Peridynamic modelling of deformation field on isotropic medium

by

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“Hayatta en hakiki mürşid ilimdir.“

(Science is the only true guide in life.)

Mustafa Kemal Atatürk

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Abstract

Designing light weight structural parts especially in the areas of aerospace, marine and automotive industries has become a must over the years to reduce energy consumption of structures and systems. To this end, numerical models enabling realistic and accurate results for deformations, damage initiations and propagations inside solid mediums constitutes a corner stone for failure prediction since they provide flexibility in optimization of design constraints such as topology, material type and reduction of structural weight.

Within the framework of continuum mechanics, classical approaches are well studied however they include the restriction of local interactions for finite element methods since classical theory of continuum mechanics assumes that each individual particle interacts with those locating in their immediate vicinity. Due to the inherent formulation of classical theory of continuum mechanics, in case of continuously transferred thermal and mechanical loadings, the governing laws that they include partial differential equations become undefined in the presence of discontinuities inside solid mediums. A study in applied mechanics called peridynamic theory introduces a new modelling concept of non-local interactions for solid structures. To be able to avoid undefined equations of associated problems, the peridynamic theory of non-local continuum mechanics replaces the spatial partial differentiations with integro-differential equations.

In the content of this study, deformation field of an isotropic plate under the effect of uni-axial stretching has been investigated by means of this relatively new approach of non-local continuum mechanics.

Deformasyon alanının çevresel-dinamik teori ile isotrop ortamlar için modellenmesi

Çağdaş Akalın

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Özet

Son yıllarda, havacılık-uzay, denizcilik ve otomotiv endüstrilerinde dinamik yükler altında çalışan hafif ve mekanik dayanımları yüksek yapısal elemanların geliştirilmesi ve bu elemanların bulundukları sistemler üzerindeki atalet etkilerinin azaltılarak enerji tüketimlerinin düşürülmesi konusu bir hayli önem kazanmıştır. Bu bağlamda, malzeme deformasyonlarının ve hasar oluşumlarının, hassas ve gerçekçi sonuçlar üretebilen sayısal yöntemlerle modellenmesi yapısal elemanların ağırlık, topoloji ve malzeme türü gibi tasarımsal sınırlamaların belirlenmesine ve bunların optimizayonuna esneklikle imkan sağladığından gelişmiş mühendislik uygulmalarının araştırma-geliştirme süreçlerinde önemli bir yer teşkil etmektedir.

Klasik süreklilik mekaniği nazarındaki yaklaşımlarda her biri sonsuz küçüklükte kabul edilen diferansiyel elemanların sadece onların bitişik komşuluğunda bulunanlarla etkileşim halinde olduklarının kabul edilmesi sonlu elemanlar yönteminin kullanıldığı saysal yöntemlerde bölgesel etkileşimlerin neden olduğu kısıtlamaları içlerinde barındırır. Klasik sürüklilik mekaniğininde ısıl ve mekanik yüklerin dağılımlarının tarifinde kullanılan denklemler doğaları gereği kat ortamda çatlak gibi süreksizlikler barındırmaları halinde tanımsız hale gelirler. Uygulamalı mekanik alanında görece yeni bir yaklaşm olarak gösterilebilecek olan çevresel-dinamik modelleme yöntemi sonsuz küçüklükte diferansiyel elemanların lokal olmayan etkileşimine izin vermektedir. Lokal süreklilik mekaniğinin yapı içi süreksizlikleriyle tanımsız hale gelen denklemleri bu yeni modelleme tekniğinde integro-diferansiyel denklemler ile değiştirilerek tanımsızlık ortadan kaldırılır.

Bölgesel olmayan süreklilik mekaniğinin bu yeni yaklaşımından yararlanlarak, isotrop malzemler üzerinde mekanik kuvvetler nedenli oluşan deformasyon alanının sayısal yöntemlerle hesaplanması çalışılmıştır.

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Çağdaş Akalın

Contents

Abstract **iv**

Özet **v**

Acknowledgements **vi**

List of Figures **x**

List of Tables **xi**

Abbreviations **xii**

Physical Constants **xiii**

Symbols **xiv**

1 Introduction **1**
1.1 Motivation and Literature Review 1
1.2 Outline of Thesis 4

2 Background **6**
2.1 Fundamentals of Classical Continuum Theory of Solids 6
2.2 Equilibrium of traction forces 7
2.3 Balance Laws in Local Theory 12
2.3.1 Conservation of Linear Momentum 12
2.3.2 Conservation of Angular Momentum 14

3 Strain Energy and Equation of Motion in Local Theory **16**
3.1 Strain energy density function for isotropic materials 16
3.2 Lagrangian formalism for equation of motion in classical interaction 21
3.3 Equations of motion in classical theory 31

4 Fundamentals of Peridynamic Modelling 33
4.1 Introduction 33
4.1.1 Hamilton's Principle 37
4.1.2 Equations of Motion in Non-Local Theory 39
4.1.3 Balance Equations 45
4.1.3.1 Global Balance of Linear Momentum 45
4.1.3.2 Global Balance of Angular Momentum 47
4.2 Alignments and magnitudes of pair-wise forces in peridynamic theory 48
4.2.1 Formulation of bond-based peridynamic 49
4.2.2 Formulation of ordinary state-based peridynamic 50
4.2.3 Stretch Notion in Peridynamic Theory 51
4.2.4 Peridynamic pair-wise force interaction 51

5 Formulations of Peridynamic Modelling for Deformation Fields 60
5.1 Strain energy and dilatation 60
5.1.1 Relating deformation constants with peridynamic parameters 70
5.2 Peridynamic parameters for three-dimensional structures 71
5.3 Peridynamic parameters for two-dimensional structures 84

6 Results and Discussion 97
6.1 Numerical Implementation 97
6.2 Definition of test case 98
6.3 Test parameters 99
6.4 Numerical results and validation 101

7 Conclusion 105

A Background 107
A.1 Introduction 107
A.1.1 Fundamentals of Tensor Transformations 107
A.1.2 Tensor Transformation 108
A.2 Isotropic Tensors 115

B Deformation Constants 124
B.1 General review on deformation 124
B.2 Decomposition of deformation 125

C Constitutive Relation for Particle Interaction in Local Theory 129
C.1 Reduction of fourth-order isotropic tensor 129
C.2 Symmetry for anisotropic materials 130
C.3 Symmetry for monoclinic materials 132

C.4 Symmetry for transversely isotropic materials . . . 135
C.5 Symmetry for isotropic materials . . . 138

D Deformation and Finite Strain Tensors 141
D.1 Deformation and Finite Strain Tensors . . . 141
D.2 Geometric Interpretation of Deformation . . . 145

Bibliography 150

List of Figures

2.1 Oblique cut of an arbitrary solid body 7
2.2 Representation of forces acting on surfaces of tetrahedron 9

3.1 Local interactions of material particles 22

4.1 Non-Local interactions of material particles 34
4.2 Peridynamic deformation vectors and particle horizons 34
4.3 Evaluation of micro-potential energy 55

5.1 Three-dimensional integration domain for volume of a material point 74
5.2 Deformation of a plate in case of pure isotropic expansion 85
5.3 Two-dimensional integration domain for volume of a material point 86
5.4 Pure shearing deformation of a plate 87

6.1 Representation of externally applied loads causing uni-axial stretching of a thin plate 98
6.2 Colour diagrams for displacement field, u_{x_1} 101
6.3 Displacement comparison through horizontal mid-line as a function of x_1 102
6.4 Colour diagrams for displacement field, u_{x_2} 102
6.5 Displacement comparison through vertical mid-line as a function of x_2 103
6.6 Colour diagrams for resultant displacement field, u 103
6.7 Resultant displacement comparison through horizontal mid-line as a function of x_1 104
6.8 Resultant displacement comparison through vertical mid-line as a function of x_2 104

A.1 Equivalent representations of a vector rotation 108

B.1 Exaggerated shearing deformation in plane stress condition 124
B.2 Exaggerated pure shear (a) and exaggerated pure tensile (b) loading of RVE 126

C.1 Representation of a transversely isotropic structure 137

D.1 General case of deformation in a two dimensional body 146

List of Tables

6.1 External dimensions of the test material 98
6.2 Test parameters 99

7.1 Comparisons of results 105

Abbreviations

CFD	**C**entral **F**inite **D**ifference
PD	**R**eri-**D**dynamic
RVE	**R**epresentative **V**olume **E**lement
FEM	**F**inite **E**lement **M**odelling
FEA	**F**inite **E**lement **A**nalysis

Physical Constants

Young's modulus of the test material	E	$=$	$192\ \times 10^9\ [Pa]$
Shear modulus of the test material	μ	$=$	$72\ \times 10^9\ [Pa]$
Density of the test material	ρ	$=$	$7800\ \left[\frac{kg}{m^3}\right]$
Poisson's ratio of the test material	ν	$=$	$1/3$

Symbols

α	coefficient of thermal expansion
a_1	generalized PD parameter
$a_{1,2D}$	PD parameter for two-dimension
a_2	generalized PD parameter
$a_{2,2D}$	PD parameter for two-dimension
a_3	generalized PD parameter
$a_{3,2D}$	PD parameter for two-dimension
A^{sb}, B^{sb}	Bond constants for ordinary state-based approximation
A^{bb}, B^{bb}	Bond constants for bond-based approximation
A^{sb}_{2D}, B^{sb}_{2D}	Bond constants for ordinary state-based approach in two-dimensions
A^{bb}_{2D}, B^{bb}_{2D}	Bond constants for bond-based approach in two-dimensions
b	generalized PD parameter
b_{2D}	generalized PD parameter for two-dimensions
c, C^p	PD bond constant for bond-based approximation
C^p_{2D}	PD bond constants for ordinary state-based approximation
C_{ijkl}	fourth-order isotropic tensor
C_{ij}	reduced stiffness matrix
d	generalized PD parameter
d_{2D}	generalized PD parameter for two-dimension
η	relative deformation amount
δ	horizon length
δ_{ij}	Kronecker-Delta operator
ΔT	temperature difference

ϵ	strain tensor
$\epsilon_{(j)ii}$	PD definition of dilatation
ϵ_{ijk}	permutation symbol
σ	Cauchy's stress tensor
ρ	density
ν	Poisson's ratio
γ_{max}	maximum shear strain
κ	bulk modulus
κ_{2D}	bulk modulus in two-dimensions
λ	Lame-constant
E	elastic modulus
G, μ	shear modulus
ϵ_{ii}	dilatation
h	thickness of the two-dimensional plate
I	identity matrix
T	kinetic energy
U	potential energy
V	volume
L	Lagrangian
$\mathbf{f}$	pair-wise force
f	magnitude of pair-wise force
$\underline{\mathbf{F}}$	force state
$\mathbf{L}$	linear momentum
$\mathbf{F}$	force
$\mathbf{H}_o$	angular momentum
$\mathbf{T}$	torque
W	classical strain energy density
$W_{(j)}$	strain energy density or strain energy of a particle
$w_{(i)(j)}$	micro-potential energy for particle (i)
ω	influence or micro-modulus function
W_{IsoExp}	strain energy in case of isotropic expansion

W_{Shear}	strain energy in case of isotropic expansion
ξ	relative displacement vector in undeformed state
$\left(y_{(k)} - y_{(j)}\right)$	relative displacement vector in deformed state
u_i	displacement vector
$\dot{u}_{(i)}$	velocity vector
x_i	position vector in undeformed configuration
y_i	position vector in deformed configuration
ζ	peridynamic strain
s	stretch

To my family,

Chapter 1

Introduction

1.1 Motivation and Literature Review

Since classical approaches to model mechanical behaviour of materials including discontinuities utilize from partial differential equations(PDEs) it is required that the governing equations have to be continuously differentiable through material domain. As explained in detail in following chapters of this present study, these PDEs become undefined when the equations of motion derived based on classical continuum mechanics are applied on a region including discontinues.

Instead of using partial differential equations, a non-local(being an particle-based method) approach, named as peridynamic theory, in which one of the fundamental concept of classical mechanics is that of a particle, [6], is considered as cornerstone, has been introduced by S.A. Silling in [32]. Non-locality of the peridynamic theory comes from interactions of each particle with others within a pre-defined sub-domain so that effects of neighbouring particles on subjected particle are applied through integro-differential equations.

In literature there can be found many research effort regarding analytical solution and numerical implementation of peridynamic theory. For instance, idealization of real structures can also be tuned by their one or two-dimensional representatives, in a sense is that dispersion of stresses along unbounded rod is examined in [36]. In addition, analytical solutions for simulation of crack initiation and propagations have been developed and numerical implementations have been proposed in [8] for KalthoffWinkler experiment. As well as deformation case of a one-dimensional string, distortion of a membrane type structure that can be approximated as a two-dimensional body is studied in [37] where numerical simulations for opening-mode resulting in plane-stress condition. Additionally, simulation of tearing mode are presented and associated results of discontinuity formations and their propagations throughout material domain are provided. In the same study, damage simulation of spherical membrane under the effect of a sharp fragment is presented as well.

Besides, coupling effect of thermo-mechanical interaction for structural deformations in presence of crack formations and propagations are modelled based upon Lagrangian formalism in peridynamic theory, [2]. Furthermore, an extensive study of bond-based peridynamic modelling capturing damage initiation phenomenon in brittle structures are studied in [14].

In particular, influence function in peridynamic theory brings the effect of neighbouring particles onto each subjected material point thus it terminates locality concern in classical approach. In this manner, effect of non-locality providing a function for propagation of waves on structures of different kinds is presented in [30]. A well-know approach using energy dissipation during propagations of cracks is based on calculation of J-Integral, in this sense adaptation of J-Integral on peridynamic theory with explicit derivation scheme is given in [15].

Moreover, time efficiency in numerical analysis for computational work is one of the fundamental concern, relating to this, efficiency of numerical implementations for peridynamic theory is analysed and propagation of discontinuities in specific type of materials are studied in [33].

Furthermore, capability of peridynamic theory on capturing stress-strain fields on bodies with discontinuities is introduced for conventional laminate composite plates configured with varying fiber orientation in [43]. The other study for deformation of non-conventional composite plates with non-ordinary state-based approach is introduced in [42]. Additionally, one another study by means of non-ordinary state-based approach in case of quasi-static loading condition has been proposed in [5] for linear elastic materials.

The study, [12], can be seen as an extensive discussion on peridynamic modelling for materials whose Poisson's ratios are different than 1/4, while fundamentals for generalization of bond-based technique leading to state-based approach in peridynamic theory has been introduced together with numerical solutions of well-known benchmark problems in literature are provided based on explicit solution scheme in [27]. Furthermore, extension of peridynamic modelling capturing plastic and permanent deformations of solid bodies is covered within the frame of state-based approach establishing a constitutive model between forces and deformations in [34]. Additionally, generalized approach of peridynamic theory being non-ordinary state-based modelling was used for solution of deformation problems of solid mechanics in [40].

Governing laws of thermodynamics can be applied to derivation scheme for governing laws of peridynamic theory leading to coupling effects of thermal and mechanical phenomenons for deformations of solid bodies, in this sense numerical solutions of thermo-mechanical problems based on peridynamic modelling are presented in [26].

Non-locality of peridynamic theory provides an inherent capability of taking effect of long-range forces into account for each material point in equations of motion which is very similar to computational structure in classical molecular-dynamics, in this sense, comparison of formulations for dynamic effects and governing equations that are consolidated by computational results have been provided in [31]. Moreover, under the effect of non-locality, analytic solutions for deformation field of a one-dimensional micro-elastic structure with dispersion relations of different kinds are presented explicitly with various examples in [41].

As well as coupling of material parameters in classical approach with those that are in peridynamic modelling, mathematical investigation upon equations of motion being a integro-differential equation with second-order time derivative for numerical solutions of benchmark problems are studied in [9]. Moreover, a new proposal for solution of peridynamic formulation with examples is introduced in [10].

As oppose to crack behaviour in ductile materials, crack branching phenomenon is more likely observed in brittle structures and capability of peridynamic modelling on capturing material fragmentation is presented in [13]. Moreover, impact studies as a result of extreme loadings on material domain within the frame of peridynamic modelling are performed in [7]. In one another impact study on damaging of circular plate with implementation of peridynamic formulation in molecular dynamic solution algorithm, numerical simulation has been presented in [28].

Heat dissipation models based on peridynamic approach for materials including discontinuities differ than results of classical continuum approaches, regarding to this, solutions of both are compared in [3]. As a result, it is emphasized that classical outcomes overlap with consequences of peridynamic modelling while limit of horizon size approaches to zero. Moreover, peridynamics is considered as an embracing formulation of those that belong to classical theory because of the fact that peridynamics stresses approaches to classical stress state depending on smoothness of motion for particles, constitutive equations and non-homogeneities of material domain in [35].

As a bridge between classical stress formalism and peridynamic pair-wise forces has been introduced in this present study, introduction of peridynamic equation of motion in terms of stress tensor can be found in [18] as well. Moreover, improvement of solution steps for elasticity problems including discontinuities within the frame of peridynamic modelling that takes long-rage effects of surrounding particles into account for each subjected particle on material domain has been introduced in [4]. Additionally, application of peridynamic theory for consideration of both thermal and mechanical effects being a challenging issue for small length scale systems such as electronic parts is presented in [17].

Specifically, application of both finite element analysis and peridynamic technique of non-local continuum mechanics are utilized in the modelling of a truss element and results are compared in [21].

A novel research on a numerical simulation for fragmentation of a isotropic tube whose damage behaviour under compressive forces are observed in [39] has been developed by implementation of both peridynamic and FEM algorithms validating upon experimental results in [19].

1.2 Outline of Thesis

In this study, balance laws for conservation of linear and angular momentums under the effect of internal stresses and resultant traction vectors are presented in an explicit manner in Chapter (2).

Afterwards, in Chapter (3), the equations of motion in local theory is obtained based on Lagrangian formalisms.

In Chapter (4), peridynamic equations of motion is derived for linear micro-elastic materials, [22], [25] while in the following sections, peridynamic definition of deformation is presented, [33].

Peridynamic parameters leading to bond-constants for three and two-dimensional isotropic materials are obtained in Chapter (5). Based on peridynamic bond-constant derived in Chapter (5) for two-dimensional structures, results of numerical simulations pertaining to deformation of an isotropic plate in plane-stress condition under the effect of uni-axial stretching are presented in Chapter (6) . Furthermore numerical results that are obtained in MatLab® Version R2016a are compared with FEA results in terms of deformation fields.

The present work is finalized in Chapter (7)with remarks based on results obtained in Chapter (7).

Additionally, Appendix (A) presents vector rotations, tensor transformations and a general review of derivation for a fourth-order isotropic tensor that is highly occupied in constitutive relations of applied mechanics. In Appendix (B), classical constants of deformation are provided under the review of simple body distortions while in Appendix (C) constitutive relations for different type of structures are introduced based on fourth-order isotropic tensor derived in Appendix (A) Furthermore, general review on classical definition of deformation tensor is presented in Appendix (D) for the purpose of establishing relations between components of finite strains and stresses.

Chapter 2

Background

2.1 Fundamentals of Classical Continuum Theory of Solids

Property of being continuous for a material medium under consideration disregards molecular structure and states it as not consisting of gaps or voids. Because of this hypothetical definition of material domain, theory is referred as theory of continuous medium or briefly continuum theory, [23].

From classical point of view of continuum mechanics for solids, a well-equilibrated body in terms of internal forces sustains stabilities of displacements between particles. Nevertheless, any disturbance against equilibrium condition of internal forces causes deformations and discontinuities such as cracks due to external forces that compels body to exceed mechanical endurance limits. These stiffness properties are prescribed in constitutive relations to be able to relate associated stress and strain components along desired directions of material domain. In local continuum theory, interactions of subjected particles presented by RVE are restricted by only neighbouring material points located in their immediate vicinities. Moreover, stress and strain components occurring on sides of each RVE play a fundamental role in terms of determining traction forces that acts on subjected particle.

2.2 Equilibrium of traction forces

Classical approach regarding interaction of material points dictates locality such a way that particles which are represented by infinitesimal RVEs interact with only others in immediate vicinity. In this sense, internal surface forces of RVEs ordinarily named as tractions appear on oblique-cut surface of RVE as shown in Figure (2.1) while they are balanced with stress components, σ_{ij}, of associated side-faces on tetrahedron that is presented in Figure (2.2).

These stress components appear as a result of balancing forces against traction exerted on oblique-cut surface of tetrahedron. Therefore, balance forces associated with their stress components acting on infinitesimal areas, dS_i on side-surfaces of tetrahedron while traction force, $t_i^{e_j}$, acts on oblique-cut surface area, dS_n and defined as follows.

$$t_i^{e_j} = \sigma_{ij} e_j \tag{2.1}$$

where e_j are basis vectors of Cartesian co-ordinate system. Also, in relation (2.1), sub-index, i, indicates surface normal in which associated component of stress tensor is applied while sub-index j denotes direction of same stress component.

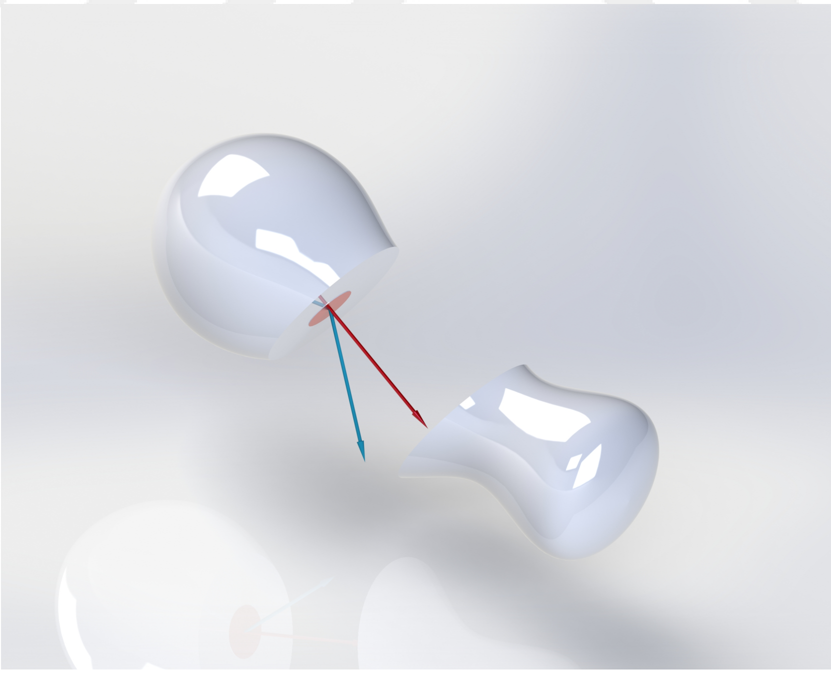


FIGURE 2.1: Oblique cut of an arbitrary solid body

As shown in Figure (2.1), red arrow represents surface normal while traction force belonging to oblique-cut is indicated by means of blue arrow. A relation between tetrahedron's side-surface areas and oblique surface area can be expressed as follows.

$$dS_n\left(n \cdot e_i\right) = dS_n \cos\left(\theta_n^{e_i}\right) = dS_n n_i = dS_i \tag{2.2}$$

in which n is normal vector belonging to oblique-cut's surface while $\theta_n^{e_i}$ represents angle between principle axes and surface normal, n. On the other hand, n_i is cosine value of this angle.

As a result of applied external forces, solid body can be expected to experience either a elastic or plastic deformation. From this point of view, stresses emerging from these applied forces through cross-sectional surface inside body can be defined.

Now, let us consider a solid body on which external forces come into existence. Splitting body into two pieces through any arbitrary oblique cut and inspecting free-body diagram of half part of RVE, one may realize that equivalent force on former contact surfaces of cut-off plane emerges. These forces applying on side-surface areas of tetrahedron allow us define stress vector as follows.

$$t_i^{e_j} = \lim_{dS_j \to 0}\left(\frac{f_i}{dS_j}\right) \tag{2.3}$$

where f_i is a force along any arbitrary direction and sub-index i indicates surface normal of an area on which f_i is applied. Similarly, traction force on oblique-cut surface of tetrahedron is defined as

$$t_i^{n} = \lim_{dS_n \to 0}\left(\frac{f_n}{dS_n}\right) \tag{2.4}$$

The Divergence Theorem [11] which conserves the fluxes of vector field inside a close surface is given by

$$\iiint\limits_V \left(\nabla \cdot \vec{f}\right) dV = \oiint\limits_V \vec{f} \cdot d\vec{S} \tag{2.5}$$

If force vector field is assumed to be divergence-free, namely $(\nabla \cdot \vec{f}) = 0$, then relation (2.5) can be reduced to

$$\oiint\limits_V \vec{f} \cdot d\vec{S} = 0 \tag{2.6}$$

Thus, equilibrium state of all forces affecting on tetrahedron can be expressed through relation (2.6). In point-wise manner, the left-hand side of relation (2.6) can be written as a sum of associated dot products. This result implies Newton's third law of motion for equilibrium state which dictates that net force on tetrahedron has to be equivalent to zero. Therefore,

$$t_i^n dS_n + t_i^{e_1} dS_1 + t_i^{e_2} dS_2 + t_i^{e_3} dS_3 = 0 \tag{2.7}$$

And according to (2.2), relation (2.7) reads

$$\begin{aligned} t_i^n dS_n + t_i^{e_1} n_1 dS_n + t_i^{e_2} n_2 dS_n + t_i^{e_3} n_3 dS_n &= 0 \\ t_i^n + t_i^{e_1} n_1 + t_i^{e_2} n_2 + t_i^{e_3} n_3 &= 0 \\ t_i^n + t_i^{e_j} n_j &= 0 \end{aligned} \tag{2.8}$$

By invoking the identity given by relation (2.1) into (2.8) equilibrium equation is obtained as

$$t_i^n + \sigma_{ij} e_j n_j = 0 \tag{2.9}$$

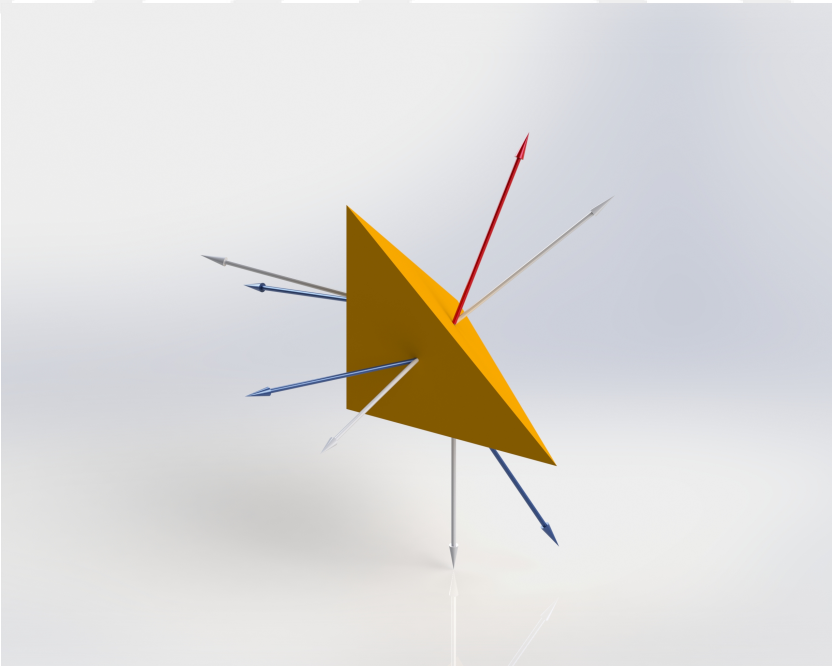


FIGURE 2.2: Representation of forces acting on surfaces of tetrahedron

It can be also shown that traction vector, $t^{(n)}$, is obtained long vector-dyadic dot product of second-order tensor,σ, and surface normal vector, n, as follows.

$$\begin{aligned} t_i^n &= \sigma \cdot n \\ &= (\sigma_{ij} e_j) \cdot (n_k e_k) \\ &= \sigma_{ij} n_k e_j \cdot e_k = \sigma_{ij} n_k \delta_{jk} \\ &= \sigma_{ij} n_j \end{aligned} \tag{2.10}$$

Traction vector is applied on an area whose surface normal is defined by n, hereafter n term can be neglected in notation. Therefore three-components of t_i^n in Cartesian co-ordinates can be written explicitly as follows.

$$\begin{aligned} t_1 &= \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\ t_2 &= \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ t_3 &= \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{aligned} \tag{2.11}$$

In matrix notation, traction vector and right-hand side of relation (2.10) can be shown as

$$t_i^n = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{2.12}$$

in which second-order tensor defines Cauchy's stress components. Moreover traction vector, $t^{(n)}$, can be decomposed into its normal and shear components.

Additionally, determination of maximum normal and maximum shear stresses that body can withstand is considered as consequential issue in terms of failure criteria of structural parts. Disappearance of shear forces results in existence of pure normal forces on oblique-cut surface or other way around. In this perspective, because of orthogonality condition between shear and normal forces, traction vector, $t^{(n)}$, can be mathematically expressed as

$$(t_i^n)^2 = \left(\lim_{dS_n \to 0} \left(\frac{\vec{f}_{normal}}{dS_n} \right) \right)^2 + \left(\lim_{dS_n \to 0} \left(\frac{\vec{f}_{shear}}{dS_n} \right) \right)^2 = (t_S)^2 + (t_N)^2 \tag{2.13}$$

or

$$\|t_S\| = \sqrt{(t^{(n)})^2 - (t_N)^2} \tag{2.14}$$

In case of non-shear force on oblique-cut surface of Cauchy's tetrahedron, namely $t_S = 0$, then relation (2.13) is reduced to

$$\begin{aligned} t^n &= t_N \\ t^n - t_N &= 0 \end{aligned} \tag{2.15}$$

By means of relation (2.10), the last line of expression (2.15) can be stated in component form as

$$\begin{aligned} \sigma_{ki} n_k - n_i &= 0 \\ \sigma_{ki} n_k - \sigma_p \delta_{ki} n_k &= 0 \\ n_k \left(\sigma_{ki} - \sigma_p \delta_{ki} \right) &= 0 \end{aligned} \tag{2.16}$$

or in matrix form

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \begin{bmatrix} \sigma_{11} - \sigma_p \delta_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_p \delta_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_p \delta_{33} \end{bmatrix} = 0 \tag{2.17}$$

Since the first vector is any arbitrary array being different than zero, then determinant of second-order tensor has to be equal to zero. Namely,

$$\left| \sigma_{ki} - \sigma_p \delta_{ki} \right| = \begin{vmatrix} \sigma_{11} - \sigma_p \delta_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_p \delta_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_p \delta_{33} \end{vmatrix} = 0 \tag{2.18}$$

which yields to following expression.

$$-\sigma_p^3 + \sigma_p^2 I_1 + \sigma_p I_2 - I_3 = 0 \tag{2.19}$$

in which I_1, I_2 and I_3 are named as stress invariants and defined respectively as

$$\begin{aligned} I_1 &= trace(\sigma) = \sigma_{ii} \\ I_2 &= \frac{1}{2} \left(trace(\sigma)^2 - trace(\sigma^2) \right) = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ji}) \\ I_3 &= det(\sigma) = |\sigma_{ij}| \end{aligned} \tag{2.20}$$

Solution to expression (2.19) results in three principle stress components that are σ_{p_1}, σ_{p_2} and σ_{p_3}. By means of kinetic equilibrium equations of tetrahedron, it can be shown that maximum shear stress is equivalent to half of difference in maximum and minimum principle stresses. Namely,

$$\tau_{max} = max\left(\tau_1, \tau_2, \tau_3\right) \tag{2.21}$$

where $\tau_1 = \left(\sigma_{p2} - \sigma_{p3}\right)/2$, $\tau_2 = \left(\sigma_{p1} - \sigma_{p3}\right)/2$ and $\tau_3 = \left(\sigma_{p1} - \sigma_{p2}\right)/2$.

2.3 Balance Laws in Local Theory

2.3.1 Conservation of Linear Momentum

Forces acting on infinitesimal area of tetrahedron's oblique-cut surface have been inspected in the very beginning of Chapter (2). Resultant vector based on tractions in addition to body forces, b_i, over entire material domain can be expressed as follows.

$$\vec{f}_{resultant} = \int\limits_V \vec{t}_i(\sigma_{ij}, n_i) \cdot d\vec{A} + \int\limits_V \rho_i \vec{b}_i dV \tag{2.22}$$

in which t_i is same traction vector appearing in very left-hand side of relation (2.12) which is a function of its stress component and its associated surface normal. This resultant force plays a role in altering linear momentum of entire body in time domain. Mathematically,

$$\frac{d}{dt}\int\limits_V \rho_i \dot{\vec{u}}_i dV \tag{2.23}$$

According to Newton's second law of motion, entire body is accelerated by resultant force as inversely proportional to its inertia being resistance against motion. Additionally, resultant force is balanced with (2.23) as inertia of entire material domain remains constant. Therefore,

$$\frac{d}{dt}\int\limits_V \rho_i \dot{\vec{u}}_i dV = \int\limits_A \vec{t}_i(\sigma_{ij}, n_i) \cdot d\vec{A} + \int\limits_V \rho_i \vec{b}_i dV \tag{2.24}$$

By means of Divergence theorem given by (2.5), the first integral on right-hand side of relation (2.24) can be converted into volume integral and can be expressed as follows.

$$\int\limits_A \left(\sigma_{ij} n_j\right) \cdot d\vec{A} = \int\limits_V \left(\partial_{,j} e_j\right) \cdot \left(\sigma_{ij}\right) dV = \int\limits_V \sigma_{ij,j} dV \tag{2.25}$$

and balance equation for linear momentum can be obtained as follows.

$$\frac{d}{dt}\int\limits_{V}\rho_i\dot{u}_i dV=\int\limits_{V}\sigma_{ij,j}dV+\int\limits_{V}\rho_i b_i dV \tag{2.26}$$

By collecting all terms under a single integral, relation (2.26) can be expressed as

$$\int\limits_{V}\left(\rho_i\frac{d}{dt}\left(\dot{u}_i\right)-\sigma_{ij,j}-\rho_i b_i\right)dV=0 \tag{2.27}$$

Since dV is an arbitrary infinitesimal volume being different than zero, integrand of integral given by relation (2.27) can be directly equalized to zero. Therefore,

$$\rho_i\ddot{u}_i-\sigma_{ij,j}-\rho_i b_i=0 \tag{2.28}$$

in which spatial derivative of second-order tensor, σ_{ij}, is $\sigma_{ij,j}$ which has unit of force per volume, $[N/m^3]$. This term can be associated with force densities emerging from strain energy between interacting particles as limit of horizon in peridynamic theory approaches to zero. From this point of view, stress statement of a body including discontinuities reveals importance of peridynamic theory in comparison to local approaches. Stress field around a crack tip can be obtained through following expression which is derived based on Airy's function.

$$\sigma=\frac{\sigma_0}{\sqrt{1-\left(\frac{a}{x}\right)^2}} \tag{2.29}$$

in which σ_0 is nominal stress applied on cross-sectional area of a two-dimensional plate including hole in centre. Substitution of relation (2.29) to equations of motion in local theory following relation is obtained.

$$\rho(x,t)\ddot{u}(x,t)=div\left(\sigma\right)+b\left(x,t\right) \tag{2.30}$$

including spatial derivatives because of which stress term becomes infinity at crack tip position, $x=a$.

2.3.2 Conservation of Angular Momentum

Ordinarily, balance of angular momentum leads to symmetry condition for Cauchy's stress tensor whose derivation has been introduced at the beginning of Chapter (2). Based on balance of linear momentum equation for whole entire material domain inside solid body (2.24), by multiplying both side by distance of rotation around centre of Cartesian co-ordinate system, $\mathbf{y}$, equation for balance of angular momentum is obtained as follows.

$$\left(\frac{d}{dt}\int_V \rho_i \dot{\vec{u}}_i dV\right) \times \vec{y}_k = \left(\int_A \vec{t}_i\left(\sigma_{ij}, n_j\right) \cdot d\vec{A}\right) \times \vec{y}_k + \left(\int_V \rho_i \vec{b}_i dV\right) \times \vec{y}_k \tag{2.31}$$

Applying explicit form of traction stress vector, t_i, which is given by relation (2.10) to the first integral in the right-hand side of relation (2.31).

$$\begin{aligned} \int_V \rho_i \ddot{u}_i e_i \times y_k e_k dV &= \int_A \left(\sigma_{ij} n_j \times y_k e_k\right) \cdot d\vec{A} + \int_V \rho_i b_i e_i \times y_k e_k dV \\ \int_V \rho_i \epsilon_{ikl} \ddot{u}_i y_k e_l dV &= \int_A \left(\epsilon_{ikl} \sigma_{ij} n_j y_k e_l\right) \cdot d\vec{A} + \int_V \rho_i \epsilon_{ikl} b_i y_k e_l dV \end{aligned} \tag{2.32}$$

Divergence theorem given by relation (2.5) can be applied to convert area integral in relation (2.32) to volume integral as follows.

$$\begin{aligned} \int_A \left(\epsilon_{ikl} \sigma_{ij} n_j y_k\right) \cdot d\vec{A} &= \int_A \left(\epsilon_{ikl} \sigma_{ij} y_k\right) n_j \cdot d\vec{A} \\ &= \int_V \left(\partial_{,j} e_j\right) \cdot \left(\epsilon_{ikl} \sigma_{ij} y_k\right) dV \\ &= \int_V \left(\epsilon_{ikl} \sigma_{ij,j} y_k + \epsilon_{ikl} \sigma_{ij} \delta_{kj}\right) e_l dV \end{aligned} \tag{2.33}$$

By rearranging all terms in relation (2.32) in an appropriate way and substituting the result obtained in relation (2.33) into relation (2.32), one may write

$$\begin{aligned} \int_V \rho_i \epsilon_{ikl} \ddot{u}_i y_k e_l dV &= \int_V \left(\epsilon_{ikl} \sigma_{ij,j} y_k + \epsilon_{ikl} \sigma_{ij} \delta_{kj}\right) e_l dV + \int_V \rho_i \epsilon_{ikl} b_i y_k e_l dV \\ \int_V \epsilon_{ikl} \sigma_{ik} dV &= \int_V \epsilon_{ikl} y_k \left(\rho_i \ddot{u}_i - \sigma_{ij,j} - \rho_i b_i\right) dV \end{aligned} \tag{2.34}$$

Because balance of linear momentum has to be satisfied according to relation (2.28), the left hand-side of expression (2.34) yields to zero. Thus,

$$\epsilon_{ikl}\sigma_{ik} = 0 \tag{2.35}$$

Expansion of result obtained in (2.35) yields to following three a set of equation.

$$\begin{aligned} \epsilon_{123}\sigma_{12} + \epsilon_{213}\sigma_{21} &= 0 \\ \epsilon_{132}\sigma_{13} + \epsilon_{312}\sigma_{31} &= 0 \\ \epsilon_{231}\sigma_{23} + \epsilon_{321}\sigma_{32} &= 0 \end{aligned} \tag{2.36}$$

According to Levi-Civita permutation symbol given in (A.22), in each line of these a set of equation, coefficients seen in front of stress components imply skew-symmetric property in permutation symbol. Therefore,

$$\begin{aligned} \sigma_{12} - \sigma_{21} &= 0 \\ \sigma_{31} - \sigma_{13} &= 0 \\ \sigma_{23} - \sigma_{32} &= 0 \end{aligned} \tag{2.37}$$

which dictates symmetry condition that is $\left(\sigma = \sigma^T\right)$ for Cauchy's stress tensor that can be also presented in a short-hand notation by using index notation as follows.

$$\sigma_{ij} = \sigma_{ji} \tag{2.38}$$

Chapter 3

Strain Energy and Equation of Motion in Local Theory

3.1 Strain energy density function for isotropic materials

Externally applied forces to linearly elastic isotropic material domain causes energy accumulation and conversely removing external forces results in release of this accumulated energy. In this manner, it can be considered that strain energy density function relates the deformation amount and internal stress components based on energy stored inside material domain.

As shown in Chapter (2) and Appendix (D), symmetry condition in stress and strain tensors given by relations (D.38) and (2.38), allows us to express them as in arrays of six-components. Namely,

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{23} & \sigma_{13} & \sigma_{12} \end{bmatrix}^T \tag{3.1}$$

and

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{22} & \epsilon_{33} & \epsilon_{23} & \epsilon_{13} & \epsilon_{12} \end{bmatrix}^T \tag{3.2}$$

As explained in Chapter (4), kinetic and potential energies of a body can be expressed along sum of individual kinetic and potential energies of each particle in material domain. Namely,

$$T = \frac{1}{2}\sum_{j=1}^{\infty} m_{(j)}\vec{\dot{u}}_{(j)} \cdot \vec{\dot{u}}_{(j)} \tag{3.3}$$

and

$$U = \sum_{j=1}^{\infty} W_{(j)}V_{(j)} - \sum_{j=1}^{\infty} \vec{u}_{(j)}\vec{b}_{(j)}V_{(j)} \tag{3.4}$$

Classically, during a simple unidirectional tensile stretching of an arbitrary body, energy that emerges from uni-axial deformation of RVE is defined as strain energy which can be obtained by calculating area under associated stress-strain curve. By same analogy for uni-axial deformation of a body, strain energy emerging from arbitrary distortion of an RVE is expressed by

$$W_{(j)} = \frac{1}{2}\sum_{m=1}^{3}\sum_{n=1}^{3} \sigma_{(j)mn}\epsilon_{(j)mn} \tag{3.5}$$

For a single material point denoted by (j), let us write normal and shear strain components given by relations (D.31) and (D.36) respectively as follows. Considering condition, $i = k$, leading to normal strains that is

$$\epsilon_{ik(j)} \Leftrightarrow \epsilon_{ii(j)} = u_{i,i(j)} = \frac{\partial u_{i(j)}}{\partial x'_i} \tag{3.6}$$

and condition $i \neq k$, leading to shear strains that is

$$\gamma_{ik(j)} = u_{i,k(j)} + u_{k,i(j)} = \frac{\partial u_{i(j)}}{\partial x'_k} + \frac{\partial u_{k(j)}}{\partial x'_i} \tag{3.7}$$

Based on array representation of stress and strain components given by relations (3.1) and (3.2) respectively, constitutive equation expressed through relation (A.1) can be expanded for explicit calculation of strain energy density function as follows.

$$\begin{aligned} W_{(j)} &= \frac{1}{2}\sum_{i=1}^{3}\sum_{k=1}^{3} \sigma^T\epsilon = \frac{1}{2}\sum_{i=1}^{3}\sum_{k=1}^{3} \left(C_{ik(j)}\epsilon_{k(j)}\right)\epsilon_{k(j)} \\ &= \sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \sigma_{33}\epsilon_{33} + \sigma_{23}\epsilon_{23} + \sigma_{13}\epsilon_{13} + \sigma_{12}\epsilon_{12} \end{aligned} \tag{3.8}$$

From constitutive relation given by (C.55) for a linearly elastic isotropic material, Cauchy's stress components are explicitly obtained as follows.

$$\begin{aligned}
\sigma_{11} &= \left(\kappa + \frac{4\mu}{3}\right)\epsilon_{11} + \left(\kappa - \frac{2\mu}{3}\right)\epsilon_{22} + \left(\kappa - \frac{2\mu}{3}\right)\epsilon_{33} \\
\sigma_{22} &= \left(\kappa - \frac{2\mu}{3}\right)\epsilon_{11} + \left(\kappa + \frac{4\mu}{3}\right)\epsilon_{22} + \left(\kappa - \frac{2\mu}{3}\right)\epsilon_{33} \\
\sigma_{33} &= \left(\kappa - \frac{2\mu}{3}\right)\epsilon_{11} + \left(\kappa - \frac{2\mu}{3}\right)\epsilon_{22} + \left(\kappa + \frac{4\mu}{3}\right)\epsilon_{33} \\
\sigma_{23} &= \mu\epsilon_{23} \\
\sigma_{13} &= \mu\epsilon_{13} \\
\sigma_{12} &= \mu\epsilon_{12}
\end{aligned} \tag{3.9}$$

Performing calculations in relation (3.8) by using explicit forms of stress components given by relation (3.9) yields to

$$\begin{aligned}
W_{(j)} = &\frac{1}{2}\left(\kappa + \frac{4\mu}{3}\right)(\epsilon_{11}\epsilon_{11} + \epsilon_{22}\epsilon_{22} + \epsilon_{33}\epsilon_{33}) + \frac{1}{2}\left(\kappa - \frac{2\mu}{3}\right)(2\epsilon_{11}\epsilon_{22} + 2\epsilon_{11}\epsilon_{33} + 2\epsilon_{22}\epsilon_{33}) \\
&+ \mu\left(\epsilon_{23}\epsilon_{23} + \epsilon_{13}\epsilon_{13} + \epsilon_{12}\epsilon_{12}\right)
\end{aligned} \tag{3.10}$$

Invoking explicit forms of strain terms given by relations (3.6) and (3.7) into relation (3.10), strain energy density function becomes

$$\begin{aligned}
W_{(j)} = &\frac{1}{2}\left(\kappa + \frac{4\mu}{3}\right)\left(u_{1,1(j)}^2 + u_{2,2(j)}^2 + u_{3,3(j)}^2\right) \\
&+ \left(\kappa - \frac{2\mu}{3}\right)\left(u_{1,1(j)}u_{2,2(j)} + u_{1,1(j)}u_{3,3(j)} + u_{2,2(j)}u_{3,3(j)}\right) \\
&+ \frac{\mu}{2}\left(\left(u_{2,3(j)} + u_{3,2(j)}\right)^2 + \left(u_{1,3(j)} + u_{3,1(j)}\right)^2 + \left(u_{1,2(j)} + u_{2,1(j)}\right)^2\right)
\end{aligned} \tag{3.11}$$

Ordinarily, an arbitrary single variable continues function, $u(x)$, can be expressed based on based on first-degree Taylor expansion as follows.

$$u(x) = \sum_{n=0}^{\infty} \frac{(x - x_i)^n}{n!}\left(\frac{\partial^n u(x)}{\partial x^n}\right) \tag{3.12}$$

Numerically, relation (3.12) can be approximated in a way that original function at points x_{i+1} and x_{i-1} by infinitesimal forward and backward incremental distance, Δx. In other words,

$$u_{i+1}(x_{i+1}) \approx u_i(x_i) + u_i'(x_i)\left(x_{i+1} - x_i\right)/1! \tag{3.13}$$

and

$$u_{i-1}(x_{i-1}) \approx u_i(x_i) + u_i'(x_i)\left(x_{i-1} - x_i\right)/1! \tag{3.14}$$

in which it is possible to write $(x_{i+1} - x_i) = \Delta x$ and $(x_{i-1} - x_i) = -\Delta x$. After multiplying both side of relation (3.14) by Δx and summing relations (3.13) and (3.14) side by side, first-order derivative of function, $u(x)$, through central finite difference is obtained as follows.

$$u_i'(x_i) \approx \frac{u_{i+1}(x_{i+1}) - u_{i-1}(x_{i-1})}{2\Delta x} \tag{3.15}$$

Second-order mixed partial derivatives of an arbitrary function, $u(x, y)$, can also be expressed by means of CFD along first-degree derivatives. Namely,

$$\frac{\partial^2 u_{i,j}(x_i, y_j)}{\partial x_i \partial y_j} \approx \frac{\partial}{\partial x_i}\left(\frac{\partial u_{i,j}(x_i, y_j)}{\partial y_j}\right) = \frac{(\partial u/\partial y_j)_{i+1,j} - (\partial u/\partial y_j)_{i-1,j}}{2\Delta x} \tag{3.16}$$

in which partial differentials with respect to variable y in numerator can also be expressed based on CFD in relation (3.16) as follows.

$$\frac{\partial u_{i+1,j}(x_{i+1}, y_j)}{\partial y_j} \approx \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\Delta y} \tag{3.17}$$

and

$$\frac{\partial u_{i-1,j}(x_{i-1}, y_j)}{\partial y_j} \approx \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\Delta y} \tag{3.18}$$

By invoking discretized equivalences of partial derivatives given by relations (3.17) and (3.18) into relation (3.16), we can expressed second-order mixed partial derivative as follows.

$$\frac{\partial^2 u_{i,j}(x_i, y_j)}{\partial x_i \partial y_j} \approx \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4\Delta x \Delta y} \tag{3.19}$$

In addition to mixed kind partial, second-order derivative u_i'' can be obtained by using forward and backward second-order Taylor expansions around points x_{i+1} and x_{i-1}. In other words, forward and backward second-order Taylor expansions are respectively

$$u_{i+1} = u_i + \frac{1}{1!}\left(\frac{\partial u_i}{\partial x}\right)(x_{i+1} - x_i) + \frac{1}{2!}\left(\frac{\partial^2 u_i}{\partial x^2}\right)(x_{i+1} - x_i)^2 \tag{3.20}$$

and

$$u_{i-1} = u_i + \frac{1}{1!}\left(\frac{\partial u_i}{\partial x}\right)(x_{i-1} - x_i) + \frac{1}{2!}\left(\frac{\partial^2 u_i}{\partial x^2}\right)(x_{i-1} - x_i)^2 \tag{3.21}$$

By summing relations (3.20) and (3.21) side by side and substituting $(x_{i+1} - x_i) = \Delta x$ and $(x_{i-1} - x_i) = -\Delta x$ in resultant line, we obtain second-order partial derivative at point x_i as

$$\frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x}\right) \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \tag{3.22}$$

Applying CFD on first-order partial derivatives in strain energy density function given by (3.11), can be approximated form of the strain energy for material point (j) and expressed as follows.

$$\begin{aligned} W_{(j)} = \frac{1}{2}\left(\kappa + \frac{4\mu}{3}\right)\Bigg(\left(\frac{u_{1(j+l)} - u_{1(j-l)}}{2\Delta x_1}\right)^2 + \left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_2}\right)^2 + \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_3}\right)^2\Bigg) \\ + \left(\kappa - \frac{2\mu}{3}\right)\Bigg(\left(\frac{u_{1(j+l)} - u_{1(j-l)}}{2\Delta x_1}\right)\left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_2}\right) \\ + \left(\frac{u_{1(j+l)} - u_{1(j-l)}}{2\Delta x_1}\right)\left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_3}\right)\Bigg) \\ + \left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_2}\right)\left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_3}\right)\Bigg) \\ + \frac{\mu}{2}\left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_3} + \frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_2}\right)^2 \\ + \frac{\mu}{2}\left(\frac{u_{1(j+l)} - u_{1(j-l)}}{2\Delta x_3} + \frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_1}\right)^2 \\ + \frac{\mu}{2}\left(\frac{u_{1(j+l)} - u_{1(j-l)}}{2\Delta x_2} + \frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_1}\right)^2 \end{aligned} \tag{3.23}$$

in which sub-indices $(j) = 1, 2, 3, 4, 5, 6$ inside brackets stand for material points around particle (j) while sub-indices outside brackets are for co-ordinate directions, (x_1, x_2, x_3) as shown in Figure (3.1). Strain energy density function given by relation (3.23) can be expressed along expansion of squares in bracket of μ, yielding to following form.

$$
\begin{aligned}
W_{(j)} = \frac{1}{2}\left(\kappa + \frac{4\mu}{3}\right) &\left(\left(\frac{u_{1(j+l)} - u_{1(j-l)}}{2\Delta x_1}\right)^2 + \left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_2}\right)^2 + \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_3}\right)^2 \right) \\
&+ \left(\kappa - \frac{2\mu}{3}\right) \left(\left(\frac{u_{1(j+l)} - u_{1(j-l)}}{2\Delta x_1}\right) \left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_2}\right) \right. \\
&+ \left. \left(\frac{u_{1(j+l)} - u_{1(j-l)}}{2\Delta x_1}\right) \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_3}\right) + \left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_2}\right) \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_3}\right) \right) \\
&+ \frac{\mu}{2} \left(\left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_3}\right)^2 + \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_2}\right)^2 \right. \\
&+ \left. 2\left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_3}\right) \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_2}\right) \right) \\
&+ \frac{\mu}{2} \left(\left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_3}\right)^2 + \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_2}\right)^2 \right. \\
&+ \left. 2\left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_3}\right) \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_2}\right) \right) \\
&+ \frac{\mu}{2} \left(\left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_3}\right)^2 + \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_2}\right)^2 \right. \\
&+ \left. 2\left(\frac{u_{2(j+m)} - u_{2(j-m)}}{2\Delta x_3}\right) \left(\frac{u_{3(j+n)} - u_{3(j-n)}}{2\Delta x_2}\right) \right)
\end{aligned}
\tag{3.24}
$$

in which the first sub-indices outside brackets indicate directions of displacement vector, u. As later remarked, strain energy of material particle (j) can be decomposed into its constituents for each interaction in its immediate vicinity as illustrated in Figure (3.1).

3.2 Lagrangian formalism for equation of motion in classical interaction

Interaction of material particles in classical approach is considered as they communicate with others that they are only in their immediate vicinity leading to locality notion. In Figure (3.1), neighbouring particles appearing in green colour around blue-colour RVE are illustrated.

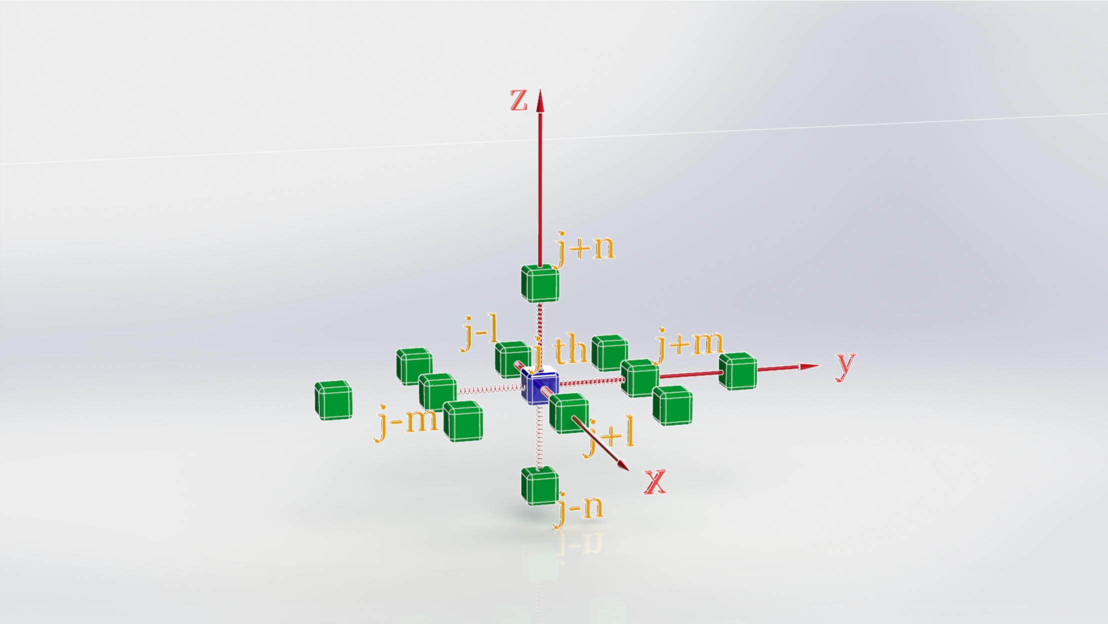


FIGURE 3.1: Local interactions of material particles

As details given in Chapter (4), using Lagrange-Euler equations of motion leads to determination of displacement field for whole material domain. From this perspective, firstly let us write Lagrange-Euler equations of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_{(j)}}\right) - \frac{\partial L}{\partial u_{(j)}} = 0 \tag{3.25}$$

in which $(j) = 1, 2, 3, ..., s$, indicates all particles in material domain and L presents Lagrangian which is given by difference between kinetic an potential energies of solid body, namely, $L = T - U$. Total kinetic and potential energies of body can be expressed as a sum of each particle's kinetic and potential energies therefore relations (3.3) and (3.4) can be expressed based on constant volumetric expansion and density of each RVE as follows.

$$\begin{aligned} T &= \frac{1}{2}\sum_{j=1}^{\infty}\left(\vec{\dot{u}}_{(j)} \cdot \vec{\dot{u}}_{(j)}\right)\rho_{(j)}\Delta V_{(j)} \\ &= \frac{1}{2}\left(\ldots + \dot{u}_{x_1(j)} \cdot \dot{u}_{x_1(j)} + \dot{u}_{x_2(j)} \cdot \dot{u}_{x_2(j)} + \dot{u}_{x_3(j)} \cdot \dot{u}_{x_3(j)} + \ldots\right)\rho_{(j)}\Delta V_{(j)} \end{aligned} \tag{3.26}$$

and

$$U = \sum_{j=1}^{\infty} \frac{1}{2} W_{(j)} \Delta V_{(j)} - \sum_{j=1}^{\infty} \vec{u}_{(j)} \vec{b}_{(j)} \Delta V_{(j)}$$
$$= \frac{1}{2} \left(\ldots + w_{(j)(j+l)} + w_{(j)(j-l)} + w_{(j)(j+m)} + w_{(j)(j-m)} + w_{(j)(j+n)} + w_{(j)(j-n)} + \ldots \right) \Delta V_{(j)}$$
$$+ \left(\ldots + u_{x_1(j)} b_{x_1(j)} + u_{x_2(j)} b_{x_2(j)} + u_{x_3(j)} b_{x_3(j)} + \ldots \right) \Delta V_{(j)} \tag{3.27}$$

As explained while writing relation (4.7), strain energy density function, $W_{(j)}$, is defined along summation of sub-strain energies, $w_{(j)(i)}$ of interacting particles. At that point, classical theory differs from peridynamic approach by including only strain energies in immediate vicinity of particle (j).

In total potential energy relation (3.27), strain energies, $w_{(j)(k+l)}$, $w_{(j)(k-l)}$, $w_{(j)(j+m)}$, $w_{(j)(j-m)}$, $w_{(j)(j+n)}$ and $w_{(j)(j-n)}$ can be expressed in their explicit form similar to (3.23). In strain energy density term, $w_{(j)(i)}$, first sub-index inside bracket indicates blue particle in the middle and second sub-indices symbolises green interaction particles as shown in Figure (3.1). Strain energies of interacting particles, $(j+l)$ and $(j-l)$, with particle (j) along x_1 direction are $w_{(j)(j+l)}$ and $w_{(j)(j-l)}$ that are respectively defined as

$$w_{(j)(j+l)} = \frac{1}{2} \left(\kappa + \frac{4\mu}{3} \right) \left(\frac{u_{1(j+l+1)} - u_{1(j)}}{2\Delta x_1} \right)^2$$
$$+ \left(\kappa - \frac{2\mu}{3} \right) \left(\left(\frac{u_{1(j+l+1)} - u_{1(j)}}{2\Delta x_1} \right) \left(\frac{u_{2(j+l+m)} - u_{2(j+l-m)}}{2\Delta x_2} \right) \right.$$
$$\left. + \left(\frac{u_{1(j+l+1)} - u_{1(j)}}{2\Delta x_1} \right) \left(\frac{u_{3(j+l+n)} - u_{3(j+l-n)}}{2\Delta x_3} \right) \right)$$
$$+ \frac{\mu}{2} \left(\left(\frac{u_{1(j+l+1)} - u_{1(j)}}{2\Delta x_2} \right)^2 + 2 \left(\frac{u_{1(j+l+1)} - u_{1(j)}}{2\Delta x_2} \right) \left(\frac{u_{2(j+l+m)} - u_{2(j+l-m)}}{2\Delta x_1} \right) \right)$$
$$+ \frac{\mu}{2} \left(\left(\frac{u_{1(j+l+1)} - u_{1(j)}}{2\Delta x_3} \right)^2 + 2 \left(\frac{u_{1(j+l+1)} - u_{1(j)}}{2\Delta x_3} \right) \left(\frac{u_{3(j+l+n)} - u_{3(j+l-n)}}{2\Delta x_1} \right) \right) \tag{3.28}$$

and

$$
\begin{aligned}
w_{(j)(j-l)} = & \frac{1}{2}\left(\kappa+\frac{4\mu}{3}\right)\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{2\Delta x_1}\right)^2 \\
& +\left(\kappa-\frac{2\mu}{3}\right)\left(\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{2\Delta x_1}\right)\left(\frac{u_{2(j-l+m)}-u_{2(j-l-m)}}{2\Delta x_2}\right)\right. \\
& \left.+\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{2\Delta x_1}\right)\left(\frac{u_{3(j-l+n)}-u_{3(j-l-n)}}{2\Delta x_3}\right)\right) \\
& +\frac{\mu}{2}\left(\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{2\Delta x_2}\right)^2+2\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{2\Delta x_2}\right)\left(\frac{u_{2(j-l+m)}-u_{2(j-l-m)}}{2\Delta x_1}\right)\right) \\
& +\frac{\mu}{2}\left(\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{2\Delta x_3}\right)^2+2\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{2\Delta x_3}\right)\left(\frac{u_{3(j-l+n)}-u_{3(j-l-n)}}{2\Delta x_1}\right)\right)
\end{aligned}
\tag{3.29}
$$

in which second-order powers and multiplications of partial differentials are negligibly small compared to first orders and its second-order power along (x_1) direction, namely,

$$\left(u_{2,2(j\mp l)}\right)^2 \ll 1 \text{ and } \left(u_{3,3(j\mp l)}\right)^2 \ll 1$$

$$\left(u_{2,3(j\mp l)}\right)^2 \ll 1 \text{ and } \left(u_{3,2(j\mp l)}\right)^2 \ll 1$$

and

$$(u_{2(j+l+m)}-u_{2(j+l-m)})/(2\Delta x_2)(u_{3(j+l+n)}-u_{3(j+l-n)})/(2\Delta x_3) \ll 1$$

$$(u_{3(j+l+n)}-u_{3(j+l-n)})/(2\Delta x_2)(u_{2(j+l+m)}-u_{2(j+l-m)})/(2\Delta x_3) \ll 1$$

following assumptions are considered as valid,

$$\left(u_{2,2(j\mp l)}\right)^2 \approx 0 \text{ and } \left(u_{3,3(j\mp l)}\right)^2 \approx 0$$

$$\left(u_{2,3(j\mp l)}\right)^2 \approx 0 \text{ and } \left(u_{3,2(j\mp l)}\right)^2 \approx 0$$

and

$$(u_{2(j+l+m)} - u_{1(j+l-m)})/(2\Delta x_2)(u_{1(j+l+n)} - u_{1(j+l-n)})/(2\Delta x_3) \approx 0$$

$$(u_{2(j+l+n)} - u_{2(j+l-n)})/(2\Delta x_2)(u_{3(j+l+n)} - u_{3(j+l-n)})/(2\Delta x_3) \approx 0$$

By means of assumptions made above, similarly, corresponding strain energies of material particles, $(j+m)$ and $(j-m)$, along (x_2) direction, $w_{(j)(j+m)}$ and $w_{(j)(j-m)}$, are written respectively as

$$\begin{aligned} w_{(j)(j+m)} = &\frac{1}{2}\left(\kappa + \frac{4\mu}{3}\right)\left(\frac{u_{2(j+m+1)} - u_{2(j)}}{2\Delta x_2}\right)^2 \\ &+ \left(\kappa - \frac{2\mu}{3}\right)\left(\left(\frac{u_{1(j+m+l)} - u_{1(j+m-l)}}{2\Delta x_1}\right)\left(\frac{u_{2(j+m+1)} - u_{2(j)}}{2\Delta x_2}\right)\right. \\ &+ \left.\left(\frac{u_{2(j+m+1)} - u_{2(j+m-1)}}{2\Delta x_2}\right)\left(\frac{u_{3(j+m+n)} - u_{3(j+m-n)}}{2\Delta x_3}\right)\right) \\ &+ \frac{\mu}{2}\left(\left(\frac{u_{2(j+m+1)} - u_{2(j)}}{2\Delta x_1}\right)^2 + 2\left(\frac{u_{2(j+m+1)} - u_{2(j)}}{2\Delta x_1}\right)\left(\frac{u_{1(j+m+l)} - u_{1(j+m-l)}}{2\Delta x_2}\right)\right) \\ &+ \frac{\mu}{2}\left(\left(\frac{u_{2(j+m+1)} - u_{2(j)}}{2\Delta x_3}\right)^2 + 2\left(\frac{u_{2(j+m+1)} - u_{2(j)}}{2\Delta x_3}\right)\left(\frac{u_{1(j+m+n)} - u_{1(j+m-n)}}{2\Delta x_2}\right)\right) \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} w_{(j)(j-m)} = &\frac{1}{2}\left(\kappa + \frac{4\mu}{3}\right)\left(\frac{u_{2(j)} - u_{2(j-m-1)}}{2\Delta x_2}\right)^2 \\ &+ \left(\kappa - \frac{2\mu}{3}\right)\left(\left(\frac{u_{1(j-m+l)} - u_{1(j-m-l)}}{2\Delta x_1}\right)\left(\frac{u_{2(j)} - u_{2(j-m-1)}}{2\Delta x_2}\right)\right. \\ &+ \left.\left(\frac{u_{2(j)} - u_{2(j-m-1)}}{2\Delta x_2}\right)\left(\frac{u_{1(j-m+n)} - u_{1(j-m-n)}}{2\Delta x_3}\right)\right) \\ &+ \frac{\mu}{2}\left(\left(\frac{u_{2(j)} - u_{2(j-m-1)}}{2\Delta x_1}\right)^2 + 2\left(\frac{u_{2(j)} - u_{2(j-m-1)}}{2\Delta x_1}\right)\left(\frac{u_{1(j-m+l)} - u_{1(j-m-l)}}{2\Delta x_2}\right)\right) \\ &+ \frac{\mu}{2}\left(\left(\frac{u_{2(j)} - u_{2(j-m-1)}}{2\Delta x_3}\right)^2 + 2\left(\frac{u_{2(j)} - u_{2(j-m-1)}}{2\Delta x_3}\right)\left(\frac{u_{1(j-m+n)} - u_{1(j-m-n)}}{2\Delta x_2}\right)\right) \end{aligned} \tag{3.31}$$

In immediate vicinity of particle (j), strain energies of material points along (x_3) direction, $w_{(j)(j+n)}$ and $w_{(j)(j-n)}$, are given respectively as

$$\begin{aligned}
w_{(j)(j+n)} &= \frac{1}{2}\left(\kappa+\frac{4\mu}{3}\right)\left(\frac{u_{3(j+n+1)}-u_{3(j)}}{2\Delta x_3}\right)^2 \\
&+\left(\kappa-\frac{2\mu}{3}\right)\left(\left(\frac{u_{1(j+n+l)}-u_{1(j+n-l)}}{2\Delta x_1}\right)\left(\frac{u_{3(j+n+1)}-u_{3(j)}}{2\Delta x_3}\right)\right. \\
&+\left.\left(\frac{u_{2(j+n+m)}-u_{2(j+n-m)}}{2\Delta x_2}\right)\left(\frac{u_{3(j+n+1)}-u_{3(j)}}{2\Delta x_3}\right)\right) \\
&+\frac{\mu}{2}\left(\left(\frac{u_{3(j+n+1)}-u_{3(j)}}{2\Delta x_1}\right)^2+2\left(\frac{u_{3(j+n+1)}-u_{3(j)}}{2\Delta x_1}\right)\left(\frac{u_{1(j+n+l)}-u_{1(j+n-l)}}{2\Delta x_3}\right)\right) \\
&+\frac{\mu}{2}\left(\left(\frac{u_{3(j+n+1)}-u_{3(j)}}{2\Delta x_2}\right)^2+2\left(\frac{u_{3(j+n+1)}-u_{3(j)}}{2\Delta x_2}\right)\left(\frac{u_{2(j+n+m)}-u_{2(j+n-m)}}{2\Delta x_3}\right)\right)
\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}
w_{(j)(j-n)} &= \frac{1}{2}\left(\kappa+\frac{4\mu}{3}\right)\left(\frac{u_{3(j)}-u_{3(j-n-1)}}{2\Delta x_3}\right)^2 \\
&+\left(\kappa-\frac{2\mu}{3}\right)\left(\left(\frac{u_{1(j-n+l)}-u_{1(j-n-l)}}{2\Delta x_1}\right)\left(\frac{u_{3(j)}-u_{3(j-n-1)}}{2\Delta x_3}\right)\right. \\
&+\left.\left(\frac{u_{2(j-n+m)}-u_{2(j-n-m)}}{2\Delta x_2}\right)\left(\frac{u_{3(j)}-u_{3(j-n-1)}}{2\Delta x_3}\right)\right) \\
&+\frac{\mu}{2}\left(\left(\frac{u_{3(j)}-u_{3(j-n-1)}}{2\Delta x_1}\right)^2+2\left(\frac{u_{3(j)}-u_{3(j-n-1)}}{2\Delta x_1}\right)\left(\frac{u_{1(j-n+l)}-u_{1(j-n-l)}}{2\Delta x_3}\right)\right) \\
&+\frac{\mu}{2}\left(\left(\frac{u_{3(j)}-u_{3(j-n-1)}}{2\Delta x_2}\right)^2+2\left(\frac{u_{3(j)}-u_{3(j-n-1)}}{2\Delta x_2}\right)\left(\frac{u_{2(j-n+m)}-u_{2(j-n-m)}}{2\Delta x_3}\right)\right)
\end{aligned} \tag{3.33}$$

Substitution of six terms for micro-potentials obtained along relations (3.28), (3.29), (3.30), (3.31), (3.32) and (3.33) into relation (3.27) leads to determination of motion for each material point, (j), by means of Lagrange-Euler equations of motion given by relation (3.25) for each co-ordinate direction, (x_1), (x_2) and (x_3).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_{x_1(j)}}\right)-\frac{\partial L}{\partial u_{x_1(j)}}=0 \tag{3.34}$$

or in explicit form

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{u}_{x_1(j)}}\left(\frac{1}{2}\left(\dot{u}_{x_1(j)}\cdot\dot{u}_{x_1(j)}\right)\rho_{(j)}\Delta V_{(j)}\right)\right)$$
$$-\frac{1}{2}\left(\frac{\partial}{\partial u_{x_1(j)}}\left(\ldots+w_{(j+l)}+w_{(j-l)}+w_{(j+m)}+w_{(j-m)}+w_{(j+n)}+w_{(j-n)}+\ldots\right)\Delta V_{(j)}\right)V_{(j)}$$
$$-\frac{\partial}{\partial u_{x_1(j)}}\left(\left(\ldots+u_{x_1(j)}b_{x_1(j)}+u_{x_2(j)}b_{x_2(j)}+u_{x_3(j)}b_{x_3(j)}+\ldots\right)\Delta V_{(j)}\right)=0 \tag{3.35}$$

or

$$\ddot{u}_{x_1(j)}\rho_{(j)}-\frac{1}{2}\left(\frac{\partial w_{(j+l)}}{\partial u_{x_1(j)}}+\frac{\partial w_{(j-l)}}{\partial u_{x_1(j)}}\right)V_{(j)}-b_{x_1(j)}\frac{\partial u_{x_1(j)}}{\partial u_{x_1(j)}}=0 \tag{3.36}$$

Similarly, equations of motion for other principle directions, x_2 and x_3 can be written respectively as follows.

$$\ddot{u}_{x_2(j)}\rho_{(j)}-\frac{1}{2}\left(\frac{\partial w_{(j+m)}}{\partial u_{x_2(j)}}+\frac{\partial w_{(j-m)}}{\partial u_{x_2(j)}}\right)V_{(j)}-b_{x_2(j)}\frac{\partial u_{x_2(j)}}{\partial u_{x_2(j)}}=0 \tag{3.37}$$

and

$$\ddot{u}_{x_3(j)}\rho_{(j)}-\frac{1}{2}\left(\frac{\partial w_{(j+n)}}{\partial u_{x_3(j)}}+\frac{\partial w_{(j-n)}}{\partial u_{x_3(j)}}\right)V_{(j)}-b_{x_3(j)}\frac{\partial u_{x_3(j)}}{\partial u_{x_3(j)}}=0 \tag{3.38}$$

Before associated substitutions, let us treat partial differentiations of local micro-potentials given through relations (3.28) and (3.29) only and substitute them in equation of motion given by relation (3.36). Additionally, remaining partial differentiations along other directions (x_2) and (x_3) are performed in the same way and substituted in (3.37) and (3.38) respectively. In this manner, (3.28) can be expressed as follows.

$$\begin{aligned}\frac{\partial w_{(j)(j+l)}}{\partial u_{x_1(j)}}=&-\left(\kappa+\frac{4\mu}{3}\right)\left(\frac{u_{1(j+l+1)}-u_{1(j)}}{4\left(\Delta x_1\right)^2}\right)\\&-\frac{1}{2\Delta x_1}\left(\kappa-\frac{2\mu}{3}\right)\left(\left(\frac{u_{2(j+l+m)}-u_{2(j+l-m)}}{2\Delta x_2}\right)+\left(\frac{u_{3(j+l+n)}-u_{3(j+l-n)}}{2\Delta x_3}\right)\right)\\&+\frac{\mu}{2}\left(-2\left(\frac{u_{1(j+l+1)}-u_{1(j)}}{4\left(\Delta x_2\right)^2}\right)-\frac{2}{2\Delta x_2}\left(\frac{u_{2(j+l+m)}-u_{2(j+l-m)}}{2\Delta x_1}\right)\right)\\&+\frac{\mu}{2}\left(-2\left(\frac{u_{1(j+l+1)}-u_{1(j)}}{4\left(\Delta x_3\right)^2}\right)-2\frac{2}{2\Delta x_3}\left(\frac{u_{3(j+l+n)}-u_{3(j+l-n)}}{2\Delta x_1}\right)\right)\end{aligned} \tag{3.39}$$

or in a more compact form

$$\begin{aligned}\frac{\partial w_{(j)(j+l)}}{\partial u_{x1(j)}} =& -\left(\kappa+\frac{4\mu}{3}\right)\left(\frac{u_{1(j+l+1)}-u_{1(j)}}{4\left(\Delta x_1\right)^2}\right)\\ &+\left(\kappa-\frac{2\mu}{3}\right)\left(-\frac{u_{2(j+l+m)}-u_{2(j+l-m)}}{4\Delta x_1\Delta x_2}-\frac{u_{3(j+l+n)}-u_{3(j+l-n)}}{4\Delta x_1\Delta x_3}\right)\\ &+\mu\left(-\frac{u_{1(j+l+1)}-u_{1(j)}}{4\left(\Delta x_2\right)^2}-\frac{u_{2(j+l+m)}-u_{2(j+l-m)}}{4\Delta x_1\Delta x_2}\right)\\ &+\mu\left(-\frac{u_{1(j+l+1)}-u_{1(j)}}{4\left(\Delta x_3\right)^2}-\frac{u_{3(j+l+n)}-u_{3(j+l-n)}}{4\Delta x_1\Delta x_3}\right)\end{aligned} \tag{3.40}$$

By combining all terms, having the same denominators, local micro-potential energy, $w_{(j)(j+l)}$, belonging to material particle at co-ordinate designated by $(j+l)$ becomes

$$\begin{aligned}\frac{\partial w_{(j)(j+l)}}{\partial u_{x1(j)}} =& -\left(\kappa+\frac{4\mu}{3}\right)\left(\frac{u_{1(j+l+1)}-u_{1(j)}}{4\left(\Delta x_1\right)^2}\right)\\ &-\left(\kappa+\frac{\mu}{3}\right)\left(\frac{u_{2(j+l+m)}-u_{2(j+l-m)}}{4\Delta x_1\Delta x_2}+\frac{u_{3(j+l+n)}-u_{3(j+l-n)}}{4\Delta x_1\Delta x_3}\right)\\ &-\mu\left(\frac{u_{1(j+l+1)}-u_{1(j)}}{4\left(\Delta x_2\right)^2}+\frac{u_{1(j+l+1)}-u_{1(j)}}{4\left(\Delta x_3\right)^2}\right)\end{aligned} \tag{3.41}$$

Differentiation of other coupling local micro-potential along (x_1) direction is performed based on (3.29) and expressed as follows.

$$\begin{aligned}\frac{\partial w_{(j)(j-l)}}{\partial u_{x1(j)}} =& \left(\kappa+\frac{4\mu}{3}\right)\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{4\left(\Delta x_1\right)^2}\right)\\ &+\frac{1}{2\Delta x_1}\left(\kappa-\frac{2\mu}{3}\right)\left(\frac{u_{2(j-l+m)}-u_{2(j-l-m)}}{2\Delta x_2}+\frac{u_{3(j-l+n)}-u_{3(j-l-n)}}{2\Delta x_3}\right)\\ &+\frac{\mu}{2}\left(2\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{4\left(\Delta x_2\right)^2}\right)+\frac{2}{2\Delta x_2}\left(\frac{u_{2(j-l+m)}-u_{2(j-l-m)}}{2\Delta x_1}\right)\right)\\ &+\frac{\mu}{2}\left(2\left(\frac{u_{1(j)}-u_{1(j-l-1)}}{4\left(\Delta x_3\right)^2}\right)+\frac{2}{2\Delta x_3}\left(\frac{u_{3(j-l+n)}-u_{3(j-l-n)}}{2\Delta x_1}\right)\right)\end{aligned} \tag{3.42}$$

As similar to procedure that is performed while obtaining relation (3.40), expression given by (3.42) can be treated as follows.

$$\begin{aligned}\frac{\partial w_{(j)(j-l)}}{\partial u_{x_1(j)}} &= \left(\kappa + \frac{4\mu}{3}\right)\left(\frac{u_{1(j)} - u_{1(j-l-1)}}{4\left(\Delta x_1\right)^2}\right) \\ &+ \left(\kappa - \frac{2\mu}{3}\right)\left(\frac{u_{2(j-l+m)} - u_{2(j-l-m)}}{4\Delta x_1 \Delta x_2} + \frac{u_{3(j-l+n)} - u_{3(j-l-n)}}{4\Delta x_1 \Delta x_3}\right) \\ &+ \mu\left(\frac{u_{1(j)} - u_{1(j-l-1)}}{4\left(\Delta x_2\right)^2} + \frac{u_{2(j-l+m)} - u_{2(j-l-m)}}{4\Delta x_1 \Delta x_2}\right) \\ &+ \mu\left(\frac{u_{1(j)} - u_{1(j-l-1)}}{4\left(\Delta x_3\right)^2} + \frac{u_{3(j-l+n)} - u_{3(j-l-n)}}{4\Delta x_1 \Delta x_3}\right)\end{aligned} \tag{3.43}$$

By rearranging all terms appearing in the right-hand side of relation (3.43), local micro-potential energy, $w_{(j)(j-l)}$, belonging to material particle at co-ordinate designated by $(j-l)$ becomes

$$\begin{aligned}\frac{\partial w_{(j)(j-l)}}{\partial u_{x_1(j)}} &= \left(\kappa + \frac{4\mu}{3}\right)\left(\frac{u_{1(j)} - u_{1(j-l-1)}}{4\left(\Delta x_1\right)^2}\right) \\ &+ \left(\kappa + \frac{\mu}{3}\right)\left(\frac{u_{2(j-l+m)} - u_{2(j-l-m)}}{4\Delta x_1 \Delta x_2} + \frac{u_{3(j-l+n)} - u_{3(j-l-n)}}{4\Delta x_1 \Delta x_3}\right) \\ &+ \mu\left(\frac{u_{1(j)} - u_{1(j-l-1)}}{4\left(\Delta x_2\right)^2} + \frac{u_{1(j)} - u_{1(j-l-1)}}{4\left(\Delta x_3\right)^2}\right)\end{aligned} \tag{3.44}$$

Sum of two partial derivatives, $\partial w_{(j)(j+l)}/\partial u_{x_1(j)}$ and $\partial w_{(j)(j-l)}/\partial u_{x_1(j)}$ is obtained as follows.

$$\begin{aligned}
\frac{\partial w_{(j)(j+l)}}{\partial u_{x_1(j)}} + \frac{\partial w_{(j)(j-l)}}{\partial u_{x_1(j)}} = &- \left(\kappa + \frac{\mu}{3}\right)\left(\frac{u_{1(j-l-1)} - 2u_{1(j)} + u_{1(j+l+1)}}{4\left(\Delta x_1\right)^2}\right) \\
&- \left(\kappa + \frac{\mu}{3}\right)\left(\frac{-u_{2(j-l+m)} + u_{2(j-l-m)} + u_{2(j+l+m)} - u_{2(j+l-m)}}{4\Delta x_1 \Delta x_2}\right) \\
&- \left(\kappa + \frac{\mu}{3}\right)\left(\frac{-u_{3(j-l+n)} + u_{3(j-l-n)} + u_{3(j+l+n)} - u_{3(j+l-n)}}{4\Delta x_1 \Delta x_3}\right) \\
&- \mu\left(\frac{u_{1(j-l-1)} - 2u_{1(j)} + u_{1(j+l+1)}}{4\left(\Delta x_1\right)^2}\right) \\
&- \mu\left(\frac{u_{1(j-l-1)} - 2u_{1(j)} + u_{1(j+l+1)}}{4\left(\Delta x_2\right)^2}\right) \\
&- \mu\left(\frac{u_{1(j-l-1)} - 2u_{1(j)} + u_{1(j+l+1)}}{4\left(\Delta x_3\right)^2}\right)
\end{aligned} \tag{3.45}$$

Substitution of equivalent terms in discrete form in accordance with relations (3.19) and (3.22) in the right-hand side of relation (3.45) leads to the following.

$$\begin{aligned}
\frac{\partial w_{(j)(j+l)}}{\partial u_{x_1(j)}} + \frac{\partial w_{(j)(j-l)}}{\partial u_{x_1(j)}} = &- \left(\kappa + \frac{\mu}{3}\right)\left(u_{(j)1,11} + u_{(j)2,21} + u_{(j)3,31}\right) \\
&- \mu\left(u_{(j)1,11} + u_{(j)1,22} + u_{(j)1,33}\right)
\end{aligned} \tag{3.46}$$

Additionally, summation of local micro-potentials energies, $w_{(j)(j+m)}$, $w_{(j)(j-m)}$ and $w_{(j)(j+n)}$, $w_{(j)(j-n)}$, can be obtained in a similar way of relation (3.46) and written respectively as follows.

$$\begin{aligned}
\frac{\partial w_{(j)(j+m)}}{\partial u_{x_1(j)}} + \frac{\partial w_{(j)(j-m)}}{\partial u_{x_1(j)}} = &- \left(\kappa + \frac{\mu}{3}\right)\left(u_{(j)1,12} + u_{(j)2,22} + u_{(j)3,32}\right) \\
&- \mu\left(u_{(j)2,11} + u_{(j)2,22} + u_{(j)2,33}\right)
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
\frac{\partial w_{(j)(j+n)}}{\partial u_{x_1(j)}} + \frac{\partial w_{(j)(j-n)}}{\partial u_{x_1(j)}} = &- \left(\kappa + \frac{\mu}{3}\right)\left(u_{(j)1,13} + u_{(j)2,23} + u_{(j)3,33}\right) \\
&- \mu\left(u_{(j)3,11} + u_{(j)3,22} + u_{(j)3,33}\right)
\end{aligned} \tag{3.48}$$

3.3 Equations of motion in classical theory

The last three equations given through relations (3.46), (3.47) and (3.48) can be invoked in (3.36), (3.37) and (3.38) respectively hence equations of motion can be expressed as follows.

$$\ddot{u}_{\alpha(j)}\rho_{(j)} = -\frac{1}{2}\Bigg(\left(\kappa + \frac{\mu}{3}\right)\left(u_{(j)1,1\alpha} + u_{(j)2,2\alpha} + u_{(j)3,3\alpha}\right) \\ + \mu\left(u_{(j)\alpha,11} + u_{(j)\alpha,22} + u_{(j)\alpha,33}\right)\Bigg)V_{(j)} + b_{\alpha(j)} \tag{3.49}$$

or in a more compact form, equations of motion can be expressed as

$$\ddot{u}_{\alpha(j)}\rho_{(j)} = -\frac{1}{2}\Bigg(\left(\kappa + \frac{\mu}{3}\right)\left(u_{(j)\beta,\beta\alpha}\right) + \mu\left(u_{(j)\alpha,\beta\beta}\right)\Bigg)V_{(j)} + b_{\alpha(j)} \tag{3.50}$$

in which sub-index, β, that is repeated, implies a summation over co-ordinates, (x_1), (x_2) and (x_3) while α term stands for free index.

Instead of using displacement related terms, equations of motion can be expressed as a functions of associated stress components in accordance with relations (C.53) and (D.25). To this end, relation (2.28) is achieved as follows.

$$\rho_{\alpha(j)}\ddot{u}_{\alpha(j)} - \left(\sigma_{(j)\alpha x_1,x_1} + \sigma_{(j)\alpha x_2,x_2} + \sigma_{(j)\alpha x_3,x_3}\right) - \rho_{\alpha(j)}b_{\alpha(j)} = 0 \tag{3.51}$$

or

$$\rho_{\alpha(j)}\ddot{u}_{\alpha(j)} = \frac{1}{2}\left(\frac{\Delta\sigma_{(j)\alpha x_1}}{\Delta x_1} + \frac{\Delta\sigma_{(j)\alpha x_2}}{\Delta x_2} + \frac{\Delta\sigma_{(j)\alpha x_3}}{\Delta x_3}\right) + \rho_{\alpha(j)}b_{\alpha(j)} \tag{3.52}$$

Relation (3.52) can be written in a discrete from based on central finite difference method given by relation (3.15) thereby three components of equations of motion can be expressed as follows.

$$\rho_{x_1(j)}\ddot{u}_{x_1(j)} = \frac{\sigma_{(j+l)x_1x_1} - \sigma_{(j-l)x_1x_1}}{2\Delta x_1} + \frac{\sigma_{(j+m)x_1x_2} - \sigma_{(j-m)x_1x_2}}{2\Delta x_2} + \frac{\sigma_{(j+n)x_1x_3} - \sigma_{(j-n)x_1x_3}}{2\Delta x_3} \\ + \rho_{x_1(j)}b_{x_1(j)} \tag{3.53}$$

$$\rho_{x_2(j)}\ddot{u}_{x_2(j)} = \frac{\sigma_{(j+l)x_2x_1} - \sigma_{(j-l)x_2x_1}}{2\Delta x_1} + \frac{\sigma_{(j+m)x_2x_2} - \sigma_{(j-m)x_2x_2}}{2\Delta x_2} + \frac{\sigma_{(j+n)x_2x_3} - \sigma_{(j-n)x_2x_3}}{2\Delta x_3} \\ + \rho_{x_2(j)}b_{\alpha(j)} \tag{3.54}$$

$$\rho_{x_3(j)}\ddot{u}_{x_3(j)} = \frac{\sigma_{(j+l)x_3x_1} - \sigma_{(j-l)x_3x_1}}{2\Delta x_1} + \frac{\sigma_{(j+m)x_3x_2} - \sigma_{(j-m)x_3x_2}}{2\Delta x_2} + \frac{\sigma_{(j+n)x_3x_3} - \sigma_{(j-n)x_3x_3}}{2\Delta x_3} + \rho_{x_3(j)}b_{\alpha(j)} \tag{3.55}$$

in which sub-indices of stress tensor, σ, indicates local neighbours of material point labelled by (j). Adding and subtracting terms that are $\sigma_{(j)x_1x_1}$, $\sigma_{(j)x_2x_2}$ and $\sigma_{(j)x_3x_3}$ seen in numerators of relations (3.53), (3.54) and (3.55) respectively enable us to rewrite equations of motion as follows.

$$\begin{aligned}\rho_{x_1(j)}\ddot{u}_{x_1(j)} = &\left(\frac{\sigma_{(j)x_1x_1} - \sigma_{(j-l)x_1x_1}}{2\Delta x_1}\right) + \left(\frac{\sigma_{(j+l)x_1x_1} - \sigma_{(j)x_1x_1}}{2\Delta x_1}\right) \\ &+ \left(\frac{\sigma_{(j)x_1x_1} - \sigma_{(j-m)x_1x_2}}{2\Delta x_2}\right) + \left(\frac{\sigma_{(j+m)x_1x_2} - \sigma_{(j)x_1x_1}}{2\Delta x_2}\right) \\ &+ \left(\frac{\sigma_{(j)x_1x_1} - \sigma_{(j-n)x_1x_3}}{2\Delta x_3}\right) + \left(\frac{\sigma_{(j+n)x_1x_3} - \sigma_{(j)x_1x_1}}{2\Delta x_3}\right) + \rho_{x_1(j)}b_{x_1(j)}\end{aligned} \tag{3.56}$$

$$\begin{aligned}\rho_{x_2(j)}\ddot{u}_{x_2(j)} = &\left(\frac{\sigma_{(j)x_2x_2} - \sigma_{(j-l)x_2x_1}}{2\Delta x_1}\right) + \left(\frac{\sigma_{(j+l)x_2x_1} - \sigma_{(j)x_2x_2}}{2\Delta x_1}\right) \\ &+ \left(\frac{\sigma_{(j)x_2x_2} - \sigma_{(j-m)x_2x_2}}{2\Delta x_2}\right) + \left(\frac{\sigma_{(j+m)x_2x_2} - \sigma_{(j)x_2x_2}}{2\Delta x_2}\right) \\ &+ \left(\frac{\sigma_{(j)x_2x_2} - \sigma_{(j-n)x_2x_3}}{2\Delta x_3}\right) + \left(\frac{\sigma_{(j+n)x_2x_3} - \sigma_{(j)x_2x_2}}{2\Delta x_3}\right) + \rho_{x_2(j)}b_{x_2(j)}\end{aligned} \tag{3.57}$$

$$\begin{aligned}\rho_{x_3(j)}\ddot{u}_{x_3(j)} = &\left(\frac{\sigma_{(j)x_3x_3} - \sigma_{(j-l)x_3x_1}}{2\Delta x_1}\right) + \left(\frac{\sigma_{(j+l)x_3x_1} - \sigma_{(j)x_3x_3}}{2\Delta x_1}\right) \\ &+ \left(\frac{\sigma_{(j)x_3x_3} - \sigma_{(j-m)x_3x_2}}{2\Delta x_2}\right) + \left(\frac{\sigma_{(j+m)x_3x_2} - \sigma_{(j)x_3x_3}}{2\Delta x_2}\right) \\ &+ \left(\frac{\sigma_{(j)x_3x_3} - \sigma_{(j-n)x_3x_3}}{2\Delta x_3}\right) + \left(\frac{\sigma_{(j+n)x_3x_3} - \sigma_{(j)x_3x_3}}{2\Delta x_3}\right) + \rho_{x_3(j)}b_{x_3(j)}\end{aligned} \tag{3.58}$$

Chapter 4

Fundamentals of Peridynamic Modelling

4.1 Introduction

As a result of particle interaction, emerging potential energy on an imaginary bond between interacting particles is attributed to deformation of that bond. This potential energy caused by restoring forces between interacting particles in deformed state of a body is called micro-potential, being strain energy of a scalar valued function, $w_{(i)(j)}$, [22].

In peridynamic theory, every particle defined on a body interact with its surrounding particles located on a spherical region as illustrated in Figure (4.1). Boundary of this region which is called horizon of subjected material point painted with red colour in Figure (4.1) is determined by a radius, δ named as horizon. From non-local approaches' point of view, locality is determined by size of horizon.

Within the realm of particle interaction on a body, all neighbouring material points denoted by sub-index (j) communicate with the subjected particle (i) that are illustrated by blue and red colour balls respectively in Figure (4.1).

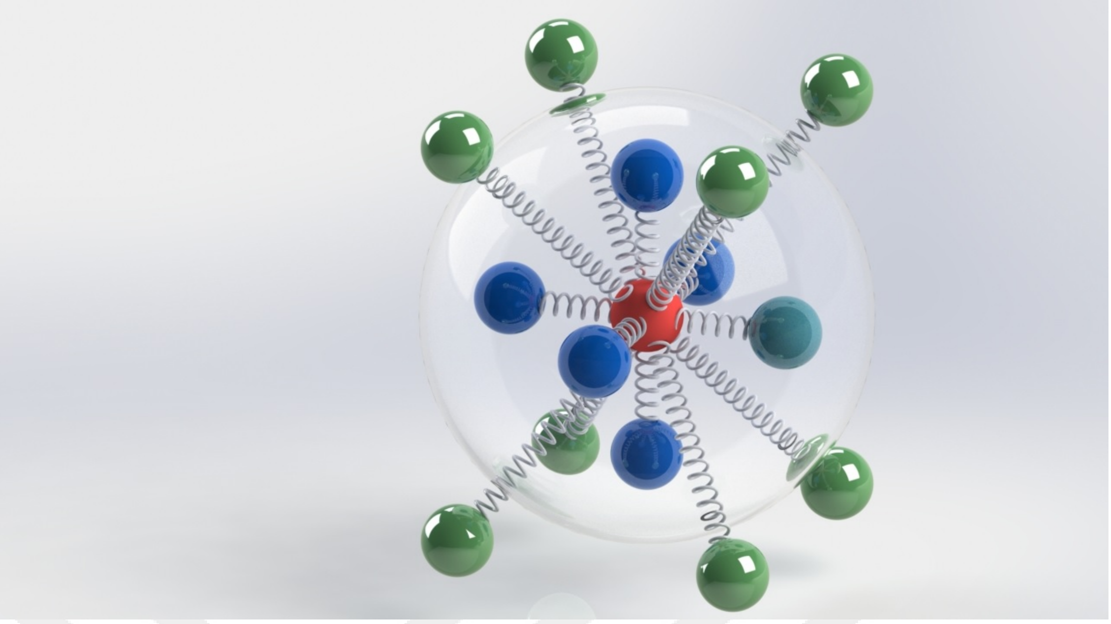


FIGURE 4.1: Non-Local interactions of material particles

As illustrated in Figure (4.2), micro-potential energy that each particle has does not have to be necessarily same since horizon of each subjected material point denoted by (j) are different.

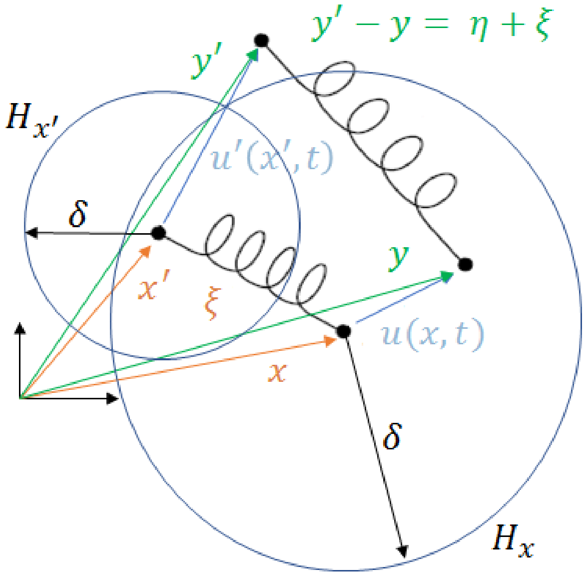


FIGURE 4.2: Peridynamic deformation vectors and particle horizons

Material points in a body are presented by position vectors in both initial and deformed states for which orange and green vectors are used in Figure (4.2).

Mathematically, the prescription that distinguishes micro-potential energies of particles (i) and (j) is stated as follows.

$$w_{(i)(j)} \neq w_{(j)(i)} \tag{4.1}$$

Beside micro-potentials that each particle has alters depending on displacement vector between interacting particles, it is defined as a function of relative position vector, $\xi = x_{(j)} - x_{(i)}$, in reference configuration as well because stretch state of imaginary bond at initial configuration contributes micro-potential energy as well.

To this end, micro-potential energy is written as a function of both relative position vectors $\eta = y_{(j)} - y_{(i)}$ and $\xi = x_{(j)} - x_{(i)}$ respectively in deformed and reference configurations of a body as follows, [25].

$$w_{(i)(j)} = w_{(i)(j)}(u_{(1)}, u_{(2)}, u_{(3)}, ..., x_{(1)}, x_{(2)}, x_{(3)}, ...) \tag{4.2}$$

which indicates micro-potential energy on particle (i) that is caused by surrounding particles (j)s On the other hand, micro-potential energy is expressed with respect to particle (j) as follows.

$$w_{(j)(i)} = w_{(j)(i)}(u_{(1)}, u_{(2)}, u_{(3)}, ..., x_{(1)}, x_{(2)}, x_{(3)}, ...). \tag{4.3}$$

in which $u_{(j)}$s are vectorial differences of position vectors that are $u_{(i)} = y_{(1)} - x_{(i)}, y_{(2)} - x_{(i)}, y_{(3)} - x_{(i)}, ...$ including all relative displacement vectors within the horizon of particle (i).

Relative position vectors in reference configuration are expressed as $\xi_{(i)(j)} = x_{(j)} - x_{(i)}$ including all associated relative position vectors that can be explicitly listed as $(x_{(1)} - x_{(i)}), (x_{(2)} - x_{(i)}), (x_{(3)} - x_{(i)}), ..., (x_{(n)} - x_{(i)})$ with respect to particle labelled by $x_{(i)}$. On the other hand, relative displacement vector in deformed state of body, $y_{(j)} - y_{(i)}$ defines deformed state of a bond between each pair of particle.

The micro-potential function can be also presented as a term of relative deformations since total deformation can be written as $y_{(i)} = x_{(i)} + u_{(i)}$ and $y_{(j)} = x_{(j)} + u_{(j)}$, [22]. To this end, expression (4.2) and (4.3) can be alternatively written as follows.

$$w_{(i)(j)} = w_{(i)(j)}(y_{(1)}, y_{(2)}, y_{(3)}, ...) \tag{4.4}$$

and

$$w_{(j)(i)} = w_{(j)(i)}(y_{(1)}, y_{(2)}, y_{(3)}, ...). \quad (4.5)$$

Alternatively relative position vectors can be expressed as $\xi = x' - x$ and $\eta = y' - y$ in undeformed and deformed configurations respectively.

Due to prescribed body forces on particle (i) caused by a potential e.g. gravitational field and restoring forces on bond connecting particle pairs, total potential energy upon particle (i) can be written as a sum of both and expressed as follows.

$$U = \sum_{i=1}^{\infty} W_{(i)} V_{(i)} - \sum_{i=1}^{\infty} \vec{u}_{(i)} \vec{b}_{(i)} V_{(i)} \quad (4.6)$$

in which strain energy density, $W_{(i)}$ accumulated on particle (i) is given by a summation of micro-potentials arising from interactions of neighbour particles denoted by (j) with subjected particle (i).

Moreover, strain energy density on pair of particles with subscripts both (i) and (j) is considered as half of that micro-potential energy on the bond, [33]. Therefore, strain energy density of particle labelled by (i) and having neighbouring particles with volume $V_{(j)}$ is given by

$$W_{(i)} = \frac{1}{2} \sum_{j=1}^{\infty} \left(w_{(i)(j)}(\vec{u}_{(1)}, \vec{u}_{(2)}, \vec{u}_{(3)}, \ldots) + w_{(j)(i)} \left(\vec{u}_{(1)}, \vec{u}_{(2)}, \vec{u}_{(3)}, \ldots \right) \right) V_{(j)} \quad (4.7)$$

in which micro-potential strain energy terms, w, are expressed as a function of relative displacement vector $\eta_{(i)(j)} = \left(u_{(j)} - u_{(i)} \right)$ only instead since relative position vector in reference configuration that is $\xi_{(i)(j)} = \left(x_{(j)} - x_{(i)} \right)$ is already an argument inside u.

Consequently, total potential energy for all particles is obtained by substituting relation (4.7) in relation (4.6) and expressed as follows.

$$U = \sum_{i=1}^{\infty} \left(\frac{1}{2} \sum_{j=1}^{\infty} \left(w_{(i)(j)}(\vec{u}_{(1)}, \vec{u}_{(2)}, \ldots) + w_{(j)(i)}(\vec{u}_{(1)}, \vec{u}_{(2)}, \ldots) \right) V_{(j)} \right) V_{(i)} - \sum_{i=1}^{\infty} \vec{u}_{(i)} \vec{b}_{(i)} V_{(i)} \quad (4.8)$$

4.1.1 Hamilton's Principle

One of the fundamental prescription in classical mechanics is known as principle of least action or Hamilton's principle which dictates best possible path a particle has to follow in its motion from one point to another in space.

Relations determining transitions between accelerations, velocities and co-ordinates of a system are called equations of motion [6]. Acceleration of a free particle can be determined if its positions and velocities at each instant of time are known. Therefore, fundamental functions leading to equations of motion have to be function of position, velocity and time, in other words, Lagrangian of system that is expressed as $L(q, \dot{q}, t)$ in which q and $\dot{q}$ indicate generalized co-ordinates and generalized velocities respectively while t symbolizes time. Specifically, notion of generalized co-ordinates of a particle is considered as the minimum number of co-ordinates by which position of a particle is sufficiently identifiable.

The other consequential parameter that controls motion of a particle in space is degrees of freedom. For example, if a single mass pendulum problem in two-dimensional space is taken into account, constrain equation for oscillating mass would be a path defined by associated circle on which mass moves on. For this specific problem of motion, one may intuitively expect that system should have only one degree of freedom. Formally degrees of freedom any arbitrary system has is determined by a generalized formula as given below.

$$s = DN - C \tag{4.9}$$

in which D, N and C indicate number of dimension, particles and constrain equations respectively. Consequently, Lagrangian of a system becomes $L(q_1, ..., q_s, \dot{q}_1, ..., \dot{q}_s, ..., t)$ in case of defining s degrees of freedom for co-ordinates.

Assuming a particle moves through space from co-ordinate q^1 to q^2 in an infinitesimal time interval, dt, then this action is defined by area under co-ordinate-time curve and denoted by S. The path of this curve can be any that ties these two co-ordinates. One of these path can be deviate from one another by δq with infinitesimal variation $\delta q(t)$ at same instant of time, t. As a result, infinitesimal difference in action, δS, can be given by terminating area between these curves.

As a result of that the best possible least action is achieved as follows.

$$\delta S = \int_{t_1}^{t_2} L\left(q(t) + \delta q(t), \dot{q}(t) + \delta\dot{q}(t), t\right) dt - \int_{t_1}^{t_2} L\left(q(t), \dot{q}(t), t\right) dt \tag{4.10}$$

As a conclusion, the right side of equation (4.10) yields to its following form.

$$\delta S = \int_{t_1}^{t_2} \delta L\left(q(t), \dot{q}(t), t\right) dt. \tag{4.11}$$

According to Taylor expansion for one variable function, e.g. $f(x)$, an expression for total differential of $f(x)$ that is $f(x+dx) - f(x) = (\partial f(x)/\partial x)\, dx$ or $df(x)$ are obtained. By means of this definition and expanding total differential inside integral in relation (4.11), we write

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q}\right) dt. \tag{4.12}$$

Manipulation of the second term in right-hand side of relation (4.12) is needed to be able to minimize variation in action term, δS. Therefore, we write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) = \frac{\partial L}{\partial \dot{q}}\left(\frac{d}{dt}\delta q\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dq \tag{4.13}$$

or

$$\left(\frac{\partial L}{\partial \dot{q}}\right)\delta\dot{q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dq \tag{4.14}$$

and substituting relation (4.14) for the second term in expression (4.12) and rearranging terms in an appropriate way variation, δS is obtained as follows.

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} dq\right) dt + \int_{t_1}^{t_2} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dq\right) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} dq\right) dt + \int_{t_1}^{t_2} d\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dq dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} dq\right) dt + \left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dq dt
\end{aligned} \tag{4.15}$$

Since all possible paths between two position end up with the same co-ordinates, namely $\delta q(t_1) = \delta q(t_2)$, middle integral in relation (4.15) yields to zero. Moreover, the second proposal that we have at the beginning was to determine the shortest path that a particle follows by minimizing its action, namely, $\delta S = 0$, therefore the last line of relation (4.15) leads to

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q dt = 0 \tag{4.16}$$

4.1.2 Equations of Motion in Non-Local Theory

The only way of satisfying condition given by relation (4.16) is to equal the terms inside brackets in relation (4.16) to zero.

To this end, Lagrange-Euler equations of motion being a set of differential equations is obtained as expressed as follows.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_{(i)}} \right) - \frac{\partial L}{\partial q_{(i)}} = 0 \tag{4.17}$$

where $i = 1, 2, 3, ..., s$, indicates number of degrees of freedom the particle has.

Displacement of a particle labelled by position vector, $\vec{x}_{(i)}$ in undeformed configuration can be denoted by $d\vec{x}_{(i)}$ or in short hand notation by $\vec{u}_{(i)}$.

Accordingly, time derivative of displacement vector field, $\vec{u}_{(i)}$, becomes $d\vec{x}_{(i)}/dt$ which is $\dot{\vec{u}}_{(i)}$ in short hand notation.

Additionally, kinetic energy of a particle in motion is given as a scalar product of forces applied on particle and distance it travels, namely, $\vec{F} \cdot d\vec{x}_{(i)}$. To this end, kinetic energy for each particle is expressed as follows.

$$T = \sum_{i=1}^{\infty} m_{(i)} \frac{d\dot{\vec{u}}_{(i)}}{dt} \cdot d\vec{x}_{(i)} = \sum_{i=1}^{\infty} m_{(i)} d\dot{\vec{u}}_{(i)} \cdot \frac{d\vec{x}_{(i)}}{dt} = \sum_{i=1}^{\infty} \frac{1}{2} m_{(i)} \dot{\vec{u}}_{(i)} \cdot \dot{\vec{u}}_{(i)} \tag{4.18}$$

Lagrangian of a system, including all particles in a body is defined as a difference of kinetic and potential energies, $L = T - U$.

Furthermore, Lagrangian is written by substituting latest statements of kinetic and potential energy terms in relation (4.8) as follows.

$$L = T - U = \sum_{i=1}^{\infty} \frac{1}{2}\rho_{(i)}\dot{\vec{u}}_{(i)} \cdot \dot{\vec{u}}_{(i)} V_{(i)} - \left(\sum_{i=1}^{\infty} \left(\frac{1}{2} \sum_{j=1}^{\infty} \left(w_{(i)(j)}(\vec{u}_{(1)}, \vec{u}_{(2)}, \vec{u}_{(3)}, ...) + w_{(j)(i)}(\vec{u}_{(1)}, \vec{u}_{(2)}, \vec{u}_{(3)}, ...) \right) V_{(j)} \right) V_{(i)} - \sum_{i=1}^{\infty} \vec{b}_{(i)} \vec{u}_{(i)} V_{(i)} \right) \tag{4.19}$$

Lagrange-Euler equations of motion might be modified by making a substitution in generalized co-ordinates such a way that $\dot{\vec{q}}_{(i)}$ and $\vec{q}_{(i)}$ vectors are substituted by $\dot{\vec{u}}_{(i)}$ and $\vec{u}_{(i)}$ respectively.

Thus, Lagrange-Euler equation is expressed as follows.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{u}}_{(i)}}\right) - \frac{\partial L}{\partial \vec{u}_{(i)}} = 0 \tag{4.20}$$

Before substituting relation (4.19) in Lagrange-Euler equation of motion given by expression (4.20), let us introduce a set of differentials that are needed while performing differentiation of Lagrangian seen in expression (4.20).

$$\frac{\partial w_{(i)(j)} \left(\vec{u}_{(1)}, \vec{u}_{(2)}, ... \right)}{\partial \dot{\vec{u}}_{(i)}} = 0 \quad , \quad \frac{\partial w_{(j)(i)} \left(\vec{u}_{(1)}, \vec{u}_{(2)}, ... \right)}{\partial \dot{\vec{u}}_{(i)}} = 0 \tag{4.21}$$

$$\frac{\partial}{\partial \dot{\vec{u}}_{(i)}} \left(\frac{1}{2}\rho_{(i)}\dot{\vec{u}}_{(i)} \cdot \dot{\vec{u}}_{(i)} V_{(i)} \right) = \rho_{(i)}\dot{\vec{u}}_{(i)} V_{(i)} \quad , \quad \frac{\partial \left(\vec{b}_{(i)} \vec{u}_{(i)} V_{(i)} \right)}{\partial \dot{\vec{u}}_{(i)}} = 0 \tag{4.22}$$

and

$$\frac{\partial}{\partial \vec{u}_{(i)}} \left(\frac{1}{2}\rho_{(i)}\dot{\vec{u}}_{(i)} \cdot \dot{\vec{u}}_{(i)} V_{(i)} \right) = 0 \quad , \quad \frac{\partial \left(\vec{b}_{(i)} \vec{u}_{(i)} V_{(i)} \right)}{\partial \vec{u}_{(i)}} = \vec{b}_{(i)} V_{(i)} \tag{4.23}$$

After performing partial differentials and using results along relations (4.21), (4.22) and (4.23), relation (4.20) becomes

$$\frac{d}{dt}\left(\sum_{i=1}^{\infty} \rho_{(i)}\dot{\vec{u}}_{(i)} V_{(i)} \right) - \sum_{i=1}^{\infty} \left(-\frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{\partial w_{(i)(j)}}{\partial \vec{u}_{(i)}} + \frac{\partial w_{(j)(i)}}{\partial \vec{u}_{(i)}} V_{(j)} \right) \right) V_{(i)} - \sum_{i=1}^{\infty} \vec{b}_{(i)} V_{(i)} = 0 \tag{4.24}$$

or

$$\sum_{i=1}^{\infty}\rho_{(i)}\ddot{\vec{u}}_{(i)}V_{(i)} = -\frac{1}{2}\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{\partial w_{(i)(j)}}{\partial\vec{u}_{(i)}}V_{(j)} + \sum_{j=1}^{\infty}\frac{\partial w_{(j)(i)}}{\partial\vec{u}_{(i)}}V_{(j)}\right)V_{(i)} + \sum_{i=1}^{\infty}\vec{b}_{(i)}V_{(i)} \tag{4.25}$$

Since all terms under summation of (i) index, relation (4.25) can be simplified as follows.

$$\sum_{i=1}^{\infty}\left(\rho_{(i)}\ddot{\vec{u}}_{(i)} + \frac{1}{2}\left(\sum_{j=1}^{\infty}\frac{\partial w_{(i)(j)}}{\partial\vec{u}_{(i)}}V_{(j)}\right) + \frac{1}{2}\left(\sum_{j=1}^{\infty}\frac{\partial w_{(j)(i)}}{\partial\vec{u}_{(i)}}V_{(j)}\right) - \vec{b}_{(i)}\right)V_{(i)} = 0 \tag{4.26}$$

As it can be immediately realized that the bracket inside relation (4.26) has to be zero in order to be able to satisfy itself so that relation (4.26) can be alternatively expressed as follows.

$$\rho_{(i)}\ddot{\vec{u}}_{(i)} + \frac{1}{2}\left(\sum_{j=1}^{\infty}\frac{\partial w_{(i)(j)}}{\partial\vec{u}_{(i)}}\right)V_{(j)} + \frac{1}{2}\left(\sum_{j=1}^{\infty}\frac{\partial w_{(j)(i)}}{\partial\vec{u}_{(i)}}\right)V_{(j)} - \vec{b}_{(i)} = 0 \tag{4.27}$$

or

$$\rho_{(i)}\ddot{\vec{u}}_{(i)} = -\frac{1}{2}\left(\sum_{j=1}^{\infty}\frac{\partial w_{(i)(j)}}{\partial\vec{u}_{(i)}}\right)V_{(j)} - \frac{1}{2}\left(\sum_{j=1}^{\infty}\frac{\partial w_{(j)(i)}}{\partial\vec{u}_{(i)}}\right)V_{(j)} + \vec{b}_{(i)} \tag{4.28}$$

Unit analysis that is $\left([kg/m^3]\,[m/s^2] = \left(\frac{\partial w_{(i)(j)}}{\partial\vec{u}_{(i)}}\right)[m^3] = \left(\frac{\partial w_{(j)(i)}}{\partial\vec{u}_{(i)}}\right)[m^3]\right)$ acquaints us with the right-hand side of relation (4.28) that appears in a unit of force per unit volume, $[N/m^3]$.

Additionally, micro-potential strain energy $w_{(j)(i)}$, has a unit of $[J/m^6]$, whose integration over a defined volume including associated material points yields to strain energy density, $W_{(i)}$, in a unit of energy per unit volume, $[J/m^3]$, for corresponding particle (i). From this point of view, the terms having unit of in force per unit volume are referred as force density in peridynamic theory.

As it can be intuitively realized that effective force applied on an arbitrary particle is affected by how much interacting particles are far away from subjected particle. To this end, force density should be a function of relative position vector in both initial and deformed states of a body.

Relative deformation vectors that are considered as deviations with respect to initial state of relative position vector can be expressed in a way around and they can be collected for each pair of particles in an array form as follows.

$$u_{(j)} - u_{(i)} = \begin{Bmatrix} u_{(1)} - u_{(i)} \\ u_{(2)} - u_{(i)} \\ u_{(3)} - u_{(i)} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} y_{(1)} - y_{(1)} - (x_{(1)} - x_{(1)}) \\ y_{(2)} - y_{(1)} - (x_{(2)} - x_{(1)}) \\ y_{(3)} - y_{(1)} - (x_{(3)} - x_{(1)}) \\ \vdots \end{Bmatrix} \tag{4.29}$$

Since strain energy on a bond arises from relative position vector, $(y' - y)$, in deformed state with respect to relative position vector, ξ, in reference configuration satisfying relation $y' - y = (u' - u) + \xi$, we can conclude that micro-potential energy is expressed as a function of either relative position vector, $(y' - y)$, in deformed configuration or relative displacement vector, $(u' - u)$. Therefore, the statements that are $w_{(i)(j)} = w_{(i)(j)}(y_{(j)} - y_{(i)})$ and $w_{(i)(j)} = w_{(i)(j)}(u_{(j)} - u_{(i)})$ become equivalent, [22] and [25].

In the scope of bond-based peridynamic, force density between interacting particles obeys Newton's third law of motion establishing a balance equation that is expressed as follows.

$$\vec{\mathbf{f}}_{(i)(j)}(u_{(j)} - u_{(i)}, x_{(j)} - x_{(i)}, t) + \vec{\mathbf{f}}_{(j)(i)}(u_{(i)} - u_{(j)}, x_{(i)} - x_{(j)}, t) = 0 \tag{4.30}$$

or

$$-\vec{\mathbf{f}}_{(i)(j)}(u_{(j)} - u_{(i)}, x_{(j)} - x_{(i)}, t) = \vec{\mathbf{f}}_{(j)(i)}(u_{(i)} - u_{(j)}, x_{(i)} - x_{(j)}, t) \tag{4.31}$$

Moreover, the terms

$$\frac{\partial w_{(i)(j)}}{\partial u_{(i)}} \quad \text{and} \quad \frac{\partial w_{(j)(i)}}{\partial u_{(i)}} \tag{4.32}$$

force per unit volume can be interpreted as force density being a function of ξ and η, $f(\eta, \xi)$. To this end, equivalence of terms that are given along relations (4.30) and (4.32) are expressed as follows.

$$\frac{\partial w_{(i)(j)}}{\partial \vec{u}_{(i)}} = \vec{\mathbf{f}}_{(i)(j)}(u_{(j)} - u_{(i)}, x_{(j)} - x_{(i)}, t) \tag{4.33}$$

and

$$\frac{\partial w_{(j)(i)}}{\partial \vec{u}_{(i)}} = -\vec{\mathbf{f}}_{(j)(i)}(u_{(i)} - u_{(j)}, x_{(i)} - x_{(j)}, t) \tag{4.34}$$

which constitute fundamental statement of bond-based peridynamic theory.

In this perspective, force densities have been associated with traction forces of classical continuum mechanics in [32].

Accordingly, equations of motion in peridynamic theory are obtained for an arbitrary particle, (i), after substituting relations (4.33) and (4.34) in Lagrangian based expression given by (4.28) as follows.

$$\rho_{(i)}\ddot{u}_{(i)} = \frac{1}{2}\sum_{j=1}^{\infty}\big(\mathbf{f}_{(i)(j)}(u_{(j)} - u_{(i)}, x_{(j)} - x_{(i)}, t) - \mathbf{f}_{(j)(i)}(u_{(i)} - u_{(j)}, x_{(i)} - x_{(j)}, t)\big) V_{(j)} + b_{(i)} \tag{4.35}$$

By means of a conversion set that is

$$\sum_{j=1}^{\infty}(\cdot)V_{(j)} \approx \int_V (\cdot)\, dV' \tag{4.36}$$

$$x_{(i)} = x \quad , \quad x_{(j)} = x' \quad , \quad u_{(i)} = u(x,t) = u \quad , \quad u_{(j)} = u(x',t) = u' \tag{4.37}$$

$$\rho_{(i)} = \rho(x,t) \quad , \quad b_{(i)} = b(x,t) \tag{4.38}$$

peridynamic equation of motion can be explicitly expressed in the following form.

$$\rho(x,t)\ddot{u}(x,t) = \frac{1}{2}\int_V \left(\mathbf{f}(u' - u, x' - x, t) - \mathbf{f}(u - u', x - x', t)\right) dV' + b\,(x,t) \tag{4.39}$$

Within the realm of particle interactions, the simplest form of equations of motion given by relation (4.35) applies to each particle labelled by (i) and determines whole displacement field for entire material domain. Regarding infinite-dimensional arrays, state notion has been developed in [34] and represented by bold and underlined capital letters, namely $\underline{\mathbf{Y}}$. One of the state notion is the deformation state that can be interpreted as a function relating relative position vectors in undeformed deformed configurations.

Moreover, force states can be defined in a similar way to definition of deformation state given by expression (4.29) and collection of pair-wise forces are symbolized by another underlined capital letters, namely $\underline{\mathbf{F}}$. All associated pair-wise forces of interacting particles can be collected inside an infinite dimensional array constituting force state as follows

$$\underline{\mathbf{F}}\big(x_{(i)}, t\big) = \frac{1}{2}\begin{Bmatrix} \vdots \\ \mathbf{f}_{(i)(j)}(u_{(j)} - u_{(i)}, x_{(j)} - x_{(i)}, t) \\ \vdots \end{Bmatrix} \tag{4.40}$$

and

$$\underline{\mathbf{F}}\big(x_{(j)}, t\big) = \frac{1}{2}\begin{Bmatrix} \vdots \\ -\mathbf{f}_{(j)(i)}(u_{(i)} - u_{(j)}, x_{(i)} - x_{(j)}, t) \\ \vdots \end{Bmatrix} \tag{4.41}$$

or in short-hand notation their equivalent representation is written as

$$\underline{\mathbf{F}}\big(x, t\big) = \frac{1}{2}\begin{Bmatrix} \vdots \\ \mathbf{f}(u' - u, x' - x, t) \\ \vdots \end{Bmatrix} \tag{4.42}$$

and

$$\underline{\mathbf{F}}\big(x', t\big) = \frac{1}{2}\begin{Bmatrix} \vdots \\ -\mathbf{f}(u - u', x - x', t) \\ \vdots \end{Bmatrix} \tag{4.43}$$

Applying force states to corresponding relative position vectors in undeformed configuration yields to force densities. This operation can be seen as decomposition of force densities into force states and its associated relative position vector in undeformed configuration. Namely,

$$\underline{\mathbf{F}}(x_{(i)}, t)\langle x_{(j)} - x_{(i)}\rangle = \frac{1}{2}\mathbf{f}_{(i)(j)}\left(u_{(j)} - u_{(i)}, x_{(j)} - x_{(i)}, t\right) \tag{4.44}$$

and

$$\underline{\mathbf{F}}(x_{(j)}, t)\langle x_{(i)} - x_{(j)}\rangle = -\frac{1}{2}\mathbf{f}_{(j)(i)}\left(u_{(i)} - u_{(j)}, x_{(i)} - x_{(j)}, t\right) \tag{4.45}$$

By substituting relations (4.44) and (4.44) in equations of motion given by expression (4.39). To this end, peridynamic equations of motion is expressed as follows.

$$\rho_{(i)}\ddot{u}_{(i)} = \sum_{j=1}^{\infty}\left(\underline{\mathbf{F}}\left(x_{(j)}, t\right)\left\langle x_{(i)} - x_{(j)}\right\rangle - \underline{\mathbf{F}}\left(x_{(j)}, t\right)\left\langle x_{(i)} - x_{(j)}\right\rangle\right) V_{(j)} + b_{(i)} \tag{4.46}$$

or in integral form,

$$\rho(x,t)\ddot{u}(x,t) = \frac{1}{2}\int\limits_V \left(\mathbf{f}(u'-u, x'-x, t) - \mathbf{f}(u-u', x-x', t)\right) dV' + b\,(x,t) \quad (4.47)$$

Force densities can be replaced with associated states and expressed as follows.

$$\rho(x,t)\ddot{u}(x,t) = \int\limits_V \left(\underline{\mathbf{F}}\,(x,t)\,\langle x'-x\rangle - \underline{\mathbf{F}}\,(x',t)\,\langle x-x'\rangle\right) dV' + b\,(x,t) \quad (4.48)$$

4.1.3 Balance Equations

As other conserved quantities in nature, e.g. energy, momentum of a system in both linear and rotational point of view has to be in balance. This part consists of derivation of balance linear and angular momentum equations to establish a set of constraint relation. Linear momentum of a free-particle in space considered as particle moves in a linear path while angular momentum designates rotational momentum of particle around a specified origin of non-linear trajectory.

4.1.3.1 Global Balance of Linear Momentum

Let us start with conservation of linear momentum for a free-particle. Linear momentum of a particle in space being a well-known notion is defined by scalar product of velocity vector with mass of the particle, namely, $\rho_{(i)} dV \dot{u}_{(i)}$. In that sense,integrating result over a volumetric domain, total momentum for a group of particle can be evaluated as follows.

$$\mathbf{L} = \sum_{i=1}^{\infty} \rho_{(i)} \dot{u}_{(i)} V_{(i)} \quad (4.49)$$

or in integral form

$$\mathbf{L} = \int\limits_V \rho(x,t)\dot{u}(x,t) dV \quad (4.50)$$

Because of particle interactions and prescribed body forces the left-hand side of relations (4.46) and (4.47) results in a net force on the particle labelled by (i). Moreover, this net force by which the particle is accelerated is obtained with time derivative of linear momentum as well. In other words, $\mathbf{F} = \partial\mathbf{L}/\partial t = \dot{\mathbf{L}}$, in which $\mathbf{F}$ is a force due to alternation in momentum with respect to change in time. Thus, the force on a group of particle is written by means of relation (4.49) as follows.

$$\mathbf{F} = \frac{\partial \mathbf{L}}{\partial t} = \sum_{i=1}^{\infty} \rho_{(i)} \ddot{u}_{(i)} V_{(i)} \tag{4.51}$$

or in integral form

$$\dot{\mathbf{L}} = \mathbf{F} = \int_V \rho(x,t)\ddot{u}(x,t)dV \tag{4.52}$$

As one may realize that the same force term can be obtained by integrating either left or right-hand side of relation (4.48) over a volumetric domain V leading to

$$\int_V \rho(x,t)\ddot{u}(x,t)dV = \int_V \int_V \left(\underline{\mathbf{F}}(x,t)\left\langle x' - x\right\rangle - \underline{\mathbf{F}}(x',t)\left\langle x - x'\right\rangle\right) dV'dV + \int_V b(x,t)dV \tag{4.53}$$

or in more compact form

$$\int_V \left(\rho(x,t)\ddot{u}(x,t) - b(x,t)\right) dV = \int_V \int_V \left(\underline{\mathbf{F}}(x,t)\left\langle x' - x\right\rangle - \underline{\mathbf{F}}(x',t)\left\langle x - x'\right\rangle\right) dV'dV \tag{4.54}$$

which constitutes a condition for conservation of linear momentum.

Global balance equation of linear momentum for entire domain is satisfied if prescription, $x \Leftrightarrow x'$, is applied on term $\underline{\mathbf{F}}\big(x,t\big)\big\langle x' - x\big\rangle$ and substituted in expression (4.48), leading to termination of force states of interacting particles.

$$\underline{\mathbf{F}}\left(x,t\right)\left\langle x' - x\right\rangle = \underline{\mathbf{F}}(x',t)\left\langle x - x'\right\rangle \tag{4.55}$$

Expressing that the second integration appearing in relation (4.53) is performed on infinitesimal volume, dV, so that linear momentum is conserved and entire body behaves under effect of prescribed body forces only. As a result, global conservation of linear momentum is obtained as follows.

$$\int_V \left(\rho\left(x,t\right)\ddot{u}\left(x,t\right) - b\left(x,t\right)\right) dV = 0. \tag{4.56}$$

4.1.3.2 Global Balance of Angular Momentum

Beside satisfying balance of linear momentum, another constrain equation being balance of angular momentum has to be conserved. If rotation of a group of particles around a co-ordinate origin whose distance vector to subjected particle labelled by $x_{(i)}$ is $y(x_{(i)}, t)$, then angular momentum over a volumetric domain V is expressed as follows.

$$\mathbf{H}_o = \sum_{i=1}^{\infty} \rho_{(i)} y(x_{(i)}, t) \times \dot{u}_{(i)} V_{(i)} \tag{4.57}$$

or in integral form

$$\mathbf{H}_o = \int_V \rho(x,t) y(x,t) \times \dot{u}(x,t) dV \tag{4.58}$$

in which subscript o under H represents origin of co-ordinate system for rotation. As similar to time derivative of linear momentum, change in angular momentum with respect to time yields to a net torque on rotating group of particle around co-ordinate origin.

$$\mathbf{T} = \frac{\partial \mathbf{H}_o}{\partial t} = \sum_{i=1}^{\infty} \rho_{(i)} y(x_{(i)}, t) \times \ddot{u}_{(i)} V_{(i)} \tag{4.59}$$

equivalently

$$\dot{\mathbf{L}}_o = \int_V \rho(x,t) y(x,t) \times \ddot{u}(x,t) dV \tag{4.60}$$

in which $\partial y(x_{(i)}, t)/\partial t = 0$. By applying vector product to both side of relation (4.48) with $y(x_{(i)}, t)$, balance equation for angular momentum is written as follows.

$$\int_V y(x,t) \times (\rho(x,t)\ddot{u}(x,t) - b(x,t))\, dV = \int_V \int_V y(x,t) \times \left(\underline{\mathbf{F}}(x,t) \langle x' - x \rangle - \underline{\mathbf{F}}(x',t) \langle x - x' \rangle\right) dV' dV \tag{4.61}$$

The same prescription used on conservation equation for linear momentum that is $x \Leftrightarrow x'$ for each particle is applied on terms $\underline{\mathbf{F}}\big(x,t\big)\big\langle x' - x\big\rangle$ and $y(x,t) \Leftrightarrow y(x',t)$, as follows.

$$y(x,t) \times \underline{\mathbf{F}}(x',t) \langle x - x' \rangle\, dV' dV = y(x',t) \times \underline{\mathbf{F}}(x,t) \langle x' - x \rangle\, dV dV' \tag{4.62}$$

Thus, global conservation of angular momentum equation is obtained as

$$\int_V y(x,t) \times (\rho(x,t)\ddot{u}(x,t) - b(x,t))dV = \int_V \int_V y(x,t) \times \underline{\mathbf{F}}(x,t) \langle x' - x \rangle \, dV' dV - \int_V \int_V y(x',t) \times \underline{\mathbf{F}}(x,t) \langle x' - x \rangle \, dV dV' \tag{4.63}$$

or in more compact form

$$\int_V y(x,t) \times (\rho(x,t)\ddot{u}(x,t) - b(x,t))dV = -\int_V \int_V \big(y(x',t) - y(x,t)\big) \times \underline{\mathbf{F}}(x,t) \langle x' - x \rangle \, dV dV' \tag{4.64}$$

in which deformation state relating relative position vectors in deformed configuration with undeformed relative position vectors such that $y' - y = y(x',t) - y(x,t) = \underline{\mathbf{Y}}(x,t) \langle x' - x \rangle$ can be substituted in conservation of angular momentum equation and it becomes

$$\int_V y(x,t) \times (\rho(x,t)\ddot{u}(x,t) - b(x,t))dV = -\int_V \int_{V'} \underline{\mathbf{Y}}(x,t) \langle x' - x \rangle \times \underline{\mathbf{F}}(x,t) \langle x' - x \rangle \, dV' dV \tag{4.65}$$

4.2 Alignments and magnitudes of pair-wise forces in peridynamic theory

The pair-wise forces are taken into account with formulations based on different configurations in terms of alignments and magnitudes of pair-wise forces. In this manner, one of which collocates pair-wise forces in same alignment with equal magnitudes and opposite directions which is named as bond-based peridynamics while alignments of pair-wise forces in deformed state become same with different magnitudes in ordinary state-based peridynamics. On the other hand, formulation structured with coupling forces of different magnitudes and alignments is named as non-ordinary state-based peridynamic, [34].

These three formulations of peridynamic theory are introduced in equations of motion by associated auxiliary parameters in front of unitary direction vectors of deformed bonds, [22], [25].

4.2.1 Formulation of bond-based peridynamic

From conservation of linear momentum for all particles over entire domain it is concluded that following condition has to be satisfied.

$$\int_V \underline{\mathbf{Y}}(x,t)\left\langle x'-x\right\rangle \times \underline{\mathbf{F}}(x,t)\left\langle x'-x\right\rangle dV' = 0 \tag{4.66}$$

The only way of satisfying condition given by relation (4.66) is that all relative position vectors in deformed configuration, $(y'-y)$, and their corresponding force density vectors have to be in same alignment.

By now, force densities of interacting particles are assumed to be in same alignment and have magnitudes provided by relations (4.31), (4.33) and (4.34) therefore this concept of the theory is referred as bond-based peridynamics.

In the scope of bond-based peridynamics, and according to magnitude and alignment configurations of force densities as explained above, firstly, following assignments are going to be valid.

$$\underline{\mathbf{F}}(x,t)\left\langle x'-x\right\rangle = \frac{1}{2}\mathbf{f}(u'-u, x'-x, t) \tag{4.67}$$

and

$$\underline{\mathbf{F}}(x',t)\left\langle x-x'\right\rangle = -\frac{1}{2}\mathbf{f}(u'-u, x'-x, t) \tag{4.68}$$

The pair-wise force function indicated by $\mathbf{f}(u'-u, x'-x, t)$ including force densities that are $f(u'-u, x'-x, t)$ and $f(u-u', x-x', t)$, has been defined as a vector-valued function by S.A Silling in [32].

Furthermore, inherent formulation of bond-based peridynamics has a restriction for modelling of materials whose Poisson's ratio, ν, different than $1/4$ [34]. In the framework of bond-based peridynamics, pair-wise forces are recast as multiple of unitary vector in deformed state with bond constant, C^p, defining direction dependent stiffness properties.

As definition of bond-based peridynamics implies, force densities given by relations (4.67) and (4.68) are alternatively formulated as follows.

$$\underline{\mathbf{F}}(x_{(j)},t)\left\langle x_{(k)}-x_{(j)}\right\rangle=\frac{1}{2}C^p\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|} \tag{4.69}$$

and

$$\underline{\mathbf{F}}(x_{(k)},t)\left\langle x_{(j)}-x_{(k)}\right\rangle=\frac{1}{2}C^p\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|}=-\frac{1}{2}C^p\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|} \tag{4.70}$$

Invoking relations (4.67) and (4.68) in expression (4.48) provides us with equations of motion for bond-based peridynamics proposed in [32].

$$\rho(x,t)\ddot{u}(x,t)=\int\limits_V \mathbf{f}(u'-u,x'-x,t)dV'+b(x,t) \tag{4.71}$$

which can be alternatively expressed by means of substitutions, $\xi=x'-x$ and $\eta=u'-u$ in relation (4.71) as follows.

$$\rho(x,t)\ddot{u}(x,t)=\int\limits_V \mathbf{f}(\eta,\xi,t)dV'+b(x,t) \tag{4.72}$$

4.2.2 Formulation of ordinary state-based peridynamic

Improvements on formulation of bond-based peridynamics resulted in ordinary state-based peridynamic approach that avoids material type restriction for those having Poisson's ratio different than 1/4, [34].

In this sense, force states in ordinary state-based approach are formulated as follows.

$$\underline{\mathbf{F}}(x_{(j)},t)\left\langle x_{(k)}-x_{(j)}\right\rangle=\frac{1}{2}A^{sb}_{(j)(k)}\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|} \tag{4.73}$$

and

$$\underline{\mathbf{F}}(x_{(k)},t)\left\langle x_{(j)}-x_{(k)}\right\rangle=-\frac{1}{2}B^{sb}_{(k)(j)}\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|} \tag{4.74}$$

4.2.3 Stretch Notion in Peridynamic Theory

As opposed to definition of classical stretch which is given by $(d\mathbf{x}/d\mathbf{X})$ in classical theory, peridynamic equivalent of stretch notion is defined by change in magnitude of relative position vector from reference to deformed configuration. Namely,

$$s_{(k)(j)} = \frac{\left|y_{(j)} - y_{(k)}\right| - \left|x_{(j)} - x_{(k)}\right|}{\left|x_{(j)} - x_{(k)}\right|} \tag{4.75}$$

or it is also possible to write relation (4.75) in short hand notation as follows.

$$s = \frac{|y' - y| - |x' - x|}{|x' - x|} = \frac{|\xi + \eta| - |\xi|}{|\xi|} \tag{4.76}$$

Consequently, it can be concluded that stretch in peridynamic theory can be more likely associated with definition of engineering strain in classical theory of elasticity which is given by ratio of change in a length element with respect to its initial magnitude.

4.2.4 Peridynamic pair-wise force interaction

As mentioned at the beginning of Chapter (4), strain energy density which arises from interactions of associated particles has been defined by summation of micro-potentials that are functions of both difference in relative displacement vector $\eta = u(x', t) - u(x, t) = u' - u$ in deformed configuration and relative position vector $\xi = x' - x$ in undeformed configuration. The summation representing strain energy of particle shown by $x_{(i)}$ inside its horizon $\mathscr{H}_x$ can be evaluated by an indefinite integral as follows, [33].

$$W_{(x)} = \frac{1}{2} \int\limits_{\mathscr{H}_x} w(\eta, \xi)\, dV' \tag{4.77}$$

in which dV' is infinitesimal volume of neighbouring particles, positioned by vector x' around particle having position vector x.

Regarding relation between pair-wise forces and micro-potential energy, pair-wise force function between interacting particles is given by $\mathbf{f}(0, \xi)$ in equilibrium or reference configuration of body when force densities are in balance or no deformation occurs in body.

Since all relative position vectors in undeformed configuration are fixed, first-order Taylor expansion around $\eta = 0$ for pair-wise force function inside integral given in relation (4.72) is written as follows.

$$\mathbf{f}(\eta, \xi) = \frac{\partial \mathbf{f}}{\partial \eta}(0, \xi)(\eta - 0) + \mathbf{f}(0, \xi) \quad \text{where} \quad \frac{\partial \mathbf{f}}{\partial \eta}(0, \xi) = \left. \frac{\partial \mathbf{f}(\eta, \xi)}{\partial \eta} \right|_{\eta=0} \tag{4.78}$$

Writing Taylor expansion for pair-wise force function around $\eta = 0$ enables us to approximates $\mathbf{f}(\eta, \xi)$ when deformation defined by $|\eta| << 1$ occur between particles.

The first term in relation (4.78) is added to pair-wise force function $f(0, \xi)$ in reference configuration hence $\mathbf{f}(\eta, \xi)$ for deformed configuration of particles is obtained.

Since micro-potential is related with the change in bond length with respect to reference relative position vector, ξ, pair-wise force function at any state including reference is defined by gradient of scalar-valued potential field, $w(\eta, \xi)$ with respect to η only.

$$\mathbf{f}(\eta, \xi) = \frac{\partial w(\eta, \xi)}{\partial \eta} \tag{4.79}$$

As stated above, if the condition given by expression (4.79) is satisfied, then materials is referred as micro-elastic [32]. In this manner, pair-wise force function for undeformed configuration of body is expressed as follows.

$$\mathbf{f}(0, \xi) = \left. \frac{\partial w(\eta, \xi)}{\partial \eta} \right|_{\eta=0} = \frac{\partial w}{\partial \eta}(0, \xi) \tag{4.80}$$

Moreover, pair-wise force function between interacting particles loses its effectiveness beyond horizon. Accordingly, interaction constraint is mathematically given by

$$\text{If} \quad |\xi| > \delta, \quad \text{then} \quad \mathbf{f}(\eta, \xi) = 0 \quad , \quad \forall \eta \tag{4.81}$$

Equivalent notation for Newton's third law of motion given by relation (4.31) has been named as linear admissibility condition, [32].

After substituting assignments given by expressions (4.67) and (4.68) into relation (4.31) linear admissibility condition is automatically satisfied. Namely,

$$f(u' - u, x' - x) + f(u - u', x - x') = 0 \quad , \quad \forall (u' - u), (x' - x) \tag{4.82}$$

leading to

$$\mathbf{f}(\eta,\xi) - \mathbf{f}(\eta,\xi) = 0 \quad , \quad \forall\, \eta, \xi \tag{4.83}$$

in which t can be hidden in the argument of $\mathbf{f}$. It is also possible to present conservation of angular momentum given by expression (4.66) in same notation used in relation (4.82).

$$(y' - y) \times \mathbf{f}(\eta,\xi) = 0 \quad \text{or} \quad (\eta + \xi) \times \mathbf{f}(\eta,\xi) = 0 \quad , \quad \forall\, \eta, \xi \tag{4.84}$$

The micro-potential energy, $w(\eta,\xi)$, between a pair of particles can be expressed as a function of magnitude in relative position vectors in deformed and undeformed configurations as proposed in relation (4.4). Namely,

$$w(\eta,\xi) = \hat{\boldsymbol{w}}(|y' - y|\,, \xi) \tag{4.85}$$

Accordingly, pair-wise function can be defined by substituting relation (4.85) in fundamental definition of pair-wise force function given in expression (4.79) as follows.

$$\mathbf{f}(\eta,\xi) = \frac{\partial}{\partial \eta}\hat{\boldsymbol{w}}(|y' - y|\,, \xi) \tag{4.86}$$

in which assigning magnitude of relative position vector to $\mathbf{y}$, relation (4.86) can be alternatively expressed as follows.

$$\mathbf{f}(\eta,\xi) = \frac{\partial}{\partial \eta}\hat{\boldsymbol{w}}(\mathbf{y},\xi). \tag{4.87}$$

Magnitude of relative position vector $\mathbf{y}$ in deformed configuration is presented as a function of relative displacement vector that is $\mathbf{y} = \mathbf{y}(\eta)$. Accordingly, applying a simple chain rule in differentiation, pair-wise force density function can be expressed as follows.

$$\mathbf{f}(\eta,\xi) = \frac{\partial}{\partial \eta}\hat{\boldsymbol{w}}(\mathbf{y},\xi) = \frac{\partial}{\partial \mathbf{y}}\hat{\boldsymbol{w}}(\mathbf{y},\xi)\frac{\partial \mathbf{y}}{\partial \eta} \tag{4.88}$$

In detail, one might expand second power of of relative position vector's magnitude in deformed configuration to be able to obtain gradient term for pair-wise force function, $|\eta + \xi|/(\eta + \xi)$.

For this purpose, magnitude of vector addition $|\xi + \eta|$ can be expanded over its second power first as follows.

$$(|\xi + \eta|)^2 = \xi \cdot \xi + \eta \cdot \eta + 2\xi \cdot \eta \tag{4.89}$$

Derivative of both hand side of expression (4.89) with respect to η can be written explicitly as

$$2\,|\xi + \eta|\,\frac{\partial(|\xi + \eta|)}{\partial \eta} = \frac{\partial \xi}{\partial \eta} \cdot \xi + \xi \cdot \frac{\partial \xi}{\partial \eta} + \frac{\partial \eta}{\partial \eta} \cdot \eta + \eta \cdot \frac{\partial \eta}{\partial \eta} + 2\frac{\partial \xi}{\partial \eta} \cdot \eta + 2\xi \cdot \frac{\partial \eta}{\partial \eta} \tag{4.90}$$

Since ξ is not a function of η, partial derivative of ξ with respect to η yields to zero. Consequently, relation (4.90) is simplified to

$$\frac{\partial(|\xi + \eta|)}{\partial \eta} = \frac{\xi + \eta}{|\xi + \eta|} \tag{4.91}$$

Substitution of unit vector instead of term $\partial(|\xi + \eta|)/\partial\eta$ in relation (4.88) leads to pair-wise force function that has been introduced in [33].

$$\mathbf{f}(\eta, \xi) = \frac{\partial}{\partial \mathbf{y}}\hat{\boldsymbol{w}}(\mathbf{y}, \xi)\frac{\partial \mathbf{y}}{\partial \eta} = \frac{\partial}{\partial \mathbf{y}}\hat{\boldsymbol{w}}(\mathbf{y}, \xi)\frac{\xi + \eta}{|\xi + \eta|} \tag{4.92}$$

The deformation gradient of scalar-potential field that was offered by Silling and Askari in [33] can be invoked in relation (4.92) and the pair-wise force function including thermal effects is expressed as follows.

$$\mathbf{f}(\eta, \xi) = c\,(s - \alpha \Delta T)\,\frac{\xi + \eta}{|\xi + \eta|} \tag{4.93}$$

in which c and α are called as bond constant and coefficient of thermal expansion respectively while s defines stretch between a pair of material points, introduced by relation (4.76).

One may establish two different analogies between linearly elastic spring and peridynamic bond. In this manner, energy accumulation on a spring can be described by area under force-displacement curve defined by $dW\,(x_0, \Delta x) = F \cdot dx$ while spring constant, k, is evaluated through $k = dF/dx$.

First analogy corresponds to statement given by expression (4.87) while relation (4.93) can be explained by the second analogy.

Moreover, total change in bond length under the effect of pair-wise forces can be expressed along $\eta = s\xi$, hence final length of the bond measures $|\eta + \xi| = (1+s)\,|\xi|$ in deformed state of body.

In Appendix (A) it has been shown that stress terms are derivable from derivatives of corresponding strain energies with respect to deformation parameters. Moreover, comparison of equations of motion obtained in classical and peridynamic theories allows us to couple stiffness related terms.

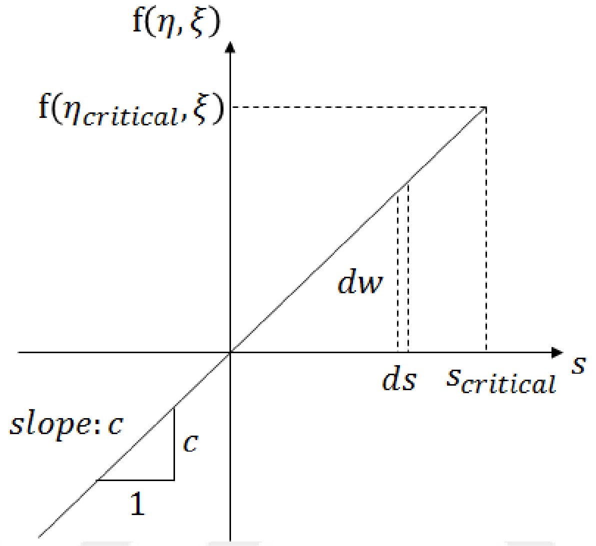


FIGURE 4.3: Evaluation of micro-potential energy

According to Figure (4.3) based on relation (4.79), strain energy on a bond can be explicitly evaluated by using $s = \eta/\xi$ and $\eta = s\xi$ in relation. Namely,

$$w(\eta, \xi) = \int_0^{\eta} cs d\eta = c\int_0^{\eta} \frac{\eta}{\xi} d\eta = \frac{c\eta^2}{2\xi} = \frac{cs^2\xi}{2} \tag{4.94}$$

Moreover, micro-potential strain energy that is calculated by means of relation (4.94) can be used to be able to calculate strain energy density for entire material domain. Alternatively, strain energy density can be evaluated according to relation (4.77). Namely,

$$W_{(x)} = \frac{1}{2}\int_0^{\delta} \frac{cs^2\xi}{2}\, 4\pi\xi^2 d\xi = cs^2\pi\delta^4/4 \tag{4.95}$$

in which a relation between volume and surface area of a spherical geometry, being $dV_{x'} = d\left(4\pi\xi^3/3\right) = 4\pi\xi^2 d\xi$ has been utilized.

In case of volumetric expansion of a linear isotropic material, classical theory of elasticity reads strain energy density function given by relation (3.5) as follows.

$$W_{IsoExp} = \frac{1}{2}\sum_{m=1}^{3} \sigma_{(j)mm}\epsilon_{(j)mm} = \frac{1}{2}\left(\sigma_{(j)11}\epsilon_{(j)11} + \sigma_{(j)22}\epsilon_{(j)22} + \sigma_{(j)33}\epsilon_{(j)33}\right) \quad (4.96)$$

in which shearing components are excluded since only isotropic expansion case is under consideration.

Classically, isotropic expansion of material domain can also be expressed along bulk modulus being volumetric stiffness coefficient of a body. Therefore, a constitutive relation for this specific type of behaviour can be expressed based on relation (B.10) as follows.

$$\tilde{\sigma} = \kappa e \quad (4.97)$$

in which $\tilde{\sigma}$ and e are respectively mean stress and dilatation which are defined as follows.

$$\begin{aligned}\tilde{\sigma} &= \left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)/3 \\ &= \sigma_{ii}/3\end{aligned} \quad (4.98)$$

and

$$\begin{aligned}e &= \Delta V_{(RVE)}/V_{0,(RVE)} \\ &= \left(1+\epsilon_{11}\right)\left(1+\epsilon_{22}\right)\left(1+\epsilon_{33}\right) - \epsilon_{11}\epsilon_{22}\epsilon_{33} \\ &= \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{ii}\end{aligned} \quad (4.99)$$

Thus relation (4.97) can be alternatively expressed for isotropic expansion of a bulk which is under stress, $\tilde{\sigma}$ as follows.

$$3\tilde{\sigma} = 3\kappa \quad (4.100)$$

The Hooke's law expressed in terms of λ and μ in relation (C.53) has been inverted to obtain expression (B.14), representing strain terms as a function of corresponding stress components. It is also convenient to express same equation in its equivalent form with addition of thermal effects for material point (j) as follows.

$$\epsilon_{(j)ij} = \frac{1}{2\mu}\left(\sigma_{(j)ij} - \frac{1}{3\kappa}\left(\kappa - \frac{2\mu}{3}\right)\delta_{(j)ij}\epsilon_{(j)ii} + 3\kappa\alpha\Delta T_{(j)}\delta_{(j)ij}\right) \quad (4.101)$$

in which by substituting equivalent terms for λ and ϵ_{ii} given by relations (B.11) and (B.13) respectively, expression (B.14) can be verified. Alternatively, dilatation can be proven by means of strain equality given by relation (4.101). To this end, trace of dilatation, $\epsilon_{(j)ij}$, is written as

$$\begin{aligned} trace\left(\epsilon_{(j)ij}\right) &= \epsilon_{(j)ii} \\ &= \left(\sigma_{(j)ii} - 3\kappa\epsilon_{(j)ii} + 2\mu\epsilon_{(j)ii} + 9\kappa\alpha\Delta T_{(j)}\right) / \left(2\mu\right) \end{aligned} \tag{4.102}$$

By leaving $\sigma_{(j)ii}$ on the left hand side, we write

$$\begin{aligned} \sigma_{(j)ii} &= 2\mu\epsilon_{(j)ii} + 3\kappa\epsilon_{(j)ii} - 2\mu\epsilon_{(j)ii} - 9\kappa\alpha\Delta T_{(j)} \\ &= 3\kappa\left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right) \end{aligned} \tag{4.103}$$

or dilatation is obtained as follows.

$$\epsilon_{(j)ii} = \sigma_{(j)ii}/(3\kappa) + 3\alpha\Delta T_{(j)} \tag{4.104}$$

Classically, strain energy density given by relation (4.96) in case of isotropic expansion is obtained as follows based on relation (4.98).

$$W_{IsoExp} = \frac{1}{2}\sigma_{ii}\epsilon_{ii} = \frac{3}{2}\kappa\epsilon_{ii}^2 \tag{4.105}$$

Specifically, isotropic expansion of a body can be expressed along $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon$. After substitution of them into classical strain energy density function, relation (4.105) becomes

$$W_{IsoExp} = \frac{3}{2}\kappa\left(\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2\right) = \frac{9}{2}\kappa\epsilon^2 \tag{4.106}$$

As noted at the beginning of Chapter (4), strain term, ϵ, seen in relation (4.106) can be replaced by peridynamic stretch, s, since they are equivalent statements. Furthermore, equating peridynamic strain energy density result from to classical one, leads to determination of peridynamic bond constant, c, being a function of material volumetric stiffness. Namely,

$$c = \frac{18\kappa}{\pi\delta^4} \tag{4.107}$$

in which, κ stands for bulk modulus of body as introduced before. The bond constant derived in relation (4.107) can be invoked in pair-wise force density function given in (4.93) and by using the explicit form of stretch given by relation (4.76), peridynamic response function is obtained as follows.

$$\mathbf{f}(\eta,\xi)=\left(\frac{18\kappa}{\pi\delta^4}\right)\left(\frac{|y'-y|-|x'-x|}{|x'-x|}-\alpha\Delta T_{(j)}\right)\frac{y'-y}{|y'-y|} \tag{4.108}$$

The force states given by relations (4.44) and (4.45) can be related with relation (4.108) as follows.

$$\begin{aligned}\underline{\mathbf{F}}(x_{(i)},t)\langle x_{(j)}-x_{(i)}\rangle &= \frac{1}{2}\mathbf{f}\left(u'-u,x'-x,t\right)\\ &=\left(\frac{9\kappa}{\pi\delta^4}\right)\left(\frac{|y'-y|-|x'-x|}{|x'-x|}-\alpha\Delta T_{(j)}\right)\frac{y'-y}{|y'-y|}\end{aligned} \tag{4.109}$$

and

$$\begin{aligned}\underline{\mathbf{F}}(x_{(j)},t)\langle x_{(i)}-x_{(j)}\rangle &= \frac{1}{2}\mathbf{f}\left(u-u',x-x',t\right)=\\ &=-\left(\frac{9\kappa}{\pi\delta^4}\right)\left(\frac{|y'-y|-|x'-x|}{|x'-x|}-\alpha\Delta T_{(j)}\right)\frac{y'-y}{|y-y'|}\end{aligned} \tag{4.110}$$

Thus, pairwise force functions for material points x and x' are obtained as follows.

$$\begin{aligned}&f_{(j)(k)}\left(u_{(k)}-u_{(j)},x_{(k)}-x_{(j)},t\right)=\\ &=\frac{18\kappa\delta}{\pi\delta^5\left|x_{(k)}-x_{(j)}\right|}\left(\left|y_{(k)}-y_{(j)}\right|-\left|x_{(k)}-x_{(j)}\right|-\alpha\left|x_{(k)}-x_{(j)}\right|\Delta T_{(j)}\right)\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|}\end{aligned} \tag{4.111}$$

and

$$\begin{aligned}&f_{(k)(j)}\left(u_{(j)}-u_{(k)},x_{(j)}-x_{(k)},t\right)=\\ &=\frac{-18\kappa\delta}{\pi\delta^5\left|x_{(j)}-x_{(k)}\right|}\left(\left|y_{(j)}-y_{(k)}\right|-\left|x_{(j)}-x_{(k)}\right|-\alpha\left|x_{(j)}-x_{(k)}\right|\Delta T_{(j)}\right)\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|}\end{aligned} \tag{4.112}$$

in which influence functions related with material points (j) and (k) are defined respectively as

$$\omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle=\frac{\delta}{\left|x_{(k)}-x_{(j)}\right|}=\frac{\delta}{\xi} \tag{4.113}$$

and

$$\omega_{(k)(j)} \left\langle x_{(j)} - x_{(k)} \right\rangle = \frac{\delta}{\left| x_{(j)} - x_{(k)} \right|} = \frac{\delta}{\xi} \tag{4.114}$$

In case of any perturbation caused by external forces on material domain distorts equilibrium condition that is given by $\mathbf{f}_{(i)(j)} + \mathbf{f}_{(j)(i)} = 0$ between material points hence peridynamic equations of motion is expressed as a function of net resultant pair-wise forces as follows.

$$\rho(x,t)\ddot{u}(x,t) = \int_V \mathbf{f}(u' - u, x' - x, t)dV' + b\,(x,t) \tag{4.115}$$

which includes integration of pair-wise forces between material points instead of spatial derivatives of stress components.

Chapter 5

Formulations of Peridynamic Modelling for Deformation Fields

5.1 Strain energy and dilatation

Within the frame of bond-based peridynamics, configuring pair-wise forces in equal magnitude with same alignment and opposite direction in any force state, components of equations of motion given along relations (3.56 - 3.58) can be expressed in a more compact form with the assistance of an appropriate index notation $f_{(j)(k)\beta}V_{(k)} = \sigma_{(j)\alpha\beta}/2\Delta\alpha$, $f_{(k)(j)\beta}V_{(k)} = \sigma_{(k)\alpha\beta}/2\Delta\alpha$ and $-f_{(j)(p)\beta}V_{(p)} = \sigma_{(j)\alpha\beta}/2\Delta\alpha$, $-f_{(p)(j)\beta}V_{(p)} = \sigma_{(p)\alpha\beta}/2\Delta\alpha$ for $\alpha, \beta = x_1, x_2, x_3$, by which each term given inside brackets along relations (3.56 - 3.58) can be transformed into force densities and restated in accordance with relation (4.115).

As summation convention applies, surrounding particles are summed over indices, k and p, as $k = (j+l), (j+m), (j+n)$ and $p = (j-l), (j-m), (j-n)$ respectively according to index notation stated above.

In conjunction with relation (B.12), isotropic expansion or dilatation of associated material point labelled by (j) with addition of thermal effects can be restated as follows.

$$\epsilon_{(j)ii} = \left(\sigma_{(j)x_1x_1} + \sigma_{(j)x_2x_2} + \sigma_{(j)x_3x_3}\right)/3\kappa + 3\alpha\Delta T_{(j)} \tag{5.1}$$

The stress vectors belonging only to material point remarked by (j) are given as $\sigma_{(j)\alpha\beta} = f_{(j)(k)\beta} \cdot \left(x_{\beta(k)} - x_{\beta(j)}\right) V_{(k)}$ and $\sigma_{(j)\alpha\beta} = -f_{(j)(p)\beta} \cdot \left(x_{\beta(p)} - x_{\beta(j)}\right) V_{(p)}$. To this end, these stress components appearing in relations (3.53 - 3.55) for $\alpha = \beta = x_1, x_2, x_3$ are expressed respectively as follows.

For the first line of relation (3.56) in the right hand side, we perform following matchings.

$$\begin{aligned} \sigma_{(j)x_1x_1} &= f_{(j)(j+l)x_1} \cdot \left(x_{1(j+l)} - x_{1(j)}\right) V_{(j+l)} \\ \sigma_{(j-l)x_1x_1} &= -f_{(j-l)(j)x_1} \cdot \left(x_{1(j)} - x_{1(j-l)}\right) V_{(j-l)} \\ \sigma_{(j+l)x_1x_1} &= f_{(j+l)(j)x_1} \cdot \left(x_{1(j)} - x_{1(j+l)}\right) V_{(j+l)} \\ \sigma_{(j)x_1x_1} &= -f_{(j)(j-l)x_1} \cdot \left(x_{1(j-l)} - x_{1(j)}\right) V_{(j-l)} \end{aligned} \tag{5.2}$$

For the second line of relation (3.56) in the right hand side, we perform following matchings.

$$\begin{aligned} \sigma_{(j)x_1x_2} &= f_{(j)(j+m)x_2} \cdot \left(x_{2(j+m)} - x_{2(j)}\right) V_{(j+m)} \\ \sigma_{(j-m)x_1x_2} &= -f_{(j-m)(j)x_2} \cdot \left(x_{2(j)} - x_{2(j-m)}\right) V_{(j-m)} \\ \sigma_{(j+m)x_1x_2} &= f_{(j+m)(j)x_2} \cdot \left(x_{2(j)} - x_{2(j+m)}\right) V_{(j+m)} \\ \sigma_{(j)x_1x_2} &= -f_{(j)(j-m)x_2} \cdot \left(x_{2(j-m)} - x_{2(j)}\right) V_{(j-m)} \end{aligned} \tag{5.3}$$

For the third line of relation (3.56) in the right hand side, we perform following matchings.

$$\begin{aligned} \sigma_{(j)x_1x_3} &= f_{(j)(j+n)x_3} \cdot \left(x_{3(j+n)} - x_{3(j)}\right) V_{(j+n)} \\ \sigma_{(j-n)x_1x_3} &= -f_{(j-n)(j)x_3} \cdot \left(x_{3(j)} - x_{(j-n)}\right) V_{(j-n)} \\ \sigma_{(j+n)x_1x_3} &= f_{(j+n)(j)x_3} \cdot \left(x_{3(j)} - x_{3(j+n)}\right) V_{(j+n)} \\ \sigma_{(j)x_1x_3} &= -f_{(j)(j-n)x_3} \cdot \left(x_{3(j-n)} - x_{3(j)}\right) V_{(j-n)} \end{aligned} \tag{5.4}$$

For the first line of relation (3.57) in the right hand side, we perform following matchings.

$$\begin{aligned} \sigma_{(j)x_2x_1} &= f_{(j)(j+l)x_1} \cdot \left(x_{1(j+l)} - x_{1(j)}\right) V_{(j+l)} \\ \sigma_{(j-l)x_2x_1} &= -f_{(j-l)(j)x_1} \cdot \left(x_{1(j)} - x_{1(j-l)}\right) V_{(j-l)} \\ \sigma_{(j+l)x_2x_1} &= f_{(j+l)(j)x_1} \cdot \left(x_{1(j)} - x_{1(j+l)}\right) V_{(j+l)} \\ \sigma_{(j)x_2x_1} &= -f_{(j)(j-l)x_1} \cdot \left(x_{1(j-l)} - x_{1(j)}\right) V_{(j-l)} \end{aligned} \tag{5.5}$$

For the second line of relation (3.57) in the right hand side, we perform following matchings.

$$\begin{aligned}
\sigma_{(j)x_2x_2} &= f_{(j)(j+m)x_2} \cdot \left(x_{2(j+m)} - x_{2(j)}\right) V_{(j+m)} \\
\sigma_{(j-m)x_2x_2} &= -f_{(j-m)(j)x_2} \cdot \left(x_{2(j)} - x_{2(j-m)}\right) V_{(k-m)} \\
\sigma_{(j+m)x_2x_2} &= f_{(j+m)(j)x_2} \cdot \left(x_{2(j)} - x_{2(j+m)}\right) V_{(j+m)} \\
\sigma_{(j)x_2x_2} &= -f_{(j)(j-m)x_2} \cdot \left(x_{2(j-m)} - x_{2(j)}\right) V_{(j-m)}
\end{aligned} \tag{5.6}$$

For the third line of relation (3.57) in the right hand side, we perform following matchings.

$$\begin{aligned}
\sigma_{(j)x_2x_3} &= f_{(j)(j+n)x_3} \cdot \left(x_{3(j+n)} - x_{3(j)}\right) V_{(j+n)} \\
\sigma_{(j-n)x_2x_3} &= -f_{(j-n)(j)x_3} \cdot \left(x_{3(j)} - x_{3(j-n)}\right) V_{(k-n)} \\
\sigma_{(j+n)x_2x_3} &= f_{(j+n)(j)x_3} \cdot \left(x_{3(j)} - x_{3(j+n)}\right) V_{(j+n)} \\
\sigma_{(j)x_2x_3} &= -f_{(j)(j-n)x_3} \cdot \left(x_{3(j-n)} - x_{3(j)}\right) V_{(j-n)}
\end{aligned} \tag{5.7}$$

For the first line of relation (3.58) in the right hand side, we perform following matchings.

$$\begin{aligned}
\sigma_{(j)x_3x_1} &= f_{(j)(j+l)x_1} \cdot \left(x_{1(j+l)} - x_{1(j)}\right) V_{(j+l)} \\
\sigma_{(j-l)x_3x_1} &= -f_{(j-l)(j)x_1} \cdot \left(x_{1(j)} - x_{1(j-l)}\right) V_{(j-l)} \\
\sigma_{(j+l)x_3x_1} &= f_{(j+l)(j)x_1} \cdot \left(x_{1(j)} - x_{1(j+l)}\right) V_{(j+l)} \\
\sigma_{(j)x_3x_1} &= -f_{(j)(j-l)x_1} \cdot \left(x_{1(j-l)} - x_{1(j)}\right) V_{(j-l)}
\end{aligned} \tag{5.8}$$

For the second line of relation (3.58) in the right hand side, we perform following matchings.

$$\begin{aligned}
\sigma_{(j)x_3x_2} &= f_{(j)(j+m)x_2} \cdot \left(x_{2(j+m)} - x_{2(j)}\right) V_{(j+m)} \\
\sigma_{(j-m)x_3x_2} &= -f_{(j-m)(j)x_2} \cdot \left(x_{2(j)} - x_{2(j-m)}\right) V_{(k-m)} \\
\sigma_{(j+m)x_3x_2} &= f_{(j+m)(j)x_2} \cdot \left(x_{2(j)} - x_{2(j+m)}\right) V_{(j+m)} \\
\sigma_{(j)x_3x_2} &= -f_{(j)(j-m)x_2} \cdot \left(x_{2(j-m)} - x_{2(j)}\right) V_{(j-m)}
\end{aligned} \tag{5.9}$$

For the third line of relation (3.58) in the right hand side, we perform following matchings.

$$\begin{aligned}
\sigma_{(j)x_3x_3} &= f_{(j)(j+n)x_3} \cdot \left(x_{3(j+n)} - x_{3(j)}\right) V_{(j+n)} \\
\sigma_{(j-n)x_3x_3} &= -f_{(j-n)(j)x_3} \cdot \left(x_{3(j)} - x_{3(j-n)}\right) V_{(k-n)} \\
\sigma_{(j+n)x_3x_3} &= f_{(j+n)(j)x_3} \cdot \left(x_{3(j)} - x_{3(j+n)}\right) V_{(j+n)} \\
\sigma_{(j)x_3x_3} &= -f_{(j)(j-n)x_3} \cdot \left(x_{3(j-n)} - x_{3(j)}\right) V_{(j-n)}
\end{aligned} \tag{5.10}$$

Three stress components inside the brackets in relation (5.1) can be obtained with the combinations of first and fourth lines from above 36 components of stress state. Therefore, summation $\left(\sigma_{(j)x_1x_1} + \sigma_{(j)x_2x_2} + \sigma_{(j)x_3x_3}\right)$ can be expressed as follows.

$$\begin{aligned}
&0.5\left(\sigma_{(j)x_1x_1} + \sigma_{(j)x_1x_1} + \sigma_{(j)x_2x_2} + \sigma_{(j)x_2x_2} + \sigma_{(j)x_3x_3} + \sigma_{(j)x_3x_3}\right) = \\
&= 0.5\left(f_{(j)(j+l)x_1} \cdot \left(x_{1(j+l)} - x_{1(j)}\right) V_{(j+l)} - f_{(j)(j-l)x_1} \cdot \left(x_{1(j-l)} - x_{1(j)}\right) V_{(j-l)}\right) \\
&+ 0.5\left(f_{(j)(j+m)x_2} \cdot \left(x_{2(j+m)} - x_{2(j)}\right) V_{(j+m)} - f_{(j)(j-m)x_2} \cdot \left(x_{2(j-m)} - x_{2(j)}\right) V_{(j-m)}\right) \\
&+ 0.5\left(f_{(j)(j+n)x_3} \cdot \left(x_{3(j+n)} - x_{3(j)}\right) V_{(j+n)} - f_{(j)(j-n)x_3} \cdot \left(x_{3(j-n)} - x_{3(j)}\right) V_{(j-n)}\right) \\
&= 0.5 \sum_{\beta=x_1,x_2,x_3} \left(\sum_{k=j+l,j+m,j+n,j-l,j-m,j-n} f_{(j)(k)\beta} \cdot \left(x_{\beta(k)} - x_{\beta(j)}\right) V_{(j)} \right)
\end{aligned} \tag{5.11}$$

Accordingly, the stress components in relation (5.1) can be replaced with the last line of relation (5.11). As a conclusion, dilation term can be obtained as follows.

$$\epsilon_{(j)ii} = \left(\frac{1}{3K}\right) \sum_{k=j+l,j+m,j+n,j-l,j-m,j-n} 0.5 f_{(j)(k)} \cdot \left(x_{(k)} - x_{(j)}\right) V_{(j)} + 3\alpha\Delta T_{(j)} \tag{5.12}$$

The pairwise force function appearing as $f_{(j)(k)\beta}$ in relation (5.12) can be substituted for its equivalent form given by relation (4.111). To this end, a discrete definition for peridynamic dilatation is obtained as follows.

$$\begin{aligned}
\epsilon_{(j)ii} = \left(\frac{3}{\pi\delta^4}\right) \sum_k \left(\frac{\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right|}{\left|x_{(k)} - x_{(j)}\right|} - \alpha\Delta T_{(j)} \right) \frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|} \\
\cdot \left(x_{(k)} - x_{(j)}\right) V_{(j)} + 3\alpha\Delta T_{(j)}
\end{aligned} \tag{5.13}$$

or

$$\epsilon_{(j)ii} = \left(\frac{3}{\pi\delta^4}\right)\sum_k \left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right)\frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|} \cdot \frac{\left(x_{(k)} - x_{(j)}\right)}{\left|x_{(k)} - x_{(j)}\right|}V_{(j)} + 3\alpha\Delta T_{(j)} \tag{5.14}$$

Based on array representation of strain components given by relation (3.2), strain energy density of corresponding material point is written by means of combinations of relations (4.101) and (3.5) as follows.

$$\begin{aligned} W_{(j)} &= 0.5\sum_{m=1}^{3}\sum_{n=1}^{3}\sigma_{(j)mn}\epsilon_{(j)mn} \\ &= 0.5\left(\sigma_{(j)x_1x_1}\epsilon_{(j)x_1x_1} + \sigma_{(j)x_2x_2}\epsilon_{(j)x_2x_2} + \sigma_{(j)x_3x_3}\epsilon_{(j)x_3x_3}\right) \\ &+ \left(\sigma_{(j)x_2x_1}\epsilon_{(j)x_2x_1} + \sigma_{(j)x_2x_2}\epsilon_{(j)x_2x_2} + \sigma_{(j)x_2x_3}\epsilon_{(j)x_2x_3}\right) \\ &= 0.5\left(\frac{\sigma^2_{(j)x_1x_1}}{2\mu} - \left(\kappa - \frac{2\mu}{3}\right)\frac{\sigma_{(j)x_1x_1}\epsilon_{(j)ii}}{2\mu} - \frac{3\sigma_{(j)x_1x_1}K\alpha\Delta T_{(j)}}{2\mu}\right) \\ &+ 0.5\left(\frac{\sigma^2_{(j)x_2x_2}}{2\mu} - \left(\kappa - \frac{2\mu}{3}\right)\frac{\sigma_{(j)x_2x_2}\epsilon_{(j)ii}}{2\mu} - \frac{3\sigma_{(j)x_2x_2}K\alpha\Delta T_{(j)}}{2\mu}\right) \\ &+ 0.5\left(\frac{\sigma^2_{(j)x_3x_3}}{2\mu} - \left(\kappa - \frac{2\mu}{3}\right)\frac{\sigma_{(j)x_3x_3}\epsilon_{(j)ii}}{2\mu} - \frac{3\sigma_{(j)x_3x_3}K\alpha\Delta T_{(j)}}{2\mu}\right) \\ &+ \frac{\sigma^2_{(j)x_2x_3}}{2\mu} + \frac{\sigma^2_{(j)x_1x_3}}{2\mu} + \frac{\sigma^2_{(j)x_1x_2}}{2\mu} \end{aligned} \tag{5.15}$$

or

$$\begin{aligned} W_{(j)} &= \frac{1}{4\mu}\left(\sigma^2_{(j)x_1x_1} + \sigma^2_{(j)x_2x_2} + \sigma^2_{(j)x_3x_3}\right) + \frac{1}{2\mu}\left(\sigma^2_{(j)x_1x_2} + \sigma^2_{(j)x_1x_3} + \sigma^2_{(j)x_2x_3}\right) \\ &- \left(\left(\kappa - \frac{2\mu}{3}\right)\frac{\epsilon_{(j)ii}}{4\mu} + \frac{3\kappa\alpha\Delta T_{(j)}}{4\mu}\right)3\kappa\left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right) \end{aligned} \tag{5.16}$$

in which right hand side of relation (4.103) has been substituted for stress terms, $\sigma_{(j)ii}$, therefore strain energy density for material point labelled as (j) is obtained as follows.

$$\begin{aligned} W_{(j)} &= \frac{1}{4\mu}\left(\sigma^2_{(j)x_1x_1} + \sigma^2_{(j)x_2x_2} + \sigma^2_{(j)x_3x_3}\right) + \frac{1}{2\mu}\left(\sigma^2_{(j)x_1x_2} + \sigma^2_{(j)x_1x_3} + \sigma^2_{(j)x_2x_3}\right) \\ &+ \frac{\kappa}{2}\left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right)^2 - \frac{3\kappa^2\epsilon^2_{(j)ii}}{4\mu} \end{aligned} \tag{5.17}$$

The first line of relation (5.17) can be modified in a way that all components of Cauchy's stress tensor are explicitly expressed instead. Namely,

$$\begin{aligned}
W_{(j)} = & \frac{1}{8\mu}\left(\sigma^2_{(j)x_1x_1} + \sigma^2_{(j)x_1x_2} + \sigma^2_{(j)x_1x_3} + \sigma^2_{(j)x_1x_1} + \sigma^2_{(j)x_1x_2} + \sigma^2_{(j)x_1x_3}\right) \\
& + \frac{1}{8\mu}\left(\sigma^2_{(j)x_2x_1} + \sigma^2_{(j)x_2x_2} + \sigma^2_{(j)x_2x_3} + \sigma^2_{(j)x_2x_1} + \sigma^2_{(j)x_2x_2} + \sigma^2_{(j)x_2x_3}\right) \\
& + \frac{1}{8\mu}\left(\sigma^2_{(j)x_3x_1} + \sigma^2_{(j)x_3x_2} + \sigma^2_{(j)x_3x_3} + \sigma^2_{(j)x_3x_2} + \sigma^2_{(j)x_3x_2} + \sigma^2_{(j)x_3x_3}\right) \\
& + \frac{K}{2}\left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right)^2 - \frac{3\kappa^2\epsilon^2_{(j)ii}}{4\mu}
\end{aligned} \tag{5.18}$$

The second-powers of stress components can be replaced with their equivalent forms of pair-wise force functions given through a set of relations (5.2 - 5.10) in which stress states are matched with pair-wise forces based on $\sigma_{(j)\alpha\beta} = f_{(j)(k)\beta}\Delta\alpha V_{(k)}$ and $\sigma_{(j)\alpha\beta} = -f_{(j)(p)\beta}\Delta\alpha V_{(p)}$ with $k = (j+l), (j+m), (j+n)$ and $p = (j-l)$, $(j-m), (j-n)$ or each stress component are explicitly written in behalf of the second line in relation (5.18) as follows.

$$\begin{aligned}
\sigma_{(j)x_1x_1} &= f_{(j)(j+l)x_1}\left(x_{1(j+l)} - x_{1(j)}\right)V_{(j+l)} \\
\sigma_{(j)x_1x_2} &= f_{(j)(j+m)x_2}\left(x_{2(j+m)} - x_{2(j)}\right)V_{(j+m)} \\
\sigma_{(j)x_1x_3} &= f_{(j)(j+n)x_3}\left(x_{3(j+n)} - x_{3(j)}\right)V_{(j+n)} \\
\sigma_{(j)x_1x_1} &= -f_{(j)(j-l)x_1}\left(x_{1(j-l)} - x_{1(j)}\right)V_{(j-l)} \\
\sigma_{(j)x_1x_2} &= -f_{(j)(j-m)x_2}\left(x_{2(j-m)} - x_{2(j)}\right)V_{(j-m)} \\
\sigma_{(j)x_1x_3} &= -f_{(j)(j-n)x_3}\left(x_{3(j-n)} - x_{3(j)}\right)V_{(j-n)}
\end{aligned} \tag{5.19}$$

for the third line in (5.18) as

$$\begin{aligned}
\sigma_{(j)x_2x_1} &= f_{(j)(j+l)x_1}\left(x_{1(j+l)} - x_{1(j)}\right)V_{(j+l)} \\
\sigma_{(j)x_2x_2} &= f_{(j)(j+m)x_2}\left(x_{2(j+m)} - x_{2(j)}\right)V_{(j+m)} \\
\sigma_{(j)x_2x_3} &= f_{(j)(j+n)x_3}\left(x_{3(j+n)} - x_{3(j)}\right)V_{(j+n)} \\
\sigma_{(j)x_2x_1} &= -f_{(j)(j-l)x_1}\left(x_{1(j-l)} - x_{1(j)}\right)V_{(j-l)} \\
\sigma_{(j)x_2x_2} &= -f_{(j)(j-m)x_2}\left(x_{2(j-m)} - x_{2(j)}\right)V_{(j-m)} \\
\sigma_{(j)x_2x_3} &= -f_{(j)(j-n)x_3}\left(x_{3(j-n)} - x_{3(j)}\right)V_{(j-n)}
\end{aligned} \tag{5.20}$$

for the fourth line in (5.18) as

$$\begin{aligned}
\sigma_{(j)x_3x_1} &= f_{(j)(j+l)x_1}\left(x_{1(j+l)} - x_{1(j)}\right)V_{(j+l)} \\
\sigma_{(j)x_3x_2} &= f_{(j)(j+m)x_2}\left(x_{2(j+m)} - x_{2(j)}\right)V_{(j+m)} \\
\sigma_{(j)x_3x_3} &= f_{(j)(j+n)x_3}\left(x_{3(j+n)} - x_{3(j)}\right)V_{(j+n)} \\
\sigma_{(j)x_3x_1} &= -f_{(j)(j-l)x_1}\left(x_{1(j-l)} - x_{1(j)}\right)V_{(j-l)} \\
\sigma_{(j)x_3x_2} &= -f_{(j)(j-m)x_2}\left(x_{2(j-m)} - x_{2(j)}\right)V_{(j-m)} \\
\sigma_{(j)x_3x_3} &= -f_{(j)(j-n)x_3}\left(x_{3(j-n)} - x_{3(j)}\right)V_{(j-n)}
\end{aligned} \tag{5.21}$$

To this end, strain energy density function given by relation (5.18) leads to

$$\begin{aligned}
W_{(j)} = &\frac{\kappa}{2}\left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right)^2 - \frac{3\kappa^2}{4\mu}\epsilon^2_{(j)ii} \\
&+ \frac{1}{8\mu}\sum_{\beta=x_1,x_2,x_3}\left(\sum_{k=j+l,j+m,j+n,j-l,j-m,j-n} f^2_{(j)(k)\beta}\left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)}\right)
\end{aligned} \tag{5.22}$$

Expanding second power of the terms in the first line of relation (5.22) and assigning coefficients in front of terms that are $\epsilon_{(j)ii}$, $\epsilon_{(j)ii}\Delta T_{(j)}$ and $\Delta T^2_{(j)}$ to a_1, a_2 and a_3 respectively allow us to write strain energy density function as follows.

$$\begin{aligned}
W_{(j)} = &a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} \\
&+ \frac{1}{8\mu}\sum_{\beta=x_1,x_2,x_3}\left(\sum_{k=j+l,j+m,j+n,j-l,j-m,j-n} f^2_{(j)(k)\beta}\left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)}\right)
\end{aligned} \tag{5.23}$$

in which coefficients a_1, a_2 and a_3 are assigned as

$$a_1 = \left(\frac{\kappa}{2} - \frac{3\kappa^2}{4\mu}\right) \tag{5.24}$$

$$a_2 = 3\alpha\kappa \tag{5.25}$$

$$a_3 = \frac{9\alpha^2\kappa}{2} \tag{5.26}$$

Lastly, the terms that are summed over k index in relation (5.23) can be decomposed into its components along $\sigma^2_{(j)x_1\beta} = f^2_{(j)(k)\beta}\left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)}$ for $\alpha = x_1$,

$\beta = x_1, x_2, x_3$ and $k = (j+l), (j+m), (j+n)$ as

$$\begin{aligned}
\sigma^2_{(j)x_1x_1} &= f^2_{(j)(j+l)x_1}\left(x_{x_1(j+l)} - x_{x_1(j)}\right)^2 V^2_{(j+l)} \\
\sigma^2_{(j)x_1x_2} &= f^2_{(j)(j+m)x_2}\left(x_{x_2(j+m)} - x_{x_2(j)}\right)^2 V^2_{(j+m)} \\
\sigma^2_{(j)x_1x_3} &= f^2_{(j)(j+n)x_3}\left(x_{x_3(j+n)} - x_{x_3(j)}\right)^2 V^2_{(j+n)}
\end{aligned} \tag{5.27}$$

with $\sigma^2_{(j)x_1\beta} = f^2_{(j)(k)\beta}\left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)}$ for $\alpha = x_1$, $\beta = x_1, x_2, x_3$ and $k = (j-l), (j-m), (j-n)$ as

$$\begin{aligned}
\sigma^2_{(j)x_1x_1} &= f^2_{(j)(j-l)x_1}\left(x_{x_1(j-l)} - x_{x_1(j)}\right)^2 V^2_{(j-l)} \\
\sigma^2_{(j)x_1x_2} &= f^2_{(j)(j-m)x_2}\left(x_{x_2(j-m)} - x_{x_2(j)}\right)^2 V^2_{(j-m)} \\
\sigma^2_{(j)x_1x_3} &= f^2_{(j)(j-n)x_3}\left(x_{x_3(j-n)} - x_{x_3(j)}\right)^2 V^2_{(j-n)}
\end{aligned} \tag{5.28}$$

with $\sigma^2_{(j)x_2\beta} = f^2_{(j)(k)\beta}\left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)}$ for $\alpha = x_2$, $\beta = x_1, x_2, x_3$ and $k = (j+l), (j+m), (j+n)$ as

$$\begin{aligned}
\sigma^2_{(j)x_2x_1} &= f^2_{(j)(j+l)x_1}\left(x_{x_1(j+l)} - x_{x_1(j)}\right)^2 V^2_{(j+l)} \\
\sigma^2_{(j)x_2x_2} &= f^2_{(j)(j+m)x_2}\left(x_{x_2(j+m)} - x_{x_2(j)}\right)^2 V^2_{(j+m)} \\
\sigma^2_{(j)x_2x_3} &= f^2_{(j)(j+n)x_3}\left(x_{x_3(j+m)} - x_{x_3(j)}\right)^2 V^2_{(j+n)}
\end{aligned} \tag{5.29}$$

with $\sigma^2_{(j)x_2\beta} = f^2_{(j)(k)\beta}\left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)}$ for $\alpha = x_2$, $\beta = x_1, x_2, x_3$ and $k = (j-l), (j-m), (j-n)$ as

$$\begin{aligned}
\sigma^2_{(j)x_2x_1} &= f^2_{(j)(j-l)x_1}\left(x_{x_1(j-l)} - x_{x_1(j)}\right)^2 V^2_{(j-l)} \\
\sigma^2_{(j)x_2x_2} &= f^2_{(j)(j-m)x_2}\left(x_{x_2(j-m)} - x_{x_2(j)}\right)^2 V^2_{(j-m)} \\
\sigma^2_{(j)x_2x_3} &= f^2_{(j)(j-n)x_3}\left(x_{x_3(j-n)} - x_{x_3(j)}\right)^2 V^2_{(j-n)}
\end{aligned} \tag{5.30}$$

with $\sigma^2_{(j)x_3\beta} = f^2_{(j)(k)\beta}\left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)}$ for $\alpha = x_3$, $\beta = x_1, x_2, x_3$ and $k = (j+l), (j+m), (j+n)$ as

$$\begin{aligned}
\sigma^2_{(j)x_3x_1} &= f^2_{(j)(j+l)x_1}\left(x_{x_1(j+l)} - x_{x_1(j)}\right)^2 V^2_{(j+l)} \\
\sigma^2_{(j)x_3x_2} &= f^2_{(j)(j+m)x_2}\left(x_{x_2(j+m)} - x_{x_2(j)}\right)^2 V^2_{(j+m)} \\
\sigma^2_{(j)x_3x_3} &= f^2_{(j)(j+n)x_3}\left(x_{x_3(j+n)} - x_{x_3(j)}\right)^2 V^2_{(j+n)}
\end{aligned} \tag{5.31}$$

with $\sigma^2_{(j)x_3\beta} = f^2_{(j)(k)\beta}\left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)}$ for $\alpha = x_3$, $\beta = x_1, x_2, x_3$ and $k = (j-l), (j-m), (j-n)$ as

$$\begin{aligned}
\sigma^2_{(j)x_3x_1} &= f^2_{(j)(j-l)x_1}\left(x_{x_3(j-l)} - x_{x_3(j)}\right)^2 V^2_{(j-l)} \\
\sigma^2_{(j)x_3x_2} &= f^2_{(j)(j-m)x_2}\left(x_{x_3(j-m)} - x_{x_3(j)}\right)^2 V^2_{(j-m)} \\
\sigma^2_{(j)x_3x_3} &= f^2_{(j)(j-n)x_3}\left(x_{x_3(j-n)} - x_{x_3(j)}\right)^2 V^2_{(j-n)}
\end{aligned} \tag{5.32}$$

Consequently, relation (5.23) can be expressed in component base explicit notation as the terms in third, fourth and fifth lines are expanded according to those that are listed through sets of relations along (5.27 - 5.32). Therefore strain energy density for material point labelled by (j) is written as follows.

$$\begin{aligned}
&W_{(j)} = a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} \\
&+ f^2_{(j)(j+l)x_1}\left(x_{x_1(j+l)} - x_{x_1(j)}\right)^2 V^2_{(j+l)}/8\mu + f^2_{(j)(j+m)x_2}\left(x_{x_2(j+m)} - x_{x_2(j)}\right)^2 V^2_{(j+m)}/8\mu \\
&+ f^2_{(j)(j+n)x_3}\left(x_{x_3(j+n)} - x_{x_3(j)}\right)^2 V^2_{(j+n)}/8\mu \\
&+ f^2_{(j)(j-l)x_1}\left(x_{x_1(j-l)} - x_{x_1(j)}\right)^2 V^2_{(j-l)}/8\mu + f^2_{(j)(j-m)x_2}\left(x_{x_2(j-m)} - x_{x_2(j)}\right)^2 V^2_{(j-m)}/8\mu \\
&+ f^2_{(j)(j-n)x_3}\left(x_{x_3(j-n)} - x_{x_3(j)}\right)^2 V^2_{(j-n)}/8\mu \\
&+ f^2_{(j)(j+l)x_1}\left(x_{x_1(j+l)} - x_{x_1(j)}\right)^2 V^2_{(j+l)}/8\mu + f^2_{(j)(j+m)x_2}\left(x_{x_2(j+m)} - x_{x_2(j)}\right)^2 V^2_{(j+m)}/8\mu \\
&+ f^2_{(j)(j+n)x_3}\left(x_{x_3(j+m)} - x_{x_3(j)}\right)^2 V^2_{(j+n)}/8\mu \\
&+ f^2_{(j)(j-l)x_1}\left(x_{x_1(j-l)} - x_{x_1(j)}\right)^2 V^2_{(j-l)}/8\mu + f^2_{(j)(j-m)x_2}\left(x_{x_2(j-m)} - x_{x_2(j)}\right)^2 V^2_{(j-m)}/8\mu \\
&+ f^2_{(j)(j-n)x_3}\left(x_{x_3(j-n)} - x_{x_3(j)}\right)^2 V^2_{(j-n)}/8\mu \\
&+ f^2_{(j)(j+l)x_1}\left(x_{x_1(j+l)} - x_{x_1(j)}\right)^2 V^2_{(j+l)}/8\mu + f^2_{(j)(j+m)x_2}\left(x_{x_2(j+m)} - x_{x_2(j)}\right)^2 V^2_{(j+m)}/8\mu \\
&+ f^2_{(j)(j+n)x_3}\left(x_{x_3(j+n)} - x_{x_3(j)}\right)^2 V^2_{(j+n)}/8\mu \\
&+ f^2_{(j)(j-l)x_1}\left(x_{x_3(j-l)} - x_{x_3(j)}\right)^2 V^2_{(j-l)}/8\mu + f^2_{(j)(j-m)x_2}\left(x_{x_3(j-m)} - x_{x_3(j)}\right)^2 V^2_{(j-m)}/8\mu \\
&+ f^2_{(j)(j-n)x_3}\left(x_{x_3(j-n)} - x_{x_3(j)}\right)^2 V^2_{(j-n)}/8\mu
\end{aligned} \tag{5.33}$$

Furthermore, discrete equivalents of pair-wise forces given by relation (4.111) can be invoked in expression (5.23) and strain energy density for material point labelled by (j) is obtained in its general form for its explicit evaluation.

$$W_{(j)} = a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} + \frac{1}{8\mu}\sum_{\beta}\sum_{k}\left(\frac{18\kappa\delta}{\pi\delta^5\left|x_{\beta(k)} - x_{\beta(j)}\right|}\right)^2 \left(\left|y_{\beta(k)} - y_{\beta(j)}\right| - \left|x_{\beta(k)} - x_{\beta(j)}\right| - \alpha\left|x_{\beta(k)} - x_{\beta(j)}\right|\Delta T_{(j)}\right)^2 \left(\frac{y_{\beta(k)} - y_{\beta(j)}}{\left|y_{\beta(k)} - y_{\beta(j)}\right|}\right)^2 \left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)} \tag{5.34}$$

in which k and β stand for $(j+l), (j-l), (j+m), (j-m), (j+n), (j-n)$ and x_1, x_2, x_3 respectively. By writing the relative position vector in denominator under summation sign, strain energy density for the material point (j) becomes

$$W_{(j)} = a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} + \frac{1}{8\mu}\left(\frac{18\kappa}{\pi\delta^4}\right)^2 \sum_{\beta}\sum_{k}\left(\frac{\left|y_{\beta(k)} - y_{\beta(j)}\right| - \left|x_{\beta(k)} - x_{\beta(j)}\right|}{\left|x_{\beta(k)} - x_{\beta(j)}\right|} - \alpha\Delta T_{(j)}\right)^2 \left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V^2_{(k)} \tag{5.35}$$

or

$$W_{(j)} = a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} + \frac{V}{8\mu}\left(\frac{18\kappa}{\pi\delta^4}\right)^2 \sum_{\beta}\sum_{k}\left(s_{(j)(k)\beta} - \alpha\Delta T_{(j)}\right)^2 \left|x_{\beta(k)} - x_{\beta(j)}\right|^2 V_{(k)} \tag{5.36}$$

For the sake of simplicity, terms appearing in front of summation signs in relation (5.36) can be assigned to a coefficient, b.

$$b = \frac{V}{8\mu}\left(\frac{18\kappa}{\pi\delta^4}\right)^2 \tag{5.37}$$

in which V represents the volume of surrounding material points as a constant parameter thus it can be taken out of the summation. As a conclusion, strain energy density function, $W_{(j)}$, becomes

$$\begin{aligned}W_{(j)} &= a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} \\ &+ b\sum_{\beta}\sum_{k}\left(\left|y_{\beta(k)} - y_{\beta(j)}\right| - \left|x_{\beta(k)} - x_{\beta(j)}\right| - \alpha\left|x_{\beta(k)} - x_{\beta(j)}\right|\Delta T_{(j)}\right)^2 \\ &\frac{\left|x_{\beta(k)} - x_{\beta(j)}\right|^2}{\left|x_{\beta(k)} - x_{\beta(j)}\right|^2}V_{(k)} \\ &= a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} \\ &+ b\sum_{\beta}\sum_{k}\left(\left|y_{\beta(k)} - y_{\beta(j)}\right| - \left|x_{\beta(k)} - x_{\beta(j)}\right| - \alpha\left|x_{\beta(k)} - x_{\beta(j)}\right|\Delta T_{(j)}\right)^2 V_{(k)}\end{aligned} \tag{5.38}$$

or

$$\begin{aligned}W_{(j)} &= a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} \\ &+ b\sum_{\beta}\sum_{k}\left(\left|y_{\beta(k)} - y_{\beta(j)}\right| - \left|x_{\beta(k)} - x_{\beta(j)}\right| - \alpha\left|x_{\beta(k)} - x_{\beta(j)}\right|\Delta T_{(j)}\right)^2 V_{(k)}\end{aligned} \tag{5.39}$$

with k and β are for $(j+l), (j-l), (j+m), (j-m), (j+n), (j-n)$ and x_1, x_2, x_3 respectively.

5.1.1 Relating deformation constants with peridynamic parameters

Deformation parameters in peridynamic theory are determined through equating definitions of dilatation and strain energy terms in classical and peridynamic theories since it is supposed that both have to yield same values, [22], [25]

Notation in classical theory to define infinitesimal differential distance between two material points, dx corresponds to $\left(x_{(k)} - x_{(j)}\right)$ in peridynamics before. On the other hand, differential distance between these two particles is given classically by dy and with its peridynamic equivalent, $\left(y_{(k)} - y_{(j)}\right)$ after deformation.

In this manner, distance between two particles in deformed configuration can be expressed through summation of initial distance and deformation amount between interacting particles. Namely,

$$dy = dx + du + \alpha\Delta T dx = \left(1 + \frac{du}{dx} + \alpha\Delta T\right)dx = \left(1 + \epsilon + \alpha\Delta T\right)dx \tag{5.40}$$

or mechanical strain with thermal effect can be expressed as

$$\frac{dy}{dx} = 1 + \epsilon + \alpha \Delta T \tag{5.41}$$

leading to

$$\varepsilon_{ii} = \frac{dy}{dx} - 1 = \frac{dy - dx}{dx} = \epsilon + \alpha \Delta T \tag{5.42}$$

On the other hand, relative position vector in deformed state of body in peridynamic notation with the addition of thermal effect can be expressed as follows.

$$\begin{aligned} y_{(k)} - y_{(j)} &= \left(x_{(k)} - x_{(j)}\right) + \left(u_{(k)} - u_{(j)}\right) + \alpha \left|x_{(k)} - x_{(j)}\right| \Delta T_{(j)} \\ &= \left(1 + \frac{u_{(k)} - u_{(j)}}{\left|x_{(k)} - x_{(j)}\right|} + \alpha \Delta T_{(j)}\right) \left(x_{(k)} - x_{(j)}\right) \\ &= \left(1 + \zeta + \alpha \Delta T_{(j)}\right) \left(x_{(k)} - x_{(j)}\right) = \left(1 + \zeta + \alpha \Delta T_{(j)}\right) \xi \end{aligned} \tag{5.43}$$

or mechanical stretch with thermal effect is expressed as follows.

$$\frac{y_{(k)} - y_{(j)}}{x_{(k)} - x_{(j)}} = 1 + \zeta + \alpha \Delta T_{(j)} \tag{5.44}$$

leading to

$$\epsilon_{(j)} = \frac{y_{(k)} - y_{(j)}}{x_{(k)} - x_{(j)}} - 1 = \frac{y_{(k)} - y_{(j)} - \left(x_{(k)} - x_{(j)}\right)}{x_{(k)} - x_{(j)}} = \zeta + \alpha \Delta T_{(j)} \tag{5.45}$$

The parameters that are associating strain energy and dilatation in classical approach with their peridynamic equivalents can be determined in a way that strain energy density function expressed by relation (4.106) is equated to the result given by relation (5.23).

Well know deformation cases that are isotropic expansion and pure shearing disturbance of an infinitesimal RVE are applied to be able to obtain peridynamic deformation parameters.

5.2 Peridynamic parameters for three-dimensional structures

Based on deformation case for pure isotropic expansion in which normal and shearing strain components can be summarized through $\epsilon_{(j)x_1x_1} = \epsilon_{(j)x_2x_2} = \epsilon_{(j)x_3x_3} =$

$\zeta + \alpha \Delta T_{(j)}$ and $\gamma_{(j)x_1x_2} = \gamma_{(j)x_1x_3} = \gamma_{(j)x_2x_3} = 0$ in classical continuum mechanics,

Based on this summary, strain energy density function and dilatation term in case of isotropic expansion are evaluated as follows.

$$W_{IsoExp} = 0.5 \sum_{m=1}^{3} \sum_{n=1}^{3} \sigma_{(j)mn} \epsilon_{(j)mn} = \begin{bmatrix} \sigma_{x_1x_1} & \sigma_{x_1x_2} & \sigma_{x_1x_3} \\ \sigma_{x_2x_1} & \sigma_{x_2x_2} & \sigma_{x_2x_3} \\ \sigma_{x_3x_1} & \sigma_{x_3x_2} & \sigma_{x_3x_3} \end{bmatrix} \begin{bmatrix} \epsilon_{x_1x_1} & \epsilon_{x_1x_2} & \epsilon_{x_1x_3} \\ \epsilon_{x_2x_1} & \epsilon_{x_2x_2} & \epsilon_{x_2x_3} \\ \epsilon_{x_3x_1} & \epsilon_{x_3x_2} & \epsilon_{x_3x_3} \end{bmatrix}$$
$$= 0.5 \left(\sigma_{11} \epsilon_{x_1x_1} + \sigma_{x_2x_2} \epsilon_{x_2x_2} + \sigma_{x_3x_3} \epsilon_{x_3x_3} \right) + \sigma_{x_1x_2} \epsilon_{x_1x_2} + \sigma_{x_1x_3} \epsilon_{x_1x_3} + \sigma_{x_2x_3} \epsilon_{x_2x_3} \tag{5.46}$$

Because of stiffness matrix for an isotropic material and strain conditions summarized above for isotropic expansion, relation (5.46) leads to

$$\begin{aligned} W_{IsoExp} = & \left(\kappa + \frac{4\mu}{3} \right) \epsilon_{x_1x_1} \epsilon_{x_1x_1} + \left(\kappa - \frac{2\mu}{3} \right) \epsilon_{x_2x_2} \epsilon_{x_1x_1} + \left(\kappa - \frac{2\mu}{3} \right) \epsilon_{x_3x_3} \epsilon_{x_1x_1} \\ & + \left(\kappa - \frac{2\mu}{3} \right) \epsilon_{x_1x_1} \epsilon_{x_2x_2} + \left(\kappa + \frac{4\mu}{3} \right) \epsilon_{x_2x_2} \epsilon_{x_2x_2} + \left(\kappa - \frac{2\mu}{3} \right) \epsilon_{x_3x_3} \epsilon_{x_2x_2} \\ & + \left(\kappa - \frac{2\mu}{3} \right) \epsilon_{x_1x_1} \epsilon_{x_3x_3} + \left(\kappa - \frac{2\mu}{3} \right) \epsilon_{x_2x_2} \epsilon_{x_3x_3} + \left(\kappa + \frac{4\mu}{3} \right) \epsilon_{x_3x_3} \epsilon_{x_3x_3} \\ & + \mu \epsilon_{x_2x_3} \epsilon_{x_2x_3} + \mu \epsilon_{x_1x_3} \epsilon_{x_1x_3} + \mu \epsilon_{x_1x_2} \epsilon_{x_1x_2} \end{aligned} \tag{5.47}$$

or

$$\begin{aligned} W_{IsoExp} = & K \left(\epsilon_{x_1x_1}^2 + \epsilon_{x_2x_2}^2 + \epsilon_{x_3x_3}^2 \right) + 2\kappa \left(\epsilon_{x_1x_1} \epsilon_{x_2x_2} + \epsilon_{x_1x_1} \epsilon_{x_3x_3} + \epsilon_{x_2x_2} \epsilon_{x_3x_3} \right) \\ & + \frac{4\mu}{3} \left(\epsilon_{x_1x_1}^2 + \epsilon_{x_2x_2}^2 + \epsilon_{x_3x_3}^2 \right) - \frac{4\mu}{3} \left(\epsilon_{x_1x_1} \epsilon_{x_2x_2} + \epsilon_{x_1x_1} \epsilon_{x_3x_3} + \epsilon_{x_2x_2} \epsilon_{x_3x_3} \right) \\ = & \frac{9}{2} \kappa \zeta^2 \end{aligned} \tag{5.48}$$

and for isotropic expansion case, dilatation becomes

$$\epsilon_{(j)ii} = \epsilon_{(j)x_1x_1} + \epsilon_{(j)x_2x_2} + \epsilon_{(j)x_3x_3} = 3 \left(\zeta + \alpha \Delta T_{(j)} \right) \tag{5.49}$$

Exact computation of dilatation and strain energy function can be performed based on their discrete formulations given by relations (5.14) and (5.39) with addition of dimensionless influence function, $\omega_{(j)(k)}$, that takes the importance of distance effect of interacting particles into account, [22]. Namely,

$$\epsilon_{(j)ii} = d \int\limits_V \omega_{(j)(k)} \left\langle x_{(k)} - x_{(j)} \right\rangle \left(\left| y_{(k)} - y_{(j)} \right| - \left| x_{(k)} - x_{(j)} \right| - \alpha \left| x_{(k)} - x_{(j)} \right| \Delta T_{(j)} \right)$$
$$\frac{y_{(k)} - y_{(j)}}{\left| y_{(k)} - y_{(j)} \right|} \cdot \frac{x_{(k)} - x_{(j)}}{\left| x_{(k)} - x_{(j)} \right|} dV_{(k)} + 3\alpha \Delta T_{(j)} \tag{5.50}$$

in which d is defined as follows.

$$d = \left(\frac{3}{\pi \delta^4} \right) \tag{5.51}$$

and strain energy density function, $W_{(j)}$ becomes

$$W_{(j)} = a_1 \epsilon_{(j)ii}^2 - a_2 \epsilon_{(j)ii} \Delta T_{(j)} + a_3 \Delta T_{(j)}^2 + b \int\limits_V \omega_{(j)(k)} \left\langle x_{(k)} - x_{(j)} \right\rangle$$
$$\left(\left| y_{(k)} - y_{(j)} \right| - \left| x_{(k)} - x_{(j)} \right| - \alpha \left| x_{(k)} - x_{(j)} \right| \Delta T_{(j)} \right)^2 dV_{(k)} \tag{5.52}$$

in which b and $\omega_{(j)(k)}$ that is assigned according to dimensional analysis in strain energy density are given by respectively as follows.

$$b = \frac{V}{8\mu} \left(\frac{18\kappa}{\pi \delta^4} \right)^2 \tag{5.53}$$

and

$$\omega_{(j)(k)} \left\langle x_{(k)} - x_{(j)} \right\rangle = \frac{\delta}{\left| x_{(k)} - x_{(j)} \right|} = \frac{\delta}{|\xi|} \tag{5.54}$$

Classical equivalence of elongation or shrinkage can be evaluated as multiple of unitary change by total initial length of directional span and change in temperature can be added to this change as well.

To this end, deformation amount to which thermal effects are added can be expressed by means of relation (5.45), in terms of classical stretch parameter ζ as follows.

$$\begin{aligned} \left| y_{(k)} - y_{(j)} \right| - \left| x_{(k)} - x_{(j)} \right| &= \left(\zeta + \alpha \Delta T_{(j)} \right) \left| x_{(k)} - x_{(j)} \right| \\ &= \left(\zeta + \alpha \Delta T_{(j)} \right) |\xi| \end{aligned} \tag{5.55}$$

in which α is defined as coefficient of thermal expansion while $\Delta T_{(j)}$ stands for temperature change on material point labelled by (j).

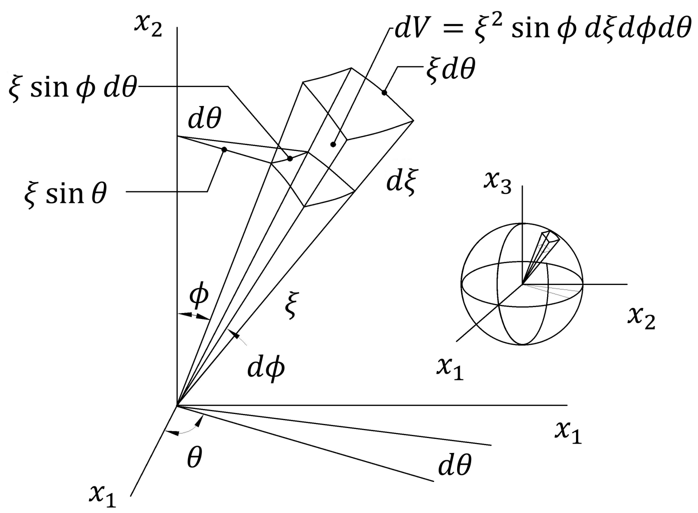


Figure 5.1: Three-dimensional integration domain for volume of a material point

Substitutions of relations (5.43), (5.54) and (5.55) in both (5.50) and (5.52) respectively enables calculating dilatation and strain energy density functions in case of isotropic expansion of a body through following integrations respectively.

$$\begin{aligned}\epsilon_{(j)ii} =& d_{3D} \int_V \frac{\delta}{|\xi|} \left(\left(\zeta + \alpha\Delta T_{(j)} \right) |\xi| - \alpha\Delta T_{(j)} |\xi| \right) \frac{\left(1+\zeta+\alpha\Delta T_{(j)}\right) \xi}{\left(1+\zeta+\alpha\Delta T_{(j)}\right) |\xi|} \cdot \frac{\xi}{|\xi|} dV_{(k)} \\ &+ 3\alpha\Delta T_{(j)}\end{aligned} \tag{5.56}$$

whose evaluation leads to

$$\begin{aligned}\epsilon_{(j)ii,IsoExp} =& d_{3D} \int_0^{2\pi}\int_0^{\pi}\int_0^{\delta} \frac{\delta}{|\xi|} \left(\left(\zeta + \alpha\Delta T_{(j)} \right) - \alpha\Delta T_{(j)} \right) |\xi| \frac{|\xi|^2 \cos(0)}{|\xi|^2} \xi^2 \sin(\phi) \, d\xi d\phi d\theta \\ &+ 3\alpha\Delta T_{(j)} \\ =& \, d_{3D}\zeta\delta \int_0^{2\pi}\int_0^{\pi}\int_0^{\delta} \sin(\phi) \, \xi^2 d\xi d\phi d\theta + 3\alpha\Delta T_{(j)} = \frac{4\pi d_{3D}\delta^4}{3}\zeta + 3\alpha\Delta T_{(j)}\end{aligned} \tag{5.57}$$

Equating results for dilatation in classical approach and peridynamics that are given by relations (5.49) and (5.57) leads to determination of coefficient d_{3D} which is

$$\begin{aligned}3\zeta + 3\alpha\Delta T_{(j)} &= \frac{4\pi d_{3D}\delta^4}{3}\zeta + 3\alpha\Delta T_{(j)} \\ d_{3D} &= \frac{9}{4\pi\delta^4}\end{aligned} \tag{5.58}$$

and exact evaluation of strain energy density function based on relation (5.52) by using definition of dilatation given by relation (5.49) leads to

$$\begin{aligned}W_{(j),IsoExp} =& a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} \\ &+ b_{3D}\int_0^{2\pi}\int_0^{\pi}\int_0^{\delta}\frac{\delta}{|\xi|}\left(\left(\zeta+\alpha\Delta T_{(j)}\right)-\alpha\Delta T_{(j)}\right)^2\xi^2\sin(\phi)\,d\xi d\phi d\theta \\ =& a_1\epsilon^2_{(j)ii} - a_2\epsilon_{(j)ii}\Delta T_{(j)} + a_3\Delta T^2_{(j)} + b_{3D}\delta\int_0^{2\pi}\int_0^{\pi}\int_0^{\delta}\zeta^2\xi^3\sin(\phi)\,d\xi d\phi d\theta \\ =& a_1\left(3\zeta+3\alpha\Delta T_{(j)}\right)^2 - a_2\left(3\zeta+3\alpha\Delta T_{(j)}\right)\Delta T_{(j)} + a_3\Delta T^2_{(j)} + b_{3D}\pi\zeta^2\delta^5\end{aligned} \tag{5.59}$$

Furthermore, following connections can be generated by means of energy coupling of classical and peridynamic definitions for isotropic expansion of a body.

$$a_1\left(3\zeta+3\alpha\Delta T_{(j)}\right)^2 - a_2\left(3\zeta+3\alpha\Delta T_{(j)}\right)\Delta T_{(j)} + a_3\Delta T^2_{(j)} + b_{3D}\pi\zeta^2\delta^5 = \frac{9}{2}K\zeta^2 \tag{5.60}$$

By means of relation (5.60), three lines of equation can be obtained as follows.

$$\begin{aligned}\left(9a_1 + b_{3D}\pi\delta^5\right)\zeta^2 &= \frac{9}{2}\kappa\zeta^2 \\ a_1 &= \frac{\kappa}{2} - \frac{b_{3D}\pi\delta^5}{9}\end{aligned} \tag{5.61}$$

$$\begin{aligned}18a_1\alpha\zeta\Delta T_{(j)} - 3a_2\zeta\Delta T_{(j)} &= 0 \\ \left(18a_1\alpha\zeta - 3a_2\zeta\right)\Delta T_{(j)} &= 0 \\ a_2 &= a_1(6\alpha)\end{aligned} \tag{5.62}$$

Consequently, parameter a_3 can be expressed in terms of other parameter a_1 by substituting result obtained from relation (5.62) as follows.

$$\begin{aligned}9a_1\alpha^2\Delta T^2_{(j)} - 3a_2\alpha\Delta T^2_{(j)} + a_3\Delta T^2_{(j)} &= 0 \\ \left(9a_1\alpha^2 - 18a_1\alpha\alpha + a_3\right)\Delta T^2_{(j)} &= 0 \\ a_3 &= a_1\left(9\alpha^2\right)\end{aligned} \tag{5.63}$$

The other fundamental type of deformation is pure in-plane shearing effect on infinitesimal cubic material element for which strain conditions can be abstracted through $\gamma_{(j)x_1x_2} = \zeta$, $\gamma_{(j)x_1x_3} = \gamma_{(j)x_2x_3} = \epsilon_{(j)x_1x_1} = \epsilon_{(j)x_2x_2} = \epsilon_{(j)x_3x_3} = 0$ and $\Delta T_{(j)} = 0$. According to classical shear strain energy density function and specific strain condition for an isotropic body in under the effect pure shearing is appraised as follows.

$$
\begin{aligned}
W_{Shear} = &\ 0.5\left(\left(\kappa+\frac{4\mu}{3}\right)\epsilon_{x_1x_1}\epsilon_{x_1x_1}+\left(\kappa-\frac{2\mu}{3}\right)\epsilon_{x_2x_2}\epsilon_{x_1x_1}+\left(\kappa-\frac{2\mu}{3}\right)\epsilon_{x_3x_3}\epsilon_{x_1x_1}\right)\\
&+0.5\left(\left(\kappa-\frac{2\mu}{3}\right)\epsilon_{x_1x_1}\epsilon_{x_2x_2}+\left(\kappa+\frac{4\mu}{3}\right)\epsilon_{x_2x_2}\epsilon_{x_2x_2}+\left(\kappa-\frac{2\mu}{3}\right)\epsilon_{x_3x_3}\epsilon_{x_2x_2}\right)\\
&+0.5\left(\left(\kappa-\frac{2\mu}{3}\right)\epsilon_{x_1x_1}\epsilon_{x_3x_3}+\left(\kappa-\frac{2\mu}{3}\right)\epsilon_{x_2x_2}\epsilon_{x_3x_3}+\left(\kappa+\frac{4\mu}{3}\right)\epsilon_{x_3x_3}\epsilon_{x_3x_3}\right)\\
&+\mu\epsilon_{x_2x_3}\epsilon_{x_2x_3}+\mu\epsilon_{x_1x_3}\epsilon_{x_1x_3}+\mu\epsilon_{x_1x_2}\epsilon_{x_1x_2}\\
&=0.5\mu\zeta^2
\end{aligned}
\tag{5.64}
$$

in which μ stands for shear modulus of medium experiencing pure shear deformation whereas zero dilatation is observed since no volume change occurs during pure shearing deformation which is proven through appropriate modification of integration given by expression (5.50) after exclusion of thermoelastic effect during shearing deformation. Namely,

$$
\epsilon_{(j)ii} = d_{3D}\int\limits_V \frac{\delta}{|\xi|}\left(\left|y_{(k)}-y_{(j)}\right|-\left|x_{(k)}-x_{(j)}\right|\right)\frac{\left|y_{(k)}-y_{(j)}\right|\left|x_{(k)}-x_{(j)}\right|\cos(\zeta)}{\left|y_{(k)}-y_{(j)}\right|\left|x_{(k)}-x_{(j)}\right|}dV_{(k)}
\tag{5.65}
$$

in which in-plane shearing deformation, the angle $\gamma_{x_1x_2} = \zeta$ between relative position vectors in reference and deformed states that are already defined as $\left(x_{(k)}-x_{(j)}\right)$ and $\left(y_{(k)}-y_{(j)}\right)$ respectively causes displacement amounts $\xi_{x_3}\sin\left(\gamma_{x_1x_2}\right)$ and $\xi_{x_3}\zeta$ along x_2 direction.

By means of polar coordinate representation for position vector between two material points in undeformed configuration, coordinate components are written instead as follows.

$$
\begin{aligned}
\xi_{x_1} &= \xi\sin(\phi)\cos(\theta)\\
\xi_{x_2} &= \xi\sin(\phi)\sin(\theta)\\
\xi_{x_3} &= \xi\cos(\phi)
\end{aligned}
\tag{5.66}
$$

To this end, vertical displacement and change in bond length can be determined for infinitesimal deformation angle ζ through following relations below.

$$
\begin{aligned}
\zeta\xi_{x_3} &= \xi\cos(\phi)\sin(\zeta)\\
&= \zeta\xi\cos(\phi)
\end{aligned}
\tag{5.67}
$$

with small angle approximation that is $\sin(\zeta)\approx\zeta$. This deformation length along shearing forces can be converted to deformation amount of relative positions by multiplying term $\zeta\xi_{x_3}$ with $(\sin(\phi)\sin(\theta))$. Therefore final length of a bond after deformation of body is expressed as follows.

$$y_{(k)} - y_{(j)} = \left(x_{(k)} - x_{(j)}\right) + \zeta\xi\cos(\phi)\sin(\phi)\sin(\theta) \tag{5.68}$$

or

$$y_{(k)} - y_{(j)} = \left(1 + \zeta\cos(\phi)\sin(\phi)\sin(\theta)\right)\xi \tag{5.69}$$

By substituting of final length for a bond given by relation (5.69) into relation (5.65), dilatation term with small degree approximation that is $\cos(\zeta) \approx 1$ can be expressed as follows.

$$\begin{aligned}\epsilon_{(j)ii,Shear} &= d_{3D}\int_0^{2\pi}\int_0^{\pi}\int_0^{\delta}\frac{\delta}{|\xi|}\zeta\,|\xi|\left(\xi\zeta\cos(\phi)\sin(\phi)\sin(\theta)\right)\cos(\zeta)\,\xi^2\sin(\phi)\,d\xi d\phi d\theta \\ &= d_{3D}\frac{\delta^4}{3}\int_0^{\pi}\cos(\phi)\sin^2(\phi)\,d\phi\int_0^{2\pi}\sin(\theta)\,d\theta = 0\end{aligned} \tag{5.70}$$

As performed while connecting strain energies in classical with peridynamic results, evaluation of strain energy density in case of pure in-plane shearing deformation is calculated by inserting deformed bond length given by relation (5.69) into relation (5.52).

$$\begin{aligned}W_{(j)PDShear} &= b_{3D}\int_0^{2\pi}\int_0^{\pi}\int_0^{\delta}\frac{\delta}{|\xi|}\left(\xi\zeta\cos(\phi)\sin(\phi)\sin(\theta)\right)^2\xi^2\sin(\phi)\,d\xi d\phi d\theta \\ &= b_{3D}\frac{\delta^5\zeta^2}{4}\int_0^{2\pi}\sin^2(\theta)\,d\theta\int_0^{\pi}\sin^3(\phi)\cos^2(\phi)\,d\phi \\ &= b_{3D}\frac{\pi\delta^5\zeta^2}{4}\left(\frac{1}{30}\cos^3(\pi)\left(3\cos(2\pi)-7\right) - \frac{1}{30}\cos^3(0)\left(3\cos(0)-7\right)\right) \\ &= \frac{b_{3D}\pi\delta^5\zeta^2}{15}\end{aligned} \tag{5.71}$$

in which temperature change terms that are associated with these volumetric alternations are disregarded since neither volumetric expansion nor contraction of material points are observed during shearing deformation.

By equating results of energy density function from classical and peridynamic formulations that are given by relations (5.64) and (5.71), the coefficient, b_2 is obtained as follows.

$$
\begin{aligned}
0.5\mu\zeta^2 &= \frac{b_{3D}\pi\delta^5\zeta^2}{15} \\
b_{3D} &= \frac{15\mu}{2\pi\delta^5}
\end{aligned}
\tag{5.72}
$$

Previously established relations between a_1, a_2, a_3 and b_2 along relations (5.61 - 5.63) enable us to determine remaining peridynamic coefficients as follows.

$$
a_1 = \left(\frac{\kappa}{2} - \frac{5\mu}{6}\right) \tag{5.73}
$$

$$
a_2 = \alpha\left(3\kappa - 5\mu\right) \tag{5.74}
$$

Consequently, a relation between a_1 and a_3 is established by means of result obtained from relation (5.62).

$$
a_3 = \alpha^2\left(\frac{9\kappa}{2} - \frac{15\mu}{2}\right) \tag{5.75}
$$

Substitution of peridynamic parameters, a_1, a_2, a_3, b_2 and d_3 in discrete forms of dilatation and strain energy density based on relations (5.50) and (5.52) respectively let us rewrite them in a general from for evaluation of differentiation given by relation (4.92). To this end, direction dependent material constants C^p in bond-based and A^p, B^p in ordinary state-based peridynamic formulations are determined as follows.

$$
\begin{aligned}
&\epsilon_{(j)ii} = \frac{9}{4\pi\delta^4}\sum_k \omega_{(j)(k)}\left\langle x_{(k)} - x_{(j)}\right\rangle\left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right) \\
&\frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|} \cdot \frac{x_{(k)} - x_{(j)}}{\left|x_{(k)} - x_{(j)}\right|} V_{(k)} + 3\alpha\Delta T_{(j)}
\end{aligned}
\tag{5.76}
$$

and

$$
\begin{aligned}
W_{(j)} &= \frac{1}{2}\left(\kappa - \frac{5\mu}{3}\right)\epsilon_{(j)ii}^{2} - \alpha\left(3\kappa - 5\mu\right)\epsilon_{(j)ii}\Delta T_{(j)} + \alpha^{2}\left(\frac{9\kappa}{2} - \frac{15\mu}{2}\right)\Delta T_{(j)}^{2} \\
&+ \frac{15\mu}{2\pi\delta^{5}}\sum_{k}\omega_{(j)(k)}\left\langle x_{(k)} - x_{(j)}\right\rangle\left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right)^{2} V_{(k)} \\
&= \frac{1}{2}\left(\kappa - \frac{5\mu}{3}\right)\left(\epsilon_{(j)ii}^{2} - 6\alpha\epsilon_{(j)ii}\Delta T_{(j)} + 9\alpha^{2}\Delta T_{(j)}^{2}\right) \\
&+ \frac{15\mu}{2\pi\delta^{5}}\sum_{k}\omega_{(j)(k)}\left\langle x_{(k)} - x_{(j)}\right\rangle\left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right)^{2} V_{(k)} \\
&= \frac{1}{2}\left(\kappa - \frac{5\mu}{3}\right)\left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right)^{2} \\
&+ \frac{15\mu}{2\pi\delta^{5}}\sum_{k}\omega_{(j)(k)}\left\langle x_{(k)} - x_{(j)}\right\rangle\left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right)^{2} V_{(k)}
\end{aligned}
\tag{5.77}
$$

Additionally, strain energy density term inside relation (4.92) can also be expressed in its non-normalized form therefore pairwise force density vector for material point (j) and surrounding particles (k) are expressed as in the frame of state-based peridynamics as follows.

$$
\begin{aligned}
\mathbf{f}_{(j)(k)}\left(u_{(k)} - u_{(j)}, x_{(k)} - x_{(j)}, t\right) &= \frac{\partial w\left(\left(y_{(k)} - y_{(j)}\right), \left(x_{(k)} - x_{(j)}\right)\right)}{\partial\left(\left|y_{(k)} - y_{(j)}\right|\right)}\frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|} \\
&= \frac{1}{V_{(j)}}\frac{\partial W\left(\left(y_{(k)} - y_{(j)}\right), \left(x_{(k)} - x_{(j)}\right)\right)}{\partial\left(\left|y_{(k)} - y_{(j)}\right|\right)}\frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|} \\
&= \frac{1}{V_{(j)}}A_{(j)(k)}^{sb}\frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|}
\end{aligned}
\tag{5.78}
$$

and

$$
\begin{aligned}
\mathbf{f}_{(k)(j)}\left(u_{(j)} - u_{(k)}, x_{(j)} - x_{(k)}, t\right) &= \frac{\partial w\left(\left(y_{(j)} - y_{(k)}\right), \left(x_{(j)} - x_{(k)}\right)\right)}{\partial\left(\left|y_{(j)} - y_{(k)}\right|\right)}\frac{y_{(j)} - y_{(k)}}{\left|y_{(j)} - y_{(k)}\right|} \\
&= \frac{1}{V_{(k)}}\frac{\partial W\left(\left(y_{(j)} - y_{(k)}\right), \left(x_{(j)} - x_{(k)}\right)\right)}{\partial\left(\left|y_{(j)} - y_{(k)}\right|\right)}\frac{y_{(j)} - y_{(k)}}{\left|y_{(j)} - y_{(k)}\right|} \\
&= \frac{1}{V_{(k)}}B_{(k)(j)}^{sb}\frac{y_{(j)} - y_{(k)}}{\left|y_{(j)} - y_{(k)}\right|}
\end{aligned}
\tag{5.79}
$$

in which partial differentiations given by relations (5.78) and (5.79) are assigned to bond constants $A_{(j)(k)}^{sb}$ and $B_{(k)(j)}^{sb}$ respectively. By this way, bond constant are determined by performing associated differentiations as follows.

$$A^{sb}_{(j)(k)} = \frac{1}{2}\left(\kappa - \frac{5\mu}{3}\right) 2\left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right) \frac{\partial\left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right)}{\partial\left(\left|y_{(k)} - y_{(j)}\right|\right)}$$
$$+ \left(\frac{15\mu}{2\pi\delta^5}\right) \sum_j \omega_{(j)(k)} \left\langle x_{(k)} - x_{(j)} \right\rangle \frac{\partial\left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right)}{\partial\left(\left|y_{(k)} - y_{(j)}\right|\right)} V_{(k)}$$
$$2\left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right) \tag{5.80}$$

or

$$A^{sb}_{(j)(k)} = \left(\kappa - \frac{5\mu}{3}\right) \left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right) \frac{\partial\epsilon_{(j)ii}}{\partial\left(\left|y_{(k)} - y_{(j)}\right|\right)}$$
$$+ \left(\frac{30\mu}{2\pi\delta^5}\right) \sum_j \omega_{(j)(k)} \left\langle x_{(k)} - x_{(j)} \right\rangle \left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right) \tag{5.81}$$

and

$$B^{sb}_{(k)(j)} = \frac{1}{2}\left(\kappa - \frac{5\mu}{3}\right) 2\left(\epsilon_{(k)ii} - 3\alpha\Delta T_{(k)}\right) \frac{\partial\left(\epsilon_{(k)ii} - 3\alpha\Delta T_{(k)}\right)}{\partial\left(\left|y_{(k)} - y_{(j)}\right|\right)}$$
$$+ \left(\frac{15\mu}{2\pi\delta^5}\right) \sum_k \omega_{(k)(j)} \left\langle x_{(j)} - x_{(k)} \right\rangle \frac{\partial\left(\left|y_{(j)} - y_{(k)}\right| - \left|x_{(j)} - x_{(k)}\right| - \alpha\left|x_{(j)} - x_{(k)}\right|\Delta T_{(k)}\right)}{\partial\left(\left|y_{(j)} - y_{(k)}\right|\right)} V_{(k)}$$
$$2\left(\left|y_{(j)} - y_{(k)}\right| - \left|x_{(j)} - x_{(k)}\right| - \alpha\left|x_{(j)} - x_{(k)}\right|\Delta T_{(k)}\right) \tag{5.82}$$

or

$$B^{sb}_{(k)(j)} = \left(\kappa - \frac{5\mu}{3}\right) \left(\epsilon_{(k)ii} - 3\alpha\Delta T_{(k)}\right) \frac{\partial\epsilon_{(k)ii}}{\partial\left(\left|y_{(k)} - y_{(j)}\right|\right)}$$
$$+ \left(\frac{30\mu}{2\pi\delta^5}\right) \sum_k \omega_{(k)(j)} \left\langle x_{(j)} - x_{(k)} \right\rangle \left(\left|y_{(j)} - y_{(k)}\right| - \left|x_{(j)} - x_{(k)}\right| - \alpha\left|x_{(j)} - x_{(k)}\right|\Delta T_{(k)}\right) \tag{5.83}$$

in which dilatations $\epsilon_{(j)ii}$ and $\epsilon_{(k)ii}$ are different and their partial differentials with respect to magnitudes of relative position vectors $|y_j - y_k|$ and $|y_k - y_j|$ respectively are needed in terms of determination of explicit forms of bond constants.

In that sense, associated differentials are expressed as follows.

$$\frac{\partial\left(\epsilon_{(j)ii}-3\alpha\Delta T_{(j)}\right)}{\partial\left(\left|y_{(k)}-y_{(j)}\right|\right)}=$$
$$=\frac{9}{4\pi\delta^4}\sum_k \omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle \frac{\partial\left(\left|y_{(k)}-y_{(j)}\right|-\left|x_{(k)}-x_{(j)}\right|-\alpha\left|x_{(k)}-x_{(j)}\right|\Delta T_{(j)}\right)}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}$$
$$\left(\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|}\cdot\frac{x_{(k)}-x_{(j)}}{\left|x_{(k)}-x_{(j)}\right|}\right)V_{(j)} \tag{5.84}$$

yielding to

$$\frac{\partial\left(\epsilon_{(j)ii}-3\alpha\Delta T_{(j)}\right)}{\partial\left(\left|y_{(k)}-y_{(j)}\right|\right)}=\frac{9}{4\pi\delta^4}\sum_k \omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle\left(\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|}\cdot\frac{x_{(k)}-x_{(j)}}{\left|x_{(k)}-x_{(j)}\right|}\right)V_{(j)} \tag{5.85}$$

and

$$\frac{\partial\left(\epsilon_{(k)ii}-3\alpha\Delta T_{(k)}\right)}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}=$$
$$=\frac{9}{4\pi\delta^4}\sum_j \omega_{(k)(j)}\left\langle x_{(j)}-x_{(k)}\right\rangle \frac{\partial\left(\left|y_{(j)}-y_{(k)}\right|-\left|x_{(j)}-x_{(k)}\right|-\alpha\left|x_{(j)}-x_{(k)}\right|\Delta T_{(k)}\right)}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}$$
$$\left(\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|}\cdot\frac{x_{(j)}-x_{(k)}}{\left|x_{(j)}-x_{(k)}\right|}\right)V_{(k)} \tag{5.86}$$

yielding to

$$\frac{\partial\left(\epsilon_{(k)ii}-3\alpha\Delta T_{(k)}\right)}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}=\frac{9}{4\pi\delta^4}\sum_j \omega_{(k)(j)}\left\langle x_{(j)}-x_{(k)}\right\rangle\left(\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|}\cdot\frac{x_{(j)}-x_{(k)}}{\left|x_{(j)}-x_{(k)}\right|}\right)V_{(k)} \tag{5.87}$$

Substitutions of results obtained from partial differentiations of associated dilatations in relations (5.86) and (5.87) into relations (5.81) and (5.83) respectively enable us to write the change in strain energy densities of subjected particle with respect to deformed bond length as follows.

$$A^{sb}_{(j)(k)}=$$
$$=\left(\kappa-\frac{5\mu}{3}\right)\left(\epsilon_{(j)ii}-3\alpha\Delta T_{(j)}\right)\frac{9}{4\pi\delta^4}\sum_k \omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle\left(\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|}\cdot\frac{x_{(k)}-x_{(j)}}{\left|x_{(k)}-x_{(j)}\right|}\right)V_{(j)}$$
$$+\left(\frac{30\mu}{2\pi\delta^5}\right)\sum_k \omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle\left(\left|y_{(k)}-y_{(j)}\right|-\left|x_{(k)}-x_{(j)}\right|-\alpha\left|x_{(k)}-x_{(j)}\right|\Delta T_{(j)}\right)V_{(j)} \tag{5.88}$$

and

$$B^{sb}_{(k)(j)} =$$
$$= \left(\kappa - \frac{5\mu}{3}\right)\left(\epsilon_{(k)ii} - 3\alpha\Delta T_{(k)}\right)\frac{9}{4\pi\delta^4}\sum_j \omega_{(k)(j)}\left\langle x_{(j)} - x_{(k)}\right\rangle\left(\frac{y_{(j)} - y_{(k)}}{\left|y_{(j)} - y_{(k)}\right|}\cdot\frac{x_{(j)} - x_{(k)}}{\left|x_{(j)} - x_{(k)}\right|}\right)V_{(k)}$$
$$+ \left(\frac{30\mu}{2\pi\delta^5}\right)\sum_j \omega_{(k)(j)}\left\langle x_{(j)} - x_{(k)}\right\rangle\left(\left|y_{(j)} - y_{(k)}\right| - \left|x_{(j)} - x_{(k)}\right| - \alpha\left|x_{(j)} - x_{(k)}\right|\Delta T_{(k)}\right)V_{(k)} \tag{5.89}$$

Bond constants, $A^{sb}_{(j)(k)}$ and $B^{sb}_{(k)(j)}$ for a single bond of interacting particles are determined by terminating summation sign in relations (5.88) and (5.89). In this way,

$$A^{bb}_{(j)(k)} =$$
$$= \left(\kappa - \frac{5\mu}{3}\right)\left(\frac{9}{4\pi\delta^4}\right)\left(\epsilon_{(j)ii} - 3\alpha\Delta T_{(j)}\right)\omega_{(j)(k)}\left\langle x_{(k)} - x_{(j)}\right\rangle\left(\frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|}\cdot\frac{x_{(k)} - x_{(j)}}{\left|x_{(k)} - x_{(j)}\right|}\right)V_{(j)}$$
$$+ \left(\frac{30\mu}{2\pi\delta^5}\right)\omega_{(j)(k)}\left\langle x_{(k)} - x_{(j)}\right\rangle\left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right)V_{(j)} \tag{5.90}$$

and

$$B^{bb}_{(k)(j)} =$$
$$= \left(\kappa - \frac{5\mu}{3}\right)\left(\frac{9}{4\pi\delta^4}\right)\left(\epsilon_{(k)ii} - 3\alpha\Delta T_{(k)}\right)\omega_{(k)(j)}\left\langle x_{(j)} - x_{(k)}\right\rangle\left(\frac{y_{(j)} - y_{(k)}}{\left|y_{(j)} - y_{(k)}\right|}\cdot\frac{x_{(j)} - x_{(k)}}{\left|x_{(j)} - x_{(k)}\right|}\right)V_{(k)}$$
$$+ \left(\frac{30\mu}{2\pi\delta^5}\right)\omega_{(k)(j)}\left\langle x_{(j)} - x_{(k)}\right\rangle\left(\left|y_{(j)} - y_{(k)}\right| - \left|x_{(j)} - x_{(k)}\right| - \alpha\left|x_{(j)} - x_{(k)}\right|\Delta T_{(k)}\right)V_{(k)} \tag{5.91}$$

Bond-based formulation of peridynamic theory is obtained as a special case of state-based approach by means of a prescription is that direction dependent material coefficients have to be equal, namely, $A^{bb}_{(j)(k)} = B^{bb}_{(k)(j)}$. Since dilatation terms that are $\epsilon_{(j)ii}$ and $\epsilon_{(j)ii}$ vary, the condition dictating equivalence of bond constants is only satisfied through following mathematical constraint.

$$\kappa = \frac{5\mu}{3} \tag{5.92}$$

from which a relation between material constants, λ and μ can be deduced by means of relation (B.11) leading to

$$\lambda = \mu \tag{5.93}$$

A well-known prescription given by relation (B.17) can be used so as to express the restriction in bond-based formulation of peridynamics in terms of Poisson's ratio which is $\nu = 1/4$.

Direction dependent material parameters or bond constants in bond-based approach between each couple of particle can be determined through substitution of relation (5.92) in both (5.90) and (5.91). The influence function given by relation (5.54) can be utilized for the explicit definition of bond constants. As a conclusion, bond constants for bond-based approach of peridynamic modelling become equal and expressed as follows.

$$\begin{aligned} A^{bb}_{(j)(k)} &= \frac{1}{2}\left(\frac{30\mu}{\pi\delta^5}\right)\frac{\delta}{|\xi|}\left(|\eta+\xi| - |\xi| - \alpha\,|\xi|\,\Delta T_{(j)}\right) \\ &= \frac{1}{2}\left(\frac{30\mu}{\pi\delta^4}\right)\left(\frac{|\eta+\xi|-|\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right) = \frac{1}{2}\left(\frac{18\kappa}{\pi\delta^4}\right)\left(\frac{|\eta+\xi|-|\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right) \end{aligned} \tag{5.94}$$

and

$$\begin{aligned} B^{bb}_{(k)(j)} &= \frac{1}{2}\left(\frac{30\mu}{\pi\delta^5}\right)\frac{\delta}{|\xi|}\left(|\eta+\xi| - |\xi| - \alpha\,|\xi|\,\Delta T_{(k)}\right) \\ &= \frac{1}{2}\left(\frac{30\mu}{\pi\delta^4}\right)\left(\frac{|\eta+\xi|-|\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right) = \frac{1}{2}\left(\frac{18\kappa}{\pi\delta^4}\right)\left(\frac{|\eta+\xi|-|\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right) \end{aligned} \tag{5.95}$$

Even though difference in dilatations, $\epsilon_{(j)ii}$ and $\epsilon_{(k)ii}$, associated with material points $x_{(j)}$ and $x_{(k)}$ respectively appear in strain energy and bond constant terms they have no effect on these because of material constrain provided by $\kappa = 5\mu/3$ or $\nu = 1/4$.

As a conclusion, pair-wise forces are obtained by inserting related bond constants in relations (4.69) and (4.70) as follows.

$$\underline{\mathbf{F}}(x_{(j)}, t)\left\langle x_{(k)} - x_{(j)}\right\rangle = \frac{1}{2}\left(\frac{18\kappa}{\pi\delta^4}\right)\left(\frac{|\eta+\xi|-|\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right)\frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|} \tag{5.96}$$

and

$$\underline{\mathbf{F}}(x_{(k)}, t)\left\langle x_{(j)} - x_{(k)}\right\rangle = \frac{1}{2}\left(\frac{18\kappa}{\pi\delta^4}\right)\left(\frac{|\eta+\xi|-|\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right)\frac{y_{(j)} - y_{(k)}}{\left|y_{(k)} - y_{(j)}\right|} \tag{5.97}$$

or in short hand notation, both are represented as follows.

$$\underline{\mathbf{F}}(x, t)\left\langle x' - x\right\rangle = \frac{1}{2}C^p\left(s - \alpha\Delta T_{(j)}\right)\frac{\eta+\xi}{|\eta+\xi|} \tag{5.98}$$

and

$$\underline{\mathbf{F}}(x',t)\left\langle x-x'\right\rangle = -\frac{1}{2}C^{p}\left(s-\alpha\Delta T_{(j)}\right)\frac{\eta+\xi}{|\eta+\xi|} \tag{5.99}$$

in which bond constant, C^p, is defined as follows.

$$C^{p} = \left(\frac{18\kappa}{\pi\delta^{4}}\right)\left(s-\alpha\Delta T_{(j)}\right) \tag{5.100}$$

5.3 Peridynamic parameters for two-dimensional structures

Three dimensional structures can be replaced with their two-dimensional representatives since they provide computational efficiency as long as they give conceivable result compared to their three-dimensional masks. In that sense, peridynamic parameters for two-dimensional spaces are derived accordingly.

Classically, it is possible to describe stress-strain relations for planar structures by means of a constitutive equation that is expressed in a matrix form as

$$\begin{bmatrix}\sigma_{x_1x_1}\\ \sigma_{x_2x_2}\\ \sigma_{x_1x_2}\end{bmatrix} = \begin{bmatrix}\kappa_{2D}+\mu & \kappa_{2D}-\mu & 0\\ . & \kappa_{2D}+\mu & 0\\ . & . & \mu\end{bmatrix}\begin{bmatrix}\epsilon_{x_1x_1}\\ \epsilon_{x_2x_2}\\ \epsilon_{x_1x_2}\end{bmatrix} \tag{5.101}$$

in which the term associated with bulk modulus differs from its three-dimensional peer while shear modulus stays as same and they are defined respectively as follows, [16].

$$\kappa_{2D} = \frac{E}{2\left(1-\nu\right)} \tag{5.102}$$

and

$$\mu = \frac{E}{2\left(1+\nu\right)} \tag{5.103}$$

As similar to determination of peridynamic parameters for three-dimensional structures, two fundamental loading cases are considered for a planar body as well. Plane deformation of a plate in case of pure isotropic expansion results in strain components that are $\epsilon_{(j)x_1x_1} = \epsilon_{(j)x_2x_2} = \zeta+\alpha\Delta T_{(j)}$ and $\epsilon_{(j)x_1x_2} = 0$ with dilatation term that is

$$\epsilon_{2D(j)ii} = 2\zeta + 2\alpha\Delta T_{(j)} \tag{5.104}$$

Classically, strain energy density based on relation (3.5) can be expressed in conjunction with constitutive equation given relation (5.101) for two-dimensional structures as follows.

$$\begin{aligned}
W_{2D,IsoExp} &= 0.5\left(\sigma_{(j)x_1x_1}\epsilon_{(j)x_1x_1} + \sigma_{x_2x_2}\epsilon_{(j)x_2x_2}\right) + \sigma_{(j)x_1x_2}\epsilon_{(j)x_1x_2} \\
&= 0.5\left(\epsilon_{(j)x_1x_1}\left(\kappa_{2D}+\mu\right) + \epsilon_{(j)x_2x_2}\left(\kappa_{2D}-\mu\right)\right)\epsilon_{(j)x_1x_1} \\
&+ 0.5\left(\epsilon_{(j)x_1x_1}\left(\kappa_{2D}-\mu\right) + \epsilon_{(j)x_2x_2}\left(\kappa_{2D}+\mu\right)\right)\epsilon_{(j)x_2x_2} + \mu\epsilon^2_{(j)x_2x_2} \\
&= 0.5\left(\epsilon_{(j)x_1x_1}\kappa_{2D} + \epsilon_{(j)x_1x_1}\mu + \epsilon_{(j)x_2x_2}\kappa_{2D} - \epsilon_{(j)x_2x_2}\mu\right)\epsilon_{(j)x_1x_1} \\
&+ 0.5\left(\epsilon_{(j)x_1x_1}\kappa_{2D} - \epsilon_{(j)x_1x_1}\mu + \epsilon_{(j)x_2x_2}\kappa_{2D} + \epsilon_{(j)x_2x_2}\mu\right)\epsilon_{(j)x_2x_2} + \mu\epsilon^2_{(j)x_1x_2} \\
&= 2\epsilon^2_{(j)x_1x_1}\kappa_{2D} + \mu\epsilon^2_{(j)x_1x_2} = 2\zeta^2\kappa_{2D}
\end{aligned} \tag{5.105}$$

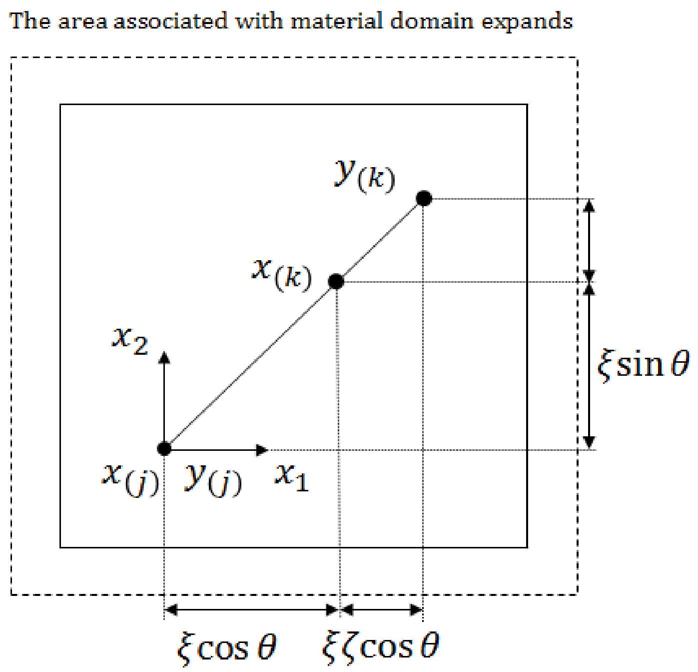


FIGURE 5.2: Deformation of a plate in case of pure isotropic expansion

Defining deformed bond length of relative position vector is required to invoke it in dilatation and strain energy density calculations for determination of peridynamic parameters in two-dimensional analysis as well.

The vector indicating relative position in deformed configuration of the body is defined according to Figure (5.2) which is the representative deformation of the medium in case of pure isotropic expansion case. To this end, deformed bond length is given by

$$y_{(k)} - y_{(j)} = \left(1 + \zeta + \alpha\Delta T_{(j)}\right)\xi \tag{5.106}$$

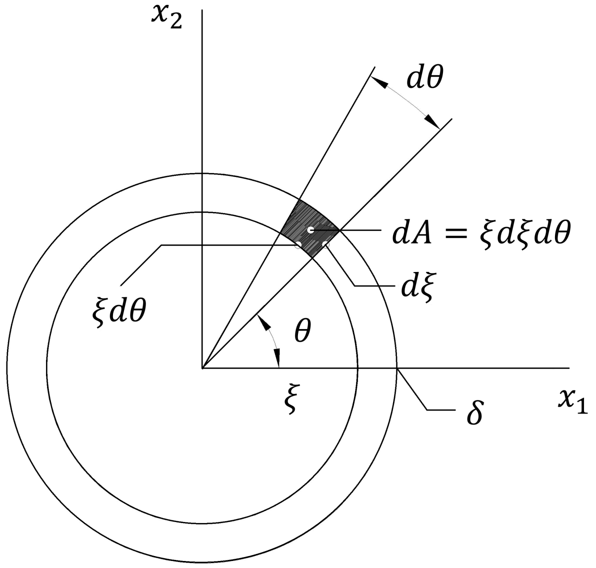


FIGURE 5.3: Two-dimensional integration domain for volume of a material point

Consequently, dilatation of the particle from peridynamic point of view for isotropic expansion is evaluated according to relation (5.50) with a slight modification on it as follows.

$$\begin{aligned}\epsilon_{2D,IsoExp(j)ii} &= d_{2D}\int_0^{\delta}\int_0^{2\pi}\frac{\delta}{|\xi|}\zeta\xi\frac{\left(1+\zeta+\alpha\Delta T_{(j)}\right)\xi}{\left(1+\zeta+\alpha\Delta T_{(j)}\right)|\xi|}\cdot\frac{\xi}{|\xi|}h\xi d\theta d\xi + 2\alpha\Delta T_{(j)}\\ &= d_{2D}h\delta\zeta\int_0^{\delta}\int_0^{2\pi}\xi d\theta d\xi + 2\alpha\Delta T_{(j)}\\ &= \pi d_{2D}h\delta^3\zeta + 2\alpha\Delta T_{(j)}\end{aligned} \tag{5.107}$$

in which infinitesimal volume element inside integration is calculated through $h\xi d\theta d\xi$. Moreover, comparison of results provided by relations (5.104) and (5.107) leads to determination of peridynamic parameter, $b_{2,2D}$ that is

$$\begin{aligned}2\zeta + 2\alpha\Delta T_{(j)} &= \pi d_{2D}h\delta^3\zeta + 2\alpha\Delta T_{(j)}\\ d_{2D} &= \frac{2}{\pi h\delta^3}\end{aligned} \tag{5.108}$$

Moreover, peridynamic strain energy density for isotropic expansion is calculated through an appropriate modification on relation (5.52) and expressed for two-dimensional medium with different peridynamic parameters, $a_{1,2D}$, $a_{2,2D}$ and $a_{3,2D}$

as follows.

$$W_{2D,Shear(j)} = a_{1,2D}\epsilon^2_{2D(j)ii} - a_{2,2D}\epsilon_{2D(j)ii}\Delta T_{(j)} + a_{3,2D}\Delta T^2_{(j)} + b_{2D}\int_0^{\delta}\int_0^{2\pi}\frac{\delta}{|\xi|}(\zeta\xi)^2 h\xi d\theta d\xi \quad (5.109)$$

Dilatation term given by relation (5.104) is invoked in relation (5.104) for exact evaluation of strain energy density function for isotropic expansion of planar body. Therefore, comparison of strain energy densities from classical and peridynamic results results in

$$a_{1,2D}(2\zeta + 2\alpha\Delta T)^2 - a_{2,2D}(2\zeta + 2\alpha\Delta T)\Delta T_{(j)} + a_{3,2D}\Delta T^2_{(j)} + \frac{2}{3}b_{2D}h\delta^4\zeta^2 = 2\zeta^2\kappa_{2D} \quad (5.110)$$

leads to following a set of equations.

$$4a_{1,2D} + \frac{2}{3}b_{2D}h\delta^4\zeta^2 = 2\kappa_{2D} \quad (5.111)$$

$$a_{2,2D} = 4\alpha a_{1,2D} \quad (5.112)$$

and

$$a_{3,2D} = 4\alpha^2 a_{1,2D} \quad (5.113)$$

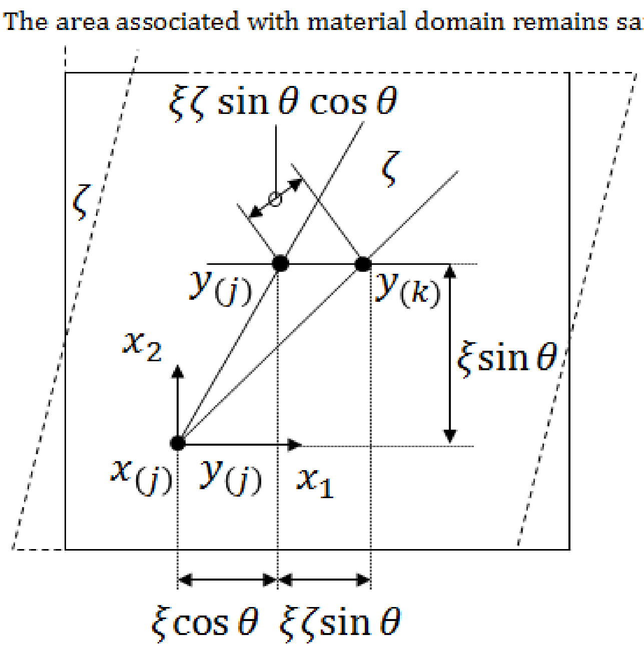


FIGURE 5.4: Pure shearing deformation of a plate

The other fundamental distortion case is pure shear deformation of a two-dimensional body. Accordingly, strain components of associated deformation are summarized along $\gamma_{(j)x_1x_2} = \zeta$, $\epsilon_{(j)x_1x_1} = \epsilon_{(j)x_2x_2} = 0$ and $\Delta T_{(j)} = 0$ for which dilatation of body becomes zero according to representative deformation of a planar body that is illustrated in Figure (5.4).

Therefore, dilatation and strain energy terms are written respectively within the frame of classical continuum mechanics as follows.

$$\epsilon_{2D,Shear(j)ii} = 0 \tag{5.114}$$

and

$$\begin{aligned} W_{2D,Shear(j)} &= 0.5 \sum_{m=1}^{2} \sum_{n=1}^{2} \sigma_{(j)mn} \epsilon_{(j)mn} = \begin{bmatrix} \sigma_{x_1x_1} & \sigma_{x_1x_2} \\ \sigma_{x_2x_1} & \sigma_{x_2x_2} \end{bmatrix} \begin{bmatrix} \epsilon_{x_1x_1} & \epsilon_{x_1x_2} \\ \epsilon_{x_2x_1} & \epsilon_{x_2x_2} \end{bmatrix} \\ &= 0.5 \left(\sigma_{x_1x_1} \epsilon_{x_1x_1} + \sigma_{x_2x_2} \epsilon_{x_2x_2} \right) + \sigma_{x_1x_2} \epsilon_{x_1x_2} \end{aligned} \tag{5.115}$$

By means of stiffness matrix for two-dimensional isotropic structures, classical strain energy density in case of in-plane shearing deformation is obtained as follows.

$$\begin{aligned} W_{2D,Shear(j)} &= 0.5 \left(\left(\kappa_{2D} + \mu \right) \epsilon_{x_1x_1}^2 + \left(\kappa_{2D} - \mu \right) \epsilon_{x_2x_2} \epsilon_{x_1x_1} \right) \\ &+ 0.5 \left(\left(\kappa_{2D} - \mu \right) \epsilon_{x_1x_1} \epsilon_{x_2x_2} + \left(\kappa_{2D} + \mu \right) \epsilon_{x_2x_2}^2 \right) + 0.5 \mu \epsilon_{x_1x_2}^2 \\ &= 0.5 \mu \zeta^2 \end{aligned} \tag{5.116}$$

According to the distortion case that is illustrated in Figure (5.4), relative position vector in deformed configuration of material domain is obtained with small degree approximation that is $\sin(\zeta) \approx \zeta$ as follows.

$$\begin{aligned} y_{(k)} - y_{(j)} &= \xi + \xi \sin(\theta) \sin(\zeta) \cos(\theta) \\ &= \xi + \xi \zeta \sin(\theta) \cos(\theta) = \left(1 + \zeta \sin(\theta) \cos(\theta) \right) \xi \end{aligned} \tag{5.117}$$

As a consequence, dilatation for a two-dimensional body in case of pure sharing distortion based on relation (5.50) is obtained with substitution of deformed bond length given by relation (5.117) as follows.

$$\begin{aligned}\epsilon_{2D,Shear(j)ii} &= d_{2D}\int_0^{\delta}\int_0^{2\pi}\frac{\delta}{|\xi|}\zeta\xi\sin(\theta)\cos(\theta)\frac{\left|y_{(k)}-y_{(j)}\right|\left|x_{(k)}-x_{(j)}\right|\cos(\zeta)}{\left|y_{(k)}-y_{(j)}\right|\left|x_{(k)}-x_{(j)}\right|}h\xi d\theta d\xi \\ &= d_{2D}h\delta\zeta\int_0^{\delta}\xi d\xi\int_0^{2\pi}\sin(\theta)\cos(\theta)\,d\theta = 0\end{aligned} \tag{5.118}$$

Accordingly, peridynamic strain energy density in case of shearing deformation of a planar body is evaluated with appropriate modifications on relation (5.52) in which relative position vector defined by relation (5.117) is substituted. Thus,

$$\begin{aligned}W_{2D,Shear(j)} &= a_{1,2D}\epsilon^2_{(j)ii} - a_{2,2D}\epsilon_{(j)ii}\Delta T_{(j)} + a_{3,2D}\Delta T^2_{(j)} \\ &+ b_{2D}\int_0^{\delta}\int_0^{2\pi}\frac{\delta}{|\xi|}\left(\zeta\xi\sin(\theta)\cos(\theta)\right)^2 h\xi d\theta d\xi \\ &= b_{2D}\zeta^2 h\delta\int_0^{\delta}\xi^2 d\xi\int_0^{2\pi}\sin^2(\theta)\cos^2(\theta)\,d\theta \\ &= b_{2D}\left(\frac{\pi h\delta^4\zeta^2}{12}\right)\end{aligned} \tag{5.119}$$

Accordingly, comparison of results obtained from classical and peridynamic approaches leads to determination of b_{2D} and express it in terms of shear modulus μ that is

$$\begin{aligned}b_{2D}\frac{\pi h\delta^4\zeta^2}{12} &= 0.5\mu\zeta^2 \\ b_{2D} &= \frac{6\mu}{\pi h\delta^4}\end{aligned} \tag{5.120}$$

By means of relations along (5.111 - 5.113) remaining peridynamic parameters for two-dimensional material domain are determined as follows.

$$a_{1,2D} = \left(\frac{\kappa_{2D}}{2} - \mu\right) \tag{5.121}$$

$$a_{2,2D} = \alpha\left(2\kappa_{2D} - 4\mu\right) \tag{5.122}$$

and

$$a_{3,2D} = \alpha^2\left(2\kappa_{2D} - 4\mu\right) \tag{5.123}$$

Generalized and discrete forms of dilatation and strain energy density for a two-dimensional structure are expressed by means of substitution of associated peridynamic parameters in relation (5.50) as follows.

$$\epsilon_{(j)ii} = \frac{2}{\pi h \delta^3} \sum_k \omega_{(j)(k)} \langle x_{(k)} - x_{(j)} \rangle \left(|y_{(k)} - y_{(j)}| - |x_{(k)} - x_{(j)}| - \alpha |x_{(k)} - x_{(j)}| \Delta T_{(j)} \right) \frac{y_{(k)} - y_{(j)}}{|y_{(k)} - y_{(j)}|} \cdot \frac{x_{(k)} - x_{(j)}}{|x_{(k)} - x_{(j)}|} V_{(k)} + 3\alpha \Delta T_{(j)} \tag{5.124}$$

and

$$\begin{aligned} W_{(j)} &= \left(\frac{\kappa_{2D}}{2} - \mu\right) \epsilon^2_{(j)ii} - \alpha \left(2\kappa_{2D} - 4\mu\right) \epsilon_{(j)ii} \Delta T_{(j)} + \alpha^2 \left(2\kappa_{2D} - 4\mu\right) \Delta T^2_{(j)} \\ &+ \frac{15\mu}{2\pi\delta^5} \sum_k \omega_{(j)(k)} \langle x_{(k)} - x_{(j)} \rangle \left(|y_{(k)} - y_{(j)}| - |x_{(k)} - x_{(j)}| - \alpha |x_{(k)} - x_{(j)}| \Delta T_{(j)} \right)^2 V_{(k)} \\ &= \left(\frac{\kappa_{2D}}{2} - \mu\right) \left(\epsilon^2_{(j)ii} - 4\alpha\epsilon_{(j)ii} \Delta T_{(j)} + 4\alpha^2 \Delta T^2_{(j)}\right) \\ &+ \frac{15\mu}{2\pi\delta^5} \sum_k \omega_{(j)(k)} \langle x_{(k)} - x_{(j)} \rangle \left(|y_{(k)} - y_{(j)}| - |x_{(k)} - x_{(j)}| - \alpha |x_{(k)} - x_{(j)}| \Delta T_{(j)} \right)^2 V_{(k)} \\ &= \frac{1}{2} \left(\kappa_{2D} - 2\mu\right) \left(\epsilon_{(j)ii} - 2\alpha \Delta T_{(j)}\right)^2 \\ &+ \frac{6\mu}{\pi h \delta^4} \sum_k \omega_{(j)(k)} \langle x_{(k)} - x_{(j)} \rangle \left(|y_{(k)} - y_{(j)}| - |x_{(k)} - x_{(j)}| - \alpha |x_{(k)} - x_{(j)}| \Delta T_{(j)} \right)^2 V_{(k)} \end{aligned} \tag{5.125}$$

Differentiation that is with respect to magnitudes of relative position vectors which are $|y_k - y_j|$ and $|y_j - y_k|$ in relation (5.125) leads to determination of bond constants for two-dimensional structures in peridynamic modelling. Namely,

$$\begin{aligned} \mathbf{f}_{(j)(k)} \left(u_{(k)} - u_{(j)}, x_{(k)} - x_{(j)}, t\right) &= \frac{\partial w \left(\left(y_{(k)} - y_{(j)}\right), \left(x_{(k)} - x_{(j)}\right)\right)}{\partial \left(|y_{(k)} - y_{(j)}|\right)} \frac{y_{(k)} - y_{(j)}}{|y_{(k)} - y_{(j)}|} \\ &= \frac{1}{V_{(j)}} \frac{\partial W \left(\left(y_{(k)} - y_{(j)}\right), \left(x_{(k)} - x_{(j)}\right)\right)}{\partial \left(|y_{(k)} - y_{(j)}|\right)} \frac{y_{(k)} - y_{(j)}}{|y_{(k)} - y_{(j)}|} \\ &= \frac{1}{V_{(j)}} A^{sb,2D}_{(j)(k)} \frac{y_{(k)} - y_{(j)}}{|y_{(k)} - y_{(j)}|} \end{aligned} \tag{5.126}$$

and

$$\begin{aligned}\mathbf{f}_{(k)(j)}\left(u_{(j)}-u_{(k)},x_{(j)}-x_{(k)},t\right)=&\frac{\partial w\left(\left(y_{(j)}-y_{(k)}\right),\left(x_{(j)}-x_{(k)}\right)\right)}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|}\\=&\frac{1}{V_{(k)}}\frac{\partial W\left(\left(y_{(j)}-y_{(k)}\right),\left(x_{(j)}-x_{(k)}\right)\right)}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|}\\=&\frac{1}{V_{(k)}}B^{sb,2D}_{(k)(j)}\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|}\end{aligned}\tag{5.127}$$

in which associated partial differentiations are accordingly expressed as bond constants for two-dimensional structures, [1].

$$\begin{aligned}A^{sb,2D}_{(j)(k)}=&\frac{1}{2}\left(\kappa_{2D}-2\mu\right)2\left(\epsilon_{(j)ii}-2\alpha\Delta T_{(j)}\right)\frac{\partial\left(\epsilon_{(j)ii}-2\alpha\Delta T_{(j)}\right)}{\partial\left(\left|y_{(k)}-y_{(j)}\right|\right)}\\&+\left(\frac{6\mu}{\pi h\delta^4}\right)\sum_k\omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle\frac{\partial\left(\left|y_{(k)}-y_{(j)}\right|-\left|x_{(k)}-x_{(j)}\right|-\alpha\left|x_{(k)}-x_{(j)}\right|\Delta T_{(j)}\right)}{\partial\left(\left|y_{(k)}-y_{(j)}\right|\right)}\\&2\left(\left|y_{(k)}-y_{(j)}\right|-\left|x_{(k)}-x_{(j)}\right|-\alpha\left|x_{(k)}-x_{(j)}\right|\Delta T_{(j)}\right)\end{aligned}\tag{5.128}$$

or

$$\begin{aligned}A^{sb,2D}_{(j)(k)}=&\left(\kappa_{2D}-2\mu\right)\left(\epsilon_{(j)ii}-2\alpha\Delta T_{(j)}\right)\frac{\partial\epsilon_{(j)ii}}{\partial\left(\left|y_{(k)}-y_{(j)}\right|\right)}\\&+\left(\frac{12\mu}{\pi h\delta^4}\right)\sum_k\omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle\left(\left|y_{(k)}-y_{(j)}\right|-\left|x_{(k)}-x_{(j)}\right|-\alpha\left|x_{(k)}-x_{(j)}\right|\Delta T_{(j)}\right)\end{aligned}\tag{5.129}$$

and

$$\begin{aligned}B^{sb,2D}_{(k)(j)}=&\frac{1}{2}\left(\kappa_{2D}-2\mu\right)2\left(\epsilon_{(k)ii}-2\alpha\Delta T_{(k)}\right)\frac{\partial\left(\epsilon_{(k)ii}-2\alpha\Delta T_{(k)}\right)}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}\\&+\left(\frac{6\mu}{\pi h\delta^4}\right)\sum_k\omega_{(j)(k)}\left\langle x_{(j)}-x_{(k)}\right\rangle\frac{\partial\left(\left|y_{(j)}-y_{(k)}\right|-\left|x_{(j)}-x_{(k)}\right|-\alpha\left|x_{(j)}-x_{(k)}\right|\Delta T_{(k)}\right)}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}\\&2\left(\left|y_{(j)}-y_{(k)}\right|-\left|x_{(j)}-x_{(k)}\right|-\alpha\left|x_{(j)}-x_{(k)}\right|\Delta T_{(k)}\right)\end{aligned}\tag{5.130}$$

or

$$\begin{aligned}B^{sb,2D}_{(k)(j)}=&\left(\kappa_{2D}-2\mu\right)\left(\epsilon_{(k)ii}-2\alpha\Delta T_{(k)}\right)\frac{\partial\epsilon_{(k)ii}}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}\\&+\left(\frac{12\mu}{\pi h\delta^4}\right)\sum_k\omega_{(j)(k)}\left\langle x_{(j)}-x_{(k)}\right\rangle\left(\left|y_{(j)}-y_{(k)}\right|-\left|x_{(j)}-x_{(k)}\right|-\alpha\left|x_{(j)}-x_{(k)}\right|\Delta T_{(k)}\right)\end{aligned}\tag{5.131}$$

in which dilatation terms being $\epsilon_{(j)ii}$ and $\epsilon_{(k)ii}$ are different. Furthermore, differentiations in the first lines of relations (5.129) and (5.131) are performed as

$$\frac{\partial\left(\epsilon_{(j)ii}-2\alpha\Delta T_{(j)}\right)}{\partial\left(\left|y_{(k)}-y_{(j)}\right|\right)}= \\ =\frac{2}{\pi h\delta^3}\sum_k \omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle \frac{\partial\left(\left|y_{(k)}-y_{(j)}\right|-\left|x_{(k)}-x_{(j)}\right|-\alpha\left|x_{(k)}-x_{(j)}\right|\Delta T_{(k)}\right)}{\partial\left(\left|y_{(k)}-y_{(j)}\right|\right)} \\ \left(\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|}\cdot\frac{x_{(k)}-x_{(j)}}{\left|x_{(k)}-x_{(j)}\right|}\right)V_{(k)} \tag{5.132}$$

yielding to

$$\frac{\partial\epsilon_{(j)ii}}{\partial\left(\left|y_{(k)}-y_{(j)}\right|\right)}=\frac{2}{\pi h\delta^3}\sum_k \omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle\left(\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|}\cdot\frac{x_{(k)}-x_{(j)}}{\left|x_{(k)}-x_{(j)}\right|}\right)V_{(j)} \tag{5.133}$$

and

$$\frac{\partial\left(\epsilon_{(k)ii}-2\alpha\Delta T_{(k)}\right)}{\partial\left(\left|y_{(k)}-y_{(j)}\right|\right)}= \\ =\frac{2}{\pi h\delta^3}\sum_j \omega_{(k)(j)}\left\langle x_{(j)}-x_{(k)}\right\rangle \frac{\partial\left(\left|y_{(j)}-y_{(k)}\right|-\left|x_{(j)}-x_{(k)}\right|-\alpha\left|x_{(j)}-x_{(k)}\right|\Delta T_{(k)}\right)}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)} \\ \left(\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|}\cdot\frac{x_{(j)}-x_{(k)}}{\left|x_{(j)}-x_{(k)}\right|}\right)V_{(k)} \tag{5.134}$$

leading to

$$\frac{\partial\epsilon_{(k)ii}}{\partial\left(\left|y_{(j)}-y_{(k)}\right|\right)}=\frac{2}{\pi h\delta^3}\sum_j \omega_{(k)(j)}\left\langle x_{(j)}-x_{(k)}\right\rangle\left(\frac{y_{(j)}-y_{(k)}}{\left|y_{(j)}-y_{(k)}\right|}\cdot\frac{x_{(j)}-x_{(k)}}{\left|x_{(j)}-x_{(k)}\right|}\right)V_{(k)} \tag{5.135}$$

Bond constants are obtained by means of substitutions of corresponding differentiations given by relations (5.133) and (5.135) in relations (5.128) and (5.130) respectively.

$$A^{sb,2D}_{(j)(k)}= \\ =\left(\kappa_{2D}-2\mu\right)\left(\epsilon_{(j)ii}-2\alpha\Delta T_{(j)}\right)\frac{2}{\pi h\delta^3}\sum_k \omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle\left(\frac{y_{(k)}-y_{(j)}}{\left|y_{(k)}-y_{(j)}\right|}\cdot\frac{x_{(k)}-x_{(j)}}{\left|x_{(k)}-x_{(j)}\right|}\right)V_{(j)} \\ +\frac{12\mu}{\pi h\delta^4}\sum_k \omega_{(j)(k)}\left\langle x_{(k)}-x_{(j)}\right\rangle\left(\left|y_{(k)}-y_{(j)}\right|-\left|x_{(k)}-x_{(j)}\right|-\alpha\left|x_{(k)}-x_{(j)}\right|\Delta T_{(j)}\right) \tag{5.136}$$

and

$$B^{sb,2D}_{(k)(j)} =$$
$$= (\kappa_{2D} - 2\mu)\left(\epsilon_{(k)ii} - 2\alpha\Delta T_{(k)}\right)\frac{2}{\pi h\delta^3}\sum_j \omega_{(k)(j)}\left\langle x_{(j)} - x_{(k)}\right\rangle\left(\frac{y_{(j)} - y_{(k)}}{\left|y_{(j)} - y_{(k)}\right|}\cdot\frac{x_{(j)} - x_{(k)}}{\left|x_{(j)} - x_{(k)}\right|}\right)V_{(k)}$$
$$+ \frac{12\mu}{\pi h\delta^4}\sum_k \omega_{(j)(k)}\left\langle x_{(j)} - x_{(k)}\right\rangle\left(\left|y_{(j)} - y_{(k)}\right| - \left|x_{(j)} - x_{(k)}\right| - \alpha\left|x_{(j)} - x_{(k)}\right|\Delta T_{(k)}\right) \tag{5.137}$$

Consequently, $C^{bb,2D}$, $A^{sb,2D}_{(j)(k)}$ and $B^{sb,2D}_{(k)(j)}$ are defined by excluding summation sign to be able to write an appropriate expression for only a couple of interacting particle as follows.

$$A^{sb,2D}_{(j)(k)} = (\kappa_{2D} - 2\mu)\left(\epsilon_{(j)ii} - 2\alpha\Delta T_{(j)}\right)\frac{2}{\pi h\delta^3}\omega_{(j)(k)}\left\langle x_{(k)} - x_{(j)}\right\rangle\left(\frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|}\cdot\frac{x_{(k)} - x_{(j)}}{\left|x_{(k)} - x_{(j)}\right|}\right)V_{(j)}$$
$$+ \frac{12\mu}{\pi h\delta^4}\omega_{(j)(k)}\left\langle x_{(k)} - x_{(j)}\right\rangle\left(\left|y_{(k)} - y_{(j)}\right| - \left|x_{(k)} - x_{(j)}\right| - \alpha\left|x_{(k)} - x_{(j)}\right|\Delta T_{(j)}\right) \tag{5.138}$$

and

$$B^{sb,2D}_{(k)(j)} = (\kappa_{2D} - 2\mu)\left(\epsilon_{(k)ii} - 2\alpha\Delta T_{(k)}\right)\frac{2}{\pi h\delta^3}\omega_{(k)(j)}\left\langle x_{(j)} - x_{(k)}\right\rangle\left(\frac{y_{(j)} - y_{(k)}}{\left|y_{(j)} - y_{(k)}\right|}\cdot\frac{x_{(j)} - x_{(k)}}{\left|x_{(j)} - x_{(k)}\right|}\right)V_{(k)}$$
$$+ \frac{12\mu}{\pi h\delta^4}\omega_{(j)(k)}\left\langle x_{(j)} - x_{(k)}\right\rangle\left(\left|y_{(j)} - y_{(k)}\right| - \left|x_{(j)} - x_{(k)}\right| - \alpha\left|x_{(j)} - x_{(k)}\right|\Delta T_{(k)}\right) \tag{5.139}$$

To be able to provide the condition $A^{bb,2D}_{(j)(k)} = B^{bb,2D}_{(k)(j)}$ for bond based approach of peridynamic modelling, the following material constraint has to be satisfied.

$$\kappa_{2D} = 2\mu \tag{5.140}$$

Let us to modify relation (B.10) for a two-dimensional body in the following way.

$$\begin{aligned}\sigma_{ii} &= 2\lambda\epsilon_{kk} + 2\mu\epsilon_{ii}\\ &= 2\left(\lambda + \mu\right)\epsilon_{ii}\end{aligned} \tag{5.141}$$

for which hydrostatic stress state is expressed as follows.

$$\sigma_{ii} = 2\kappa_{2D}\epsilon_{ii} \tag{5.142}$$

As a result, a relation including terms that are λ, μ and κ_{2D} is obtained by equating relation (5.141) and (5.142) to each others.

$$\kappa_{2D} = \lambda + \mu \tag{5.143}$$

Relation (C.53) can also be modified to be able to establish a correlation between stress and strain terms for two-dimensional case. In this manner, associated strain components can be expressed as follows.

$$\epsilon_{ij} = -\frac{\lambda\delta_{ij}}{2\mu}\epsilon_{kk} + \frac{\sigma_{ij}}{2\mu} \tag{5.144}$$

in which dilatation term can be replaced by its equivalent term based on relation (5.142), therefore strain components are expressed as follows.

$$\epsilon_{ij} = -\frac{\lambda\delta_{ij}}{4\mu\left(\lambda+\mu\right)}\sigma_{ii} + \frac{\sigma_{ij}}{2\mu} \tag{5.145}$$

Considering a plate that is subjected to uni-axial stress state, stress components, ϵ_{11} and ϵ_{22}, can be expressed based on relation (5.145) as follows.

$$\epsilon_{11} = -\frac{\lambda\delta_{11}}{4\mu\left(\lambda+\mu\right)}\sigma_{11} + \frac{\sigma_{11}}{2\mu} = \frac{2\mu+\lambda}{4\mu\left(\lambda+\mu\right)}\sigma_{11} \tag{5.146}$$

and

$$\epsilon_{22} = -\frac{\lambda\delta_{22}}{4\mu\left(\lambda+\mu\right)}\sigma_{11} + \frac{\sigma_{22}}{2\mu} = \frac{\lambda}{4\mu\left(\lambda+\mu\right)}\sigma_{11} \tag{5.147}$$

in which $trace[\sigma] = \sigma_{ii} = \sigma_{11} + \sigma_{22} = \sigma_{11}$. Based on relation (5.146), elastic modulus can be obtained for two-dimensional case as given below.

$$E = \frac{4\mu\left(\lambda+\mu\right)}{\lambda+2\mu} \tag{5.148}$$

Moreover, Poisson's ratio is determined by using relations (5.146) and (5.147) for two-dimensional case as follows.

$$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{-\dfrac{\lambda}{4\mu\left(\lambda+\mu\right)}\sigma_{11}}{\dfrac{\lambda+2\mu}{4\mu\left(\lambda+\mu\right)}\sigma_{11}} = \frac{\lambda}{\lambda+2\mu} \tag{5.149}$$

accordingly, Lame constant, λ, and shear modulus, μ, can be obtained based on relation (5.149) as follows.

$$\lambda = \frac{2\mu\nu}{(1-\nu)} \tag{5.150}$$

and

$$\mu = \frac{\lambda\,(1-\nu)}{2\nu} \tag{5.151}$$

Substitution of Lame constant, λ allows us to express shear modulus, μ, as a function of Poisson's ratio, ν and elastic modulus, E.Namely,

$$E = \frac{\left(\dfrac{2\mu\nu}{1-\nu} + \mu\right) 4\mu}{\dfrac{2\mu\nu}{1-\nu} + 2\mu} = \frac{(2\mu\nu + \mu)\,4\mu}{2\mu} \tag{5.152}$$

or

$$\mu = \frac{E}{2\,(1+\nu)} \tag{5.153}$$

Equating results obtained in (5.151) and (5.153) to each others leads to write Lame constant in terms of elastic modulus and Poisson's ratio as follows.

$$\frac{\lambda\,(1-\nu)}{2\nu} = \frac{E}{2\,(1+\nu)} \tag{5.154}$$

or

$$\lambda = \frac{E\nu}{(1-\nu)\,(1+\nu)} \tag{5.155}$$

The material constraint provided by relation (5.140) satisfying condition $A^{bb,2D}_{(j)(k)} = B^{bb,2D}_{(k)(j)}$ for bond based approach can be substituted in relation (5.143) and it leads to a relation between Lame constant, λ and shear modulus, μ, that is

$$\lambda = \mu \tag{5.156}$$

On the other hand, relation (5.143) can be utilized to obtain bulk modulus of a two-dimensional body as a function of elastic modulus, E and Poisson's ratio, ν based on relations (5.153) and (5.155). Namely,

$$\kappa_{2D} = \lambda + \mu = \frac{2E\nu + E - E\nu}{2\,(1-\nu)\,(1+\nu)} = \frac{E\,(1+\nu)}{2\,(1-\nu)\,(1+\nu)} = \frac{E}{2\,(1-\nu)} \tag{5.157}$$

By means of relation (5.156), Poisson's ratio given by relation (5.149) is determined as $\nu = 1/3$ in the frame of bond-based approach of peridynamic modelling for a two-dimensional structure. As a conclusion, bond constants provided by relations

(5.138) and (5.139) become

$$A^{bb,2D}_{(j)(k)} = \frac{1}{2}\left(\frac{24\mu}{\pi h\delta^4}\right)\frac{\delta}{|\xi|}\left(|\eta+\xi| - |\xi| - \alpha\,|\xi|\,\Delta T_{(j)}\right)$$
$$= \frac{1}{2}\left(\frac{24\mu}{\pi h\delta^3}\right)\left(\frac{|\eta+\xi| - |\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right) = \frac{1}{2}\left(\frac{12\kappa_{2D}}{\pi h\delta^3}\right)\left(\frac{|\eta+\xi| - |\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right) \tag{5.158}$$

and

$$B^{bb,2D}_{(k)(j)} = \frac{1}{2}\left(\frac{24\mu}{\pi h\delta^4}\right)\frac{\delta}{|\xi|}\left(|\eta+\xi| - |\xi| - \alpha\,|\xi|\,\Delta T_{(j)}\right)$$
$$= \frac{1}{2}\left(\frac{24\mu}{\pi h\delta^3}\right)\left(\frac{|\eta+\xi| - |\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right) = \frac{1}{2}\left(\frac{12\kappa_{2D}}{\pi h\delta^3}\right)\left(\frac{|\eta+\xi| - |\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right) \tag{5.159}$$

Accordingly, force states are expressed as follows.

$$\underline{\mathbf{F}}(x_{(j)},t)\left\langle x_{(k)} - x_{(j)}\right\rangle = \frac{1}{2}\left(\frac{12\kappa_{2D}}{\pi h\delta^3}\right)\left(\frac{|\eta+\xi| - |\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right)\frac{y_{(k)} - y_{(j)}}{\left|y_{(k)} - y_{(j)}\right|} \tag{5.160}$$

and

$$\underline{\mathbf{F}}(x_{(k)},t)\left\langle x_{(j)} - x_{(k)}\right\rangle = \frac{1}{2}\left(\frac{12\kappa_{2D}}{\pi h\delta^3}\right)\left(\frac{|\eta+\xi| - |\xi|}{|\xi|} - \alpha\Delta T_{(j)}\right)\frac{y_{(j)} - y_{(k)}}{\left|y_{(k)} - y_{(j)}\right|} \tag{5.161}$$

or in short hand notation

$$\underline{\mathbf{F}}(x,t)\left\langle x' - x\right\rangle = \frac{1}{2}C^{p}_{2D}\left(s - \alpha\Delta T_{(j)}\right)\frac{\eta+\xi}{|\eta+\xi|} \tag{5.162}$$

and

$$\underline{\mathbf{F}}(x',t)\left\langle x - x'\right\rangle = -\frac{1}{2}C^{p}_{2D}\left(s - \alpha\Delta T_{(j)}\right)\frac{\eta+\xi}{|\eta+\xi|} \tag{5.163}$$

in which bond constant, C^p, is defined as follows.

$$C^{p}_{2D} = \left(\frac{12\kappa_{2D}}{\pi h\delta^3}\right)\left(s - \alpha\Delta T_{(j)}\right) \tag{5.164}$$

Chapter 6

Results and Discussion

6.1 Numerical Implementation

For many cases, it can be practical to solve the equations of a system in point-wise manner however it becomes inappropriate to apply governing equations and obtain free-body diagram to each element in the system. In this sense, numerical approaches play important roles to be able to obtain required outputs since they drastically reduce time and effort that should have been normally spent on.

The integration operator in peridynamic equation of motion given by (4.115) can be replaced with the summation sign to express that it is going to be be solved for a finite number of neighbouring particles inside the horizon of each material point throughout whole material domain.

Under the consideration of finite number of material points within horizon of subjected particle which has position vector $x_{(j)}$ in initial state, the peridynamic equation of motion for these materials is written as follows.

$$\rho(x_{(j)}, t)\ddot{u}(x_{(j)}, t) = \sum_{i=1}^{N} \mathbf{f}(u_{(k)} - u_{(j)}, x_{(k)} - x_{(j)}, t)dV' + b\left(x_{(j)}, t\right) \tag{6.1}$$

6.2 Definition of test case

In this study, displacement-control tensile stretching of square plate whose geometric properties with number of elements and particles that are used in numerical implementations are introduced in Table (6.1) while Young's modulus, shear modulus, Poisson's ratio and material density values for the plate are respectively $192 \times 10^{09} [Pa]$, 72×10^{09}, $7800 [kg/m^3]$ and 1/3.

As given in Table (6.1), number of finite elements in ABAQUS and number of particles in peridynamic model are set to 500. Monitorization of the displacement

TABLE 6.1: External dimensions of the test material

Body	Length [m]	Width [m]	Depth [m]	Number of elements	Number of particles
Thin plate	$5.0 * 10^{-02}$	$5.0 * 10^{-02}$	$1.0 * 10^{-04}$	250000	250000

field pertaining to isotropic plate under the effect of applied boundary conditions in x_1 and x_2 directions is performed by means of both bond-based peridynamic model and ABAQUS, being a FEM solver and accordingly outputs of these two method are compared based on displacements through mid-line surfaces along longitudinal and vertical directions.

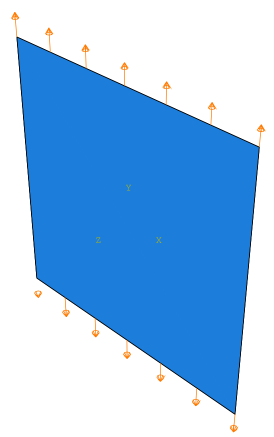


FIGURE 6.1: Representation of externally applied loads causing uni-axial stretching of a thin plate

The simulation test is performed under the effect of externally applied forces that are illustrated in Figure (6.1) in displacement control mode for both peridynamic code and FEM solver. Additionally, test speed and number of iterations are set

Table 6.2: Test parameters

Deformation	Discrete time [s]	Iteration No	Test speed [m/s]	Elongation [m]	Half elongation [m]	Strain [mm/mm]
Plastic	$1.3667 * 10^{-08}$	1050	40	$1.1228 * 10^{-03}$	$5.6141 * 10^{-04}$	0.0225

to 40 $[m/s]$ and 1050 respectively to be able to reach total displacement amount $1.1228 * 10^{-03}[m]$ causing elastic deformation of the test material with ultimate strain $0.0225[mm/mm]$.

6.3 Test parameters

The equation of motion of peridynamic modelling expressed in a discrete form for numerical solution of any particular problem is given as

$$\rho(x_{(j)},t)\ddot{u}(x_{(j)},t) = \sum_{k=1}^{N} \mathbf{f}(u_{(k)} - u_{(j)}, x_{(k)} - x_{(j)}, t)dV' + b\left(x_{(j)},t\right) \tag{6.2}$$

in which N number of interacting material points around the subjected particles labelled by (j) are located in a circular region encapsulated by the radius of horizon, $\delta = 4 \times \left|x_{(j)} - x_{(k)}\right|$.

In the numerical implementation of the peridynamic equation of motion, a stability condition is required to be able to obtain convergent results. Thus, the numerical stability condition in the implementation scheme of peridynamic equation of motion that has been introduced in [33] is given as follows.

$$\Delta t < \left(\frac{2\rho\left(x_{(j)}\right)}{C_{2D}^{p} \sum_{k=1}^{N} \left(\frac{V_{(k)}}{\left|x_{(j)} - x_{(k)}\right|} \right)} \right)^{0.5} \tag{6.3}$$

The smallest time increment is determined as $1.3367 \times 10^{-}8[s]$ according to stability stability condition given by relation (6.3).

In the n-th cycle of time integration, peridynamic equation of motion including pair-wise forces of interacting material points with the horizon of particle (j) can be implemented as follows.

$$\rho(x_{(j)},t)\ddot{u}^n(x_{(j)},t) = \mathbf{f}^n_{(j)(j+1)}V_{(j+1)} + \mathbf{f}^n_{(j)(j+2)}V_{(j+2)} + ... + \mathbf{f}^n_{(j)(j+N)}V_{(j+N)} + b^n\left(x_{(j)},t\right) \tag{6.4}$$

in which pair-wise force function and stretch are expressed in compliance with the numerical method respectively as follows.

$$\mathbf{f}^n_{(j)(k)} = \frac{\xi^n_{(j)(k)} + \eta^n_{(j)(k)}}{\left|\xi^n_{(j)(k)} + \eta^n_{(j)(k)}\right|} C^p_{2D} s^n_{(j)(k)} \tag{6.5}$$

and

$$\mathbf{s}^n_{(j)(k)} = \frac{\left|y^n_{(j)} - y^n_{(k)}\right| - \left|x^{n-1}_{(j)} - x^{n-1}_{(k)}\right|}{\left|x^{n-1}_{(j)} - x^{n-1}_{(k)}\right|} \tag{6.6}$$

in which $\xi_{(j)(k)}$ and $\eta_{(j)(k)}$ are relative position vector in undeformed and relative deformation vectors of material points (j) in deformed states respectively and can be expressed as $\xi_{(j)(k)} = x_{(j)} - x_{(k)}$ and $\eta_{(j)(k)} = u_{(j)} - u_{(k)}$.

Applying displacement amount to the top and bottom edges of the plate illustrated in Figure (6.1) simultaneously initiates stretches between interacting particles as a function of relative position and deformation vectors at initial and deformed configuration according to relation (4.75). Since pair-wise forces are the function of stretches occurring between interacting particles, change in $s^n_{(j)(k)}$ leads to determination of $\mathbf{f}^n_{(j)(k)}$ for each material point.

By invoking associated accelerations $\ddot{u}^n(x_{(j)})$ as a result of solution of peridynamic equation of motion given by (6.4) in previous time step of numerical implementation leads to the determination of velocities that are calculated through

$$\dot{u}^{n+1}(x_{(j)},t) = \ddot{u}^n(x_{(j)},t)\Delta t + \dot{u}^n(x_{(j)},t) \tag{6.7}$$

and displacements which is calculated by means of

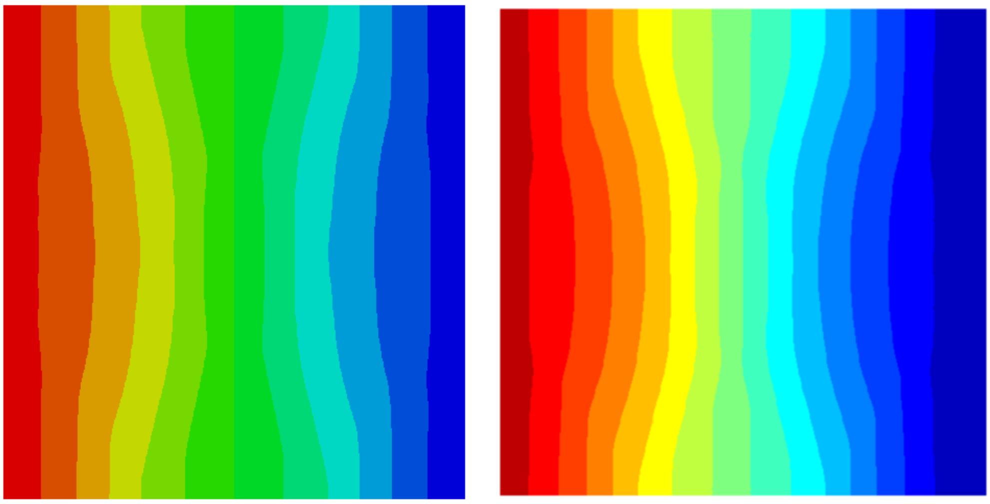
$$u^{n+1}(x_{(j)},t) = \dot{u}^n(x_{(j)},t)\Delta t + u^n(x_{(j)},t) \tag{6.8}$$

for each material point in the body. The displacement vectors obtained from relation (6.8) for each material point are used to calculate the stretches for the following time step, $(n+1)$, until specified number of iteration is reached in peridynamic code.

In ABAQUS, the same displacement amount which is $5.6141 * 10^{-04}$ has been applied to the top and bottom edges of the plate as illustrated in Figure (6.1). Furthermore, the both longitudinal and horizontal mid-line nodes for corresponding plots such as position-displacement has been determined in path manager tab as (250501 : 251001 : 1) and (251 : 501251 : 1002) in ABAQUS.

6.4 Numerical results and validation

The numerical data obtained from ABAQUS are used for comparison of peridynamic modelling results. The deformation fields in u_{x_1} and u_{x_2} throughout horizontal and vertical mid-lines are selected for validation of peridynamic code in comparison with ABAQUS outputs.



Abaqus *Peridynamic*

FIGURE 6.2: Colour diagrams for displacement field, u_{x_1}

The colour diagrams in Figure (6.3) of deformation field along x_1 direction pertaining to peridynamic and Abaqus outputs can be compared based on displacement values along transverse line. To this end, associated plots from both method are compared in Figure (6.3).

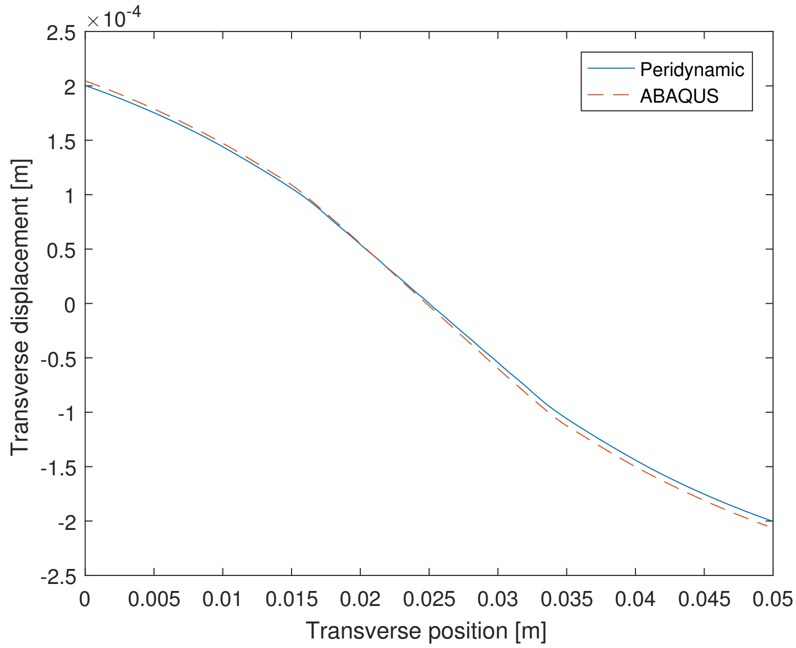


FIGURE 6.3: Displacement comparison through horizontal mid-line as a function of x_1

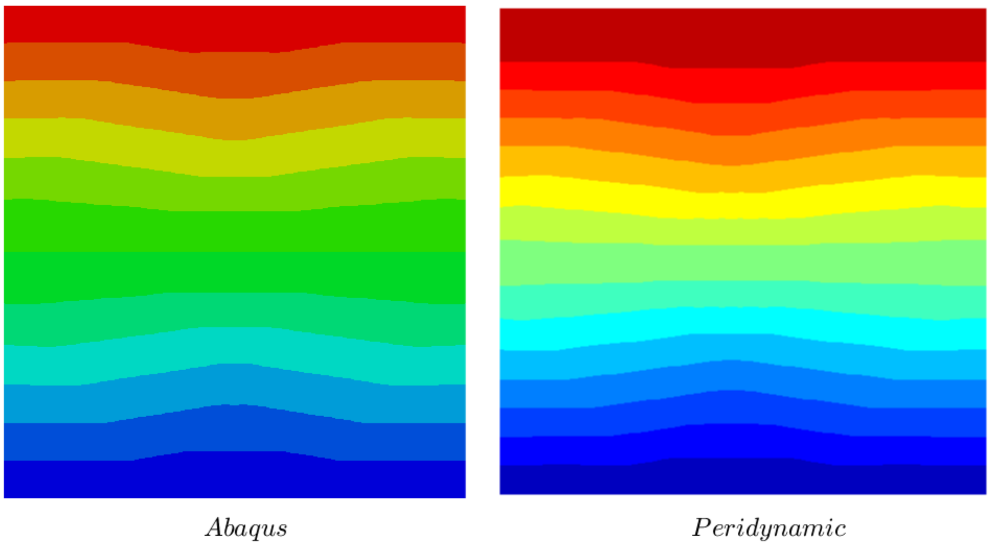


FIGURE 6.4: Colour diagrams for displacement field, u_{x_2}

As similar to comparison of deformation field for u_{x_1}, plots that are used in comparison of displacement field in x_2 direction along a vertical mid-line is given in Figure (6.5) based on peridynamics and Abaqus outputs.

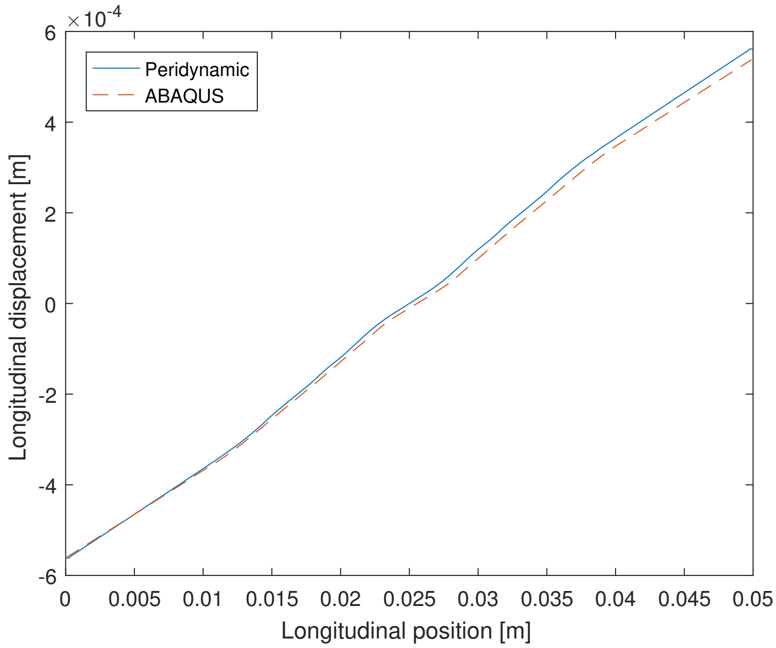
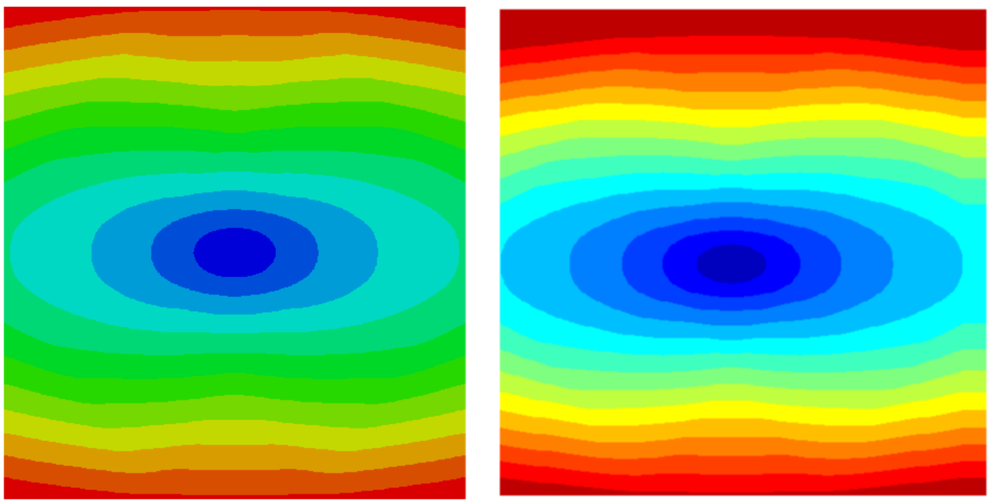


FIGURE 6.5: Displacement comparison through vertical mid-line as a function of x_2

Total displacement field under uni-axial stretching can also be compared based on colour diagrams of both method that are given in Figure (6.6).



Abaqus *Peridynamic*

FIGURE 6.6: Colour diagrams for resultant displacement field, u

Based on colour diagrams of resultant displacement field introduced in Figure (6.6), displacement values along horizontal and vertical mid-lines are compared in Figures (6.6) and (6.6) respectively.

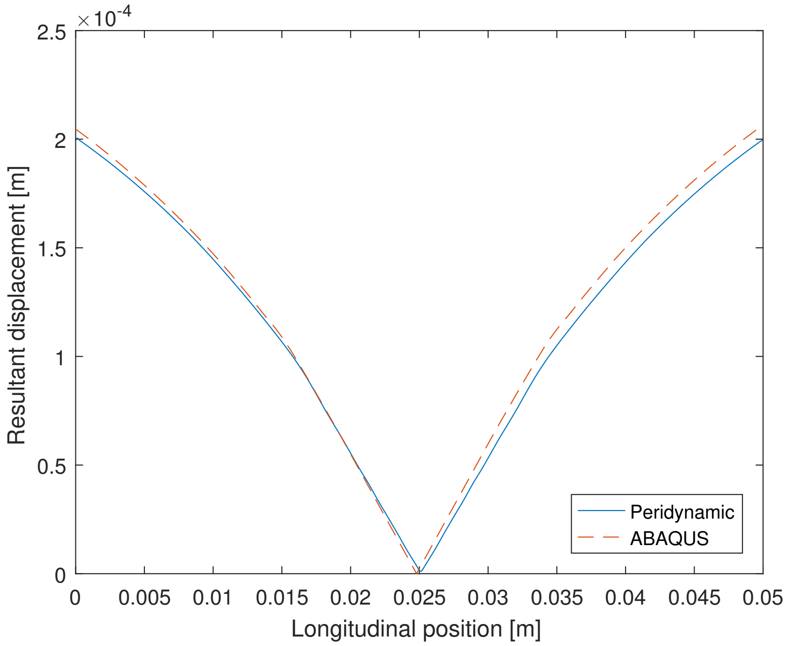


FIGURE 6.7: Resultant displacement comparison through horizontal mid-line as a function of x_1

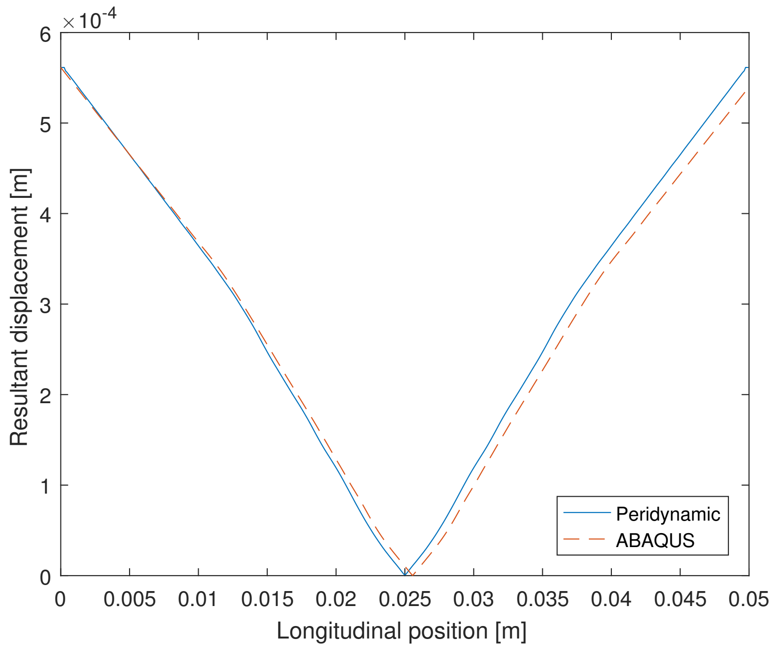


FIGURE 6.8: Resultant displacement comparison through vertical mid-line as a function of x_2

Chapter 7

Conclusion

Even though peridynamic theory fundamentally has been introduced for modelling of initiations, propagations of discontinuities, its formulations can be utilized to simulate deformation fields without any discontinuity formation.

In this study, fundamental equations in classical solid mechanics are aimed to be derived in a comprehensive way. Classical and peridynamic definitions of kinetic and kinematic relations are compared to be able to establish a bridge between these two approach.

In Chapter (6), deformation field of an isotropic medium is modelled by means of a this relatively new approach, peridynamic theory. Results pertaining to peridynamic model are compared with outputs of FEA method. Deviations with respect

TABLE 7.1: Comparisons of results

Points/Deviations	Deviation[%] u_{x_1} along x_1	Deviation [%] u_{x_2} along x_2	Deviation [%] u along x_1	Deviation [%] u along x_2
Point 01	2.67	2.77	2.19	0.32
Point 02	5.21	0.46	3.27	0.46
Point 03	3.36	2.99	5.35	2.99
Point 04	2.23	0.16	2.72	0.16
Average	3.37	1.59	3.38	0.98

to outputs of FEA analysis are used to validate peridynamic results in Table (7.1). Peridynamic outputs for u_{x_1} deviates in average by 3.37% based on displacement values along horizontal mid-line while deviation in u_{x_2} field reaches up to 1.59% in average along vertical mid-line. On the other hand, resultant displacement fields along horizontal and vertical mid-lines respectively are obtained as 3.38% and 0.98%. To this end, developed code in this study for simulation of displacement field shows consisting results compared to outputs of FEA analysis.

As a future work, analytical and numerical solution of anisotropic macro-structures including discontinuities can also be developed for other specific type of topologies by means of peridynamic theory based on accumulated knowledge throughout this study.

Appendix A

Background

A.1 Introduction

Linear algebra is highly utilized to be able to solve a system of linear equations of n number of equations with n number of unknowns. Within the frame of solid mechanics, rotations of vectors and transformations of matrices can be considered as practical and efficient mapping operations for vector fields and material properties between initial and rotated states of material domain and they enable us to express constitutive relations in compact forms.

Furthermore, a fourth-order isotropic tensor is inherently required to be able to establish a constitutive relation between stress and strain fields of any arbitrary states of a three-dimensional material domain. After introducing fundamental transformation rules for vectors and higher order tensors in index notation, derivation scheme for fourth-order tensor are presented as well.

A.1.1 Fundamentals of Tensor Transformations

In linear elasticity theory, material response such as deformation against applied loads are defined by constitutive equations. From the classical point of view, the responsive behaviour of body is characterized by internal constitution of material disregarding atomistic structure instead considering it as continuous medium, [23].

From the classical continuum perspective, the constitutive equation that relates stress and strain is known as Hooke's law and its generalized state for a linear

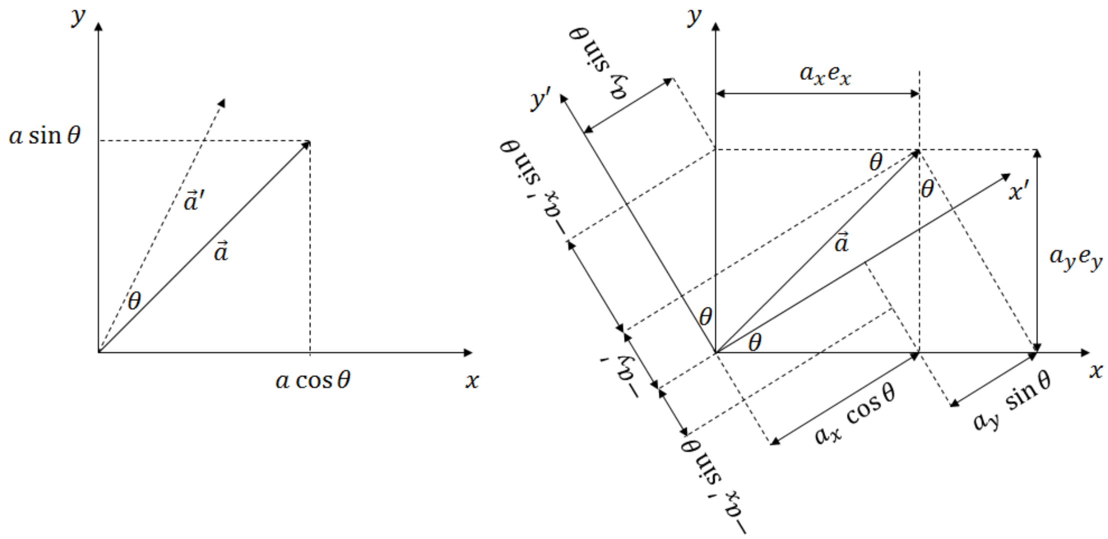


FIGURE A.1: Equivalent representations of a vector rotation

elastic solid body is given as

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{A.1}$$

in which C_{ijkl} is named as stiffness matrix including information regarding mechanical resistance of material against applied forces in specified directions.

A.1.2 Tensor Transformation

Before studying concept of isotropy, let us introduce tensor transformation which is a fundamental tool for rotations of associated co-ordinates. By considering an arbitrary vector $a = a_x e_i + a_y e_j$, defining a point in xy plane, and infinitesimal rotation of xy plane by $\delta\theta_k$ about co-ordinate axis z. Then, the same point can also be defined by using basis vectors of rotated co-ordinates as follows.

$$a'_x = (a_x \cos\delta\theta_z + a_y \sin\delta\theta_z)e'_i \tag{A.2}$$

$$a'_y = (a_y \cos\delta\theta_z - a_x \sin\delta\theta_z)e'_j \tag{A.3}$$

Moreover, relations (A.2) and (A.3), stating components in rotated co-ordinate system, can also presented in a matrix form as follows.

$$\begin{bmatrix} a'_x \\ a'_y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90-\theta) \\ \cos(90+\theta) & \cos\theta \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} \tag{A.4}$$

or in index notation we can write it as

$$a'_i = B_{ij} a_j \tag{A.5}$$

in which the first matrix in the right-hand side of relation (A.4) is called as transformation matrix whose inverse is always equal to its transpose, in other words, $B^T = B^{-1}$. Beside that it is also expressed is a way that components of the transformation matrix are directional cosines of angles between reference and rotated co-ordinate axes.

For infinitesimal angles, the condition that is $\sin \delta\theta_k \simeq \delta\theta_k$, is always valid, therefore the second term in the right-hand side of rellation (A.2) can be approximated as $\epsilon_{ijk} a_j \delta\theta_k$ which can be defined as cross product of an arbitrary position vector a with rotation vector $\delta\theta$, namely, $a \times \delta\theta$. By expressing that in the right-hand of relation (A.2) sub-index of the first term has the same argument with the sub-index in the left-hand side, one may write the components of the same vector a in the basis of rotated co-ordinate system in short-hand notation as

$$a' = a + a \times \delta\theta \tag{A.6}$$

or in index notion

$$a'_i = a_i + \epsilon_{ijk} a_j \delta\theta_k \tag{A.7}$$

In the most general case of transformation around each co-ordinate axes for a vector, a, can be considered as well. In the reference co-ordinate configuration which is ordinarily called material co-ordinates, a is presented as

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 \tag{A.8}$$

while in rotated co-ordinate configuration, which is called spatial co-ordinates representing current state, we express the same vector as

$$a' = a'_1 e'_1 + a'_2 e'_2 + a'_3 e'_3 \tag{A.9}$$

Moreover, a matrix transformation can be defined to be able to rotate xy plane to $x'y'$ plane around co-ordinate axis, z around origin and algebraically a sets of equations can be expressed as follows,

$$\begin{bmatrix} a'_x \\ a'_y \\ a'_z \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \tag{A.10}$$

whose compact form can be expressed as follows with the help of index notation.

$$a'_i = A_{ij} a_j \tag{A.11}$$

in which $A_{ij} = cos(e_i, e'_j)$. As similar to statement of a vector in rotated co-ordinate system, basis vectors of reference co-ordinate system can also be expressed by using same transformation matrix that is

$$e'_i = A_{ij} e_j \tag{A.12}$$

Furthermore, same vector a can be expressed by primed and un-primed co-ordinate symbols respectively as

$$a = a'_i e'_i \tag{A.13}$$

$$a = a_j e_j \tag{A.14}$$

By invoking relation (A.12) into relation (A.13) we write

$$a = a'_i A_{ij} e_j \tag{A.15}$$

By inspecting relations (A.14) and (A.15), it is realized that inverted form of relation (A.11) is obtained as follows.

$$a_j = A_{ij} a'_i \tag{A.16}$$

By invoking relation (A.11) into (A.16) and changing dummy index j by k, vector a_j is expressed as

$$a_j = A_{ij} A_{ik} a_k \tag{A.17}$$

To be able to satisfy the condition given by relation (A.17), tensor multiplication has to yield into δ_{jk}. Namely,

$$\delta_{jk} = A_{ij} A_{ik} \tag{A.18}$$

or in matrix notation

$$I = A^T A \tag{A.19}$$

in which Kronecker-Delta operator, δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{A.20}$$

or in matrix form

$$I = \begin{bmatrix} \mathit{1} & \mathit{0} & \mathit{0} \\ \mathit{0} & \mathit{1} & \mathit{0} \\ \mathit{0} & \mathit{0} & \mathit{1} \end{bmatrix} \tag{A.21}$$

By means of relation (A.19), it is also concluded that inverse of matrix A has to equal to transpose of itself, in other words, $A^{-1} = A^T$ which is known as orthogonality or ortho-normality condition for transformation of matrix A.

Another important operator is needed for easiness of vector multiplications of vector entities. To this end, cross product of vectors in the right-hand side of relation (A.7) can be stated equivalently as $-\epsilon_{kij}\delta\theta_k a_j$ based on Levi-Civita symbol which is given as permutation of natural numbers yielding to either plus, minus or zero depending on sequence of successive numbers, in other words

$$\epsilon_{ijk} = \begin{cases} \mathit{+1}, & \textit{if (i,j,k) is (1,2,3), (2,3,1), (3,1,2)} \\ \mathit{-1}, & \textit{if (i,j,k) is (3,2,1), (1,3,2), (2,1,3)} \\ \mathit{0}, & \textit{if i=j or j=k or k=i} \end{cases} \tag{A.22}$$

$$\begin{aligned} a'_i &= a_i - \epsilon_{kij}\delta\theta_k a_j \\ &= \delta_{ij} a_j - \epsilon_{kij}\delta\theta_k a_j \\ &= (\delta_{ij} - \epsilon_{kij}\delta\theta_k) a_j \\ &= R_{ij} a_j \end{aligned} \tag{A.23}$$

in which R obeys the fact that is $R^{-1} = R^T$ since R is an orthogonal matrix. By multiplying both side inverse of R, we obtain $a'R^{-1} = RaR^{-1}$ or $a'R^{-1} = a$ and since $R^{-1} = R^T$ is always valid, equations given by relation (A.23) are inverted and takes the form of

$$a_i = R_{ji} a'_j \tag{A.24}$$

Direct generalization of equation (A.23) for transformations second and third-order tensors can be easily made as follows.

$$a'_{ij} = R_{ik}R_{jl}a_{kl} \tag{A.25}$$

$$a'_{ijk} = R_{ik}R_{jl}R_{km}a_{klm} \tag{A.26}$$

Similarly, generalization of relation (A.24) is written by the same analogy that is established when writing expressions (A.25) and (A.26). Accordingly,

$$a_{ij} = R_{ki}R_{lj}a'_{kl} \tag{A.27}$$

$$a_{ijk} = R_{ki}R_{lj}R_{mk}a'_{klm} \tag{A.28}$$

Specifically, it can also be shown that relation (A.18) is satisfied for R matrix. Let us consider R_{ij} and R_{ik} and change the dummy indices k to m and l respectively in each.

$$\begin{aligned} R_{ij}R_{ik} &= (\delta_{ij} - \delta\theta_m\epsilon_{mij})(\delta_{ik} - \delta\theta_l\epsilon_{lik}) \\ &= \delta_{ij}\delta_{ik} - \delta_{ij}\delta\theta_l\epsilon_{lik} - \delta_{ik}\delta\theta_m\epsilon_{mij} + \delta\theta_m\delta\theta_l\epsilon_{mij}\epsilon_{lik} \end{aligned} \tag{A.29}$$

Since multiplication of infinitesimal angles, $\delta\theta_m$ and $\delta\theta_l$ can be approximated as $\delta\theta_m\delta\theta_l \simeq 0$. By applying Kronecker-Delta operators on corresponding Levi-Civita operators and changing dummy index l by m, relation (A.29) is simplified as

$$R_{ij}R_{ik} = \delta_{jk} - \delta\theta_m\epsilon_{kmj} - \delta\theta_m\epsilon_{jmk} \tag{A.30}$$

Because of the fact that is $\epsilon_{jkm} = -\epsilon_{kjm}$ according to relation (A.22), expression (A.30) can be further simplified to

$$R_{ij}R_{ik} = \delta_{jk} \tag{A.31}$$

Similar procedure can be applied on $R_{ji}R_{ki}$ to be able to show that it yields to δ_{jk}. Namely,

$$\begin{aligned} R_{ji}R_{ki} &= (\delta_{ji} - \delta\theta_m\epsilon_{mji})(\delta_{ki} - \delta\theta_l\epsilon_{lki}) \\ &= \delta_{ji}\delta_{ki} - \delta_{ji}\delta\theta_l\epsilon_{lki} - \delta\theta_m\epsilon_{mji}\delta_{ki} + \delta\theta_m\epsilon_{mji}\delta\theta_l\epsilon_{lki} \\ &= \delta_{jk} - \theta_m(\epsilon_{mkj} - \epsilon_{mjk}) = \delta_{jk} \end{aligned} \tag{A.32}$$

In further, tensor transformation that is $\epsilon_{ijk}R_{li}R_{mj}R_{nk} = \epsilon_{lmn}$ can be proven by substituting $\delta_{ij} - \delta\theta_k\epsilon_{kij}$ for each R having different dummy indices as follows.

$$\begin{aligned}
\epsilon_{ijk}R_{li}R_{mj}R_{nk} &= \epsilon_{ijk}(\delta_{li} - \delta\theta_a\epsilon_{ali})(\delta_{mj} - \delta\theta_b\epsilon_{bmj})(\delta_{nk} - \delta\theta_c\epsilon_{cnk}) \\
&= \epsilon_{lmn} - \epsilon_{ijk}\delta\theta_a(\delta_{li}\delta_{mj}\epsilon_{cnk} + \delta_{li}\delta_{nk}\epsilon_{bmj} + \delta_{mj}\delta_{nk}\epsilon_{ali}) \\
&= \epsilon_{lmn} - \delta\theta_a(\epsilon_{lmk}\epsilon_{cnk} + \epsilon_{ljn}\epsilon_{bmj} + \epsilon_{imn}\epsilon_{ali}) \\
&= \epsilon_{lmn} - \delta\theta_a(\epsilon_{lmk}\epsilon_{ank} + \epsilon_{ljn}\epsilon_{amj} + \epsilon_{imn}\epsilon_{ali}) \\
&= \epsilon_{lmn} - \delta\theta_a(\epsilon_{imn}\epsilon_{ial} + \epsilon_{jnl}\epsilon_{jam} + \epsilon_{klm}\epsilon_{kan}) \\
&= \epsilon_{lmn} - \delta\theta_a(\delta_{ma}\delta_{nl} - \delta_{ml}\delta_{na} + \delta_{na}\delta_{lm} - \delta_{nm}\delta_{la} + \delta_{la}\delta_{mn} - \delta_{ln}\delta_{ma}) \\
&= \epsilon_{lmn}
\end{aligned} \tag{A.33}$$

in which assumptions that are $\delta\theta_a = \delta\theta_b = \delta\theta_c$ and $\delta\theta_b\delta\theta_c = \delta\theta_a\delta\theta_c = \delta\theta_a\delta\theta_c = 0$ have been made and contracted epsilon identity $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, is applied to the fifth line of relation (A.33) as well.

Furthermore, scalar product of two arbitrary vectors in rotated frame of reference that is given by relation (A.23) can be calculated by means of relation (A.31).

$$\begin{aligned}
a'_i b'_i &= R_{il}R_{im}a_l b_m \\
&= \delta_{lm}a_l b_m \\
&= a_m b_m = a_l b_l
\end{aligned} \tag{A.34}$$

For a given statement $a_i b_{jk} = c_{ijk}$, recasting relations (A.23) and (A.25), a similar relation with (A.26) or inverted version of (A.28) can also be obtained.

$$\begin{aligned}
a'_i b'_{jk} &= R_{il}R_{jm}R_{kn}a_l b_{mn} \\
c'_{ijk} &= R_{il}R_{jm}R_{kn}c_{lmn}
\end{aligned} \tag{A.35}$$

For arbitrary tensors a_{ik} and a_{jk}, according to relation (A.18) apparently operation $a_{ik}b_{jk} = c_{ij}$ which is usually known as outer product of two tensors can be written in general.

Let us consider following outer product of two tensors in rotated frame of reference and prove it in an appropriate way, utilizing from the rules that have been obtained so far.

$$c'_{ij} = a'_{ik}b'_{jk} \tag{A.36}$$

By means of relation (A.25) and invoking $a_{kp}b_{lp}$ in c_{kl} we write

$$\begin{aligned} c'_{ij} &= R_{ik}R_{jk}c_{kl} = R_{ik}R_{jl}a_{kp}b_{lp} \\ &= R_{ik}R_{jl}a_{kp}R_{ml}R_{np}b'_{mn} \\ &= R_{ik}R_{np}\delta_{jm}a_{kp}b'_{mn} \\ &= R_{ik}R_{np}\delta_{jm}a_{kp}b'_{mn} = R_{ik}R_{np}a_{kp}b'_{jn} \end{aligned} \tag{A.37}$$

Combining relations (A.36) and (A.37) leads to

$$\begin{aligned} a'_{in}b'_{jn} &= R_{ik}R_{np}a_{kp}b'_{jn} \\ 0 &= (a'_{in} - R_{ik}R_{np}a_{kp})b'_{jn} \end{aligned} \tag{A.38}$$

Finally, it can be concluded that the condition $a'_{in} = R_{ik}R_{np}a_{kp}$ has to be satisfied. This result complies with relation (A.25).

Additionally, relation (A.25) can be generalized for Kronecker-Delta operator and by means of $R_{ij} = \delta_{ij} - \epsilon_{kij}\delta\theta_k$ we write

$$\begin{aligned} \delta'_{ij} &= R_{ik}R_{jl}\delta_{kl} = R_{ik}R_{jk} \\ &= (\delta_{ik} - \epsilon_{mik}\delta\theta_m)(\delta_{jk} - \epsilon_{njk}\delta\theta_n) \\ &= \delta_{ik}\delta_{jk} - \delta_{ik}\epsilon_{njk}\delta\theta_n - \delta_{jk}\epsilon_{mik}\delta\theta_m + \epsilon_{mik}\delta\theta_m\epsilon_{njk}\delta\theta_n \\ &= \delta_{ij} - \delta\theta_m(\epsilon_{mji} - \epsilon_{mij}) = \delta_{ij} \end{aligned} \tag{A.39}$$

Lastly, transformation of Levi-Civita operator from reference to rotated co-ordinate system is introduced as follows.

$$\begin{aligned} \epsilon'_{ijk} &= R_{il}R_{jm}R_{kn}\epsilon_{lmn} \\ &= (\delta_{il} - \delta\theta_a\epsilon_{ail})(\delta_{jm} - \delta\theta_b\epsilon_{bjm})(\delta_{kn} - \delta\theta_c\epsilon_{ckn})\epsilon_{lmn} \\ &= (\delta_{il}\delta_{jm}\delta_{kn} - \delta_{il}\delta_{jm}\delta\theta_c\epsilon_{ckn} - \delta_{il}\delta\theta_b\epsilon_{bjm}\delta_{kn} - \delta\theta_a\epsilon_{ail}\delta_{jm}\delta_{kn})\epsilon_{lmn} \\ &= \epsilon_{ijk} - \delta\theta_a(\delta_{il}\delta_{jm}\epsilon_{akn}\epsilon_{lmn} + \delta_{il}\delta_{kn}\epsilon_{ajm}\epsilon_{lmn} + \delta_{jm}\delta_{kn}\epsilon_{ail}\epsilon_{lmn}) \\ &= \epsilon_{ijk} - \delta\theta_a(\epsilon_{nak}\epsilon_{nij} + \epsilon_{maj}\epsilon_{mki} + \epsilon_{lai}\epsilon_{ljk}) \\ &= \epsilon_{ijk} - \delta\theta_a(\delta_{ai}\delta_{kj} - \delta_{aj}\delta_{ki} + \delta_{ak}\delta_{ji} - \delta_{ai}\delta_{jk} + \delta_{aj}\delta_{ik} - \delta_{ak}\delta_{ij} \\ &= \epsilon_{ijk} \end{aligned} \tag{A.40}$$

A.2 Isotropic Tensors

Being rotationally invariant of any n-th order tensor leads to an isotropic kind whose all components are not affected from rotation of co-ordinate system and stay constant as being in reference frame. Let us present two obvious example for isotropic tensors to be able to clarify definition of isotropy. As shown in relations (A.39) and (A.40) respectively, the second-order Kronecker-Delta and third-order Levi-Civita tensors remain same after their transformation from one to an another frame of reference.

In this manner, isotropy condition stipulates that the condition $a_{ij} = a'_{ij}$ has to be satisfied. By means of relation (A.25) and combining it with a tensor in rotated frame of reference we can show that

$$\begin{aligned} a'_{ij} &= R_{ip}R_{jq}a_{pq} = a_{ij} \\ &= (\delta_{ip} - \delta\theta_m\epsilon_{mip})(\delta_{jq} - \delta\theta_m\epsilon_{mjq})a_{pq} \\ &= (\delta_{ip}\delta_{ip} - \delta_{ip}\delta\theta_m\epsilon_{mjq} - \delta_{jq}\theta_m\epsilon_{mip})a_{pq} \\ &= a_{ij} - \delta\theta_m(a_{iq}\epsilon_{mjq} + a_{pj}\epsilon_{mip}) = a_{ij} \end{aligned} \tag{A.41}$$

Following condition might be directly concluded from the last line of relation (A.41).

$$a_{iq}\epsilon_{mjq} + a_{pj}\epsilon_{mip} = 0 \tag{A.42}$$

By multiplying both sides of relation (A.42) by ϵ_{mik} and applying contracted epsilon identity we write

$$\begin{aligned} \epsilon_{mik}\epsilon_{mjq}a_{iq} + \epsilon_{mik}\epsilon_{mip}a_{pj} &= 0 \\ (\delta_{ii}\delta_{kp} - \delta_{ip}\delta_{ki})a_{pj} + (\delta_{ij}\delta_{kq} - \delta_{iq}\delta_{kj})a_{iq} &= 0 \\ \delta_{ii}a_{kj} - a_{ij}\delta_{ki} + \delta_{ij}a_{ki} - a_{ii}\delta_{kj} &= 0 \\ \delta_{ii}a_{kj} - a_{kj} + a_{kj} - a_{ii}\delta_{kj} &= 0 \\ \delta_{ii}a_{kj} &= \delta_{kj} \\ a_{kj} &= \lambda\delta_{kj} \\ a_{ij} &= \lambda\delta_{ij} \end{aligned} \tag{A.43}$$

in which λ can be considered as a constant value which implies mean value of orthogonal component of a_{ij}. In other words we can express it as $trace(a_{ij})/3 =$

$a_{ii}/3$.

Let us now consider a third-order isotropic tensor a_{ijk} and show that it is also invariant under rotation of co-ordinate system. Then, transformation relation to be considered becomes $a'_{ijk} = R_{ip}R_{jq}R_{kr}a_{pqr}$. To this end, by applying explicit form of rotation tensor which is in general $(\delta_{ij} - \delta\theta_k\epsilon_{kij})$ we write

$$\begin{aligned} a'_{ijk} &= R_{ip}R_{jq}R_{kr}a_{pqr} \\ &= (\delta_{ip} - \delta\theta_a\epsilon_{aip})(\delta_{jq} - \delta\theta_a\epsilon_{ajq})(\delta_{kr} - \delta\theta_a\epsilon_{akr})a_{pqr} \\ &= (\delta_{ip} - \delta\theta_a\epsilon_{aip})(\delta_{jq}\delta_{kr}a_{pqr} - \delta_{jq}\delta\theta_a\epsilon_{akr}a_{pqr} - \delta_{kr}\delta\theta_a\epsilon_{ajq}a_{pqr}) \\ &= \delta_{ip}\delta_{jq}\delta_{kr}a_{pqr} - \delta_{ip}\delta_{jq}\delta\theta_a\epsilon_{akr} - \delta_{ip}\delta_{kr}\delta\theta_a\epsilon_{ajq}a_{pqr} - \delta_{jq}\delta_{kr}\delta\theta_a\epsilon_{aip}a_{pqr} \\ &= a_{ijk} - \delta\theta_a(\delta_{ip}\delta_{jq}\epsilon_{akr} + \delta_{ip}\delta_{kr}\epsilon_{ajq} + \delta_{jq}\delta_{kr}\epsilon_{aip})a_{pqr} \end{aligned} \tag{A.44}$$

To be able to satisfy isotropy condition for a third-order tensor a_{ijk}, the coefficient in front of $\delta\theta_a$ has to be equal to zero. As a consequence , we write

$$(\delta_{ip}\delta_{jq}\epsilon_{akr} + \delta_{ip}\delta_{kr}\epsilon_{ajq} + \delta_{jq}\delta_{kr}\epsilon_{aip})a_{pqr} = 0 \tag{A.45}$$

Multiplying relation (A.45) by ϵ_{ait} and using contracted epsilon identity first set of equations are obtained as follows.

$$\begin{aligned} (\delta_{ip}\delta_{jq}\epsilon_{ait}\epsilon_{akr} + \delta_{ip}\delta_{kr}\epsilon_{ait}\epsilon_{ajq} + \delta_{jq}\delta_{kr}\epsilon_{ait}\epsilon_{aip})a_{pqr} &= 0 \\ (\delta_{ip}\delta_{jq}(\delta_{ik}\delta_{tr} - \delta_{ir}\delta_{tk}) + \delta_{ip}\delta_{kr}(\delta_{ij}\delta_{tq} - \delta_{iq}\delta_{tj}) + \delta_{jq}\delta_{kr}(\delta_{ii}\delta_{tp} - \delta_{ip}\delta_{ti}))\, a_{pqr} &= 0 \\ (\delta_{ip}\delta_{jq}\delta_{ik}\delta_{tr} - \delta_{ip}\delta_{jq}\delta_{ir}\delta_{tk} + \delta_{ip}\delta_{kr}\delta_{ij}\delta_{tq} - \delta_{ip}\delta_{kr}\delta_{iq}\delta_{tj} + \delta_{jq}\delta_{kr}\delta_{ii}\delta_{tp} - \delta_{jq}\delta_{kr}\delta_{ip}\delta_{ti})a_{pqr} &= 0 \\ (\delta_{kp}\delta_{jq}\delta_{tr} - \delta_{rp}\delta_{jq}\delta_{tk} + \delta_{jp}\delta_{kr}\delta_{tq} - \delta_{qp}\delta_{kr}\delta_{tj} + 3\delta_{jq}\delta_{kr}\delta_{tp} - \delta_{jq}\delta_{kr}\delta_{tp})a_{pqr} &= 0 \end{aligned} \tag{A.46}$$

Rearranging the terms in last line of relation (A.46)

$$\begin{aligned} (2\delta_{jq}\delta_{kr}\delta_{tp} + \delta_{kp}\delta_{jq}\delta_{tr} + \delta_{jp}\delta_{kr}\delta_{tq})a_{pqr} &= (\delta_{rp}\delta_{jq}\delta_{tk} + \delta_{qp}\delta_{kr}\delta_{tj})a_{pqr} \\ 2a_{tjk} + a_{kjt} + a_{jtk} &= a_{rjr}\delta_{tk} + a_{ppk}\delta_{tj} \end{aligned} \tag{A.47}$$

Since t is an arbitrary index, it can be replaced by i, namely, $t = i$

$$2a_{ijk} + a_{kji} + a_{jik} = a_{rjr}\delta_{ik} + a_{ppk}\delta_{ij} \tag{A.48}$$

Multiplying both side of relation (A.48) with term that is δ_{jk} and rearranging terms we obtain

$$2a_{ijk}\delta_{jk} + a_{kji}\delta_{jk} + a_{jik}\delta_{jk} = a_{rjr}\delta_{ik}\delta_{jk} + a_{ppk}\delta_{ij}\delta_{jk} \tag{A.49}$$

or

$$2a_{ijj} + a_{jji} + a_{jij} = a_{rir} + a_{ppi} \tag{A.50}$$

and since q, r and j are dummy indices in the right and left-hand sides respectively, they can be replaced by the index s, thus relation (A.50) yields to

$$2a_{iss} + a_{ssi} + a_{sis} = a_{sis} + a_{ssi} \tag{A.51}$$

To be able to obtain the second and third set of equations, both side of relation (A.45) are multiplied by entities ϵ_{ajt} and ϵ_{akt} respectively. To this end, by applying contracted epsilon identity to both one may write them down as follows.

$$(\delta_{ip}\delta_{jq}\delta_{kj}\delta_{rt} - \delta_{ip}\delta_{jq}\delta_{kt}\delta_{rj} + \delta_{ip}\delta_{kr}\delta_{jj}\delta_{qt} - \delta_{ip}\delta_{kr}\delta_{jt}\delta_{qj} + \delta_{jq}\delta_{kr}\delta_{ij}\delta_{pt} - \delta_{jq}\delta_{kr}\delta_{it}\delta_{pj})a_{pqr} = 0 \tag{A.52}$$

and

$$(\delta_{ip}\delta_{jq}\delta_{kk}\delta_{rt} - \delta_{ip}\delta_{jq}\delta_{kt}\delta_{rt} + \delta_{ip}\delta_{kr}\delta_{jk}\delta_{qt} - \delta_{ip}\delta_{kr}\delta_{jt}\delta_{qk} + \delta_{jq}\delta_{kr}\delta_{ik}\delta_{pt} - \delta_{jq}\delta_{kr}\delta_{it}\delta_{pk})a_{pqr} = 0 \tag{A.53}$$

They can be reduced to

$$2a_{itk} + a_{ikt} + a_{tik} = a_{irr}\delta_{kt} + a_{ppk}\delta_{it} \tag{A.54}$$

and

$$2a_{ijt} + a_{itj} + a_{tji} = a_{irr}\delta_{jt} + a_{rjr}\delta_{it} \tag{A.55}$$

respectively. Since t is an any arbitrary index in relations (A.54) and (A.55), they can be replaced by j and k respectively. As a result,

$$2a_{ijk} + a_{ikj} + a_{jik} = a_{irr}\delta_{kj} + a_{ppk}\delta_{ij} \tag{A.56}$$

$$2a_{ijk} + a_{ikj} + a_{kji} = a_{irr}\delta_{jk} + a_{rjr}\delta_{ik} \tag{A.57}$$

Moreover, multiplying both sides of relations (A.56) and (A.57) by δ_{ik} and δ_{ij} respectively leads to

$$2a_{ijk}\delta_{ik} + a_{ikj}\delta_{ik} + a_{jik}\delta_{ik} = a_{irr}\delta_{kj}\delta_{ik} + a_{ppk}\delta_{ij}\delta_{ik} \tag{A.58}$$

and

$$2a_{ijk}\delta_{ij} + a_{ikj}\delta_{ij} + a_{kji}\delta_{ij} = a_{irr}\delta_{jk}\delta_{ij} + a_{rjr}\delta_{ik}\delta_{ij} \tag{A.59}$$

Accordingly, relations (A.58) and (A.59) are reduced to

$$2a_{kjk} + a_{jkk} + a_{kkj} = a_{jkk} + a_{ppj} \tag{A.60}$$

and

$$2a_{jjk} + a_{jkj} + a_{kjj} = a_{krr} + a_{rkr} \tag{A.61}$$

For the sake of completeness of final results, relations (A.60) and (A.61) can be presented by using same alphabetic indices appearing in relation (A.51). All in all, three sets of equations are obtained as

$$\begin{aligned} 2a_{iss} + a_{ssi} + a_{sis} &= a_{sis} + a_{ssi} \\ 2a_{sis} + a_{iss} + a_{ssi} &= a_{iss} + a_{ssi} \\ 2a_{ssi} + a_{sis} + a_{iss} &= a_{iss} + a_{sis} \end{aligned} \tag{A.62}$$

It is concluded that these three sets of equations have to satisfy the condition that is

$$a_{iss} = a_{sis} = a_{ssi} = 0 \tag{A.63}$$

Since the right hand side of relations (A.48), (A.56) and (A.57) yield to zero as deduced from (A.63), it is concluded that

$$\begin{aligned} 2a_{ijk} + a_{kji} + a_{jik} &= 0 \\ 2a_{ijk} + a_{ikj} + a_{jik} &= 0 \\ 2a_{ijk} + a_{ikj} + a_{kji} &= 0 \end{aligned} \tag{A.64}$$

By combining first set of relation (A.64) with second and third sets respectively, following condition is obtained.

$$a_{kji} = a_{ikj} = a_{jik} \tag{A.65}$$

Invoking a_{jik} for a_{kji} into the first line of relation (A.64) leads to

$$a_{ijk} = -a_{kji} = -a_{ikj} = -a_{jik} \tag{A.66}$$

Due to same cyclic permutation rule defined by Levi-Civita operator given by relation (A.22), the third-order isotropic tensor can be expressed as a multiple of ϵ_{ijk} through a constant value assigned as μ. Accordingly, we have

$$a_{ijk} = \mu\epsilon_{ijk} \tag{A.67}$$

While constructing isotropic fourth-order tensor, an extra orthogonal term R_{ls} is needed to add it in relation (A.44) and transformation is expressed as $a'_{ijkl} = R_{ip}R_{jq}R_{kr}R_{ls}a_{pqrs}$. Moreover, isotropy condition requires that fourth-order tensor has to yield to itself in reference configuration. In this manner, transformation relation can be proven through

$$\begin{aligned}
a'_{ijkl} &= R_{ip}R_{jq}R_{kr}R_{ls}a_{pqrs} \\
&= (\delta_{ip} - \delta\theta_a\epsilon_{aip})(\delta_{jq} - \delta\theta_a\epsilon_{ajq})(\delta_{kr} - \delta\theta_a\epsilon_{akr})(\delta_{ls} - \delta\theta_a\epsilon_{als})a_{pqrs} \\
&= (\delta_{ip}\delta_{jq} - \delta_{ip}\delta\theta_a\epsilon_{ajq} - \delta_{jq}\delta\theta_a\epsilon_{aip})(\delta_{kr}\delta_{ls} - \delta_{kr}\delta\theta_a\epsilon_{als} - \delta_{ls}\delta\theta_a\epsilon_{akr})a_{pqrs} \\
&= (\delta_{ip}\delta_{jq}\delta_{kr}\delta_{ls} - \delta_{ip}\delta_{jq}\delta_{kr}\delta\theta_a\epsilon_{als} - \delta_{ip}\delta_{jq}\delta_{ls}\delta\theta_a\epsilon_{akr} - \delta_{kr}\delta_{ls}\delta_{ip}\delta\theta_a\epsilon_{ajq} - \delta_{kr}\delta_{ls}\delta_{jq}\delta\theta_a\epsilon_{aip})a_{pqrs} \\
&= \delta_{ip}\delta_{jq}\delta_{kr}\delta_{ls}a_{pqrs} - \delta\theta_a(\delta_{ip}\delta_{jq}\delta_{kr}\epsilon_{als} + \delta_{ip}\delta_{jq}\delta_{ls}\epsilon_{akr} + \delta_{kr}\delta_{ls}\delta_{ip}\epsilon_{ajq} + \delta_{kr}\delta_{ls}\delta_{jq}\epsilon_{aip})a_{pqrs} \\
&= \delta_{ip}\delta_{jq}\delta_{kr}\delta_{ls}a_{pqrs} - \delta\theta_a(\delta_{ip}\delta_{jq}\delta_{kr}\epsilon_{als} + \delta_{ip}\delta_{jq}\delta_{ls}\epsilon_{akr} + \delta_{kr}\delta_{ls}\delta_{ip}\epsilon_{ajq} + \delta_{kr}\delta_{ls}\delta_{jq}\epsilon_{aip})a_{pqrs}
\end{aligned} \tag{A.68}$$

If isotropy condition, $a'_{ijkl} = a_{ijkl}$ holds, it is concluded that

$$(\delta_{ip}\delta_{jq}\delta_{kr}\epsilon_{als} + \delta_{ip}\delta_{jq}\delta_{ls}\epsilon_{akr} + \delta_{kr}\delta_{ls}\delta_{ip}\epsilon_{ajq} + \delta_{kr}\delta_{ls}\delta_{jq}\epsilon_{aip})a_{pqrs} = 0 \tag{A.69}$$

Through a similar procedure when deriving isotropic third-order tensor, all terms of relation (A.69) are multiplied by ϵ_{ait}, ϵ_{ajt}, ϵ_{akt} and ϵ_{alt} respectively and subindices are rearranged in an appropriate way as follows.

$$\begin{aligned}
0 =& a_{ijks}\epsilon_{als} + a_{ijrl}\epsilon_{akr} + a_{iqkl}\epsilon_{ajq} + a_{pjkl}\epsilon_{aip} \\
0 =& a_{ijks}\epsilon_{als}\epsilon_{ait} + a_{ijrl}\epsilon_{akr}\epsilon_{ait} + a_{iqkl}\epsilon_{ajq}\epsilon_{ait} + a_{pjkl}\epsilon_{aip}\epsilon_{ait} \\
0 =& a_{ijks}(\delta_{li}\delta_{st} - \delta_{lt}\delta_{si}) + a_{ijrl}(\delta_{ki}\delta_{rt} - \delta_{kt}\delta_{ri}) + a_{iqkl}(\delta_{ji}\delta_{qt} - \delta_{jt}\delta_{qi}) + a_{pjkl}(\delta_{ii}\delta_{pt} - \delta_{it}\delta_{pi}) \\
0 =& a_{ljkt} - a_{ijki}\delta_{lt} + a_{kjtl} - a_{ijil}\delta_{kt} + a_{jtkl} - a_{iikl}\delta_{jt} + 3a_{tjkl} - a_{ijkl}\delta_{it}
\end{aligned} \tag{A.70}$$

$$\begin{aligned}
0 =& a_{ijks}\epsilon_{als} + a_{ijrl}\epsilon_{akr} + a_{iqkl}\epsilon_{ajq} + a_{pjkl}\epsilon_{aip} \\
0 =& a_{ijks}\epsilon_{als}\epsilon_{ajt} + a_{ijrl}\epsilon_{akr}\epsilon_{ajt} + a_{iqkl}\epsilon_{ajq}\epsilon_{ajt} + a_{pjkl}\epsilon_{aip}\epsilon_{ajt} \\
0 =& a_{ijks}(\delta_{lj}\delta_{st} - \delta_{lt}\delta_{sj}) + a_{ijrl}(\delta_{kj}\delta_{rt} - \delta_{kt}\delta_{rj}) + a_{iqkl}(\delta_{jj}\delta_{qt} - \delta_{jt}\delta_{qj}) + a_{pjkl}(\delta_{ij}\delta_{pt} - \delta_{it}\delta_{pj}) \\
0 =& a_{ilkt} - a_{ijkj}\delta_{lt} + a_{iktl} - a_{ijjl}\delta_{kt} + 3a_{itkl} - a_{ijkl}\delta_{jt} + a_{tikl} - a_{jjkl}\delta_{it}
\end{aligned} \tag{A.71}$$

$$\begin{aligned}
0 =& a_{ijks}\epsilon_{als} + a_{ijrl}\epsilon_{akr} + a_{iqkl}\epsilon_{ajq} + a_{pjkl}\epsilon_{aip} \\
0 =& a_{ijks}\epsilon_{als}\epsilon_{akt} + a_{ijrl}\epsilon_{akr}\epsilon_{akt} + a_{iqkl}\epsilon_{ajq}\epsilon_{akt} + a_{pjkl}\epsilon_{aip}\epsilon_{akt} \\
0 =& a_{ijks}(\delta_{lk}\delta_{st} - \delta_{lt}\delta_{sk}) + a_{ijrl}(\delta_{kk}\delta_{rt} - \delta_{kt}\delta_{rk}) + a_{iqkl}(\delta_{jk}\delta_{qt} - \delta_{jt}\delta_{qk}) + a_{pjkl}(\delta_{ik}\delta_{pt} - \delta_{it}\delta_{pk}) \\
0 =& a_{ijlt} - a_{ijkk}\delta_{lt} + 3a_{ijtl} - a_{ijtl} + a_{itjl} - a_{ikkl}\delta_{jt} + a_{tjil} - a_{kjkl}\delta_{it}
\end{aligned} \tag{A.72}$$

$$\begin{aligned}
0 =& a_{ijks}\epsilon_{als} + a_{ijrl}\epsilon_{akr} + a_{iqkl}\epsilon_{ajq} + a_{pjkl}\epsilon_{aip} \\
0 =& a_{ijks}\epsilon_{als}\epsilon_{alt} + a_{ijrl}\epsilon_{akr}\epsilon_{alt} + a_{iqkl}\epsilon_{ajq}\epsilon_{alt} + a_{pjkl}\epsilon_{aip}\epsilon_{alt} \\
0 =& a_{ijks}(\delta_{ll}\delta_{st} - \delta_{lt}\delta_{sl}) + a_{ijrl}(\delta_{kl}\delta_{rt} - \delta_{kt}\delta_{rl}) + a_{iqkl}(\delta_{jl}\delta_{qt} - \delta_{jt}\delta_{ql}) + a_{pjkl}(\delta_{il}\delta_{pt} - \delta_{it}\delta_{pl}) \\
0 =& 3a_{ijkt} - a_{ijkt} + a_{ijtk} - a_{ijll}\delta_{kt} + a_{itkj} - a_{ilkl}\delta_{jt} + a_{tjki} - a_{ljkl}\delta_{it}
\end{aligned} \tag{A.73}$$

The last lines of relations (A.70), (A.71), (A.72) and (A.73) can be simplified as follows.

$$\begin{aligned}
3a_{tjkl} - a_{tjkl} + a_{ljkt} + a_{kjtl} + a_{jtkl} &= a_{ijki}\delta_{lt} + a_{iikl}\delta_{jt} + a_{ijil}\delta_{kt} \\
2a_{tjkl} + a_{ljkt} + a_{kjtl} + a_{jtkl} &= a_{ijki}\delta_{lt} + a_{iikl}\delta_{jt} + a_{ijil}\delta_{kt}
\end{aligned} \tag{A.74}$$

$$\begin{aligned}
a_{ilkt} + a_{iktl} + 3a_{itkl} - a_{itkl} + a_{tikl} &= a_{ijkj}\delta_{lt} + a_{ijjl}\delta_{kt} + a_{jjkl}\delta_{it} \\
2a_{itkl} + a_{iktl} + a_{ilkt} + a_{tikl} &= a_{ijkj}\delta_{lt} + a_{ijjl}\delta_{kt} + a_{jjkl}\delta_{it}
\end{aligned} \tag{A.75}$$

$$\begin{aligned} a_{ijlt} + 3a_{ijtl} + -a_{ijtl} + a_{itjl} + a_{tjil} &= a_{ijkk}\delta_{lt} + a_{ikkl}\delta_{jt} + a_{kjkl}\delta_{it} \\ 2a_{ijtl} + a_{ijlt} + a_{itjl} + a_{tjil} &= a_{ijkk}\delta_{lt} + a_{ikkl}\delta_{jt} + a_{kjkl}\delta_{it} \end{aligned} \tag{A.76}$$

$$\begin{aligned} 3a_{ijkt} - a_{ijkt} + a_{ijtk} + a_{itkj} + a_{tjki} &= a_{ijll}\delta_{kt} + a_{ilkl}\delta_{jt} + a_{ljkl}\delta_{it} \\ 2a_{ijkt} + a_{ijtk} + a_{itkj} + a_{tjki} &= a_{ijll}\delta_{kt} + a_{ilkl}\delta_{jt} + a_{ljkl}\delta_{it} \end{aligned} \tag{A.77}$$

Since t, appearing in the left-hand side is an arbitrary index, we can invoke i, j, k and l respectively to the last lines of above four sets of equations and the tensors having repeating sub-indices e.g. a_{iikl}, a_{ijil} and a_{ijki} yield to second-order tensors and can presented as $\lambda\delta_{kl}$, $\mu\delta_{jl}$ and $\nu\delta_{jk}$ respectively. As a result, following four sets of equations are obtained as follows.

$$\begin{aligned} 2a_{ijkl} + a_{ljki} + a_{kjil} + a_{jikl} &= \nu\delta_{jk}\delta_{li} + \lambda\delta_{kl}\delta_{ji} + \mu\delta_{jl}\delta_{ki} \\ 2a_{ijkl} + a_{ikjl} + a_{ilkj} + a_{jikl} &= \mu\delta_{ik}\delta_{lj} + \nu\delta_{il}\delta_{kj} + \lambda\delta_{kl}\delta_{ij} \\ 2a_{ijkl} + a_{ijlk} + a_{ikjl} + a_{kjil} &= \lambda\delta_{ij}\delta_{lk} + \nu\delta_{il}\delta_{jk} + \mu\delta_{jl}\delta_{ik} \\ 2a_{ijkl} + a_{ijlk} + a_{ilkj} + a_{ljki} &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{jk}\delta_{il} \end{aligned} \tag{A.78}$$

Possible three combinations of above four sets of equations can be obtained from pairs of first and second lines and pairs of third and fourth lines. Namely,

$$\begin{aligned} &2a_{ijkl} + a_{ljki} + a_{kjil} + a_{jikl} + 2a_{ijkl} + a_{ikjl} + a_{ilkj} + a_{jikl} = \\ &= 2(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{jk}\delta_{li}) \\ &- 2a_{ijkl} - a_{ijlk} - a_{ikjl} - a_{kjil} - 2a_{ijkl} - a_{ijlk} - a_{ilkj} - a_{ljki} \quad = \\ &= -2(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{jk}\delta_{li}) \\ a_{kjil} &= a_{ilkj} \end{aligned} \tag{A.79}$$

from pair of first and third lines and pair of second and fourth lines

$$\begin{aligned} &2a_{ijkl} + a_{ljki} + a_{kjil} + a_{jikl} + 2a_{ijkl} + a_{ijlk} + a_{ikjl} + a_{kjil} = \\ &= 2(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{jk}\delta_{li}) \\ &- 2a_{ijkl} - a_{ikjl} - a_{ilkj} - a_{jikl} - 2a_{ijkl} - a_{ijlk} - a_{ilkj} - a_{ljki} \quad = \\ &= -2(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{jk}\delta_{li}) \\ a_{ljki} &= a_{ikjl} \end{aligned} \tag{A.80}$$

from pair of first and fourth lines and pair of second and third lines

$$\begin{aligned}
& 2a_{ijkl} + a_{ljki} + a_{kjil} + a_{jikl} + 2a_{ijkl} + a_{ijlk} + a_{ilkj} + a_{ljki} = \\
& = 2(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{jk}\delta_{li}) \\
& - 2a_{ijkl} - a_{ikjl} - a_{ilkj} - a_{jikl} - 2a_{ijkl} - a_{ijlk} - a_{ikjl} - a_{kjil} \quad = \\
& = -2(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{jk}\delta_{li}) \\
a_{jikl} &= a_{ijlk}
\end{aligned} \tag{A.81}$$

Last three lines of relations (A.79), (A.80) and (A.81) provide us with alternative forms of the terms appearing in the left-hand side of relation (A.78). Therefore, the first line of (A.78) can be expressed as

$$2a_{ijkl} + a_{ikjl} + a_{ilkj} + a_{ijlk} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} \tag{A.82}$$

Moreover, remaining two sets of equations which are required to be able to write isotropic fourth-order tensor can be generated through permutation of indices j, k and l while keeping the index i fixed.

$$2a_{iklj} + a_{ijlk} + a_{ikjl} + a_{ilkj} = \lambda\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} + \nu\delta_{ij}\delta_{kl} \tag{A.83}$$

$$2a_{iljk} + a_{ilkj} + a_{ijlk} + a_{ikjl} = \lambda\delta_{il}\delta_{jk} + \mu\delta_{ij}\delta_{kl} + \nu\delta_{ik}\delta_{jl} \tag{A.84}$$

As a consequence, summing up (A.82), (A.83) and (A.84) side by side leads to

$$2(a_{ijkl} + a_{iklj} + a_{iljk}) + 3(a_{ikjl} + a_{ilkj} + a_{ijlk}) = (\lambda + \mu + \nu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl}) \tag{A.85}$$

If a symmetry condition which requires $a_{ijkl} = a_{ijlk}$, $a_{iklj} = a_{ilkj}$ and $a_{iljk} = a_{ikjl}$ holds, then from relation (A.85), following statement is concluded.

$$a_{ikjl} + a_{ilkj} + a_{ijlk} = \frac{1}{5}(\lambda + \mu + \nu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl}) \tag{A.86}$$

Substitution of the part $(a_{ikjl} + a_{ilkj} + a_{ijlk})$ into the right hand side of (A.82) gives

$$2a_{ijkl} + \frac{1}{5}(\lambda + \mu + \nu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl}) = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} \tag{A.87}$$

Rearranging terms in an appropriate way simplifies relation (A.87) as

$$\begin{aligned}2a_{ijkl} &= \delta_{ij}\delta_{kl}(\lambda - \frac{1}{5}\lambda - \frac{1}{5}\mu - \frac{1}{5}\nu) + \delta_{ik}\delta_{jl}(\mu - \frac{1}{5}\lambda - \frac{1}{5}\mu - \frac{1}{5}\nu) \\ &+ \delta_{il}\delta_{jk}(\nu - \frac{1}{5}\lambda - \frac{1}{5}\mu - \frac{1}{5}\nu)\end{aligned} \tag{A.88}$$

or in more compact form, relation (A.88) can be written as follows.

$$a_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \tag{A.89}$$

in which α, β and γ are respectively equal to $(4\lambda - \mu - \nu)/10$, $(4\mu - \lambda - \nu)/10$ and $(4\nu - \lambda - \mu)/10$.

Appendix B

Deformation Constants

B.1 General review on deformation

For the sake of simplicity, let us consider two-dimensional deformation of RVE introduced in Figure (B.1) and assume that infinitesimal side lengths dx_1 and dx_2 are same, $dx_1 = dx_2$ in un-deformed state. As shown in Figure (B.1), shearing angle γ is measured by how much top side of RVE is slid by distorting its shape.

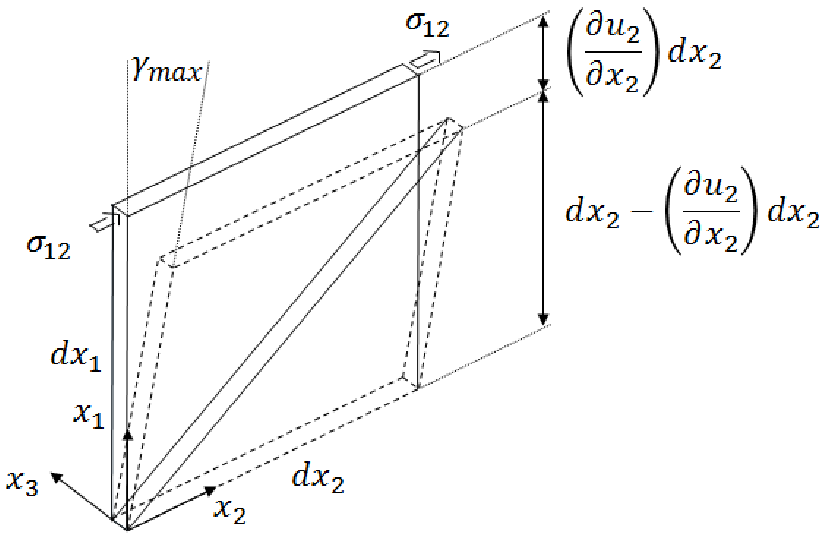


Figure B.1: Exaggerated shearing deformation in plane stress condition

Tangent of angle $(\pi/4 - \gamma_{max}/2)$ can be expressed in two different ways. One way is to use its geometric definition which is given by the ratio of orthogonal edges while the other way is to utilize from trigonometric identity for tangent function.

Namely,

$$\tan(\pi/4 - \gamma_{max}/2) = \frac{dx'_2 - \left(\frac{\partial u_2}{\partial x'_2}\right) dx'_2}{dx'_1 + \left(\frac{\partial u_1}{\partial x'_1}\right) dx'_1} = \frac{(1-\epsilon_{22})\, dx'_2}{(1+\epsilon_{11})\, dx'_1} \approx \frac{(1-\nu\epsilon_{11})}{(1+\epsilon_{11})} \tag{B.1}$$

or equivalently,

$$\tan(\pi/4 - \gamma_{max}/2) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\gamma_{max}}{2}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\gamma_{max}}{2}\right)} \approx \frac{1-\gamma_{max}/2}{1+\gamma_{max}/2} \tag{B.2}$$

By combining relations (B.1) and (B.2), we write

$$\frac{(1-\nu\epsilon_{11})}{(1+\epsilon_{11})} = \frac{1-\gamma_{max}/2}{1+\gamma_{max}/2} \tag{B.3}$$

and by further simplification on relation (B.3), we write

$$\gamma_{max} = (1+\nu)\,\epsilon_{11} \tag{B.4}$$

in which ν is known as Poisson's ratio which defines the rate of normalized change in length along transverse directions with respect to normalized length change along axial direction, $\nu = -d\epsilon_{jj}/d\epsilon_{ii}$ in which $i \neq j$.

B.2 Decomposition of deformation

According to Figure (B.2.b) presenting exaggerated simple shear and uni-axial tensile deformations, body's mechanical resistance against shear and tensile stresses can be expressed based on Hooke's law given by relation (A.1), and assuming ($\sigma_{p2} = \sigma_{p2} = 0$) we write

$$\epsilon_{11} = \frac{\sigma_{p1}}{E} \tag{B.5}$$

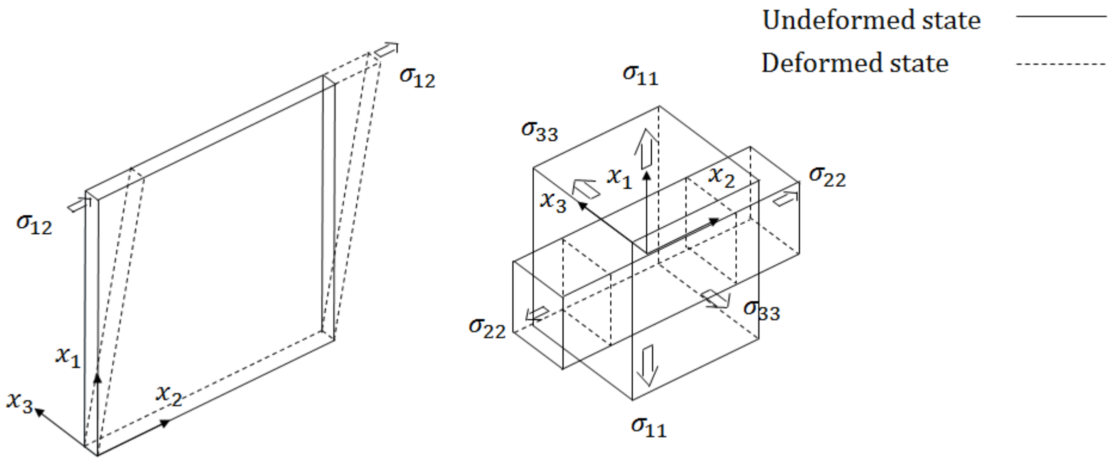


FIGURE B.2: Exaggerated pure shear (a) and exaggerated pure tensile (b) loading of RVE

According to relation (2.21), maximum shear stress, τ_1, is expressed as follows.

$$\tau_1 = \frac{\sigma_{p1}}{2} \tag{B.6}$$

in which E and G are respectively elastic and shear Modulus values.

$$\gamma_{max} = \frac{\tau_1}{G} \tag{B.7}$$

By substituting relation (B.6) in expression (B.7) we obtain

$$\gamma_{max} = \frac{\sigma_{p1}}{2G} \tag{B.8}$$

Invoking relations (B.5) and (B.8) into relation (B.4), shear modulus, G is obtained as follows.

$$G = \frac{E}{2\left(1+\nu\right)} \tag{B.9}$$

Moreover, as illustrated in Figure (B.2.b), cubical dilatation (relative variation of volume) occurs apparently in case of hydrostatic stresses applied on the body. Namely, setting $i = j$ in relation (C.53) leads to

$$\begin{aligned} \sigma_{ii} &= \lambda\delta_{ii}\epsilon_{kk} + 2\mu\epsilon_{ii} \\ &= 3\lambda\epsilon_{kk} + 2\mu\epsilon_{ii} = \left(3\lambda + 2\mu\right)\epsilon_{ii} \end{aligned} \tag{B.10}$$

in which bulk expansion of RVE allows us to assign $3\lambda+2\mu$ to a three-dimensional expansion coefficient, κ which is known as bulk modulus of the isotropic medium within the scope of homogeneous isotropic linear elasticity. Therefore, bulk modulus is defined as follows.

$$\kappa = \frac{3\lambda + 2\mu}{3} = \lambda + \frac{2\mu}{3} \tag{B.11}$$

Based on relation (B.11), it can be realized that relation (B.10) can be stated in its equivalent form as follows.

$$\sigma_{ii} = 3\kappa\epsilon_{ii} \tag{B.12}$$

with the addition of thermal effects, ϵ_{ii} reads

$$\epsilon_{ii} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3\kappa} + 3\alpha\Delta T \tag{B.13}$$

in which ϵ_{ii} defines a volumetric stretch or dilatation of representative volume element while α is defined as coefficient of thermal expansion for body.

Relation (C.53) which gives stress state of an isotropic elastic material body can be inverted to one that provides strain components. By applying basic algebraic manipulations on relation (C.53) and using definition of bulk modulus given by relation (B.10), we obtain

$$\epsilon_{ij} = \frac{\sigma_{ij} - \lambda\delta_{ij}\epsilon_{kk}}{2\mu} = -\frac{\lambda\delta_{ij}}{2\mu\left(3\lambda + 2\mu\right)}\sigma_{ii} + \frac{\sigma_{ij}}{2\mu} \tag{B.14}$$

Under consideration of one-dimensional stress state of an isotropic elastic body, the only non-zero component in stress tensor becomes σ_{11} with remaining stress components that are zero, namely $\sigma_{ii} = trace[\sigma] = \sigma_{11}+\sigma_{22}+\sigma_{33} = \sigma_{11}$. Therefore, ϵ_{11}, ϵ_{22} and ϵ_{33} are respectively obtained as follows.

$$\epsilon_{11} = -\frac{\lambda\delta_{11}}{2\mu\left(3\lambda + 2\mu\right)}\sigma_{11} + \frac{\sigma_{11}}{2\mu} = \frac{\lambda + \mu}{\mu\left(3\lambda + 2\mu\right)}\sigma_{11} \tag{B.15}$$

$$\epsilon_{22} = \epsilon_{33} = -\frac{\lambda\delta_{22}}{2\mu\left(3\lambda + 2\mu\right)}\sigma_{11} = -\frac{\lambda}{2\mu\left(3\lambda + 2\mu\right)}\sigma_{11} \tag{B.16}$$

Thus, Poison's ratio for isotropic elastic materials can be recast by using strain values given by relations (B.15) and (B.16) as follows.

$$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{\epsilon_{33}}{\epsilon_{11}} = \frac{\lambda}{2\left(\lambda + \mu\right)} \tag{B.17}$$

In relation (B.15), the coefficient that relates stress component σ_{11} with strain component ϵ_{11} is known as modulus of elasticity or Young' s modulus and denoted as E. Thus, we conclude that E can also be expressed in terms of λ and μ as well.

$$E = \frac{(3\lambda + 2\mu)\,\mu}{\lambda + \mu} \tag{B.18}$$

By invoking $\lambda = (2\mu\nu)\,/\,(1 - 2\nu)$ derived in relation (B.17) into (B.18), μ can be expressed in a form that confirms the result obtained in relation (B.9). Namely,

$$E = \frac{\left(\dfrac{6\mu\nu}{1 - 2\nu} + 2\mu\right)\mu}{\dfrac{2\mu\nu}{1 - 2\nu} + \mu} = \frac{\mu\,(2\mu\nu + 2\mu)}{\mu} = 2\mu\,(1 + \nu) \tag{B.19}$$

or

$$\mu = \frac{E}{2\,(1 + \nu)} \tag{B.20}$$

Invoking $\mu = (\lambda\,(1 - 2\nu))\,/\,(2\nu)$ obtained from relation (B.17) in (B.20) and rearranging terms in an appropriate way lead to determination of λ as a function of modulus of elasticity and Poisson's ratio.

$$\frac{\lambda\,(1 - 2\nu)}{2\nu} = \frac{E}{2\,(1 + \nu)} \tag{B.21}$$

or

$$\lambda = \frac{E\nu}{(1 + \nu)\,(1 - 2\nu)} \tag{B.22}$$

Lastly, bulk modulus of an isotropic medium defined by relation (B.11) can be expressed by substituting the terms λ and μ given by relations (B.9) and (B.20) respectively in expression (B.11) as follows.

$$\kappa = \frac{E}{3\,(1 - 2\nu)} \tag{B.23}$$

Appendix C

Constitutive Relation for Particle Interaction in Local Theory

C.1 Reduction of fourth-order isotropic tensor

Directional properties of a body that shows mechanically isotropic behaviour can be defined through a fourth-order tensor given by relation (A.89) whose derivation has been introduced in Appendix (A.2). The Hooke's law given by expression (A.1) constitutes a relation between stress and strain states of a material under consideration. Moreover, nine components of both stress and strain tensors result in $9^2 = 81$ components if and only if symmetry condition is applied to both tensors, accordingly, stiffness matrix is obtained with its $6^2 = 36$ components in the following way.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix} \tag{C.1}$$

in which each component of stiffness matrix relates each stress term with associated strains in directions prescribed by sub-indices. For instance, component C_{12} relates strain in the direction of x_2 with stress along x_1 direction.

For the sake of simplicity in sub-index notation, according to relation (C.1), Hooke's law can also be expressed as follows.

$$\sigma_i = C_{ij}\epsilon_j \tag{C.2}$$

Inspection of partial derivative of strain energy of RVE given by relation (3.5) with respect to strain tensor ϵ_j leads to further reduction in stiffness tensor. Restating strain energy given by expression (3.5) and combining it with constitutive relation introduced by relation (C.2), the following expression is obtained.

$$\partial W = C_{ij}\epsilon_j \partial\epsilon_j \tag{C.3}$$

and by performing integration in both side, relation (C.3) becomes

$$\begin{aligned} W &= C_{ij}\epsilon_j\epsilon_j/2 \\ &= \sigma_{ij}\epsilon_j/2 \end{aligned} \tag{C.4}$$

Let us evaluate second-order partial derivative of strain energy that will lead to further simplification on stiffness tensor in the sense that first-order differentiation defines stress state and second-order differential enables us to come up with reduced stiffness tensor as follows.

$$\begin{aligned} \frac{\partial}{\partial\epsilon_k\partial\epsilon_l}\left(C_{ij}\epsilon_j\epsilon_j/2\right) &= \frac{1}{2}\frac{\partial}{\partial\epsilon_k}\left(C_{ij}\frac{\partial\epsilon_j}{\partial\epsilon_l}\epsilon_i + C_{ij}\epsilon_j\frac{\partial\epsilon_i}{\partial\epsilon_l}\right) \\ &= \frac{1}{2}\frac{\partial}{\partial\epsilon_k}\left(C_{ij}\delta_{jl}\epsilon_i + C_{ij}\delta_{il}\epsilon_j\right) \\ &= \left(C_{ij}\delta_{jl}\delta_{ki} + C_{ij}\delta_{il}\delta_{kj}\right)/2 \\ &= \left(C_{kl} + C_{lk}\right)/2 \end{aligned} \tag{C.5}$$

C.2 Symmetry for anisotropic materials

In the way of deriving a constitutive relation for isotropic materials, expression (C.5) leads to C_{ij}, if and only if C_{kl} or C_{lk} is a symmetric tensor, $C = C^T$. Symmetry condition for C_{ij} implies that 36 material constants is reduced to 21 independent elastic coefficients and as a result of that constitutive relation for anisotropic materials is expressed as follows.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ . & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ . & . & C_{33} & C_{34} & C_{35} & C_{36} \\ . & . & . & C_{44} & C_{45} & C_{46} \\ . & sym & . & . & C_{55} & C_{56} \\ . & . & . & . & . & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} \tag{C.6}$$

in which $[C]$ is known as stiffness matrix for general anisotropic or triclinic structures. As introduced before, transformation of any first-order tensor given by relation (A.5) from reference to a rotated co-ordinate system can be directly generalized for a fourth-order stiffness tensor, $[C]$. Accordingly, it is expressed as follows.

$$C_{ijkl} = l_{ip} l_{jq} l_{kr} l_{ls} C_{pqrs} \tag{C.7}$$

in which l terms are called transformation matrices, determining rotational or directional invariance of mechanical structure with respect to symmetry planes. By means of defining symmetry planes, number of elastic coefficient can be reduced for the purpose of obtaining more simpler structure in stiffness matrix, $[C]$, which includes information regarding directional stiffness properties of material. The transformation matrix, satisfying orthogonality condition $\left(L^{-1} = L^T\right)$, also allows us to determine stress and strain components through any specific directions of rotated co-ordinate system as explained in [29]. Namely,

$$\sigma' = L^T \sigma L \tag{C.8}$$

and

$$\epsilon' = L^T \epsilon L \tag{C.9}$$

or in index notation, they are respectively

$$\sigma'_{ij} = l_{ip} l_{jq} \sigma_{pq} \tag{C.10}$$

and

$$\epsilon'_{ij} = l_{ip} l_{jq} \epsilon_{pq} \tag{C.11}$$

C.3 Symmetry for monoclinic materials

Let us consider a transformation condition which leads to stiffness matrix for monoclinic structures whose symmetry plane is on z axis. In this sense, orthogonal transformation matrix becomes

$$[L]^{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{C.12}$$

By applying transformation matrix given by relation (C.12) onto relations (C.8) and (C.9), an expression can be obtained between associated stress and strain components in rotated co-ordinates respectively as follows.

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ . & \sigma'_{22} & \sigma'_{23} \\ . & . & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{13} \\ . & \sigma_{22} & -\sigma_{23} \\ . & . & \sigma_{33} \end{bmatrix} \tag{C.13}$$

and similarly,

$$\begin{bmatrix} \epsilon'_{11} & \epsilon'_{12} & \epsilon'_{13} \\ . & \epsilon'_{22} & \epsilon'_{23} \\ . & . & \epsilon'_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & -\epsilon_{13} \\ . & \epsilon_{22} & -\epsilon_{23} \\ . & . & \epsilon_{33} \end{bmatrix} \tag{C.14}$$

From relations (C.13) and (C.14), it can be directly concluded that $\sigma'_{13} = -\sigma_{13}$, $\sigma'_{23} = -\sigma_{23}$ and $\epsilon'_{13} = -\epsilon_{13}$, $\epsilon'_{23} = -\epsilon_{23}$. Writing the first stress-strain equation by means of constitutive relation given by expression (C.6) leads to further reduction in stiffness tensor by symmetry condition defined by a transformation matrix, L. Explicit form of the first stress-strain relation is

$$\sigma_{11} = C_{1111}\epsilon_{11} + C_{1122}\epsilon_{22} + C_{1133}\epsilon_{33} + C_{1123}\epsilon_{23} + C_{1113}\epsilon_{13} + C_{1112}\epsilon_{12} \tag{C.15}$$

Stress and strain tensors in reference configuration of a co-ordinate system can be replaced by corresponding stress and strain components in rotated frame of reference according to relations (C.13) and (C.14) while stiffness matrix, being rotational invariant, does not alter. Thus, we write first stress-strain equation based on constitutive relation for rotated co-ordinate system as follows.

$$\begin{aligned} \sigma'_{11} &= C_{1111}\epsilon'_{11} + C_{1122}\epsilon'_{22} + C_{1133}\epsilon'_{33} + C_{1123}\epsilon'_{23} + C_{1113}\epsilon'_{13} + C_{1112}\epsilon'_{12} \\ &= C_{1111}\epsilon_{11} + C_{1122}\epsilon_{22} + C_{1133}\epsilon_{33} - C_{1123}\epsilon_{23} - C_{1113}\epsilon_{13} + C_{1112}\epsilon_{12} \end{aligned} \tag{C.16}$$

in which we substituted $\epsilon'_{13} = -\epsilon_{13}$ and $\epsilon'_{23} = -\epsilon_{23}$ in equation (C.16). In further, since we can write $\sigma_{11} = \sigma'_{11}$ according to relations (C.13) and (C.14), it allows us to combine relations (C.15) and (C.16). Therefore,

$$C_{1123}\epsilon_{23} + C_{1113}\epsilon_{13} = -C_{1123}\epsilon_{23} - C_{1113}\epsilon_{13} \tag{C.17}$$

or

$$2C_{1123}\epsilon_{23} + 2C_{1113}\epsilon_{13} = 0 \tag{C.18}$$

As a result, it is concluded that $C_{1123} = C_{1113} = 0$. Accordingly, this result allows us make further reduction on C_{ijkl}. The second stress-strain relation from constitutive equation in reference and rotated co-ordinates respectively are expressed as follows.

$$\sigma_{22} = C_{2211}\epsilon_{11} + C_{2222}\epsilon_{22} + C_{2233}\epsilon_{33} + C_{2223}\epsilon_{23} + C_{2213}\epsilon_{13} + C_{2212}\epsilon_{12} \tag{C.19}$$

and

$$\begin{aligned} \sigma'_{22} &= C_{2211}\epsilon'_{11} + C_{2222}\epsilon'_{22} + C_{2233}\epsilon'_{33} + C_{2223}\epsilon'_{23} + C_{2213}\epsilon'_{13} + C_{2212}\epsilon'_{12} \\ &= C_{2211}\epsilon_{11} + C_{2222}\epsilon_{22} + C_{2233}\epsilon_{33} - C_{2223}\epsilon_{23} - C_{2213}\epsilon_{13} + C_{2212}\epsilon_{12} \end{aligned} \tag{C.20}$$

Combining relations (C.19) and (C.20) leads to

$$C_{2223}\epsilon_{23} + C_{2213}\epsilon_{13} = -C_{2223}\epsilon_{23} - C_{2213}\epsilon_{13} \tag{C.21}$$

or

$$2C_{2223}\epsilon_{23} + 2C_{2213}\epsilon_{13} = 0 \tag{C.22}$$

concluding that condition $C_{2223} = C_{2213} = 0$ is always valid since all coefficients of stiffness tensor have to be positive in any case. The fourth relation which defines a stress-strain pair for remaining normal direction.

$$\sigma_{33} = C_{3311}\epsilon_{11} + C_{3322}\epsilon_{22} + C_{3333}\epsilon_{33} + C_{3323}\epsilon_{23} + C_{3313}\epsilon_{13} + C_{3312}\epsilon_{12} \tag{C.23}$$

and

$$\sigma'_{33} = C_{3311}\epsilon_{11} + C_{3322}\epsilon_{22} + C_{3333}\epsilon_{33} - C_{3323}\epsilon_{23} - C_{3313}\epsilon_{13} + C_{3312}\epsilon_{12} \tag{C.24}$$

By considering directionality condition on stress and strain components in the way that symmetry condition implies $\sigma_{33} = \sigma'_{33}$, $\epsilon_{23} = -\epsilon'_{23}$ and $\epsilon_{13} = -\epsilon'_{13}$ based on transformation matrix, L. Accordingly, relations (C.23) and (C.24) can be combined as follows.

$$C_{3323}\epsilon_{23} + C_{3313}\epsilon_{13} = -C_{3323}\epsilon_{23} - C_{3313}\epsilon_{13} \tag{C.25}$$

or

$$2C_{3323}\epsilon_{23} + 2C_{3313}\epsilon_{13} = 0 \tag{C.26}$$

Therefore, condition $C_{3323} = 0$ and $C_{3323} = 0$ has to be valid. On the other hand, shear stress components, σ_{23} and σ_{13}, do not lead to any further simplification on elastic coefficients in stiffness matrix. Namely,

$$\sigma_{23} = C_{2311}\epsilon_{11} + C_{2322}\epsilon_{22} + C_{2333}\epsilon_{33} + C_{2323}\epsilon_{23} + C_{2313}\epsilon_{13} + C_{3312}\epsilon_{12} \tag{C.27}$$

$$\sigma'_{23} = C_{2311}\epsilon_{11} + C_{2322}\epsilon_{22} + C_{2333}\epsilon_{33} - C_{2323}\epsilon_{23} - C_{2313}\epsilon_{13} + C_{3312}\epsilon_{12} \tag{C.28}$$

or

$$C_{2323}\epsilon_{23} + C_{2313}\epsilon_{13} = C_{2323}\epsilon_{23} + C_{2313}\epsilon_{13} \tag{C.29}$$

As similar to the fourth stress-strain relation, the fifth one also does not provide any reduction in number of coefficient in stiffness tensor, because of

$$C_{1323}\epsilon_{23} + C_{1313}\epsilon_{13} = C_{1323}\epsilon_{23} + C_{1313}\epsilon_{13} \tag{C.30}$$

Lastly, by proceeding on writing stress-strain relations for σ_{12} and σ'_{12}, following relation is obtained.

$$2C_{1223}\epsilon_{23} + 2C_{1213}\epsilon_{13} = 0 \tag{C.31}$$

which leads to conditions $C_{1223} = 0$ and $C_{1213} = 0$ that are valid. Invoking associated zero coefficients into the stiffness matrix given for an anisotropic material reduces 21 independent component to 13 that define directional properties for monoclinic structures as follows.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ . & C_{22} & C_{23} & 0 & 0 & C_{26} \\ . & . & C_{33} & 0 & 0 & C_{36} \\ . & . & . & C_{44} & C_{46} & 0 \\ . & sym & . & . & C_{55} & 0 \\ . & . & . & . & . & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix} \tag{C.32}$$

C.4 Symmetry for transversely isotropic materials

For the case of an orthotropic structure, there are three planes of symmetry defined by three transformation matrices that are

$$[L]^{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, [L]^{13} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [L]^{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(C.33)}$$

in which superscripts indicate the planes of symmetry for orthotropic structures. In addition to the symmetry condition of monoclinic structure, applying transformation matrices, $[L]^{13}$ and $[L]^{23}$, to elements of constitutive equation given in (C.32) reduces it further and results in constitutive stress-strain relation for orthotropic structures.

Specifically, stress component in rotated frame of reference by the second symmetry plane defined by transformation matrix, $[L]^{13}$, can be calculated as follows. By using (C.10), the relations for first three stress components respectively are

$$\begin{aligned} \sigma'_{11} &= L^{13}_{1p} L^{13}_{1q} \sigma_{pq} \\ &= (-\delta_{1p})(-\delta_{1p})\,\sigma_{pq} = \sigma_{11} \end{aligned} \quad \text{(C.34)}$$

$$\begin{aligned} \sigma'_{22} &= L^{13}_{2p} L^{13}_{2q} \sigma_{pq} \\ &= (\delta_{2p})(\delta_{2p})\,\sigma_{pq} = \sigma_{22} \end{aligned} \quad \text{(C.35)}$$

$$\begin{aligned} \sigma'_{33} &= L^{13}_{3p} L^{13}_{3q} \sigma_{pq} \\ &= (\delta_{3p})(\delta_{3p})\,\sigma_{pq} = \sigma_{33} \end{aligned} \quad \text{(C.36)}$$

Beside these, remaining three shear stress components can be expressed as follows.

$$\begin{aligned} \sigma'_{23} &= L^{13}_{2p} L^{13}_{3q} \sigma_{pq} \\ &= (\delta_{2p})(\delta_{3p})\,\sigma_{pq} = \sigma_{23} \end{aligned} \quad \text{(C.37)}$$

$$\begin{aligned} \sigma'_{13} &= L^{13}_{1p} L^{13}_{3q} \sigma_{pq} \\ &= (-\delta_{1p})(\delta_{3p})\,\sigma_{pq} = -\sigma_{13} \end{aligned} \quad \text{(C.38)}$$

$$\begin{aligned}\sigma'_{12} &= L^{13}_{1p}L^{13}_{2q}\sigma_{pq} \\ &= (-\delta_{1p})(\delta_{2p})\,\sigma_{pq} = -\sigma_{12}\end{aligned} \tag{C.39}$$

Other than stress tensor, same transformation procedure can be applied to second-order strain tensor, ϵ_{ij} as similar to (13) plane symmetry condition between stress components. Accordingly, symmetry condition by means of (13) plane leading to $\epsilon'_{11} = \epsilon_{11}$, $\epsilon'_{22} = \epsilon_{22}$, $\epsilon'_{33} = \epsilon_{33}$, $\epsilon'_{23} = \epsilon_{23}$, $\epsilon'_{13} = -\epsilon_{13}$ and $\epsilon'_{12} = -\epsilon_{12}$ is utilized as well.

In the way of obtaining stiffness tensor for an orthotropic material, number of elastic coefficients in (C.32) can be further reduced through following transformations.

$$\begin{aligned}C_{1112} &= L^{13}_{1p}L^{13}_{1p}L^{13}_{1p}L^{13}_{2p}C_{pqrs} \\ &= (-\delta_{1p})(-\delta_{1p})(-\delta_{1p})(\delta_{2p})\,C_{pqrs} = -C_{1112}\end{aligned} \tag{C.40}$$

The only condition that makes relation (C.40) valid is $C_{1112}+C_{1112} = 0$. Therefore, it is concluded that elastic coefficient defined by C_{1112} is zero, namely, $C_{1112} = 0$. Accordingly, transformation of coefficient C_{2212} is the following.

$$\begin{aligned}C_{2212} &= L^{13}_{2p}L^{13}_{2p}L^{13}_{1p}L^{13}_{2p}C_{pqrs} \\ &= (\delta_{2p})(\delta_{2p})(-\delta_{1p})(\delta_{2p})\,C_{pqrs} = -C_{2212}\end{aligned} \tag{C.41}$$

Similar to former calculations, $C_{2212} = 0$ is obtained. Transformation of coefficients that are C_{3312} and C_{2313} lead to zero, $C_{3312} = 0$ and $C_{2313} = 0$, as well because of

$$\begin{aligned}C_{3312} &= L^{13}_{3p}L^{13}_{3p}L^{13}_{1p}L^{13}_{2p}C_{pqrs} \\ &= (\delta_{3p})(\delta_{3p})(-\delta_{1p})(\delta_{2p})\,C_{pqrs} = -C_{3312}\end{aligned} \tag{C.42}$$

and

$$\begin{aligned}C_{2313} &= L^{13}_{2p}L^{13}_{3p}L^{13}_{1p}L^{13}_{3p}C_{pqrs} \\ &= (\delta_{2p})(\delta_{3p})(-\delta_{1p})(\delta_{3p})\,C_{pqrs} = -C_{2313}\end{aligned} \tag{C.43}$$

Transformations of remaining elastic coefficients, C_{1111}, C_{2222}, C_{3333}, C_{2323}, C_{1313}, C_{1212}, C_{1122}, C_{1133} and C_{2233} with respect to (13) symmetry plane result in themselves. Also, applying transformation matrix $[L]^{23}$ to stiffness tensor does not provide further reduction in terms of elastic coefficients. As a result above calculations, following constitutive relation is obtained for an orthotropic structure with its symmetric stiffness tensor, having nine non-zero components as follows.

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ . & C_{22} & C_{23} & 0 & 0 & 0\\ . & . & C_{33} & 0 & 0 & 0\\ . & . & . & C_{44} & 0 & 0\\ . & sym & . & . & C_{55} & 0\\ . & . & . & . & . & C_{66}\end{bmatrix} \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \epsilon_{23}\\ \epsilon_{13}\\ \epsilon_{12}\end{bmatrix} \tag{C.44}$$

Transversely isotropic materials, having a structure similar to one illustrated in Figure (C.1) requires application of an additional transformation matrix, which defines rotational symmetry around one of the axis in Cartesian co-ordinate system, to the stiffness tensor given by relation (C.44) and stress-strain tensors.

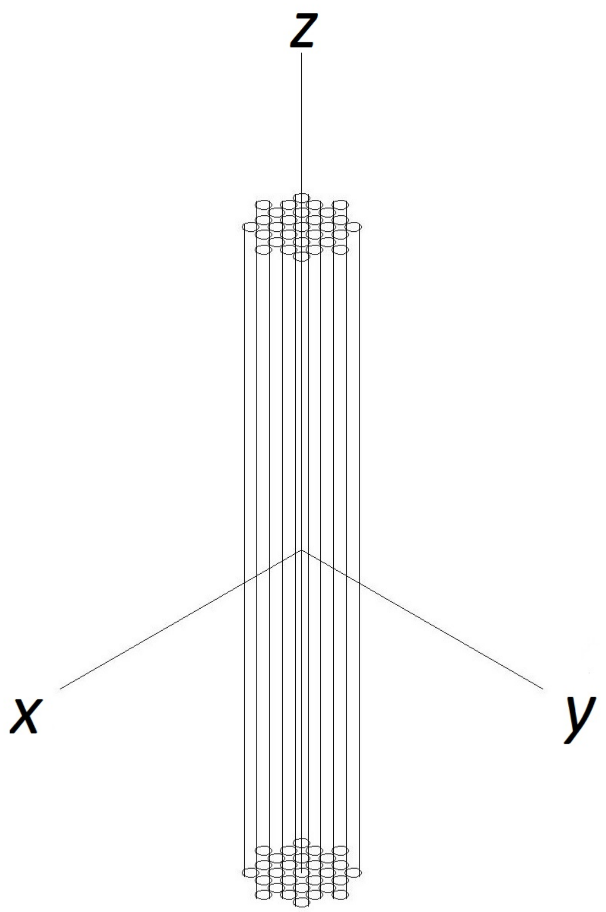


FIGURE C.1: Representation of a transversely isotropic structure

The additional rotation tensor to those given by relation (C.33) is

$$[L]^3 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{C.45}$$

in which the superscript 3 indicates rotational symmetry around z axis.

Moreover, by applying transformation matrix given by relation (C.45), stress and strain states of deformed configuration is obtained for transversely isotropic structures accordingly. Then, expressing invariance of strain energy density function in case of elastic deformation inside a continuum medium ($W' = W$) and equalizing the coefficients of corresponding strain terms, e.g. $(\epsilon'_{11})^2 = \epsilon_{11}$, in undeformed and deformed states result in constitutive relation for a transversely isotropic structure.

To this end, whole process of derivation of Hooke's law for a transversely isotropic structures leads to

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ . & C_{11} & C_{13} & 0 & 0 & 0 \\ . & . & C_{33} & 0 & 0 & 0 \\ . & . & . & C_{44} & 0 & 0 \\ . & sym & . & . & C_{44} & 0 \\ . & . & . & . & . & (C_{11} - C_{12})/2 \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix} \tag{C.46}$$

whose stiffness matrix has five independent elastic coefficients as shown in [20].

C.5 Symmetry for isotropic materials

In addition to the symmetry condition for a transversely isotropic material, applying rotation matrix, $[L]^1$, around the co-ordinate axis x provides us with constitutive equation for an isotropic structure. As a result, Hooke's law for isotropic structures is obtained as follows.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ . & C_{11} & C_{12} & 0 & 0 & 0 \\ . & . & C_{11} & 0 & 0 & 0 \\ . & . & . & \left(C_{11}-C_{12}\right)/2 & 0 & 0 \\ . & sym & . & . & \left(C_{11}-C_{12}\right)/2 & 0 \\ . & . & . & . & . & \left(C_{11}-C_{12}\right)/2 \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix} \tag{C.47}$$

whose number of independent elastic coefficients is only two, being C_{11} and C_{12}. As concluded in Appendix (A.2), relation (A.89) defines the general form of a fourth-order isotropic tensor. Based on this, the stiffness tensor, C_{ijkl}, for an isotropic structure can be obtained by modifying the right-hand side of relation (A.89) in an appropriate way.

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \kappa\delta_{il}\delta_{jk} \tag{C.48}$$

in which λ, μ and κ are known as Lame constants which are different from the coefficients defined at the end of Chapter (A.2). Because of the symmetry condition, $C_{ijkl} = C_{ijlk}$, proposed when reducing the number of independent elastic coefficient from 81 to 36, enables us to write

$$\begin{aligned} C_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \kappa\delta_{il}\delta_{jk} \\ C_{ijlk} &= \lambda\delta_{ij}\delta_{lk} + \mu\delta_{il}\delta_{jk} + \kappa\delta_{ik}\delta_{lj} \end{aligned} \tag{C.49}$$

Multiplying both sides of two lines in (C.49) with δ_{jk} and subtracting the former line from the later leads to

$$C_{ijjl} - C_{ijlj} = (\mu - \kappa)\left(\delta_{il}\delta_{lj} - \delta_{il}\delta_{lj}\right) \tag{C.50}$$

and owing to the symmetry condition that is $C_{ijjl} = C_{ijlj}$, it is concluded that

$$(\mu - \kappa)\left(\delta_{il}\delta_{lj} - \delta_{il}\delta_{kk}\right) = 0 \tag{C.51}$$

has to be satisfied. Accordingly, $\mu = \kappa$ becomes valid. So that the forth-order isotropic stiffness tensor, C_{ijkl}, is obtained as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{C.52}$$

and invoking the result, found in (C.52), into (A.1), we obtain the generalized Hooke's law for an isotropic medium as

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\epsilon_{kl} \\ &= \lambda\delta_{ij}\delta_{kl}\epsilon_{kl} + \mu\left(\delta_{ik}\delta_{jl}\epsilon_{kl} + \delta_{il}\delta_{jk}\right)\epsilon_{kl} \\ &= \lambda\delta_{ij}\delta_{kl}\epsilon_{kl} + \mu\delta_{ik}\delta_{jl}\epsilon_{kl} + \mu\delta_{il}\delta_{jk}\epsilon_{kl} \\ &= \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \end{aligned} \tag{C.53}$$

or constitutive relation (C.53) can be expressed in matrix form as follows.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ . & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ . & . & \lambda + 2\mu & 0 & 0 & 0 \\ . & . & . & \mu & 0 & 0 \\ . & sym & . & . & \mu & 0 \\ . & . & . & . & . & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix} \tag{C.54}$$

in which μ corresponds to shear modulus whose details are explained in Appendix (B), thus we write $\mu = G$. By using relations (B.23), (B.20) and (B.22), it can be shown that terms, $\lambda + 2\mu$ and λ, are equivalently expressed as $\kappa + (4\mu/3)$ and $\kappa - (2\mu/3)$ respectively. Therefore, the constitutive relation (C.54) for isotropic mediums can be alternatively expressed as follows.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \kappa + (4\mu/3) & \kappa - (2\mu/3) & \kappa - (2\mu/3) & 0 & 0 & 0 \\ . & \kappa + (4\mu/3) & \kappa - (2\mu/3) & 0 & 0 & 0 \\ . & . & \kappa + (4\mu/3) & 0 & 0 & 0 \\ . & . & . & \mu & 0 & 0 \\ . & sym & . & . & \mu & 0 \\ . & . & . & . & . & \mu \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix} \tag{C.55}$$

Appendix D

Deformation and Finite Strain Tensors

D.1 Deformation and Finite Strain Tensors

As a result of applied forces on a solid body, local theory uses infinitesimal elements for measuring deformation and strain between material particles. By means of vector addition, deformation vector, **u** can be equivalently expressed as,

$$\mathbf{u} = \mathbf{b} + \mathbf{x}' - \mathbf{x} \tag{D.1}$$

or in component form

$$u_i = b_i + x_i' - x_i \tag{D.2}$$

in which **x** expresses position of a material particle in reference configuration while position vector in deformed state is expressed by $\mathbf{x}'$. Under superimposition of co-ordinate systems for reference and deformed configurations, vector **b** is neglected thus equivalent deformation vector addition is simplified as

$$u_i = x_i' - x_i \tag{D.3}$$

whose differentiation with respect to co-ordinates, dx', of deformed configuration leads to

$$\frac{\partial u_i}{\partial x_j'} = \frac{\partial x_i'}{\partial x_j'} - \frac{\partial x_i}{\partial x_j'} = \delta_{ij} - \frac{\partial x_i}{\partial x_j'} \tag{D.4}$$

which is named as deformation gradient in co-ordinates of deformed configuration or shortly in spatial co-ordinates. It is also possible to express differentiation of deformation vector with respect to co-ordinates of reference configuration or shortly, material co-ordinates, dx.

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial x'_i}{\partial x_j} - \frac{\partial x_i}{\partial x_j} = \frac{\partial x'_i}{\partial x_j} - \delta_{ij} \tag{D.5}$$

According to vector transformation rule given in expression (A.11) which relates position vector of an arbitrary material point in reference configuration with its position in deformed configuration, relations (D.4) and (D.5) can be rewritten respectively as follows.

$$\frac{\partial u_i}{\partial x'_j} = \delta_{ij} - \frac{\partial}{\partial x'_j}\left(M_{ij}x'_j\right) = \delta_{ij} - M_{ij}\left(\frac{\partial x'_j}{\partial x'_j}\right) = \delta_{ij} - M_{ij} \tag{D.6}$$

and

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(N_{ij}x_j\right) - \delta_{ij} = N_{ij}\left(\frac{\partial x_j}{\partial x_j}\right) - \delta_{ij} = N_{ij} - \delta_{ij} \tag{D.7}$$

By comparing relations (D.4) and (D.5) with expressions (D.2) and (2.23) respectively, it can be noted that the second-order tensors are equivalent terms of material and spatial deformation gradients. Accordingly, they can be represented in component form as follows.

$$M_{ij} = \frac{\partial x_i}{\partial x'_j} = \begin{bmatrix} \partial x_1/\partial x'_1 & \partial x_1/\partial x'_2 & \partial x_1/\partial x'_3 \\ \partial x_2/\partial x'_1 & \partial x_2/\partial x'_2 & \partial x_2/\partial x'_3 \\ \partial x_3/\partial x'_1 & \partial x_3/\partial x'_2 & \partial x_3/\partial x'_3 \end{bmatrix} \tag{D.8}$$

and

$$N_{ij} = \frac{\partial x'_i}{\partial x_j} = \begin{bmatrix} \partial x'_1/\partial x_1 & \partial x'_1/\partial x_2 & \partial x'_1/\partial x_3 \\ \partial x'_2/\partial x_1 & \partial x'_2/\partial x_2 & \partial x'_2/\partial x_3 \\ \partial x'_3/\partial x_1 & \partial x'_3/\partial x_2 & \partial x'_3/\partial x_3 \end{bmatrix} \tag{D.9}$$

The small deformations in classical theory are given by difference in magnitudes of position vectors in reference and deformed configurations. Because square of any arbitrary vector can be considered to give its magnitude, taking square of infinitesimal length elements in both material and spatial co-ordinates, differencing them from each others leads to infinitesimal strain tensors which constitute fundamental of small deformation theory in classical continuum mechanics.

Accordingly, square of infinitesimal length elements in material and spatial co-ordinates are given respectively in component form as

$$(d\mathbf{x})^2 = (d\mathbf{x} \cdot d\mathbf{x}) = (dx_i e_i \cdot dx_j e_j) = dx_i dx_j e_i \cdot e_j = dx_i dx_j \delta_{ij} \tag{D.10}$$

and

$$(d\mathbf{x}')^2 = (d\mathbf{x}' \cdot d\mathbf{x}') = \left(dx'_i e'_i \cdot dx'_j e'_j\right) = dx'_i dx'_j e'_i \cdot e'_j = dx'_i dx'_j \delta_{ij} \tag{D.11}$$

Since material co-ordinates can be expressed as a function of spatial co-ordinates or other way around, namely, $x_i = x_i\left(x'_j\right)$ and $x'_i = x'_i\left(x_j\right)$ respectively, total differentials of infinitesimal length elements in material and spatial co-ordinates are

$$dx_i = \left(\frac{\partial x_i}{\partial x'_j}\right) dx'_j \tag{D.12}$$

and

$$dx'_i = \left(\frac{\partial x'_i}{\partial x_j}\right) dx_j \tag{D.13}$$

Substituting equivalent total differentials given by relations (D.12) and (D.13) respectively in square of infinitesimal length elements leads to

$$(d\mathbf{x})^2 = dx_i dx_j \delta_{ij} = \left(\frac{\partial x_k}{\partial x'_i}\right) dx'_i \left(\frac{\partial x_k}{\partial x'_j}\right) dx'_j = \left(\frac{\partial x_k}{\partial x'_i}\frac{\partial x_k}{\partial x'_j}\right) dx'_i dx'_j \tag{D.14}$$

and

$$(d\mathbf{x}')^2 = dx'_i dx'_j \delta_{ij} = \left(\frac{\partial x'_k}{\partial x_i}\right) dx_i \left(\frac{\partial x'_k}{\partial x_j}\right) dx_j = \left(\frac{\partial x'_k}{\partial x_i}\frac{\partial x'_k}{\partial x_j}\right) dx_i dx_j \tag{D.15}$$

As it can be intuitively understood, amount of deformation can be measured by difference between magnitudes of infinitesimal length elements in final and reference states of a body under consideration.

In other words, by means of relations (D.10) and (D.15), th,s difference can be expressed as follows.

$$(d\mathbf{x}')^2 - (d\mathbf{x})^2 = \left(\frac{\partial x'_k}{\partial x_i}\frac{\partial x'_k}{\partial x_j}\right) dx_i dx_j - \delta_{ij} dx_i dx_j = \left(\frac{\partial x'_k}{\partial x_i}\frac{\partial x'_k}{\partial x_j} - \delta_{ij}\right) dx_i dx_j \tag{D.16}$$

or by using a pair of relations that are (D.11) and (D.14), same difference in expression (D.16) can be equivalently expressed as

$$\begin{aligned}(d\mathbf{x}')^2 - (d\mathbf{x})^2 &= \delta_{ij}dx'_i dx'_j - \left(\frac{\partial x_k}{\partial x'_i}\frac{\partial x_k}{\partial x'_j}\right)dx'_i dx'_j \\ &= \left(\delta_{ij} - \frac{\partial x_k}{\partial x'_i}\frac{\partial x_k}{\partial x'_j}\right)dx'_i dx'_j\end{aligned} \tag{D.17}$$

The terms inside the brackets in relations (D.16) and (D.17) define respectively Lagrangian (or Green's) and Eulerian (or Almansi's) finite strain tensors, [24], that are expressed in a way that is

$$l_{ij} = \frac{1}{2}\left(\frac{\partial x'_k}{\partial x_i}\frac{\partial x'_k}{\partial x_j} - \delta_{ij}\right) \tag{D.18}$$

and

$$\epsilon_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial x_k}{\partial x'_i}\frac{\partial x_k}{\partial x'_j}\right) \tag{D.19}$$

Accordingly, difference in square of an infinitesimal length element takes a from either as

$$(d\mathbf{x}')^2 - (d\mathbf{x})^2 = 2l_{ij}dx_i dx_j \tag{D.20}$$

or

$$(d\mathbf{x}')^2 - (d\mathbf{x})^2 = 2\epsilon_{ij}dx'_i dx'_j \tag{D.21}$$

The relations given by expressions (D.5) and (D.4) can be rearranged in an appropriate way and then substituted in Lagrangian and Eulerian descriptions of finite strain tensors. Moreover, they can be appropriately manipulated and written in terms of deformation vector between material particles.

$$\begin{aligned}L_{ij} &= \frac{1}{2}\left(\left(\frac{\partial u_k}{\partial x_i} + \delta_{ki}\right)\left(\frac{\partial u_k}{\partial x_j} + \delta_{kj}\right) - \delta_{ij}\right) \\ &= \frac{1}{2}\left(\delta_{ki}\delta_{kj} + \delta_{ki}\frac{\partial u_k}{\partial x_j} + \delta_{kj}\frac{\partial u_k}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} - \delta_{ij}\right) \\ &= \frac{1}{2}\left(\delta_{ij} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} - \delta_{ij}\right) \\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)\end{aligned} \tag{D.22}$$

and

$$\begin{aligned} E_{ij} &= \frac{1}{2}\left(\delta_{ij} - \left(\delta_{ki} - \frac{\partial u_k}{\partial x'_i} +\right)\left(\delta_{kj} - \frac{\partial u_k}{\partial x'_j}\right)\right) \\ &= \frac{1}{2}\left(\delta_{ij} - \delta_{ki}\delta_{kj} + \delta_{ki}\frac{\partial u_k}{\partial x'_j} + \delta_{kj}\frac{\partial u_k}{\partial x'_i} - \frac{\partial u_k}{\partial x'_i}\frac{\partial u_k}{\partial x'_j}\right) \\ &= \frac{1}{2}\left(\delta_{ij} - \delta_{ij} + \frac{\partial u_i}{\partial x'_j} + \frac{\partial u_j}{\partial x'_i} - \frac{\partial u_k}{\partial x'_i}\frac{\partial u_k}{\partial x'_j}\right) \\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x'_j} + \frac{\partial u_j}{\partial x'_i} - \frac{\partial u_k}{\partial x'_i}\frac{\partial u_k}{\partial x'_j}\right) \end{aligned} \tag{D.23}$$

If deformation gradients are small enough, namely $\partial u_i/\partial x_j \ll 1$ and $\partial u_i/\partial x'_j \ll 1$, products of differential terms in the last lines of relations (D.22) and (D.23) can be neglected, thus infinitesimal Lagrangian and Eulerian strain tensors respectively become

$$l_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{D.24}$$

and

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x'_j} + \frac{\partial u_j}{\partial x'_i}\right) \tag{D.25}$$

Under consideration of both infinitesimal deformation gradients constituting tiny strains, no difference between Lagrangian and Eulerian infinitesimal strain tensors appear. By arbitrary selection, Eulerian infinitesimal strain tensor in explicit component form reads

$$\epsilon_{ij} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial u_1}{\partial x'_1} + \frac{\partial u_1}{\partial x'_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x'_2} + \frac{\partial u_2}{\partial x'_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x'_3} + \frac{\partial u_3}{\partial x'_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x'_1} + \frac{\partial u_1}{\partial x'_2}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x'_2} + \frac{\partial u_2}{\partial x'_2}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x'_3} + \frac{\partial u_3}{\partial x'_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x'_1} + \frac{\partial u_1}{\partial x'_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x'_2} + \frac{\partial u_2}{\partial x'_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x'_3} + \frac{\partial u_3}{\partial x'_3}\right) \end{bmatrix} \tag{D.26}$$

D.2 Geometric Interpretation of Deformation

As partially stated above, strain is a measure of deformation representing displacement between material points relative to reference length.

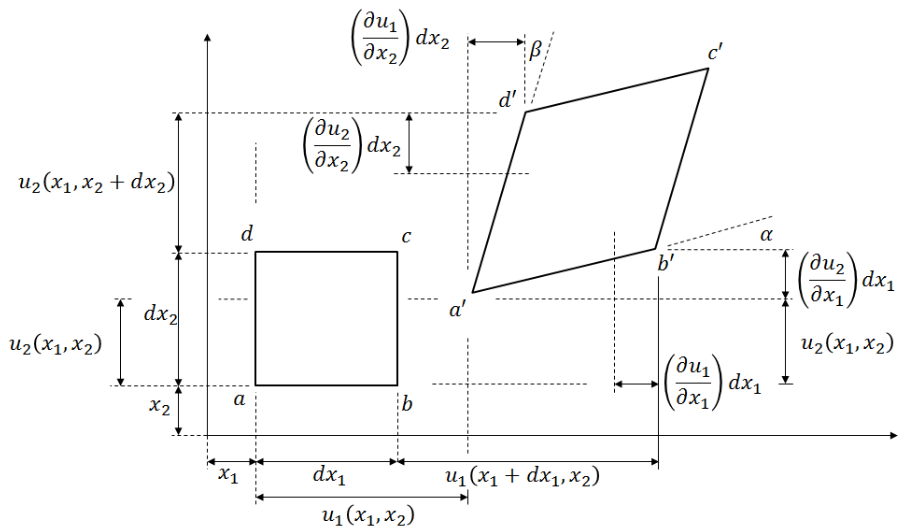


FIGURE D.1: General case of deformation in a two dimensional body

Instead of using three-dimensional strain tensor, interpretation of strain analysis for two-dimensional geometry can be performed for the sake of simplicity in which each term can be match with their dimensional equivalents.

Since displacement vectors can be presented as a function of spatial co-ordinates such a way that are $u_1(x'_1, x'_2)$ and $u_2(x'_1, x'_2)$, their total differentials respectively become

$$du_1(x'_1, x'_2) = \left(\frac{\partial u_1}{\partial x'_1}\right) dx'_1 + \left(\frac{\partial u_1}{\partial x'_2}\right) dx'_2 \tag{D.27}$$

and

$$du_2(x'_1, x'_2) = \left(\frac{\partial u_2}{\partial x'_1}\right) dx'_1 + \left(\frac{\partial u_2}{\partial x'_2}\right) dx'_2 \tag{D.28}$$

From a classical point of view for strain, let us consider uni-axial elongation of a bar, basic definition of normal strain is given by change in length of the bar which is normalized with respect to its initial length. Namely,

$$\epsilon_{normal} = \frac{l_{final} - l_{initial}}{l_{initial}} \tag{D.29}$$

By this interpretation of normal strain, edge length of square element in undeformed configuration, $a-b$, illustrated in Figure (D.1) extends to its final length

, $a' - b'$, definition of normal strain given by relation (D.29) becomes

$$\begin{aligned}
\epsilon_{11} &= \left(\sqrt{\left(dx'_1 + \left(\frac{\partial u_1}{\partial x'_1} \right) dx'_1 \right)^2 + \left(\left(\frac{\partial u_2}{\partial x'_1} \right) dx'_1 \right)^2} - dx'_1 \right) \frac{1}{dx'_1} \\
&= \left(\sqrt{(dx'_1)^2 + \left(\frac{\partial u_1}{\partial x'_1} \right)^2 (dx'_1)^2 + 2 \left(\frac{\partial u_1}{\partial x'_1} \right) (dx'_1)^2 + \left(\frac{\partial u_2}{\partial x'_1} \right)^2 (dx'_1)^2} - dx'_1 \right) \frac{1}{dx'_1} \\
&= \left(\sqrt{(dx'_1)^2 + \left(\frac{\partial u_1}{\partial x'_1} \right)^2 (dx'_1)^2 + 2 \left(\frac{\partial u_1}{\partial x'_1} \right) (dx'_1)^2} - dx'_1 \right) \frac{1}{dx'_1} \\
&= \frac{1}{dx'_1} \sqrt{(dx'_1)^2 \left(1 + \frac{\partial u_1}{\partial x'_1} \right)^2} - \frac{dx'_1}{dx'_1} = \frac{\partial u_1}{\partial x'_1}
\end{aligned} \tag{D.30}$$

in which square of partial derivative has been neglected, $(\partial u_2 / \partial x'_1)^2 \approx 0$, because displacement gradients of deformations are small enough, $(\partial u_2 / \partial x'_1) \ll 1$.

By generalizing the result obtained in expression (D.30) for other directions, all normal strain components in index notation are expressed as follows.

$$\epsilon_{ii} = \left(\frac{\partial u_i}{\partial x'_i} \right) \tag{D.31}$$

On the other hand, shearing strain of two-dimensional rectangular plate is measured by how much edges of a rectangle element illustrated in Figure (D.1) deviates from their initial alignments with respect to co-ordinate axes.

Accordingly, total shear deformity is stated by sum of angles α and β, namely

$$\gamma_{12} = \alpha + \beta \tag{D.32}$$

For the purpose of deriving shear components in strain tensor, let us consider tangent of these angles. Small angle approximation for tangent function allow us to write the identities that are $\tan \alpha \approx \alpha$ and $\tan \beta \approx \beta$, and based on Figure (D.1). To this end, shearing strain given by relation (D.32) can be extended as follows.

$$\tan \alpha = \frac{\left(\frac{\partial u_2}{\partial x'_1} \right) dx'_1}{dx'_1 + \left(\frac{\partial u_1}{\partial x'_1} \right) dx'_1} \tag{D.33}$$

and

$$\tan\beta = \frac{\left(\dfrac{\partial u_1}{\partial x_2'}\right) dx_2'}{dx_2' + \left(\dfrac{\partial u_2}{\partial x_2'}\right) dx_2'} \tag{D.34}$$

By approximation of small deformation gradients, $(\partial u_1/\partial x_1') \ll 1$ and $(\partial u_2/\partial x_2') \ll 1$, relation (D.32) is reduced to

$$\begin{aligned} \gamma_{12} &= \tan\alpha + \tan\beta \\ &\approx \alpha + \beta \\ &= \left(\frac{\partial u_2}{\partial x_1'}\right) + \left(\frac{\partial u_1}{\partial x_2'}\right) \end{aligned} \tag{D.35}$$

By generalizing the result obtained from relation (D.35) for other directions, all shearing components becomes

$$\gamma_{ij} = \left(\frac{\partial u_i}{\partial x_j'}\right) + \left(\frac{\partial u_j}{\partial x_i'}\right) \tag{D.36}$$

By substituting relations (D.31) and (D.36) into relation (D.26) and rearranging all terms, infinitesimal strain tensor ϵ_{ij} becomes

$$\epsilon_{ij} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1'} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2'} + \dfrac{\partial u_2}{\partial x_1'}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3'} + \dfrac{\partial u_3}{\partial x_1'}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_1'} + \dfrac{\partial u_1}{\partial x_2'}\right) & \dfrac{\partial u_2}{\partial x_2'} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3'} + \dfrac{\partial u_3}{\partial x_2'}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_1'} + \dfrac{\partial u_1}{\partial x_3'}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_2'} + \dfrac{\partial u_2}{\partial x_3'}\right) & \dfrac{\partial u_3}{\partial x_3'} \end{bmatrix} \tag{D.37}$$

$$= \begin{bmatrix} \epsilon_{11} & \gamma_{12}/2 & \gamma_{13}/2 \\ \gamma_{21}/2 & \epsilon_{22} & \gamma_{23}/2 \\ \gamma_{31}/2 & \gamma_{32}/2 & \epsilon_{33} \end{bmatrix}$$

In implicit form, the infinitesimal strain tensor can be expressed as follows.

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \tag{D.38}$$

in which $\epsilon_{ij} = \gamma_{ij}/2$.

Moreover, from the explicit form of infinitesimal Eulerian strain tensor given by relation (D.37), it can be seen that symmetry condition appears because of terms other than in diagonal. Accordingly, symmetry condition can be expressed as follows.

$$\epsilon = \epsilon^T \tag{D.39}$$

or based on component notation

$$\epsilon_{ij} = \epsilon_{ji}. \tag{D.40}$$

Moreover, spatial-partial differential terms appearing in relations (D.27) and (D.28) constitute deformation tensor which is given as

$$e_{ij} = \left(\frac{\partial u_i}{\partial x'_j}\right) \tag{D.41}$$

The strain tensor based on deformation illustrated in Figure (B.1), can be decomposed into symmetric and anti-symmetric parts. Namely,

$$\begin{aligned} e_{ij} &= sym\,(e_{ij}) + anti.sym\,(e_{ij}) \\ &= \frac{1}{2}\,(e_{ij} + e_{ji}) + \frac{1}{2}\,(e_{ij} - e_{ji}) \\ &= \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} + \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{21} & 0 & \omega_{23} \\ -\omega_{31} & -\omega_{32} & 0 \end{pmatrix} \\ &= \epsilon_{ij} + \omega_{ij} \end{aligned} \tag{D.42}$$

in which ω_{ij} corresponds to pure rotational motion of RVE while ϵ_{ij} is associated with pure shear deformation given by relation (D.38).

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