Stone theorem stated in the first section of this chapter.

First, we state and prove a simple consequence of being σ-regular for a pair (Y, Z). Let $B \subseteq Z$, and suppose that B is not so small. Then, most of the vertices in Y have more or less the expected number of neighbors in B:

Lemma 2.7 *Let (Y, Z) be a σ-regular pair, and let $B \subseteq Z$ be such that $|B| \geq \sigma|Z|$. Suppose that the density of (Y, Z) is d. Then, except for at most $\sigma|Y|$ vertices of Y, the number of neighbors in B of (almost) each of the vertices of Y is at least $(d - \sigma)|B|$.*

Proof. Let $A \subseteq Z$ be the vertex set such that each vertex $\mu \in A$ have less than $(d-\sigma)|B|$ neighbors in B. Then, the number of $A - B$ edges satisfies the following inequalities:

$$\|A, B\| < |A|(d - \sigma)|B|.$$

Therefore, we have:

$$d(A, B) = \frac{\|A, B\|}{|A||B|} < d - \sigma = d(Y, Z) - \sigma \implies d(Y, Z) - d(A, B) > \sigma$$

$$\implies |d(A, B) - d(Y, Z)| > \sigma$$

Therefore, since (Y, Z) is σ-regular and $|B| \geq \sigma|Z|$, $|A| < \sigma|Y|$. (By the definition of σ-regularity, for each pair $A \subseteq Z$ and $B \subseteq Y$ with $|A| \geq \sigma|Y|$ and $|B| \geq \sigma|Z|$, we have: $|d(A, B) - d(Y, Z)| \leq \sigma$. Therefore, since $|B| \geq \sigma|Y|$, $|A|$ must be less than $\sigma|Z|$.) ■

Definition 2.10 Let Γ be a graph, and let $\{P_0, P_1, \ldots, P_\iota\}$ be an σ-r.p. of V, where P_0 is an exceptionnal set and $|P_1| = |P_2| = \ldots = |P_\iota| = l$. Let $d \in [0, 1]$ be a given fixed real number, and let $\mathcal{R}$ be the graph on $\{P_1, \ldots, P_\iota\}$ such that there exists an edge between the vertices P_i and P_j iff P_i and P_j form a σ-regular pair in Γ with density $\geq d$. Then $\mathcal{R}$ is called a *regularity graph* of Γ, and its parameters are σ, l, d.

Now, let r be a given natural number, and replace each vertex set P_i of the graph $\mathcal{R}$ by a vertex set, denoted by P_i^r, containing r vertices; and replace each edge between P_i and

P_j by a complete bipartite graph between P_i^r and P_j^r. Denote the resulting graph by $\mathcal{R}_r$. (For example, if $\mathcal{R} = \mathbb{K}^s$, then $\mathcal{R}_r = \mathbb{K}_r^s$.)

The next lemma states that if the density $d > 0$, σ is sufficiently small, and P_i s are sufficiently large, then we can also find subgraphs of the graph $\mathcal{R}_r$ in the graph Γ. In fact, the required values of σ and l depend upon only (the density d and) the maximum degree of the considered subgraph:

Lemma 2.8 *Let $d \in (0, 1]$ be a given real number and $\Delta \geq 1$ be a given natural number. Then, there is an $\sigma_0 > 0$ satisfying the following property: Let Γ and Π be any two given graphs such that $\Delta(\Pi) \leq \Delta$, and let r be a given natural number. Also let $\mathcal{R}$ be a regularity graph of the graph Γ such that the parameters of $\mathcal{R}$ satisfy $\sigma \leq \sigma_0$, $l \geq \frac{2s}{d^\Delta}$ and d. Then we have:*

$$\Pi \subseteq \mathcal{R}_r \Longrightarrow \Pi \subseteq \Gamma.$$

Now, we have ready to show the Erdös-Stone theorem. First, let us state the theorem once again:

Theorem 2.16 (Erdös and Stone 1946) *Let $s \geq 2$ and $t \geq 1$ be integers, and let $\sigma > 0$ be a positive real number. Then there is a natual number n_0 such that every graph Γ with the number of vertices $n \geq n_0$ and the number of edges*

$$\|\Gamma\| \geq \theta_{s-1}(n) + \sigma n^2$$

contains a copy of $\mathbb{K}_t^s = \mathbb{K}_{t,t,\ldots,t} = \overline{\mathbb{K}_t} * \overline{\mathbb{K}_t} * \ldots * \overline{\mathbb{K}_t}$ (s copies of $\overline{\mathbb{K}_t}$).

Proof of Erdös-Stone Theorem. First, for $t = 1$, the statement is true by Turan's Theorem. Therefore, we suppose that $t \geq 2$.
Let $\rho > 0$ be a given real number; ρ plays the role of σ given in the statement of the theorem. Let Γ be a graph with order $|\Gamma| = n$. Then if Γ contains

$$\|\Gamma\| \geq \theta_{s-1}(n) + \rho n^2$$

edges, then $\rho < 1$. We want to prove that Γ contains $\mathbb{K}_t^s$ if n is sufficiently large.

To prove the theorem, our strategy is the following: First, we use the Regularity Lemma to prove that Γ contains a regularity graph $\mathcal{R}$ such that $\mathcal{R}$ is sufficiently dense to have a $\mathbb{K}^s$ by Turan's theorem. Second, since $\mathcal{R}_r$ has a $\mathbb{K}_r^s$, we may use the previous lemma to conclude that Γ does contain a $\mathbb{K}_t^s$.

Step 1. [By using the Regularity Lemma, we prove that Γ contains a sufficiently dense regularity graph $\mathcal{R}$.] For $d = \rho$ and $\Delta = \Delta(\mathbb{K}_t^s)$, let σ_0 be a corresponding positive real number determined by the previous lemma. To use the Regularity Lemma, let $\upsilon > \dfrac{1}{\rho}$ be a natural number, and choose $\sigma > 0$ sufficiently small such that $\sigma \leq \sigma_0$,

$$\sigma < \frac{\rho}{2} < 1,$$

and

$$\zeta = 2\rho - \sigma^2 - 4\sigma - d - \frac{1}{\upsilon} > 0.$$

Note that this is possible, because $2\rho - d - \dfrac{1}{\upsilon} > 0$. ($2\rho - d - \dfrac{1}{\upsilon} = 2\rho - \rho - \dfrac{1}{\upsilon} = \rho - \dfrac{1}{\upsilon}$; and since $\upsilon > \dfrac{1}{\rho}$, $\dfrac{1}{\upsilon} < \rho$. Also, $\sigma \leq \sigma_0$ is chosen sufficiently small so that the last inequality is satisfied and $2\rho - d - \dfrac{1}{\upsilon} - \sigma^2 - 4\sigma > 0$.)

Now, for this σ and υ, the Regularity Lemma gives us an integer Υ such that each graph of order $\geq \upsilon$ has an σ-r.p. $\{P_0, P_1, \ldots, P_\iota\}$ with $\upsilon \leq \iota \leq \Upsilon$. And suppose that

$$n \geq \frac{2\Upsilon r}{d^\Delta(1-\sigma)}.$$

Then note that $\dfrac{2\Upsilon r}{d^\Delta(1-\sigma)} \geq \upsilon$:

$$\frac{2\Upsilon r}{d^\Delta(1-\sigma)} \underset{(\rho/2<1-\sigma)}{>} \frac{2\Upsilon r}{d^\Delta \rho/2} \underset{(\rho=d)}{=} \frac{4\Upsilon r}{d^{\Delta+1}} \underset{(d=\rho<1)}{>} 4\Upsilon r \underset{(r\geq 2)}{>} 4\Upsilon \geq 4\upsilon \geq \upsilon.$$

Therefore, since $n \geq \upsilon$, the Regularity Lemma ensures the existence of a σ-r.p. $\{P_0, P_1, \ldots, P_\iota\}$ of the graph Γ (we know that $\upsilon \leq \iota \leq \Upsilon$). Let $|P_1| = |P_2| = \ldots = |P_\iota| = l$. Then, we have :

$$n \geq \iota l \qquad [n = |V| = |P_0| + \iota l]$$

and

$$l = \frac{n - |P_0|}{\iota} \underset{(|P_0|\leq\sigma|V|)}{\geq} \frac{n - \sigma n}{\Upsilon} = n\frac{1-\sigma}{\Upsilon} \underset{(n\geq\frac{2\Upsilon r}{d^\Delta(1-\sigma)})}{\geq} \frac{2r}{d^\Delta}.$$

Now, let $\mathcal{R}$ be the regularity graph of Γ such that the parameters of $\mathcal{R}$ are σ, l, d corresponding to the partition $\{P_0, P_1 \ldots, P_\iota\}$ of Γ. And if $\mathbb{K}^s \subseteq \mathcal{R}$, then $\mathbb{K}^s_r \subseteq \mathcal{R}_r$. Therefore, by the previous lemma, if $\mathbb{K}^s \subseteq \mathcal{R}$, then $\mathbb{K}^s_t \subseteq \Gamma$ as required.

Step 2. [By using Turan's Theorem, we prove the following: The graph $\mathcal{R}$ found in the previous step contains an isomorphic copy of $\mathbb{K}^s$.] Since we want to prove that $\mathbb{K}^s \subseteq \mathcal{R}$ by using Turan's Theorem, we must prove that $\mathcal{R}$ contains sufficient number of edges, that is, that sufficient number of σ-regular pairs (P_i, P_j) have density $\geq d$. This should be concluded from the hypothesis that the number of edges of $\mathcal{R}$ is $\geq \theta_{s-1}(n) + \rho n^2$, that is, an edge density of around $\dfrac{s-2}{s-1} + 2\rho$:

The edge density of $\Gamma = \dfrac{\|\Gamma\|}{\binom{|\Gamma|}{2}} \geq \dfrac{\theta_{s-1}(n) + \rho n^2}{n(n-1)/2} \approx \dfrac{\frac{1}{2}n^2\frac{s-2}{s-1} + \rho n^2}{\frac{n^2}{2}} = \dfrac{s-2}{s-1} + 2\rho.$

But note that an edge density of around $\dfrac{s-2}{s-1} + 2\rho$ lies considerably above the edge density of the Turan graph $\Theta^{s-1}(\iota)$, which is approximately $\dfrac{s-2}{s-1}\cdot$ (The edge density of $\Theta^{s-1}(\iota) = \dfrac{\|\Theta^{s-1}(\iota)\|}{\binom{|\Theta^{s-1}(\iota)|}{2}} = \dfrac{\theta_{s-1}(n)}{\binom{|\Theta^{s-1}(\iota)|}{2}} \leq \dfrac{\frac{1}{2}\frac{s-2}{s-1}}{n(n-1)/2}\cdot$) And therefore, considerably above any density that Γ can arise out of $\theta_{s-1}(\iota)$ dense pairs alone, even if each of these pairs does have density 1.

Now, let us estimate $\|\mathcal{R}\|$ more exactly.

How many edges of Γ does not lie inside any σ-regular pairs?

First of all, the number of edges lying inside the e.s. P_0 is $\leq \binom{|P_0|}{2}$, and by condition (1) given in the definition of a σ-r.p. of Γ, this number is $\leq \dfrac{1}{2}(\sigma n)^2$:

By mentioned condition (1), $|P_0| \leq \sigma|V| = \sigma n$. Therefore, since this number $\leq \binom{|P_0|}{2}$, it is $\leq \dfrac{|P_0|(|P_0|-1)}{2} \leq \dfrac{|P_0||P_0|}{2} \leq \dfrac{\sigma n \sigma n}{2} = \dfrac{1}{2}(\sigma n)^2$.

Secondly, the number of edges between P_0 and other partition sets is $\leq |V_0|\iota l \leq \sigma|V|\iota l = \sigma n(\iota l) \leq \sigma n n = \sigma n^2$.

Thirdly, by condition (3) in the definition of a σ-r.p. of Γ, there are at most $\sigma\iota^2$ pairs (P_i, P_j) which are not σ-regular, and the number of edges between each such pairs is $\leq l^2$. Therefore, the total number of such edges is $\leq \sigma\iota^2 l^2$.

Fourtly, the number of edges between the σ-regular pairs having inadequate density, that is, having density $< d$, is $\leq dl^2$. Therefore, the total number of such edges is $\leq \binom{\iota}{2} dl^2 = \dfrac{\iota(\iota-1)}{2} dl^2 \leq \dfrac{1}{2}\iota^2 dl^2$.

Finally, the number of edges inside each of the partition sets is $\leq \binom{l}{2} = \dfrac{l(l-1)}{2} \leq \dfrac{l.l}{2} = \dfrac{l^2}{2}$. Therefore, the total number of such edges is $\leq \iota\binom{l}{2} \leq \dfrac{1}{2}l^2\iota$.
Each of the other edges of Γ lies between σ-regular pairs with density $\geq d$, and therefore, each such edge adds an edge to the regularity graph $\mathcal{R}$ of Γ. And for each edge of $\mathcal{R}$, the number of corresponding edges in Γ is $\leq l^2$. Therefore, the number of such corresponding edges in Γ is $\leq \|\mathcal{R}\|l^2$.
As a result, the total number of edges in Γ is :

$$\|\Gamma\| \leq \frac{1}{2}\sigma^2n^2 + \sigma n^2 + \sigma\iota^2l^2 + \frac{1}{2}\iota^2dl^2 + \frac{1}{2}l^2\iota + \|\mathcal{R}\|l^2.$$

Therefore, for all large enough n values, we have :

$$\begin{aligned}
\|\mathcal{R}\| &\geq \frac{1}{2}\iota^2\left(\frac{\|\Gamma\| - \frac{1}{2}\sigma^2n^2 - \sigma n^2 - \sigma\iota^2l^2 - \frac{1}{2}d\iota^2l^2 - \frac{1}{2}\iota l^2}{\frac{1}{2}\iota^2l^2}\right)\\
&= \frac{1}{2}\iota^2\left(\frac{\|\Gamma\| - \frac{1}{2}\sigma^2n^2 - \sigma n^2}{\frac{1}{2}\iota^2l^2} - 2\sigma - d - \frac{1}{\iota}\right)\\
&\underset{(\|\Gamma\|\geq\theta_{s-1}(n)+\rho n^2)}{\geq} \frac{1}{2}\iota^2\left(\frac{\theta_{s-1}(n) + \rho n^2 - \frac{1}{2}\sigma^2n^2 - \sigma n^2}{\frac{1}{2}(\iota l)^2} - 2\sigma - d - \frac{1}{\iota}\right)\\
&\underset{(n\geq\iota l)}{\geq} \frac{1}{2}\iota^2\left(\frac{\theta_{s-1}(n) + \rho n^2 - \frac{1}{2}\sigma^2n^2 - \sigma n^2}{\frac{1}{2}n^2} - 2\sigma - d - \frac{1}{\iota}\right)\\
&= \frac{1}{2}\iota^2\left(\frac{\theta_{s-1}(n)}{n^2/2} + 2\rho - \sigma^2 - 2\sigma - 2\sigma - d - \frac{1}{\iota}\right)\\
&\underset{(\iota\geq m)}{\geq} \frac{1}{2}\iota^2\left(\frac{\theta_{s-1}(n)}{n^2/2} + 2\rho - \sigma^2 - 4\sigma - d - \frac{1}{m}\right)\\
&= \frac{1}{2}\iota^2\left(\theta_{s-1}(n)\frac{2}{n^2} + \zeta\right)\left[\frac{2}{n^2} = \frac{2}{n(n-1)}\frac{n-1}{n} = \binom{n}{2}^{-1}\left(1-\frac{1}{n}\right)\right]\\
&= \frac{1}{2}\iota^2\left(\theta_{s-1}(n)\binom{n}{2}^{-1}\left(1-\frac{1}{n}\right) + \zeta\right)\\
&> \frac{1}{2}\iota^2\left(\frac{s-2}{s-1} + \frac{\zeta}{2}\right) > \frac{1}{2}\iota^2\,\frac{s-2}{s-1} \geq \theta_{s-1}(\iota).
\end{aligned}$$

[The first inequality of the last line is due to the fact $\lim_{n\to\infty}\theta_{s-1}(n)\binom{n}{2}^{-1} = \frac{s-2}{s-1}$: It is obvious that $\lim_{n\to\infty}\theta_{s-1}(n)\binom{n}{2}^{-1}(1-\frac{1}{n}) = \frac{s-2}{s-1}$. $\therefore \mid \theta_{s-1}(n)\binom{n}{2}^{-1}(1-\frac{1}{n}) - \frac{s-2}{s-1} \mid < \frac{\zeta}{2}$. $\therefore\ \theta_{s-1}(n)\binom{n}{2}^{-1}(1-\frac{1}{n}) > \frac{s-2}{s-1} - \frac{\zeta}{2}$.]
As a result, as required, $\mathbb{K}^s \subseteq \mathcal{R}$ by Turan's Theorem. ■

3. LITERATURE REVIEW

The classical textbook for extremal graph theory is Bollobas (2004). A survey about this area is given by Simonovits (2013).

Turan's theorem is the most influential result in extremal graph theory. The original proof of this theorem is fundamentally the first proof given in the thesis.

Theorem 2.7 was first demonstrated by Bollobas and Thomason (1998), and severally by Komlos and Szemeredi (1996).

For more information about Theorem 2.9, Mader (1998) can be consulted. The girth assumption of 8k+3 has been cut down to about 4k by Kühn and Osthus (2003), and it is conjectured that this lower bound is the best possible.

Hadwiger (1943) has demonstrated his conjecture for the $r = 4$ case. Proof of the $r = 5$ case is given in Bollobas (2004), where this case is proved as a corollary of Wagner's Theorem. Robertson *et al.* (1993) proved the $r = 6$ case.

Thomassen (2007) proved Hajos's conjecture for line graphs and for graphs of large girth. It was demonstrated by Erdös and Fajtlowicz that Hajos's conjecture is false for 'almost all' graphs, on the other hand It was demonstrated by Bollobas, Catlin and Erdös that Hadwiger's conjecture is true for 'almost all graphs'.

Szemeredi (1976) proved the regularity lemma. There is a survey about this famous lemma and its applications in Komlos and Simonovits (1996).

4. CONCLUSIONS AND RECOMMENDATION

The extremal graph theory is one of the important and fundamental topics in graph theory, and there are still lots of open problems related to this area. Thus, it is one of the active research areas in mathematics.

In this thesis, since we introduced the basic notions and theorems about the extremal graph theory, and since we surveyed some of the recent results about this area; on one side this thesis is an introduction source related to this area, but on the other side it presents some of the recent results and developments in this area in a very compact form.

The purpose of this thesis is to introduce this rich and active research area at a basic level, and our purpose is not to solve an open problem in this area or not to obtain an original result about this area. But we aim to present a survey or a review/tutorial source about the extremal graph theory for non-experts in this area. Therefore, we hope this thesis will be useful for students and researchers who want to learn the basics of this influential area of mathematics.

REFERENCES

Bollobas, B. 2004. Extremal Graph Theory. Dover Publications, 512 pages, New York.

Bollobas, B. and Thomason, A.G. 1998. Proof of a conjecture of Mader, Erdös and Hajnal on topological complete subgraphs. Eur. J. Comb., 19: 883–887.

Diestel, R. 2017. Graph Theory, Fifth Edition (Graduate Texts in Mathematics, 173). Springer, 448 pages, Berlin.

Erdös, P. and Stone, A. H. 1946. On the structure of linear graphs. Bulletin of the American Mathematical Society, 52 (12): 1087–1091.

Hadwiger, H. 1943. Über eine Klassifikation der Streckenkomplexe. Vierteljschr. Naturforsch. Ges. Zürich, 88: 133–143.

Hajós, G. 1961. Über eine Konstruktion nicht n-färbbarer Graphen. Wiss. Z. Martin Luther Univ. Halle-Wittenberg Math.-Natur. Reihe, 10: 116–117.

Komlos, J. and Simonovits, M. 1996. Paul Erdös is 80, Vol. 2 (Miklos, D., Sos, V.T. and Szönyi, T. eds.). Proc. Colloq. Math. Soc. Janos Bolyai, 516 pages, Budapest.

Komlos, J. and Szemeredi, E. 1996. Topological cliques in graphs II. Comb. Probab. Comput., 5: 79–90.

Kostochka, A. V. 1982 The minimum Hadwiger number for graphs with a given mean degree of vertices. Metody Diskret. Analiz., 38: 37-58.

Kühn, D. and Osthus, D. 2002. Topological Minorsin Graphs of Large Girth. J. Comb. Theory Ser. B, 86: 364–380.

Kühn, D. and Osthus, D. 2003. Minors in graphs of large girth. Random Struct. Alg., 22: 213–225.

Kühn, D. and Osthus, D. 2005. Complete minors in $K_{s,s}-$free graphs, Combinatorica, 25: 49–64.

Mader, W. 1972. Hinreichende Bedingungen fiir die Existenz von Teilgraphen, die zu einem vollstiindigen Graphen homiiomorph sind. Math. Nachr. 53: 145-150.

Mader, W. 1998. Topological subgraphs in graphs of large girth. Combinatorica, 18: 405–412.

Robertson, N., Seymour, P.D. and Thomas R. 1993. Hadwiger's conjecture for K_6-free graphs. Combinatorica, 13: 279–361.

Simonovits, M. 2013. The Mathematics of Paul Erdös 2 (2nd edition) (Graham, R.L., Nesetril, J. and Butler, S. eds.). Springer, 626 pages, Berlin.

Szemeredi, E. 1976. Regular partitions of graphs. Colloques Internationaux C.N.R.S. No: 260 - Problemes Combinatoires et Theorie des Graphes, Orsay, 399–401.

Thomas, R. and Wollan, P. 2005. An improved linear edge bound for graph linkages. European Journal of Combinatorics, 26 (3–4): 309-324.

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