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GEBZE TECHNICAL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ZERO FORCING IN GRAPHS

ASLIHAN GÜR

A THESIS SUBMITTED FOR THE DEGREE OF

MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

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T.C

GEBZE TEKNİK ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

ÇİZGELERDE SIFIR ZORLAMA

ASLIHAN GÜR

YÜKSEK LİSANS TEZİ

MATEMATİK ANABİLİM DALI

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2022

GEBZE TEKNİK ÜNİVERSİTESİ	**YÜKSEK LİSANS JÜRİ ONAY FORMU**

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ONAY

Gebze Teknik Üniversitesi Fen Bilimleri Enstitüsü Yönetim Kurulu'nun ……………………………. tarih ve ………/……... sayılı kararı.

İMZA/MÜHÜR

SUMMARY

For given a graph G, the color-change rule is a graph coloring rule which satisfies that if all vertices of G are colored black or white and a black vertex u of G has a unique white neighbor v, then the color of v is changed to black. The zero forcing number of G is the minimum number of black vertices of G that must be taken initially to color all white vertices of G to black. In this thesis, we study the zero forcing numbers of some special graphs with together the works in the literature. In particular, we focus on finding the zero forcing numbers of the strong product of some special graphs using case analysis and induction.

Keywords: Zero Forcing Set, Zero Forcing Number, Graph, Strong Product Graph.

ÖZET

Verilen bir G çizgesi için, renk değiştirme kuralı, G'nin tüm köşeleri siyah veya beyaz renkliyse ve G'nin siyah bir köşesi olan u'nun yalnızca bir beyaz komşusu v varsa, v'nin renginin siyaha değiştirileceğini söyleyen bir çizge renklendirme kuralıdır. G'nin sıfır zorlama sayısı, G'nin tüm beyaz köşelerini siyaha boyamak için başlangıçta alınması gereken G'nin minimum siyah köşe sayısıdır. Bu tezde, bazı özel çizgelerin sıfır zorlama sayıları literatürde yer alan çalışmalarla birlikte incelenmiştir. Özellikle, durum analizi ve tümevarım kullanarak bazı özel çizgelerin güçlü çarpımlarının sıfır zorlama sayılarını bulmaya odaklanıyoruz.

Anahtar Kelimeler: Sıfır Zorlama Kümesi, Sıfır Zorlama Sayısı, Çizge, Güçlü Çarpım Çizgesi.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
$G = G(V, E)$	:	Graph.
$V = V(G)$	:	The set of vertices of G.
$E = E(G)$	:	The set of edges of G.
$u \sim v$	:	u and v are neighbors.
$u \nsim v$	:	u and v are not neighbors.
$\|G\| = n(G)$	:	The order of G.
$d_G(v)$	:	The degree of the vertex v of G.
$\Delta(G)$	:	The maximum vertex degree in G.
$\delta(G)$	:	The minimum vertex degree in G.
$G[V(H)]$	:	An induced subgraph of G by $V(H) \subseteq V(G)$.
$N_G(u)$	:	The open neighborhood of u.
$N_G[u]$	:	The closed neighborhood of u.
$IR(G)$	:	The largest cardinality of an irredundant set of G.
$G \cong H$	:	G is isomorphic to H.
N_n	:	A null graph with n vertices.
P_n	:	A path graph with n vertices.
C_n	:	A cycle graph with n vertices.
K_n	:	A complete graph with n vertices.
W_n	:	A wheel graph with n vertices.
$K_{p,q}$	:	A complete bipartite graph with $p + q$ vertices.
$K_{n_1,n_2,\dots,n_k}$	:	A complete multipartite graph with $n_1 + n_2 + \cdots + n_k$ vertices.
P	:	Petersen graph.
$\bar{G}$	:	The complement of G.
$G + H$	:	The union of G and H.
$G \vee H$	:	The join of G and H.
$G \,\square\, H$	:	The cartesian product of G and H.
$G \times H$	:	The direct product of G and H.

$G \boxtimes H$	:	The strong product of G and H.
$G \odot H$	:	The corona product of G and H.
$u \rightarrow v$	:	u forces v.
$Z(G)$	:	Zero forcing number of G.
$\ker(A)$	:	The kernel of the matrix A.
$\mathrm{corank}(A)$	:	The nullity of the matrix A.
$\mathrm{rank}(A)$	:	The rank of the matrix A.
$\mathrm{Sym}_n(\mathbb{R})$	:	The set of $n \times n$ symmetric matrices over $\mathbb{R}$.
$\mathcal{G}(A)$	:	The graph of the matrix A.
$\mathcal{S}(G)$	:	The set of symmetric matrices of the graph G.
$\mathrm{mr}(G)$	:	The minimum rank of the graph G.
$\mathrm{M}(G)$	:	The maximum nullity of the graph G.
$\mathrm{supp}(\mathbf{x})$	:	The set that consists of each index i such that x_i is non-zero.
$\gamma_{gr}(G)$	:	Grundy domination number of G.
$\gamma_{gr}^{Z}(G)$	:	$Z -$ Grundy domination number of G.

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1. INTRODUCTION

The foundation of graph theory is based on the solution of a problem in the article "Seven Bridges of Königsberg" published by Swiss mathematician and physicist Leonhard Euler in 1736. See [1] and [2] for more information on this topic.

Graph theory is a field of study in mathematics that examines relations between objects with the help of a diagram in which the objects are shown by points and the relations of the objects are shown by lines. Since graphs provide significant convenience in modeling and solving problems, they are used not only in mathematics but also in numerous fields such as chemistry, physics, biology, geography, computer science, and social networks (see for instance, [3], [4], [5], [6], [7], and [8], respectively).

In chapter 2, we will present basic definitions from graph theory, we will mention the structure of several special graphs and several special graph operations, and we will give some examples of all of these. All of these definitions will help us throughout this thesis.

The concept of a zero forcing set of a simple graph that is the main focus of this thesis was introduced by "AIM Minimum Rank-Special Graphs Work Group (Barioli et al.)" in [9]. Their main purpose was to find an upper bound for the maximum nullity of the family of real symmetric matrices whose non-zero entries not on the diagonal are described by a simple graph. Also, as a graph parameter zero forcing number of a simple graph that is the minimum cardinality of its zero forcing sets was introduced again in [9]. Moreover, the zero forcing numbers of several special graphs were found in this article. See in [10] for more results on the zero forcing numbers of several special graphs. We will examine most of them.

Zero forcing has attracted a lot of attention from many researchers and has become the topic of their works for almost fifteen years. Zero forcing which is a great topic of graph theory has wide applications such as quantum physics, power network monitoring, and logic (see for instance, [11], [12], and [13], respectively). Also, some bounds for the zero forcing number of a graph have been studied concerning some other graph parameters such as maximum or minimum vertex degree, girth number, and path cover number (see for instance, [14], [15] and [16]).

In chapter 3, firstly we will give a graph coloring rule that is required for a zero forcing process on a graph G. Subsequently, we will present basic definitions and examples about zero forcing, and we will elaborate on several known results from [9] and [10].

There are many results for the zero forcing number of any product of any two special graphs (see for instance, [9], [17], [18] and [19]). We will mention the upper or lower bounds and certain results for the zero forcing number of a special product of any two graphs in the following. For example, Taklimi, in her Ph.D. thesis (see [20]), found a fact regarding the zero forcing number of the join product of any two connected graphs. AIM Minimum Rank-Special Graphs Work Group provided an upper bound for the cartesian product of any two graphs in [9]. Huang et al. provided an upper bound for the direct product of any two graphs in [17]. Javaid et al. found a fact regarding the zero forcing number of the corona product of any two connected graphs in [21]. We will refer to all of them in the later stages of this thesis.

We will mainly work on the zero forcing number of the strong product of any two special graphs. Some studies that have been done on this topic are the following. AIM Minimum Rank-Special Graphs Work Group found the zero forcing number of the strong product of any two path graphs in [9]. Huang et al. provided an upper bound for the strong product of any two graphs in [17]. They showed that the zero forcing number of the strong product of two graphs in the set $\{C_{3n}, P_{3n-1}, K_{n+1} : n \geq 1\} \cup P$ is equal to the upper bound. Moreover, they showed that it holds for the set $\{C_n, P_k : n \geq 5, k \geq 2\}$. Brešar et al. introduced a graph parameter called Grundy domination number of a graph G that is the maximum length of a special sequence in [22]. We will explain this sequence. After that Brešar et al. obtained a relation between the Grundy domination number and the zero forcing number of a graph G in [18]. This connection provided an upper bound for the zero forcing number of any graph without isolated vertices. Also, they introduced a new graph parameter $Z-$Grundy domination number of a graph G in this paper and they presented a connection between $Z-$Grundy domination number and zero forcing number of a graph G. This connection explained the zero forcing number of any graph G without isolated vertices based on $Z-$Grundy domination number of G. Also, they found the zero forcing number of the strong product of a path graph and a cycle graph in [18].

In chapter 4, we will examine all of the above-mentioned works and we will present the main contributions of this thesis. Firstly, we will present a lower bound related to the zero forcing number of the strong product of a complete graph and any connected graph. We will present an exact result for the zero forcing number of the strong product of a complete graph and a cycle graph. Subsequently, we will put forward an upper bound for the zero forcing number of the strong product of a complete bipartite graph and any connected graph. After that, we will present the exact result for the zero forcing number of the strong product of a complete graph and a complete bipartite graph. Finally, based on works in the literature and our results, we will present the zero forcing number of the strong product of a path graph and a complete graph, and the zero forcing number of the strong product of a path graph and a complete bipartite graph as corollaries.

2. PRELIMINARIES FOR GRAPH THEORY

2.1. Basic Definitions

In this section, the basic concepts of graph theory are mentioned and several examples are given.

Definition 2.1: [1] A graph, denoted by G or $G(V, E)$, is a diagram comprised of a vertex set $V = V(G)$, an edge set $E = E(G)$, and a connection that associates with each edge two vertices (not necessarily distinct).

Also, if there exists an edge between two vertices in a graph G, then the two vertices are called endpoints of the edge.

Definition 2.2: [1] An edge with the same endpoints in a graph G is called a loop.

Definition 2.3: [1] Edges having the same pair of endpoints in a graph G are called multiple edges.

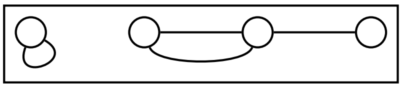


Figure 2.1: A graph G with a loop and multiple edges.

Definition 2.4: [1] If a graph G has no loops and multiple edges, then G is called a simple graph.

From now on, we will consider that each graph G we use in this work is a simple and finite graph.

Definition 2.5: [23] If there exists an edge between two vertices of a graph G, say u and v, then these vertices are called neighbors, or we say that u adjacent to v. Then it is denoted by $u \sim v$; otherwise, the vertices u and v are not neighbors or u is not adjacent to v. Then it is denoted by $u \nsim v$.

Definition 2.6: [1] The order of a graph is the number of its vertices. For a graph G, the order of G is denoted by $|G|$ or $n(G)$.

Definition 2.7: [1] Let v be a vertex of a graph G. The degree of the vertex v is the number of vertices that is adjacent to v. It is denoted by $d_G(v)$.

Also, the maximum degree is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$ in a graph G.

Let G be a graph. For representing, $V(G)$ usually equals to the following set $\{v_1, v_2, \cdots, v_n\}$ or its subset, also $E(G)$ usually equals to the following set $\{e_k := \{v_i, v_j\} : v_i \sim v_j;\ 1 \leq i \leq n;\ 1 \leq j \leq n;\ k \leq \frac{n(n-1)}{2}\}$ or its subset where the order of the vertices v_i and v_j is not important.

Now, we will explain these definitions with the help of the following example.

Example 2.1: Let G be a graph as follows.

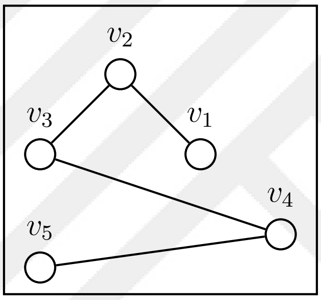


Figure 2.2: A graph G.

Here, $V = V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $|G| = 5$, $d_G(v_4) = d_G(v_2) = d_G(v_3) = 2$, and $d_G(v_1) = d_G(v_5) = 1$. Then $\Delta(G) = 2$ and $\delta(G) = 1$. Also, $E = E(G) = \{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_4\}, e_4 = \{v_4, v_5\}\}$ since $v_1 \sim v_2, v_2 \sim v_3, v_3 \sim v_4, v_4 \sim v_5$.

Definition 2.8: [1] Let $G = G(V(G), E(G))$ be a graph. A subgraph $H = H(V(H), E(H))$ of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We then write $H \subseteq G$.

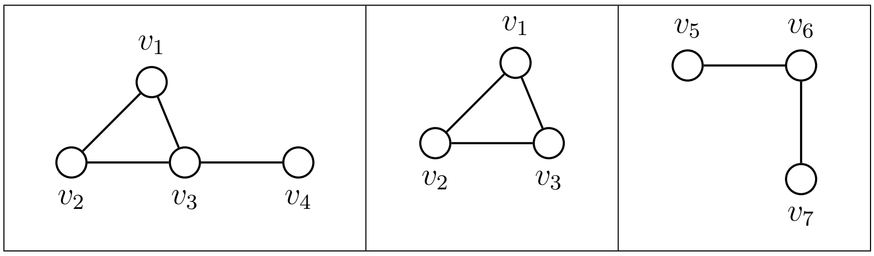
Definition 2.9: [23] Let G and H be two graphs with the vertex sets $V(G)$ and $V(H)$, respectively. H is called an induced subgraph of G if H is a subgraph of G and its edge set consists of the edges whose both endpoints are in $V(H)$. Then H is denoted by $G[V(H)]$.

Definition 2.10: [23] Two graphs G and H with the vertex sets $V(G)$ and $V(H)$, respectively are called disjoint if $V(G) \cap V(H) = \emptyset$.

Now, we will explain these definitions with the following example.

Example 2.2: Let G_i be a graph in the following, where $1 \leq i \leq 4$.

Table 2.1: A graph G_i for all $1 \leq i \leq 3$, respectively.



Clearly, one can easily see that G_2 is a subgraph of G_1 and the reason is $V(G_2) = \{v_1, v_2, v_3\} \subseteq V(G_1) = \{v_1, v_2, v_3, v_4\}$ and $E(G_2) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\} \subseteq E(G_1) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}\}$.

Also, G_2 is an induced subgraph of G_1 by $V(G_2)$ since $G_2 \subseteq G_1$ and the end-points of each edge of G_2 is contained in $V(G_2)$.

Moreover, we see that G_2 and G_3 are disjoint since $V(G_2) \cap V(G_3) = \emptyset$.

Definition 2.11: [24] Let G be a graph with the vertex set $V(G)$. For any $u \in V(G)$, the open neighborhood (or just neighborhood) of the vertex u, denoted by $N_G(u)$, is the following set $N_G(u) = \{v \in V(G) : u \sim v\}$. Also, the closed neighborhood of the vertex u, denoted by $N_G[u]$, is the following set $N_G[u] = N_G(u) \cup \{u\}$.

Definition 2.12: [25] Let G be a graph with the vertex set $V(G)$ and let $u, v \in V(G)$. The vertices u and v are called false twins if they have the same open neighborhood; that is, $N_G(u) = N_G(v)$. Also, the vertices u and v are called true twins if they have the same closed neighborhood; that is $N_G[u] = N_G[v]$.

Definition 2.13: [2] An isolated vertex v in a graph G is a vertex with $N_G(v) = \emptyset$.

Definition 2.14: [26] Let G be a graph with the vertex set $V(G)$. A set $S \subseteq V(G)$ is called an irredundant set if $N_G[S - \{v\}] \neq N_G[S]$ for all $v \in S$. Also, the upper irredundance number of G, denoted by IR(G), is the maximum cardinality of an irredundant set of G.

Definition 2.15: [1] Let G and H be two graphs with the vertex sets $V(G)$ and $V(H)$, respectively. We say that G is isomorphic to H if there exists a bijection (onto and one to one mapping) f from $V(G)$ to $V(H)$ such that for any vertices $u, v \in V(G)$, $u \sim v$ if and only if $f(u) \sim f(v)$ for $f(u), f(v) \in V(H)$.

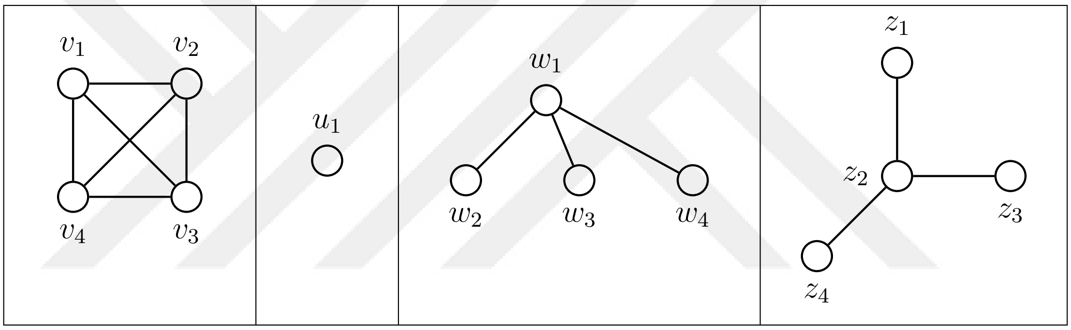
Definition 2.16: [24] A graph G is called a regular graph if all of its vertices have the same degree.

Also, if every vertex of G has degree r, then G is called $r-$regular.

Now, we will explain these definitions with the following example.

Example 2.3: Let G_i be a graph in the following, where $1 \leq i \leq 4$.

Table 2.2: A graph G_i for all $1 \leq i \leq 4$, respectively.



One can easily see that G_1 is $3-$regular graph because of $d_{G_1}(v_j) = 3$ for all $1 \leq j \leq 4$.

Clearly, $N_{G_1}(v_j) = V(G_1)\backslash\{v_j\}$ and $N_{G_1}[v_j] = V(G_1)$ for all $1 \leq j \leq 4$. Since $N_{G_1}[v_j] = N_{G_1}[v_k]$ for each $1 \leq j \leq 4$ and $1 \leq k \leq 4$, all vertices are true twins each other in G_1. Moreover, $S = \{v_1, v_2, v_3, v_4\}$ is an irredundant set with maximum cardinality of G_1 since $N_{G_1}[S_j] = V(G_1) - \{v_j\} \neq V(G_1) = N_{G_1}[S]$, where $S_j = S - \{v_j\}$ and $1 \leq j \leq 4$. Then IR$(G_1) = 4$.

G_2 consists of an isolated vertex u_1.

In Table 2.2 we also see that $G_3 \cong G_4$. The reason of this is the following. Let a bijection $f : V(G_3) \longrightarrow V(G_4)$ with $f(w_1) = z_2, f(w_2) = z_1, f(w_3) = z_3$ and $f(w_4) = z_4$. Clearly, we know that $w_1 \sim w_2, w_1 \sim w_3$, and $w_1 \sim w_4$, also $f(w_1) \sim f(w_2), f(w_1) \sim f(w_3)$, and $f(w_1) \sim f(w_4)$.

2.2. Several Special Graphs

Many graphs are specially named because of their structure or their different properties. Some of them are introduced and their graphs are given in this section.

Definition 2.17: [2] A null graph is a graph whose each vertex is isolated.

Also, a null graph with $n \in \mathbb{N}^+$ vertices is denoted by N_n.

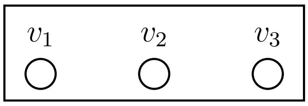


Figure 2.3: Graph of N_3.

Definition 2.18: [1] A path is a graph whose vertices can be ordered as a list so that two vertices are neighbors if and only if they are consecutive in the list.

Also, a path with n vertices is denoted by P_n, where $n \geq 2$. Moreover, P_1 is a path graph that is isomorphic to N_1.

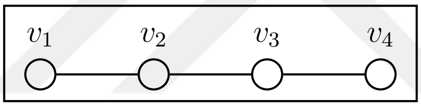


Figure 2.4: Graph of P_4.

Definition 2.19: [1] If each pair of vertices in a graph G belongs to a path, then G is called connected; otherwise, G is called disconnected.

Definition 2.20: [2] A cycle graph is a connected graph such that the degree of each vertex of the graph is 2.

Also, a cycle with n vertices is denoted by C_n, where $n \geq 3$.

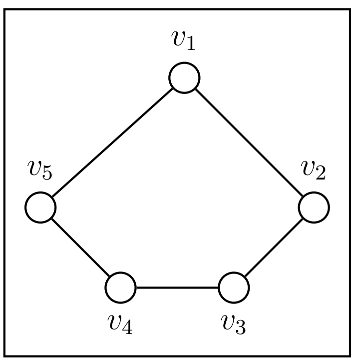


Figure 2.5: Graph of C_5.

Definition 2.21: [23] A tree is a connected graph that has no cycles.

Definition 2.22: [2] If each pair of distinct vertices is adjacent in a graph G, then G is called a complete graph.

A complete graph with n vertices is denoted by K_n, where $n \geq 2$. Moreover, K_1 is a complete graph that is isomorphic to N_1.

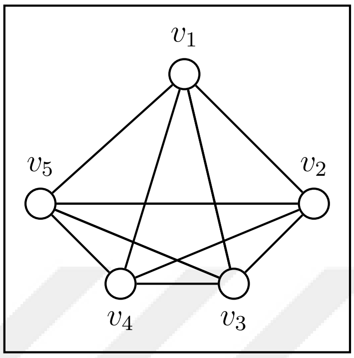


Figure 2.6: Graph of K_5.

Definition 2.23: [1] Let G be a graph. If the neighborhood of a vertex v of G induces a complete graph, then v is called the simplicial vertex.

Definition 2.24: [24] If a graph G with n vertices contains C_{n-1} and a vertex u so that u is adjacent to each vertex of C_{n-1}, then G is called a wheel graph.

Also, it is denoted by W_n, where $n \geq 4$.

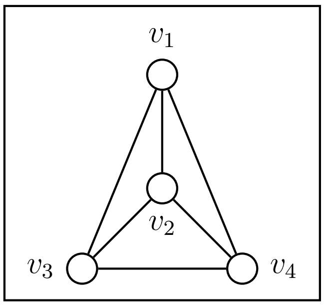


Figure 2.7: Graph of W_4.

Definition 2.25: [23] Let G be a graph. A set $S \subseteq V(G)$ is called an independent set if the induced subgraph $G[S]$ of G has no edges.

Definition 2.26: [1] Let G be a graph with the vertex set $V(G)$. If $V(G)$ can be partitioned into non-empty two disjoint independent sets U_1 and U_2, then G is called a bipartite ($2-$partite) graph. Also, U_1 and U_2 are called partite sets of G.

Definition 2.27: [2] A complete bipartite graph is a bipartite graph which satisfies that each vertex of the partite set U_1 and each vertex of the partite set U_2 are neighbors.

If $1 \leq |U_1| = p, |U_2| = q$, then a complete bipartite graph is denoted by $K_{p,q}$, where $p, q \in \mathbb{N}^+$.

In particular, the graph $K_{1,n}$ is called a "$star\ graph$" and it is denoted by S_n for $n \geq 3$. Also, the star graph S_3 is called a "$claw\ graph$". See Figure 2.8.

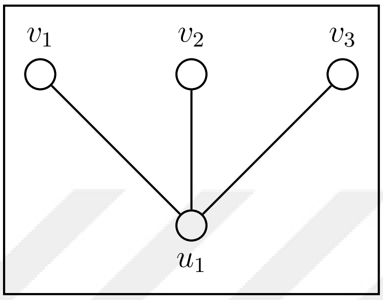


Figure 2.8: Graph of $K_{1,3}$.

Definition 2.28: [26] Let G be a graph with the vertex set $V(G)$. If $V(G)$ can be partitioned into non-empty $k-$disjoint independent sets U_1, U_2, $\cdots$, U_k, then G is called a multipartite ($k-$partite) graph. Also, U_1, U_2, $\cdots$, U_k are called partite sets of G, where $k > 2$.

Definition 2.29: [24] A complete multipartite graph is a multipartite graph which satisfies that each vertex of the partite set U_i and each vertex of the partite set U_j are neighbors for all $1 \leq i \leq k$ and $1 \leq j \leq k$, where $i \neq j$.

If $1 \leq |U_1| = n_1, |U_2| = n_2, \cdots, |U_k| = n_k$, then a complete multipartite graph is denoted by $K_{n_1,n_2,\cdots,n_k}$, where $k > 2$.

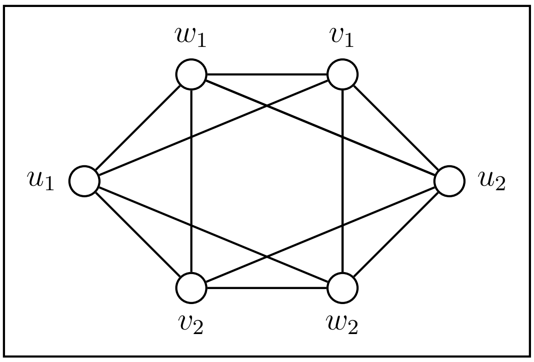


Figure 2.9: Graph of $K_{2,2,2}$.

Definition 2.30: [24] If a graph G is a $3-$regular graph and G has 10 vertices, then G is called the Petersen graph. It is denoted by P.

2.3. Basic Graph Operations

In this section, some graph operations are introduced and some special examples are given.

Definition 2.31: [1] The complement of a graph $G = G(V(G), E(G))$, denoted by $\overline{G}$, is a graph with the vertex set $V(\overline{G})$ equals to $V(G)$ and the edge set $E(\overline{G})$ equals to $\{\{u, v\} : \{u, v\} \notin E(G); u, v \in V(G)\}$.

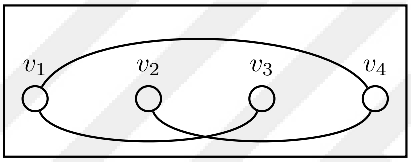


Figure 2.10: Graph of $\overline{P_4}$.

Definition 2.32: [24] Let $G = G(V(G), E(G))$ and $H = H(V(H), E(H))$ be two disjoint graphs. Then the union of G and H, denoted by $G + H$, is a graph with the vertex set $V = V(G) \cup V(H)$ and the edge set $E = E(G) \cup E(H)$.

Definition 2.33: [24] Let $G = G(V(G), E(G))$ and $H = H(V(H), E(H))$ be two disjoint graphs. Then the join of G and H, denoted by $G \vee H$, is a graph with the vertex set $V = V(G) \cup V(H)$ and the edge set $E = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$.

Example 2.4: We consider the graph in Figure 2.5 and the null graph N_1 with the vertex set $V(N_1) = \{u\}$. Then the join of C_5 and N_1 is the following graph.

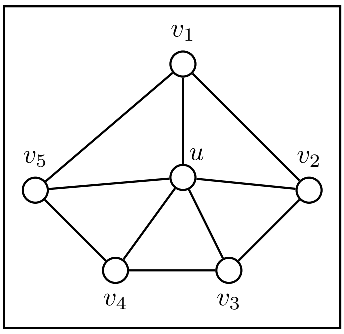


Figure 2.11: Graph of $C_5 \vee N_1 \cong W_6$.

Generally, a wheel graph with n vertices is the join of C_{n-1} and N_1. That is, $W_n \cong C_{n-1} \vee N_1$.

Definition 2.34: [24] Let $G = G(V(G), E(G))$ *and* $H = H(V(H), E(H))$ *be two disjoint graphs. Then the cartesian product of* G *and* H*, denoted by* $G \Box H$*, is a graph with the vertex set* $V = V(G) \times V(H)$ *and the edge set* $E = \{\{(u, v), (x, y)\} : [u = x, v \sim y] \vee [v = y, u \sim x]$ *for all* $(u, v), (x, y) \in V\}$.

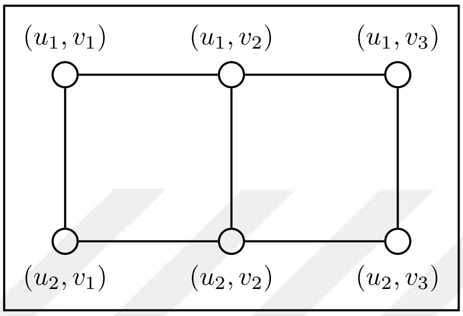


Figure 2.12: Graph of $P_2 \Box P_3$.

Definition 2.35: [27] Let $G = G(V(G), E(G))$ *and* $H = H(V(H), E(H))$ *be two disjoint graphs. Then the direct product of* G *and* H*, denoted by* $G \times H$*, is a graph with the vertex set* $V = V(G) \times V(H)$ *and the edge set* $E = \{\{(u, v), (x, y)\} : [u \sim x, v \sim y]$ *for all* $(u, v), (x, y) \in V\}$.

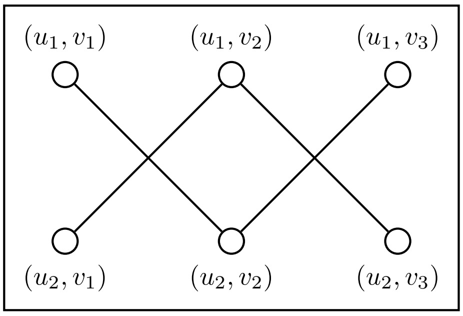


Figure 2.13: Graph of $P_2 \times P_3$.

Definition 2.36: [27] Let $G = G(V(G), E(G))$ *and* $H = H(V(H), E(H))$ *be two disjoint graphs. Then the strong product of* G *and* H*, denoted by* $G \boxtimes H$*, is a graph with the vertex set* $V = V(G) \times V(H)$ *and the edge set* $E = \{\{(u, v), (x, y)\} : [u \sim x, v \sim y] \vee [u = x, v \sim y] \vee [v = y, u \sim x]$ *for all* $(u, v), (x, y) \in V\}$.

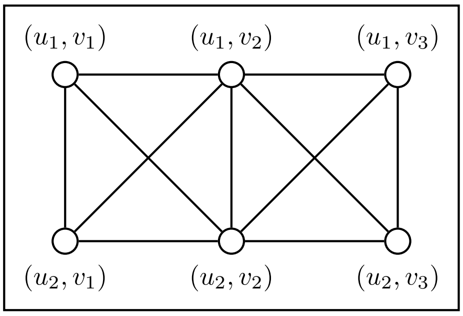


Figure 2.14: Graph of $P_2 \boxtimes P_3$.

Definition 2.37: [28] The corona product of two graphs G and H, denoted by $G \odot H$, is a graph obtained by taking one copy of G and $|G|$ copies of H and joining by an edge each vertex from the ith copy of H with the ith vertex of G.

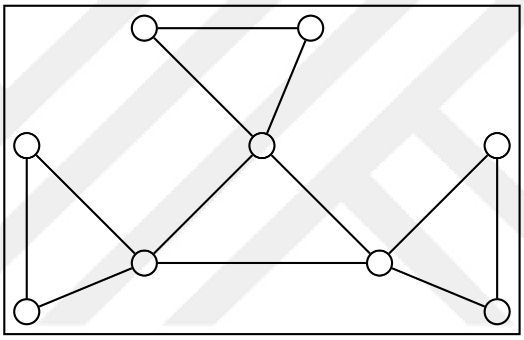


Figure 2.15: Graph of $C_3 \odot P_2$.

Remark 2.1: In some sources, the union of G and H is denoted by $G \cup H$, the join of G and H is denoted by $G + H$, the cartesian product of G and H is denoted by $G \times H$, the direct product of G and H is denoted by $G \cdot H$, and the corona product of G and H is denoted by $G \circ H$. But throughout this thesis, we use the above notations.

3. ZERO FORCING SETS

3.1. An Upper Bound for the Maximum Nullity of a Simple Graph

In this section, a graph coloring rule is introduced first. It is required for a zero forcing process on G. Then, the concept of zero forcing number associated with this rule is given with some examples. We know that the starting point of the concept of a zero forcing set of a simple graph was actually to find an upper bound for the maximum nullity of a family of some special real symmetric matrices (see [9]). To associate the concept of nullity with a graph, it is necessary to find the matrices corresponding to a graph G. Likewise, the opposite can also be done. That is, the graph of a matrix can be found. Several linear algebra concepts have been mentioned to do these. Some concepts related to the zero forcing process are given and explained with examples and some important results have been examined from [9]. Finally, an upper bound for the maximum nullity of a graph is given (see [9]).

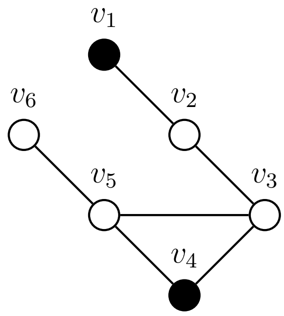
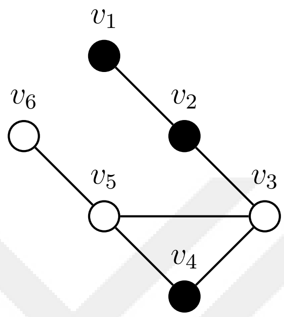
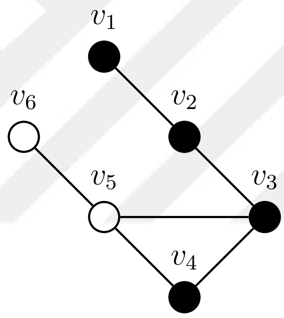
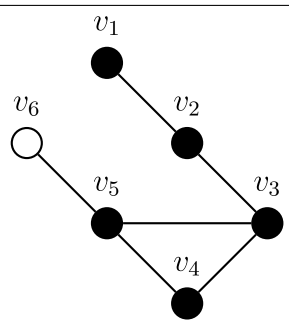
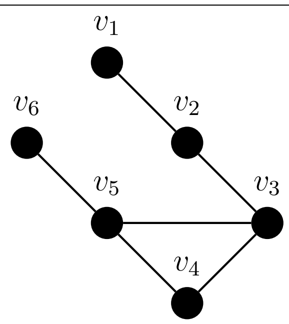
Definition 3.1: [9] (Color-Change Rule) Let G be a graph whose each vertex is colored either white or black. If u is a black vertex of G with exactly one white neighbor is v, then the color of v is changed to black. Then we say that 'u forces v' and it is denoted by $u \to v$. This rule is called the color-change rule.

Also, the set of all black vertices of a graph G before applying any color changing process is called an initial set of black vertices of G or briefly an initial set of G.

Moreover, the rule of color changing process is applied to a graph G until no possible coloring. This procedure is called a zero forcing process on G.

Now, we will explain this definition with the following table.

Table 3.1: A zero forcing process of a graph G.

Graph	Zero Forcing Process
	The graph G with the initial set $\{v_1, v_4\}$
	The graph G after the first iteration $(v_1 \to v_2)$
	The graph G after the second iteration $(v_2 \to v_3)$
	The graph G after the third iteration $(v_4 \to v_5)$
	The graph G after the fourth iteration (The final version of G) $(v_5 \to v_6)$

Example 3.1: We consider the graph G with the initial set $Z = \{v_1, v_4\}$ in Table 3.1 and we start a zero forcing process on G. In the first iteration, v_1 forces v_2 since v_1 has a unique white neighbor v_2. However, v_4 can not force any vertex of G since it has two white neighbors v_3 and v_5. Now, v_2 has a unique white neighbor v_3. Then in the second iteration, v_2 forces v_3. Also, v_4 forces v_5 in the following step since v_4 has a unique white neighbor v_5. Finally, in the last step, v_5 forces v_6 since v_5 has a unique white neighbor v_6. So, the zero forcing process on G is completed.

As a result, at the end of the zero forcing process applying according to the initial set $Z = \{v_1, v_4\}$, all white vertices of the graph G are colored black. However, it is not always all of the white vertices of a graph G are changed to black at the end of a zero forcing process on G, the condition depends on the taken initial set of G.

Definition 3.2: [9] A zero forcing set of a graph G is an initial set of black vertices of G which satisfies that after applying a zero forcing process according to the initial set, all white vertices of G turn to black.

Also, a minimum zero forcing set of a graph G is a zero forcing set of G with the minimum cardinality.

Definition 3.3: [9] The zero forcing number of a graph G, denoted $Z(G)$, is the number of vertices of a minimum zero forcing set of G.

Example 3.2: In Example 3.1, $Z = \{v_1, v_4\}$ is a zero forcing set of the graph G since we apply the zero forcing process according to the initial set Z, all white vertices of G turn to black. Moreover, Z is a minimum zero forcing set of G.

Now, we will examine whether there exists a zero forcing set of G whose cardinality is smaller than 2. Let $Z_1 = \{v_2\}$ be an initial set of G. Then the black vertex v_2 has two white neighbors v_1 and v_3. Then no force occurs. That is, Z_1 is not a zero forcing set of G. Similarly, it holds for $Z_2 = \{v_3\}$, $Z_3 = \{v_4\}$, and $Z_4 = \{v_5\}$.

Now, let $Z_5 = \{v_1\}$ be an initial set of G. See Figure 3.1.

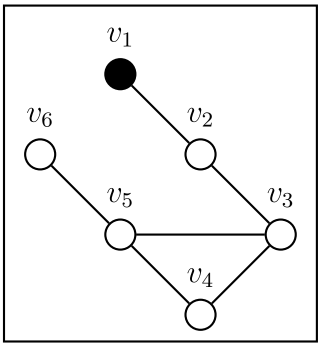


Figure 3.1: The graph G with the initial set Z_5.

$v_1 \to v_2$ in the first iteration and $v_2 \to v_3$ in the second iteration. However, the black vertex v_3 has two white neighbors v_4 and v_5 in this step. Then no force occurs. That is, Z_5 is not a zero forcing set of G. See Figure 3.2.

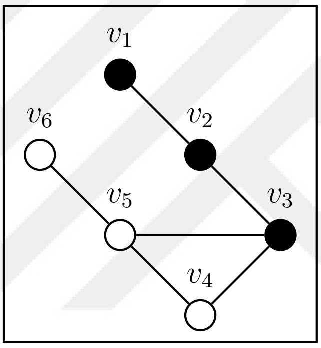


Figure 3.2: The final version of the graph G.

Similarly, it holds for $Z_6 = \{v_6\}$. As a result, there is no such an initial set S of G such that $|S| < |Z|$. That is, there is no such a zero forcing set of G with the cardinality of 1. *Thus, $Z(G) = 2$.*

Another minimum zero forcing set of G can be found. For example, the initial set $\{v_1, v_6\}$ also is a minimum zero forcing set of G.

Since we will work with $n \times n$ matrices over $\mathbb{R}$ for the matrix representation of a graph G, we have adapted all of the definitions contained in this section accordingly.

Also, the set of $n \times n$ matrices over $\mathbb{R}$ is denoted by $\mathbb{R}^{n\times n}$.

Definition 3.4: [23]

i) The kernel of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by ker$(A)$, is the set of all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a vector and $\mathbf{0} \in \mathbb{R}^n$ is a zero vector.

ii) The kernel of the matrix A is also called the null space of A, and its dimension (the number of vectors contained in any basis of the null space of A) is called the nullity of A and it is denoted by null(A). The nullity of A also is called the corank of A, and it is denoted by corank(A).

iii) The rank of the matrix A, denoted by rank(A), is the number of leading entries (pivots) in the reduced row echelon form of A.

Also, well-known equality $\text{rank}(A) + \text{corank}(A) = n$ holds for each matrix $A \in \mathbb{R}^{n\times n}$.

Definition 3.5: [29] The set of $n \times n$ symmetric matrices over $\mathbb{R}$, denoted by $Sym_n(\mathbb{R})$, is a set of $n \times n$ square matrices over $\mathbb{R}$ which are equal to their transpose.

That is, if the matrix $A = [a_{ij}] \in Sym_n(\mathbb{R})$, then $A = [a_{ij}] = [a_{ji}] = A^T$.

One can see [29] and [23] for more information on Linear Algebra.

Definition 3.6: [23] For a matrix $A = [a_{ij}] \in Sym_n(\mathbb{R})$, the graph of A is a graph with the vertex set $\{v_1, v_2, \cdots, v_n\}$ and the edge set $\{\{v_i, v_j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$, and the graph is denoted by $\mathcal{G}(A)$.

We see that the diagonal of A is ignored while determining the graph $\mathcal{G}(A)$. If we do not ignore the diagonal of A, then $\mathcal{G}(A)$ may contain loops when all the entries in the diagonal of A are not zero. However, we do not want this since we are working with simple graphs.

Example 3.3: Let A be a matrix as follows.

$$A = \begin{bmatrix} 2 & 3 & 0 & -5 \\ 3 & -1 & 9 & 4 \\ 0 & 9 & 2 & 8 \\ -5 & 4 & 8 & 0 \end{bmatrix} \in Sym_4(\mathbb{R}). \tag{3.1}$$

The graph corresponding to A is the following.

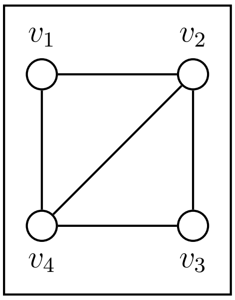


Figure 3.3: The graph $\mathcal{G}(A)$.

For a symmetric matrix A, we can find exactly one corresponding graph G, but the converse is not true.

Example 3.4: Let G be a graph as follows.

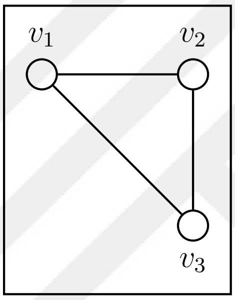


Figure 3.4: A graph G.

There exist infinitely many matrices corresponding to G. Two of them are the following.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 9 \\ 3 & 9 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 8 \\ 1 & 8 & 3 \end{bmatrix} \in Sym_3(\mathbb{R}). \tag{3.2}$$

Definition 3.7: [23]

i) The set of symmetric matrices of a graph G, denoted by $\mathcal{S}(G)$, is a set which contains all symmetric matrices whose graphs are the same as G.

That is, $\mathcal{S}(G) = \{A \in Sym_n(\mathbb{R}) : \mathcal{G}(A) = G\}$.

ii) The minimum rank of a graph G equals to the following set $mr(G) = min\{rank(A) : A \in \mathcal{S}(G)\}$.

iii) The maximum nullity (corank) of a graph G equals to the following set $M(G) = max\{corank(A) : A \in \mathcal{S}(G)\}$.

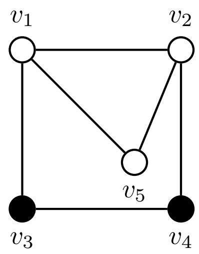
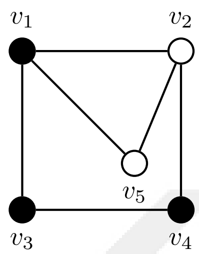
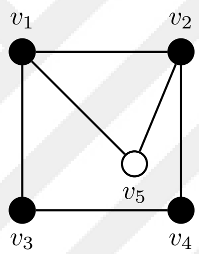
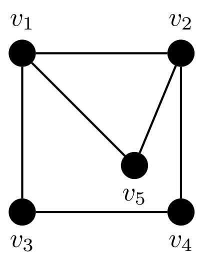
Let $n \in \mathbb{N}^+$. Now, we will consider a vector $\mathbf{x}$ with n components as follows.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{3.3}$$

And we will consider a graph G with n vertices and its vertex set is $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let a relation between the component of nth coordinate of $\mathbf{x}$ and the nth vertex of G be the following. If S is an initial set of black vertices of G, then the components of the coordinates of $\mathbf{x}$ corresponding to the vertices in S are zero and the remaining components of $\mathbf{x}$ are non-zero. Subsequently, we will start a zero forcing process on G. Also, let a component of $\mathbf{x}$ change to the following after each forcing. If the vertex v_i of G forces the vertex v_j, then the component of the coordinate of $\mathbf{x}$ corresponding to v_j is changed to be zero, where $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$. This process is completed when the zero forcing process is completed on G. As a result, this vector is well-matched to the final version of G.

Example 3.5: We consider the graph G with the initial set $Z = \{v_3, v_4\}$ in Table 3.2. Then the vector corresponding to G is $\mathbf{x} \in \mathbb{R}^{5\times 5}$ whose the components of 3rd and 4th coordinates are zero and the remaining components are non-zero. We begin a zero forcing process on G. In the first iteration, if $v_3 \to v_1$, then the component x_1 of the first coordinate of $\mathbf{x}$ becomes 0. Subsequently, $v_4 \to v_2$ in the second iteration, then the component x_2 of the second coordinate $\mathbf{x}$ becomes 0. In the last step, if $v_1 \to v_5$, then the component x_5 of the fifth coordinate of $\mathbf{x}$ becomes 0. As a result, the final version of $\mathbf{x}$ corresponding to the final version of G is a zero vector.

Table 3.2: Corresponding vectors to a colored graph G.

Graph	Corresponding Vector
	$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ x_5 \end{bmatrix}$ $; x_1, x_2, x_5 \neq 0$
	$\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ x_5 \end{bmatrix}$ $; x_2, x_5 \neq 0$
	$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x_5 \end{bmatrix}$ $; x_5 \neq 0$
	$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Definition 3.8: [23] The support of a vector $\mathbf{x} = [x_i] \in \mathbb{R}^{n \times n}$*, denoted by supp*$(\mathbf{x})$*, is the set of index* i *for all* $1 \leq i \leq n$ *such that* x_i *is non-zero.*

AIM Minimum Rank-Special Graphs Work Group in [9] established a connection between the vectors described above and the vectors in the kernel of a symmetric matrix of a graph G.

Proposition 3.1: [9] If $A \in \mathbb{R}^{n\times n}$ and corank$(A) > k$ for $k \in \mathbb{Z}^+$, then there exists a non-zero vector $\mathbf{x}$ in the kernel of A whose components in any k specified positions are zero. In other words, if W is a set of k indices, then there exists a non-zero vector $\mathbf{x}$ in the kernel of A such that supp$(\mathbf{x}) \cap W = \emptyset$.

Proof: Let $A \in \mathbb{R}^{n\times n}$ and let corank$(A) > k$ for $k \in \mathbb{Z}^+$. Also, let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and let $V_k = \{\mathbf{x} = [x_i] \in \mathbb{R}^n : x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\}$. It is clear that V_k is the set of all vectors in $\mathbb{R}^n$ whose components in specified $k-$coordinates are zero.

Claim: There exists a non-zero vector in V_k which contained in the kernel of the matrix A. That is, ker$(A) \cap V_k \neq \{\mathbf{0}\}$.

It is clear that dim$(V_k) = n - k$. Since V_k and ker(A) are subspaces of $\mathbb{R}^n$, $V_k +$ ker(A) is a subspace of $\mathbb{R}^n$. Then dim$(V_k +$ ker$(A)) \leq$ dim$(\mathbb{R}^n) = n$. Also, we know that dim(ker$(A)) =$ corank$(A) > k$. If we consider these informations and the equation in (3.4) is known to all, then the following is obtained.

$$\mathit{dim}(V_k \cap N) = \mathit{dim}(V_k) + \mathit{dim}(\mathit{ker}(A)) - \mathit{dim}(V_k + \mathit{ker}(A)) \tag{3.4}$$

$$> n - k + k - n = 0. \tag{3.5}$$

As a result, ker$(A) \cap V_k \neq \{\mathbf{0}\}$ is obtained. □

Lemma 3.1: [9] Let G be a graph with n vertices, $v_i \in V(G)$ and let $A \in \mathcal{S}(G)$, $\mathbf{x} \in \mathbb{R}^n$. Then ith entry of the matrix $A\mathbf{x}$ is as follows.

$$(A\mathbf{x})_i = a_{ii}x_i + \sum_{v_j \sim v_i} a_{ij}x_j. \tag{3.6}$$

Proof: Let G be a graph with n vertices, $v_i \in V(G)$ and let $A = [a_{ij}] \in \mathcal{S}(G)$, $\mathbf{x} = [x_i] \in \mathbb{R}^n$. Since $A \in \mathcal{S}(G)$, we know that A is symmetric and if $v_j \sim v_i$, then $a_{ij} \neq 0$, where $i \neq j$. From all of these and the definition of matrix multiplication, we obtain the following.

$$(A\mathbf{x})_i = a_{1i}x_1 + a_{2i}x_2 + \cdots + a_{ni}x_n \tag{3.7}$$

$$= \sum_{j=1}^{n} a_{ij}x_j \tag{3.8}$$

$$= a_{ii}x_i + \sum_{v_j \sim v_i} a_{ij}x_j. \tag{3.9}$$

We know that there is no condition on the entries of the diagonal of A. That is, the entries in the diagonal of A may be 0 or not. So, the entry a_{ii} must be written as well. □

Proposition 3.2: [9] Let Z be a zero forcing set of a graph G with n vertices and let $K = \{i : v_i \in Z, 1 \leq i \leq n\}$, $A \in \mathcal{S}(G)$. If $\mathbf{x} \in ker(A)$ and supp$(\mathbf{x}) \cap K = \emptyset$, then $\mathbf{x} = \mathbf{0}$. (supp$(\mathbf{x}) \cap K = \emptyset$ means that $\mathbf{x} \in ker(A)$ vanishes on K.)

Proof: Let $\mathbf{x} = [x_i] \in ker(A)$ and supp$(\mathbf{x}) \cap K = \emptyset$.

Case 1: Let $Z = V(G)$. Then $K = \{1, 2, \cdots, n\}$. Since supp$(\mathbf{x}) \cap K = \emptyset$, then supp$(\mathbf{x}) = \emptyset$ which implies that $\mathbf{x} = \mathbf{0}$.

Case 2: Let $Z \subset V(G)$. Since Z is a zero forcing set, there exists a black vertex v_i such that it has a unique white neighbor v_j. Then $x_i = 0$, and x_j is not yet zero in the initial vector $\mathbf{x}$.

By Lemma 3.1, the following is obtained.

$$\mathbf{x} \in ker(A) \Rightarrow (A\mathbf{x})_i = \mathbf{0} \tag{3.10}$$

$$\Rightarrow a_{ii}x_i + a_{ij}x_j = 0 \tag{3.11}$$

$$\Rightarrow a_{ii}0 + a_{ij}x_j = 0 \tag{3.12}$$

$$\Rightarrow a_{ij}x_j = 0. \tag{3.13}$$

Hence, we see that the equation in (3.13) implies that $x_j = 0$ since $v_i \sim v_j$. That is, v_i forces v_j which means that v_j is colored black and then $x_j = 0$. This process continues until all components of the coordinates of $\mathbf{x}$ corresponding to the elements in the set supp$(\mathbf{x})$ vanish since Z is a zero forcing set. Then $\mathbf{x} = \mathbf{0}$. □

Again, AIM Minimum Rank-Special Graphs Work Group in [9] found the following important result, which is the origin of the concept of a zero forcing set of any graph G.

Proposition 3.3: [9] Let G be a graph and Z be a zero forcing set of G. Then $M(G) \leq |Z|$. Moreover, $M(G) \leq Z(G)$.

Proof: Assume that $M(G) > |Z|$ and let $A \in \mathcal{S}(G)$ with corank$(A) > |Z|$. By Proposition 3.1, there exists a non-zero vector $\mathbf{x}$ in ker(A) such that it vanishes on $K = \{i : v_i \in Z, 1 \leq i \leq n\}$. That is, supp$(\mathbf{x}) \cap K = \emptyset$. By Proposition 3.2, $\mathbf{x} = \mathbf{0}$ which gives us a contradiction. Then $M(G) \leq |Z|$. Since $|Z| \geq Z(G)$, it is also obtained that $M(G) \leq Z(G)$. □

3.2. Zero Forcing Numbers of Some Special Graphs

In this section, based on [10], [20], and our works, the zero forcing number of a null graph, a path graph, a cycle graph, a complete graph, a wheel graph, a complete bipartite graph, and a complete multipartite graph is given, respectively.

Proposition 3.4: $Z(N_n) = n$ for $n \geq 1$.

Proof: Since a null graph has no edges, the initial set of black vertices of the graph contains all vertices of the graph. Then $Z(N_n) = n$ for $n \geq 1$. □

Proposition 3.5: [10] $Z(P_n) = 1$ for $n \geq 2$.

Proof: Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and let $Z_1 = \{v_1\}$ be an initial set of P_n for $n \geq 2$. Clearly, after $n-1$ steps all white vertices of P_n turn to black. Then $0 < Z(P_n) \leq 1$ which implies that $Z(P_n) = 1$.

Similarly, the initial set $Z_2 = \{v_n\}$ is also a minimum zero forcing set of P_n. However, if we consider a black vertex v_i which is not v_1 or v_n for $1 \leq i \leq n$, then v_i has 2 white neighbors and no force occurs. That is, the minimum zero forcing sets of P_n are $\{v_1\}$ and $\{v_n\}$ for $n \geq 2$. □

Proposition 3.6: [10] $Z(C_n) = 2$ for $n \geq 3$.

Proof: Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and let $Z_1 = \{v_i, v_{i+1}\}$ be an initial set of black vertices of C_n for $n \geq 3$, where $1 \leq i \leq n-1$. It can be seen that after $n-2$ steps all white vertices of C_n turn to black from the definition of a cycle graph. Then $Z(C_n) \leq 2$. Suppose that $Z_2 = \{v_i\}$ is an initial set of C_n, where $1 \leq i \leq n$ and $n \geq 3$. Then the vertex v_i has 2 white neighbors since C_n is a $2-$regular graph. Then no force occurs so this implies that $Z(C_n) > 1$. As a result, $Z(C_n) = 2$.

Also, if we consider 2 black vertices which are not adjacent as an initial set of C_n for $n > 3$, then these vertices have 2 white neighbors and no force occurs still. That is, the minimum zero forcing sets of C_n are only its pairwise adjacent vertices. □

Remark 3.1: If Z is a minimum zero forcing set of a graph G and $S \supset Z$ is an initial set of G, then S is a zero forcing set of G.

Proposition 3.7: [10] $Z(K_n) = n - 1$ for $n \geq 2$.

Proof: Firstly, let $n = 2$. In this case, $K_2 \cong P_2$. We know that $Z(P_2) = 1$ which implies that $Z(K_2) = 1$. Now let $n > 2$, $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and let $Z_1 = V(K_n)\backslash\{v_i\}$ be an initial set of black vertices of K_n, where $1 \leq i \leq n$. Since each vertex of K_n has $n - 1$ neighbors, after one step the white vertex v_i turns to black. Then $Z(K_n) \leq n - 1$. Suppose that $Z_2 = V(K_n)\backslash\{v_i, v_j\}$ is an initial set, where $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$. Then all black vertices have 2 white neighbors v_i and v_j. Then no force occurs so this implies that $Z(K_n) > n-2$. Therefore, we obtain that $Z(K_n) = n - 1$. □

Proposition 3.8: [10] $Z(W_n) = 3$ for $n \geq 4$.

Proof: Let $V(W_n) = \{v_1, v_2, \cdots, v_{n-1}, w\}$ and let $Z_1 = \{v_i, v_j, w\}$ be an initial set of black vertices of W_n for $n \geq 4$, where $1 \leq i \leq n-1$, $1 \leq j \leq n-1$, $v_i \sim v_j$, and w is adjacent to all vertices of W_n. After $n - 3$ steps all white vertices of W_n turn to black since $W_n \cong C_{n-1} \vee N_1$ and $\{v_i, v_j\}$ is the minimum zero forcing set of C_{n-1} since $v_i \sim v_j$. Thus, $Z(W_n) \leq 3$.

Case 1: Suppose that $Z_1 = \{v_i, v_j\}$ is an initial set of W_n, where $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$. It does not matter whether $v_i \sim v_j$ or $v_i \nsim v_j$ here. It can be seen that v_i and v_j have at least 2 white neighbors from the definition of a wheel graph. Then no force occurs so this implies that $Z(W_n) > 2$. As a result, $Z(W_n) = 3$.

Case 2: Suppose that $Z_1 = \{v_i, w\}$ is an initial set of W_n, where $1 \leq i \leq n-1$. We can observe that w has $n - 2$ white neighbors and v_i has 2 white neighbors from the definition of a wheel graph. Then no force occurs so this implies that $Z(W_n) > 2$.

As a result, $Z(W_n) = 3$. □

Lemma 3.2: If there exists a pair of true (false) twins in a graph G, then at least one of them is contained in all zero forcing sets of G.

Proof: Let G be a graph that contains true twins, say u and v. Then $N_G[u] = N_G[v]$ *from the definition of true twins. Let S be a zero forcing set of G. Suppose on the contrary that let* $u, v \notin S$.

Case 1: $N_G[u] = N_G[v] = \{u, v\}$. *Since G is a simple graph,* $G \cong P_2$. *This gives us a contradiction since u or v must contain in S.*

Case 2: Let $w \in N_G[u] = N_G[v]$, *where the vertex w is different from u and v.*

Then no vertex of G can force u or v. We know that u and v is forced by a vertex that is adjacent to u and v. Also, $w \sim u$ *if and only if* $w \sim v$. *If w is black vertex of G, then it has always 2 white neighbors u and v. As a result, u and v are not forced by any vertex of G with the fact that S is a zero forcing set of G. So u or v must contain in S.*

Similarly, it holds for any pair of false twins of G. □

Proposition 3.9: [10] For $p = q = 1$, $Z(K_{p,q}) = 1$; *otherwise,* $Z(K_{p,q}) = p + q - 2$.

Proof: Let $V(K_{p,q}) = \{u_1, u_2, \cdots, u_p, v_1, v_2, \cdots, v_q\}$ *and let* U_1 *and* U_2 *be* $2-$*partite subsets of* $V(K_{p,q})$ *such that these are consisting of the vertices* $u_1, u_2, \cdots, u_p$ *and* $v_1, v_2, \cdots, v_q$, *respectively. Then* $|U_1| = p, |U_2| = q$. *Let* $p \leq q$, *without loss of the generality.*

i) Let $p = 1, q \geq 1$.

Case 1: Let $p = q = 1$. *Then* $K_{1,1} \cong P_2$. *We know that* $Z(P_2) = 1$ *which implies that* $Z(K_{1,1}) = 1$ *by Proposition 3.5.*

Case 2: Let $p = 1, q = 2$. *Then* $K_{1,2} \cong P_3$. *We know that* $Z(P_3) = 1$ *which implies that* $Z(K_{1,2}) = 1$ *by Proposition 3.5.*

Case 3: Let $p = 1, q > 2$ *and let* $Z_1 = U_2 \backslash \{v_j\}$ *be an initial set of black vertices of* $K_{p,q}$, *where* $1 \leq j \leq q$. *After 2 steps all white vertices of* $K_{p,q}$ *turn to black from the definition of a complete bipartite graph. Then* $Z(K_{p,q}) \leq q - 1$.

Suppose that $Z_2 = U_2 \backslash \{v_i, v_j\}$ *is an initial set of* $K_{p,q}$, *where* $1 \leq i \leq q$, $1 \leq j \leq q$, *and* $i \neq j$. *Then after one step* u_1 *turns to black. And then* u_1 *has 2 white* *neighbors v_i and v_j. Then no force occurs so this implies that $Z(K_{p,q}) > q - 2$. As a result, $Z(K_{p,q}) = q - 1$.*

Suppose that $Z_3 = U_1 \cup (U_2 \backslash \{v_i, v_j, v_k\})$ is an initial set of $K_{p,q}$, where $1 \leq i \leq q$, $1 \leq j \leq q$, $1 \leq k \leq q$, and they are not the same vertices. Then the vertex u_1 has 3 white neighbors and no force occurs so this implies that $Z(K_{p,q}) > q - 2$. As a result, $Z(K_{p,q}) = q - 1$.

ii) Let $p \geq 2$, $q \geq 2$.

Let $Z_1 = (U_1 \backslash \{u_i\}) \cup (U_2 \backslash \{v_j\})$ be an initial set of black vertices of $K_{p,q}$, where $1 \leq i \leq p$, $1 \leq j \leq q$. Clearly, after 2 steps all white vertices of $K_{p,q}$ turn to black from the definition of a complete bipartite. Then $Z(K_{p,q}) \leq p + q - 2$.

By Lemma 3.2, a zero forcing set of $K_{p,q}$ must contain at least $p - 1$ vertices from U_1 and at least $q - 1$ vertices from U_2 since the vertices in U_1 are false twins to each other and the vertices in U_2 are false twins to each other. Then $Z(K_{p,q}) \geq (p - 1) + (q - 1) = p + q - 2$. As a result, $Z(K_{p,q}) = p + q - 2$. □

Proposition 3.10: [20] Let $k > 2$. If $n_i = 1$, then $Z(K_{n_1,n_2,\cdots,n_k}) = k - 1$ for all $1 \leq i \leq k$; otherwise, $Z(K_{n_1,n_2,\cdots,n_k}) = (n_1 + n_2 + \cdots + n_k) - 2$.

Proof: Without loss of the generality, let $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$ and let $k > 2$. Let $V(K_{n_1,n_2,\cdots,n_k}) = \bigcup_{i=1}^{k} U_i$, where U_i for each $1 \leq i \leq k$ be $k-$partite subsets of vertices of $K_{n_1,n_2,\cdots,n_k}$. Then $|U_i| = n_i$ for each $1 \leq i \leq k$.

i) Let at least one of n_i be 1 for $1 \leq i \leq k$.

Case 1: Let $n_1 = n_2 = \cdots = n_k = 1$. Then $K_{1,1,\cdots,1} \cong K_k$. We know that $Z(K_k) = k - 1$ by Proposition 3.18.

Case 2: Let at least one of n_j be not 1 for $1 \leq j \leq k$. Let $Z_1 = \left(\bigcup_{i=1}^{k} U_i\right) \backslash \{v, w\}$ be an initial set of $K_{n_1,n_2,\cdots,n_k}$ such that the vertex v is contained in any $k-$partite set of $V(K_{n_1,n_2,\cdots,n_k})$ with the cardinality of 1, say U_a, and the vertex w is contained in any $k-$partite set of $V(K_{n_1,n_2,\cdots,n_k})$ with the cardinality of not 1, say U_b, where $1 \leq a \leq k$ and $1 \leq b \leq k$. From the definition of a complete multipartite graph, any black vertex in U_b has a unique white neighbor w and forces it. Subsequently, w has a unique white neighbor v and forces it. Then the following is obtained.

$$Z(G) \leq (n_1 + n_2 + \cdots + n_k) - 2. \tag{3.14}$$

Suppose that $Z_2 = \left(\bigcup_{i=1}^{k} U_i\right) \setminus \{u, v, w\}$ *is an initial set of* $K_{n_1,n_2,\cdots,n_k}$.

If $n_l > 2$ *for some* l*, then the vertices* u*,* v *and* w *can be taken from the same partite set* U_l*, where* $1 \leq l \leq k$*. Then, except* U_l*, all black vertices in other partite sets have* 3 *white neighbors* u*,* v *and* w *and no force occurs.*

Let $u \in U_i$*,* $v \in U_j$*, and* $w \in U_l$*, where* $1 \leq i \leq k$*,* $1 \leq j \leq k$*,* $1 \leq l \leq k$*, and* $i \neq j \neq l$*. Then each black vertex of* $K_{n_1,n_2,\cdots,n_k}$ *has at least* 2 *white neighbors and no force occurs from the definition of a complete multipartite graph.*

We know that there is an m *such that* $n_m > 1$*, then the vertices* u*,* v *can be taken from the same partite set* U_m *and* w *can be taken from another partite set. Then each black vertex of* $K_{n_1,n_2,\cdots,n_k}$ *has at least* 2 *white neighbors and no force occurs from the definition of a complete multipartite graph.*

As a result, Z_2 *is not a zero forcing set of* $K_{n_1,n_2,\cdots,n_k}$*. Then we have the following.*

$$Z(G) \geq (n_1 + n_2 + \cdots + n_k) - 2. \tag{3.15}$$

By the inequalities in (3.14) and (3.15), we obtain $Z(G) = (n_1 + n_2 + \cdots + n_k) - 2$.

ii) Let all $n_i > 1$ *for* $1 \leq i \leq k$.

Let $Z_1 = \left(\bigcup_{i=1}^{k} U_i\right) \setminus \{v, w\}$ *be an initial set of* $K_{n_1,n_2,\cdots,n_k}$ *such that the vertex* $v \in U_i$ *and the vertex* $w \in U_j$ *for any* $1 \leq i \leq k$ *and* $1 \leq j \leq k$*, where* $i \neq j$*. From the definition of a complete multipartite graph, any black vertex in* U_i *has a unique white neighbor* w *and forces it.*

Similarly, any black vertex in U_j *has a unique white neighbor* v *and forces it. Then the following is obtained.*

$$Z(G) \leq (n_1 + n_2 + \cdots + n_k) - 2. \tag{3.16}$$

We know that from the definition of a complete multipartite graph and by Lemma 3.2, a zero forcing set of $K_{n_1,n_2,\cdots,n_k}$ *must contain at least* $n_1 - 1$ *vertices from* U_1*, at least* $n_2 - 1$ *vertices from* U_2*,* $\cdots$*, and at least* $n_k - 1$ *vertices from* U_k *since the vertices in* U_i *are false twin vertices of* $K_{n_1,n_2,\cdots,n_k}$ *for each* $1 \leq i \leq k$*. Then a zero* *forcing set of* $K_{n_1,n_2,\cdots,n_k}$ *contains at least* $(n_1+n_2+\cdots+n_k)-k$ *vertices. However, every black vertex has* $k-1$ *white neighbors and* $k>2$*. As a result, for any forcing, at least* $k-2$ *more vertices must be added in a zero forcing set of* $K_{n_1,n_2,\cdots,n_k}$*.*

Then the following is obtained.

$$Z(G) \geq (n_1+n_2+\cdots+n_k)-k+k-2 \tag{3.17}$$

$$= (n_1+n_2+\cdots+n_k)-2. \tag{3.18}$$

By the inequalities in (3.16) and (3.18), $Z(K_{n_1,n_2,\cdots,n_k}) = (n_1+n_2+\cdots+n_k)-2$*.* □

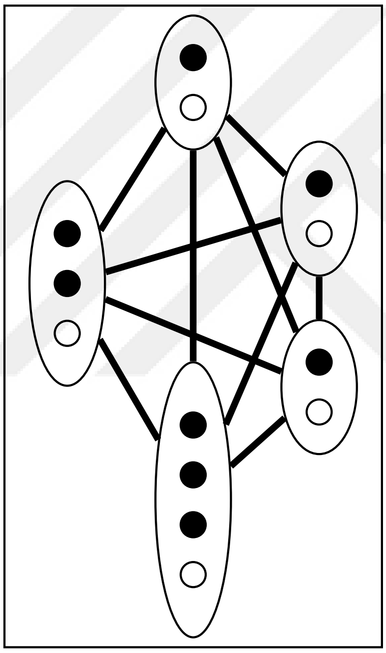


Figure 3.5: The graph $K_{4,3,2,2,2}$ with an initial set of black vertices.

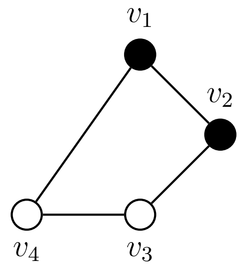
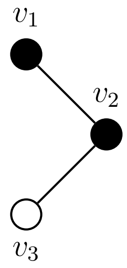
In Figure 3.5 each thick line between two partite sets says that each of the vertex of the first partite set is adjacent to each of the vertex of the second partite set.

We will use this representation from now on.

Observation 3.1: Let G be a graph, and let H *and* S *be its subgraph and its minimum zero forcing set, respectively. Then* $S \cap V(H)$ *does not have to be a minimum zero forcing set of* H*.*

Example 3.6: We consider the following graph G *with a minimum zero forcing set* $S=\{v_1,v_2\}$ *and we consider its subgraph* H *as follows.*

Table 3.3: An example for Observation 3.1.

Graph G	Graph H
	

One can easily see that $S \cap V(H) = S$ is not a minimum zero forcing set of the graph H. The reason is that $H \cong P_3$ and we know that $Z(P_3) = 1$ by Proposition 3.5 even though $|S| = 2$.

4. ZERO FORCING NUMBERS OF PRODUCT GRAPHS

4.1. Grundy and Z-Grundy Domination Number

In this section, the concepts of Grundy domination number and $Z-$Grundy domination number of a graph G have been introduced and the Grundy domination numbers of several special graphs have been found. These concepts will help us to find the zero forcing numbers of some graphs.

Let G be a graph. A vertex v of G "$dominates$" a vertex u of G if $u \in N_G[v]$. Then also we say that the vertex u is "$dominated$" by the vertex v.

Definition 4.1: [22] Let $k \geq 1$. Let $S = (v_1, v_2, \cdots, v_k)$ be an ordered sequence of distinct vertices of a graph G with $n \geq k$ vertices and $\hat{S}_k = \{v_1, v_2, \cdots, v_k\}$ be an ordered set of vertices in S. Then S is called a legal (closed) dominating sequence in G if for each $2 \leq i \leq k$, we have the following.

$$N_G[v_i] \backslash \bigcup_{j=1}^{i-1} N_G[v_j] = N_G[v_i] \backslash N_G[\hat{S}_{i-1}] \neq \emptyset. \quad (4.1)$$

That is, every vertex v_i in the sequence S dominates at least one vertex that is not dominated by any vertex preceding it for each $2 \leq i \leq k$.

The Grundy domination number of G, denoted by $\gamma_{gr}(G)$, is the length of the longest legal dominating sequence in G. The corresponding sequence is called a Grundy dominating sequence in G.

Example 4.1: Let G be a graph as follows.

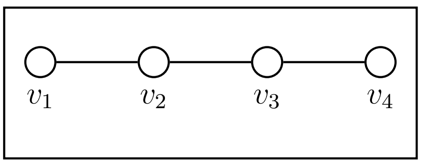


Figure 4.1: The graph $G \cong P_4$.

In this example, the sequence $S = (v_1, v_2, v_3)$ is a legal dominating sequence in G since $N_G[v_2]\backslash N_G[\hat{S}_1] = \{v_3\} \neq \emptyset$ and $N_G[v_3]\backslash N_G[\hat{S}_2] = \{v_4\} \neq \emptyset$. Also, S has the maximum length in G. Hence, we can see that $\gamma_{gr}(P_4) = 3$.

Definition 4.2: [18] Let $k \geq 1$. Let G be a graph without isolated vertices, let $S = (v_1, v_2, \cdots, v_k)$ be an ordered sequence of distinct vertices of G with $n \geq k$ vertices and let $\hat{S}_k = \{v_1, v_2, \cdots, v_k\}$ be an ordered set of vertices in S. Then S is called a legal $Z-$sequence in G if for each $2 \leq i \leq k$, we have the following.

$$N_G(v_i)\backslash \bigcup_{j=1}^{i-1} N_G[v_j] = N_G(v_i)\backslash N_G[\hat{S}_{i-1}] \neq \emptyset. \tag{4.2}$$

The $Z-$Grundy domination number of G, denoted by $\gamma_{gr}^Z(G)$, is the length of the longest legal $Z-$sequence in G.

Example 4.2: Let G be a graph as follows.

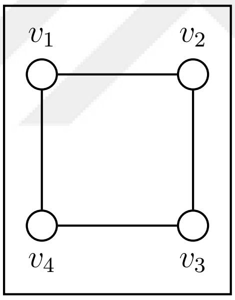


Figure 4.2: The graph $G \cong C_4$.

The sequence $S = (v_1, v_2)$ is a legal $Z-$sequence in G since $N_G(v_2)\backslash N_G[\hat{S}_1] = \{v_3\} \neq \emptyset$. Also, S has the maximum length in G. Then $\gamma_{gr}^Z(C_4) = 2$.

The following substantial Theorem was presented at [22] by Brešar et al. It is the origin of the concept of the Grundy domination number of a graph G that is a graph parameter.

Theorem 4.1: [22]

i) For any arbitrary graph G, $\gamma_{gr}(G) \leq |V(G)| - \delta(G)$.

ii) For any arbitrary graph G, $\gamma_{gr}(G) \geq IR(G)$.

The following important Theorem was presented at [18] by Brešar et al. Theorem 4.2 which provides a connection among the zero forcing number, the Grundy domination number, and the $Z-$Grundy domination number of a graph G that has no any isolated vertices, which will help us next works.

Theorem 4.2: [18]

i) If G is a graph without isolated vertices, then $\gamma_{gr}(G) + Z(G) \geq |V(G)|$. In particular, G has a Grundy dominating sequence that is also $Z-$sequence, then $\gamma_{gr}(G) + Z(G) = |V(G)|$.

ii) If G is a graph without isolated vertices, then $\gamma_{gr}^{Z}(G) + Z(G) = |V(G)|$.

Now, based on previous studies that are shown above, we will examine the Grundy domination numbers of the graphs that will be useful to us. Also, these results have already been known.

Lemma 4.1: $\gamma_{gr}(K_n) = 1$ for $n \geq 2$.

Proof: Let $n \geq 2$. We know that $\gamma_{gr}(K_n) + Z(K_n) \geq |V(K_n)|$ from Theorem 4.2. Then by Proposition 3.7, the following is obtained.

$$\gamma_{gr}(K_n) \geq 1. \tag{4.3}$$

We know that $\gamma_{gr}(K_n) \leq |V(K_n)| - \delta(K_n)$ from Theorem 4.1. Also, since $\delta(K_n) = n - 1$, the following is obtained.

$$\gamma_{gr}(K_n) \leq 1. \tag{4.4}$$

As a result, we get $\gamma_{gr}(K_n) = 1$ by (4.3) and (4.4). $\square$

Lemma 4.2: $\gamma_{gr}(P_n) = n - 1$ for $n \geq 2$.

Proof: We know that $\gamma_{gr}(P_n) + Z(P_n) \geq |V(P_n)|$ from Theorem 4.2. Then by Proposition 3.5, the following is obtained.

$$\gamma_{gr}(P_n) \geq n - 1. \tag{4.5}$$

We know that $\gamma_{gr}(P_n) \leq |V(P_n)| - \delta(P_n)$ *from Theorem 4.1. Also, since* $\delta(P_n) = 1$*, the following is obtained.*

$$\gamma_{gr}(P_n) \leq n - 1. \tag{4.6}$$

As a result, we get $\gamma_{gr}(P_n) = n - 1$ *by (4.5) and (4.6).* □

Lemma 4.3: $\gamma_{gr}(K_{p,q}) = q$ *for* $q \geq p \geq 1$*.*

Proof: We know that by Theorem 4.1, $\gamma_{gr}(K_{p,q}) \leq |V(K_{p,q})| - \delta(K_{p,q})$*. Also, since* $\delta(K_{p,q}) = p$ *from the definition of a complete bipartite graph, the following is obtained.*

$$\gamma_{gr}(K_{p,q}) \leq q. \tag{4.7}$$

Also, we know that $IR(K_{p,q}) = q$*. Then by Theorem 4.1, the following is obtained.*

$$\gamma_{gr}(K_{p,q}) \geq q. \tag{4.8}$$

As a result, we get $\gamma_{gr}(K_{p,q}) = q$ *by (4.7) and (4.8).* □

4.2. Some Results on Graph Products

In this section, several important results regarding the zero forcing numbers of some graph products have been examined, which exist in the literature.

Firstly, we will present an observation.

Observation 4.1: Let G *and* H *be two distinct graphs with the zero forcing numbers* k *and* m*, respectively. Since the graphs are not joined by any vertex,* $Z(G + H) = Z(G) + Z(H) = k + m$*.*

The following Theorem related to the zero forcing number of a join product graph was presented at [20] by Taklimi.

Theorem 4.3: [20] If G *and* H *are any two connected graphs, then* $Z(G \vee H) = min\{|H| + Z(G), |G| + Z(H)\}$*.*

The following Proposition was presented at [9] to provide an upper bound for the zero forcing number of a cartesian product graph by AIM Minimum Rank-Special Graphs Work Group.

Proposition 4.1: *[9] If G and H are any two graphs, then $Z(G\Box H) \leq min\{|H|Z(G),$ $|G|Z(H)\}$.*

Huang et al. found an upper bound for the zero forcing number of a direct product graph in [17]. This result is given in the following.

Lemma 4.4: *[17] If G and H are any two graphs, $Z(G\times H) \leq |G|Z(H)+Z(G)|H|-$ $Z(G)Z(H)$.*

The following Theorem was presented at [21] by Javaid et al., which gives us the zero forcing number of a corona product graph.

Theorem 4.4: *[21] If G and H are two connected graphs, then $Z(G \odot H) = Z(G) +$ $|G|Z(H)$.*

For more results on this topic, one can also see [31] and [32].

4.3. Strong Product Graph

In this section with the main lines, some results regarding the zero forcing numbers of strong product graphs are given from the literature. Then, a lower bound related to the zero forcing number of the strong product of a complete graph and any connected graph has been presented. An upper bound has been shown for the zero forcing number of the strong product of a complete bipartite graph and any connected graph. Later, by combining these results, we study the zero forcing number of the strong product of a complete graph and a cycle graph, the zero forcing number of the strong product of a path graph and a complete bipartite graph, the zero forcing number of the strong product of a path graph and a complete graph, and the zero forcing number of the strong product of a complete graph and a complete bipartite graph. Moreover, we examine several corollaries.

The following Proposition was presented at [9] by AIM Minimum Rank-Special Graphs Work Group, which gives us the zero forcing number of the strong product of two path graphs.

Proposition 4.2: *[9] If $s \geq 2$ and $t \geq 2$, then $Z(P_s \boxtimes P_t) = s + t - 1$.*

The following important Theorem was shown by Huang et al. in [17]. They found an upper bound for the zero forcing number of the strong product of any two graphs. Also, they showed that the zero forcing number of the strong product of any two graphs equals to the upper bound in some special cases.

Theorem 4.5: *[17]*

i) If G and H are any two graphs and $s = |G|Z(H) + Z(G)|H| - Z(G)Z(H)$, then $Z(G \boxtimes H) \leq s$.

ii) Let graphs G and H be isomorphic to one of the graphs in the set $K = \{C_{3n}, P_{3n-1}, K_{n+1} : n \geq 1\} \cup P$, where P is the Petersen graph and let $s = |G|Z(H) + Z(G)|H| - Z(G)Z(H)$. Then $Z(G \boxtimes H) = s$. Also, the equality holds the graphs in the set $\{C_n, P_k : n \geq 5, k \geq 2\}$.

Observation 4.2: *There exist two graphs G and H such that the zero forcing number of $G \boxtimes H$ is not equal to $s = |G|Z(H) + Z(G)|H| - Z(G)Z(H)$.*

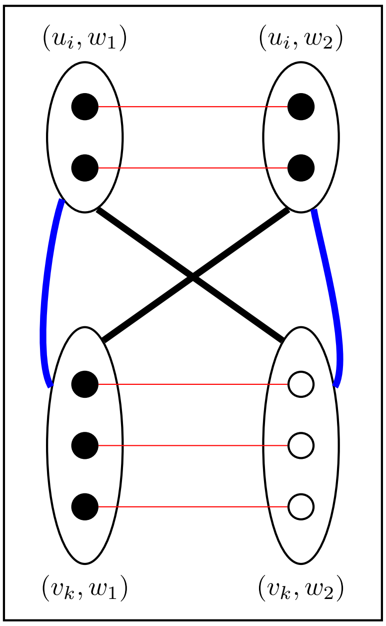


Figure 4.3: The graph $K_{2,3} \boxtimes P_2$ with the zero forcing set S.

We consider the graph $K_{2,3} \boxtimes P_2$. See Figure 4.3. The zero forcing number of $K_{2,3} \boxtimes P_2$ was found as $Z(K_{2,3} \boxtimes P_2) = 7$ by use of Sage zero forcing software [30]; that is, $Z(K_{2,3} \boxtimes P_2) \neq 8$.

Let $V(K_{2,3}) = \{u_1, u_2, v_1, v_2, v_3\}$ and $V(P_2) = \{w_1, w_2\}$. Then a zero forcing set of $K_{2,3} \boxtimes P_2$ is $S = \{(u_1, w_1), (u_2, w_1), (u_1, w_2), (u_2, w_2), (v_1, w_1), (v_2, w_1), (v_3, w_1)\}$.

We obtain that $Z(K_{2,3} \boxtimes P_2) \leq 7$, but by Proposition 3.5 and Proposition 3.9, we obtain the following.

$$s = |K_{2,3}|Z(P_2) + Z(K_{2,3})|P_2| - Z(K_{2,3})Z(P_2) \tag{4.9}$$

$$= 5 \cdot 1 + 3 \cdot 2 - 3 \cdot 1 = 8. \tag{4.10}$$

The following Proposition was presented at [18] by Brešar et al.

Proposition 4.3: [18] If $s \geq 3$, $t \geq 2$, then $Z(C_s \boxtimes P_t) = 2t + s - 2$.

Corollary 4.1: Let H be a connected graph with k vertices. Then $k(n-1) \leq Z(K_n \boxtimes H) \leq k(n-1) + Z(H)$ for $n \geq 2$, $k \geq 2$.

Proof: Let $V(K_n) = \{u_1, u_2, \cdots, u_n\}$ and $V(H) = \{v_1, v_2, \cdots, v_k\}$, where $n \geq 2$ and $k \geq 2$. Then $V(K_n \boxtimes H) = V(K_n) \times V(H)$. One can easily see that each pair of vertices in the set $\{(u_i, v_j) : 1 \leq i \leq n\}$ for all $j \in \{1, 2, \cdots, k\}$ is true twins because of $N_{K_n \boxtimes H}[(u_i, v_1)] = N_{K_n \boxtimes H}[(u_j, v_1)]$, $N_{K_n \boxtimes H}[(u_i, v_2)] = N_{K_n \boxtimes H}[(u_j, v_2)]$, $\cdots$, $N_{K_n \boxtimes H}[(u_i, v_k)] = N_{K_n \boxtimes H}[(u_j, v_k)]$ from the definitions of strong product and complete graph, where $1 \leq i \leq n$ and $1 \leq j \leq n$. From Lemma 3.2, every zero forcing set of $K_n \boxtimes H$ must contain $n-1$ vertices from each set $\{(u_i, v_j) : 1 \leq i \leq n\}$ for all $j \in \{1, 2, \cdots, k\}$. Therefore, we have the following.

$$Z(K_n \boxtimes H) \geq k(n-1). \tag{4.11}$$

Also, the following is obtained by Theorem 4.5 and Proposition 3.7.

$$Z(K_n \boxtimes H) \leq |K_n|Z(H) + Z(K_n)|H| - Z(K_n)Z(H) \tag{4.12}$$

$$= nZ(H) + (n-1)k - (n-1)Z(H) \tag{4.13}$$

$$= k(n-1) + Z(H). \tag{4.14}$$

By (4.11) and (4.14), we get $k(n-1) \le Z(K_n \boxtimes H) \le k(n-1) + Z(H)$. □

Proposition 4.4: $Z(K_n \boxtimes C_k) = k(n-1) + 2$ *for* $n \ge 2$, $k \ge 3$.

Proof: Let $n \ge 2$, $k \ge 3$. *Let* $V(K_n) = \{u_1, u_2, \cdots, u_n\}$ *and* $V(C_k) = \{v_1, v_2, \cdots, v_k\}$. *Then* $V(K_n \boxtimes C_k) = V(K_n) \times V(C_k)$.

By Corollary 4.1 and Proposition 3.6, the following is obtained.

$$Z(K_n \boxtimes C_k) \le k(n-1) + Z(C_k) \tag{4.15}$$

$$= k(n-1) + 2. \tag{4.16}$$

Assume that $Z(K_n \boxtimes C_k) \le k(n-1) + 1$. *Then there exists a zero forcing set* S *of* $K_n \boxtimes C_k$ *such that* S *has* $k(n-1) + 1$ *vertices.*

For convenience, we arrange the vertices of $K_n \boxtimes C_k$ *to form* k *columns and* n *rows. For instance, see Figure 4.4. We know that each pair of the vertices of* $K_n \boxtimes C_k$ *on the same column is true twins. Then* S *must contain* $n-1$ *vertices from each column by Lemma 3.2. Without loss of generality, let* S *contain the first* $n-1$ *vertices of each column.*

Let $S = \left(\bigcup_{i=1}^{n-1} S_i\right) \cup \{w\}$, *where* $S_i = \{(u_i, v_j) : 1 \le j \le k\}$ *for* $1 \le i \le n-1$ *and* $w \in \mathcal{A} = \{(u_n, v_j) : 1 \le j \le k\}$. *Clearly,* $\mathcal{A} - \{w\}$ *contains all white vertices of* $K_n \boxtimes C_k$. *By the definition of strong product,* $\mathcal{A}$ *induces a subgraph of* $K_n \boxtimes C_k$ *which is isomorphic to* C_k. *Hence, the black vertex* w *cannot force any white vertex of* $K_n \boxtimes C_k$. *Can any black vertex in* $\bigcup_{i=1}^{n-1} S_i$ *force any white vertex of* $K_n \boxtimes C_k$*? It is sufficient to examine the black vertices in any set* S_i *because of definition of true twins, where* $1 \le i \le n-1$. *The reason is that black true twins either both can force a white vertex or neither can. Then for* $l \in 1 \le i \le n-1$, *we will examine the black vertices in the set* $S_l = \{(u_l, v_j) : 1 \le j \le k\}$.

Case 1: $j = 1$. *In this case, the black vertex* (u_l, v_1) *is adjacent to the vertices* $(u_l, v_2), (u_l, v_k)$, *and* (u_n, v_1) *in* $\mathcal{A}$. *Even if the black vertex* w *is one of these vertices,* (u_l, v_1) *will have still* 2 *white neighbors. Then it can not force any white vertex.*

Case 2: $j = k$. *In this case, the black vertex* (u_l, v_k) *is adjacent to the vertices* $(u_l, v_{k-1}), (u_l, v_1)$, *and* (u_n, v_k) *in* $\mathcal{A}$. *Even if the black vertex* w *is one of these vertices,* (u_l, v_k) *will have still* 2 *white neighbors. Then it can not force any white vertex.*

Case 3: $1 < j = m < k$. *In this case, the black vertex* (u_l, v_m) *is adjacent to the vertices* $(u_l, v_{m-1}), (u_l, v_{m+1})$, *and* (u_n, v_m) *in* $\mathcal{A}$. *Even if the black vertex* w *is one of these vertices,* (u_l, v_m) *will have still* 2 *white neighbors. Then it can not force any white vertex.*

Since none of the white vertices of $K_n \boxtimes C_k$ *are forced, it contradicts with the fact that* S *is a zero forcing set of* $K_n \boxtimes C_k$. *Then the following is obtained.*

$$Z(K_n \boxtimes C_k) \geq k(n-1) + 2. \tag{4.17}$$

Thus, we obtain $Z(K_n \boxtimes C_k) = k(n-1) + 2$ *by (4.16) and (4.17).* □

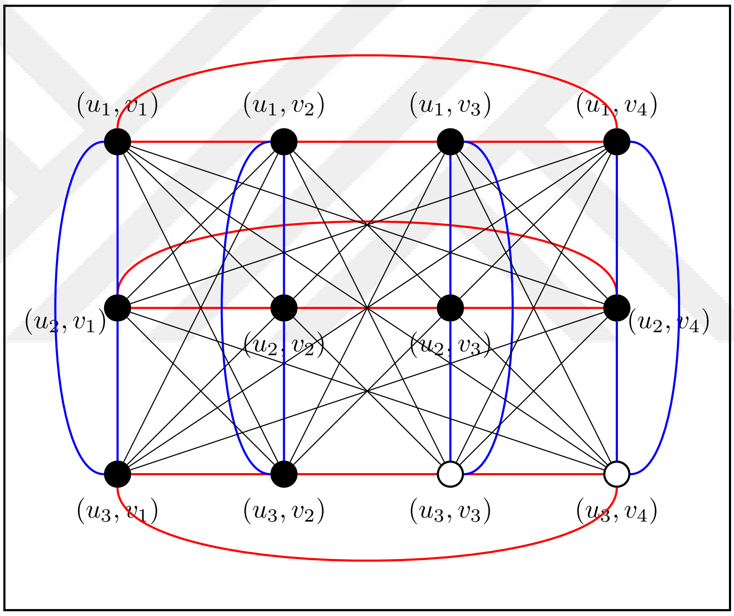


Figure 4.4: The graph $K_3 \boxtimes C_4$ with an initial set S.

Theorem 4.6: Let H *be a connected graph with* n *vertices. Then* $Z(K_{p,q} \boxtimes H) \leq np + qZ(H)$ *for* $q \geq p \geq 1$ *and* $n \geq 2$.

Proof: For $q \geq p \geq 1$ *and* $|H| = n \geq 2$, *let* $V(K_{p,q}) = \{u_1, u_2, \cdots, u_p, v_1, v_2, \cdots, v_q\}$ *and* $V(H) = \{w_1, w_2, \cdots, w_n\}$.

From the definition of strong product, we know that $V(K_{p,q} \boxtimes H) = V(K_{p,q}) \times V(H)$ *and* $E(K_{p,q} \boxtimes H) = \{\{(u,v),(x,y)\} :$ *i)* $[u \sim x, v \sim y] \vee$ *ii)* $[u = x, v \sim y] \vee$ *iii)* $[u \sim x, v = y]$ *for all* $(u,v),(x,y) \in V(K_{p,q}) \times V(H)\}$.

After that let $A_i = \{(u_i, w_j) : 1 \leq j \leq n\}$ *for all* $i \in \{1, 2, \cdots, p\}$ *and* $B_k = \{(v_k, w_j) : 1 \leq j \leq n\}$ *for all* $k \in \{1, 2, \cdots, q\}$*. Clearly,* $V(K_{p,q} \boxtimes H) = \left(\bigcup_{1\leq i\leq p} A_i\right) \cup \left(\bigcup_{1\leq k\leq q} B_k\right)$.

Suppose that we arrange the vertices of $K_{p,q} \boxtimes H$ *to form* $p+q$ *rows. Let the vertices of the set* A_i *be on the* i*th row and the vertices of the set* B_k *be on the* $(p+k)$*th row, where* $1 \leq i \leq p, 1 \leq k \leq q$*. Then clearly* n *columns appear. See Figure 4.5.*

There exists a graph in each row that is isomorphic to H *from ii), and there exists a graph in each column that is isomorphic to* $K_{p,q}$ *from iii), which is a subgraph of* $K_{p,q} \boxtimes H$*. Different subgraphs will appear depending on the graph* H *from i).*

Let $S = (\bigcup_{i=1}^{p} A_i) \cup (\bigcup_{k=1}^{q} Z_k)$ *be an initial set of black vertices of* $K_{p,q} \boxtimes H$*, where* Z_k *is a minimum zero forcing set of the subgraph of* $K_{p,q} \boxtimes H$ *in the* $(p+k)$*th row for each* $1 \leq k \leq q$*. We know that every row induces* H*. From the definitions of strong product and complete bipartite graph, and choosing* S*, each black vertex of the induced subgraph* $(K_{p,q} \boxtimes H)[B_k]$ *in* $(p+k)$*th row has no white neighbors from the other rows, where* $1 \leq k \leq q$*. Since* Z_k *is a minimum zero forcing set for each induced subgraph* $(K_{p,q} \boxtimes H)[B_k]$*, all white vertices on each row turn to black, where* $1 \leq k \leq q$*. Then* S *is a zero forcing set of* $K_{p,q} \boxtimes H$*. Then* $Z(K_{p,q} \boxtimes H) \leq |S| = np + qZ(H)$. □

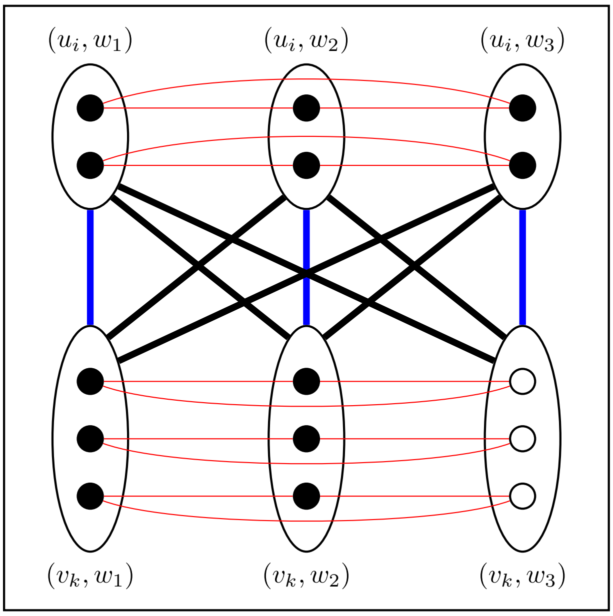


Figure 4.5: The graph $K_{2,3} \boxtimes C_3$ with an initial set S.

Observation 4.3: There exists any connected graph H *with* n *vertices such that zero forcing number of* $K_{p,q} \boxtimes H$ *is not equal to* $s = np + qZ(H)$ *for* $q \geq p \geq 1$ *and* $n \geq 2$.

We consider the graph $K_{2,3} \boxtimes P_2$. See Figure 4.6. The zero forcing number of $K_{2,3} \boxtimes S_3$ was found as $Z(K_{2,3} \boxtimes S_3) = 11$ by use of the Sage zero forcing software [30]; that is, $Z(K_{2,3} \boxtimes P_2) \neq 14$.

Let $V(K_{2,3}) = \{u_1, u_2, v_1, v_2, v_3\}$ and $V(S_3) = \{w_1, w_2, w_3, w_4\}$. A zero forcing set of $K_{2,3} \boxtimes S_3$ equals to the following set $S = \{(u_1, w_1), (u_2, w_1), (u_1, w_2), (u_2, w_2),$ $(u_2, w_3), (u_2, w_4), (v_1, w_1), (v_2, w_1), (v_3, w_1), (v_1, w_3), (v_2, w_4)\}$.

We obtain that $Z(K_{2,3} \boxtimes S_3) \leq 11$, but by Proposition 3.9, the following is obtained.

$$s = np + qZ(S_3) \tag{4.18}$$
$$= np + qZ(K_{1,3}) \tag{4.19}$$
$$= 4 \cdot 2 + 3 \cdot 2 = 14. \tag{4.20}$$

Also, $(v_1, w_1) \to (v_1, w_2)$, $(v_2, w_1) \to (v_2, w_2)$, $(v_3, w_1) \to (v_3, w_2)$, $(v_1, w_3) \to (u_1, w_3)$, $(v_2, w_4) \to (u_1, w_4)$, $(v_2, w_2) \to (v_2, w_3)$, $(u_2, w_3) \to (v_3, w_3)$, $(v_1, w_2) \to (v_1, w_4)$, $(v_3, w_2) \to (v_3, w_4)$ are the forces in which they are performed in the order.

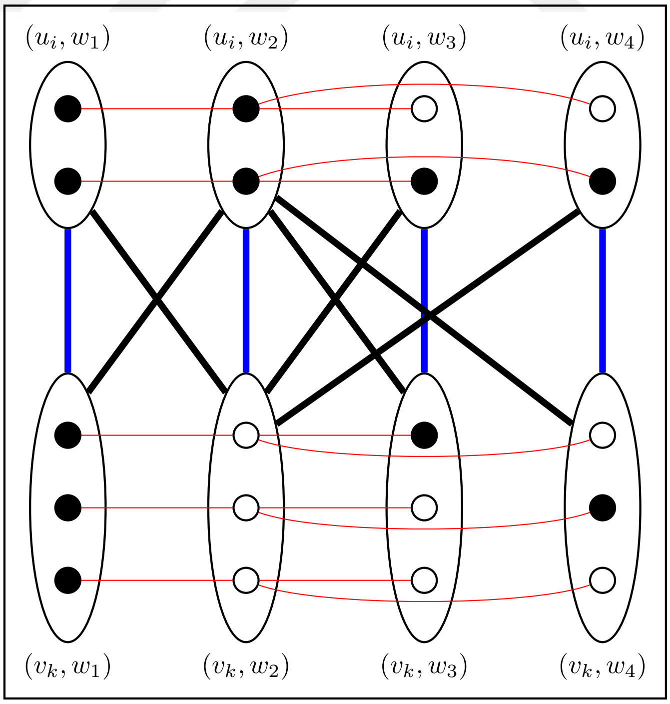


Figure 4.6: The graph $K_{2,3} \boxtimes S_3$ with the initial set S.

Proposition 4.5: $Z(K_n \boxtimes K_{p,q}) = (p+q)(n-1) + p$ *for* $q \geq p \geq 1$ *and* $n \geq 2$.

Proof: *Let* $q \geq p \geq 1$ *and* $n \geq 2$. *Let* $V(K_n) = \{u_1, u_2, \cdots, u_n\}$ *and* $V(K_{p,q}) = \{v_1, v_2, \cdots, v_p, w_1, w_2, \cdots, w_q\}$. *Then* $V(K_n \boxtimes K_{p,q}) = V(K_n) \times V(K_{p,q})$.

From Theorem 4.6 and Proposition 3.7, the following is obtained.

$$Z(K_n \boxtimes K_{p,q}) \leq np + qZ(K_n) \tag{4.21}$$

$$= np + q(n-1) \tag{4.22}$$

$$= (p+q)(n-1) + p. \tag{4.23}$$

Assume that $Z(K_n \boxtimes K_{p,q}) \leq (p+q)(n-1) + p - 1$. *Then there exists a zero forcing set* S *of* $K_n \boxtimes K_{p,q}$ *such that* S *has* $(p+q)(n-1)+p-1$ *vertices.*

For convenience, we arrange the vertices of $K_n \boxtimes K_{p,q}$ *to form* $p+q$ *columns and* n *rows. For instance, see Figure 4.7. We know that each pair of the vertices of* $K_n \boxtimes K_{p,q}$ *in the same column is true twins. Without loss of generality, let* S *contain the first* $n-1$ *vertices of each column.*

Let $S = \left(\bigcup_{i=1}^{n-1}(S_i \cup K_i)\right) \cup ((\mathcal{A} \cup \mathcal{B}) - \{x_1, x_2, \cdots, x_{q+1}\})$, *where* $S_i = \{(u_i, v_j) : 1 \leq j \leq p\}$ *for* $1 \leq i \leq n-1$, $K_i = \{(u_i, w_l) : 1 \leq l \leq q\}$ *for* $1 \leq i \leq n-1$, $\mathcal{A} = \{(u_n, v_j) : 1 \leq j \leq p\}$, $\mathcal{B} = \{(u_n, w_l) : 1 \leq l \leq q\}$, *and* $x_1, x_2, \cdots, x_{q+1} \in \mathcal{A} \cup \mathcal{B}$. *Clearly,* $\{x_1, x_2, \cdots, x_{q+1}\}$ *is the set of all white vertices of* $K_n \boxtimes K_{p,q}$. *By the definition of strong product,* $\mathcal{A} \cup \mathcal{B}$ *induces a subgraph of* $K_n \boxtimes K_{p,q}$ *which is isomorphic to* $K_{p,q}$. *Hence, if there exists, the black vertices in* $(\mathcal{A} \cup \mathcal{B}) - \{x_1, x_2, \cdots, x_{q+1}\}$ *can not force any white vertex of* $K_n \boxtimes K_{p,q}$. *Can any black vertex in* $\bigcup_{i=1}^{n-1}(S_i \cup K_i)$ *force any white vertex of* $K_n \boxtimes K_{p,q}$? *It is sufficient to examine the black vertices in any* $S_i \cup K_i$ *because of definition of true twins, where* $1 \leq i \leq n-1$. *Then for* $m \in 1 \leq i \leq n-1$, *we will examine the black vertices in the set* $S_m \cup K_m = \{(u_m, v_j) : 1 \leq j \leq p\} \cup \{(u_m, w_l) : 1 \leq l \leq q\}$.

Case 1: For the black vertices in S_m, *any black vertex* (u_m, v_j) *has* $q+1$ *neighbors from the set* $\mathcal{A} \cup \mathcal{B}$ *by the definition of strong product. Even if the* $p-1$ *black vertices in the set* $(\mathcal{A} \cup \mathcal{B}) - \{x_1, x_2, \cdots, x_{q+1}\}$ *is some of these* $q+1$ *vertices,* (u_m, v_j) *will have at least* $q+1-(p-1) = q-p+2 \geq 2$ *white neighbors. Then it can not force any white vertex of* $K_n \boxtimes K_{p,q}$.

Case 2: For the black vertices in K_m, any black vertex (u_m, w_l) has $p+1$ neighbors from the set $\mathcal{A} \cup \mathcal{B}$ by the definition of strong product. Even if the $p-1$ black vertices in the set $(\mathcal{A} \cup \mathcal{B}) - \{x_1, x_2, \cdots, x_{q+1}\}$ is some of these $p+1$ vertices, (u_m, w_l) will have at least $p+1-(p-1) = 2$ white neighbors. Then it can not force any white vertex of $K_n \boxtimes K_{p,q}$.

Since none of the white vertices of $K_n \boxtimes K_{p,q}$ are forced, it gives us a contradiction with the fact that S is a zero forcing set of $K_n \boxtimes K_{p,q}$. Then the following is obtained.

$$Z(K_n \boxtimes K_{p,q}) \geq (p+q)(n-1) + p. \tag{4.24}$$

Thus, we obtain that $Z(K_n \boxtimes K_{p,q}) = (p+q)(n-1) + p$ by the inequalities in (4.23) and (4.24). □

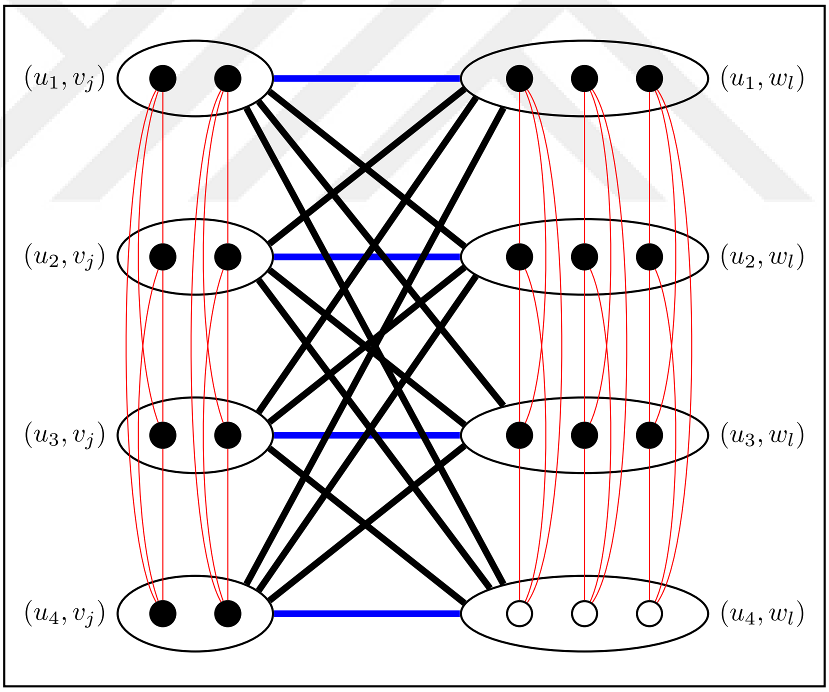


Figure 4.7: The graph $K_4 \boxtimes K_{2,3}$ with an initial set S.

Lemma 4.5: $Z(K_{p,q} \boxtimes P_2) = 2p + q$ for $q \geq p \geq 1$.

Proof: Let $q \geq p \geq 1$. By Theorem 4.6 and Proposition 3.5, we obtain the following inequality.

$$Z(K_{p,q} \boxtimes P_2) \leq 2p + q. \tag{4.25}$$

Let $V(K_{p,q}) = \{u_1, u_2, \cdots, u_p, v_1, v_2, \cdots, v_q\}$ *and* $V(P_n) = \{w_1, w_2\}$.

Let $C_1 = \{(u_i, w_1) : 1 \le i \le p\}$, $C_2 = \{(u_i, w_2) : 1 \le i \le p\}$ *and let* $D_1 = \{(v_k, w_1) : 1 \le k \le q\}$, $D_2 = \{(v_k, w_2) : 1 \le k \le q\}$, *say* $p-$*sets and* $q-$*sets, respectively. Clearly,* $V(K_{p,q} \boxtimes P_2) = (C_1 \cup C_2) \cup (D_1 \cup D_2)$. *For convenience, we will array all vertices of* $V(K_{p,q} \boxtimes P_2)$ *as in proof of Theorem 4.6.*

Let S *be a minimum zero forcing set of* $K_{p,q} \boxtimes P_2$. *From the definition of strong product, the vertices that are on the same row in the* $p-$*sets are true twins to each other. Similarly, this situation holds for the* $q-$*sets. We know that at least one of the true twins must be contained in all zero forcing sets of* $K_{p,q} \boxtimes P_2$ *by Lemma 3.2. Then at least* p *vertices from* $p-$*sets and at least* q *vertices from* q *sets are contained in the set* S. *Then* S *has at least* $p + q$ *vertices (see Figure 4.8).*

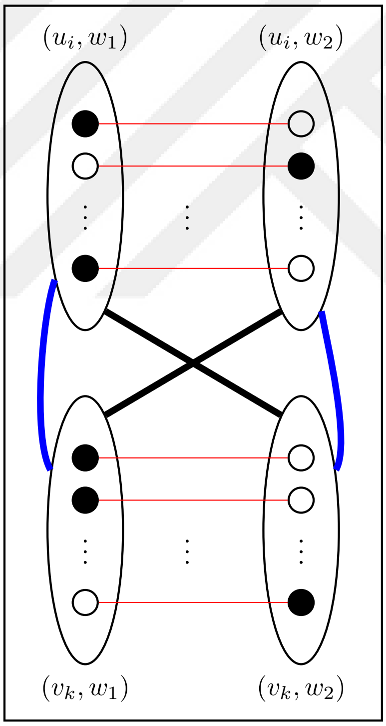


Figure 4.8: Representing graph for $1 \le i \le p$, $1 \le k \le q$.

However, at least another p *vertices must be contained in* S *to do any forcing. The reason is that when we consider any black vertex that has the least number of white neighbors (black vertices in q-sets), we see that it has* $p + 1$ *white neighbors. So, we have the following.*

$$Z(K_{p,q} \boxtimes P_2) \ge 2p + q. \tag{4.26}$$

As a result, we obtain that $Z(K_{p,q} \boxtimes P_2) = 2p + q$ by the inequalities in (4.25) and (4.26). □

The following significant proposition was presented at [33] by Brešar et al. It will help us next works.

Proposition 4.6: [33]

i) For any graphs G and H, $\gamma_{gr}(G \boxtimes H) \geq \gamma_{gr}(G)\gamma_{gr}(H)$.

ii) Let G and H be arbitrary graphs. If v is a simplicial vertex in G, then $\gamma_{gr}(G \boxtimes H) \leq \gamma_{gr}(H) + \gamma_{gr}((G - \{v\}) \boxtimes H)$.

The following pretty important Theorem was shown in [34] by Bell et al.

Theorem 4.7: [34] For any tree G and graph H, $\gamma_{gr}(G \boxtimes H) = \gamma_{gr}(G)\gamma_{gr}(H)$.

Now, we will prove a special case of the above theorem.

Corollary 4.2: [34] $\gamma_{gr}\left(K_{p,q} \boxtimes P_n\right) = q(n-1)$ for $q \geq p \geq 1$, $n \geq 2$.

Proof: Let $q \geq p \geq 1$ and $n \geq 2$. By Proposition 4.6, Lemma 4.2, and Lemma 4.3, the following is obtained.

$$\gamma_{gr}(K_{p,q} \boxtimes P_n) \geq \gamma_{gr}(K_{p,q})\gamma_{gr}(P_n) = q(n-1). \tag{4.27}$$

Now, we will show that $\gamma_{gr}\left(K_{p,q} \boxtimes P_n\right) \leq \gamma_{gr}\left(K_{p,q}\right)\gamma_{gr}\left(P_n\right) = q(n-1)$. We prove this inequality by the induction on n.

Basis step: $n = 2$.

Let $V\left(K_{p,q}\right) = \{u_1, u_2, \cdots, u_p, v_1, v_2, \cdots, v_q\}$ and $V(P_2) = \{w_1, w_2\}$. Clearly, the ordered sequence $S = ((v_k, w_1) : 1 \leq k \leq q)$ is a Grundy dominating sequence that is also a legal $Z-$sequence in the graph $K_{p,q} \boxtimes P_2$. Then by Theorem 4.2, $\gamma_{gr}(K_{p,q} \boxtimes P_2) + Z(K_{p,q} \boxtimes P_2) = |V(K_{p,q} \boxtimes P_2)|$. We know that $Z(K_{p,q} \boxtimes P_2) = 2p+q$ by Lemma 4.5. So, we obtain $\gamma_{gr}(K_{p,q} \boxtimes P_2) = 2p + 2q - 2p - q = q \leq q$.

Induction step: For $n - 1$, let our assumption be straight. Then by Lemma 4.2 and Lemma 4.3, the following is obtained.

$$Y_{gr}\left(K_{p,q} \boxtimes P_{n-1}\right) \leq \gamma_{gr}\left(K_{p,q}\right)\gamma_{gr}\left(P_{n-1}\right) = q(n-2). \tag{4.28}$$

Now, let $V(P_n) = \{w_1, w_2, \cdots, w_n\}$. We know that w_1 and w_n are simplicial vertices in P_n since $P_n[\{w_2\}] \cong P_n[\{w_{n-1}\}] \cong K_1$.

Also, we know that $K_{p,q} \boxtimes P_n \cong P_n \boxtimes K_{p,q}$. Then by Proposition 4.6 and the inequality in (4.28), the following is obtained.

$$\gamma_{gr}(P_n \boxtimes K_{p,q}) \leq \gamma_{gr}(K_{p,q}) + \gamma_{gr}((P_n - \{w_1\}) \boxtimes K_{p,q}) \tag{4.29}$$

$$= \gamma_{gr}(K_{p,q}) + \gamma_{gr}(P_{n-1} \boxtimes K_{p,q}) \tag{4.30}$$

$$\leq q + q(n-2). \tag{4.31}$$

So, the following is obtained.

$$\gamma_{gr}(K_{p,q} \boxtimes P_n) \leq q(n-1). \tag{4.32}$$

As a result, we obtain that $\gamma_{gr}(K_{p,q} \boxtimes P_n) = q(n-1)$ by the inequalities in (4.27) and (4.32). □

Corollary 4.3: $Z(K_{p,q} \boxtimes P_n) = np + q$ for $q \geq p \geq 1$ and $n \geq 2$.

Proof: Let $q \geq p \geq 1$ and $n \geq 2$. The following inequality holds from Theorem 4.6.

$$Z(K_{p,q} \boxtimes P_n) \leq np + q. \tag{4.33}$$

Also, the following is obtained by Theorem 4.2 and Corollary 4.2.

$$Z(K_{p,q} \boxtimes P_n) \geq |V(K_{p,q} \boxtimes P_n)| - \gamma_{gr}(K_{p,q} \boxtimes P_n) \tag{4.34}$$

$$= np + nq - nq + q \tag{4.35}$$

$$= np + q. \tag{4.36}$$

As a result, we get $Z(K_{p,q} \boxtimes P_n) = np + q$ by the inequalities in (4.33) and (4.36). □

Corollary 4.4: Let H be a connected graph with k vertices. Then $Z(P_n \boxtimes H) \geq nk - (n-1)\gamma_{gr}(H)$ for $n \geq 2$.

Proof: Let $n \geq 2$ and H be a connected graph with k vertices. We know that every path graph is a tree.

By Theorem 4.7 and Lemma 4.2, the following is obtained.

$$\gamma_{gr}(P_n \boxtimes H) = \gamma_{gr}(P_n)\gamma_{gr}(H) \tag{4.37}$$

$$= (n-1)\gamma_{gr}(H). \tag{4.38}$$

Also, the following is obtained from Theorem 4.2 and the equation in (4.38).

$$Z(P_n \boxtimes H) \geq |V(P_n \boxtimes H)| - \gamma_{gr}(P_n \boxtimes H) \tag{4.39}$$

$$= nk - (n-1)\gamma_{gr}(H). \tag{4.40}$$

The proof is completed. □

Corollary 4.5: $Z(K_n \boxtimes P_k) = k(n-1) + 1$ *for* $n \geq 2$, $k \geq 2$.

Proof: *Let* $n \geq 2$, $k \geq 2$. *By Corollary 4.1 and Proposition 3.5, the following is obtained.*

$$Z(K_n \boxtimes P_k) \leq k(n-1) + Z(P_k) \tag{4.41}$$

$$= k(n-1) + 1. \tag{4.42}$$

Also, the following is obtained from Theorem 4.7, Lemma 4.1, and Lemma 4.2.

$$\gamma_{gr}(K_n \boxtimes P_k) = \gamma_{gr}(K_n)\gamma_{gr}(P_k) \tag{4.43}$$

$$= k - 1. \tag{4.44}$$

Again by the equality (4.44) and Theorem 4.2, we get the following.

$$Z\left(K_n \boxtimes P_k\right) \geq \left|V\left(K_n \boxtimes P_k\right)\right| - \gamma_{gr}\left(K_n \boxtimes P_k\right) \tag{4.45}$$

$$= nk - k + 1 \tag{4.46}$$

$$= k(n-1) + 1. \tag{4.47}$$

As a result, we get $Z(K_n \boxtimes P_k) = k(n-1) + 1$ *by the inequalities in (4.42) and (4.47).* □

Corollary 4.6: Let H be a connected graph with n vertices. Then $\gamma_{gr}^{Z}(K_{p,q} \boxtimes H) \geq q(n - Z(H))$ for $q \geq p \geq 1$ and $n \geq 2$.

Proof: Let $q \geq p \geq 1$ and $n \geq 2$.

The following is obtained from Theorems 4.6 and 4.2 .

$$\gamma_{gr}^{Z}(K_{p,q} \boxtimes H) = V(K_{p,q} \boxtimes H) - Z(K_{p,q} \boxtimes H) \tag{4.48}$$

$$\geq n(p+q) - np - qZ(H) \tag{4.49}$$

$$= q(n - Z(H)). \tag{4.50}$$

The proof is completed. □

Corollary 4.7: $\gamma_{gr}^{Z}(K_n \boxtimes C_k) = k - 2$ for $n \geq 2$ and $k \geq 2$.

Proof: Let $n \geq 2$ and $k \geq 2$. The following is obtained from Theorem 4.2 and Proposition 4.4 .

$$\gamma_{gr}^{Z}(K_n \boxtimes C_k) = V(K_n \boxtimes C_k) - Z(K_n \boxtimes C_k) \tag{4.51}$$

$$= nk - nk + k - 2 \tag{4.52}$$

$$= k - 2. \tag{4.53}$$

The proof is completed. □

Corollary 4.8: $\gamma_{gr}^{Z}(K_n \boxtimes P_k) = k - 1$ for $n \geq 2$ and $k \geq 2$.

Proof: Let $n \geq 2$ and $k \geq 2$. The following is obtained from Theorem 4.2 and Corollary 4.5.

$$\gamma_{gr}^{Z}(K_n \boxtimes P_k) = V(K_n \boxtimes P_k) - Z(K_n \boxtimes P_k) \tag{4.54}$$

$$= nk - nk + k - 1 \tag{4.55}$$

$$= k - 1. \tag{4.56}$$

The proof is completed. □

Corollary 4.9: $\gamma^{Z}_{gr}(K_n \boxtimes K_{p,q}) = q$ *for* $q \geq p \geq 1$ *and* $n \geq 2$.

Proof: *Let* $q \geq p \geq 1$ *and* $n \geq 2$. *The following is obtained from Theorem 4.2 and Proposition 4.5.*

$$\gamma^{Z}_{gr}(K_n \boxtimes K_{p,q}) = V(K_n \boxtimes K_{p,q}) - Z(K_n \boxtimes K_{p,q}) \tag{4.57}$$

$$= n(p+q) - (n-1)(p+q) - p \tag{4.58}$$

$$= q. \tag{4.59}$$

The proof is completed. □

5. CONCLUSION

In this thesis, we have worked on the zero forcing number of some special graphs such as a path graph, a cycle graph, a complete graph, etc. Especially, as we expressed in the introduction section, we have focused on finding the zero forcing number of the strong product of any two special graphs. We have considered the zero forcing number of the strong product of a complete bipartite graph and any special graph; namely, a complete graph and a cycle graph or a complete graph and a path graph. Moreover, we considered the zero forcing number of the strong product of a complete bipartite graph and any special graph; namely, a complete bipartite graph and a complete graph or a complete bipartite graph and a path graph. To obtain these studies, we have used the works in the literature, and Lemma 3.2 and Theorem 4.6 that we have proved.

REFERENCES

[1] West D. B., (2001), “Introduction to Graph Theory”, 2nd Edition, Prentice Hall.

[2] Wilson R. J., (2010), “Introduction to Graph Theory”, 5th Edition, Pearson.

[3] Balaban A. T., (1985), “Applications of graph theory in chemistry”, Journal of Chemical Information and Computer Sciences, 25 (3), 334-343.

[4] Kozuch S., (2015), “Steady state kinetics of any catalytic network: graph theory, the energy span model, the analogy between catalysis and electrical circuits, and the meaning of “mechanism””, ACS Catalysis, 5 (9), 5242-5255.

[5] Eghdami H., Darehmiraki M., (2012), “Application of DNA computing in graph theory”, Artificial Intelligence Review, 38 (3), 223-235.

[6] Mackaness W. A., Beard K. M., (1993), “Use of graph theory to support map generalization”, Cartography and Geographic Information Systems, 20 (4), 210-221.

[7] Harary F., Gupta G., (1997), “Dynamic graph models”, Mathematical and Computer Modelling, 25 (7), 79-87.

[8] Kossinets G., (2006), “Effects of missing data in social networks”, Social Networks, 28 (3), 247-268.

[9] Work A. M. R. S. G., (2008), “Zero forcing sets and the minimum rank of graphs”, Linear Algebra and its Applications, 428 (7), 1628-1648.

[10] Web 4, (2022), http://aimath.org/pastworkshops/catalog2.html, (Erişim Tarihi: 18/05/2022).

[11] Burgarth D., Maruyama K., (2009), “Indirect Hamiltonian identification through a small gateway”, New Journal of Physics, 11 (10), 103019.

[12] Bozeman C., Brimkov B., Erickson C., Ferrero D., Flagg M., Hogben L., (2019), “Restricted power domination and zero forcing problems”, Journal of Combinatorial Optimization, 37 (3), 935-956.

[13] Burgarth D., Giovannetti V., Hogben L., Severini S., Young M., (2015), “Logic circuits from zero forcing”, Natural Computing, 14 (3), 485-490.

[14] Davila R., Kenter F., (2015), “Bounds for the zero forcing number of graphs with large girth”, Theory and Applications of Graphs, 2 (2), 1.

[15] Amos D., Caro Y., Davila R., Pepper R., (2015), “Upper bounds on the k-forcing number of a graph” Discrete Applied Mathematics, 181, 1-10.

[16] Davila R., Kalinowski T., Stephen S., (2018), “A lower bound on the zero forcing number”, Discrete Applied Mathematics, 250, 363-367.

[17] Huang L. H., Chang G. J., Yeh H. G., (2010), “On minimum rank and zero forcing sets of a graph”, Linear Algebra and its Applications, 432, 2961–2973.

[18] Brešar B., Bujtás C., Gologranc T., Klavžar S., Košmrlj G., Patkós B., Tuza Z., Vizer M., (2017), “Grundy dominating sequences and zero forcing sets”, Discrete Optimization, 26, 66–77.

[19] Benson K. F., Ferrero D., Flagg M., Furst V., Hogben L., Vasilevska V., Wissman B., (2018), “Zero forcing and power domination for graph products”, Australasian Journal of Combinatorics, 70 (2), 221–235.

[20] Taklimi F. A., (2013), “Zero forcing sets for graphs”, Ph.D. Thesis, University of Regina.

[21] Javaid I., Irshad I., Batool M., Raza Z., (2016), “On the zero forcing number of corona and lexicographic product of graphs”, arXiv preprint, arXiv:1607.04071.

[22] Brešar B., Gologranc T., Milanič M., Rall D. F., Rizzi R., (2014), “Dominating sequences in graphs”, Discrete Mathematics, 336, 22–36.

[23] Hogben L., (2006), “Handbook of Linear Algebra”, 1st edition, Chapman & Hall/CRC.

[24] Chartrand G., Zhang P., (2009), “Chromatic Graph Theory”, 1st Edition, Chapman & Hall/CRC.

[25] Burlet M., Uhry J. P., (1982), “Parity graphs”, Annals of Discrete Mathematics, 16, 1-26.

[26] Chartrand G., Lesniak L., (1996), “Graphs & Digraphs”, 3rd Edition, Chapman & Hall.

[27] Hammack R. H., Imrich W., Klavžar S., (2011), “Handbook of product graphs”, 2nd Edition, CRC press.

[28] Frucht R., Harary F., (1970), “On the corona two graphs”, Aequationes Math., 4, 322–325.

[29] Roman S., (2005), “Advanced Linear Algebra”, 3rd Edition, Springer.

[30] Web 4, (2022), https://github.com/jasongrout/minimum_rank, (Erişim Tarihi: 01/07/2022).

[31] Edholm C. J., Hogben L., Huynh M., LaGrange J., Row D. D., (2012), “Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph”, Linear Algebra and its Applications, 436 (12), 4352-4372.

[32] Benson K., Ferrero D., Flagg M., Furst V., Hogben L., Vasilevska V., Wissman B., (2018), “Zero forcing and power domination for graph products”. Australasian Journal of Combinatorics, 70 (2), 221.

[33] Brešar B., Bujtás C., Klavžar S., Košmrlj G., Patkós B., Tuza Z., Vizer M., (2016), “Dominating sequences in grid-like and toroidal graphs”, The Electronic Journal of Combinatorics, 23, P4.34.

[34] Bell K., Driscoll K., Krop E., Wolff K., (2021), "Grundy domination of forests and the strong product conjecture", arXiv preprint, arXiv:2104.05665.

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