CONSERVED CHARGES IN VARIOUS THEORIES OF GRAVITY

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

HIKMET ÖZŞAHIN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
PHYSICS

SEPTEMBER 2022

Approval of the thesis:

CONSERVED CHARGES IN VARIOUS THEORIES OF GRAVITY

submitted by **HIKMET ÖZŞAHIN** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Physics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** ______________

Prof. Dr. Seçkin Kürkçüoğlu
Head of Department, **Physics** ______________

Prof. Dr. Bayram Tekin
Supervisor, **Physics Department, METU** ______________

Examining Committee Members:

Prof. Dr. Atalay Karasu
Physics Department, METU ______________

Prof. Dr. Bayram Tekin
Physics Department, METU ______________

Prof. Dr. Ali Murat Güler
Physics Department, METU ______________

Prof. Dr. Tahsin Çağrı Şişman
Astronautical Engineering, UTAA ______________

Assoc. Prof. Dr. Suat Dengiz
Mechanical Engineering, UTAA ______________

Date: 02.09.2022

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Surname: Hikmet Özşahin

Signature :

ABSTRACT

CONSERVED CHARGES IN VARIOUS THEORIES OF GRAVITY

Özşahin, Hikmet
Ph.D., Department of Physics
Supervisor: Prof. Dr. Bayram Tekin

SEPTEMBER 2022, 102 pages

The first law of black hole thermodynamics in the presence of a cosmological constant Λ can be generalized by introducing a term containing the variation $\delta\Lambda$. Similar to other terms in the first law, which are variations of some conserved charges like mass, entropy, angular momentum, electric charge etc and it has been shown in [1] that the new term has the same structure: Λ is a conserved charge associated with a gauge symmetry. In this work, first we propose and prove the generalized Smarr relation in this new paradigm. Then, we reproduce systematically the “effective volume" of a black hole which has been introduced before in the literature as the conjugate of pressure. Our construction removes the ambiguity in the definition of volume. Finally, we apply and investigate this formulation of “Λ as a charge" on a number of solutions to different models of gravity for different spacetime dimensions. Specially, we investigate the applicability and validity of the analysis for black branes, whose enclosed volume is not well-defined in principle

Keywords: Black Holes, thermodynamics of black holes, covariant phase space formalism, Wald entropy

ÖZ

ÇEŞİTLİ KÜTLEÇEKİM TEORİLERİNDE KORUNUMLU YÜKLER

Özşahin, Hikmet
Doktora, Fizik Bölümü
Tez Yöneticisi: Prof. Dr. Bayram Tekin

Eylül 2022 , 102 sayfa

Kara delik termodinamiğinin birinci yasası, kozmolojik sabit Λ'nın mevcut olduğu durumlarda yeni bir $\delta\Lambda$ terimi ile genelleştirilebilir. [1]'de gösterildiği üzere kara delik termodinamiğinin birinci yasasındaki bu yeni terim, diğer terimlerde olduğu gibi korunumlu bir yükün varyasyonu şeklinde yazılır. Bu formulasyonda Λ ayar simetrilerine karşılık gelen korunumlu yük iken, birinci yasadaki rolü elektrik kuvvetiyle benzerdir. Bu çalışmada, öncelikle genelleştirilmiş Smarr eşitliğini ispatladık ve litatürde daha önceden tanımlanmış olan "efektif hacim"i yeniden çıkardık. Bu şekilde yapıldığında hacimin tanımındaki belirsizliklerden kurtulduğunu gösterdik. Son olarak, kozmolojik sabiti korunumlu yük olarak tanımlayan bu formulasyonu çeşitli kütleçekim teorilerinin çözümlerine uyguladık.

Anahtar Kelimeler: Karadelikler, karadelik termodinamiği, kovaryant faz uzayı formalizmi, Wald entropisi

To my family

ACKNOWLEDGMENTS

First and foremost, i would like to thank my supervisor Prof. Dr. Bayram Tekin. Without his guidance this dissertation and works it relies on would be impossible to finish. I was not one of his students form the first day of my doctoral program but still, i have learned so much from his during this brief yet intense period under his guidance.

Special thanks to my friends Dr. Gökhan Alkaç, Dr. Merve Demirtaş, Dr. Mustafa Tek, Dr. Deniz Özen, Dr. Deniz Devecioğlu and Özcan Arslan. Their friendship is what gave me the perseverance i needed to push through all the hardship i have endured.

I would like to reserve my final words to my lovely wife, Seyhan Çelik Özşahin, whose continuous support and encouragement helped me during all my academic career. Knowing how hard of a person can i be, i am grateful for her support and patience.

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

2D	2 Dimensional
3D	3 Dimensional
GR	General Relativity
$g_{\mu\nu}$	Generic Metric
g	Metric Determinant
$d\Omega_d$	Line element on d dimensional unit sphere
∂_μ	Partial derivative
∇_μ	Metric compatible covariant derivative
$\mathcal{L}$	Lagrangian density

CHAPTER 1

INTRODUCTION

Einstein introduced the cosmological constant Λ in order to explain, within General Relativity, the "apparent" staticity nature of the universe [2]. But, after the discovery of the expansion of the universe in the late 1920s, the idea of a static universe was essentially put to rest together with the cosmological constant by the majority of the researchers in the field. However, in a rather ironic piece of scientific history, Λ has taken center stage in cosmology since the discovery of accelerating expansion of the universe [3, 4] and the AdS/CFT correspondence [5, 7]. In the Einstein's method of introducing cosmological constant, Λ is considered as a constant parameter in the Lagrangian, *i.e.* as a part of the definition of the theory. Mathematically, denoting the Lagrangian by $\mathcal{L}$, Λ is incorporated in the Lagrangian via the shift $\mathcal{L} \to \mathcal{L} - \frac{\Lambda}{8\pi G}$ where G is the Newton constant. Alternatively, in a less known route, one can introduce a new gauge field in the Lagrangian [8, 9] in which makes Λ to be a free parameter in the solution. This approach was introduced in early 80s, and was studied in more details in a series of papers by M. Henneaux and C. Teitelboim [10, 11, 12, 14] who studied Hamiltonian dynamics of this new gauge field and identified its canonical variables (canonical field and its momentum conjugate), and constants of integration. Continuing this research-line, it is shown that not only Λ is a constant of integration in the solution, but also (its square-root) is a conserved charge (denoted by C) associated with the global part of the gauge symmetry of this gauge field [1]. In addition, its conjugate chemical potential associated with a black hole horizon (denoted by Θ_H) was introduced for the first time. This formulation brings a new perspective to Λ: it becomes a conserved charge as a property of the solution; and can naturally contribute to the first law of black hole thermodynamics just like other conserved charges. This can also be considered as a continuation of the seminal work by R.M. Wald [15, 16] who recognized the entropy in the black hole thermodynamics is a conserved charge. Consequently, this approach resolves some conceptual, physical and mathematical issues in regard to the generalization of first law with variation of cosmological constant and the issues with the Smarr formula. We will come back and summarize these issues later in this section.

For the sake of completeness, in what follows, we briefly review the "Λ as a conserved charge" approach [1]. We shall use the following conventions: $[\mu_1\mu_2\dots\mu_p]$ will be used to denote anti-symmetrization over the set of indices within the bracket normalized by the factor $\frac{1}{p!}$. The exterior derivative of a p-form $\boldsymbol{a} = \frac{1}{p!}a_{\rho_1\dots\rho_p}dx^{\rho_1}\wedge\dots\wedge dx^{\rho_p}$ is defined as

$$d\boldsymbol{a} \equiv (p+1)\,\partial_{[\mu_1}a_{\mu_2\dots\mu_{p+1}]}\,dx^{\mu_1}\wedge\dots\wedge dx^{\mu_{p+1}}.$$

Considering a gravitational theory described by a Lagrangian $\mathcal{L}$ without cosmological constant in D dimensional spacetime, the action and gravitational equation of motion can be represented as

$$I = \int d^Dx\sqrt{-g}\mathcal{L}, \qquad\qquad E_{\mu\nu} \equiv \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} = 0, \tag{1.1}$$

in which $\delta g^{\mu\nu}$ is variations of the inverse metric. In order to introduce a cosmological constant, one can add a gauge field Lagrangian (a term similar to the electromagnetic Lagrangian) to the gravity sector as

$$\mathcal{L} \to \mathcal{L} \mp \frac{1}{8\pi G}F^2 \quad \Rightarrow \quad I = \int d^Dx\sqrt{-g}\Big(\mathcal{L} \mp \frac{1}{8\pi G}F^2\Big), \tag{1.2}$$

where $F^2 \equiv \frac{1}{D!}F_{\mu_1\dots\mu_D}F^{\mu_1\dots\mu_D}$. F is a top-form (i.e. having D antisymmetric indices), and is the field strength of a gauge field $F = dA$, i.e.

$$\frac{1}{D!}F_{\mu_1\dots\mu_D} = \partial_{[\mu_1}A_{\mu_2\dots\mu_D]}. \tag{1.3}$$

We note that the new term in the Lagrangian (1.2) is quite similar to Maxwell Lagrangian, and the only difference is that A and F have $D-1$ and D indices (instead of 1 and 2 indices) respectively. In general, the top-form F can be an arbitrary scalar function times the volume form, i.e. $F_{\mu_1\dots\mu_D} = \phi(x^\mu)\sqrt{-g}\epsilon_{\mu_1\dots\mu_D}$, with the convention $\epsilon_{01\dots D-1} = +1$ for the Levi-Civita tensor density. In another words, the most generic F is Hodge dual to a scalar field ϕ. Variation of the action (1.2) with respect to $g_{\mu\nu}$ and F,one finds the following two field equations:

$$E_{\mu\nu} = \frac{\pm 1}{8\pi G(D-1)!}\Big(F_{\mu\rho_2\ldots\rho_D}F_{\nu}{}^{\rho_2\ldots\rho_D} - \frac{(D-1)!}{2}F^2 g_{\mu\nu}\Big), \tag{1.4}$$

$$\nabla_\mu F^{\mu\mu_2\ldots\mu_D} = 0. \tag{1.5}$$

The latter equation is easy to solve, and the result is

$$F_{\mu_1\ldots\mu_D} = c\sqrt{-g}\,\epsilon_{\mu_1\ldots\mu_D} \tag{1.6}$$

for a constant c. We assume $0 \leq c$ for later convenience; and c should not be confused with the speed of light which is set to 1. It is easy to see why (1.6) is the generic solution for the equation of motion (1.5), because in terms of the Hodge dual field $\phi(x^\mu)$, (1.5) is simply $d\phi(x^\mu) = 0$ which admits $\phi(x^\mu) = c =$ constant as its most generic solution.

The solution (1.6) can be put in the field equation (1.4) in order to reproduce the standard field equation with a cosmological constant,

$$E_{\mu\nu} + \frac{1}{16\pi G}\Lambda g_{\mu\nu} = 0, \qquad\qquad \Lambda = \pm c^2. \tag{1.7}$$

To derive the above equation, the identities $\epsilon_{\mu_1\ldots\mu_D}\epsilon^{\mu_1\ldots\mu_D} = -D!$ and $\epsilon_{\mu\rho_2\ldots\rho_D}\epsilon_{\nu}{}^{\rho_2\ldots\rho_D} = -(D-1)!g_{\mu\nu}$ have been used, in which $\epsilon^{01\ldots D-1} = -1$. This procedure of introducing Λ as a parameter of the solution (instead of a constant in the Lagrangian) can be applied in any gravity theory, *i.e.* it is independent of the $\mathcal{L}$ in the analysis above.

In a $U(1)$ gauge theory with the gauge symmetry $A_\mu \to A_\mu + \partial_\mu\lambda(x^\mu)$, the conserved charge (such as the electric charge) is associated with the global part of the symmetry $\partial_\mu\lambda = 0$. Similarly, the Lagrangian (1.2) has a gauge symmetry

$$A_{\mu_1\ldots\mu_{D-1}} \to A_{\mu_1\ldots\mu_{D-1}} + \partial_{[\mu_1}\lambda_{\mu_2\ldots\mu_{D-1}]}. \tag{1.8}$$

It was shown in [1] that the conserved charge of the global part of this symmetry $\partial_{[\mu_1}\lambda_{\mu_2\ldots\mu_{D-1}]} = 0$, which we denote it as C, is equal to

$$C = \pm\frac{\sqrt{|\Lambda|}}{4\pi G}. \tag{1.9}$$

The signs correspond to those in the Lagrangian (1.2), and are associated with de Sitter (upper sign which here is plus) and Anti de Sitter (lower sign which here is minus) sectors. These $\pm$ signs (upper/lower signs) and their correspondence with the dS and AdS sectors will continue to be valid in the rest of this work. We shall call the conserved charge as *cosmological charge* in order to distinguish C from Λ (which is called cosmological constant). Moreover, we shall call the *cosmological gauge field* and the *cosmological field strength* for $A_{\mu_1 \dots \mu_{D-1}}$ and $F_{\mu_1 \dots \mu_D}$ respectively.

Identification of C as the cosmological charge turns out to be very useful in the black hole thermodynamics. The first law of thermodynamics for an electrically charged black hole in asymptotic flat spacetimes reads as $\delta M = T_{\rm H} \delta S + \Omega_{\rm H} \delta J + \Phi_{\rm H} \delta Q$ [18], where (M, S, J, Q) are mass/energy, entropy, angular momentum and electric charge of the black hole respectively. All of these quantities, whose variations appear in the first law, are conserved charges associated with a symmetry. In addition, these quantities are all extensive thermodynamic quantities. On the other hand, $(T_{\rm H}, \Omega_{\rm H}, \Phi_{\rm H})$ are the temperature, angular velocity and the electric potential of the black hole all of which can be calculated using the metric on the black hole horizon, hence the subscript H. These quantities are all intensive quantities. Let us note that the electric potential is defined with $\Phi_{\rm H} \equiv \langle \xi_{\rm H}, A \rangle = \xi_{\rm H} \cdot A$ calculated on the horizon, in which $\xi_{\rm H}$ is horizon-generating null Killing vector field and A is the electromagnetic gauge field A_μ.

According to the analysis above, C is a new conserved charge for black hole solutions in asymptotically (A)dS spacetimes, which naturally should appear in the first law in equal footing with the other charges. This generalization has been elaborated in [1] and the modified first law reads

$$\delta M = T_{\rm H} \delta S + \Omega_{\rm H} \delta J + \Phi_{\rm H} \delta Q + \Theta_{\rm H} \delta C, \tag{1.10}$$

with

$$\Theta_{\rm H} \equiv \oint_{\rm H} \xi_{\rm H} \cdot A, \tag{1.11}$$

where A is the cosmological gauge field $A_{\mu_1 \dots \mu_{D-1}}$, and the integration is taken over the horizon which is a codimension 2 null hypersurface. In (1.11) one has $(\xi_H \cdot A)_{\mu_1 \dots \mu_{D-2}} = \xi_H^\mu A_{\mu \mu_1 \dots \mu_{D-2}}$. This definition was inspired by the definition of the electric potential Φ_H which was given above. So, it is appropriate to use the name *cosmological potential* for $\Theta_{\rm H}$.

The new term $\Theta_{\rm H} \delta C$ in the first law resolves some issues related to the volume-pressure term $V\delta P$ which has been used before in the literature [12, 14](review [19]). Let us elaborate on this.

- Θ_{H} is a property of the event horizon similar to the other horizon parameters $(T_{\text{H}}, \Omega_{\text{H}}, \Phi_{\text{H}})$, in contrast with the volume V which conceptually cannot be a property of the horizon, if it is considered to be some volume inside the black hole.
- δC is variation of a charge which is a parameter in the solution similar to (M, S, J, Q), and in contrast with δP which has been considered to be proportional to $\delta\Lambda$, i.e. variation of a parameter in Lagrangian.
- Θ_{H} and C are intensive and extensive quantities respectively, and they are on the same foot as other terms in the first law. This is in contrast with $V\delta P$ where V and P are extensive and intensive respectively.
- Noting the order of intensive and extensive quantities in $\Theta_H\delta C$, the M in the first law (1.10) would be the energy/mass, in agreement with being conserved charge associated with time translation. This resolves the problem of promoting M to be enthalpy [23, 20] (as a result of the inverse order of extensive/intensivity of $V\delta P$) which is inconsistent with M as the conserved charge of the time translation symmetry.
- The conceptual problem with the *negative* pressure for de-Sitter spacetime is resolved, because the charge C, which is conceptually and mathematically similar to the electric charge, can be positive or negative.

The layout of this thesis is as follows: In the first two chapter we discuss the covariant phase space formalism and the black hole thermodynamics. These chapters will serve as a background information on what follows. In the following chapters, we continue analysis in [1] in three aspects: Firstly, we revisit Smarr formula in the presence of the cosmological charge C. Secondly, we show that the definition of Θ_{H} in (1.11) reproduces successfully an ad-hoc (but successful) volume term introduced in Ref.[23] called effective volume. And finally, we fix a freedom/ambiguity in the definition of effective volume in the literature, which will be discussed in details, by fixing the gauge freedom in the cosmological gauge field A such that mass and other charge variations are reproduces correctly when the solution is perturbed by $\Lambda \to \Lambda + \delta\Lambda$. The rest of the paper is devoted to case study of different black hole solutions in different dimensions and theories. Using examples, we examine the reliability of the $\Theta_{\rm H}\delta C$ as a universal generalization of the first law. Besides, we enhance all black hole solutions by finding the cosmological gauge field $A_{\mu_1\dots\mu_{D-1}}$ for them, and presenting complete solutions as a reference for the interested readers. We will also see that studying these examples sheds light on the universality of the Smarr formula for all $D \geq 3$ dimensions.

CHAPTER 2

COVARIANT PHASE SPACE FORMALISM

2.1 Symplectic Structure

In this chapter, an overview of the symplectic structure in the Hamiltonian formulation will be given. We first investigate the symplectic structure in the particle mechanics and define basic concepts. Definitions of Symplectic manifold and how it is related to the phase space of mechanical systems will be studied and properties of symplectic form will be studied. [1]

2.1.1 The Symplectic Manifold

Definition: Let M be a manifold of dimension $2n$, with $n \geq 1$. A symplectic structure on M can be defined by giving a closed and non-degenerate 2-form $\omega \in \Lambda_2(M)$

$$d\omega = 0 \quad \text{(closed)},$$

$$\forall \xi \neq 0, \exists \xi : \omega(\varepsilon, \eta) \neq 0 \quad \text{(non-degenerate)}.$$

The manifold M endowed with this symplectic structure (M, w) is called a symplectic manifold.

The symplectic structure defined in this way has its use in both finding the Hamiltonian vector flow (which we will define momentarily) and providing a natural choice

[1] This section is based on [80], and we claim no novelty.

for the volume element in the symplectic manifold. If one considers n times the exterior product of the two-form w, the object obtained this way can provide a $2n$- form on M, that is $w \wedge w \cdots \equiv w^n$

Let's us recall the definition of the cotangent bundle [76] on the manifold as

$$T^*M = \bigcup_{x \in M} T_x^* M,$$

where T_x^*M denotes the cotangent space at point x. The contangent bundle constructed this way is a differential manifold of dimension $2n$.

Elements of T^*M are one-forms on T_x^*M, the tangent space at the point x. If we choose $q = (q_1, q_2, .., q_n)$ to be the local coordinate of the configuration manifold M, then the n components of a such one-form can be given with respect to this choice of coordinates $p = (p_1, p_2, ..., p_n)$. Recalling that T^*M is a cotangent bundle, $2n$ coordinates (p, q) provides a local coordinate system for this manifold.

An example of the symplectic structure can be given, following the discussion above, by considering the vector space R^{2n} with coordinates (p_i, q_i)

$$w = \sum_{i=1}^{N} dp_i \wedge dq_i.$$

It is clear that this two-form is both closed and non-degenerate.

The cotangent bundle T^*M defined above is naturally endowed with a symplectic structure. In local coordinates, it is given by

$$w = d\vec{p} \wedge d\vec{q}, \tag{2.1}$$

$$= dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n. \tag{2.2}$$

To show this, one needs to observe that on T^*M, one can always take a one-form of the following:

$$\Omega = \vec{p}.d\vec{q}$$

Then defining the symplectic structure as $\omega = d\Omega$ guarantees that w is closed and non-degenerate.

Let us consider (M, w) which is assumed to be a connected manifold. The dimension of this manifold is $2n$. Any point of M has an open neignorhood which is the domain of a chart whose local coordinates, denoted as $(p_1, p_2, ..., p_n, q_n)$ are such that the form w has the expression

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i, \tag{2.3}$$

such a chart is called canonical or Darboux.From this one can infer that in the neighborhood of each point, the symplectic manifold is isomorphic to a cotangent bundle and two symplectic manifolds of the same dimension are locally isomorphic to each other.

We have defined the symplectic manifold and the symplectic structure. From the above discussion, we can see that the cotangent bundle provides a natural setting for a system evolving along its position and momenta. In order to understand how these change, however, we will need the generator of these 'evolutions'.

Collection of all possible positions, or configurations, constitute the configuration space which is a manifold called the configuration manifold.
A trajectory t on this manifold can be defined as a function $t : R \longrightarrow M$. At each point x on t, velocity is an element of the T_xM, the tangent space to M at x. In the local coordinates, velocity can be expanded in the basis $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{x^n}\right)$. The momentum, however, is a function of velocity, so it is an element of the cotangent space.

Just as the metric on a spacetime manifold provides an isomorphism between tangent vectors and one-forms, the symplectic structure ω also provides a similar isomorphism. For each tangent vector ξ of a symplectic manifold (M, ω) at point x, there is an associated one-form ω_ξ on TM_x, which is giving by the formula

$$w_\xi(\eta) = \omega(\eta, \xi) \tag{2.4}$$

For any $\eta \in T_xM$. In other words, just as one can use the metric tensor to raise and lower spacetime indices, the symplectic structure can be used to raise and lower indices in the phase space manifold as well.

We denote the isomorphism provided by the symplectic structure as $I : T_x^*M \to T_xM$. Now, let us consider a differentiable function $\mathcal{H} : M \to \mathbb{R}$, a one-form $d\mathcal{H}$ on M can be obtained and we can find the associated tangent vector at every point using the isometry provided by the symplectic structure. Since ω is non-degenerate, the vector field defined in this way is unique.

$$I(d\mathcal{H}) = X_{\mathcal{H}}. \tag{2.5}$$

The vector field $I(d\mathcal{H})$ is called a Hamiltonian vector field, $\mathcal{H}$ is called a Hamiltonian function. If we wanted to find the $\mathcal{H}$ in terms of $X_{\mathcal{H}}$ and ω, we integrate from $t = 0$ to $t = 1$.

$$d\mathcal{H}(tx)x = w^2\left(X_{\mathcal{H}}(tx), x\right),$$

from this, we obtain the following

$$\mathcal{H}(x) - \mathcal{H}(0) = \int_0^1 \omega\left(X_{\mathcal{H}}(tx), x\right) dt. \tag{2.6}$$

Having defined both the Hamiltonian vector field and symplectic structure, we now turn our attention to the effect of transformations of these Hamiltonian vector fields. Consider a symplectic manifold (M, ω) and a function $\mathcal{H} : M \to \mathbb{R}$. The vector field $I(d\mathcal{H}) = X_{\mathcal{H}}$ corresponding to $\mathcal{H}$ gives a group of diffeomorphisms:

$$\begin{aligned} g^t &: M \to M, \\ \frac{d}{dt} g^t(x) &= X_H\left(g^t(x)\right), \\ g^0 &= id_M. \end{aligned}$$

The group g^t is called the Hamiltonian phase flow of the Hamiltonian function $\mathcal{H}$. The generator of this group, the Hamiltonian vector X_H, is tangent to the curve g^t at every point. One can write the group element in terms of the generator as

$$g^t = e^{tX_H}. \tag{2.7}$$

A Hamiltonian phase flow preserves the symplectic structure.

$$\left(g^t\right)^* \omega = \omega,$$

where $(g^t)^*$ denotes the pullback of ω under the action of the group element g^t. This means that the ω is independent of the group parameter t.

$$\frac{d}{dt}\left\{\left(g^t\right)^* \omega^2\right\} = \frac{d}{dt}\omega^2 = 0.$$

This result is a generic case of the Liouville theorem which states that, in the phase manifold, Hamiltonian flow preserves the volume. Here we can see how this works. We note again that ω is invariant under the action of the group. So using the symplectic structure, one can construct the invariant volume element by taking n exterior multiplication of w. In order to show this, we start by the following definition:

Definition: A differential form w of degree k is called invariant under the action of the group g if the integrals of w on any k-chain and on its image by g are the same.

$$\int_{gc} \omega = \int_c w.$$

The two-form ω^2 giving the symplectic structure is an integral invariant of a Hamiltonian phase flow gt.

$$\int_{g^t c} \omega = \int_c \left(g^t\right)^* \omega = \int_c \omega.$$

By the same token, one can show that exterior power of ω, $(\omega)^2 = \omega \wedge \omega$ etc, are also integral invariants of the Hamiltonian phase flow. Therefore, one can obtain an invariant volume form on M by taking the n^{th} exterior power of ω.

$$V = (\omega)^{\wedge n} = \omega \wedge \ldots \wedge \omega.$$

2.2 Covariant Form of the Symplectic Structure

After finishing the summary of the phase space and its symplectic structure in particle mechanics, we can now turn our attention to the covariant phase space method. One might notice that our treatment in the previous section is not covariant. Indeed, looking at the Hamilton's equations of motion

$$\begin{aligned} \dot{q}^a &= \frac{\partial H}{\partial p_a}, \\ \dot{p}_a &= -\frac{\partial H}{\partial q^a}, \end{aligned}$$

one can see that split of coordinates into momentum and spatial coordinates, and choosing a time frame on the phase space breaks the covariance of the theory. So reformulation the phase space and its symplectic structure in such a way that no mention of having a preferred time frame or split of the coordinates is desired. Another aspect of the treatment we gave in the previous section is that it deals with a system with many-particles. Meaning that this naive formulation would not explain the geometry of the field theories. Hence, our task in this section is to discuss and construct the covariant symplectic geometry for field theories.

On the face value, promoting the naively constructed phase space formulation for the field theories seems trivial. Fields are considered as a continuum limit for many-particle systems and the value of each field of a given position is interpreted as the density of particles at that point. So the field theory generalization seems to be a simple procedure where one takes everything to field limit. However, looking at the underlying mathematical structures, one sees that this is not a trivial job. For example, a trajectory $q : \mathbb{R} \rightarrow M$ for a particle takes a single parameter and maps out position on the configuration manifold M. A field, however, takes the spacetime points as parameters and its output is field values on C. [2]

So how one can generalize the phase space method in this desired way while retaining the manifest covariance? The answer to this is to re-interpret the phase space in the following way [70]: Phase space of a dynamical system is the set of solutions to the equations of motion. While traditional definition sees the phase space as a set of initial conditions (q_0s and p_0s) on a given time slice, this new definition does not mention nor require any chosen time. For the initial value problem, these two definitions of the phase space are in one-to-one correspondence. One thing to note is that this construction suffers when the theory possesses local continuous symmetries. For the current discussion, however, we will assume no such symmetry is present and postpone the discussion about gauge degrees of freedom until section 2.2.1.

After having laid down the basic groundwork, we are now in a position to make the discussion more concrete. Starting with the following action

$$S = \int_M L(\phi, \chi) + \int_{\partial M} l(\phi, \chi). \tag{2.8}$$

Where L is a top-from obtained from the Lagrangian density as $L = *\mathcal{L}$. ϕ denotes all the dynamical fields whose variations are of interest, while χ denotes the rest of the non-dynamical parts. A boundary term l, which is a $(d-1)$-form over the boundary ∂M, is included for generality. By saying L or l is a form, we note that they transform as differential forms under the diffeomorphisms which act on both the dynamical and non-dynamical fields.

The basic principle of the Lagrangian formalism is that the action defined in (2.8) must be stationary for a set of field configurations ϕ_s under arbitrary variations. In addition to this, one must make the variation principle well defined, so the appropiate boundary conditions are required.

Boundary of the spacetime manifold M can be decomposed as $\partial M = \Sigma_\pm \cup \Gamma$ where $\Sigma_\pm$ are the temporal boundaries while Γ denotes the spatial boundary. We note that the boundary conditions on $\Sigma_\pm$ and Γ have different meanings, in which the former

[2] For example $C = \mathbb{R}$ for a real scalar theory

determines a state or solution within the theory while the latter is a part of the definition of the theory. Therefore, imposing boundary conditions on $\Sigma_{\pm}$ would be too much of a restriction while on Γ one needs boundary conditions to specify the theory. Therefore, the action must be stationary up to boundary conditions at future and past boundaries $\Sigma_{\pm}$.

$$\delta S = \int_{\Sigma_+} \psi - \int_{\Sigma_-} \psi. \tag{2.9}$$

Where ψ is a local function of field variables ϕ and χ at $\Sigma_{\pm}$. This is also a good place to define the configuration space $\mathcal{C}$, which is a set of dynamical field configurations that obey the boundary conditions at Γ, but not necessarily the equations of motion. The configuration space $\mathcal{C}$ is the domain in which the variational principle works, and where the path integral quantization of fields takes place.

Considering the variation of the Lagrangian in (2.8), one can always put it in the following form

$$\delta L = E_a \delta\phi^a + d\Theta. \tag{2.10}$$

Where $\delta\phi^a$ is the variation of the dynamical field in $\mathcal{C}$, Θ is a local functional of fields and their derivatives, and field variations $\delta\phi$. The index a runs over the dynamical fields ϕ^a. The boundary term $d\Theta$ is a $(d-1)$ form over the spacetime and it is the symplectic potential. Then, the variation of (2.8) becomes,

$$\delta S = \int_M E_a \delta\phi^a + \int_{\partial M} \{\Theta + \delta l\}, \tag{2.11}$$

where the use of Stokes' theorem on the second term is understood. For this to obey the (2.9), we see that the field configurations inside M must obey the equations of motion

$$E_a = 0. \tag{2.12}$$

Also, for the variation to be stationary up to boundary terms at $\Sigma_\pm$, the second term in the equation (2.11) must only have a contribution from $\Sigma_\pm$. It seems that requiring the term $\Theta + \delta l$ to vanish at Γ might be necessary for this, but it turns out that this requirement is too strong to begin with. From (2.10) we see that the boundary term $\Theta(\phi, \delta\phi)$ has an ambiguity up to a total differential term in which we can redefine as $\Theta \to \Theta + dY$. Hence given this freedom of shift in Θ, it is more natural to require that the second term of (2.11) be equal to a total derivative at Γ

$$(\Theta + \delta l)_\Gamma = dC, \tag{2.13}$$

where C is the $d-2$ form constructed from both dynamical and background fields, their variations and derivatives. It is also clear that C has the same shift ambiguity as Θ. Having found the suitable boundary conditions, we put (2.13) use in (2.11).

$$\delta S = \int_M E_a \delta\phi^a + \int_{\Sigma_\pm} (\Theta + \delta l) + \int_\Gamma (\Theta + \delta l) \tag{2.14}$$

$$= \int_M E_a \delta\phi^a + \int_{\Sigma_\pm} (\Theta + \delta l) + \int_{\partial\Gamma} C. \tag{2.15}$$

Although C is defined on the spatial boundary Γ, we can arbitrarily extend it into $\Sigma_\pm$ since only the values of C on $\partial\Sigma_\pm$ contribute. Hence, we can put the action variation into the form of (2.10)

$$\delta S = \int_M E_a \delta\phi^a + \int_{\Sigma_\pm} (\Theta + \delta l - dC), \tag{2.16}$$

where $E_a = 0$ and $\psi = \Theta + \delta l - dC$ is understood.

At this point we have a well defined action principle with the stated suitable boundary conditions. In order to switch to the Hamilton formalism and the symplectic structure, we now need to define the phase space without breaking the covariance. In accordance with our previous discussion, we define the phase space $\mathcal{P}$ to be the set of field configurations ϕ that satisfy the equations of motion. We note that by this definition, phase space $\mathcal{P}$ is a sub-space of the configuration space $\mathcal{C}$ and therefore has no reference to preferred time slice.

In defining the symplectic form, it will be very useful to first note that there is a convenient change of notation which allows us to re-interpret quantities like Θ and C as one-forms on $\mathcal{C}$. The idea is that we view quantities like $\delta\phi$ not like as infinitesimal transformations, but as like coordinate differentials on $\mathcal{C}$. With this new interpretation at hand, δ denotes the exterior derivative for differential forms defined over $\mathcal{C}$. The action of $\delta\phi^a(x)$ on a vector field is given as

$$\delta\phi^a(x)\left(\int d^dx' f^b\left(\phi, x'\right)\frac{\delta}{\delta\phi^b\left(x'\right)}\right) = f^a(\phi, x). \tag{2.17}$$

With this construction at hand, we are now ready to define the pre-symplectic current as the pullback of $\delta\psi$ to $\mathcal{P}$

$$\omega \equiv \delta\psi|_{\mathcal{P}} = \delta(\Theta - dC)|_{\mathcal{P}}\,. \tag{2.18}$$

Where we used $\delta^2 = 0$ to eliminate the δl term. ω is bi-closed, meaning that it is closed both on $\mathcal{P}$ and on the spacetime manifold $\mathcal{M}$. The former property is self-evident from the definition of ω as the variation of Θ on $\mathcal{P}$. For the latter, it is easy to show

$$\begin{aligned} d\omega &= d\{\delta(\Theta - dC)\} \\ &= \delta d\Theta \\ &= \delta\left(\delta L - E_a\delta\phi^a\right) \\ &= -\delta E_a \wedge \delta\phi^a = 0, \end{aligned} \tag{2.19}$$

where we used the fact that the exterior derivatives d and δ commute and $\delta L = E_a\delta\phi^a + d\Theta$. In the last line, $E_a = 0$ since ω is defined on $\mathcal{P}$ in which the equations of motion are satisfied. We can finally define the symplectic form over $\mathcal{P}$

$$\Omega \equiv \int_{\Sigma} \omega, \tag{2.20}$$

where Σ is a Cauchy surface of M. From (2.18) one can show that Ω defined this way is independent of choice of the Cauchy surface. To show this, we consider the difference of two symplectic form, evaluated over two different Cauchy surfaces

$$\Delta\Omega = \int_{\Sigma-\Sigma'} w.$$

To show this, consider a volume of spacetime M bounded by $\partial M = \Sigma \cup \Sigma' \cup \Gamma$. By the Stokes' theorem

$$\int_{\Sigma-\Sigma\prime} \omega = \int_{M} d\omega - \int_{\Gamma} \omega. \tag{2.21}$$

The first integral is zero by (2.19) and the second integral vanishes since w vanishes on Γ by definition (2.13).

We have now finished the basic definitions of the covariant phase space method. This formalism is rather useful when the theory in consideration possesses invariance under some continuous group of diffeomorphisms. Let us then consider a variation of a general tensor field under some diffeomorphism generated by the vector field ξ^{μ}

$$\delta_{\xi}\phi = \mathcal{L}_{\xi}\phi, \tag{2.22}$$

$$\delta_{\xi}\phi^{a}(x) = \mathcal{L}_{X_{\xi}}\phi^{a}(x) = X_{\xi} \cdot \delta\phi^{a}(x). \tag{2.23}$$

More generally, infinitesimal diffeomorphisms of any tensor defined on the configuration space is given by

$$\delta_{\xi}T = \mathcal{L}_{X_{\xi}}T \tag{2.24}$$

where in contrast with the notation defined earlier, the vector field X_ξ on configuration space can be written as

$$X_\xi \equiv \int d^d x \mathcal{L}_\xi \phi^a(x) \frac{\delta}{\delta \phi^a}. \tag{2.25}$$

We are now ready to introduce covariant tensors as follows: Consider a tensor $T(\phi, \xi)$, defined on the configuration space $\mathcal{C}$ and constucted out of dynamical and non-dynamical fields. This is said to be covariant under the action of diffeomorphism generated by the vector field ξ if

$$\delta_\xi T = \mathcal{L}_\xi T, \tag{2.26}$$

where the Lie derivative $\mathcal{L}_\xi$ is performed over the spacetime and needs to be distinguished from the configuration-space Lie derivative $\mathcal{L}_{X_\xi}$. The action of the two operations are different in the sense that $\mathcal{L}_\xi$ implements diffeomorphisms on dynamical and non-dynamical fields while the configuration-space version do the same only on the dynamical fields.This distinction is important since the symmetry variations are only allowed to act upon dynamical fields. So an arbitrary tensor T constructed out of both dynamical and non-dynamical field must have the correct transformation rule for it to be covariant.

There are couple of ways for a generic tensor, which is constructed out of dynamical and non-dynamical fields ϕ and χ, $T(\phi, \chi)$ to be covariant under the diffeomorphism generated by the vector field ξ: The simplest way for this to happen is all the non-dynamical fields χ^a to be individually invariant under the said diffeomorphisms. In the mathematical form, this corresponds to the following

$$\mathcal{L}_\xi \chi^a = 0, \tag{2.27}$$

where index a runs over all the non-dynamical fields. Another example of covariance is the case in which the Lagrangian or the tensor, here $T(\phi, \chi)$ in this discussion, is trivially independent of such non-dynamical fields. A detailed example of this case will be shown in section (2.2.2.2). More generally, it will be enough for $T(\phi, \chi)$ to be invariant if the combination of χ^as that appear in $T(\phi, \chi)$ is invariant under ξ.

Now, let's consider the variation of the action (2.8) under an infinitesimal diffeomorphism generated by ξ

$$\delta_\xi S = \int_M \delta_\xi L + \int_{\partial M} \delta_\xi l$$

$$= \int_{\partial M} (\xi \cdot L + \delta_\xi l) \,, \tag{2.28}$$

where (2.26) and the Cartan's formula $\mathcal{L}_\xi L = \xi \cdot dL + d(\xi \cdot L)$ and the fact that L is a top-form is used. Covariance requirement of the Lagrangian dictates that variation (2.28) be invariant up to a boundary term at $\Sigma_\pm$. But this is not sufficient for the diffeomorphism generated by the ξ to be a symmetry of the theory. The variation must also respect the boundary conditions, especially at the spatial boundary Γ:

$$\delta_\xi S = \int_{\Sigma_\pm} (\xi \cdot L + \delta_\xi l) + \int_\Gamma (\xi.L + \delta_\xi l) \,. \tag{2.29}$$

The first integral is the allowed contribution at the temporal boundaries $\Sigma_\pm$ while the second integral must vanish if ξ is to be a symmetry generator. So we first require that the normal component of ξ be vanish at Γ. This ensures that the term $\xi \cdot L$ vanishes. We also require l to be covariant under ξ so that at Γ no contributions present and boundary rules of the variation are respected. With these constraints imposed, the variation (2.29) is stationary up to a boundary term at $\Sigma_\pm$, just as discussed before.

We are now ready to finally construct the Hamiltonian H_ξ which is the generator of the flow in the phase space that corresponds to the diffeomorphism symmetry generated by ξ. We begin by finding a function H_ξ on the phase space that satisfies

$$\delta H_\xi = -X_{\xi}. \cdot \Omega. \tag{2.30}$$

We note that since Ω is non-degenerate we can write

$$X_\xi(f) = \Omega^{-1}\left(\delta f, \delta H_\xi\right), \tag{2.31}$$

where f is a function over the phase space $\mathcal{P}$. Note that this the Hamiltonian equation. We now need to compute the right-hand side of the Eq. (2.30), but before that if will be useful to define the Noether current

$$J_\xi \equiv X_\xi \cdot \Theta - \xi \cdot L. \tag{2.32}$$

The Noether current J_ξ is a closed $(d-1)$ form over the spacetime if the Lagrangian L is covariant under ξ

$$dJ_\xi = d\left\{X_\xi \cdot \Theta - \xi \cdot L\right\}.$$

Using (2.10), invariance of L and the Cartan's formula

$$\begin{aligned} dJ_\xi &= X_\xi \cdot \left(\delta L - E_a \delta\phi^\circ\right) - \mathcal{L}_\xi L \\ &= X_\xi \cdot \delta L - \mathcal{L}_\xi L + \xi \cdot dL \\ &= 0. \end{aligned} \tag{2.33}$$

We are now ready to compute the right hand side of (2.30)

$$\begin{aligned} X_\xi \cdot \omega &= X_\xi \cdot \delta(\Theta - dC) \\ &= X_\xi \cdot \{\Theta - d\delta C\}. \end{aligned} \tag{2.34}$$

Using the Cartan's formula $\mathcal{L}_{X_\xi}\Theta = X_\xi \cdot \delta\theta + \delta\left(X_\xi \cdot \theta\right)$ on $\mathcal{P}$ in the first term

$$X_\xi \cdot \omega = \mathcal{L}_{X_\xi}\Theta - \delta\left(X_\xi \cdot \Theta\right) - d\left\{X_\xi \cdot \delta C\right\}$$
$$= \mathcal{L}_{X_\xi}\Theta - \delta\left(X_\xi \cdot \Theta\right) - d\left\{\mathcal{L}_{X_\xi}C - \delta\left(X_\xi \cdot C\right)\right\}. \qquad (2.35)$$

Where we again made use of the Cartan's formula on M. Using the covariance of Θ and substituting (2.32)

$$X_\xi \cdot \omega = \mathcal{L}_\xi\Theta - \delta J_\xi - \delta(\xi.L) - d\left\{\mathcal{L}_{X_\xi}C - \delta\left(X_\xi \cdot C\right)\right\}$$
$$= \xi \cdot d\theta + d(\xi \cdot \Theta) - \delta J_\xi - \xi \cdot \delta L - d\left\{\mathcal{L}_{X_\xi}C - \delta\left(X_\xi \cdot C\right)\right\}$$
$$= -\delta J_\xi - d\left\{\mathcal{L}_{X_\xi}C - \delta\left(X_\xi \cdot C\right) - \xi \cdot \Theta\right\}, \qquad (2.36)$$

where we yet again used the Cartan's formula on M and made use of the fact that on $\mathcal{P}$ equations of motion are satisfied, i.e $E_a = 0$. We are beginning to see that the right hand side can be put into the form of variation δ of something. Let us continue with the computation and use (2.20)

$$-X_\xi \cdot \Omega = \int_\Sigma \delta J_\xi + \int_{\partial\Sigma}\left\{-\xi \cdot \Theta + \mathcal{L}_{X_\xi}C - \delta\left(X_\xi \cdot C\right)\right\}$$
$$= \int_\Sigma \delta J_\xi + \int_{\partial\Sigma}\left\{-\xi \cdot \theta + \xi \cdot dC + d(\xi \cdot C) - X_\xi \cdot \delta C\right\}$$
$$= \delta\int_\Sigma J_\xi + \int_{\partial\Sigma}\left\{\xi \cdot (dC - \Theta) - X_\xi \cdot \delta C\right\}$$
$$= \delta\left\{\int_\Sigma J_\xi + \int_{\partial\Sigma}\left(\xi \cdot l - X_\xi \cdot C\right)\right\}, \qquad (2.37)$$

where we used the covariance of C on Γ, Cartan's formula on M and $(\Theta + \delta l)|_\Gamma = dC$. Hence, finally we obtain the Hamiltonian in term of the Noether charge with an additional boundary term

$$H_\xi = \int_\Sigma J_\xi + \int_{\partial\Sigma} (\xi \cdot l - X_\xi \cdot C)\,. \tag{2.38}$$

2.2.1 Gauge Degrees of Freedom

Phase space constructed in section 2.2 makes use of the one-to-one correspondence betwen solutions to the equations of motion and the initial values of a physical system. In the case of gauge freedom, this correspondence breaks down since the phase space will be degenerate, in the sense that there will be more than a single field configuration that correspond to the same physical state. Assuming field configurations ϕ and ϕ' describe the same physical state, a vector field V spanning between these two point in the phase space is said to be a degeneracy direction. Then the Hamiltonian flow generated by this vector will be unphysical in the following sense

$$V \cdot \Omega = 0. \tag{2.39}$$

Therefore we see that when there are gauge degrees of freedom, the phase space and the symplectic manifold have degeneracy. We will refer this initial construction as the pre-phase space $\widehat{\mathcal{P}}$ and every object defined on this space will be denoted by a hat.

Solution to this degeneracy problem comes from the observation that the "zero modes" of Ω, i.e vectors $V \in \top\mathcal{P}$ that satisfy (2.39) form a Lie algebra which generates a group of unphsyical degrees of freedom. If $\widehat{X}$ and $\widehat{Y}$ are vectors fields which are annihilated by $\widehat{\Omega}$, then their commutator $[\widehat{X}, \widehat{Y}] = \mathcal{L}_{\widehat{X}}\widehat{Y}$ will also be a zero-mode of $\widehat{\Omega}$;

$$\begin{aligned} \mathcal{L}_{\widehat{X}}\widehat{Y} &= \mathcal{L}_{\widehat{X}}(\widehat{Y} \cdot \widehat{\Omega}) - \widehat{Y} \cdot \mathcal{L}_{\widehat{X}}\widehat{\Omega} \\ &= 0. \end{aligned} \tag{2.40}$$

Hence zero modes of vector fields of $\widehat{\Omega}$ will form a Lie algebra.Then, the physical phase space $\mathcal{P}$ can be obtained by taking the quotient of this group of zero-modes from the pre-phase space $\widehat{\mathcal{P}}$

$$\mathcal{P} \equiv \widehat{\mathcal{P}}/\widehat{\mathcal{G}}. \tag{2.41}$$

Thus the action of group of gauge symmetries $\widehat{\mathcal{G}}$ leaves no trace in the actual phase space $\mathcal{P}$. The work is not done, as we also need a symplectic form Ω. In order to do this, we proceed as follows: Let $\pi : \widehat{\mathcal{P}} \to \mathcal{P}$ be a map that sends all points q in $\widehat{\mathcal{P}}$ to its $\widehat{\mathcal{G}}$ orbit. Let us consider a point p in $\mathcal{P}$ and vectors $X,\ Y \in T\mathcal{P}$. Then we can always find a point q in $\widehat{\mathcal{P}}$ and a pair of vectors $\widehat{X}$ and $\widehat{Y}$ such that X and Y are pushforwards of $\widehat{X}$ and $\widehat{Y}$ by the map π. Hence we can define

$$\Omega(X, Y) = \widehat{\Omega}(\widehat{X}, \widehat{Y}). \tag{2.42}$$

For (2.42) to be the true symplectic form, it must be both independent of the choice of vectors $\widehat{X}$, $\widehat{Y}$ and non-degenerate. We can prove the former by considering two vectors $\widehat{X}$ and $\widehat{X'}$ which map to the same vector X in $T\mathcal{P}$. Then, from (2.39) these two vectors $\widehat{X}$ and $\widehat{X'}$ are connected along the degeneracy direction by another vector $\widehat{Z}$, which is annihilated by $\widehat{\Omega}$. Thus, this ambiguity has no imprint on the actual symplectic form (2.42).

For the non-degeneracy of the symplectic form, let us assume that for some p in $\mathcal{P}$, there exists a vector $X \neq 0$ for which $X \cdot \Omega = 0$. Then X must be a pushforward of a vector $\widehat{X} \in T\widehat{\mathcal{P}}$ by the map π and by (2.42) we have $\widehat{X} \cdot \widehat{\Omega}$. We can then extent $\widehat{X}$ to a vector field that is annihilated by $\widehat{\Omega}$. The pushforward of this vector field by the map π must vanish which contradicts our initial assumption. So the symplectic form is non-degenerate.

2.2.2 Examples

2.2.2.1 Maxwell Theory

The first example is the Maxwell theory in which the theory has unphysical degrees of freedom. Here quotient from pre-phase to phase space is non-trivial. The Lagrangian of the Maxwell theory, written in the differential form language is

$$L = -\frac{1}{2} F \wedge *F, \tag{2.43}$$

where F is the field strength tensor defined as $F = dA$ and A is the one-form potential. Taking the variation of (2.43)

$$\begin{aligned} \delta L &= -\frac{1}{2}\delta\{F \wedge *F\} \\ &= -\delta A \wedge d * F - d(\delta A \wedge *F), \end{aligned} \tag{2.44}$$

Here the first term is the obvious equations of motion, while the second term is the boundary term in (2.11) with $l = 0$. Hence, we have

$$\Theta = -\delta A \wedge *F. \tag{2.45}$$

Recalling the stationarity requirement (2.13), it is obvious that fixing the value of the one-form potential A at the boundary Γ with $C = 0$ would be sufficient. Then, the symplectic potential and the pre-symplectic form will be as follows

$$\begin{aligned} w &= \delta\Theta \\ &= \delta A \wedge *\delta F, \end{aligned} \tag{2.46}$$

and,

$$\widetilde{\Omega} = \int_{\Sigma} (\delta A \wedge *\delta F). \tag{2.47}$$

This pre-symplectic form constructed in this way would have zero modes because of the degrees of freedom the theory possesses. In order to find the quotient, we need to find the vector field X_λ which generates the flows corresponding to gauge transformations.

$$X_\lambda = \int d^d x \, (\partial_\mu \lambda) \, \frac{\delta}{\delta A_\mu}, \tag{2.48}$$

and

$$\begin{aligned} X_\lambda \cdot \tilde{\Omega} &= \int_\Sigma (d\lambda \wedge *\delta F) \\ &= \int_\Sigma d(\lambda \wedge *\delta F) \\ &= \int_{\partial\Sigma} \lambda \wedge *\delta F, \end{aligned} \tag{2.49}$$

where we used $dF = 0$ in the second line. The boundary condition we imposed, i.e A is constant at Γ, dictates that $d\lambda$ be zero at Γ. Therefore λ must be a constant on Γ. Since $*F$ is allowed to vary on Γ, X_λ will be a zero mode of $\widetilde{\Omega}$ if and only if λ is zero on Γ. Therefore, in order to construct the true phase space, we must only quotient those gauge transformations which vanish at the spatial boundary Γ.

2.2.2.2 Gravity

For the next example, we now consider the General Relativity with the following action

$$\begin{aligned} S &= \int_M L + \int_{\partial M} l \\ &= \frac{1}{16\pi G} \int_M (R - 2\Lambda)\epsilon_M + \frac{1}{8\pi G} \int_{\partial M} K \epsilon_{\partial M} \end{aligned} \tag{2.50}$$

here R is the scalar curvature, Λ is the cosmological constant, K is trace of the extrinsic curvature and ϵ is the volume form. Using $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$, variations of volume forms can be written as

$$\delta\epsilon_M = \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}\epsilon_M,$$

$$\delta\epsilon_{\partial M} = \frac{1}{2}\gamma^{\mu v}\delta g_{\mu v}\epsilon_{\partial M}. \tag{2.51}$$

The other relevant variations are [77]

$$\begin{aligned}
\delta\Gamma^{\mu}_{\alpha\beta} &= \frac{1}{2}g^{\mu\nu}\left\{\nabla_\alpha\delta g_{\beta\nu} + \nabla_\beta\delta g_{\alpha\nu} - \nabla_\nu\delta g_{\alpha\beta}\right\},\\
\delta R &= -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - \nabla_\lambda\nabla^\lambda g^{\mu v}\delta g_{\mu v},\\
\delta\eta_\mu &= \frac{1}{2}\eta^\alpha\left(\delta^\beta_\mu - \gamma^\beta_\mu\right)\delta g_{\alpha\beta},\\
\delta K &= -\frac{1}{2}K^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2}g^{\mu\nu}n^\lambda\nabla_\lambda\delta g_{\mu\nu} - \frac{1}{2}\eta^\alpha\nabla^\beta\delta g_{\alpha\beta} - \frac{1}{2}D_\mu\left(\gamma^{\mu\nu}\eta^\alpha\delta g_{\nu\alpha}\right),
\end{aligned} \tag{2.52}$$

where D_μ is the covariant derivative defined on the hypersurface ∂M. With these, the action variation can be put into the following form:

$$\delta L = E^{\mu\nu}\delta g_{\mu\nu} + d\Theta, \tag{2.53}$$

where E_a and Θ are as follows

$$E_{\mu\nu} = -\frac{1}{16\pi G}\left\{R^{\mu v} - \frac{1}{2}Rg^{\mu\nu} - \Lambda g^{\mu\nu}\right\}E_M, \tag{2.54}$$

and,

$$\Theta = \theta\cdot\epsilon_M, \theta^\mu = \frac{1}{16\pi G}\left\{g^{\mu\alpha}\nabla^\nu\delta g_{\alpha\nu} - g^{\alpha\beta}\nabla^\mu\delta g_{\alpha\beta}\right\}, \tag{2.55}$$

with the choice of orientation of the hypersurface ∂M as $\epsilon_M = \eta \wedge \epsilon_{\partial M}$ we can write Θ on the hypersurface:

$$\begin{aligned} \Theta|_\Gamma &= \theta \cdot \epsilon_M \\ &= \eta_\mu \theta^\mu \epsilon_{\partial M}. \end{aligned} \tag{2.56}$$

Similarly, for the boundary term l

$$\delta l = \frac{1}{8\pi G} \delta \left\{ K \epsilon_{\partial M} \right\} \tag{2.57}$$

$$\begin{aligned} \delta l = \frac{1}{16\pi G} \Big\{ & \left(K \gamma^{\mu\nu} - K^{\mu\nu} \right) \delta g_{\mu\nu} + \left(g^{\alpha\beta} \eta^\lambda \nabla_\lambda - \eta^\alpha \nabla^\beta \right) \delta g_{\alpha\beta} \\ & - D_\mu \left(\gamma^{\mu\nu} \eta^\alpha \delta g_{\nu\alpha} \right) \Big\} \epsilon_{\partial M} \end{aligned} \tag{2.58}$$

Using (2.58) and (2.56), we compute (2.13) as follows:

$$\begin{aligned} \Theta|_\Gamma + \delta l = & \frac{1}{16\pi G} \Big\{ \eta^\alpha \nabla^\nu \delta g_{\alpha\nu} - g^{\alpha\beta} \eta_\mu \nabla^\mu \delta g_{\alpha\beta} - \left(K \gamma^{\mu\nu} - K^{\mu\nu} \right) \delta g_{\mu\nu} \\ & + g^{\alpha\beta} \eta^\lambda \nabla_\lambda \delta g_{\alpha\beta} - n^\alpha \nabla^\beta \delta g_{\alpha\beta} - D_\mu \left(\delta^{\mu\nu} n^\alpha \delta g_{\nu\alpha} \right) \Big\} \epsilon_{\partial M} \\ = & \frac{-1}{16\pi G} \left\{ \left(K \gamma^{\mu\nu} - K^{\mu\nu} \right) \delta g_{\mu v} + D_\mu \left(\gamma^{\mu\nu} \eta^\alpha \delta g_{\nu\alpha} \right) \right\} \epsilon_{\partial M}, \end{aligned} \tag{2.59}$$

which can be written as

$$\Theta|_\Gamma + \delta l = -\frac{1}{16\pi G} \left(K^{\mu\nu} - K \gamma^{\mu\nu} \right) \epsilon_{\partial M} \delta g_{\mu\nu} + dC, \tag{2.60}$$

where

$$C = c \cdot \epsilon_{\partial M}, \; c^\mu = -\frac{1}{16\pi G} \gamma^{\mu v} n^\alpha \delta g_{\nu\alpha}. \tag{2.61}$$

For (2.60) to satisfy (2.13), the first term must vanish at the spatial boundary Γ

$$(K^{\mu\nu} - K\gamma^{\mu\nu})\, \delta g_{\mu\nu}|_{\Gamma} = 0. \tag{2.62}$$

Then, according to this analyses we have two options: one would be fixing the value of $g_{\mu\nu}$ at Γ. This corresponds to requiring the tangential components of the metric variation be zero

$$\gamma_{\mu}^{\alpha}\gamma_{\nu}^{\beta}\delta g_{\alpha\beta}\big|_{\Gamma} = 0. \tag{2.63}$$

We make note of the (2.63) only restricts the tangential components of the metric variation, while boundary term C in (2.61) contains variation of the metric with mixed components. Hence we have non zero C term. With the choice of (2.63) as a boundary condition, let us now find the Hamiltonian H_{ξ}. We begin by computing each term appearing in (2.38)

Using the results of [79], J_{ξ} is a closed form and hence can be written as $J_{\xi} = dQ_{\xi}$, where

$$Q_{\xi} = -\frac{1}{16\pi G} \star d\xi, \tag{2.64}$$

here ξ is a one-form over spacetime and $\star$ is the Hodge duality operator. In component form,

$$\begin{aligned} \star d\xi &= \frac{1}{2}\epsilon^{\mu\nu}_{\lambda\sigma}(d\xi)_{\mu\nu} \\ &= -\frac{1}{16\pi G}\epsilon_{\mu\nu\lambda\sigma}\left(\nabla^{\mu}\xi^{\nu} - \nabla^{\nu}\xi^{\mu}\right). \end{aligned} \tag{2.65}$$

Using yet again $\epsilon_M = \eta \wedge \tau \wedge \epsilon_{\partial M}$, one has

$$Q_z = -\frac{1}{16\pi G}\left\{\tau^{\alpha}\eta^{\beta} - \tau^{\beta}\eta^{\alpha}\right\}\nabla_{\alpha}\xi_{\beta}\epsilon_{\partial M}. \tag{2.66}$$

Next, for the term $X_\xi \cdot C$, we note that the dot $\cdot$ here indicates the inner product in the phase space $\mathcal{P}$, in contrast with the usual meaning of the term in which it indicates the contraction with the spacetime index. Using (2.61) and $\epsilon_{\partial M} = -\tau \wedge \epsilon_{\partial\Sigma}$

$$\begin{aligned} C &= c \cdot (-\tau \wedge \epsilon_{\partial\Sigma}) \\ &= -c^\mu \tau_\mu \epsilon_{\partial\Sigma} \\ &= \frac{1}{16\pi G} \gamma^{\mu\nu} \eta^\alpha \tau_\mu \epsilon_{\partial\Sigma} \delta g_{\nu\alpha}. \end{aligned} \tag{2.67}$$

As per our earlier discussion, the vector field generator of diffeomorphisms X_ξ

$$X_\xi = \mathcal{L}_\xi g, \tag{2.68}$$

or in component form:

$$X_\xi = \mathcal{L}_\xi g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}, \tag{2.69}$$

then we finally have

$$\begin{aligned} X_\xi \cdot C &= \mathcal{L}_\xi g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \cdot \left(\frac{1}{16\pi G} \gamma^{\mu\nu} \eta^\alpha \tau_\mu \delta g_{\nu\alpha} \epsilon_{\partial\Sigma} \right) \\ &= \frac{1}{16\pi G} \gamma^{\mu\nu} \eta^\alpha \tau_\mu \mathcal{L}_\xi g_{\nu\alpha} \epsilon_{\partial\Sigma} \\ &= \frac{1}{16\pi G} \left(\tau^\mu \eta^\nu + \tau^\nu \eta^\mu \right) \epsilon_{\partial\Sigma} \nabla_\mu \xi_\nu. \end{aligned} \tag{2.70}$$

And for the last term, one would obtain

$$\xi \cdot l = -\frac{1}{8\pi G} K \xi^\mu \tau_\mu \epsilon_{\partial\Sigma}. \tag{2.71}$$

Collecting all these we obtain,

$$\begin{aligned}
H_\xi &= -\frac{1}{16\pi G}\int_{\partial\Sigma}\left\{\left(\tau^\alpha n^\beta - \tau^\beta n^\alpha\right)\nabla_\alpha\xi_\beta + 2\xi^\mu\tau_\mu K + \left(\tau^\alpha n^\beta + \tau^\beta n^\alpha\right)\nabla_\alpha\xi_\beta\right\}\epsilon_{\partial\Sigma} \\
&= -\frac{1}{8\pi G}\int_{\partial\Sigma}\left\{\tau^\alpha\eta^\beta\nabla_\alpha\xi_\beta + \xi^\mu\tau_\mu K\right\}\epsilon_{\partial\Sigma} \\
&= -\frac{1}{8\pi G}\int_{\partial\Sigma}\left\{-\tau^\alpha\xi^\beta\nabla_\alpha\eta_\beta + \xi^\mu\tau_\mu K\right\}\epsilon_{\partial\Sigma}.
\end{aligned}$$

$$H_\xi = -\frac{1}{8\pi G}\int_{\partial\Sigma}\tau^\alpha\xi^\beta\left\{-K_{\alpha\beta} + \gamma_{\alpha\beta}K\right\}\epsilon_{\partial\Sigma}. \tag{2.72}$$

CHAPTER 3

BLACK HOLE THERMODYNAMICS

The aim of this chapter is to provide a brief, yet comprehensive overview of the thermodynamics of black holes. At a first glance, thermodynamics and black hole mechanics seem to be two subjects that have no particular connection between them. Thermodynamics is a field of study that describes thermal systems in terms of their macroscopic properties and investigates how these thermal systems undergo changes accordingly. The laws of thermodynamics describes the state of such systems in equilibrium.

The zeroth law of thermodynamics defines a state function called temperature θ and proposes a transitivity property between thermal systems which are in thermal contact. If A and B are two thermal systems which are in thermal equilibrium with a third system C, then this law states that both system A and B are in thermal equilibrium as well.

The first law of thermodynamics is energy conservation law which describes how the energy of a system is allowed to change. The change in the energy of the system can be attributed to different ways of transfer such as heat transfer, thermodynamic work or matter transfer. In general, for an infinitesimal change in the energy of the system we have

$$dE = dW + dQ, \tag{3.1}$$

where dW is the work done on the system while dQ denotes the heat transfer into the system.

The second law of thermodynamics defines the entropy (S) of a system and dictates the change in the entropy and in turn, direction of heat transfer in various cases.

$$dQ = TdS. \tag{3.2}$$

And finally, the third law of thermodynamics states that the entropy of an isolated system at equilibrium tends to a constant value as the temperature approaches to absolute zero.

With this brief reminder, we note that thermodynamic laws are observations rather than mathematical proofs or derivations from more simple principles. When we talk about thermodynamics of a system, we often think of the temperature, volume, pressure or various other internal and external properties of the system, which emerge from the collective motion of its constituents.

Then how, can we apply these laws to the mechanics of black holes or other objects that appear in the study of gravitation? Indeed, the laws of thermodynamics summarized above consider the thermal system in a fixed spacetime whereas it becomes much more complex if one takes gravity into account.

In this chapter, we review generalized laws of thermodynamics for Black Holes. We begin by briefly summarizing black hole solutions and investigate their properties such as Killing horizons, and some important theorems like no-hair theorem, which will become important in our understanding of the Black Hole Thermodynamics.

3.1 Review of Black Holes

We begin reviewing black holes by considering the simplest black hole solution, which is namely the Schwarzschild solution. The metric of the Schwarzschild solution of General Relativity without the cosmological constant is

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{3.3}$$

where Ω is the line element of 2-Sphere and

$$f(r) = \left(1 - \frac{2M}{r}\right).$$

We immediately note that $r = 2M$ is a singularity where the metric blows up. However, this is only a coordinate singularity and can be removed by the choice of appropriate coordinates.

Let us, for the purpose of this discussion, adopt the Kruskal-Szekeres coordinates [6]. Defining

$$\begin{aligned} u &= t + r_*, \\ v &= t - r_*. \end{aligned} \tag{3.4}$$

r_* is the so-called tortoise coordinate defined as

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right),\ dr_* = \frac{dr}{f(r)}. \tag{3.5}$$

The Kruskal-Szekeres coordinates then are defined as follows:

$$\begin{aligned} U &= e^{u/4M} = \left(\frac{r}{2m} - 1\right)^{1/2} e^{(r+t)/4M}, \\ V &= -e^{-v/4M} = \left(\frac{r}{2m} - 1\right)^{1/2} e^{(r-t)/4M}. \end{aligned} \tag{3.6}$$

The metric can be then be written as

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU dV + r^2 d\Omega_2, \tag{3.7}$$

with,

$$UV = -e^{r_*/2M}, or\ UV = -\left(\frac{r - 2M}{2M}\right) e^{r/2M}. \tag{3.8}$$

(3.7) is the Schwarzschild metric written in the Kruskal-Szekeres coordinates (U, V, θ, ϕ). The coordinate singularity at $r = 2M$ corresponds to $UV = 0$ and we see that the metric is no longer singular at $r = 2M$. The singularity at $r = 0$, on the other hand corresponds to $UV = 1$, which is still a singularity even in these coordinates. One important property of these coordinates is that they are maximal, in other words all the geodesics can be extended to infinity or end at the curvature singularity at $r = 0$.

The black hole solutions are characterized by their one important property which we will discuss. A black hole is a localized region of spacetime from which nothing can escape. This implies the notion of a boundary which acts like a one-way membrane, allowing anything from the exterior of the black hole to pass though but not vice versa. This boundary, the horizon as it is generally called, must be a hypersurface in the spacetime. Then the question of what kind of hypersurface would the black hole horizon be arises. It turns out that the black hole horizon must be a null hypersurface.

Let us then discuss the geometries of null hypersurfaces. We begin by defining the null hypersurfaces as follows: Let $S(x)$ be a smooth function of space time coordinates x^μ. Then, $S(x)$ =constant defines a family of hypersurfaces. We then define the following vector fields

$$l = f(x) g^{\mu\nu} \partial_\nu \frac{\partial}{\partial x^\mu}, \tag{3.9}$$

where $f(x)$ is an arbitrary function [22]. Vector fields l will be, by construction, normal to the hypersurface. If, for any given hypersurface $\mathcal{N}$ the vector l satisfies $l^2 = 0$ then the hypersurface $\mathcal{N}$ is said to be a null hypersurface. Null hypersurfaces have the strange property that their normal vectors are also tangent to the hypersurface as well. Consider a vector t tangent to $\mathcal{N}$, which is to say that $t \cdot l = 0$ is true. Since $l^2 = 0$ by definition, meaning that l is also the tangent vector for the null hypersurface and can be written as

$$l^\mu = \frac{dx^\mu}{d\lambda}, \tag{3.10}$$

where $x^\mu(\lambda)$ is an arbitrary null curve in $\mathcal{N}$. Curves $x^\mu(\lambda)$ are said to be the generators of null hypersurface $\mathcal{N}$. In our example of Schwarzschild metric (3.7), the null hypersurface is the defined by $U =$ constant and its normal vector (3.9) becomes

$$l = f(x) g^{V\nu}(\partial_\nu U) \frac{\partial}{\partial V},$$

$$= f(x) \left(-\frac{32M^3}{r} e^{-r/2M} \right) \frac{\partial}{\partial V}. \tag{3.11}$$

At the event horizon $r = 2M$:

$$l = -\frac{fe}{16M^2} \frac{\partial}{\partial V}, \tag{3.12}$$

and choosing the function $f(x) = -32M^3 e^{-1}$, we get

$$l = \frac{\partial}{\partial V}. \tag{3.13}$$

Here V is the affine parameter of the generator of this null hypersurface. We also note that it easy to see that at the horizon, $l^2 = 0$ as expected.

A special type of null hypersurfaces, which are called Killing horizons will be special interest of us; If ξ is Killing vector which is normal to the null hypersurface $\mathcal{N}$, then the null hypersurface is said to be a Killing Horizon [22]. Vectors normal to the null hypersurfaces satisfy the geodesics equation

$$l \cdot D l^\mu = 0, \tag{3.14}$$

where D is the covariant derivative defined on the hypersurface and affine parametrization is assumed. If $\mathcal{N}$ is a Killing horizon we can write in general

$$\xi = f l, \tag{3.15}$$

where f is an arbitrary function of spacetime coordinates. Substituting this into (3.14)

$$\xi \cdot D\xi^{\mu} + \xi \cdot (f\partial f^{-1}) = 0,$$
$$\xi \cdot D\xi^{\mu} = \kappa\xi^{\mu}, \tag{3.16}$$

where we identified $\kappa = \xi \cdot \partial lnf$. This is the geodesic equation for the Killing vector field ξ and is true over the entire horizon $\mathcal{N}$. The quantity κ is called surface gravity and has a physical meaning that it is the gravitational acceleration, observed by an observer at infinity, which is required to hold a particle at the horizon. Surface gravity has also the property of being constant on the orbits of ξ. To show this, we make use of the following identity

$$\kappa^2 = -\frac{1}{2}(D^{\mu}\xi^{\nu})(D_{\mu}\xi_{\nu}). \tag{3.17}$$

Then, we look at the change of κ^2 along the direction of ξ

$$\begin{aligned} \xi \cdot D\kappa^2 &= -\left(D^{\mu}\xi^{\nu}\right)\xi^{\lambda}D_{\lambda}D_{\mu}\xi_{\nu}\big|_{\mathcal{N}} \\ &= -\left(D^{\mu}\xi^{\nu}\right)\xi^{\alpha}\xi^{\beta}R_{\mu\nu\alpha\beta} \\ &= 0, \end{aligned} \tag{3.18}$$

where in the last line we used $R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}$. Hence we see that κ does not change along the orbits generated by the Killing vector ξ. Now let us consider an orbit of ξ on which the surface gravity κ is non-zero. This orbit covers only a part of the generators of $\mathcal{N}$. We can see this by choosing a coordinate system on which we have

$$\xi = \frac{\partial}{\partial\alpha}. \tag{3.19}$$

In other words, the group parameter α coincides with one of the coordinates. In terms of the affine parameter λ

$$\xi = \frac{d\lambda}{d\alpha}\frac{d}{d\lambda}. \tag{3.20}$$

By comparing with (3.15), identify

$$f = \frac{d\lambda}{d\alpha},$$
$$l = \frac{d}{d\lambda}. \tag{3.21}$$

Surface gravity in this context is

$$\kappa = \frac{\partial}{\partial \alpha} ln|f|, \tag{3.22}$$

then we have

$$\frac{d^2\lambda}{d\alpha^2} = \kappa \frac{d\lambda}{d\alpha}, \tag{3.23}$$

and,

$$\frac{d\lambda}{d\alpha} = f_0 e^{\kappa\alpha}, \tag{3.24}$$

with the ambiguity in the choice of origin in α, we can set the $f_0 = \pm\kappa$. Then finally we have

$$\frac{d\lambda}{d\alpha} = \pm\kappa e^{\kappa\alpha},\ \lambda = \pm e^{\kappa\alpha}. \tag{3.25}$$

As it is evident from (3.25), specific orbit of ξ does not cover the generators of $\mathcal{N}$. As α varies from $-\infty$ to ∞, it covers either of two regions $\lambda < 0$ or $\lambda > 0$. The point where $\lambda = 0$ corresponds to a special region of spacetime called bifurcation 2-sphere.

After this brief review of the horizon geometry, we can apply the results we have summarized to the example of (3.7). In the Kruskal-Szekeres coordinates, the Killing horizon is the union of the two hypersurfaces $U = 0$ and $V = 0$

$$\mathcal{N} = \{U = 0\} \cup \{V = 0\}. \tag{3.26}$$

Since the (3.7) is a static solution, the Killing vector on the horizon corresponds to one which generates time translations. To find the form of Killing vector, we consider $t \to t+a$ where a is a constant. Under the infinitesimal version of this transformation, Kruskal-Szekeres coordinates (3.6) transform as

$$\delta_t U = \left(\frac{r}{2M} - 1\right)^{1/2} \frac{\epsilon e^{(r+t)/4M}}{4M}, \ \delta U = \frac{\epsilon}{4M} U,$$

$$\delta_t V = -\left(\frac{r}{2M} - 1\right)^{1/2} \frac{\epsilon e^{(r-t)/4M}}{4M}, \ \delta V = \frac{-\epsilon}{4M} V. \tag{3.27}$$

The Killing vector which generates these transformations

$$\xi = \frac{1}{4M} \left(U \frac{\partial}{\partial U} - V \frac{\partial}{\partial V} \right). \tag{3.28}$$

Again, by comparing with (3.15), we identify

$$U = 0; \ \xi = -\frac{V}{4M} \frac{\partial}{\partial V},$$

$$V = 0; \ \xi = \frac{U}{4M} \frac{\partial}{\partial U}. \tag{3.29}$$

The surface gravity κ

$$\begin{aligned} \kappa|_{U=0} &= \xi^V \partial_V ln|f|, \\ \kappa|_{V=0} &= \xi^U \partial_U ln|f|. \end{aligned} \tag{3.30}$$

Thus we find

$$\begin{aligned} \kappa|_{U=0} &= -\frac{1}{4M}, \\ \kappa|_{V=0} &= \frac{1}{4M}. \end{aligned} \tag{3.31}$$

And we obtain $\kappa^2 = 1/(4M)^2$ which is indeed constant.

This concludes the brief discussion of the horizon properties of black holes. We see that the black holes possess a closed region of spacetime called horizons from which nothing can escape. On horizons, the surface gravity is constant for a stationary black hole. This gives us the first clue about the thermodynamical properties of black holes.

3.2 Thermodynamics of Black Holes

Properties of a classical black hole have similarities to thermodynamics. The horizon radius of a black hole is proportional to the mass of the black hole. The Schwarzschild solution, for example has the radius $r_H = 2M$. The horizon area then

$$A = 16\pi M^2 \tag{3.32}$$

For matter falling into the horizon of a black hole, nothing can escape from it and consequently the mass of the black hole must only increase. Then, by (3.32) horizon area of a black hole must be a non-decreasing quantity. This property suggests that black holes can have a notion of entropy.

Existence of an entropy like quantity is not the only similarity between black hole mechanics and thermodynamics. Indeed, there exists a theorem, called no-hair theorem, stating that stationary black holes can be described only by few parameters such as mass or angular momenta. This is again in contrast with the thermodynamics of a system which describes it by its macroscopic quantities.

3.2.1 The Zeroth Law

We saw in section 3.1 that the surface gravity κ is constant over the horizon for a stationary black hole. A thermal system in equilibrium also possesses constant temperature everywhere. The surface gravity κ can then be thought of a temperature parameter of the black hole in equilibrium. Also, both temperature and κ are non-negative parameters. Hence, we can consider the surface gravity defined over the horizon as a temperature. Therefore, the zeroth law of black hole thermodynamics can be stated as follows:

The Zeroth Law: The surface gravity κ is a constant over the horizon of stationary black holes [18].

3.2.2 The First Law

We already see that the mass of a black hole is proportional to its horizon area. Relation (3.32) indicates that if a black hole with mass M were to be perturbed and settles into another black hole with mass $M + \delta M$, the horizon area must also change in proportion to the mass. i,e $\delta M \approx \delta A$. Then we expect the first law of black hole thermodynamics to tell us how changes in the parameters of the black hole happens.

To obtain a general relation, consider a stationary black hole with mass M and angular momentum J. By the no-hair theorem [17], we have

$$M = M(A, J). \tag{3.33}$$

Both A and J have dimensions of M^2 so $M(A, J)$ must be a homogeneous function of degree $1/2$. By the Euler's theorem

$$\begin{aligned} \frac{1}{2}M &= A\frac{\partial M}{\partial A} + J\frac{\partial M}{\partial J} \\ &= \frac{\kappa}{8\pi}A + \Omega_H J, \end{aligned} \tag{3.34}$$

where we used the Smarr relation [21] in the second line. Thus, we have

$$A\left(\frac{\partial M}{\partial A} - \frac{\kappa}{8\pi}\right) + J\left(\frac{\partial M}{\partial J} - \Omega_H\right) = 0. \tag{3.35}$$

Since A and J are free parameters of the black hole, we identify

$$\frac{\partial M}{\partial A} = \frac{\kappa}{8\pi}, \ \frac{\partial M}{\partial J} = \Omega_H. \tag{3.36}$$

Then, the general form of the first law can be written as:

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ. \tag{3.37}$$

3.2.3 The Second Law

We have stated the zeroth and first laws of black hole thermodynamics, but the treatment given is nothing more than making observations and analogies. We observed the similarities between thermodynamics and black hole mechanics to establish these but one must be careful to take these proposed laws to literal meaning. Indeed, we immediately encounter problems if we consider black holes as thermal systems. For one, black holes must have thermal radiation if black holes really possess temperature. But this contradicts the notion of a classical black hole from which nothing can escape. Problems also arises when we assign the horizon area to the entropy, since their dimensions don't match. Entropy is a dimensionless quantity while the horizon area is not.

Solutions to these problems come through by considering the quantum nature of black holes. Black holes are not isolated objects and made by the interactions of matter, and matter obeys quantum mechanics on microscopic scale. These considerations lead to the celebrated area theorem which states that black holes indeed emit thermal radiation, whose temperature is given as [53]

$$k_\beta T = \frac{\hbar\kappa}{2\pi}. \tag{3.38}$$

In order to derive this we assume a smooth Euclidean spacetime with periodic imaginary time [13]. Temperature is then defined as the inverse of this periodicity β. Taking $t_E = it$, a generic metric of a stationary black hole becomes

$$ds_E^2 = f(r)dt_E^2 + \frac{dr^2}{f(r)} + \cdots \tag{3.39}$$

Near the Horizon $r \simeq r_H$, the metric above can be expanded as

$$dS_{\bar{E}}^2 = f'(r_H)(r - r_H)\, dt_E^2 + \frac{dr^2}{f'(r_H)(r - r_H)} \cdots \tag{3.40}$$

To simplify this, let us introduce a coordinate transform $\rho = 2\sqrt{\frac{(r-r_H)}{f'(r_H)}}$

$$dS_E^2 = d\rho^2 + \rho^2 d\left(\frac{f'(r_H)}{2} t_E\right)^2 . \tag{3.41}$$

In this coordinate the metric takes the form of plane in polar coordinates with the following identification of period

$$\frac{f'(r_H)}{2} t_E = 2\pi, \tag{3.42}$$

this leads to

$$\beta = \frac{4\pi}{f'(r_H)},\ or\ T = \frac{f'(r_H)}{4\pi}, \tag{3.43}$$

substituting $\kappa = \frac{f'(r_H)}{2}$ would yield (3.38).

The relation (3.38) is clearly a quantum effect since temperature is proportional to $\hbar$. Using this temperature definition we can find an explicit relation between the entropy and horizon area. From (3.37)

$$\begin{aligned} dM &= \frac{\kappa}{8\pi} dA \\ &= T \frac{k_\beta}{4\hbar} dA. \end{aligned} \tag{3.44}$$

Comparing this with $dE = TdS$, we identify the entropy of a black hole as

$$S = \frac{A}{4\hbar}\frac{1}{k_\beta} = \frac{A}{4l_{pl}^2}\frac{1}{k_\beta}, \tag{3.45}$$

which is the celebrated Hawking-Bekenstein entropy [53].

3.2.4 The Third Law

In contrast with the mechanic counterpart, which states that the entropy of a systems goes to a constant as the temperature approaches to zero, third law of black hole thermodynamics [18] states that it is impossible to reduce the surface gravity κ to zero by a finite sequence of operations.

CHAPTER 4

SMARR FORMULA AND THE EFFECTIVE VOLUME

4.1 Smarr formula in the presence of a cosmological charge

The first law of black hole thermodynamics is a universal constraint between the variations of conserved charges. It is universal, in the sense that it is independent of the spacetime dimension, theory and the Lagrangian, asymptotic conditions and the topology of the black hole. There is another constraint in black hole thermodynamics, a constraint between conserved charges (not their variations) which is called the Smarr relation [21]. This relation is not a universal one. Specially, it explicitly depends on the dimensions of spacetime. Here, we show that in the presence of a cosmological charge, this relation becomes

$$(D-3)M = (D-2)T_{\text{H}}S + (D-2)\Omega_{\text{H}}J + (D-3)\Phi_{\text{H}}Q - \Theta_{\text{H}}C. \tag{4.1}$$

In order to obtain this relation, we use the "scaling method" which is well-known way to derive the Smarr formula (see e.g. [22] or [23]). In this method, M is considered to be a homogeneous function of other charges (S, J, Q, C). Using the Euler theorem, for a function $f(p_1, p_2, \dots)$ homogeneous in the variables $(p_1, p_2, \dots)$, *i.e.* for a constant α one has $\alpha^r f(p_1, p_2, ...) = f(\alpha^{q_1} p_1, \alpha^{q_2} p_2, \dots)$, one can show that

$$rf(p_1, p_2, \dots) = \sum_i q_i \left(\frac{\partial f}{\partial p_i}\right) p_i, \qquad i = 1, 2, \dots. \tag{4.2}$$

We can find the degree of homogeneity in $M = M(S, J, Q, C)$ (*i.e.* the r and q_i in the above equation) using dimensional analysis. The Newton's constant is dimensionful and hence plays a role in the scaling of charges, but as a convenient convention, we set $G = 1$ hereafter. By dimensional analysis, $M \sim l^{D-3}$, $S \sim l^{D-2}$, $J \sim l^{D-2}$, $Q \sim l^{D-3}$, and $C \sim l^{-1}$ where l is a length scale. Therefore, after scaling $l \to \alpha l$, one has

$$\alpha^{D-3} M(S, J, Q, C) = M\left(\alpha^{D-2} S, \alpha^{D-2} J, \alpha^{D-3} Q, \alpha^{-1} C\right). \tag{4.3}$$

Using (4.2) and (4.3), one gets

$$(D-3)M = (D-2)\left(\frac{\partial M}{\partial S}\right) S + (D-2)\left(\frac{\partial M}{\partial J}\right) J + (D-3)\left(\frac{\partial M}{\partial Q}\right) Q - \left(\frac{\partial M}{\partial C}\right) C. \tag{4.4}$$

Finally, using the first law (1.10), we find the Smarr relation (4.1). Needless to say that the analysis above is not a rigorous proof but only a heuristic justification. The Smarr relation may fail in some cases, especially if there are dimensionful quantities other than the conserved charges, as we shall see in some massive gravity theories below.

4.2 Reproducing the effective volume

Since the realization of Λ as a pressure term in the first law, it has been a challenge how to find its thermodynamic conjugate, a "volume" for a black hole. One way to circumvent this problem in the literature has been defining the volume by the first law itself, sometimes called "thermodynamic volume." However, in [23] an ad-hoc but successful (and, importantly, independent from the first law) definition for a viable black hole volume, called "the effective volume" was proposed. It is defined at the horizon by the formula

$$V_{\text{eff}} \equiv \oint_H \star\omega, \qquad \nabla_\mu \omega^{\mu\nu} \equiv \xi_H^\nu. \tag{4.5}$$

Notice that ω is defined by the latter equation, *i.e.* $\xi_H^\nu = \nabla_\mu \omega^{\mu\nu}$, and is ambiguous;

one can deform it by $\omega \to \omega + \omega'$ with an arbitrary divergence free term: $\nabla_\mu \omega'^{\mu\nu} = 0$ (see examples in [24]). ξ_H^ν is the Killing vector at the horizon, and the 2-form $\omega_{\mu\nu}$ is called "the Killing potential," and its Hodge dual $\star\omega$ is a $(D-2)$-form which appears in the integrand of (4.5).

Here, we show how the potential Θ_H in (1.11) reproduces the V_{eff} via the equation

$$\boxed{\Theta_H = \pm\sqrt{|\Lambda|}V_{\text{eff}}.} \tag{4.6}$$

To this end, we begin from the definition of Θ_H in (1.11), denoting the Hodge dual of the integrand in it by $\tilde{\omega}$, *i.e.*

$$\star\tilde{\omega} \equiv \xi_H \cdot A. \tag{4.7}$$

By taking an exterior derivative of both sides,

$$d\star\tilde{\omega} = d(\xi_H \cdot A) \tag{4.8}$$

$$= \mathcal{L}_{\xi_H} A - \xi_H \cdot dA \tag{4.9}$$

$$= -\xi_H \cdot dA. \tag{4.10}$$

In the first equation, we used the Cartan identity $\mathcal{L}_\xi \boldsymbol{a} = \xi \cdot d\boldsymbol{a} + d(\xi \cdot \boldsymbol{a})$ which is correct for any differential form $\boldsymbol{a}$ and any vector field ξ. In the second equation, the isometry/Killing relation $\mathcal{L}_{\xi_H} A = 0$ was used. Using $F = dA$, the on-shell relation (1.6), and definition of the Hodge duality, we find from (4.10)

$$d\star\tilde{\omega} = \pm\sqrt{|\Lambda|}(\star\xi_H). \tag{4.11}$$

Applying the Hodge duality to both sides, and using the identities $(\star d\star\tilde{\omega})^\nu = (-1)^D \nabla_\mu \tilde{\omega}^{\mu\nu}$ and $\star^2 \xi_H = (-1)^D \xi_H$ (see (A.19) and (A.29) in the Appendix A of [25]), then

$$\nabla_\mu \tilde{\omega}^{\mu\nu} = \pm\sqrt{|\Lambda|}\xi_H^\nu. \tag{4.12}$$

Comparing this result with the “Killing potential" $\xi^{\nu}_{H} = \nabla_{\mu}\omega^{\mu\nu}$ in [23], one finds

$$\tilde{\omega}^{\mu\nu} = \pm\sqrt{|\Lambda|}\omega^{\mu\nu} \qquad \Rightarrow \qquad \oint_{\mathrm{H}} \star\tilde{\omega} = \pm\sqrt{|\Lambda|}\oint_{\mathrm{H}} \star\omega. \tag{4.13}$$

From this result, and using (1.11), (4.5) and (4.7), one arrives at the desired result:

$$\Theta_{H} = \pm\sqrt{|\Lambda|}V_{\text{eff}}.$$

An astute reader might wonder about the extra factor $\pm\sqrt{|\Lambda|}$ appearing in the equation above. This factor is not unexpected because the charge C and Λ are quadratically related in (1.9), so

$$\delta C = \frac{\pm\delta\Lambda}{8\pi\sqrt{|\Lambda|}}. \tag{4.14}$$

By the relation $\frac{\delta\Lambda}{8\pi} \equiv \delta P$, we realize that $\delta C = \frac{\pm\delta P}{\sqrt{|\Lambda|}}$. So, the extra factor $\pm\sqrt{|\Lambda|}$ in $\Theta_{H} = \pm\sqrt{|\Lambda|}V_{\text{eff}}$ is cancelled with the extra factor $\frac{\pm1}{\sqrt{|\Lambda|}}$ in δC, yielding same final result, i.e. $\Theta_{\mathrm{H}}\delta C = V_{\text{eff}}\delta P$.

4.3 Fixing the gauge freedom

As was mentioned in the previous section, the effective volume has an ambiguity: a divergence-free 2-form can be added to the Killing potential

$$\omega_{\mu\nu} \to \omega_{\mu\nu} + \omega'_{\mu\nu}, \qquad \nabla_{\mu}\omega'^{\mu\nu} = 0. \tag{4.15}$$

Using the $\tilde{\omega}$ to relate the equations (4.7) and (4.13), it is easy to see that this ambiguity is related to the gauge freedom in $A \to A + d\lambda$ as

$$\omega' = \star\left(\frac{\xi_{H}\cdot d\lambda}{\pm\Lambda}\right). \tag{4.16}$$

As a result, one can fix λ in the “charge formulation of Λ" in order to remove the ω'

ambiguity in the definition effective volume. To this end, we notice that the cosmological gauge field and its variations appear explicitly in the covariant formulation of charges (see Appendix A.1). In order to reproduce the variations of mass, angular momentum and other conserved charges with respect to $\delta\Lambda$, the gauge fixing plays an important role, as we will clarify this issue with different examples.

Summarizing this section, we generalized the Smarr relation to include the contribution from the cosmological conserved charge $\Theta_H C$. Moreover, it was shown how $\Theta_{\rm H}\delta C$ in the cosmological charge formulation reproduces $V_{\rm eff}\delta P$, while resolving its conceptual and computational problems, as well as removing its ambiguity by gauge fixing. In particular, the Θ_H reproduces the $V_{\rm eff}$ as the potential associated with the gauge field A on the horizon. In the rest of the paper, we examine the power of this formulation by studying different examples explicitly. Importantly, we provide the cosmological gauge field $A_{\mu_1\cdots\mu_{D-1}}$ and the corresponding black hole cosmological potential $\Theta_{\rm H}$ for all of these black hole/brane solutions, and check the first law and the Smarr relation for all of them.

CHAPTER 5

EXAMPLES: SOLUTIONS IN 3 DIMENSIONS

We start our analysis of explicit solutions in $3+1$ dimensions with the simplest example, the BTZ black hole [26]. We give the details of the calculations for the BTZ black hole, but we will only give the results of the computations for other examples to avoid repetition.

5.1 BTZ black hole in the cosmological Einstein gravity

Theory: Einstein-Λ theory in 3-dim

$$\mathcal{L} = \frac{1}{16\pi}(R - 2\Lambda). \tag{5.1}$$

Solution: The metric in the coordinates $x^\mu = (t, r, \varphi)$ is [26]

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2(d\varphi - \omega dt)^2, \qquad \Delta \equiv -m + \frac{r^2}{\ell^2} + \frac{j^2}{4r^2}, \qquad \omega \equiv \frac{j}{2r^2}, \tag{5.2}$$

where $\Lambda = \frac{-1}{\ell^2}$. The outer and inner horizons are located at $2r_\pm^2 = \ell^2(m \pm \sqrt{m^2 - \frac{j^2}{\ell^2}})$. The cosmological gauge field for this black hole solution can be found to be (see Appendix A for more details)

$$A = -\frac{r^2}{2\ell} dt \wedge d\varphi. \tag{5.3}$$

Notice that one can add a term $f(m, j, \ell) dt \wedge d\varphi$ to A, which clearly does not change the field strength $F = dA$ if f is not a function of space-time coordinates. This is a simple example of the gauge freedom that we have discussed before. Nonetheless, the A and its variations with respect to m, j, ℓ appear explicitly in the charge calculations (mass, angular momentum and cosmological charge). To see this, the Appendix-A.1 is provided.. Requesting the charges to be reproduced correctly in the new paradigm (in comparison to the usual paradigm of Λ being a constant in Lagrangian) fixes the gauge freedom for our example to be what it is already in Eq.(5.3).

Properties:

$$M = \frac{m}{8}, \qquad J = \frac{j}{8}, \qquad \Omega_\pm = \frac{r_\mp}{\ell r_\pm}, \qquad T_\pm = \frac{r_\pm^2 - r_\mp^2}{2\pi\ell^2 r_\pm}, \qquad S_\pm = \frac{\pi r_\pm}{2}. \quad (5.4)$$

The horizon Killing vectors are $\xi_\pm = \partial_t + \Omega_\pm \partial_\varphi$. Using A from (5.3) in the definition of $\Theta_{\rm H}$ in (1.11), we get

$$\begin{aligned} \Theta_\pm &= \int_{r_\pm} (\partial_t + \Omega_\pm \partial_\phi) \cdot \left(\frac{-r^2}{2\ell} dt \wedge d\phi \right) \\ &= \int_{r_\pm} \frac{-r^2}{2\ell} d\varphi - \int_{r_\pm} \Omega_\pm \frac{-r^2}{2\ell} dt. \end{aligned} \quad (5.5)$$

However, the last integral vanishes because the pull-back of the dt to the surface of integration (which is the bifurcation point of the horizon parametrized by the coordinate φ) vanishes. So,

$$\Theta_\pm = \int_{r_\pm} \frac{-r^2}{2\ell} d\varphi = -\frac{\pi r_\pm^2}{\ell}. \quad (5.6)$$

The cosmological charge C can be read from (1.9) with the lower sign (which is the one for negative Λ) to be

$$C = -\frac{1}{4\pi\ell}. \quad (5.7)$$

The first law and the Smarr formula:

The generalized first law and the Smarr formula for the BTZ black hole read

$$\delta M = T_\pm \delta S_\pm + \Omega_\pm \delta J + \Theta_\pm \delta C, \tag{5.8}$$

$$0 = T_\pm S_\pm + \Omega_\pm J - \Theta_\pm C \tag{5.9}$$

respectively. One can check the validity of these two relations explicitly which we do next. Let us check the Smarr formula first. Substituting (5.4) and (5.6) to (5.9) one has

$$0 = \frac{r_\pm^2 - r_\mp^2}{2\pi \ell^2 r_\pm} \frac{\pi r_\pm}{2} + \frac{r_\mp}{\ell r_\pm} \frac{j}{8} - \frac{-\pi r_\pm^2}{\ell} \frac{-1}{4\pi \ell} \qquad \Rightarrow \qquad 0 = r_\mp \left(\frac{-r_\mp}{4\ell^2} + \frac{j}{8\ell r_\pm} \right), \tag{5.10}$$

which is satisfied for $2r_\pm^2 = \ell^2 \left(m \pm \sqrt{m^2 - \frac{j^2}{\ell^2}} \right)$. Hence, the Smarr formula holds.

Now, let us look at the validity of the first law of black hole thermodynamics (5.8). This solution has three independent parameters m, j, ℓ. Notice that ℓ is a free parameter of the solution, if the Lagrangian (1.2) is the Lagrangian describing the theory of gravity. In other words, ℓ is related to Λ by $\Lambda = \frac{-1}{\ell^2}$, and Λ is related to c in (1.6) (which is a free parameter of the solution), by the relation (1.7). We calculate variations to nearby black hole solutions with respect to each of these three parameters. This method of variation can be called parametric variations [27]. We can begin with variation in m parameter

$$\delta_m M = \frac{\delta m}{8}, \qquad \delta_m S_\pm = \frac{\pi}{2} \delta_m r_\pm, \qquad \delta_m J = 0, \qquad \delta_m C = 0, \tag{5.11}$$

where $\delta_m r_\pm$ reads as follows

$$\delta_m r_\pm = \frac{\partial r_\pm}{\partial m} \delta m = \pm \frac{\sqrt{2\ell^3 (m\ell \pm \sqrt{m^2 \ell^2 - j^2})}}{4\sqrt{m^2 \ell^2 - j^2}} \delta m. \tag{5.12}$$

Substituting (5.11), (5.12) in the first law (5.8), one finds

$$\frac{\delta m}{8} = \left(\frac{r_\pm^2 - r_\mp^2}{2\pi\ell^2 r_\pm}\right)\left(\pm\frac{\pi}{2}\frac{\sqrt{2\ell^3(m\ell \pm \sqrt{m^2\ell^2 - j^2})}}{4\sqrt{m^2\ell^2 - j^2}}\delta m\right) \tag{5.13}$$

$$\Rightarrow \frac{r_\pm}{r_\pm^2 - r_\mp^2} = \pm\frac{\sqrt{\frac{\ell^2}{2}(m \pm \sqrt{m^2 - \frac{j^2}{\ell^2}})}}{\ell^2\sqrt{m^2 - \frac{j^2}{\ell^2}}}, \tag{5.14}$$

which is satisfied by $r_\pm$. Similarly, for the variation of j

$$\delta_j M = 0, \qquad \delta_m S_\pm = \frac{\pi}{2}\delta_j r_\pm, \qquad \delta_j J = \frac{\delta j}{8}, \qquad \delta_j C = 0, \tag{5.15}$$

in which

$$\delta_j r_\pm = \mp\frac{\sqrt{2\ell}\, j\delta j}{4\sqrt{m^2\ell^2 - j^2}(m\ell - \sqrt{m^2\ell^2 - j^2})}. \tag{5.16}$$

Inserting in the first law (5.8), we find

$$0 = \left(\frac{r_\pm^2 - r_\mp^2}{2\pi\ell^2 r_\pm}\right)\left(\mp\frac{\pi}{2}\frac{\sqrt{2\ell}\, j\delta j}{4\sqrt{m^2\ell^2 - j^2}(m\ell \pm \sqrt{m^2\ell^2 - j^2})}\right) + \frac{r_\mp}{\ell r_\pm}\frac{\delta j}{8},$$

$$\Rightarrow \qquad \frac{r_\mp}{r_\pm^2 - r_\mp^2} = \pm\frac{\sqrt{2\ell}\, j}{4\ell(\sqrt{m^2\ell^2 - j^2})\sqrt{m\ell \pm \sqrt{m^2\ell^2 - j^2}}}. \tag{5.17}$$

Using the relations $\pm\ell\sqrt{m^2\ell^2 - j^2} = (r_\pm^2 - r_\mp^2)$ and $\ell\sqrt{m \pm \sqrt{m^2 - j^2/\ell^2}} = \sqrt{2}r_\pm$ in the denominator of the right hand side, the result in (5.17) simplifies to $4r_\pm r_\mp = 2\ell j$ which is the correct equation, admitting the first law to be satisfied.

We should also check the first law for the variation with respect to ℓ. To this end we have

$$\delta_\ell M = 0, \qquad \delta_\ell S_\pm = \frac{\pi}{2}\delta_\ell r_\pm, \qquad \delta_\ell J = 0, \qquad \delta_\ell C = \frac{\delta\ell}{4\pi\ell^2}, \tag{5.18}$$

in which

$$\delta_\ell r_\pm = \pm \frac{(2m^2\ell^2 - j^2 \pm 2m\ell\sqrt{m^2\ell^2 - j^2})\delta\ell}{4\sqrt{\frac{\ell}{2}(m^2\ell^2 - j^2)(m\ell \pm \sqrt{m^2\ell^2 - j^2})}}. \tag{5.19}$$

Putting these in the first law (5.8), it follows that

$$0 = (\frac{r_\pm^2 - r_\mp^2}{2\pi\ell^2 r_\pm})\left(\pm\frac{\pi}{2}\frac{(2m^2\ell^2 - j^2 \pm 2m\ell\sqrt{m^2\ell^2 - j^2})\delta\ell}{4\sqrt{\frac{\ell}{2}(m^2\ell^2 - j^2)(m\ell \pm \sqrt{m^2\ell^2 - j^2})}}\right) + (-\frac{\pi r_\pm^2}{\ell})(\frac{\delta\ell}{4\pi\ell^2}),$$

$$\Rightarrow \quad \frac{r_\pm^3}{r_\pm^2 - r_\mp^2} = \pm\frac{\ell(2m^2\ell^2 - j^2 \pm 2m\ell\sqrt{m^2\ell^2 - j^2})}{4\sqrt{\frac{\ell}{2}(m^2\ell^2 - j^2)(m\ell \pm \sqrt{m^2\ell^2 - j^2})}}. \tag{5.20}$$

Using the relations $\pm\ell\sqrt{m^2\ell^2 - j^2} = (r_\pm^2 - r_\mp^2)$ and $\ell\sqrt{m \pm \sqrt{m^2 - j^2/\ell^2}} = \sqrt{2}r_\pm$ in the denominator of the right hand side, it reduces to $4r_\pm^4 = \ell^2(2m^2\ell^2 - j^2 \pm 2m\ell\sqrt{m^2\ell^2 - j^2})$, which is the correct equation.

According to the analysis above, we deduce that the generalized first law in (5.8) and the Smarr formula in (5.9), which include the new term $\Theta_H \delta C$ and $\Theta_H C$, are correct relations for this example and confirm the results of the analysis in this paper.

5.2 Charged static BTZ black hole

Theory: Einstein-Maxwell-Λ theory in $2+1$ dimensions

$$\mathcal{L} = \frac{1}{16\pi}(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}). \tag{5.21}$$

Solution: The metric and the Maxwell gauge field in the coordinates $x^\mu = (t, r, \varphi)$ are [28]

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\varphi^2, \qquad \Delta \equiv -m + \frac{r^2}{\ell^2} - \frac{q^2}{2}\log\frac{r}{\ell}, \tag{5.22}$$

$$A = -\frac{q}{2}\log(\frac{r}{\ell})\, dt, \tag{5.23}$$

with $\Lambda = \frac{-1}{\ell^2}$. Horizons are at $\Delta = 0$. For this solution, the cosmological gauge field $\boldsymbol{A}$ (denoted bold in order to be distinguished from the Maxwell field A_μ) in an appropriate gauge is (see Appendix A)

$$\boldsymbol{A} = -(\frac{4r^2 - q^2\ell^2}{8\ell})dt \wedge d\varphi. \tag{5.24}$$

The gauge freedom of $\boldsymbol{A}$ is fixed such that it reproduces the variation of the mass and the other charges with respect to ℓ correctly. To see this, one can use the covariant phase space formulation of charges. The details of the formulation are described in [29, 30, 1]. However, for the sake of completeness, we have added Appendix-A.1 which provides the final formula to perform such charge calculations.

Properties: Horizon Killing vectors are $\xi_H = \partial_t$. Using $\boldsymbol{A}$ from (5.24) in the definition of Θ_{H} in (1.11), we get

$$\begin{aligned}\Theta_H &= \int_{r_H} (\partial_t) \cdot \left(\frac{-(4r^2 - q^2\ell^2)}{8\ell} dt \wedge d\varphi\right) \\ &= \int_{r_H} (\frac{-(4r^2 - q^2\ell^2)}{8\ell})d\varphi \\ &= -\frac{\pi(4r_H^2 - q^2\ell^2)}{4\ell}.\end{aligned} \tag{5.25}$$

For the other properties, including C from (1.9), we find

$$M = \frac{m}{8}, \qquad Q = \frac{q}{4}, \qquad C = -\frac{1}{4\pi\ell}, \qquad \Theta_H = -\frac{\pi(4r_H^2 - q^2\ell^2)}{4\ell}$$

$$\Phi_H = -\frac{q}{2}\log\frac{r_H}{\ell}, \qquad T_H = \frac{4r_H^2 - q^2\ell^2}{8\pi r_H \ell^2}, \qquad S_H = \frac{\pi r_H}{2}. \tag{5.26}$$

The generator of the entropy as a conserved charge is $\eta_{\text{H}} = \frac{1}{T_{\text{H}}}\{\partial_t, -\Phi_{\text{H}}\}$ [29, 30].

The first law and the Smarr formula:

The generalized first law and the Smarr formula for this solution are

$$\delta M = T_H \delta S_H + \Phi_H \delta Q + \Theta_H \delta C, \tag{5.27}$$

$$0 = T_H S_H - \Theta_H C \tag{5.28}$$

respectively. The Smarr relation can be checked easily using (5.26). To check the first law, notice that the solution has three free parameters m,q and ℓ. Using the relations

$$\delta_m r_H = \frac{2\ell^2 r_H \delta m}{4r_H^2 - q^2\ell^2}, \qquad \delta_q r_H = \frac{2q\ell^2 r_H \log(\frac{r_H}{\ell})\delta q}{4r_H^2 - q^2\ell^2}, \qquad \delta_\ell r_H = \frac{r_H}{\ell}\delta\ell, \tag{5.29}$$

and following the same steps as in the example 3.1, the first law can also be checked. The result is affirmative, and the first law holds for the charged static BTZ black hole.

5.3 Lifshitz $z = 3$ black hole

Theory: New Massive Gravity (NMG) theory in $2 + 1$ dimensions [31]

$$\mathcal{L} = \frac{1}{16\pi}\left(R - 2\Lambda + \frac{1}{\mathfrak{m}^2}(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2)\right). \tag{5.30}$$

Solution: The metric in the coordinates $x^\mu = (t, r, \varphi)$ is [32, 33]

$$ds^2 = -(\frac{r}{\ell})^{2z}(1 - \frac{m\ell^2}{r^2})\, dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2}(1 - \frac{m\ell^2}{r^2})} + r^2 d\varphi^2 \tag{5.31}$$

for $z = 3$, one has $\Lambda = -\frac{13}{2\ell^2}$ and $\mathfrak{m}^2 = \frac{1}{2\ell^2}$. Notice that $\mathfrak{m}$ and m are different parameters: the former is a parameter in the Lagrangian, and the latter is a parameter of the solution. The event horizon is at $r_H = \sqrt{m\ell^2}$. The Cosmological gauge field in an appropriate gauge for this solution is (see appendix A)

$$A = \sqrt{|\Lambda|}\left(\frac{3m^2}{8\Lambda} - \frac{r^4}{4\ell^2}\right) dt \wedge d\varphi. \tag{5.32}$$

Properties:

For this black hole, one can find [34, 35, 36, 37, 30]

$$M = \frac{m^2}{4}, \qquad C = -\sqrt{\frac{13}{2}}\frac{1}{4\pi\ell}, \qquad T_H = \frac{r_H^3}{2\pi\ell^4}, \qquad S_H = 2\pi r_H. \tag{5.33}$$

Using the horizon Killing vector $\xi_H = \partial_t$ and A from (5.32) in (1.11), we find

$$\begin{aligned}\Theta_H &= \int_{r_H} (\partial_t) \cdot \left(\sqrt{|\Lambda|}(\frac{3m^2}{8\Lambda} - \frac{r^4}{4\ell^2}) dt \wedge d\varphi\right) \\ &= \int_{r_H} \sqrt{|\Lambda|}(\frac{3m^2}{8\Lambda} - \frac{r^4}{4\ell^2}) d\varphi \\ &= -\sqrt{\frac{2}{13}} 4\pi m^2 \ell, \end{aligned} \tag{5.34}$$

where, in the last equality, we used $\Lambda = -\frac{13}{2\ell^2}$ and $r_H = \sqrt{m\ell^2}$.

The First law and the Smarr formula:

This solution has two free parameters m and ℓ. The horizon radius in terms of these two parameters is $r_H = \sqrt{m\ell^2}$, which makes the calculations very simple. Using (5.33) and (5.34), the generalized first law and Smarr formula for this solution

$$\delta M = T_H \delta S_H + \Theta_H \delta C, \tag{5.35}$$

$$0 = T_H S_H - \Theta_H C \tag{5.36}$$

can be checked easily for variations with respect to m and ℓ. Hence, for this example the first law and Smarr formula hold.

5.4 BTZ black hole in the New Massive Gravity

Theory: The theory is again the NMG theory in 3 dimensions [31]

$$\mathcal{L} = \frac{1}{16\pi}\left(R - 2\Lambda + \frac{1}{\mathfrak{m}^2}(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2)\right). \tag{5.37}$$

Solution: The solution is exactly the same BTZ solution reviewed in Example 5.1, i.e. in coordinates $x^\mu = (t, r, \varphi)$ the metric is [26, 38]

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2(d\varphi - \omega dt)^2, \; \Delta \equiv -m + \frac{r^2}{\ell^2} + \frac{j^2}{4r^2}, \; \omega \equiv \frac{j}{2r^2}, \tag{5.38}$$

but for $\Lambda = \frac{-1}{\ell^2} + \frac{1}{4\ell^4\mathfrak{m}^2}$ and we assume also $\Lambda < 0$. Horizons are at $2r_\pm^2 = \ell^2(m \pm \sqrt{m^2 - \frac{j^2}{\ell^2}})$. Cosmological gauge field for this black hole solution can be found to be (see appendix A)

$$A = -\sqrt{|\Lambda|}\left(\frac{r^2}{2} - \frac{m\ell^2}{2(1 - 2\mathfrak{m}^2\ell^2)}\right) dt \wedge d\varphi. \tag{5.39}$$

The gauge freedom (i.e. the second term in the parenthesis) is fixed such that using the covariant phase space formulation of charges (see Appendix-A.1), or other methods such as the ADT formulation [39, 40, 41], yields correct mass variations with respect to ℓ as well as other solution parameters.

Properties: Although this black hole is exactly the same as the BTZ black hole in Example 3.1, it is solution to a different theory which affects the charges M, J, and S [38, 42]

$$M = (1 + \frac{1}{2\ell^2 \mathfrak{m}^2})\frac{m}{8}, \qquad J = (1 + \frac{1}{2\ell^2 \mathfrak{m}^2})\frac{j}{8},$$

$$\Omega_\pm = \frac{r_\mp}{\ell r_\pm}, \qquad T_\pm = \frac{r_\pm^2 - r_\mp^2}{2\pi\ell^2 r_\pm}, \qquad S_\pm = (1 + \frac{1}{2\ell^2 \mathfrak{m}^2})\frac{\pi r_\pm}{2}. \tag{5.40}$$

Horizon Killing vectors are $\xi_\pm = \partial_t + \Omega_\pm \partial_\varphi$. Cosmological charge and horizon potential for this solution are

$$C = -\frac{\sqrt{|\Lambda|}}{4\pi}, \qquad \Theta_\pm = -\pi\sqrt{|\Lambda|}\left(r_\pm^2 - \frac{m\ell^2}{1 - 2\mathfrak{m}^2\ell^2}\right). \tag{5.41}$$

First law and Smarr formula:
The generalized first law for this solution is

$$\delta M = T_\pm \delta S_\pm + \Omega_\pm \delta J + \Theta_\pm \delta C, \tag{5.42}$$

which can be checked to be a correct relation by using variations with respect to three free parameters of this solution m, j and ℓ. On the other hand, the generalized Smarr formula is *not* satisfied for this solution. To satisfy the Smarr formula one needs to take into account the dimensionful quantity $\mathfrak{m}$ in a suitable way which we have not been able to do so far.

5.5 Horndeski BTZ-like black hole

Theory: A Horndeski gravity in 3 dimensions [44] has the Lagrangian

$$\mathcal{L} = \frac{1}{16\pi}\Big(R - 2\Lambda - 2(\alpha g_{\mu\nu} - \gamma G_{\mu\nu})\nabla^\mu\phi\nabla^\nu\phi\Big), \tag{5.43}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor.

Solution: The metric in the coordinates $x^\mu = (t, r, \varphi)$ is [45]

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\varphi^2 - \frac{j}{r^2}dt),$$

$$f = -m + \frac{\alpha r^2}{\gamma} + \frac{j^2}{r^2}, \quad d\phi = \sqrt{\frac{-(\alpha + \gamma\Lambda)}{2\alpha\gamma f}}\, dr \tag{5.44}$$

where $\gamma < 0$ and (m, j) are free parameters of the solution. The cosmological gauge field for this solution can be found to be (see appendix A)

$$A = -\sqrt{|\Lambda|}\left(\frac{r^2}{2} - \frac{\gamma m}{4\alpha}\right) dt \wedge d\varphi. \tag{5.45}$$

The gauge is fixed such that the covariant formulation of conserved charges (see Appendix A.1) produces correct mass variation with respect to Λ, i.e. the $\delta_\Lambda M$.

Properties: For this solution, the charges and the chemical potentials are computed to be [46]

$$M = \frac{(\alpha - \Lambda\gamma)m}{16\alpha}, \quad J = \frac{(\alpha - \Lambda\gamma)j}{8\alpha}, \quad r_\pm^2 = \frac{\gamma m \mp \sqrt{\gamma^2 m^2 - 4\gamma\alpha j^2}}{2\alpha},$$

$$\Omega_\pm = \frac{j}{r_\pm^2}, \quad \kappa_\pm = \frac{\alpha(r_+^2 - r_-^2)}{\gamma r_\pm}, \quad T_\pm = \left(\frac{\alpha - \Lambda\gamma}{4\pi\alpha}\right)\kappa_\pm, \quad S_\pm = \frac{\pi r_\pm}{2}. \tag{5.46}$$

Notice that $\alpha < 0$ in order to have finite and positive horizon radii. Note also that the temperature is different from the usual $\frac{\kappa}{2\pi}$ (i.e. the standard Hawking temperature) by a factor $\frac{\alpha-\Lambda\gamma}{4\pi\alpha}$ which is a result of the fact that in Horndeski gravities, the effective speed of the graviton can be (as in our example here) different from 1 [46]. The cosmological charge and the horizon potential, using (1.11) and (5.45), are

$$C = -\frac{\sqrt{|\Lambda|}}{4\pi}, \qquad \Theta_{\pm} = -\pi\sqrt{|\Lambda|}(r_{\pm}^2 - \frac{\gamma m}{2\alpha}). \tag{5.47}$$

First law and Smarr formula:

This solution has three free parameters m, j and Λ. The generalized first law for this solution is

$$\delta M = T_{\pm}\delta S_{\pm} + \Omega_{\pm}\delta J + \Theta_{\pm}\delta C, \tag{5.48}$$

which can be checked to be a correct relation by using variations with respect to three free parameters of this solution. For this solution, the generalized Smarr formula is *not* satisfied as in the previous example. So one should find the correct formula taking into account all the dimensionful parameters in the theory.

CHAPTER 6

SOLUTIONS IN 4 DIMENSIONS

6.1 (A)dS-Kerr-Newman black hole

Theory: Einstein-Maxwell-Λ theory in 4 dimensions [47, 48, 49, 50, 51, 52]

$$\mathcal{L} = \frac{1}{16\pi}(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}). \tag{6.1}$$

Solution: The metric in the coordinates $x^\mu = (t, r, \theta, \varphi)$ is

$$\mathrm{d}s^2 = -\Delta_\theta\left(\frac{1-\frac{\Lambda r^2}{3}}{\Xi} - \Delta_\theta f\right)\mathrm{d}t^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 - 2\Delta_\theta f a \sin^2\theta\,\mathrm{d}t\mathrm{d}\varphi$$

$$+ \left(\frac{r^2+a^2}{\Xi} + f a^2 \sin^2\theta\right)\sin^2\theta\,\mathrm{d}\varphi^2\,, \tag{6.2}$$

where

$$\rho^2 \equiv r^2 + a^2\cos^2\theta\,, \qquad \Delta_r \equiv (r^2+a^2)(1-\frac{\Lambda r^2}{3}) - 2mr + q^2\,,$$

$$\Delta_\theta \equiv 1 + \frac{\Lambda a^2}{3}\cos^2\theta\,, \qquad \Xi \equiv 1 + \frac{\Lambda a^2}{3}\,, \qquad f \equiv \frac{2mr - q^2}{\rho^2\Xi^2}\,.$$

In these coordinates, the Maxwell gauge field is

$$A = \frac{qr}{\rho^2\Xi}(\Delta_\theta dt - a\sin^2\theta\, d\varphi)\,. \tag{6.3}$$

For positive and negative signs of Λ, the solution is de Sitter or Anti de Sitter Kerr-Newman black hole respectively. The analysis here is independent of this sign, and we leave it to be either positive or negative. We denote the cosmological gauge field by $\boldsymbol{A}$, in order to distinguish it from the Maxwell gauge field A. For this solution, $\boldsymbol{A}$ can be found to be (see appendix A)

$$\boldsymbol{A} = -\frac{\sqrt{|\Lambda|}(r^3 + 3ra^2\cos^2\theta + \frac{ma^2}{\Xi})\sin\theta}{3\Xi} dt \wedge d\theta \wedge d\varphi. \tag{6.4}$$

Similar to the other solutions described above, the gauge is fixed if one demands that the mass, angular momentum and other charges to be reproduced correctly by the covariant formulation of charges.

Properties: One can find the thermodynamic variables for this solution irrespective of the sign of Λ as [54, 55]

$$M = \frac{m}{\Xi^2}, \qquad J = \frac{ma}{\Xi^2}, \qquad Q = \frac{q}{\Xi}, \qquad \Phi_H = \frac{qr_H}{r_H^2 + a^2},$$

$$\Omega_H = \frac{a(1 - \frac{\Lambda r_{\text{H}}^2}{3})}{r_{\text{H}}^2 + a^2}, \; T_H = \frac{r_{\text{H}}(1 - \frac{\Lambda a^2}{3} - \Lambda r_{\text{H}}^2 - \frac{a^2}{r_{\text{H}}^2})}{4\pi(r_{\text{H}}^2 + a^2)}, \; S_{\text{H}} = \frac{\pi(r_{\text{H}}^2 + a^2)}{\Xi}, \tag{6.5}$$

in which r_{H} is the radius of the considered horizon. The cosmological charge and potential can also be found by the equations (1.9) and (1.11)

$$C = \pm\frac{\sqrt{|\Lambda|}}{4\pi}, \qquad \Theta_H = -\frac{\sqrt{|\Lambda|}4\pi(r_H^3 + r_H a^2 + \frac{ma^2}{\Xi})}{3\Xi}. \tag{6.6}$$

The upper and lower signs are for de Sitter and Anti de Sitter black holes respectively.

First law and Smarr formula:

This solution has four free parameters (m, a, q, Λ). Using (6.5) and (6.6), the generalized first law and Smarr formula for this solution

$$\delta M = T_H \delta S_H + \Omega_H \delta J + \Phi_H \delta Q + \Theta_H \delta C, \tag{6.7}$$

$$M = 2T_H S_H + 2\Omega_H J + \Phi_H Q - \Theta_H C, \tag{6.8}$$

can be checked for variations with respect to the parameters $p_i \in \{m, a, q, \Lambda\}$. Hence, for this example the generalized first law and Smarr formula hold. In appendix A.2 the method of checking the first law and Smarr formula are described, if the horizon radii cannot be found explicitly in terms of the parameters p_i of the solution.

6.2 A black hole in Horndeski gravity

Theory: The Lagrangian of the theory is [44]

$$\mathcal{L} = \frac{1}{16\pi}\Big((1+\beta\sqrt{-X})R - 2\Lambda + \eta X - \frac{\beta}{2\sqrt{-X}}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2]\Big) \tag{6.9}$$

β, η are constants. A black hole solution for this theory is introduced in [56] with the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{6.10}$$

and

$$f = 1 - \frac{2m}{r} - \frac{\beta^2}{2\eta r^2} - \frac{\Lambda r^2}{3}, \quad d\phi = \frac{\sqrt{2}\beta}{\eta r^2\sqrt{f}}dr. \tag{6.11}$$

The cosmological gauge field for this solution is (see appendix A)

$$A = -\frac{\sqrt{|\Lambda|}}{3} r^3 \sin\theta \, dt \wedge d\theta \wedge d\varphi, \tag{6.12}$$

which is fixed in a gauge such that it reproduces the mass correctly using the covariant formulation of charges in appendix A.1.

Properties: The mass, temperature and entropy for this solution are [46]

$$M = m, \qquad T_{\text{H}} = \frac{\beta^2 + 2\eta(r_{\text{H}}^2 - \Lambda r_{\text{H}}^4)}{8\pi\eta r_{\text{H}}^3}, \qquad S = \pi r_{\text{H}}^2, \tag{6.13}$$

and the cosmological charge and potential are

$$C = -\frac{\sqrt{|\Lambda|}}{4\pi}, \qquad \Theta_H = -\frac{\sqrt{|\Lambda|}4\pi r_H^3}{3}. \tag{6.14}$$

First law and Smarr formula:
This solution has two free parameters m, Λ. The generalized first law for this solution is

$$\delta M = T_H \delta S_H + \Theta_H \delta C, \tag{6.15}$$

which can be checked to be a correct relation by using variations with respect to the two free parameters of this solution. For this solution, the generalized Smarr formula is *not* satisfied as in some of the examples above.

6.3 A black brane in Horndeski gravity

Theory: The Lagrangian of the theory is [44]

$$\mathcal{L} = \frac{1}{16\pi}\Big(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu} - 2(\alpha g_{\mu\nu} - \gamma G_{\mu\nu})\nabla^\mu\phi\nabla^\nu\phi\Big) \tag{6.16}$$

in which $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor.

Solution: The metric in the coordinates $x^\mu = (t, r, x, y)$ is

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \tag{6.17}$$

$$h = \frac{r^2}{\ell^2} - \frac{m}{r} + \frac{4q^2}{(4+\beta)r^2} - \frac{4q^4\ell^2}{15(4+\beta)^2 r^6},$$

$$f = \frac{(4+\beta)^2 r^8 h}{\left(\frac{2q^2\ell^2}{3} - (4+\beta)r^4\right)^2},$$

$$d\phi = \sqrt{\frac{\beta - \frac{2q^2\ell^2}{3r^4}}{4\gamma f}}dr, \quad A = \big(\frac{q}{r} - \frac{2q^3\ell^2}{15(4+\beta)r^5}\big)dt. \tag{6.18}$$

with [57]

$$\Lambda = -\frac{3(1+\frac{\beta}{2})}{\ell^2}, \qquad \alpha = \frac{3\gamma}{\ell^2}. \tag{6.19}$$

It is easy to see that in order to vary Λ while keeping the α fixed, one can simply take variations with respect to β. So, in order to check the first law we will use variations with respect to β which appears explicitly in the solution, instead of the Λ. The cosmological gauge field for this solution is (see appendix A

$$\boldsymbol{A} = -\sqrt{|\Lambda|}\left(\frac{r^3}{3} + \frac{2q^2\ell^2}{3r(4+\beta)} - \frac{m\ell^2}{6}\right) dt \wedge d\theta \wedge d\varphi. \tag{6.20}$$

The first two terms in the parenthesis are determined by the equation $F = d\boldsymbol{A}$ and Eq.(1.6), while the last term in the parenthesis is a gauge fixing term, i.e. it does not contribute to F by the equation $F = d\boldsymbol{A}$. This gauge fixing term is determined by putting the $\boldsymbol{A}$ and its variations into the covariant formulation of charges to reproduce mass correctly.

Properties: The mass, electric charge, and entropy “densities" for this solution are

[46]

$$M=\frac{(4+\beta)m}{32\pi},\qquad Q=\frac{q}{4\pi},\qquad S=\frac{r_H^2}{4}. \tag{6.21}$$

By densities it is understood that the charges are calculated without performing the integration over the x and y coordinates. Besides, surface gravity and electric potential on the horizon are

$$\kappa=\frac{3r_{\text{H}}}{2\ell^2}-\frac{q^2}{(4+\beta)r_{\text{H}}^3},\qquad \Phi_{\text{H}}=\frac{q}{r_{\text{H}}}-\frac{2q^3\ell^2}{15(4+\beta)r_{\text{H}}^5}. \tag{6.22}$$

This example is a very special example in this work, because the standard (as well as the generalized) first law and Smarr formula do not hold if one uses the Hawking temperature $T_0=\frac{\kappa}{2\pi}$ as the temperature of the black brane. However, in the Ref.[46] it is shown that this a generic feature in Horndeski gravity (and any model of gravity in which the speed of graviton differs from $c=1$). The physical temperature in Hawking radiation is dominated by the gravitons, and it is related to the Hawking temperature by an overall factor which is a function of the parameters of the solution. The interested reader is invited to study the original paper [46] for the details. Here, we only report the final result for the example under considerations. The physical temperature is related to the T_0 by

$$T_H=\left(\frac{3(4+\beta)r_H^4-2q^2\ell^2}{12r_{\text{H}}^4}\right)T_0. \tag{6.23}$$

The cosmological charge and potential for this solution are

$$C=-\frac{\sqrt{|\Lambda|}}{4\pi},\qquad \Theta_H=-\sqrt{|\Lambda|}\left(\frac{r_H^3}{3}+\frac{2q^2\ell^2}{3r_H(4+\beta)}-\frac{m\ell^2}{6}\right). \tag{6.24}$$

First law and Smarr formula:

This solution has three free parameters m, q, and β. This latter parameter is representative of the Λ in the solution. The generalized first law for this solution is

$$\delta M = T_H \delta S_H + \Phi_H \delta Q + \Theta_H \delta C, \tag{6.25}$$

which can be checked to be a correct relation by using variations with respect to the three free parameters of this solution. For this solution, the generalized Smarr formula is *not* satisfied as like as some of the previous examples.

6.4 MTZ black hole

Theory: The Lagrangian has the metric $g_{\mu\nu}$, a scalar field ϕ, and the Maxwell gauge field A_μ as dynamical fields [58, 60]

$$\mathcal{L} = \frac{1}{16\pi} \left(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu} - 2\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{3}R\phi^2 - \alpha\phi^4 \right). \tag{6.26}$$

Solution: The dynamical fields in the coordinates $x^\mu = (t, r, \theta, \varphi)$ are [58, 60]

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad f = (1 - \frac{m}{r})^2 - \frac{r^2}{\ell^2},$$

$$A = \frac{q}{r}, \qquad \phi = \frac{\sqrt{3(m^2 - q^2)}}{r - m}, \tag{6.27}$$

where

$$\Lambda = \frac{3}{\ell^2}, \qquad q^2 = m^2(1 + \frac{2\Lambda}{9\alpha}). \tag{6.28}$$

Horizon radii are at $r_\pm = \frac{\ell}{2}(\pm 1 \mp \sqrt{1 \mp \frac{4m}{\ell}})$, and cosmological horizon is at $r_c = \frac{\ell}{2}(1 + \sqrt{1 - \frac{4m}{\ell}})$. It is clear that in order to have black holes, the conditions $0 < m < \frac{\ell}{4}$ and $\alpha < \frac{-2\Lambda}{9}$ should be satisfied. Moreover, $0 < \Lambda$ to have de Sitter asymptotics for this solution. In our analysis, we will focus on $r_H = r_+$, i.e. the black hole event horizon. However, the analysis applies to the other horizons by inserting an appropriate sign for the temperature. The cosmological gauge field $\boldsymbol{A}$ in a gauge which is fixed similar to the other examples mentioned above, can be found to be (see appendix A)

$$\boldsymbol{A} = -\frac{\sqrt{\Lambda} r^3}{3}\, dt \wedge d\theta \wedge d\varphi. \tag{6.29}$$

Properties: The mass, electric charge and horizon potential, temperature and entropy of MTZ black hole can be found respectively as [58, 59, 60]

$$\begin{aligned} &M = m, \qquad Q = q, \qquad \Phi_H = \frac{q}{r_H}, \\ &T_H = \frac{m(r_H - m)}{2\pi r_H^3} - \frac{\Lambda r_H}{6\pi}, \qquad S_H = \pi r_H^2 \left(1 - \frac{m^2 - q^2}{(r_H - m)^2}\right). \end{aligned} \tag{6.30}$$

We notice that the temperature is the standard Hawking temperature which can be found by the relation $T_H = \frac{1}{4\pi}\frac{df}{dr}$ on the horizon, while the entropy is the Bekenstein-Hawking entropy $\frac{A_H}{4}$ multiplied by the factor of scalar curvature R in the Lagrangian, i.e. $1 - \frac{\phi^2}{3}$. The cosmological charge and potential are

$$C = \frac{\sqrt{\Lambda}}{4\pi}, \qquad \Theta_H = -\frac{\sqrt{\Lambda} 4\pi r_H^3}{3}. \tag{6.31}$$

First law and Smarr formula:

This solution has three parameters m, q, and ℓ, but q is not an independent parameter, and is related to the other two parameters by the relation (6.28). The generalized first law for this solution is

$$\delta M = T_H \delta S_H + \Phi_H \delta Q + \Theta_H \delta C, \tag{6.32}$$

which can be checked to be a correct relation by using variations with respect to the two free parameters of this solution. For this solution, the generalized Smarr formula is *not* satisfied.

CHAPTER 7

SOLUTIONS IN 5 AND HIGHER DIMENSIONS

7.1 (A)dS-Myers-Perry black hole

The (A)dS-Myers-Perry black hole solution is generalization of (A)dS-Kerr black hole to 5 (and higher dimensions) [61].

Theory: Einstein-Λ gravity in 5 dimension

$$\mathcal{L} = \frac{1}{16\pi}(R - 2\Lambda). \tag{7.1}$$

Solution: The metric in the coordinates $x^\mu = (t, r, \theta, \varphi, \psi)$ with $\theta \in [0, \frac{\pi}{2}]$ and $\varphi, \psi \in [0, 2\pi]$ is

$$\mathrm{d}s^2 = -\frac{\Delta_\theta(1 - \frac{\Lambda r^2}{6})dt^2}{\Xi_a \Xi_b} + \frac{2m}{\rho^2}(\frac{\Delta_\theta dt}{\Xi_a \Xi_b} - a^2 \sin^2\theta \frac{d\varphi}{\Xi_a} - b^2 \cos^2\theta \frac{d\psi}{\Xi_b})^2$$

$$+ \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{r^2 + a^2}{\Xi_a} \sin^2\theta d\varphi^2 + \frac{r^2 + b^2}{\Xi_b} \cos^2\theta d\psi^2, \tag{7.2}$$

where

$$\Delta_r = \frac{(r^2 + a^2)(r^2 + b^2)(1 - \frac{\Lambda r^2}{6})}{r^2} - 2m, \quad \Delta_\theta = 1 + \frac{a^2\Lambda}{6}\cos^2\theta + \frac{b^2\Lambda}{6}\sin^2\theta,$$
$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \qquad \Xi_a = 1 + \frac{a^2\Lambda}{6}, \qquad \Xi_b = 1 + \frac{b^2\Lambda}{6}. \tag{7.3}$$

Horizons of Myers-Perry black hole are situated at r_H which are the roots of $\Delta_r = 0$. The cosmological gauge field can be found to be (see appendix A)

$$A = -\frac{\sqrt{|\Lambda|}\sin\theta\cos\theta}{\Xi_a\Xi_b}\left\{\frac{r^4+2r^2(a^2\cos^2\theta+b^2\sin^2\theta)}{4}+\alpha_0\right\} dt\wedge d\theta\wedge d\varphi\wedge d\psi, \tag{7.4}$$

where

$$\alpha_0 = \frac{a^2b^2}{4}+\frac{m(a^2+b^2+\frac{a^2b^2\Lambda}{3})}{6\Xi_a\Xi_b}.$$

The constant α_0, which is a gauge fixing term, is determined by the covariant formulation of charges in the appendix A.1.

Properties: Denoting the angular momenta associated with the axial symmetries of the coordinates φ and ψ by J_φ and J_ψ:

$$M = \frac{\pi m(2\Xi_a+2\Xi_b-\Xi_a\Xi_b)}{4\Xi_a^2\Xi_b^2}, \qquad J_\varphi = \frac{\pi a m}{2\Xi_a^2\Xi_b}, \qquad J_\psi = \frac{\pi b m}{2\Xi_a\Xi_b^2},$$

$$\Omega_H^\varphi = \frac{a(1-\frac{\Lambda r_H^2}{6})}{(r_H^2+a^2)}, \qquad \Omega_H^\psi = \frac{b(1-\frac{\Lambda r_H^2}{6})}{(r_H^2+b^2)},$$

$$T_H = \frac{r_H^4[1-\frac{\Lambda}{6}(2r_H^2+a^2+b^2)]-a^2b^2}{2\pi r_H[(r_H^2+a^2)(r_H^2+b^2)]}, \; S_H = \frac{\pi^2[(r_H^2+a^2)(r_H^2+b^2)]}{2\Xi_a\Xi_b r_H}. \tag{7.5}$$

The cosmological charge and potential can be read from (1.9) and (1.11):

$$C = \pm\frac{\sqrt{|\Lambda|}}{4\pi}, \tag{7.6}$$

$$\Theta_H = -\frac{\sqrt{|\Lambda|}\pi^2}{\Xi_a\Xi_b}\left(\frac{(r_H^2+a^2)(r_H^2+b^2)}{2}+\frac{m(a^2+b^2+\frac{\Lambda a^2b^2}{3})}{3\Xi_a\Xi_b}\right), \tag{7.7}$$

with the positive and negative C for the solutions with dS and AdS asymptotics.

First law and Smarr formula:

This solution has four free parameters (m, a, b, Λ). Using (7.5) and (7.7), the generalized first law and Smarr formula for this solution

$$\delta M = T_H \delta S_H + \Omega_H^{\varphi} \delta J_{\varphi} + \Omega_H^{\psi} \delta J_{\psi} + \Theta_H \delta C, \tag{7.8}$$

$$2M = 3T_H S_H + 3\Omega_H^{\varphi} J_{\varphi} + 3\Omega_H^{\psi} J_{\psi} - \Theta_H C, \tag{7.9}$$

can be checked for variations with respect to the parameters $p_i \in \{m, a, b, \Lambda\}$. Hence, for this example the generalized first law and Smarr formula hold. For the solutions whose horizon may not be found analytically in terms of the parameters of the solution (like Myers-Perry solutions), we refer the reader to the appendix A.2, in order to find how to check the first law and Smarr formula easily.

7.2 (A)dS-Reisner-Nordström-Tangherlini black hole

This family of black holes is generalization of the (A)dS-Reisner-Nordström black hole to higher D dimensions, which are spherically symmetric solutions with electric charges.

Theory: With the dynamical fields as the metric $g_{\mu\nu}$ and Maxwell gauge field A_μ, the theory is described by the Lagrangian of Einstein-Maxwell-Λ gravity in D dimensions, the metric reads

$$\mathcal{L} = \frac{1}{16\pi}(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}), \tag{7.10}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength.

Solution: Denoting the time and radius coordinates by t and r, for these black holes in D dimensions

$$\begin{aligned} \mathrm{d}s^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2_{{}_{D-2}}, \qquad A = \sqrt{\frac{(D-2)}{2(D-3)}}\frac{q}{r^{D-3}}dt, \\ f &= 1 - \frac{2m}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}} - \frac{2\Lambda r^2}{(D-1)(D-2)}, \qquad \Omega_{{}_{D-2}} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}, \end{aligned} \tag{7.11}$$

where $\Omega_{{}_{D-2}}$ is the area of the $D-2$ dimensional unit sphere, and Γ is the gamma function. Horizons are situated at the radii which can be found as roots of $f(r_H) = 0$.

The cosmological gauge field for this family of solutions is (see appendix A)

$$\boldsymbol{A} = -\frac{\sqrt{|\Lambda|}}{D-1} r^{D-1}\, dt \wedge d\Omega_{_{D-2}}. \tag{7.12}$$

Properties: For these black holes, mass, electric charge and potential, temperature and entropy are

$$M = \frac{(D-2)\Omega_{_{D-2}} m}{8\pi}, \qquad Q = \frac{\sqrt{\frac{(D-2)(D-3)}{2}}\Omega_{_{D-2}} q}{4\pi},$$

$$T_H = \frac{1}{4\pi}\left(\frac{2(D-3)m}{r_H^{D-2}} - \frac{2(D-3)q^2}{r_H^{2(D-3)+1}} - \frac{4\Lambda r_H}{(D-1)(D-2)}\right),$$

$$\Phi_H = \sqrt{\frac{D-2}{2(D-3)}}\frac{q}{r_H^{D-3}}, \qquad S_H = \frac{r_H^{D-2}\Omega_{_{D-2}}}{4}. \tag{7.13}$$

The temperature is the standard Hawking temperature which can be found by the relation $T_H = \frac{1}{4\pi}\frac{df}{dr}$ on the horizon, while the entropy is the Bekenstein-Hawking entropy $\frac{A_H}{4}$. Using (1.9) and (1.11), the cosmological charge and potential are found as:

$$C = \pm\frac{\sqrt{|\Lambda|}}{4\pi}, \qquad \Theta_H = -\frac{\sqrt{|\Lambda|}}{D-1} r_H^{D-1}\Omega_{_{D-2}}, \tag{7.14}$$

with the positive and negative C for the solutions with dS and AdS asymptotics.

First law and Smarr formula:

The RNT black holes have three free parameters (m, q, Λ). Using (7.13) and (7.14), and variations with respect to three parameters, the generalized first law and Smarr formula for this family of solutions are satisfied as

$$\delta M = T_H \delta S_H + \Phi_H \delta Q + \Theta_H \delta C, \tag{7.15}$$

$$(D-3)M = (D-2)T_H S_H + (D-3)\Phi_H Q - \Theta_H C. \tag{7.16}$$

For these solutions the horizon radii may not be found analytically in terms of the parameters of the solution. We refer the reader to the appendix A.2, in order to find how to check the first law and Smarr formula without having the explicit form of r_H.

7.3 Charged rotating black hole in minimal gauged supergravity

Theory: Lagrangian of the minimal gauged supergravity in 5 dimensions is

$$\mathcal{L} = \frac{1}{16\pi}(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu} + \frac{2}{3\sqrt{3}}\epsilon^{\mu_1\mu_2\ldots\mu_5}F_{\mu_1\mu_2}F_{\mu_3\mu_4}A_{\mu_5}), \tag{7.17}$$

where $\epsilon_{\mu_1\mu_2\ldots\mu_5}$ is the 5 dimensional Levi-Civita symbol with components $+1$ or -1. The last term in the Lagrangian above is the Chern-Simons term.

Solution: The metric in the coordinates $x^\mu = (t, r, \theta, \varphi, \psi)$ with $\theta \in [0, \frac{\pi}{2}]$ and $\varphi, \psi \in [0, 2\pi]$ is [62]

$$\mathrm{d}s^2 = -\frac{\Delta_\theta[(1-\frac{\Lambda r^2}{6})\rho^2 dt + 2q\nu]dt}{\Xi_a\Xi_b\rho^2} + \frac{2q\nu\omega}{\rho^2} + \frac{f}{\rho^4}(\frac{\Delta_\theta dt}{\Xi_a\Xi_b} - \omega)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta}$$

$$+ \frac{r^2+a^2}{\Xi_a}\sin^2\theta d\varphi^2 + \frac{r^2+b^2}{\Xi_b}\cos^2\theta d\psi^2, \tag{7.18}$$

where

$$\nu = b\sin^2\theta d\varphi + a\cos^2\theta d\psi, \qquad \omega = a\sin^2\theta\frac{d\varphi}{\Xi_a} + b\cos^2\theta\frac{d\psi}{\Xi_b},$$

$$\Delta_r = \frac{(r^2+a^2)(r^2+b^2)(1-\frac{\Lambda r^2}{6}) + q^2 + 2abq}{r^2} - 2m,$$

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \qquad \Xi_a = 1 + \frac{a^2\Lambda}{6}, \qquad \Xi_b = 1 + \frac{b^2\Lambda}{6},$$

$$f = 2m\rho^2 - q^2 - \frac{\Lambda}{3}abq\rho^2,\ \Delta_\theta = 1 + \frac{a^2\Lambda}{6}\cos^2\theta + \frac{b^2\Lambda}{6}\sin^2\theta. \tag{7.19}$$

The Maxwell gauge field in this solution is

$$A = \frac{\sqrt{3}q}{2\rho^2}(\frac{\Delta_\theta dt}{\Xi_a\Xi_b} - \omega). \tag{7.20}$$

In the special cases of $q = 0$ and $a = b = 0$, one recovers the (A)dS-Myers-Perry and the (A)dS-Reisner-Nordström-Tangherlini black holes in 5 dimensions. However, in its general form, it is not a solution to Einstein-Maxwell-Λ theory. Instead, it is a solution to a theory which is supersymmetric, and has a Chern-Simons term in it.

Having in mind that conserved charges depend on the Lagrangian, it is worth studying this solution separately. The horizon radii are

$$r_{\pm}^2 = m - \frac{a^2+b^2}{2} \pm \sqrt{(m - \frac{(a-b)^2}{2} + q)(m - \frac{(a+b)^2}{2} - q)}. \tag{7.21}$$

The cosmological gauge field can be found to be (see appendix A)

$$\boldsymbol{A} = -\frac{\sqrt{|\Lambda|}\sin\theta\cos\theta}{\Xi_a\Xi_b}\left(\frac{r^4 + 2r^2(a^2\cos^2\theta + b^2\sin^2\theta)}{4} + \sigma_0\right) dt\wedge d\theta\wedge d\varphi\wedge d\psi, \tag{7.22}$$

$$\sigma_0 = \frac{a^2b^2}{4} + \frac{m(a^2+b^2+\frac{a^2b^2\Lambda}{3})}{6\Xi_a\Xi_b} + \frac{abq(\Xi_a+\Xi_b)}{3\Xi_a\Xi_b}.$$

The constant α_0, which is a gauge fixing term, is determined by the covariant formulation of charges in the appendix A.1.

Properties:[62]

$$\begin{aligned}
M &= \frac{\pi m(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b) - 2\pi abq\frac{\Lambda}{6}(\Xi_a+\Xi_b)}{4\Xi_a^2\Xi_b^2}, \qquad Q = \frac{\sqrt{3}\pi q}{2\Xi_a\Xi_b},\\
J_\varphi &= \frac{\pi(2am + qb(1-\frac{a^2\Lambda}{6}))}{4\Xi_a^2\Xi_b}, \qquad J_\psi = \frac{\pi(2bm + qa(1-\frac{b^2\Lambda}{6}))}{4\Xi_a\Xi_b^2},\\
\Omega_H^\varphi &= \frac{a(r_H^2+b^2)(1-\frac{\Lambda r_H^2}{6}) + bq}{\sigma}, \qquad \Omega_H^\psi = \frac{b(r_H^2+a^2)(1-\frac{\Lambda r_H^2}{6}) + aq}{\sigma},\\
T_H &= \frac{r_H^4[1-\frac{\Lambda}{6}(2r_H^2+a^2+b^2)] - (ab+q)^2}{2\pi r_H\sigma}, \qquad S_H = \frac{\pi^2\sigma}{2\Xi_a\Xi_b r_H},\\
\Phi_H &= \frac{\sqrt{3}qr_H^2}{2\sigma}.
\end{aligned} \tag{7.23}$$

and $\sigma = (r_H^2+a^2)(r_H^2+b^2) + abq$. The cosmological charge and potential by the equations (1.9) and (1.11) are:

$$\begin{aligned}
C &= \pm\frac{\sqrt{|\Lambda|}}{4\pi},\\
\Theta_H &= -\frac{\sqrt{|\Lambda|}\pi^2}{\Xi_a\Xi_b}\left(\frac{(r_H^2+a^2)(r_H^2+b^2)}{2} + \frac{m(a^2+b^2+\frac{\Lambda a^2b^2}{3}) + 2abq(\Xi_a+\Xi_b)}{3\Xi_a\Xi_b}\right),
\end{aligned} \tag{7.24}$$

with the positive and negative C for the solutions with dS and AdS asymptotics.

First law and Smarr formula:

This solution has five free parameters (m, a, b, q, Λ). Using (7.23) and (7.24), the generalized first law and Smarr formula for this solution

$$\delta M = T_H \delta S_H + \Omega_H^{\varphi} \delta J_{\varphi} + \Omega_H^{\psi} \delta J_{\psi} + \Phi_H \delta Q + \Theta_H \delta C, \tag{7.25}$$

$$2M = 3T_H S_H + 3\Omega_H^{\varphi} J_{\varphi} + 3\Omega_H^{\psi} J_{\psi} + 2\Phi_H Q - \Theta_H C, \tag{7.26}$$

can be checked for variations with respect to the parameters $p_i \in \{m, a, b, q, \Lambda\}$. Hence, for this example the generalized first law and Smarr formula hold.

7.4 Lifshitz $z = 2$ black brane

Theory: The Lagrangian contains second order terms in curvature as follows:

$$\mathcal{L} = \frac{1}{16\pi} \left(R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma (R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho}) \right). \tag{7.27}$$

The last term is the Gauss-Bonnet term, and the coupling constants are

$$\Lambda = -\frac{2197}{551\ell^2}, \quad \alpha = -\frac{16\ell^2}{725}, \quad \beta = \frac{1584\ell^2}{13775}, \quad \gamma = \frac{2211\ell^2}{11020}. \tag{7.28}$$

Solution: The metric in the coordinates $x^\mu = (t, r, x, y, z)$ is [33, 36]

$$ds^2 = -(\frac{r}{\ell})^{2z} (1 - \frac{m\ell^{\frac{5}{2}}}{r^{\frac{5}{2}}})\, dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2}(1 - \frac{m\ell^{\frac{5}{2}}}{r^{\frac{5}{2}}})} + r^2(dx^2 + dy^2 + dz^2) \tag{7.29}$$

for $z = 2$. The horizon is a brane located at $r_H = m^{\frac{2}{5}}\ell$. The cosmological gauge field for this solution is (see appendix A)

$$A = -\sqrt{|\Lambda|}(\frac{r^5}{5\ell} - \frac{13121 m^2 \ell^4}{87880})\, dt \wedge dx \wedge dy \wedge dz. \tag{7.30}$$

The last term is a gauge fixing term which is fixed by using the covariant charge method of charges (see appendix A.1) to reproduce the $\delta_\ell M$ correctly.

Properties: Using the solution phase space method in [29, 30] or other methods [34, 35, 36, 37] we find

$$M = \frac{297 m^2 \ell^2}{17632\pi}, \qquad T_H = \frac{5 m^{\frac{4}{5}}}{8\pi\ell}, \qquad S_H = \frac{99 m^{\frac{6}{5}} \ell^3}{2204}. \tag{7.31}$$

M and S_H denote mass and entropy densities of the black brane. Using the equations (1.9) and (1.11), the cosmological charge and potential are:

$$C = -\frac{\sqrt{-\Lambda}}{4\pi}, \qquad \Theta_H = -\sqrt{|\Lambda|}(\frac{r_H^5}{5\ell} - \frac{13121m^2\ell^4}{87880}). \tag{7.32}$$

First law and Smarr formula:

This solution has two free parameters m, ℓ. The generalized first law and Smarr formula for this solution are

$$\delta M = T_H \delta S_H + \Theta_H \delta C, \tag{7.33}$$

$$2M = 3T_H S_H - \Theta_H C, \tag{7.34}$$

which can be checked to be a correct relation by using variations with respect to the two free parameters of this solution. We note that the couplings (α, β, γ) are not independent from the Λ. So, we could expect to have the Smarr formula without contributions from these parameters.

7.5 AdS-Schwarzschild black holes in higher curvature gravity

Theory: The Lagrangian which we consider as the last example in this work is the Einstein-Λ gravity with higher curvature terms in arbitrary $D > 2$ dimensions

$$\mathcal{L} = \frac{1}{16\pi}\Big(R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\Big), \tag{7.35}$$

in which α and β are arbitrary constants.

Solution: The metric is simply generalization of AdS-Schwarzschild black hole to D dimensions, which is

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2_{_{D-2}}, \qquad f = 1 - \frac{2m}{r^{D-3}} + \frac{r^2}{\ell^2}, \tag{7.36}$$

where ℓ satisfies $\Lambda = \frac{-\ell^2(D^2-3D+2)+(\alpha D+\beta)(D-4)(D-1)^2}{2\ell^4}$. The cosmological gauge field for this family of solutions is (see appendix A)

$$A = -\sqrt{|\Lambda|}\left(\frac{r^{D-1}}{D-1} + \sigma_0\right) dt \wedge d\Omega_{_{D-2}},$$

$$\sigma_0 = \frac{4m\ell^2(\alpha D + \beta)}{2(D-1)(D-4)(\alpha D+\beta) - (D-2)\ell^2}. \tag{7.37}$$

The σ_0 is a gauge fixing term which can be fixed by covariant formulation of charges which is described in the appendix A.1.

Properties: Conserved charges, such as the mass and entropy, depend on the solution as well as the theory. As a result, although these black holes are simply the AdS-Schwarzschild solutions, but the theory differs from the Einstein-Λ theory. The new charges associated with these solutions are different, and can be found to be [30]

$$M = \mathcal{X} \times \frac{(D-2)\Omega_{D-2}}{8\pi} m, \qquad T_H = \frac{(D-1)r_{\text{H}}^{D-2} + (D-3)\ell^2\, r_{\text{H}}^{D-4}}{4\pi\ell^2\, r_{\text{H}}^{D-3}},$$
$$S_{\text{H}} = \mathcal{X} \times \frac{r_{\text{H}}^{D-2}\Omega_{D-2}}{4}. \tag{7.38}$$

in which

$$\mathcal{X} = \frac{\ell^2 - 2D(D-1)\alpha - 2(D-1)\beta}{\ell^2}, \qquad \Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}, \tag{7.39}$$

and horizons are determined by the equation $r_{\text{H}}^{D-1} + \ell^2 r_{\text{H}}^{D-3} - 2m\ell^2 = 0$. By the equations (1.9) and (1.11), the cosmological charge and potential are:

$$C = -\frac{\sqrt{-\Lambda}}{4\pi}, \qquad \Theta_H = -\sqrt{|\Lambda|}\left(\frac{r_H^{D-1}}{D-1} + \sigma_0\right)\Omega_{D-2}. \tag{7.40}$$

First law and Smarr formula:

This family of solutions have two free parameters in the solution, (m, ℓ). These parameters should not be confused with the (α, β) which are free parameters/couplings in the Lagrangian. In case of $\alpha = \beta = 0$, we recover the AdS-Schwarzschild black holes in Einstein-Λ theory which we have already studied in the example 7.2 (by setting $q = 0$). So, in this case, we have already shown that the generalized first law and Smarr formula hold. If at least one of the α or β is non-zero, one can check that the first law is still satisfied, using (7.39) and (7.40) and the method which is described in the appendix A.2 as

$$\delta M = T_H \delta S_H + \Theta_H \delta C. \tag{7.41}$$

However, the Smarr formula fails to be satisfied which is to be expected as one needs to incorporate the other dimensionful parameters α and/or β which is an outstanding problem at this stage.

CHAPTER 8

UNIVERSALITY OF THE SMARR FORMULA

In the black hole physics literature, the Smarr formula is not considered as a universal relation. Clearly, it does depend on the dimension of space-time D. However, one can still inquire if the Smarr relation (4.1) is a generic relation. In spite of the fact that in some of the examples that we have analyzed, this relation fails, one can see a suggestive pattern in it: this relation only fails for the Lagrangians which contain at least one free dimensionful parameter/coupling constant (in addition to the Λ). This observation suggests that this generalized Smarr formula should be extended such that it contains the contributions from those dimensionful parameters. In this regard, and based on our case-by-case study and the proof in 4.1, we would like to put forward the following conjecture.

Conjecture: *The Smarr formula in* (4.1) *can always be generalized to include contributions from dimensionful coupling constants in the Lagrangian.*

In order to do this generalization, one may probably use a similar method as the one used for Λ. However, this is a subject of research beyond the scope of this paper and needs more investigations. Some guidelines for such an approach could be: 1) if the dimensionful parameter is a parameter in the Lagrangian, it should be promoted to be a parameter in the solution (not in the Lagrangian), probably as a conserved charge, 2) its conjugate chemical potential in the first law should be a (well)-defined property of the horizon, *i.e.* it could be found using only the information in the vicinity of the horizon.

Let us assume that such an analysis has been successfully done, yielding new conserved charges K_i with dimensionality $K_i \sim l^{k(i)}$ and their associated chemical potentials Ψ^i_H, with the following contribution to the first law:

$$\delta M = T_{\rm H}\delta S + \Omega_{\rm H}\delta J + \Phi_{\rm H}\delta Q + \Theta_{\rm H}\delta C + \Psi^i_H \delta K_i. \tag{8.1}$$

Following the steps in section 4.1 verbatim, after scaling $l \to \alpha l$, one has

$$\alpha^{D-3} M(S,J,Q,C,K_i) = M\left(\alpha^{D-2}S, \alpha^{D-2}J, \alpha^{D-3}Q, \alpha^{-1}C, \alpha^{k^{(i)}}K_i\right). \tag{8.2}$$

Using the Euler relation (4.2) and the equation (8.2), one gets

$$\begin{aligned}(D-3)M =&(D-2)\left(\frac{\partial M}{\partial S}\right)S + (D-2)\left(\frac{\partial M}{\partial J}\right)J \\ &+ (D-3)\left(\frac{\partial M}{\partial Q}\right)Q - \left(\frac{\partial M}{\partial C}\right)C + k^{(i)}\left(\frac{\partial M}{\partial K^i}\right)K_i\end{aligned} \tag{8.3}$$

in which the sum over i is understood. At the end, using the generalized first law (8.1), we find the generalized Smarr relation

$$(D-3)M = (D-2)T_{\rm H}S + (D-2)\Omega_{\rm H}J + (D-3)\Phi_{\rm H}Q - \Theta_{\rm H}C + k^{(i)}\Psi^i_H K_i. \tag{8.4}$$

Having the general structure of the generalized Smarr formula, one may be interested to investigate and find Ψ^i and K_i for the examples which failed to satisfy the non-generalized Smarr relation 4.1. This is a very interesting subject for research in the future, and is beyond the scope of this paper. Nonetheless it is important to emphasize, in order to find the correct contributions from dimensionful parameters to the Smarr relation, one needs to find a systematic and a precise description of these parameters as conserved charges (or at least as parameters of the solution); this is because:

- variation of a Lagrangian coupling constant in the first law is conceptually problematic,
- the dimensional analysis may not determine K_i uniquely. As an example, we remind the reader the difference of pressure P in $V_{\rm eff}\delta P$ compared to C in $\Theta_H\delta C$. The pressure (which is proportional to Λ) has dimension l^{-2}, while C (which is proportional to $\sqrt{\Lambda}$) is of dimension l^{-1}. Nonetheless, both of $V_{\rm eff}\delta P$ and $\Theta_H\delta C$ are allowed by the dimensional analysis,
- in the absence of a precise definition for the Ψ^i_H, the first law (and consequently, the Smarr relation) could act only as a definition for it. Therefore, such relations would be trivially satisfied.

Accordingly, generalization of the first law and the Smarr relation for the problematic examples in this paper (without systematic notion of charges and chemical potentials) can yield misleading outcomes, and thus we postpone their full study to later investigations.

CHAPTER 9

CONCLUSIONS

The cosmological constant Λ can be considered as a conserved charge C associated with the gauge symmetry of a gauge field A. The conserved charge C is analogous to electric charge: 1) it is a parameter of the solution, 2) it is extensive, and 3) can be positive or negative. Besides, its conjugate Θ_H is a property of the horizon. These properties resolve problems with the $V\delta P$ formulation of Λ in the first law of black hole thermodynamics. In this paper, we generalized the Smarr formula to include a contribution from the $\Theta_H C$ term, and provided a proof for it. However, the proof which is based on dimensional analysis, does not capture the free dimensionful parameters in the Lagrangian. We analyzed a handful number of examples to study this issue case-by-case.

In addition, we showed that the Θ_H reproduces the “effective volume" successfully. Besides, we showed how the ambiguity of the effective volume can be removed by the role of gauge fixing in determination of Θ_H. Studying different examples in this paper collects a fair number of black holes with non-zero Λ, and can provide a reference for the readers about the cosmological gauge field A as a part of the black hole solutions.

The successful generalization of the first law for all of the examples, not only supports the $\Theta_H \delta C$ formulation of Λ, but also it confirms the “modified temperature" for Horndeski gravities which has been recently proposed in [46].

REFERENCES

[1] D. Chernyavsky and K. Hajian, “Cosmological constant is a conserved charge,” Class. Quant. Grav. **35**, no. 12, 125012 (2018) [arXiv:1710.07904].

[2] A. Einstein, “Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie,” Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), Seite 142-152, (1917).

[3] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], “Measurements of Omega and Lambda from 42 high redshift supernovae,” Astrophys. J. **517**, 565 (1999), [arXiv:astro-ph/9812133].

[4] A. G. Riess *et al.* [Supernova Search Team], “Observational evidence from supernovae for an accelerating universe and a cosmological constant,” Astron. J. **116**, 1009 (1998), [arXiv:astro-ph/9805201].

[5] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] [arXiv:hep-th/9711200].

[6] E. Poisson, A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics. Cambridge: Cambridge University Press, 2004. doi: 10.1017/CBO9780511606601.

[7] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” Commun. Math. Phys. **104**, 207 (1986).

[8] A. Aurilia, H. Nicolai and P. K. Townsend, “Hidden Constants: The Theta Parameter of QCD and the Cosmological Constant of N=8 Supergravity,” Nucl. Phys. B **176**, 509 (1980).

[9] M. J. Duff and P. van Nieuwenhuizen, “Quantum Inequivalence of Different Field Representations,” Phys. Lett. **94B**, 179 (1980).

[10] M. Henneaux and C. Teitelboim, “The Cosmological Constant As A Canonical Variable,” Phys. Lett. **143B**, 415 (1984).

[11] M. Henneaux and C. Teitelboim, “Asymptotically anti-De Sitter Spaces,” Commun. Math. Phys. **98** (1985), 391-424

[12] C. Teitelboim, “The Cosmological Constant As A Thermodynamic Black Hole Parameter,” Phys. Lett. **158B**, 293 (1985).

[13] Natsuume, M. (2015). AdS/CFT duality user guide (Vol. 903). Springer.

[14] M. Henneaux and C. Teitelboim, "The Cosmological Constant and General Covariance," Phys. Lett. B **222**, 195 (1989).

[15] R. M. Wald, "Black hole entropy is the Noether charge", *Phys. Rev. D*, **48**, 3427–3431, (1993), [arXiv:gr-qc/9307038].

[16] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy", *Phys. Rev. D*, **50**, 846–864, (1994), [arXiv:gr-qc/9403028].

[17] Thorne, K. S., Misner, C. W., & Wheeler, J. A. (2000). Gravitation. San Francisco, CA: Freeman.

[18] J. M. Bardeen, B. Carter and S. W. Hawking, "The Four laws of black hole mechanics," Commun. Math. Phys. **31**, 161-170, (1973).

[19] D. Kubiznak, R. B. Mann and M. Teo, "Black hole chemistry: thermodynamics with Lambda," Class. Quant. Grav. **34** (2017) no.6, 063001 [arXiv:1608.06147].

[20] B. P. Dolan, "The cosmological constant and the black hole equation of state," Class. Quant. Grav. **28** (2011), 125020 [arXiv:1008.5023].

[21] L. Smarr, "Mass formula for Kerr black holes," Phys. Rev. Lett. **30**, 71-73, (1973).

[22] P. K. Townsend, "Black holes: Lecture notes," [arXiv:gr-qc/9707012].

[23] D. Kastor, S. Ray and J. Traschen, "Enthalpy and the Mechanics of AdS Black Holes," Class. Quant. Grav. **26**, 195011 (2009) [arXiv:0904.2765].

[24] M. Cvetic, G. W. Gibbons, D. Kubiznak and C. N. Pope, "Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume," Phys. Rev. D **84** (2011), 024037 [arXiv:1012.2888].

[25] K. Hajian, "On Thermodynamics and Phase Space of Near Horizon Extremal Geometries", Ph.D thesis, (2015), [arXiv:1508.03494].

[26] M. Banados, C. Teitelboim and J. Zanelli, "The Black hole in three-dimensional space-time," *Phys. Rev. Lett.*, **69**, 1849, (1992), [arXiv:hep-th/9204099].

[27] K. Hajian, A. Seraj and M. M. Sheikh-Jabbari, "Near Horizon Extremal Geometry Perturbations: Dynamical Field Perturbations vs. Parametric Variations," JHEP **10**, 111 (2014) [arXiv:1407.1992].

[28] C. Martinez, C. Teitelboim and J. Zanelli, "Charged rotating black hole in three space-time dimensions," *Phys. Rev. D*, **61**, 104013 (2000), [arXiv:hep-th/9912259].

[29] K. Hajian and M. M. Sheikh-Jabbari, "Solution Phase Space and Conserved Charges: A General Formulation for Charges Associated with Exact Symmetries", *Phys. Rev. D*, **93**, 4044074, (2016), [arXiv:1512.05584].

[30] M. Ghodrati, K. Hajian and M. R. Setare, “Revisiting Conserved Charges in Higher Curvature Gravitational Theories,” Eur. Phys. J. C **76**, no. 12, 701 (2016) [arXiv:1606.04353].

[31] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “Massive Gravity in Three Dimensions,” *Phys. Rev. Lett.*, **102**, 201301, (2009), [arXiv:0901.1766].

[32] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, “Lifshitz Black Hole in Three Dimensions,” *Phys. Rev. D*, **80**, 104029, (2009), [arXiv:0909.1347].

[33] E. Ayon-Beato, A. Garbarz, G. Giribet, and M. Hassaine, “Analytic Lifshitz black holes in higher dimensions”, *JHEP*, **04**, 030, (2010), [arXiv:1001.2361].

[34] O. Hohm and E. Tonni, “A boundary stress tensor for higher-derivative gravity in AdS and Lifshitz backgrounds,” *JHEP*, **1004**, 093, (2010), [arXiv:1001.3598].

[35] H. A. Gonzalez, D. Tempo and R. Troncoso, “Field theories with anisotropic scaling in 2D, solitons and the microscopic entropy of asymptotically Lifshitz black holes,” JHEP **11** (2011), 066 [arXiv:1107.3647].

[36] Y. Gim, W. Kim, and Sang-Heon Yi, “The first law of thermodynamics in Lifshitz black holes revisited”, *JHEP*, **07**, 002, (2014), [arXiv:1403.4704].

[37] E. Ayón-Beato, M. Bravo-Gaete, F. Correa, M. Hassaïne, M. M. Juárez-Aubry and J. Oliva, “First law and anisotropic Cardy formula for three-dimensional Lifshitz black holes,” Phys. Rev. D **91**, no.6, 064006 (2015) [arXiv:1501.01244].

[38] G. Clement, “Warped AdS(3) black holes in new massive gravity,” Class. Quant. Grav. **26** (2009), 105015 [arXiv:0902.4634].

[39] L. F. Abbott and S. Deser, “Stability of Gravity with a Cosmological Constant”, *Nucl. Phys. B*, **195**, 76–96, (1982).

[40] S. Deser and B. Tekin, “Gravitational energy in quadratic curvature gravities”, *Phys. Rev. Lett.*, **89**, 101101, (2002), [arXiv:hep-th/0205318].

[41] S. Deser and B. Tekin, “Energy in generic higher curvature gravity theories”, *Phys. Rev. D*, **67**, 084009, (2003), [arXiv:hep-th/0212292].

[42] G. Alkac and D. O. Devecioglu, “Covariant Symplectic Structure and Conserved Charges of New Massive Gravity,” Phys. Rev. D **85** (2012), 064048 [arXiv:1202.1905].

[43] S. Detournay and C. Zwikel, “Phase transitions in warped AdS_3 gravity,” *JHEP*, **1505**, 074, (2015), [arXiv:1504.00827].

[44] G. W. Horndeski, “Second-order scalar-tensor field equations in a four-dimensional space,” Int. J. Theor. Phys. **10** (1974), 363-384

[45] F. F. Santos, "Rotating black hole with a probe string in Horndeski Gravity," Eur. Phys. J. Plus **135** (2020) no.10, 810, [arXiv:2005.10983].

[46] K. Hajian, S. Liberati, M. M. Sheikh-Jabbari and M. H. Vahidinia, "On Black Hole Temperature in Horndeski Gravity," Phys. Lett. B **812** (2020), 136002, [arXiv:2005.12985].

[47] Roy P. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys.Rev.Lett.*, 11:237–238, 1963.

[48] E.T. Newman and A.I. Janis. Note on the Kerr spinning particle metric. *J.Math.Phys.*, 6:915–917, 1965.

[49] E T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash, et al. Metric of a Rotating, Charged Mass. *J.Math.Phys.*, 6:918–919, 1965.

[50] B. Carter. Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations. *Commun.Math.Phys.*, 10:280, 1968.

[51] B. Carter, "The commutation property of a stationary, axisymmetric system," *Commun. Math. Phys.*, **17**, 233–238, (1970).

[52] B. Carter; in: C. DeWitt, BS DeWitt (Eds.), Les Astre Occlus, Proceedings of 1972 Les Houches Summer School (2nd ed.), Gordon and Breach, New York (1973).

[53] S. W. Hawking, "Particle Creation by Black Holes," Commun. Math. Phys., vol. 43, pp. 199–220, 1975, doi: 10.1007/BF02345020.

[54] G. W. Gibbons and S. W. Hawking, "Cosmological Event Horizons, Thermodynamics, and Particle Creation," *Phys. Rev. D*, **15**, 2738–2751, (1977).

[55] K. Hajian, "Conserved charges and first law of thermodynamics for Kerr–de Sitter black holes," Gen. Rel. Grav. **48**, no. 8, 114 (2016), [arXiv:1602.05575].

[56] E. Babichev, C. Charmousis and A. Lehébel, "Asymptotically flat black holes in Horndeski theory and beyond," JCAP **04** (2017), 027 [arXiv:1702.01938].

[57] X. H. Feng, H. S. Liu, H. Lü and C. N. Pope, "Thermodynamics of Charged Black Holes in Einstein-Horndeski-Maxwell Theory," Phys. Rev. D **93**, no. 4, 044030 (2016) [arXiv:1512.02659].

[58] C. Martinez, R. Troncoso and J. Zanelli, "De Sitter black hole with a conformally coupled scalar field in four-dimensions," Phys. Rev. D **67** (2003), 024008 [arXiv:hep-th/0205319].

[59] E. Winstanley, "Classical and thermodynamical aspects of black holes with conformally coupled scalar field hair," Conf. Proc. C **0405132** (2004), 305-323 [arXiv:gr-qc/0408046].

[60] A. M. Barlow, D. Doherty and E. Winstanley, "Thermodynamics of de Sitter black holes with a conformally coupled scalar field," Phys. Rev. D **72** (2005), 024008 [arXiv:gr-qc/0504087].

[61] R. C. Myers and M. J. Perry, “Black Holes in Higher Dimensional Space-Times,” Annals Phys. **172** (1986), 304

[62] Z.-W. Chong, M. Cvetic, H. Lu and C. N. Pope, “General non-extremal rotating black holes in minimal five-dimensional gauged supergravity,” Phys. Rev. Lett. **95**, 161301 (2005) [arXiv:hep-th/0506029].

[63] R. L. Arnowitt, S. Deser, and C. W. Misner, “Dynamical Structure and Definition of Energy in General Relativity”, *Phys. Rev.*, **116**, 1322–1330, (1959).

[64] R. L. Arnowitt, S. Deser, and C. W. Misner, “Canonical variables for general relativity”, *Phys. Rev.*, **117**, 1595–1602, (1960).

[65] R. L. Arnowitt, S. Deser, and C. W. Misner, “The Dynamics of general relativity”, *Gen. Rel. Grav.*, **40**, 1997–2027, (2008), [arXiv:gr-qc/0405109].

[66] T. Regge and C. Teitelboim, “Role of Surface Integrals in the Hamiltonian Formulation of General Relativity”, *Annals Phys.*, **88**, 286, (1974).

[67] J. D. Brown and J. W. York Jr., “Quasilocal energy and conserved charges derived from the gravitational action”, *Phys. Rev. D*, **47**, 1407–1419, (1993), [arXiv:gr-qc/9209012].

[68] A. Ashtekar, L. Bombelli, and R. Koul, “Phase space formulation of general relativity without a 3+1 splitting”, *Lect. Notes Phys.*, **278**, 356–359, (1987).

[69] A. Ashtekar, L. Bombelli, and O. Reula, “The covariant phase space of asymptotically flat gravitational fields”, in M. Francaviglia (ed.), *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, 417-450, (1990).

[70] C. Crnkovic and E. Witten, “Covariant Description Of Canonical Formalism In Geometrical Theories”, In Hawking, S.W. (ed.), Israel, W. (ed.): *Three hundred years of gravitation*, 676-684, (1987).

[71] J. Lee and R. M. Wald, “Local symmetries and constraints”, *J. Math. Phys.*, **31**, 725–743, (1990).

[72] R. M. Wald and A. Zoupas, “A General definition of ’conserved quantities’ in general relativity and other theories of gravity”, *Phys. Rev. D*, **61**, 084027, (2000), [arXiv:gr-qc/9911095].

[73] A. Seraj, “Conserved charges, surface degrees of freedom, and black hole entropy”, Ph.D thesis, (2016), [arXiv:1603.02442].

[74] H. Adami, M. R. Setare, T. C. Sisman and B. Tekin, “Conserved Charges in Extended Theories of Gravity,” Phys. Rept. **834** (2019), 1

[75] A. Corichi, I. Rubalcava-García, and T. Vukasinac, “Actions, topological terms and boundaries in first-order gravity: A review”, *Int. J. Mod. Phys. D*, **25**, 041630011, (2016), [arXiv:1604.07764].

[76] M. Nakahara, Geometry, Topology and Physics, 2nd ed. Boca Raton: CRC Press, 2017. doi: 10.1201/9781315275826.

[77] S. M. Carroll, Spacetime and Geometry. Cambridge University Press, 2019.

[78] Wald, R. M. (2010). General relativity. University of Chicago press.

[79] Iyer, V., & Wald, R. M. (1994). Some properties of the Noether charge and a proposal for dynamical black hole entropy. Physical review D, 50(2), 846.

[80] Harlow, D., & Wu, J. Q. (2020). Covariant phase space with boundaries. Journal of High Energy Physics, 2020(10), 1-52.

APPENDIX A

HOW TO FIND THE COSMOLOGICAL GAUGE FIELD

In this section, we present a heuristic method to find the cosmological gauge field. Let us denote the coordinates by $(t, r, x^1, \dots, x^{D-2})$ for the time, radius, and some other coordinates x^i. For black hole solutions which are stationary, components of the metric $g_{\mu\nu}$ can be chosen to be independent of t. So, the determinant of the metric g, could be a function of coordinates (r, x^i). According to the equation (1.6), the cosmological gauge field strength is equal to

$$F = \sqrt{|\Lambda|}\sqrt{-g}\, dt \wedge dr \wedge dx^1 \wedge \cdots \wedge dx^{D-2}. \tag{A.1}$$

The question is how to find a gauge field A such that $F = dA$. Up to a gauge transformation, the cosmological gauge field A can be suggested to be

$$A = -\sqrt{|\Lambda|}\tilde{g}\ dt \wedge dx^1 \wedge \cdots \wedge dx^{D-2}, \qquad \tilde{g} = \int dr \sqrt{-g}. \tag{A.2}$$

It can be easily checked that $F = dA$ is satisfied. Besides, the constant of integration in $\tilde{g}$, which can be a function of parameters of the solutions as well as all coordinates except the r, is a part of the gauge freedom. This gauge freedom can be fixed by the covariant method of charges which is described in the next section.

One could ask about other components for A, which are in general a linear combination of terms $dt \wedge dr \wedge dx^1 \wedge \cdots \wedge dx^{D-2}$ with a missed dx^i, and the term $dr \wedge dx^1 \wedge \cdots \wedge dx^{D-2}$. The short answer is that such a component does not contribute to the Θ_H defined in equation (1.11), because pull back of such a term in the expression $\xi_H \cdot A$ to the horizon vanishes, because such a term inevitably misses either a direction along dt to be contracted by ξ_H, or one of dx^i to be integrated over the horizon.

A.1 Covariant calculation of charges

In gravity theories, there are different methods for calculating conserved charges. Amongst the methods, one can mention some of the well-established methods like the ADM formulation [63, 64, 65] continued by Regge-Teitelboim [66], Brown-York formulation [67], and ADT formulation of charges [39, 40, 41]. In this paper, we have used a method which is called “covariant formulation of charges" and has been introduced in the late 80s - early 90s [70, 68, 69, 71, 15, 16, 72]. Interested reader can find reviews on this method in e.g. [25, 73, 74, 75]. In this appendix, we briefly review the basics of this method, and provide the final formula by which the charges are calculated.

Phase space is a manifold with a 2-form, which is called symplectic form and is donoted by Ω. The covariant phase space formulation of charges is based on a phase space which is built covariantly; instead of fields and their momentum conjugates in a time slice, the phase space is built by the fields over all of the spacetime which we denote them collectively by $\Phi(x^\mu)$. So, we do not need to consider their momentum conjugates in the phase space. The symplectic 2-form of such a phase space is built as follows. Given a Lagrangian density $\mathcal{L}$, the surface term $\mathbf{\Theta}$ can be read by the variation of the Lagrangian dual $\mathbf{L}$

$$d\mathbf{L} = (\text{E.o.M})\delta\Phi + d\mathbf{\Theta}(\delta\Phi, \Phi), \tag{A.3}$$

in which E.o.M denotes the equations of motions. Having the $\mathbf{\Theta}$ as a 1-form on the space of fields, and a $D-1$-form on space time, the symplectic current $\boldsymbol{\omega}$ is defined by

$$\boldsymbol{\omega}(\delta_1\Phi, \delta_2\Phi, \Phi) = \delta_1\mathbf{\Theta}(\delta_2\Phi, \Phi) - \delta_2\mathbf{\Theta}(\delta_1\Phi, \Phi)\,, \tag{A.4}$$

which is just the exterior derivative of $\mathbf{\Theta}$ on the field configuration space. The the symplectic 2-form which makes the field configuration space a phase space is

$$\Omega(\delta_1\Phi, \delta_2\Phi, \Phi) \equiv \int_\Sigma \boldsymbol{\omega}(\delta_1\Phi, \delta_2\Phi, \Phi) \tag{A.5}$$

where Σ is a Cauchy surface. It can be shown that using appropriate boundary conditions, the result would not depend on the choice of this hypersurface.

On the covariant phase space which is built by the procedure above, one can associate a charge variation δH_ϵ to a generator ϵ. The generator can be a combination of diffeomorphisms and gauge transformations $\epsilon \equiv \{\xi^\mu, \lambda, \boldsymbol{\lambda}\}$. The diffeomorphism is $x^\mu \to x^\mu - \xi^\mu$, while $A \to A + d\lambda$ and $\boldsymbol{A} \to \boldsymbol{A} + d\boldsymbol{\lambda}$ are gauge transformations of the Maxwell field and cosmological gauge field respectively. Using standard definition of charge variations in a phase space which is $\delta H_\epsilon \equiv \delta_\epsilon \Phi \cdot \Omega$,

$$\delta H_\epsilon(\Phi) \equiv \int_\Sigma \big(\delta^{[\Phi]}\boldsymbol{\Theta}(\delta_\epsilon\Phi, \Phi) - \delta_\epsilon \boldsymbol{\Theta}(\delta\Phi, \Phi)\big) = \int_\Sigma d\boldsymbol{k}_\epsilon(\delta\Phi, \Phi) = \oint_{\partial\Sigma} \boldsymbol{k}_\epsilon(\delta\Phi, \Phi)\,. \tag{A.6}$$

In the equations above, the first equation is a result of $d\boldsymbol{\omega} = 0$ (on-shell and for linearized perturbations), and the Poincare Lemma which admits $\boldsymbol{\omega} = d\boldsymbol{k}$ for some $\boldsymbol{k}$. The last equation is the Stokes' theorem. The last equation is practically the most useful term for charge calculation in covariant formulation: for any solution $\Phi(x^\mu)$ in any given theory $\mathcal{L}$, and for any generator ϵ and linearized perturbation $\delta\Phi$, the $\boldsymbol{k}_\epsilon(\Phi, \delta\Phi)$ can be found. Then, $\oint_{\partial\Sigma} \boldsymbol{k}_\epsilon(\delta\Phi, \Phi)$ gives the $\delta H_\epsilon(\Phi)$ as the charge variation inside the hypersurface Σ. If $\partial\Sigma$ is chosen to be the asymptotics, then δH_ϵ would be the charge variation associated with the whole geometry.

The charge variation δH_ϵ in (A.6) may or may not be integrable, conserved, and finite. These conditions are fully discussed in the literature (e.g. see [29]). Here we report only the $\boldsymbol{k}_\epsilon$ for the Lagrangian densities we studied in this paper, which is the most important tensor for performing the calculations. The details can be found in [30]. Let us consider the following Lagrangian density as the theory under considerations.

$$\begin{aligned}\mathcal{L} = &\frac{1}{16\pi}\Big(f(R,\phi) + \mathrm{a}R_{\mu\nu}R^{\mu\nu} + \mathrm{b}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\Big)\\ &- \frac{1}{16\pi}\Big(\mathrm{c}_{ab}F^a_{\mu\nu}F^{b\,\mu\nu} + 2\mathrm{d}_{IJ}\nabla^\mu\phi^I\nabla_\mu\phi^J \mp 2\boldsymbol{F}^2\Big).\end{aligned} \tag{A.7}$$

In this Lagrangian, $R^\mu{}_{\nu\alpha\beta}$, $R_{\mu\nu}$, and R are Riemann tensor, Ricci tensor, and Ricci scalar, respectively. $F^a = dA^a$ are some Maxwell fields labeled by index a. The ϕ^I are some scalar fields labeled by I, and $\boldsymbol{F}$ is the cosmological field strength. The coefficients $\mathrm{a}(\phi)$, $\mathrm{b}(\phi)$, $\mathrm{c}_{ab}(\phi)$, and $\mathrm{d}_{IJ}(\phi)$ can be arbitrary functions of ϕ^I. For clarity, let us give a name for each one of the six parts in the Lagrangian respectively as:

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_\mathrm{a} + \mathcal{L}_\mathrm{b} + \mathcal{L}_\mathrm{c} + \mathcal{L}_\mathrm{d} + \mathcal{L}_{\boldsymbol{F}}. \tag{A.8}$$

Using the notation $\boldsymbol{k}_\epsilon = \star k_\epsilon$, then k_ϵ has a contribution from each one of these parts:

$$k^{\mu\nu}_\epsilon = k^{\mu\nu}_{\epsilon f} + k^{\mu\nu}_{\epsilon \mathrm{a}} + k^{\mu\nu}_{\epsilon \mathrm{b}} + k^{\mu\nu}_{\epsilon \mathrm{c}} + k^{\mu\nu}_{\epsilon \mathrm{d}} + k^{\mu\nu}_{\epsilon \boldsymbol{F}}, \tag{A.9}$$

where can be calculated to be found as

$$
\begin{aligned}
k^{\mu\nu}_{f\,\epsilon}(\delta\Phi,\Phi) = \frac{1}{16\pi}\Big[&\Big(h^{\mu\alpha}\nabla_\alpha\xi^\nu - \nabla^\mu h^{\nu\alpha}\xi_\alpha - \frac{1}{2}h\nabla^\mu\xi^\nu\Big)f' + 2\Big(R^{\mu\alpha}\nabla_\alpha h - \nabla_\alpha R h^{\mu\alpha} \\
&- \Box\nabla^\mu h + \nabla_\alpha\nabla^\mu\nabla_\beta h^{\alpha\beta} - \nabla^\mu(R_{\alpha\beta}h^{\alpha\beta}) + \frac{1}{2}\nabla^\mu R\, h\Big)\xi^\nu f'' \\
&+2(\nabla^\mu\delta\phi^I - h^\mu{}_\alpha\nabla^\alpha\phi^I + \frac{1}{2}h\nabla^\mu\phi^I)\xi^\nu\frac{\partial f'}{\partial\phi^I} - \delta\phi^I\nabla^\mu\xi^\nu\frac{\partial f'}{\partial\phi^I} \\
&+\Big(R_{\alpha\beta}h^{\alpha\beta} - \nabla_\alpha\nabla_\beta h^{\alpha\beta} + \Box h\Big)(\nabla^\mu\xi^\nu f'' - 2\nabla^\mu R\,\xi^\nu f''' - 2\nabla^\mu\phi^I\,\xi^\nu\frac{\partial f''}{\partial\phi^I}) \\
&+ 2\delta\phi^I\nabla^\mu\phi^J\,\xi^\nu\frac{\partial^2 f'}{\partial\phi^I\partial\phi^J} + 2\delta\phi^I\nabla^\mu R\,\xi^\nu\frac{\partial f''}{\partial\phi^I} \\
&- \big(f'(\nabla_\alpha h^{\mu\alpha} - \nabla^\mu h) - \nabla_\alpha f' h^{\mu\alpha} + \nabla^\mu f' h\big)\xi^\nu\Big] - [\mu\leftrightarrow\nu],
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
k^{\mu\nu}_{\mathrm{a}\,\epsilon}(\delta\Phi,\Phi) = \frac{\mathrm{a}}{16\pi}\Big[&\Big(\nabla^\alpha R_\alpha^\mu h - \nabla_\alpha R h^{\mu\alpha} - \nabla^\mu(R_{\alpha\beta}h^{\alpha\beta}) + \nabla^\mu\nabla_\alpha\nabla_\beta h^{\alpha\beta} - \nabla^\mu\Box h\Big)\xi^\nu \\
&- 2R^{\mu\beta}\nabla_\beta h^\nu{}_\alpha - 2\nabla^\mu R_{\alpha\beta}h^{\nu\beta} - \nabla^\mu(\nabla_\alpha\nabla^\nu h - \nabla_\beta\nabla_\alpha h^{\nu\beta} + \Box h^\nu{}_\alpha - \nabla^\beta\nabla^\nu h_{\alpha\beta}) \\
&+ \nabla^\mu R^\nu{}_\alpha h + 2R^{\mu\beta}\nabla^\nu h_{\alpha\beta}\Big)\xi^\alpha + \Big(\nabla_\alpha\nabla^\mu h - \nabla_\beta\nabla_\alpha h^{\mu\beta} - \nabla^\beta\nabla^\mu h_{\alpha\beta} + \Box h^\mu{}_\alpha \\
&+ 2(R_{\alpha\beta}h^{\mu\beta} + R^{\mu\beta}h_{\alpha\beta}) - R^\mu{}_\alpha h\Big)\nabla^\alpha\xi^\nu + \frac{2}{\mathrm{a}}(\nabla^\mu R^\nu{}_\alpha\xi^\alpha - \nabla^\alpha R^\nu{}_\alpha\xi^\mu)\frac{\partial\mathrm{a}}{\partial\phi^I}\delta\phi^I \\
&- \big(2R_{\alpha\beta}\nabla^\alpha h^{\beta\mu} - R^\mu{}_\alpha\nabla^\alpha h + \nabla^\alpha R^\mu{}_\alpha h - R_{\alpha\beta}\nabla^\mu h^{\alpha\beta} + \nabla^\mu R_{\alpha\beta}h^{\alpha\beta}\big)\xi^\nu\Big] \\
&- [\mu\leftrightarrow\nu],
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
k^{\mu\nu}_{\mathrm{b}\,\epsilon}(\delta\Phi,\Phi) = \frac{\mathrm{b}}{8\pi}\Big[&\Big(2(R^\mu{}_{\alpha\beta\gamma} - R^\mu{}_{\beta\alpha\gamma})h^{\nu\gamma} + R^\mu{}_\alpha{}^\nu{}_\beta h - R^\mu{}_\alpha{}^\nu{}_\gamma h^\gamma_\beta - \nabla^\mu\nabla_\alpha h^\nu{}_\beta + \nabla^\mu\nabla_\beta h^\nu{}_\alpha\Big)\nabla^\beta\xi^\alpha \\
+ \Big(&R^{\mu\beta}(\nabla_\alpha h^\nu{}_\beta - \nabla_\beta h^\nu{}_\alpha) + R^\mu{}_\beta{}^\nu{}_\gamma\nabla^\gamma h^\beta_\alpha + \frac{1}{2}R^{\mu\nu}{}_{\alpha\gamma}(\nabla_\beta h^{\beta\gamma} - \nabla^\gamma h) \\
+ 2&(\nabla_\beta R^\mu{}_\alpha - \nabla^\mu R_{\alpha\beta})h^{\nu\beta} + \nabla^\mu\nabla_\beta\nabla_\alpha h^{\nu\beta} - \nabla^\mu\Box h^\nu{}_\alpha + \nabla^\mu R^\nu{}_\alpha h + \nabla^\mu R^\nu{}_\beta h^\beta_\alpha \\
- \nabla&^\mu(R^\nu{}_{\beta\alpha\gamma}h^{\beta\gamma})\Big)2\xi^\alpha + \frac{2}{\mathrm{b}}(\nabla^\alpha R^{\mu\nu}{}_{\alpha\beta}\,\xi^\beta - R^{\mu\alpha\nu\beta}\nabla_\alpha\xi_\beta)\frac{\partial\mathrm{b}}{\partial\phi^I}\delta\phi^I \\
- 2&\big(\nabla^\nu R^\mu{}_{\alpha\nu\beta}h^{\alpha\beta} - R^\mu{}_{\alpha\nu\beta}\nabla^\nu h^{\alpha\beta}\big)\xi^\nu\Big] - [\mu\leftrightarrow\nu],
\end{aligned} \tag{A.12}
$$

$$
\begin{aligned}
k^{\mu\nu}_{\mathrm{c}\,\epsilon}(\delta\Phi,\Phi) = \frac{1}{8\pi}\Big[&(\frac{-h}{2}\,\mathrm{c}_{ab}\,F^{a\,\mu\nu} + 2\,\mathrm{c}_{ab}\,F^{a\,\mu\sigma}h_\sigma{}^\nu - \mathrm{c}_{ab}\,\delta F^{a\,\mu\nu} - \frac{\partial\,\mathrm{c}_{ab}}{\partial\phi^I}\,F^{a\,\mu\nu}\delta\phi^I)(\xi^\alpha A^b_\alpha + \lambda^b) - \\
\mathrm{c}_{ab}\,F^{a\,\mu\nu}\xi^\alpha\delta A^b_\alpha &- 2\,\mathrm{c}_{ab}\,F^{a\,\alpha\mu}\xi^\nu\delta A^b_\alpha\Big] - [\mu\leftrightarrow\nu]\,,
\end{aligned} \tag{A.13}
$$

$$
k^{\mu\nu}_{\mathrm{d}\,\epsilon}(\delta\Phi,\Phi) = \frac{1}{4\pi}\Big[\xi^\nu\,\mathrm{d}_{IJ}\,\nabla^\mu\phi^I\,\delta\phi^J\Big] - [\mu\leftrightarrow\nu] \tag{A.14}
$$

$$
\begin{aligned}
k^{\mu\nu}_{\boldsymbol{F}\epsilon} = \frac{\pm1}{8\pi(D-2)!}\Big[&(\frac{-h^\alpha_\alpha}{2}\boldsymbol{F}^{\mu\nu\rho_3\ldots\rho_D} + 2h^{\mu\beta}\boldsymbol{F}_\beta{}^{\nu\rho_3\ldots\rho_D} - \delta\boldsymbol{F}^{\mu\nu\rho_3\ldots\rho_D})(\xi^\sigma\boldsymbol{A}_{\sigma\rho_3\ldots\rho_D} + \boldsymbol{\lambda}_{\rho_3\ldots\rho_D}) \\
- \boldsymbol{F}^{\mu\nu\rho_3\ldots\rho_D}\xi^\sigma\delta\boldsymbol{A}_{\sigma\rho_3\ldots\rho_D} &+ (D-2)h^{\alpha\beta}\boldsymbol{F}_\alpha{}^{\mu\nu\rho_4\ldots\rho_D}(\xi^\sigma\boldsymbol{A}_{\sigma\beta\rho_4\ldots\rho_D} + \boldsymbol{\lambda}_{\beta\rho_4\ldots\rho_D}) \\
+ \frac{2}{D-1}\boldsymbol{F}^{\mu\rho_2\ldots\rho_D}\xi^\nu\delta\boldsymbol{A}_{\rho_2\ldots\rho_D}&\Big] - [\mu\leftrightarrow\nu],
\end{aligned} \tag{A.15}
$$

with the notation $h^{\mu\nu} = \delta g^{\mu\nu} \equiv g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$, $\delta \boldsymbol{F}^{\rho_1 \cdots \rho_D} \equiv g^{\rho_1\mu_1} \ldots g^{\rho_D\mu_D} \delta \boldsymbol{F}_{\mu_1 \cdots \mu_D}$ and $\delta F^{\mu\nu} \equiv g^{\mu\alpha} g^{\nu\beta} \delta F_{\alpha\beta}$ for the metric, cosmological and Maxwell field strength variations respectively. Besides, the notations $h \equiv h^{\mu}_{\mu}$ abd $f' \equiv \frac{\partial f}{\partial R}$ have been used. We notice that the cosmological gauge field appears explicitly in (A.15) and its gauge fixing is important for calculation of charges like mass.

A.2 How to check first law and Smarr formula if r_H is not known

Whenever the r_H is not known in terms of the parameters of the solution p_i, one may find checking the Smarr formula and the first law to be difficult, because the entropy is usually an explicit function of r_H. Here, we describe how to check these equations, for black hole solutions whose r_H is not explicitly known in terms of the free parameters p_i of the solution. The horizon radii are the roots of the equation $\Delta_r \equiv g^{rr} = 0$. In order to check the Smarr formula, instead of solving $\Delta_r = 0$ to find r_H as a function of p_i, one can solve this equation to find the parameter m as a function of the $\{r_H, \tilde{p}_i\}$, which is simpler to be solved. By the $\tilde{p}_i$ we mean all parameters p_i except the m. Then in the Smarr formula, the parameter m is replaced by its dependency on $\{r_H, \tilde{p}_i\}$, and the formula can be checked to hold or not. In order to check the first law, in addition to this procedure, one need to know variations of r_H w.r.t the parameters p_i, i.e. the $\delta_{p_i} r_H$. This can also be found easily by the relation $\delta_{p_i} \Delta_r = 0$ (at the horizon), which provides $\delta_{p_i} r_H = -\frac{\partial \Delta_r}{\partial p_i} / \frac{\partial \Delta_r}{\partial r}$ calculated on the horizon (so r will be replaced eventually by r_H).

CURRICULUM VITAE

PERFONAL INFORMATION

Surname, Name: Özşahin, Hikmet
Nationality: Turkish
Date and Place of Birth: 18.01.1986, Ankara/TURKEY

EDUCATION

Degree	Institution	Year of Graduation
M.S.	Physics, Middle East Technical University	2015
B.S.	Physics, Niğde University	2011

PROFESSIONAL EXPERIENCE

Year	Place	Enrollment
2012-2022	Middle East Technical University	Research Assistant

PUBLICATIONS

- Aliev, T. M., Özşahin, H., & Savcı, M. (2015). *More about the mass of the new charmonium states*. Advances in High Energy Physics, 2015.

- Hajian, K., Özşahin, H., & Tekin, B. (2021). *First law of black hole thermodynamics and Smarr formula with a cosmological constant*. Physical Review D, 104(4), 044024.

MS. THESIS

- *Properties of charmoniumlike states*. Hikmet Özşahin, (2015). Supervised by Prof. Dr. Tahmasib Aliyev.