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**DYNAMIC STABILITY ANALYSIS OF ROTATING
PRETWISTED AEROFOIL CROSS SECTION
BLADE PACKETS UNDER IN-PLANE PERIODIC
LOADING**

**A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of Dokuz Eylul University
In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy
in Mechanical Engineering, Machine Theory and Dynamics Program**

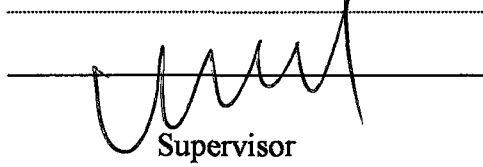
**by
Gürkan ŞAKAR**

**February, 2005
İZMİR**

Ph.D. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled “**DYNAMIC STABILITY ANALYSIS OF ROTATING PRETWISTED AEROFOIL CROSS SECTION BLADE PACKETS UNDER IN-PLANE PERIODIC LOADING**” completed by **Gürkan ŞAKAR** under supervision of **Prof. Dr. Mustafa SABUNCU** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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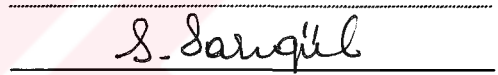
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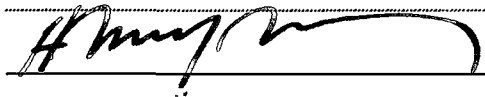
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ACKNOWLEDGEMENTS

First of all, I would like to thank my supervisor Prof. Dr. Mustafa SABUNCU for his help, guidance, criticism, very kind interest and encouragement throughout the course of this work.

I would also like to thank Prof. Dr. Hikmet ÇATAL, Prof. Dr. Saide SARIGÜL for their help with valuable suggestions and discussions that they have provided me during this research.

I am also thankful to my colleagues for their moral support throughout this study.

I am grateful to the Atatürk University of Erzurum for the opportunity which enabled me to undertake this research.

I would like to thank my parents for their supports during my education life.

Finally, I appreciate my wife Arzu Şakar for her encouragements and tolerance during this study with my daughters Beyza and Burcu.

Gürkan ŞAKAR

DYNAMIC STABILITY ANALYSIS OF ROTATING PRETWISTED AEROFOIL CROSS SECTION BLADE PACKETS UNDER IN-PLANE PERIODIC LOADING

ABSTRACT

In this study, the static and dynamic stability analysis of aerofoil cross section straight and pretwisted rotating blades and blade packets under in-plane periodic loading are carried out using the finite element method.

In the first part of the thesis the static and dynamic stability analysis of a rotating aerofoil cross section blade is carried out by assuming that the blade is fixed at the periphery of a rigid disc rotating at a constant angular velocity. The effects of pretwist, stagger angle, rotational speed, disc radius and the distance of center of flexure from the centroid on the static and dynamic stability are investigated.

To investigate the dynamic behavior of blade packets on infinitely rigid discs the shroud is assumed to have a constant curvature and is analyzed by using curved beam finite elements.

Finally the static and dynamic stability of rotating asymmetric cross section straight and pretwisted two bladed packets on rigid discs are carried out. The effects of stagger angle, shroud thickness, shroud width, rotational speed, disc radius and the distance of center of flexure from centroid on the static and dynamic stability of two bladed packets are investigated.

In the study, by applying Lagrange's principles to the energy expressions, the equation of dynamic equilibrium of the system is obtained and the problem is reduced to an eigenvalue problem. By solving eigenvalue problem, natural

frequencies, critical buckling loads and instability regions are found. Bolotin's approach is used to determine the instability regions.

In order to verify the accuracy of the model developed the present results are compared with the theoretical and experimental results of other investigators in existing literature.

Keywords : Dynamic stability, buckling, asymmetric cross section, rotating, blade packet, finite element.



DÜZLEM İÇİ PERİYODİK YÜKLEMESİNE MARUZ ÖN BURULMALI AEROFOİL KESİTLİ DÖNEN KANAT GRUPLARININ (PAKETLERİN) DİNAMİK KARARLILIK ANALİZİ

ÖZET

Bu çalışmada düzlem içi periyodik yük altındaki aerofoil kesitli düz ve ön burulmalı dönen kanatların ve kanat paketlerin statik ve dinamik kararlılığı sonlu elemanlar yöntemi kullanılarak incelenmiştir.

Tezin birinci bölümünde sabit hızla dönen rijit bir diske sabitlendiği kabul edilen aerofoil kesitli kanadın statik ve dinamik kararlılık analizi yapılmıştır. Ön burulma açısının, konum açısının, dönme hızının ve eğilme merkezinin ağırlık merkezine olan uzaklığının statik ve dinamik kararlılık üzerine olan etkileri incelenmiştir.

Rijit diskler üzerindeki kanat paketlerin dinamik davranışını incelemek için çubuk ucu bağlantı elemanının (shroud) sabit bir eğrilik yarıçapına sahip olduğu kabul edilmiş ve eğri çubuk sonlu elemanlar modeli kullanılarak analiz yapılmıştır.

Son olarak sabit hızla dönen disk üzerindeki asimetrik kesitli düz ve ön burulmalı iki kanattan oluşan paketlerin statik ve dinamik kararlılığı incelenmiştir. Konum açısının, bağlantı elemanının kalınlığının, genişliğinin, dönme hızının, disk yarıçapının ve eğilme merkezinin ağırlık merkezine olan uzaklığının paketin statik ve dinamik kararlılığı üzerine olan etkileri araştırılmıştır.

Çalışmada enerji ifadelerine Lagrange bağıntısının uygulanması ile sistemin dinamik hareket denklemi elde edilmiş ve problem özdeğer problemine indirgenmiştir. Özdeğer probleminin çözülmesi ile doğal frekanslar, kritik burkulma yükleri ve kararsızlık bölgeleri belirlenmiştir. Kararsızlık bölgelerinin belirlenmesinde Bolotin yaklaşımı kullanılmıştır.

Geliştirilen modelin doğruluğunu kontrol etmek için elde edilen sonuçlar literatürde mevcut diğer araştırmacıların teorik ve deneysel sonuçları ile karşılaştırılmıştır.

Anahtar sözcükler: Dinamik kararlılık, burkulma, asimetrik kesit, dönme, kanat grubu, sonlu elemanlar.



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CHAPTER ONE

INTRODUCTION

In recent years a new branch of the applied theory of elasticity has evolved, the theory of the dynamic stability of elastic systems. The problems which are examined in this branch of elasticity are related to those in the theory of vibrations and the stability of elastic systems. As in many other areas of learning that lie on the borderline of two fields, the theory of dynamic stability is now going through a period of intensive development.

The dynamic behavior of blades, subjected to axial periodic forces, is of great practical interest in steam and gas turbine blades and helicopter rotor blades. These axial periodic forces are the axial components of aerodynamic or hydrodynamic forces acting on the blades and may cause parametric vibrations with large amplitudes thus theoretical and experimental investigations to determination of vibrational characteristics of blading, blade packets or whole assembly is of considerable importance in the design of turbo machinery. It is well understood from investigations that the failure of the system is quite often due to fatigue of blading and it occurs when vibrations take place at or near resonant conditions. Consequently much attention has been directed, in the past to the studies of the vibration characteristics of single blades.

The problem of determining the natural frequencies and mode shapes of vibration of rotating or non rotating straight, pretwisted, aerofoil cross section blades is of importance in the design of turbine and compressor blading and aircraft and helicopter blades.

Turbo machinery blades execute either uncoupled bending or coupled bending-torsional vibrations depending on the blade characteristics. Coupled bending-torsional vibrations take place when the center of flexure (shear center) of the blade cross section is not coincident with its centroid. When the blade is pretwisted

vibrations are further coupled between the bendings in two planes. Also the natural frequencies are influenced by shear deformation, rotary inertia, fibre bending in torsion, warping in the cross section and coriolis component of acceleration. In practice the blades are attached to each other at their tips or some intermediate points by lacing or shrouding, the shroud having an initial curvature and small thickness. In present analysis the shroud is idealized by a curved beam.

The theory of dynamic stability has already opened the way for direct engineering applications. Parametrically excited vibrations are similar in appearance to the accompanying forced vibrations and can therefore qualify as ordinary resonance vibrations, by practical engineering standards. In a number of cases, however as in the presence of periodic vibrations the usual methods of damping on vibration isolation may break down and even bring about the opposite results. Although the vibrations may not threaten the structure or its normal operation, they can cause fatigue failure if they continue the act. Therefore, the study of the formation of parametric vibrations and the methods for the prevention of their occurrence is necessary in the various areas of mechanics, transportation and industrial construction.

Stability analysis of multiple degrees of freedom in parametric dynamic systems with periodic coefficients is a subject of recent interest, since the problem arises naturally in many physical situations; for example the dynamic stability of structures, under periodically varying loads, the stability of mechanisms with periodically varying inertia and stiffness coefficients and the stability of the steady state response of nonlinear systems.

Many investigations about the vibration characteristics of asymmetrical blades under rotating and non-rotating conditions, in-plane and out-of-plane vibration of curved beams and blade packets have also been carried out.

Carnegie (1957), examined the static bending of pretwisted cantilever blading. By the application of variational calculus, static equilibrium equations are derived from

expressions for the total potential energy of blades subject to either concentrated or uniformly distributed bending loads. Solution of the equilibrium equations yields expressions for the static deflections.

Carnegie & Dawson (1969), gave theoretical and experimental natural frequencies and modal shapes up to fifth mode of vibration for a straight blade of asymmetrical aerofoil cross section. The theoretical procedure consists essentially of transforming the differential equation of motions into a set of simultaneous first order equations and solving them by a step by step finite difference procedure.

Rao & Carnegie (1970), applied the Galerkin procedure to determine the natural frequencies of a straight cantilever blade with an asymmetric aerofoil cross section executing coupled bending-torsion vibrations.

Carnegie & Dawson (1971), obtained the natural frequencies and mode shapes of vibration of cantilever aerofoil cross section blades of pretwist angle 0 to 90 degrees. The theoretical results were compared to corresponding experimental results.

Carnegie (1959), derived the equations of motion of a rotating aerofoil cross section blading allowing for pretwist and stagger angle and used the Rayleigh's method to find fundamental frequency.

Carnegie, Dawson & Thomas (1966), dealt with the vibration characteristics of cantilever type turbine blading. They firstly examined the effects of taper on the vibration of pretwisted turbine blading and then as a further study vibration characteristics of non-rotating straight and pretwisted cantilever blading of uniform cross section were also examined.

Carnegie, Stirling & Fleming (1966), presented experimental results for rotating rectangular and aerofoil cross section blading. They considered the effect of stagger angle on the frequencies of vibration of aerofoil blading rotating at various speeds.

Rao & Banerjee (1977), developed a polynomial frequency equation method to determine the natural frequencies of a cantilever blade with an asymmetric cross-section mounted on a rotating disc.

Swaminathan & Rao (1977), derived the potential and kinetic energy of rotating, pretwisted, and tapered cantilever beams and obtained Lagrangian function. They used the Ritz process to obtain the natural frequencies.

Thomas & Sabuncu (1979), presented a finite element model for the dynamic analysis of an asymmetric cross section blade. The dynamic stiffness for free vibration of the blade is derived from the strain and kinetic energies using Lagrange's equation.

Fu (1974), presented a lumped parameter approach suitable for a computer analysis of the coupled bending-bending-torsion vibration of turbo machine blades.

Subrahmanyam, Kulkarni & Rao (1981), presented theoretical natural frequencies and modal shapes of the first five modes of vibration for a rotating blade of asymmetrical aerofoil cross section, with allowance for rotary inertia and shear deflection using Reissner method and classical potential energy method.

Choi & Chou (2001), proposed the modified differential quadrature method to study vibration analysis of elastically supported turbo machinery blades.

Rao & Gupta (2001), derived the stiffness and mass matrices of a rotating twisted and tapered beam element. They examined the vibration analysis of beam by the finite element method.

Petyt & Fleischer (1971), investigated three finite element models for determining the radial vibrations of a curved beam.

Sabir & Ashwell (1971), compared four shape functions by considering an internal node at the mid-point of each element in the case of in-plane vibration to obtain natural frequencies of circular arches.

Liu & Wu (2001), investigated in-plane free vibrations of circular arches using the generalized differential quadrature rule. The Kirchhoff assumptions for the these beams were considered and the neutral axis was taken as inextensible. They presented several examples of arches with uniform, continuously varying and stepped cross sections.

Yıldırım (1996), presented an effective computer program for both in-plane and out-of-plane free vibration analysis of elastic uniform arches having any double-symmetrical cross- sections. The transfer matrix method was employed in the analysis.

Ojalvo & Newman (1964), presented the frequencies of a cantilever ring segments.

Oh, B. K. Lee & I. W. Lee (1999), derived the differential equations governing free, in-plane vibrations of non circular arches, including the effects of rotary inertia, shear deformation and axial deformation and solved numerically to obtain frequencies and mode shapes.

Wu & Chiang (2003), investigated the natural frequencies and mode shapes for the radial bending vibrations of uniform circular arches by means of curved beam elements. The standard techniques were used to determine the natural frequencies and mode shapes for the curved beam with various boundary conditions and subtended angles.

Cottney & Ewins (1975), investigated the effect of shrouding by using a receptance coupling procedure. In their investigations the effect of disc, on the vibration characteristics of shrouded blades were considered.

Belek (1977), studied the vibration characteristics of a symmetrical cross section curved packet. The effect of disc curvature on the frequencies of vibration was investigated and was shown that the vibration characteristics of a multi-bladed packet can be predicted from the in-plane and out-of-plane inference diagrams of a symmetric cross section two bladed curved packet.

Thomas & Sabuncu (1979), presented a finite element model for the analysis of vibration characteristics of asymmetric cross section blade packets in a centrifugal field.

Vorobev, Kanilo & Nikulina (1997), carried out numerical vibration analysis of blade packets using detailed three-dimensional finite element models and analyzed the numerical efficiency of the approach.

There have been many studies about the static and dynamic stability of mechanical systems such as blades and curved beams.

Bolotin (1964), studied the dynamic stability problems of various kinds of structural components.

Thomas & Abbas (1976), developed a finite element model for the stability analysis of Timoshenko beams subjected to periodic axial loads. The effect of shear deformation and the effect of rotary inertia on the static buckling loads and the regions of dynamic instability was studied by the finite element method.

Abbas (1986), investigated the effects of rotational speed and root flexibilities on the static buckling loads and on the regions of dynamic instability of a Timoshenko beam by the finite element method.

Abbas & Thomas (1978), developed a finite element model for the stability analysis of a Timoshenko beam resting on an elastic foundation and subjected to a periodic axial loads. The effect of an elastic foundation on the natural frequencies

and static buckling loads of hinged-hinged and fixed free Timoshenko beams was investigated.

Chen & Peng (1995), studied the dynamic stability behavior of a rotating blade subjected to axial periodic forces by the Lagrange's equation and a Galerkin finite element method. The effects of geometric non-linearity, shear deformation and rotary inertia were considered.

Chen & Ku (1990), studied the dynamic stability of a rotating shaft subjected to axial periodic forces varying with time by the finite element method. By using Timoshenko beam theory the transverse shear effect was included in the shape functions and Bolotin's method was used to construct the dynamic instability diagrams.

Şakar & Sabuncu (2003, 2004), applied the finite element method to study the static and dynamic stability of straight and pretwisted aerofoil cross section rotating blades subjected to axial periodic forces.

Briseghella, Majarona & Pellegrino (1998), used the finite element method to find the regions of dynamic stability of beams and frames. A suitable numerical procedure was applied to obtain these regions and vibration frequency of the structures.

Tan, Lee & Leng (1997), derived the equations of motion of a rotating cantilever beam subjected to base excitations using the Euler beam theory and assumed mode method.

Steinman, Cleghorn & Tabarrok (1995), considered the free vibration of axially-loaded pretwisted rods and, a superset of the buckling and vibration equations were obtained and then solved analytically.

Gürgöze (1985), investigated the stability behavior of a pretwisted beam, subjected to a pulsating axial force. The stability of the solutions of this system was investigated by the method of Mettler.

Liao & Huang (1995), studied parametric instability of a cantilever pretwisted beam rotating around its longitudinal axis under a time-dependent conservative end axial force which contains a steady state part and a small periodically fluctuating component. The equations of motion were solved by the method of multiple scales, yielding the system response and expressions for the boundaries of the unstable regions.

Yang & Tsao (1997), presented an analytical model to investigate the vibration and stability of a pretwisted blade under nonconstant rotating speed which is characterized by a periodic perturbation.

Yokoyama (1988), studied the parametric instability behavior of a Timoshenko beam resting on an elastic foundation of the Winkler type by the finite element technique. The effect of elastic foundation on the natural frequencies, the static buckling loads and the regions of dynamic instability of the Timoshenko beams with different end conditions was investigated. In this study the natural frequency and buckling load parameters for a Bernoulli-Euler beam indicated excellent agreement with the available results.

Chen & Chern (1993), presented coupled bending-bending-torsion vibrations of a pretwisted rotating cantilever beams of fibre-reinforced material including shear deformation and rotary inertia. The free vibration and dynamic stability problems were discussed.

Chen & Shen (1997), applied a finite element model to study the dynamic stability aspects of a cracked rotating beam of general orthotropy. In this discrete model, both the transverse shear deformation effect and the rotary inertia effect were considered.

Lee (1994), formulated the equations of motion of a spinning pretwisted beam under compressive axial loads using Euler beam theory and assumed mode method. The equations of motion were transformed to the standard form of an eigenvalue problem for determining the critical buckling loads and instability regions.

Lee (1995), linearized equation of motion in matrix form of an Euler-Bernoulli inextensible beam with initial curvature and a tip mass subjected to an axial pulsating loads based on Lagrangian approach and assumed mode method. Using Bolotin's method, the linearized equation of motion is converted to the standard form of an eigenvalue problem for computing the principal instability regions.

Papangelis & Trahair (1987), developed a flexural-torsional buckling theory for circular arches of doubly symmetric cross section. Non linear expressions for the axial and shear strains were derived, and these were substituted into the second variation of the total potential to obtain the buckling equation.

Takahashi, Natsuaki & Konishi (1991), analyzed the dynamic instability of a circular arch subjected to an in-plane sinusoidal varying load by using a multiple-degrees-of-freedom approach, the Galerkin method and the harmonic balance method.

Yoo, Kang & Davidson (1996), presented the derivation of the elastic stiffness and geometric stiffness matrices for a curved beam element and verified the rigor and validity of the derivation by a series of numerical examples.

Yang & Kuo (1987), derived the nonlinear differential equations of equilibrium for a horizontally curved I-beam. Based on the principal of virtual displacements, the equilibrium of a bar was established for its deformed or buckled configuration using a Lagrangian approach.

Kang & Bert (1997), applied the differential quadrature method for the computation of the eigenvalues of flexural-torsional buckling of arches including a

warping contribution. Critical loads were calculated for a single-span wide flange arch subjected to equal and opposite in-plane bending moments or uniform distributed radial loads with various end conditions and opening angles.

Şakar, Öztuk & Sabuncu (2001), investigated the effects of variations of subtended angle and curvature of an arch, having in-plane curvature, on the natural frequencies, static and dynamic instability.

Yoo & Pfeiffer (1983), presented a general solution method of a system having coupled differential equations governing the elastic buckling of thin walled curved members. The finite element displacement method was formulated based on a variational principle.

Timoshenko & Gere studied the buckling of various types of structures in their book.

There are many investigations about static and dynamic stability of symmetric cross section (uncoupled) rotating and non-rotating blades, but no work has been done on the static and dynamic stability of asymmetric cross section blades and asymmetric cross section blade packets in existing literature. This study intends to point out the effects of coupling due to center of flexure distance from the centroid, rotational speed, stagger angle, pretwist angle and shroud dimensions on the static buckling loads and dynamic instability regions of aerofoil cross section blades and blade packets.

Chapter 2 deals with the theories used to analyze the dynamic stability of elastic systems.

Chapter 3 contains the vibration and stability analysis of non-rotating and rotating asymmetrical straight and pretwisted blades. In this chapter the blade is modeled as an Euler-Bernoulli beam and strain and kinetic energy functions of non-rotating and rotating asymmetrical straight and pretwisted blades are

summarized. Then using the finite element method the elastic stiffness, geometric stiffness and mass matrices are evaluated. The free vibration, static and dynamic stability analysis of non-rotating and rotating asymmetrical straight and pretwisted blades are given for various pretwist angles, stagger angles, rotational speeds, disc radius and shear center distances from the centroid. All results are presented in tabular and/or graphical forms. The obtained results are discussed.

Chapter 4 covers the vibration and stability analysis of a curved beam. The beam is assumed to have a rectangular cross section. It is also assumed that curved beams are having with small width/thickness ratio. As a result thin beam theory is applied by neglecting shear deformation and rotary inertia effects. The geometric stiffness, elastic stiffness and mass matrices of curved beams, representing stability and vibrations in the in-plane and out-of-plane directions, are obtained by using energy functions and given separately, then combined together to form the shroud element before the shroud blade assembly is made. The effects of the subtended angle, shroud length, disc radius and shroud dimensions on the vibration and stability analysis of a shroud are investigated. All results are presented in tabular and/or graphical forms. The obtained results are discussed.

In chapter 5 the blades are joined together by a shroud, forming a packet and rigidly fixed at the periphery of a disc rotating at a constant angular velocity. The blades and shroud are divided into a number of elements and the elastic stiffness, geometric stiffness and mass matrices of the k^{th} element are developed. Then the static and dynamic stability of rotating asymmetric cross section straight and pretwisted two bladed packets on rigid discs are carried out. The effects of stagger angle, pretwist angle, shroud thickness, shroud width, rotational speed, disc radius and the distance of center of flexure from centroid on the static and dynamic stability of two bladed packets are investigated. All results are presented in tabular and/or graphical forms. The obtained results are discussed.

Finally in chapter 6, conclusions drawn from the study are presented.

CHAPTER TWO

THEORY OF STABILITY ANALYSIS

2.1 Static stability

The modern use of steel and high-strength alloys in engineering structures, especially in bridges, ships and aircraft, has made elastic instability a problem of great importance. Urgent practical requirements have given rise in recent years to extensive theoretical investigations of the conditions governing the stability of beams, plates and shells.

The first problems of elastic instability, concerning lateral buckling of compressed members, were solved about 200 years ago by L. Euler. At that time the relatively low strength of materials necessitated stout structural members for which the question of elastic stability is not of primary importance. Thus Euler's theoretical solution, developed for slender bars, remained for a long time without a practical application. Only with the beginning of extensive steel constructions did the question of buckling of compression members become of practical importance. The use of steel led naturally to types of structures embodying slender compression members, thin plates and thin shells.

Stability problems can be treated in a general manner using the energy methods. As an introduction to such methods, the basic criteria for determining the stability of equilibrium is derived in this study for, conservative linearly elastic systems.

To establish the stability criteria, a function Π , called the potential of the system must be formulated. This function is expressed as the sum of the internal potential energy U (strain energy) and the potential energy Λ of the external forces that act on a system, i.e.,

$$\Pi = U + \Lambda \quad (2.1)$$

Disregarding a possible additive constant, $\Lambda = -W_e$, i.e., the loss of potential energy during the application of forces is equal to the work done on the system by external forces. Hence, equation (2.1) can be rewritten as

$$\Pi = U - W_e \quad (2.2)$$

As is known from classical mechanics, for equilibrium the total potential Π must be stationary, therefore its variation $\delta\Pi$ must equal zero,

$$\delta\Pi = \delta U - \delta W_e = 0 \quad (2.3)$$

For conservative, elastic systems this relation agrees with $\delta W_e = \delta W_{ei}$ equation (δW_{ei} : the external work on the internal elements of a body, δW_e : the total work), which states the virtual work principle. This condition can be used to determine the position of equilibrium. However, equation (2.3) cannot discern the type of equilibrium and there by establish the condition for the stability of equilibrium. Only by examining the higher order terms in the expression for increment in Π as given by Taylor's expansion must be examined. Such an expression is

$$\Delta\Pi = \delta\Pi + \frac{1}{2!}\delta^2\Pi + \frac{1}{3!}\delta^3\Pi + \dots \quad (2.4)$$

Since for any type of equilibrium $\delta\Pi = 0$, it is the first nonvanishing term of this expansion that determines the types of equilibrium. For linear elastic systems the second term suffices. Thus, from equation (2.4), the stability criteria are

$\delta^2\Pi > 0$ for stable equilibrium

$\delta^2\Pi < 0$ for unstable equilibrium

$\delta^2\Pi = 0$ for neutral equilibrium associated with the critical load

The meaning of these expressions may be clarified by examining the simple example shown in figure 2.1, where the shaded surfaces represent three different types of Π functions. It can be concluded at once that the ball on the concave spherical surface (a) is in stable equilibrium, while the ball on the convex spherical surface (b) is in unstable equilibrium. The ball on the horizontal plane (c) is said to be in different or neutral equilibrium. The type of equilibrium can be ascertained by considering the energy of the system. In the first case (figure 2.1(a)) any displacement of the ball from its position of equilibrium will raise the center of gravity. A certain amount of work is required to produce such a displacement; thus the potential energy of the system increases for any small displacement from the position of equilibrium. In the second case (figure 2.1 (b)), any displacement from the position of equilibrium will decrease the potential energy of the system. Thus in the case of stable equilibrium the energy of the system is a minimum and in the case of unstable equilibrium it is a maximum. If the equilibrium is indifferent (figure 2.1 (c)), there is no change in energy during a displacement.

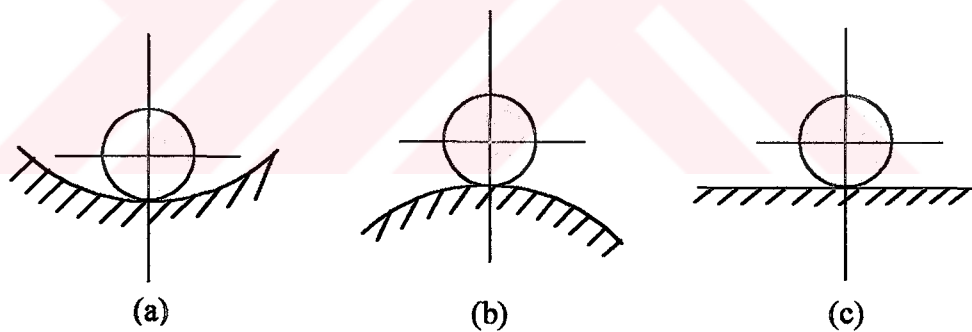


Figure 2.1 Three cases of equilibrium

For each of the systems shown in figure 2.1 stability depends only on the shape of the supporting surface and does not depend on the weight of the ball. In the case of a compressed column or plate it is found that the column or plate may be stable or unstable, depending on the magnitude of the axial load.

2.1.1 The formulation of static stability

If the displacements are large, then the deformed geometry will obviously differ significantly from the undeformed geometry. This results in a nonlinear strain-displacement relationship. Large displacement problems of this type are said to be “geometrically nonlinear” which is a feature of elastic instability problems. From the design point of view calculation of the critical loads of structures is of considerable importance. In general case the strain energy of a system,

$$U = \frac{1}{2} \{q\}^T [K_e] \{q\} \quad (2.5)$$

The additional strain energy which is function of applied external load

$$U_g = \frac{1}{2} \{q\}^T [K_g] \{q\} \quad (2.6)$$

In which $[K_e]$ and $[K_g]$ are elastic stiffness and geometric stiffness matrices.

The total potential energy of a system in equilibrium is constant when small displacements are given to the system. So

$$\delta(U + U_g) = 0 \quad (2.7)$$

$(U + U_g)$ and δ define the total potential energy and the change of the virtual displacements. Applying the above formulation to equations (2.5) and (2.6)

$$[[K_e] - P[K_g]] \{q\} = 0 \quad (2.8)$$

The roots of the eigenvalue equation (2.8) gives the buckling loads and the eigenvectors of this equation are the buckling mode shapes.

2.2 Dynamic stability

If the loading is nonconservative the loss of stability may not show up by the system going into another equilibrium state but by going into unbounded motion. To encompass this possibility we must consider the dynamic behavior of the system because stability is essentially a dynamic concept.

Whenever static loading of a particular kind causes a loss of static stability, vibrational loading of the same kind will cause a loss of dynamic stability. Such a loading is characterized by the fact that it is contained as a parameter on the left hand side of the equations of perturbed equilibrium (or motion). We will call such loading parametric; this term is more appropriate because it indicates the relation to the phenomenon of parametric resonance.

In the mechanical systems, parametric excitation occurs due to the following reasons;

- a) periodic change in rigidity
- b) periodic change in inertia
- c) periodic change in the loading of the system.

In this section firstly the differential equation related with dynamic stability is introduced and then, the determination of boundaries of the regions of instability and the amplitudes of parametrically excited vibrations for multi-degrees of freedom systems is presented.

An important special case of linear variational equations with variable coefficients occurs when the coefficient functions are periodic. Owing to their great practical importance in the theory of vibrations, a special theory has even been developed for the systems of differential equations with periodic coefficients are known as Mathieu-Hill differential equation. The Hill differential equation is in the following form,

$$\ddot{y} + [a - b f(t)]y = 0 \quad (2.9)$$

in which a and b are constant parameters, and $f(t)$ is a function having the period T . The prime denotes differentiation with respect to time. If $f(t) = 2 \cos 2t$ substituted into the Hill differential equation, the Mathieu differential equation which may be described a system that is subjected to parametric excitation is obtained in the standard form as

$$\ddot{y} + [a - 2b \cos 2t]y = 0 \quad (2.10)$$

The results of solving Mathieu's equation (2.10) for two different combinations of a and b are shown in figure 2.2. Although the parameter b of the system is the same in both cases ($b=0,1$), the vibrations are greatly different because of the difference between the values of the parameter a ($a=1$; $a=1,2$). In the first case, they increase, i.e., the system is dynamically unstable, while in the second case they remain bounded, i.e., the system is dynamically stable.

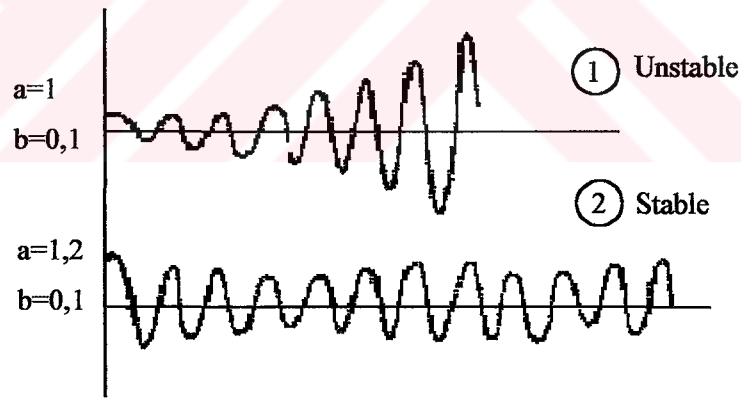


Figure 2.2 Two solutions of Mathieu's equation

The greatest importance, for practical purpose, is attached to the boundaries between the regions of stable and unstable solutions. This problem has been well studied, and the final results have been presented in the form of a diagram plotted in the plane of the parameters a and b . It is called the Haines-Strett diagram. Figure 2.3 shows part of a Haines-Strett diagram for small values of the parameter b . Any given system having the parameters a and b corresponds to the point with the co-ordinates

a and b on the Haines-Strett diagram. If the representative point is in the shaded parts of the diagram, the system is dynamically unstable, while stable systems correspond to representative points in the unshaded parts. The shaded regions are called the regions of dynamic instability.

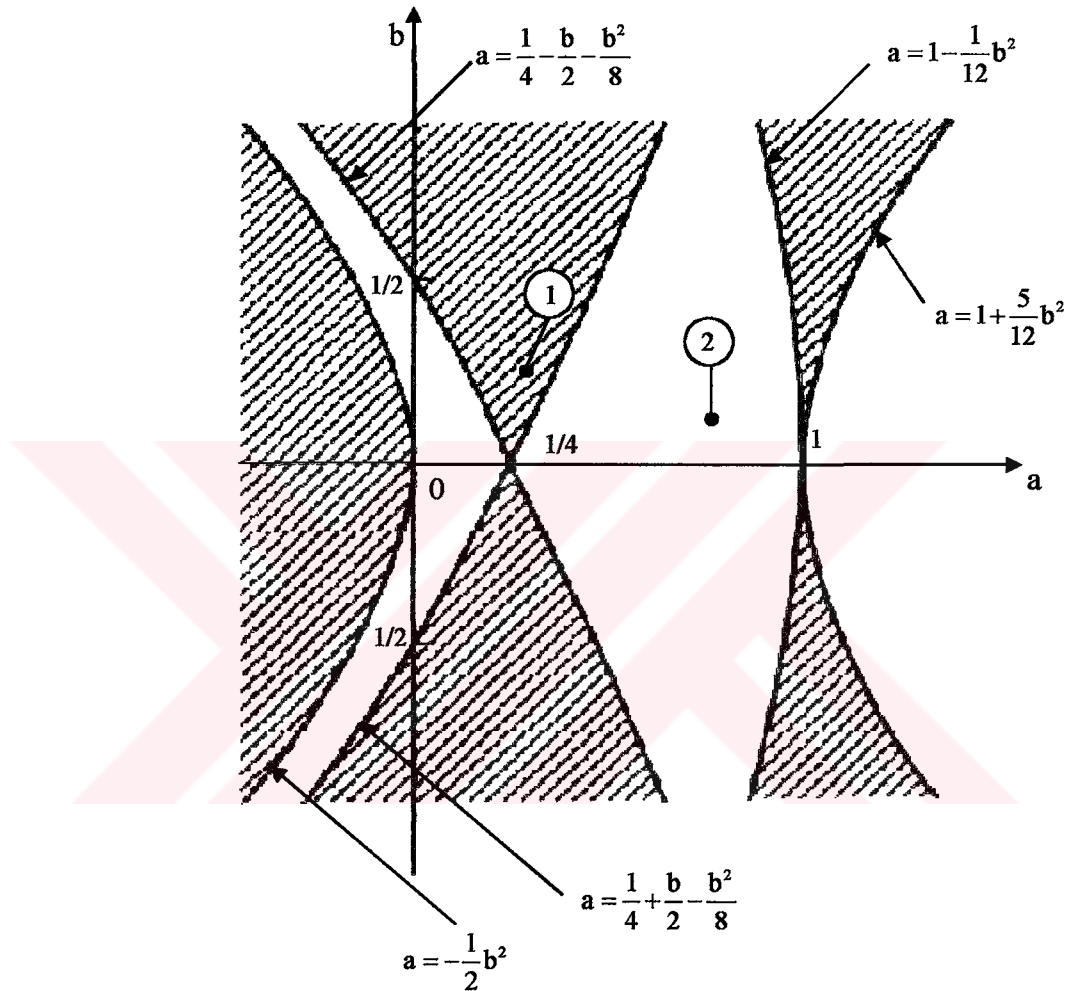


Figure 2.3 Part of Haines-Strett diagram the points ① and ② correspond to the solutions 1 and 2 in figure 2.2

As an example, the diagram in figure 2.3 shows the points 1 and 2 corresponding to the parameter $a_1=1$ and $b_1=0,1$, and $a_2=1,2$ and $b_2=0,1$. The point 1 is in the region of dynamic instability and the vibration occurs with increasing amplitude as shown in figure 2.2. The point 2 is in the stable region and it corresponds to motion with a limited amplitude.

2.2.1 The formulation of dynamic stability

The matrix equation for the free vibration of an axially loaded system can be written as:

$$[M]\{\ddot{q}\} + [K_e]\{q\} - [K_g]\{q\} = 0 \quad (2.11)$$

where

$\{q\}$ is the generalized coordinates

$[M]$ is the mass matrix

$[K_e]$ is the elastic stiffness matrix

$[K_g]$ is the geometric stiffness matrix, which is a function of the compressive axial load $P(t)$.

For a system subjected to a periodic force

$$P(t) = P_o + P_t f(t) \quad (2.12)$$

The static and time dependent components of the load can be represented as a fraction of the fundamental static buckling load P^* , in which $P_o = \alpha P^*$, $P_t = \beta P^*$. By writing $P = \alpha P^* + \beta P^* f(t)$ then the matrix equation K_g becomes

$$K_g = \alpha P^* [K_{gs}] + \beta P^* [K_{gt}] \quad (2.13)$$

where the matrices $[K_{gs}]$ and $[K_{gt}]$ reflect the influence of P_o and P_t respectively.

Substituting equation (2.13) into equation (2.11), the following system of n second order differential equations with a periodic coefficient of the known Mathieu-Hill type is obtained;

$$[M]\{\ddot{q}\} + [K_e] - \alpha P^* [K_{gs}] - \beta P^* f(t) [K_{gt}] \{q\} = 0 \quad (2.14)$$

$f(t)$ is a periodic function with period T . Therefore

$$f(t+T)=f(t) \quad (2.15)$$

Equation (2.14) is a system of n second order differential equations which may be written as

$$\{\ddot{q}(t)\} + [Z]\{q(t)\} = 0 \quad (2.16)$$

where

$$[Z] = [M]^{-1} \left[[K_e] - \alpha P^* [K_{gs}] - \beta P^* [K_{gt}] \right] \quad (2.17)$$

It is convenient to replace the n second order equations with $2n$ first order equations by introducing

$$\{h\} = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} \quad (2.18)$$

and

$$[\phi] = \begin{bmatrix} 0 & -[I] \\ [Z] & 0 \end{bmatrix} \quad (2.19)$$

then, equation (2.16) becomes

$$\{\dot{h}(t)\} + [\phi(t)]\{h(t)\} = \begin{Bmatrix} \dot{q} \\ \ddot{q} \end{Bmatrix} + \begin{bmatrix} 0 & -[I] \\ [Z] & 0 \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix} = 0 \quad (2.20)$$

Equation (2.19) needs not be solved completely in order to determine the stability of the system. It is merely necessary to determine whether the solution is bounded or unbounded.

It is assumed that the $2n$ linearly independent solutions of equation (2.20) are known over the interval $t = 0$ to $t = T$. Then they may be represented in matrix form as

$$[H(t)] = \begin{bmatrix} h_{1,1} & . & . & . & h_{1,2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ h_{2n,1} & . & . & . & h_{2n,2n} \end{bmatrix} \quad (2.21)$$

Since $f(t)$, and therefore $[\phi(t)]$ is periodic with period T , then the substitution $t = t + T$ will not alter the form of the equations, and the matrix solutions, at time $t + T$, $[H(t + T)]$ may be obtained from $[H(t)]$ by a linear transformation

$$[H(t + T)] = [R][H(t)] \quad (2.22)$$

where $[R]$ is the transformation matrix and is composed only of constant coefficients.

It is desirable to find a set of solutions for which the matrix $[R]$ can be diagonalized. Hence the i^{th} solution vector after period T , $\{\bar{h}(t + T)\}_i$ may be determined from $\{\bar{h}(t)\}_i$ using the simple expression

$$\{\bar{h}(t + T)\}_i = \rho_i \{\bar{h}(t)\}_i \quad (2.23)$$

The behavior of the solution is determined by ρ_i .

If $\rho_i > 1$, then the amplitude of vibration will increase with time. If $\rho_i < 1$, then the amplitude will decrease. For $\rho_i = 1$, the amplitude will remain unchanged, and this represents the stable boundary.

In order to diagonalize the matrix $[R]$, the characteristic equation

$$|[R] - \rho[I]| = 0 \quad (2.24)$$

must be solved for its $2n$ roots, where $[I]$ is the identity matrix. The roots of the equations, ρ_i , are eigenvalues, each having a corresponding eigenvector.

The $2n$ resulting eigenvectors are chosen as the $2n$ solutions to equation (2.20). They can be placed in a matrix, $[\bar{H}(t)]$, which will then satisfy the expression

$$[\bar{H}(t)] = [\bar{R}][\bar{H}(t+T)] \quad (2.25)$$

where

$$[\bar{R}] = \begin{bmatrix} \rho_1 & 0 & . & . & 0 \\ 0 & \rho_2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & 0 & \rho_{2n} \end{bmatrix} \quad (2.26)$$

$[\bar{R}]$ is the diagonalized matrix of $[R]$ composed of the $2n$ eigenvalues of equation (2.24).

The periodic vector, $\{Z(t)\}_i$, with period T is introduced so that

$$\{\bar{h}(t)\}_i = \{Z(t)\}_i e^{(t/T)\ln\rho_i} \quad (2.27)$$

For an even function of time like $[\phi(t)]$, it is true that

$$[\phi(t)] = [\phi(-t)] \quad (2.28)$$

Hence equation (2.27) can be written as

$$\{\bar{h}(-t)\}_i = \{Z(-t)\}_i e^{-(t/T)\ln \rho_i} \quad (2.29)$$

then

$$\{\bar{h}(-t)\}_i = \{Z(-t)\}_i e^{(t/T)\ln(1/\rho_i)} \quad (2.30)$$

It is clear from (2.30) that $1/\rho_i$ is also an eigenvalue. This property is not restricted to even functions, but is also preserved in the case of arbitrary periodic functions as shown by Bolotin, (1964).

In general, the eigenvalues ρ_i are complex numbers of the form

$$\rho_i = a_i + j b_i \quad (2.31)$$

and the natural logarithm of a complex number is given by

$$\ln \rho_i = \ln |\rho_i| + j (\text{argument } \rho) \quad (2.32)$$

or in this case

$$\ln \rho_i = \ln \sqrt{a_i^2 + b_i^2} + j \tan^{-1}(b_i / a_i) \quad (2.33)$$

where $j = \sqrt{-1}$

From equation (2.27), it is clear that if the real part of $\log \rho_i$ is positive for any of the solutions, then that solution will be unbounded with time. A negative real part

means that the corresponding solution will damp out with time. It therefore follows that the boundary case for a given solution is that for which the characteristic exponent has a zero real part. This is identical to saying that absolute value of ρ_i is unity. For the system to remain stable, every one of the solutions must remain bounded. If even one of the solutions has a characteristic exponent which is positive, then the corresponding solution is unbounded and therefore the system is unstable.

It has been shown that if ρ_i is a solution, then $1/\rho_i$ is also a solution. These two solutions can be written as

$$\rho_i = a_i + j b_i \quad (2.34)$$

$$\rho_{i+n} = (a_i - j b_i) / (a_i^2 + b_i^2) \quad (2.35)$$

Another restriction on the solutions of the characteristic equation is that the complex eigenvalues must occur in complex conjugate pairs. Hence it follows that ρ_{i+1} and ρ_{i+n+1} are also solutions where

$$\rho_{i+1} = a_i - j b_i \quad (2.36)$$

$$\rho_{i+n+1} = (a_i + j b_i) / (a_i^2 + b_i^2) \quad (2.37)$$

These solutions are presented in figure 2.4 which shows a unit circle in the complex plane. The area inside the unit circle represents stable or bounded solutions, while the area outside the unit circle represents unstable or unbounded solutions. For each stable solution which lies inside the circle, there corresponds an unstable solution outside the circle due to the reciprocity constraint. Therefore the only possible stable solutions must lie on the unit circle.

Points on this unit circle may be represented in polar co-ordinates by $r = 1$ and $\theta = \tan^{-1}b/a$ where $-\pi \leq \theta \leq \pi$. For each root on the upper semicircle, there is a

corresponding root on the lower semicircle due to the fact that the roots occur in complex conjugate pairs. The logarithm of ρ_i , when ρ_i lies on the unit circle will be

$$\ln \rho_i = j\theta \quad (2.38)$$

and equation (2.27) becomes

$$\{\bar{h}(t)\}_i = \{Z(t)\}_i e^{j\left(\frac{\theta t}{T}\right)} \quad (2.39)$$

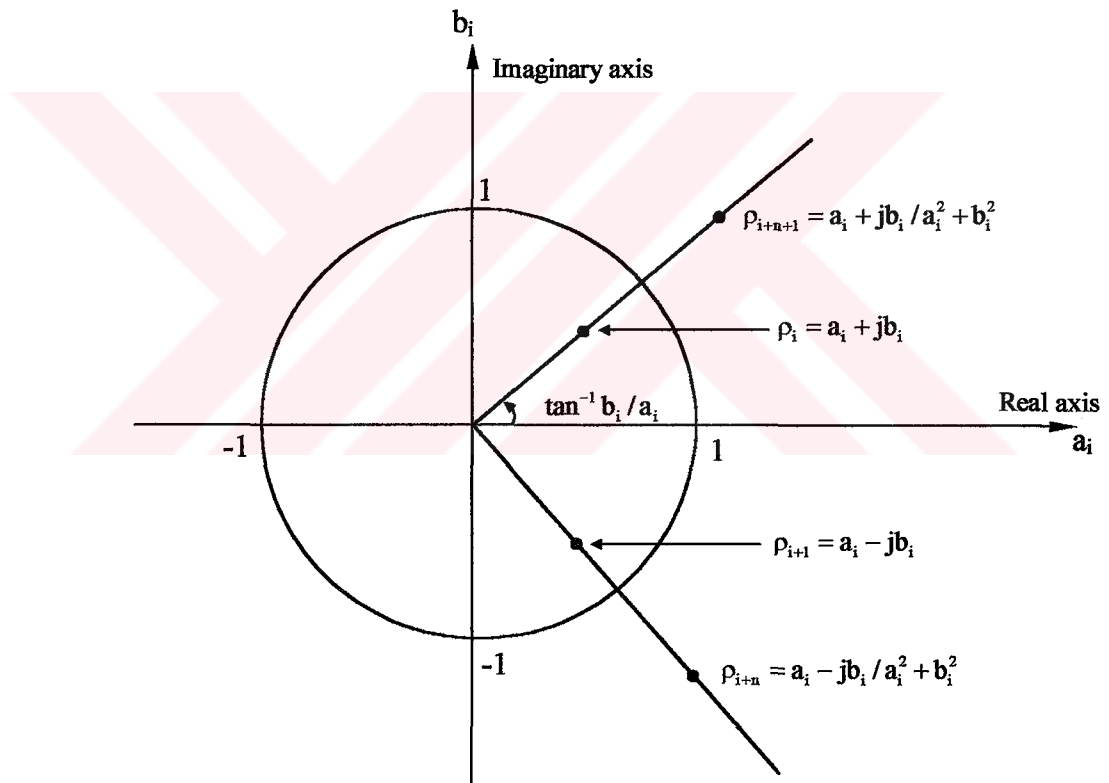


Figure 2.4 Unit circle in the complex plane

Since the eigenvalues occur in complex conjugate pairs, the limiting values of θ are zero and π .

When $\theta = 0$, equation (2.39) becomes

$$\{\bar{h}(t)\}_i = \{Z(t)\}_i \quad (2.40)$$

and, therefore, the solution $\{\bar{h}(t)\}$ is periodic with period T when $\theta = \pi$, equation (2.39) becomes

$$\{\bar{h}(t)\}_i = \{Z(t)\}_i e^{j\left(\frac{\pi t}{T}\right)} \quad (2.41)$$

$$\{\bar{h}(t+2T)\}_i = \{Z(t+2T)\}_i e^{j\left(\frac{\pi(t+2T)}{T}\right)} = \{\bar{h}(t)\}_i \quad (2.42)$$

It is clear from equation (2.42) that the solution $\{\bar{h}(t)\}$ is also periodic with a period $2T$.

It can be concluded that equation (2.11) has periodic solutions of period T and $2T$. Also the boundaries between stable and unstable regions are formed by periodic solutions of period T and $2T$.

For a system subjected to the periodic force

$$P = P_0 + P_t \cos \omega t \quad (2.43)$$

Where ω is the disturbing frequency, equation (2.11) becomes

$$[M]\{\ddot{q}\} + \left[[K_e] - \alpha P^* [K_{gs}] - \beta P^* \cos \omega t [K_{gt}] \right] \{q\} = 0 \quad (2.44)$$

Now we seek periodic solutions of period T and $2T$ of equation (2.44) where $T = 2\pi/\omega$.

When a solution of period $2T$ exists, it may be represented by the Fourier series

$$\{q\} = \sum_{k=1,3,5}^{\infty} \left[\{a\}_k \sin \frac{k\omega t}{2} + \{b\}_k \cos \frac{k\omega t}{2} \right] \quad (2.45)$$

Where $\{a\}_k$ and $\{b\}_k$ are time-independent vectors. Differentiating equation (2.45) twice with respect to time yields

$$\{\ddot{q}\} = \sum_{k=1,3,5}^{\infty} -\left(\frac{k\omega}{2}\right)^2 \left[\{a\}_k \sin \frac{k\omega t}{2} + \{b\}_k \cos \frac{k\omega t}{2} \right] \quad (2.46)$$

Substituting equations (2.45) and (2.46) into equation (2.44) and using the trigonometric relations

$$\begin{aligned} \sin A + \sin B &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\ \sin A - \sin B &= 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \\ \cos A + \cos B &= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \\ \cos A - \cos B &= 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \end{aligned} \quad (2.47)$$

and comparing the coefficients of $\sin \frac{k\omega t}{2}$ and $\cos \frac{k\omega t}{2}$ lead to the following matrix equations relating the vectors $\{a\}_k$ and $\{b\}_k$.

$$\begin{bmatrix} [K_e] - \alpha P^* [K_{\mathcal{B}}] + \frac{1}{2} \beta P^* [K_{\mathcal{B}}] - \frac{\omega^2}{4} [M] & -\frac{1}{2} \beta P^* [K_{\mathcal{B}}] & 0 \\ -\frac{1}{2} \beta P^* [K_{\mathcal{B}}] & [K_e] - \alpha P^* [K_{\mathcal{B}}] - \frac{9\omega^2}{4} [M] & -\frac{1}{2} \beta P^* [K_{\mathcal{B}}] \\ 0 & -\frac{1}{2} \beta P^* [K_{\mathcal{B}}] & [K_e] - \alpha P^* [K_{\mathcal{B}}] - \frac{25\omega^2}{4} [M] \end{bmatrix} \cdot \begin{Bmatrix} \{a\}_1 \\ \{a\}_3 \\ \{a\}_5 \end{Bmatrix} = 0 \quad (2.48)$$

and

$$\begin{bmatrix} [K_e] - \alpha P^* [K_{gs}] - \frac{1}{2} \beta P^* [K_{gt}] - \frac{\omega^2}{4} [M] & -\frac{1}{2} \beta P^* [K_{gs}] & 0 & \cdot \\ -\frac{1}{2} \beta P^* [K_{gt}] & [K_e] - \alpha P^* [K_{gs}] - \frac{9\omega^2}{4} [M] & -\frac{1}{2} \beta P^* [K_{gt}] & \cdot \\ 0 & -\frac{1}{2} \beta P^* [K_{gs}] & [K_e] - \alpha P^* [K_{gs}] - \frac{25\omega^2}{4} [M] & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \{b\}_1 \\ \{b\}_3 \\ \{b\}_5 \\ \cdot \end{Bmatrix} = 0 \quad (2.49)$$

The orders of matrices in equations (2.48) and (2.49) are infinite. If solutions of period $2T$ exist, then the determinants of these matrices must zero. Combining these two determinants, the condition may be written as

$$\begin{vmatrix} [K_e] - \alpha P^* [K_{gs}] \pm \frac{1}{2} \beta P^* [K_{gt}] - \frac{\omega^2}{4} [M] & -\frac{1}{2} \beta P^* [K_{gt}] & 0 & \cdot \\ -\frac{1}{2} \beta P^* [K_{gt}] & [K_e] - \alpha P^* [K_{gs}] - \frac{9\omega^2}{4} [M] & -\frac{1}{2} \beta P^* [K_{gt}] & \cdot \\ 0 & -\frac{1}{2} \beta P^* [K_{gs}] & [K_e] - \alpha P^* [K_{gs}] - \frac{25\omega^2}{4} [M] & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 \quad (2.50)$$

If a solution to equation (2.44) exists with a period $T=2\pi/\omega$ then it may be expressed as Fourier series

$$\{q\} = \frac{1}{2} b_0 + \sum_{k=2,4,6}^{\infty} \left[\{a\}_k \sin \frac{k\omega t}{2} + \{b\}_k \cos \frac{k\omega t}{2} \right] \quad (2.51)$$

Differentiating equation (2.51) twice with respect to time yields

$$\{\ddot{q}\} = \sum_{k=2,4,6}^{\infty} -\left(\frac{k\omega}{2}\right)^2 \left[\{a\}_k \cos \frac{k\omega t}{2} + \{b\}_k \sin \frac{k\omega t}{2} \right] \quad (2.52)$$

Substituting equations (2.51) and (2.52) into equation (2.44), the following condition for the existence of solution with period T is obtained;

$$\begin{bmatrix} [K_e] - \alpha P^*[K_{gs}] - \omega^2[M] & -\frac{1}{2}\beta P^*[K_{gt}] & 0 & \cdot \\ -\frac{1}{2}\beta P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - 4\omega^2[M] & -\frac{1}{2}\beta P^*[K_{gt}] & \cdot \\ 0 & -\frac{1}{2}\beta P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - 9\omega^2[M] & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \{a\}_2 \\ \{a\}_4 \\ \{a\}_6 \\ \cdot \end{Bmatrix} = 0 \quad (2.53)$$

and

$$\begin{bmatrix} \frac{1}{2}\{[K_e] - \alpha P^*[K_{gs}]\} & -\frac{1}{2}\beta P^*[K_{gt}] & 0 & 0 & \cdot \\ -\frac{1}{2}\beta P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - \omega^2[M] & -\frac{1}{2}\beta P^*[K_{gt}] & 0 & \cdot \\ 0 & -\frac{1}{2}\beta P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - 4\omega^2[M] & -\frac{1}{2}\beta P^*[K_{gt}] & \cdot \\ 0 & \frac{1}{2}\beta P^*[K_{gt}] & [K_e] - \alpha P^*[K_{gs}] - 9\omega^2[M] & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \{b\}_0 \\ \{b\}_2 \\ \{b\}_4 \\ \{b\}_6 \\ \cdot \end{Bmatrix} = 0 \quad (2.54)$$

It has been shown by Bolotin (1964), that solutions with period 2T are the ones of the greatest practical importance and that as a first approximation the boundaries of the principal regions of dynamic instability can be determined from the equation

$$\left[[K_e] - \alpha P^*[K_{gs}] \pm \frac{1}{2}\beta P^*[K_{gt}] - \frac{\omega^2}{4}[M] \right] \{q\} = 0 \quad (2.55)$$

The two matrices $[K_{gs}]$ and $[K_{gt}]$ will be identical if the static and time dependent components of the loads are applied in the same manner. If $[K_{gs}] \equiv [K_{gt}] \equiv [K_g]$, then equation (2.56) becomes

$$\left[[K_e] - (\alpha \pm \frac{1}{2}\beta) P^* [K_g] - \frac{\omega^2}{4} [M] \right] \{q\} = 0 \quad (2.56)$$

Equation (2.56) represents solutions to three related problems

(i) Free vibration with $\alpha = 0$, $\beta = 0$ and $p = \omega/2$ the natural frequency

$$[[K_e] - p^2 [M]] \{q\} = 0 \quad (2.57)$$

(ii) Static stability with $\alpha = 1$, $\beta = 0$ and $\omega = 0$

$$[[K_e] - P^* [K_g]] \{q\} = 0 \quad (2.58)$$

(iii) Dynamic stability when all terms are present

$$\left[[K_e] - (\alpha \pm \frac{1}{2}\beta) P^* [K_g] - \frac{\omega^2}{4} [M] \right] \{q\} = 0 \quad (2.59)$$

CHAPTER THREE

THEORATICAL CONSIDERATION OF A SINGLE BLADE

Vibration induce fatigue failure of rotor blades is a problem of major concern to designers of turbo machinery. The critical nature of the problem becomes apparent when one notes that there can be as many as thousands blades many of them with different characteristics in one engine. Therefore, the determination of dynamic characteristics of blading becomes important.

In the past, several methods of analysis have been applied to the study of vibration characteristics of turbine blades. The blades used in actual turbines and compressors are of aerofoil cross section and the vibration coupled type. The coupling varies depending upon the configuration of the blade.

A blade can be straight or pretwisted along its length. When a straight beam of asymmetrical aerofoil cross section, as shown in figure 3.1 is subjected to vibrating movement, the motion consists of simultaneous displacements in two perpendicular directions coupled with torsion displacement. This is referred to as bending-bending-torsion vibration and each normal mode consists of simultaneous bending displacements in two perpendicular directions, together with torsional movement.

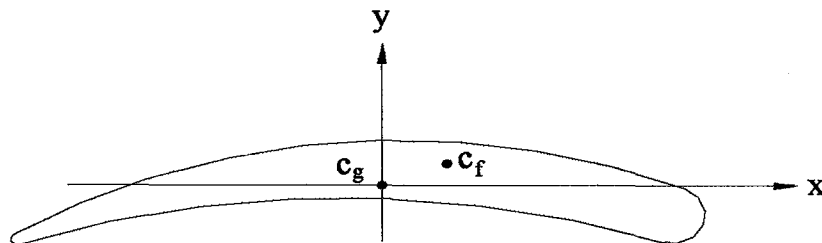


Figure 3.1 Asymmetric aerofoil cross section blade

When a beam has one axis of symmetry, as shown in figure 3.2 the bending vibration in x direction is an independent bending vibration, where as the bending vibration in the y direction occurs simultaneously with a torsional vibration and is

referred to as bending-torsion vibration. When the center of flexure and mass axis are collinear, as for example in rectangular cross-section beams, the bending vibrations in the two perpendicular principal planes occur independently and in addition there is an independent torsional one.

Pretwisted symmetrical cross section beams execute coupled bending-bending vibrations but the torsional vibration remains uncoupled. Pretwisted beam with one axis of symmetry causes the bending-torsion vibration to couple with independent bending vibration, resulting in coupled vibration of the bending-bending-torsion type. Pretwisted and asymmetrical aerofoil cross section beams vibrate in coupled bending-bending-torsion mode.

The finite element method is ideal in analyzing systems of such complex configuration. In this work a turbo machinery blade is idealized as a rotating cantilever beam. For this particular blade as mentioned by Carnegie & Dawson, (1971) due to the width/thickness ratio, shear deformation and rotary inertia effects are neglected.

An asymmetric cross section blade of length L subjected to an axial periodic force $P(t)$ and rotating speed of ω is illustrated in figure 3.3. This periodic force is assumed to be the combination of the axial component of aerodynamic or hydrodynamic forces acting on the blade. It is also assumed that this force is concentrated and applied through the centroid of the aerofoil cross section. The blade is divided into a number of elements and then stiffness matrix, mass matrix and stability matrix of the k^{th} element from the fixed end are developed.

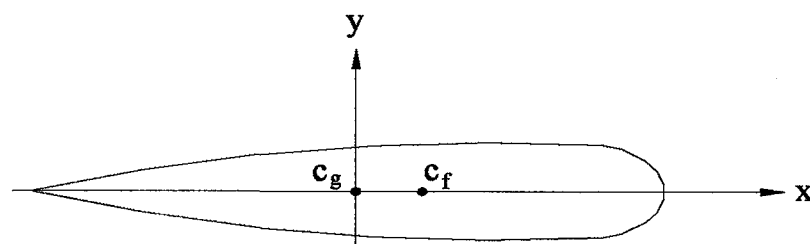


Figure 3.2 Blade with one axis of symmetry

3.1 Strain energy for a rotating blade

The equations for strain and kinetic energies of a asymmetric cross section uniform beam with or without pretwisted are given by Carnegie (1969).

When the cantilever blade rotates its potential energy is increased by the rotation, additional work being performed due to the inward motion of the blade elements in the centrifugal field.

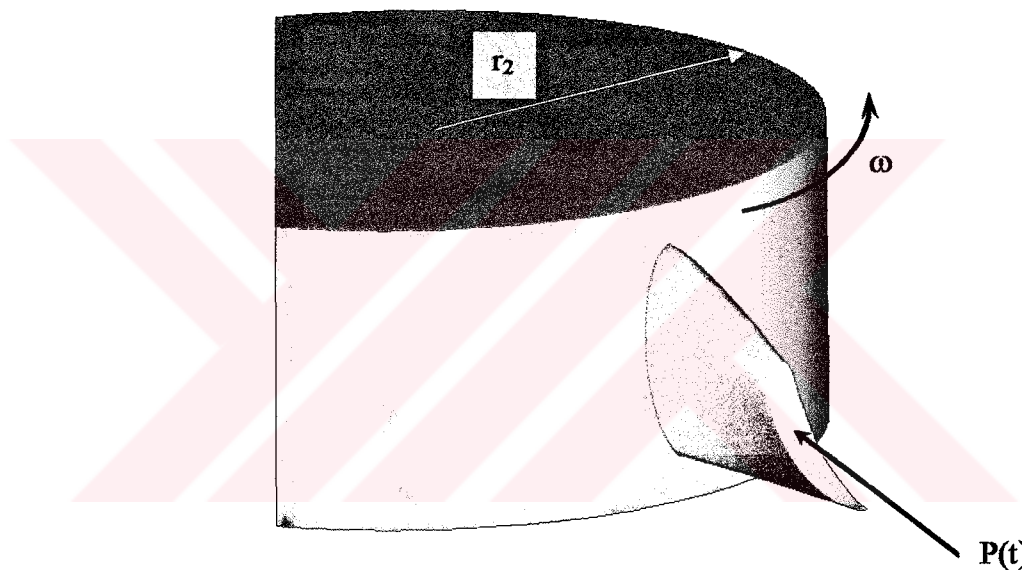


Figure 3.3 Asymmetric aerofoil cross section rotating blade-disc configuration.

The strain and kinetic energy equations of a rotating pretwisted beam are known

$$V = V_s + U_f \quad (3.1)$$

The strain energy V_s of a stationary blade vibrating in combined bending-bending-torsion mode is given by Carnegie, (1969).

$$V_s = \int_0^L \left\{ \frac{EI_{yy}}{2} \left(\frac{\partial^2 u}{\partial z^2} \right)^2 + EI_{xy} \left(\frac{\partial^2 u}{\partial z^2} \right) \left(\frac{\partial^2 v}{\partial z^2} \right) + \frac{EI_{xx}}{2} \left(\frac{\partial^2 v}{\partial z^2} \right)^2 + \frac{C}{2} \left(\frac{\partial \theta}{\partial z} \right)^2 + \frac{EA}{2} \left(\frac{\partial w}{\partial z} \right)^2 \right\} dz \quad (3.2)$$

The additional strain energy U_f due to rotation of a blade vibrating in combined bending-bending-torsion mode is (Carnegie, 1959)

$$U_f = \frac{\omega^2}{2} \int_0^L \left[W_b (r_2 + z) \int_0^z \left\{ \left(\frac{\partial u_1}{\partial z} \right)^2 + \left(\frac{\partial v_1}{\partial z} \right)^2 \right\} dz - W_b (u_1 \cos^2 \phi - 2u_1 v_1 \sin \phi \cos \phi + v_1^2 \sin^2 \phi) \right] dz \quad (3.3)$$

These equations can be written in non-dimensional form for the k^{th} element by using the following equations.

$$z = \eta \ell, \quad \text{for } z = 0 \quad \eta = 0; \quad z = \ell \quad \eta = 1 \quad (3.4)$$

$$V_s = \int_0^1 \left\{ \frac{EI_{yy}}{2\ell^3} \left(\frac{\partial^2 u}{\partial \eta^2} \right)^2 + \frac{EI_{xy}}{\ell^3} \left(\frac{\partial^2 u}{\partial \eta^2} \right) \left(\frac{\partial^2 v}{\partial \eta^2} \right) + \frac{EI_{xx}}{2\ell^3} \left(\frac{\partial^2 v}{\partial \eta^2} \right)^2 + \frac{C}{2\ell} \left(\frac{\partial \theta}{\partial \eta} \right)^2 + \frac{EA}{2\ell} \left(\frac{\partial w}{\partial \eta} \right)^2 \right\} d\eta \quad (3.5)$$

$$U_f = \frac{\omega^2 \rho A \ell}{2} \int_0^1 \left[\frac{r_2}{\ell} + k - 1 + \eta_k \right] \left[\int_0^1 \left(\frac{\partial u_1}{\partial \eta_1} \right)^2 d\eta_1 + \int_0^1 \left(\frac{\partial u_1}{\partial \eta_2} \right)^2 d\eta_2 + \dots + \int_0^{\eta_k} \left(\frac{\partial u_1}{\partial \eta_k} \right)^2 d\eta_k \right] d\eta_k +$$

$$\frac{\omega^2 \rho A \ell}{2} \int_0^1 \left[\frac{r_2}{\ell} + k - 1 + \eta_k \right] \left[\int_0^1 \left(\frac{\partial v_1}{\partial \eta_1} \right)^2 d\eta_1 + \int_0^1 \left(\frac{\partial v_1}{\partial \eta_2} \right)^2 d\eta_2 + \dots + \int_0^{\eta_k} \left(\frac{\partial v_1}{\partial \eta_k} \right)^2 d\eta_k \right] d\eta_k -$$

$$\frac{\omega^2 A \ell \rho}{2} \int_0^1 (u_1^2 \cos^2 \phi - 2u_1 v_1 \sin \phi \cos \phi + v_1^2 \sin^2 \phi) d\eta_k \quad (3.6)$$

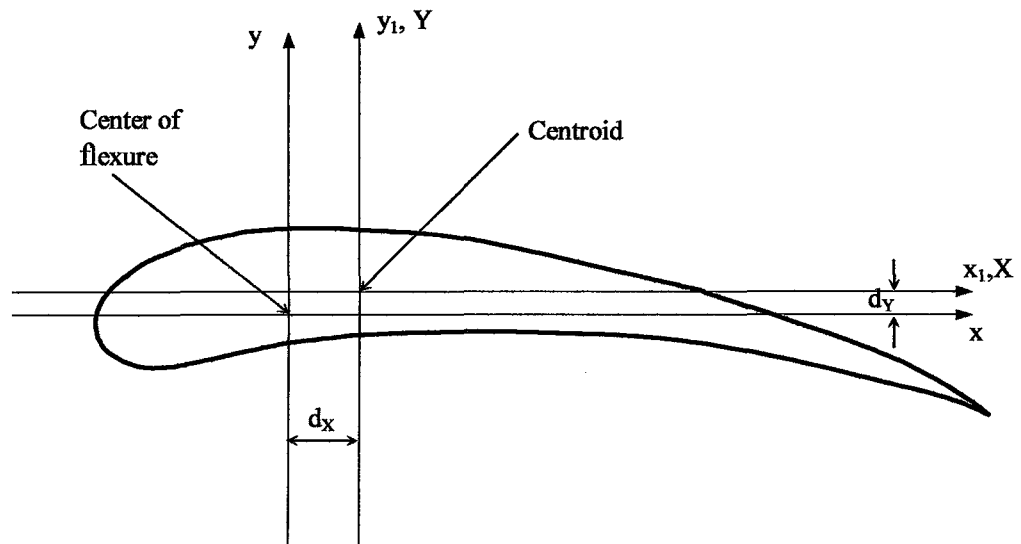


Figure 3.4 Cross section of an asymmetrical straight aerofoil blade

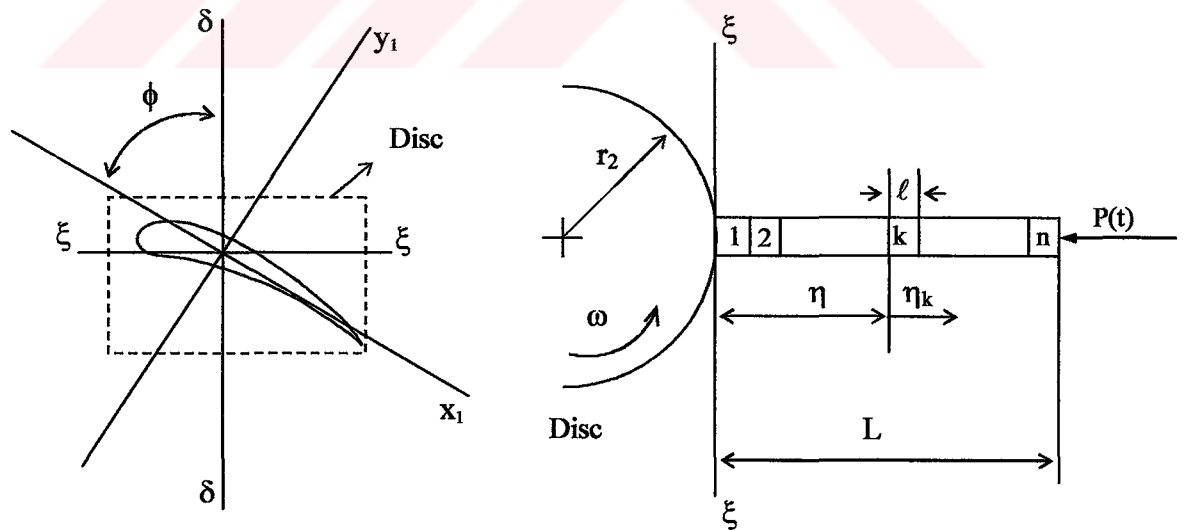


Figure 3.5 Cantilever beam diagram; element and rotational geometry

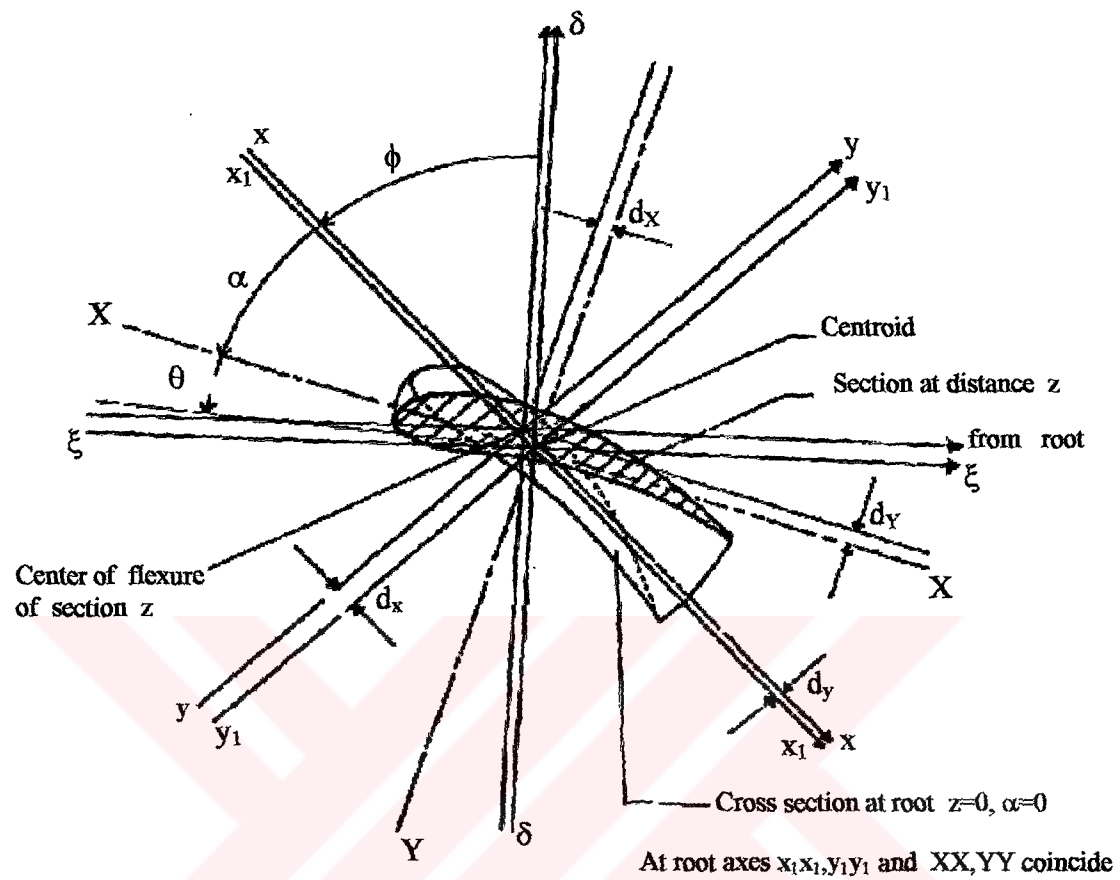


Figure 3.6 Cross section of an asymmetrical pretwisted aerofoil blade

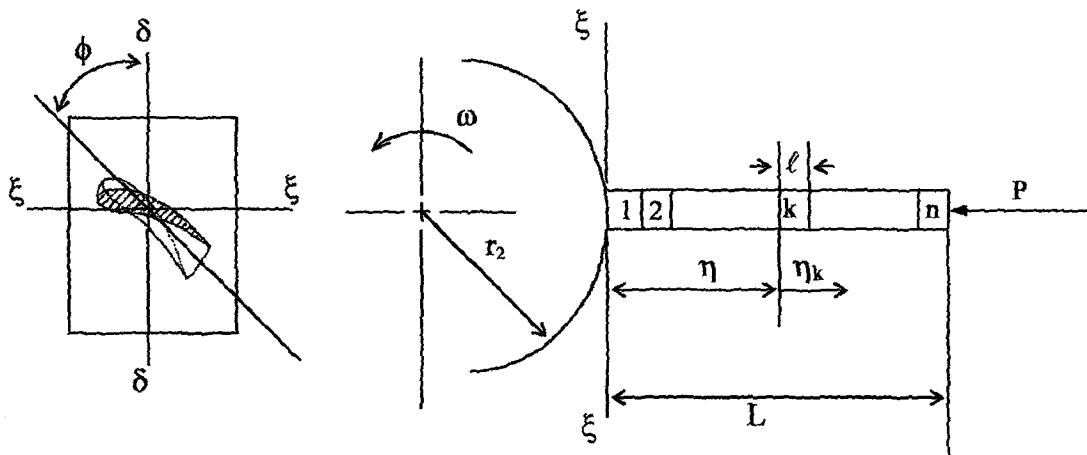


Figure 3.7 Elemental length of a pretwisted beam

The relationship between the second moment of areas,

$$\begin{aligned}
 I_{xx} &= I_{YY} \sin^2(\alpha_b \eta + \phi_i) + I_{XX} \cos^2(\alpha_b \eta + \phi_i) \\
 I_{yy} &= I_{YY} \cos^2(\alpha_b \eta + \phi_i) + I_{XX} \sin^2(\alpha_b \eta + \phi_i) \\
 I_{xy} &= \frac{(I_{YY} - I_{XX}) \sin 2(\alpha_b \eta + \phi_i)}{2}
 \end{aligned} \tag{3.7}$$

The relationship between u_1 and u , v_1 and v , being given by

$$u_1 = u + d_y \theta, \quad v_1 = v + d_x \theta \tag{3.8}$$

$$\begin{aligned}
 d_x &= d_X \cos(\alpha_b \eta + \phi_i) - d_Y \sin(\alpha_b \eta + \phi_i) \\
 d_y &= d_X \sin(\alpha_b \eta + \phi_i) + d_Y \cos(\alpha_b \eta + \phi_i)
 \end{aligned} \tag{3.9}$$

In a straight blade since there is no pretwist, so $\alpha_b = 0$, $\phi_i = 0$. These values are substituted in equations 3.7 and 3.9 then $d_x = d_X$, $d_y = d_Y$ and $I_{xx} = I_{XX}$, $I_{yy} = I_{YY}$, $I_{xy} = 0$ are obtained. Using this relations in the energy expressions strain, mass and geometric matrices of straight rotating blade are obtained. Then the free vibration analysis, static buckling and dynamic stability analysis are performed.

For a pretwisted blade with α_b , ϕ_i values. By taking these values into account the same procedure is applied to obtain strain, mass and geometric matrices of a pretwisted rotating beam. Then the free vibration analysis, static buckling and dynamic stability analysis are realized.

Assuming cubic polynomial expansions for u , v and linear for θ , w to be of the form

$$u = \sum_{r=0}^3 a_r \eta^r, \quad v = \sum_{r=0}^3 b_r \eta^r, \quad \theta = \sum_{r=0}^1 c_r \eta^r, \quad w = \sum_{r=0}^1 d_r \eta^r \tag{3.10}$$

and substituting into equations (3.5) and replacing a_r, b_r ($r = 0,1,2,3$) and c_r, d_r ($r = 0,1$) and after integration carried out then the strain energy expression of a stationary beam element becomes

$$V_s = \frac{EI_{xx}}{2\ell^3} \{q\}^T [k_{st}] \{q\} \quad (3.11)$$

The strain energy equation of a beam element due to rotation is given in equation (3.6). Using equation (3.8) and substituting the cubic polynomial expansion for u and v and linear expression for θ and w given in equation (3.10) into equation (3.6) and replacing the coefficients by a_r, b_r ($r = 0, 1, 2, 3$) and c_r, d_r ($r = 0, 1$) and after integration carried out it becomes

$$U_f = \frac{1}{2} \frac{\omega^2 \rho A \ell}{420} \{q\}^T [k_n] \{q\} \quad (3.12)$$

using equation (3.1), (3.11), (3.12) the strain energy equation becomes

$$V = \frac{EI_{xx}}{2\ell^3} \{q\}^T [k_{st}] \{q\} + \frac{1}{2} \frac{\omega^2 \rho A \ell}{420} \{q\}^T [k_n] \{q\} \quad (3.13)$$

where

$$[k_{st}] = [C_*]^T [k_s] [C_*]^{-1} \quad (3.14)$$

If strain energy due to rotation of equation (3.13) is written as

$$[k_n] = \frac{\omega^2 \rho A \ell}{420 EI_{xx}} [C_*]^T [k_r] [C_*]^{-1} \quad (3.15)$$

Then equation (3.13) can be written as

$$V = \frac{EI_{xx}}{2\ell^3} \{q\}^T [K_{e_b}] \{q\} \quad (3.16)$$

where

$$[K_{e_b}] = [C_*]^T [[k_s] + [k_r]] [C_*]^{-1} \quad (3.17)$$

Nodal coordinate vector

$$\{q\}^T = \begin{bmatrix} v_1 & \dot{v}_1 & u_1 & \dot{u}_1 & \theta_1 & w_1 & v_2 & \dot{v}_2 & u_2 & \dot{u}_2 & \theta_2 & w_2 \end{bmatrix} \quad (3.18)$$

and

In equation (3.17) $[C_*]^{-1}$ is a coefficient matrix and given in appendix 1. In equation (3.17) $[k_s]$ and $[k_r]$ are the stiffness matrices of non-rotating and rotating blade element respectively. By using appendix 3 whose elements are given as follows

$k_s = \text{zeros}(12,12)$

$k_s(3,3) = AA$

$k_s(3,4) = BB$

$k_s(3,7) = FF$

$k_s(3,8) = GG$

$k_s(4,4) = DD$

$k_s(4,7) = GG$

$k_s(4,8) = GG$

$k_s(7,7) = PP$

$k_s(7,8) = QQ$

$k_s(8,8) = RR$

$k_s(10,10) = CT \ A1$

$k_s(12,12) = CL \ A1$

$$\begin{aligned}
k_r(1,1) &= SS \\
k_r(1,2) &= SS/2 \\
k_r(1,3) &= SS/3 \\
k_r(1,4) &= SS/4 \\
k_r(1,5) &= CS \\
k_r(1,6) &= CS/2 \\
k_r(1,7) &= CS/3 \\
k_r(1,8) &= CS/4 \\
k_r(1,9) &= SS R15 + CS I15 \\
k_r(1,10) &= SS R16 + CS I16
\end{aligned}$$

$$\begin{aligned}
k_r(2,1) &= k_r(1,2) \\
k_r(2,2) &= SS/3 + \psi + A23 \\
k_r(2,3) &= SS/4 + \psi + A34 \\
k_r(2,4) &= SS/5 + \psi + A45 \\
k_r(2,5) &= CS/2 \\
k_r(2,6) &= CS/3 \\
k_r(2,7) &= CS/4 \\
k_r(2,8) &= CS/5 \\
k_r(2,9) &= H(\Lambda G25 + \Gamma FI25 + FS25) + SS R25 + CS I25 \\
k_r(2,10) &= H(\Lambda G26 + \Gamma FI26 + FS26) + SS R26 + CS I26
\end{aligned}$$

$$\begin{aligned}
k_r(3,1) &= k_r(1,3) \\
k_r(3,2) &= k_r(2,3) \\
k_r(3,3) &= (4/3)(\psi + A45) + SS/5 \\
k_r(3,4) &= 1.5(\psi + A56) + SS/6 \\
k_r(3,5) &= CS/3 \\
k_r(3,6) &= CS/4 \\
k_r(3,7) &= CS/5 \\
k_r(3,8) &= CS/6 \\
k_r(3,9) &= H(\Lambda G35 + \Gamma FI35 + FS35) + SS R35 + CS I35 \\
k_r(3,10) &= H(\Lambda G36 + \Gamma FI36 + FS36) + SS R36 + CS I36
\end{aligned}$$

$$k_r(4,1) = k_r(1,4)$$

$$k_r(4,2) = k_r(2,4)$$

$$k_r(4,3) = k_r(3,4)$$

$$k_r(4,4) = (9/5)(\psi + A67) + SS/7$$

$$k_r(4,5) = CS/4$$

$$k_r(4,6) = CS/5$$

$$k_r(4,7) = CS/6$$

$$k_r(4,8) = CS/7$$

$$k_r(4,9) = H(\Lambda G45 + \Gamma FI45 + FS45) + SS R45 + CS I45$$

$$k_r(4,10) = H(\Lambda G46 + \Gamma FI46 + FS46) + SS R46 + CS I46$$

$$k_r(5,1) = k_r(1,5)$$

$$k_r(5,2) = k_r(2,5)$$

$$k_r(5,3) = k_r(3,5)$$

$$k_r(5,4) = k_r(4,5)$$

$$k_r(5,5) = CC$$

$$k_r(5,6) = CC/2$$

$$k_r(5,7) = CC/3$$

$$k_r(5,8) = CC/4$$

$$k_r(5,9) = CC I15 + CS R15$$

$$k_r(5,10) = CC I16 + CS R16$$

$$k_r(6,1) = k_r(1,6)$$

$$k_r(6,2) = k_r(2,6)$$

$$k_r(6,3) = k_r(3,6)$$

$$k_r(6,4) = k_r(4,6)$$

$$k_r(6,5) = k_r(5,6)$$

$$k_r(6,6) = CC/3 + \psi + A23$$

$$k_r(6,7) = CC/4 + \psi + A34$$

$$k_r(6,8) = CC/5 + \psi + A45$$

$$k_r(6,9) = H(\Lambda Z25 + \Gamma TI25 + TS25) + CC I25 + CS R25$$

$$k_r(6,10) = H(\Lambda Z26 + \Gamma I26 + TS26) + CC I26 + CS R26$$

$$k_r(7,1)=k_r(1,7)$$

$$k_r(7,2)=k_r(2,7)$$

$$k_r(7,3)=k_r(3,7)$$

$$k_r(7,4)=k_r(4,7)$$

$$k_r(7,5)=k_r(5,7)$$

$$k_r(7,6)=k_r(6,7)$$

$$k_r(7,7)=(4/3)(\psi+A45)+CC/5$$

$$k_r(7,8)=1.5(\psi+A56)+CC/6$$

$$k_r(7,9)=H(\Lambda Z35+\Gamma TI35+TS35)+CC I35+CS R35$$

$$k_r(7,10)=H(\Lambda Z36+\Gamma TI36+TS36)+CC I36+CS R36$$

$$k_r(8,1)=k_r(1,8)$$

$$k_r(8,2)=k_r(2,8)$$

$$k_r(8,3)=k_r(3,8)$$

$$k_r(8,4)=k_r(4,8)$$

$$k_r(8,5)=k_r(5,8)$$

$$k_r(8,6)=k_r(6,8)$$

$$k_r(8,7)=k_r(7,8)$$

$$k_r(8,8)=(9/5)(\psi+A67)+CC/7$$

$$k_r(8,9)=H(\Lambda Z45+\Gamma TI45+TS45)+CC I45+CS R45$$

$$k_r(8,10)=H(\Lambda Z46+\Gamma TI46+TS46)+CC I46+CS R46$$

$$k_r(9,1)=k_r(1,9)$$

$$k_r(9,2)=k_r(2,9)$$

$$k_r(9,3)=k_r(3,9)$$

$$k_r(9,4)=k_r(4,9)$$

$$k_r(9,5)=k_r(5,9)$$

$$k_r(9,6)=k_r(6,9)$$

$$k_r(9,7)=k_r(7,9)$$

$$k_r(9,8)=k_r(8,9)$$

$$k_r(9,9)=H(\Lambda(Z55+G55)+\Gamma(TI55+FI55)+TS55+FS55)+CC I55+SS R55+2CSY55$$

$$k_r(9,10)=H(\Lambda(Z56+G56)+\Gamma(TI56+FI56)+TS56+FS56)+CC I56+SS R56+2CS Y56$$

$$k_r(10,1)=k_r(1,10)$$

$$k_r(10,2)=k_r(2,10)$$

$$k_r(10,3)=k_r(3,10)$$

$$k_r(10,4)=k_r(4,10)$$

$$k_r(10,5)=k_r(5,10)$$

$$k_r(10,6)=k_r(6,10)$$

$$k_r(10,7)=k_r(7,10)$$

$$k_r(10,8)=k_r(8,10)$$

$$k_r(10,9)=k_r(9,10)$$

$$k_r(10,10)=H(\Lambda(Z66+G66)+\Gamma(TI66+FI66)+TS66+FS66)+CC\ I66+SS\ R66+2CSY66$$

$$H = \frac{\omega^2 \rho A \ell}{EI_{xx}}$$

$$CC = -H \cos^2 \phi$$

$$SS = -H \sin^2 \phi$$

$$CS = H \cos \phi \sin \phi$$

$$\Gamma = \frac{r_2}{\ell} + k - 1$$

$$\Lambda = \sum_{k=2}^n \left(\frac{r_2}{\ell} + k - 0.5 \right)$$

$$\psi = \Lambda H$$

$$A_{23} = H(\Gamma/2 + 1/3)$$

$$A_{34} = H(\Gamma/3 + 1/4)$$

$$A_{45} = H(\Gamma/4 + 1/5)$$

$$A_{56} = H(\Gamma/5 + 1/6)$$

$$A_{67} = H(\Gamma/6 + 1/7)$$

3.2 External work done of a blade under axial loading

External work will be done when the beam is subjected to a periodic axial force $P(t)$. The additional energy V_p due to the periodic force can be expressed as (Chen & Chern, 1993)

$$V_p = \frac{P(t)}{2} \int_0^L \left(\left(\frac{\partial u_1}{\partial z} \right)^2 + \left(\frac{\partial v_1}{\partial z} \right)^2 \right) dz \quad (3.19)$$

This equation can be written in non-dimensional form for the k^{th} element by using equation (3.4)

$$V_p = \frac{P}{2\ell} \int_0^1 \left(\left(\frac{\partial u_1}{\partial \eta} \right)^2 + \left(\frac{\partial v_1}{\partial \eta} \right)^2 \right) d\eta \quad (3.20)$$

If the relationship between u_1 and u , v_1 and v being given by equation (3.8) is used and substituted the cubic polynomial expansion for u and v and linear expression for θ given in equation (3.10) into equation (3.20) and replaced the coefficients by a_r , b_r ($r = 0, 1, 2, 3$) and c_r ($r = 0, 1$) and after integration carried out then equation (3.20) becomes

$$V_p = \frac{P}{2\ell} \{q\}^T [K_{gb}] \{q\} \quad (3.21)$$

where

$$[K_{gb}] = [C_*]^T [k_g] [C_*]^{-1} \quad (3.22)$$

The details used to calculate the variables are given in appendix 3.

$$[k_g] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 420 & 420 & 420 & 0 & 0 & 0 & 0 & 420 & G25 & 420 & G26 & 0 & 0 \\ & & 560 & 630 & 0 & 0 & 0 & 0 & 420 & G35 & 420 & G36 & 0 & 0 \\ & & & 756 & 0 & 0 & 0 & 0 & 420 & G45 & 420 & G46 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 420 & 420 & 420 & 420 & Z25 & 420 & Z26 & 0 & 0 \\ & & & & & & 560 & 630 & 420 & Z35 & 420 & Z36 & 0 & 0 \\ & & & & & & & 756 & 420 & Z45 & 420 & Z46 & 0 & 0 \\ & & & & & & & & 420(G55+Z55) & 420(G56+Z56) & 0 & 0 \\ & & & & & & & & & 420(Z66+G66) & 0 & 0 \\ & & & & & & & & & & 0 & 0 \\ & & & & & & & & & & & 0 \end{bmatrix}$$

3.3 Kinetic energy for a rotating blade

The kinetic energy T of a blade vibrating in combined bending-bending-torsion mode is given as

$$T = \frac{\rho}{2} \int_0^L \left\{ I_{cg} \dot{\theta}^2 + A (\dot{u}_1^2 + \dot{v}_1^2 + \dot{w}^2) \right\} dz \quad (3.23)$$

By differating equations (3.8) with respect to time t , and inserting into equation (3.23), we get

$$\dot{u}_1 = \dot{u} + d_y \dot{\theta}, \quad \dot{v}_1 = \dot{v} + d_x \dot{\theta} \quad (3.24)$$

$$T = \frac{1}{2} \int_0^L \rho \left[I_{cg} \dot{\theta}^2 + A(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + A(d_x^2 + d_y^2) \dot{\theta}^2 + 2A(d_y \dot{u} \dot{\theta}) + 2A(d_x \dot{v} \dot{\theta}) \right] dz \quad (3.25)$$

$$I_{cf} = I_{cg} + AR^2 \quad (3.26)$$

where $R^2 = d_x^2 + d_y^2$

Using the equation (3.26) and writing equation (3.25) in non-dimensional form for the k^{th} element by using equation (3.4)

$$T = \frac{\rho \ell}{2} \int_0^1 I_{\text{cr}} \dot{\theta}^2 d\eta + \frac{\rho A \ell}{2} \int_0^1 (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) d\eta + \rho A \ell \int_0^1 d_Y \dot{u} \dot{\theta} d\eta + \rho A \ell \int_0^1 d_X \dot{v} \dot{\theta} d\eta \quad (3.27)$$

Substituting the cubic polynomial expansion for u and v and linear expression for θ and w given in equation (3.10) into equation (3.27) and replacing the coefficients a_r, b_r ($r = 0, 1, 2, 3$) and c_r, d_r ($r = 0, 1$) and after integration carried out then kinetic energy expression becomes

$$T = \frac{\rho A \ell}{420} \frac{1}{2} \{\dot{q}\}^T [M_b] \{\dot{q}\} \quad (3.28)$$

where

$$[M_b] = [C_*]^T [m_b] [C_*]^{-1} \quad (3.29)$$

$$[m_b] = \begin{bmatrix} 420 & 210 & 140 & 105 & 0 & 0 & 0 & 0 & A3S 420 & B3S 420 & 0 & 0 \\ & 140 & 105 & 84 & 0 & 0 & 0 & 0 & B3S 420 & C3S 420 & 0 & 0 \\ & & 84 & 70 & 0 & 0 & 0 & 0 & C3S 420 & D3S 420 & 0 & 0 \\ & & & 60 & 0 & 0 & 0 & 0 & D3S 420 & E3S 420 & 0 & 0 \\ & & & & 420 & 210 & 140 & 105 & AS 420 & BS 420 & 0 & 0 \\ & & & & & 140 & 105 & 84 & BS 420 & CS 420 & 0 & 0 \\ & & & & & & 84 & 70 & CS 420 & DS 420 & 0 & 0 \\ & & & & & & & 60 & DS 420 & ES 420 & 0 & 0 \\ & & & & & & & & TO 420 & TO 210 & 0 & 0 \\ & & & & & & & & & TO 140 & 0 & 0 \\ & & & & & & & & & & 420 & 210 \\ & & & & & & & & & & & 140 \end{bmatrix}$$

Symmetric

The details used to calculate the variables are given in appendix 3.

3.4 General dynamic stability equation for blades

Global mass and stiffness matrices are formed by using individual mass and stiffness matrices of each beam element. The dynamic response of a beam for a conservative system can be formulated by means of Lagrange's equation of motion in which the external forces are expressed in terms of time-dependent potentials, and then performing the required operations the entire system leads to the governing matrix equation of motion

$$[M]\{\ddot{q}\} + \left[[K_e] - P(t)[K_g] \right] \{q\} = 0 \quad (3.30)$$

The periodic axial force $P(t) = P_0 + P_1 \cos \Omega t$ where Ω is the disturbing frequency, the static and time dependent components of the load can be represented as a fraction of the fundamental static buckling load P^* hence putting $P(t) = \alpha P^* + \beta P^* \cos \Omega t$ equation (3.30) becomes

$$[M]\{\ddot{q}\} + \left[[K_e] - \alpha P^* [K_{gs}] \pm \frac{1}{2} \beta P^* \cos \Omega t [K_{gt}] \right] \{q\} = 0 \quad (3.31)$$

Where the matrices $[K_{gs}]$ and $[K_{gt}]$ in equation (3.31) reflect the influence of the static and time-dependent components of the load, respectively. Equation (3.31) represents a system of second order differential equation with periodic coefficients of the Mathieu- Hill type. From the theory of linear equations with periodic coefficients, the boundaries between stable and unstable solutions of equation (3.31) are formed by periodic solutions of period T and $2T$ where $T = 2\pi/\Omega$. It has been shown by Bolotin, (1964) that solutions with period $2T$ are the ones of greatest practical importance and that as a first approximation the boundaries of the principal regions of the dynamic instability can be determined from the equation

$$\left[[K_e] - \alpha P^* [K_{gs}] \pm \frac{1}{2} \beta P^* [K_{gt}] - \frac{\Omega^2}{4} [M] \right] \{q\} = 0 \quad (3.32)$$

The two matrices $[K_{gs}]$ and $[K_{gt}]$ will be identical if the static and time dependent component of the loads are applied in the same manner. If $[K_{gs}] \equiv [K_{gt}] \equiv [K_g]$ then equation (3.32) becomes

$$\left[[K_e] - \left(\alpha \pm \frac{1}{2} \beta \right) P^* [K_g] - \frac{\Omega^2}{4} [M] \right] \{q\} = 0 \quad (3.33)$$

In which $[K_e]$ represents the both stationary and rotating beam stiffness matrices. This equation represents the solution of the three related problems:

(i) Free vibration with $\alpha = 0, \beta = 0$, and $p = \Omega/2$ the natural frequency

$$[[K_e] - p^2 [M]] \{q\} = 0 \quad (3.34)$$

(ii) Static stability with $\alpha = 1, \beta = 0$ and $\Omega = 0$

$$[[K_e] - P^* [K_g]] \{q\} = 0 \quad (3.35)$$

(iii) Dynamic stability when all terms are present

$$\left[[K_e] - \left(\alpha \pm \frac{1}{2} \beta \right) P^* [K_g] - \frac{\Omega^2}{4} [M] \right] \{q\} = 0 \quad (3.36)$$

3.5 Results

3.5.1 For straight blades

The finite element model is very effective for the dynamic analysis of rotating asymmetrical cross section blading. This element exhibits a fast convergence characteristics.

The finite element model is applied to the vibration analysis and static buckling analysis of nonrotating and rotating asymmetrical straight blades in the first instance to compare the results obtained with those of other investigators in order to accuracy of the program. Then the instability regions are found for various parameters.

Table 3.1 The physical properties of the blade used in the analysis

I_{xx} (m^4)	I_{yy} (m^4)	L_b (m)	E (Pa)	ρ (kg/m^3)	A (m^2)	C(Nm^2/rad)
34,96e-12	2,792e-9	0,1524	2,139e11	7859	58,97e-6	9,14

Table 3.2 Comparison of natural frequency results for a straight uniform blade

		Analytical results (Hz) (Carnegie & Dawson, 1969)		Present results (Hz) (8 Elements)			Exp. reslt. (Carnegie &Dawson 1969)
Mode	Main mode shape for uncoupled motion	Uncoupled $d_x=0$ $d_y=0$	Coupled $d_x=0,193e-3$ m $d_y=1,193e-3$ m	Uncoupled $d_x=0$ $d_y=0$	Coupled $d_x=0,193e-3$ m $d_y=1,193e-3$ m	Coupled $d_x=0,193e-3$ m $d_y=2,54e-3$ m	
1	YY Bending	96,9	96,9	96,78	96,78	96,78	97
2	YY Bending	607	606,5	606,6	606,55	606,55	610
3	XX Bending	869	841,2	865,02	843,52	791,45	790
4	Torsion	1048,5	1072,9	1053,7	1079,12	1144,47	1102
5	YY Bending	1699	1699	1699,3	1699,21	1699,21	1693

Table 3.2 shows the comparison between the theoretical and experimental results obtained by Carnegie & Dawson, (1969) and the results obtained by the authors of an aerofoil cross section blading under non-rotating conditions. It can be seen that obtained results agree very closely with the analytical as well as experimental results.

As can be observed from table 3.2 the coupling effect is not significant in the first three YY bending modes, because the value of d_x is small. The frequency of the first XX bending dominated coupled mode has been reduced considerably from the value of the uncoupled mode frequency, because this mode is affected by d_y that is much larger than d_x . On the other hand, the frequency of the first torsional dominated mode has increased from the value of the uncoupled mode frequency. Therefore, the effect of asymmetry is such that the natural frequencies of bending vibration decrease and natural frequencies of the torsional vibration increase (Choi & Chou, 2001).

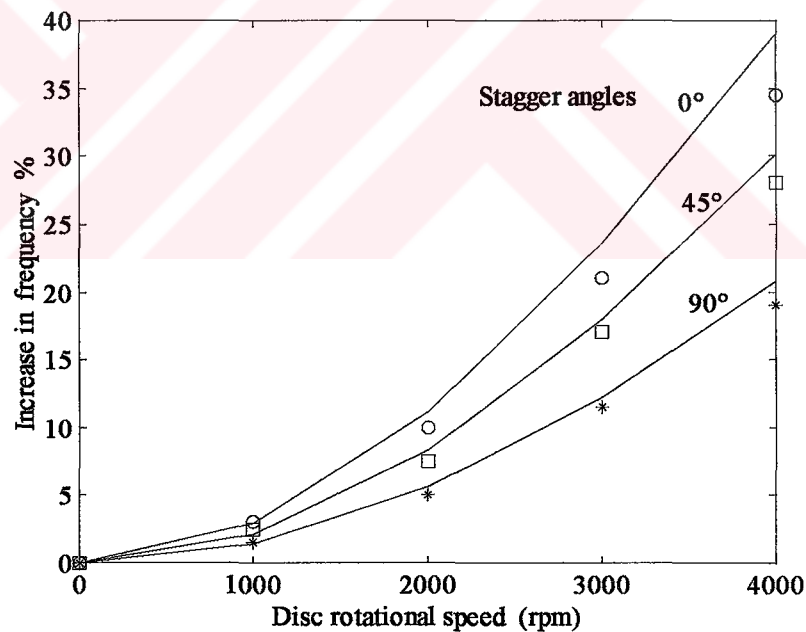


Figure 3.8 Percentage increase in frequency against disc rotational speed (Exp. results $\circ$, $\square$, $*$ stagger angles 0° , 45° , 90° Carnegie, Stirling & Fleming, (1966), — Present results)

Table 3.3 Buckling load parameters (λ_b) of an aerofoil cross section cantilever blade

		Analytical results (Yokoyama, 1988)	Present results (8 elements)		
Mode	Main buckling mode shape	Uncoupled $d_x=0, d_y=0$	Uncoupled $d_x=0, d_y=0$	Coupled $d_x=0,193e-3$ m $d_y=1,193e-3$ m	Coupled $d_x=0,193e-3$ m $d_y=2,54e-3$ m
1	YY Bending	2,467	2,467	2,467	2,467
2	YY Bending	22,21	22,21	22,20	22,20
3	YY Bending	61,69	61,76	61,75	61,75
4	YY Bending	120,9	121,44	121,43	121,43
5	XX Bending	199,9	197,11	195,18	188,68

Figure 3.8 shows the increase in the frequencies of the first mode of vibration of an aerofoil blade as the speed of rotation increases. The figure also shows the comparison of theoretical and experimental results. In the theoretical results the fundamental frequency of vibration of the blade was 96,78 Hz but in the experimental results the fundamental frequency of the same or identical blade was noted to be 106 Hz. This explains adequately the difference between the experimental results and authors' theoretical results (Thomas & Sabuncu, 1979).

The coupling effect is not significant in the first four YY bending static buckling loads modes. The critical static buckling load of the fifth XX bending dominated coupled mode has been reduced considerably from the value of the uncoupled mode, because this mode is affected by the distance d_y which is much larger than d_x . Therefore, the effect of asymmetry is such that the static buckling loads of XX bending mode decrease as in the case of natural frequency as shown in table 3.3.

Figure 3.9 shows the change in mode shapes of buckling in the first five modes. Figure 3.10 shows the effect of coupling on the mode shapes of buckling. The coupling has no effect on the buckling loads along the YY axis but the mode shapes give component (torsion) along the axis.

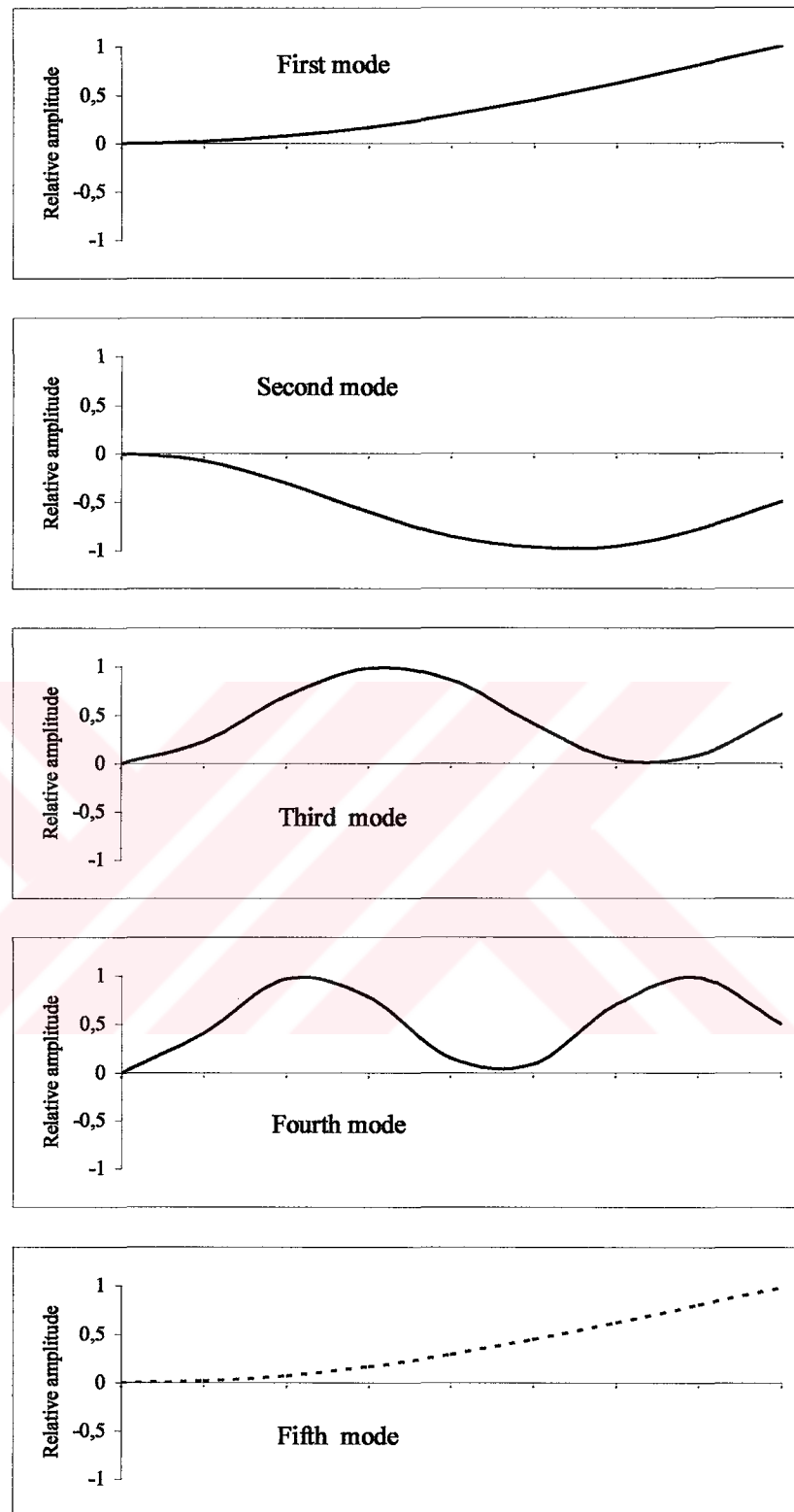


Figure 3.9 Buckling mode shapes for an aerofoil blade ($d_x=0$, $d_y=0$, $\alpha_p=0^\circ$, $\phi=0^\circ$, $\omega=0$ rpm uncoupled mode, —YY, ---- XX)

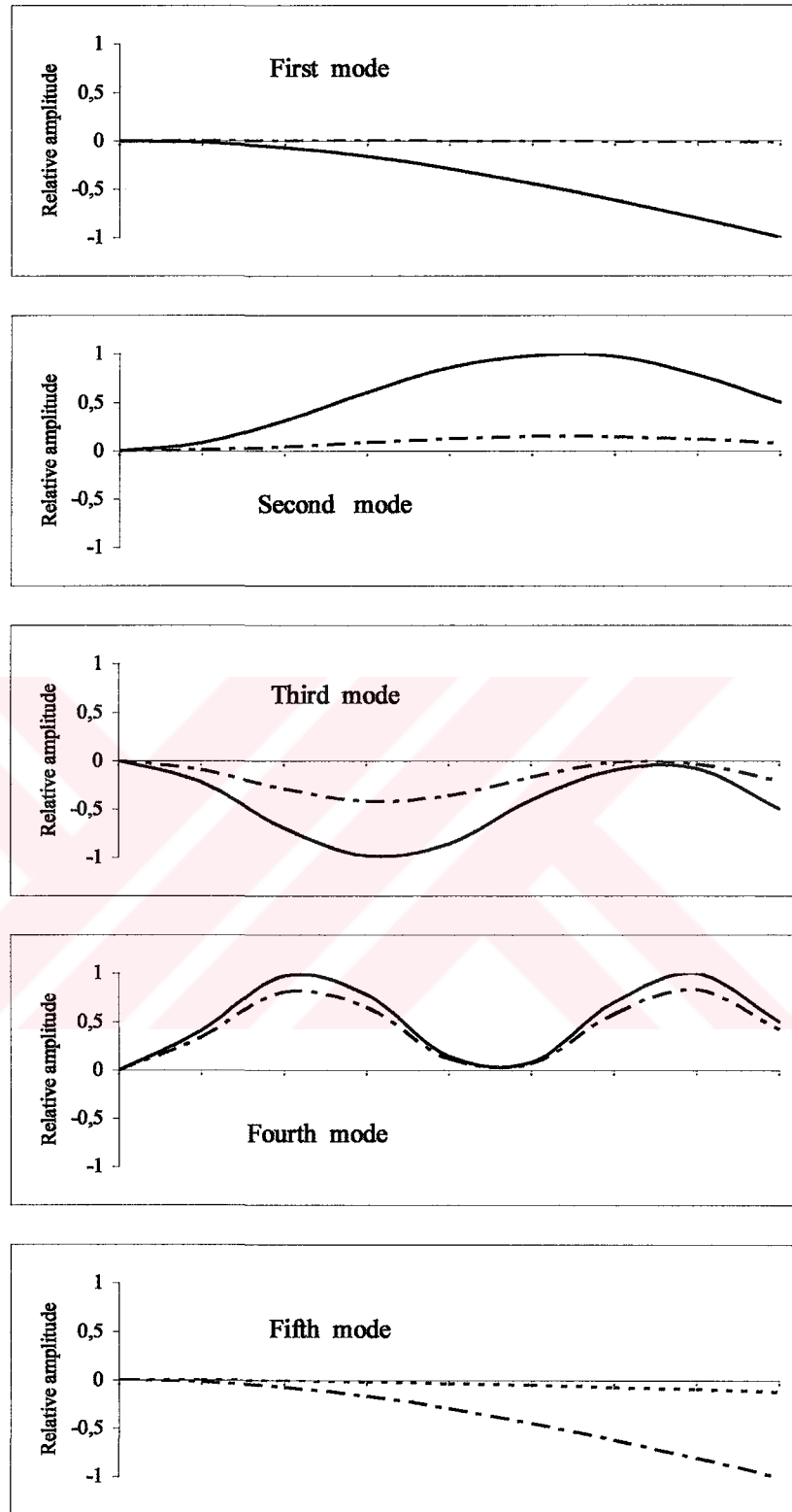


Figure 3.10 Buckling mode shapes of an aerofoil blade ($d_x=0,193e-3$ m, $d_y=1,1938e-3$ m, $\alpha_p=0^\circ$, $\phi=0^\circ$, $\omega=0$ rpm, coupled mode, — YY, ---- XX, -.-.- θ)

Table 3.4 Buckling load parameters (λ_b) against rotational speed for various d_x and d_y distances

$d_x=0, d_y=0, \phi=0^\circ$					
Mode	$\omega=0$ rpm	$\omega=2000$ rpm	$\omega=4000$ rpm	$\omega=6000$ rpm	$\omega=8000$ rpm
1	2,467	3,01	4,61	7,07	10,18
2	22,21	23,03	25,54	29,83	35,96
3	61,76	62,60	65,14	69,42	75,47
4	121,44	122,29	124,85	129,12	135,14
5	197,11	197,39	198,26	199,7	201,72
$d_x=0,193e-3$ m, $d_y=1,1938e-3$ m, $\phi=0^\circ$					
1	2,467	3,01	4,61	7,07	10,18
2	22,20	23,03	25,54	29,83	35,96
3	61,75	62,60	65,14	69,41	75,47
4	121,43	122,28	124,83	129,11	135,13
5	195,18	195,47	196,33	197,78	199,8
$d_x=0,193e-3$ m, $d_y=2,54e-3$ m, $\phi=0^\circ$					
1	2,467	3,01	4,61	7,07	10,18
2	22,21	23,03	25,54	29,83	35,96
3	61,76	62,60	65,14	69,41	75,47
4	121,43	122,28	124,83	129,11	135,13
5	188,68	188,97	189,83	191,27	193,26

Table 3.4 shows the increase in the buckling load parameters of the five buckling modes of an aerofoil blade as the speed of rotation increases. It is seen that the critical buckling loads increase with an increase in the rotational speed for all d_x and d_y distances, but in the fifth mode the rate of increase decreases as the d_y distance increase.

Table 3.5 Buckling load parameters (λ_b) against rotational speed for various stagger angles.

$d_x=0,193e-3$ m, $d_y=1,1938e-3$ m, $\phi=0^\circ$					
Mode	$\omega=0$ rpm	$\omega=2000$ rpm	$\omega=4000$ rpm	$\omega=6000$ rpm	$\omega=8000$ rpm
1	2,467	3,01	4,61	7,07	10,18
2	22,20	23,03	25,54	29,83	35,96
3	61,75	62,60	65,14	69,41	75,47
4	121,43	122,28	124,83	129,11	135,13
5	195,18	195,47	196,33	197,78	199,8
$d_x=0,193e-3$ m, $d_y=1,1938e-3$ m, $\phi=45^\circ$					
1	2,467	2,88	4,11	6,07	8,64
2	22,20	22,9	24,99	29,49	33,4
3	61,75	62,57	65,01	69,11	74,88
4	121,43	122,26	124,75	128,92	134,78
5	195,18	195,6	196,88	199	201,99
$d_x=0,193e-3$ m, $d_y=1,1938e-3$ m, $\phi=90^\circ$					
1	2,467	2,75	3,6	4,99	6,88
2	22,20	22,77	24,46	27,25	31,09
3	61,75	62,54	64,89	68,83	74,34
4	121,43	122,24	124,67	128,73	134,45
5	195,18	195,74	197,41	200,2	204,09

Table 3.5 shows the effect of stagger angle on the critical buckling load of vibration at various speeds of rotation. It can be seen that the critical buckling load increases for all stagger angles as the speed of rotation increases, but the rate of increase, in the first four modes, decreases as the stagger angle increases whereas in the fifth mode critical buckling load parameter increases as the stagger angle increases.

Figure 3.11 shows the accuracy of the present finite element model for the dynamic instability regions for first and second modes. As seen from the figure, the agreement between the results of Chen & Ku, (1990) and the present results is very good. So the present method can be applied to study the effects of coupling, rotational speed, stagger angle, disc radius on the dynamic stability regions.

Figure 3.12 shows the effect of coupling caused by the distance of center of flexure from the centroid on the regions of dynamic instability. The first and second regions don't change for various center of flexure and centroid distances. The third region is affected by d_Y and as the value of d_Y increases the regions shift to the left towards static vertical axis. Thus the effect is one of the destabilization. The fourth region shifted to the right with an increase in d_Y and it makes the blade more stable.

Figure 3.13 shows the effect of stagger angle for various rotational speeds on the regions of dynamic instability for the first mode. As stagger angle increases the regions are shifted towards the static buckling axis, and the width of these regions increases, thus the effect of increase in stagger angle makes the beam more sensitive to periodic excitation. Similar cases can be seen for the other modes.

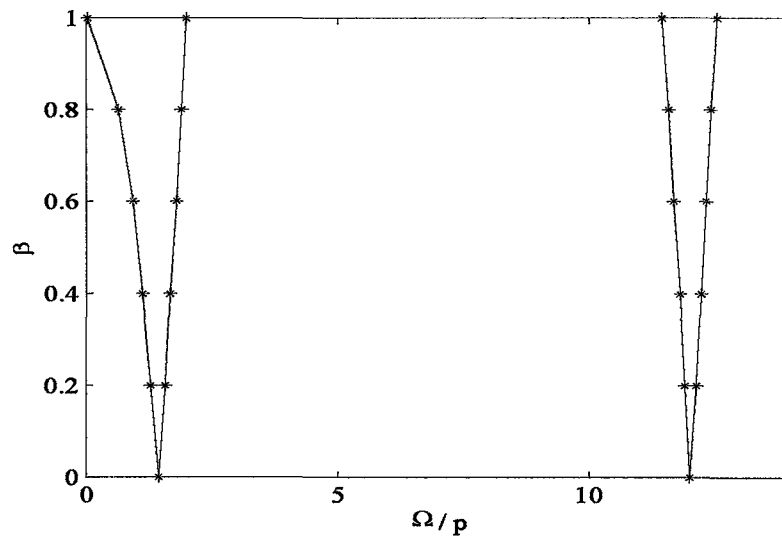


Figure 3.11 Comparison of dynamic instability regions (* Present, — Chen & Ku (1990) , $\alpha=0.5$ $d_X=0$, $d_Y=0$)

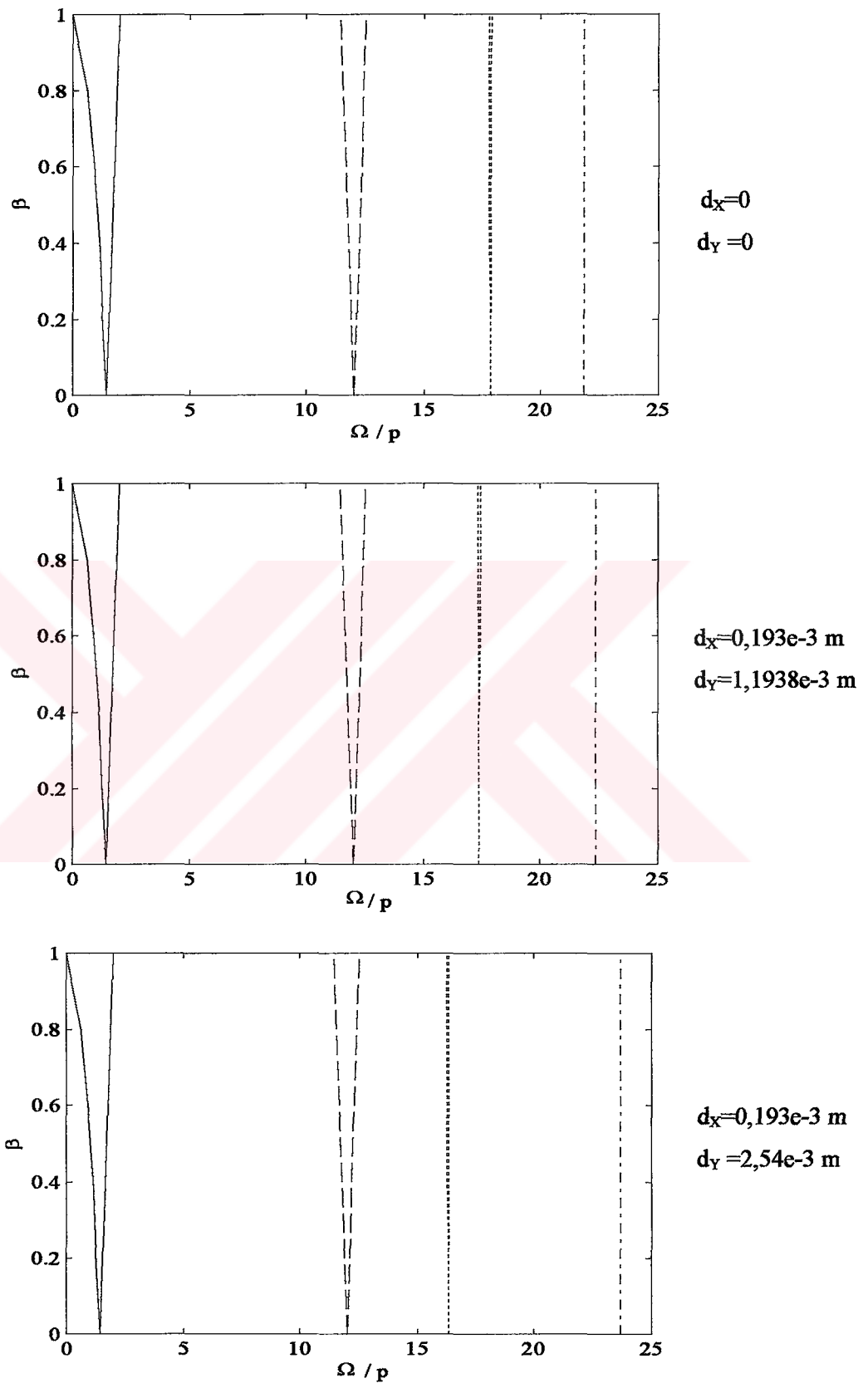


Figure 3.12 Effect of coupling on the regions of dynamic instability of an aerofoil blade

($\alpha=0,5$, $\phi=0^\circ$, $\omega=0$, — first mode, --- second mode, ... third mode, -.-.- fourth mode)

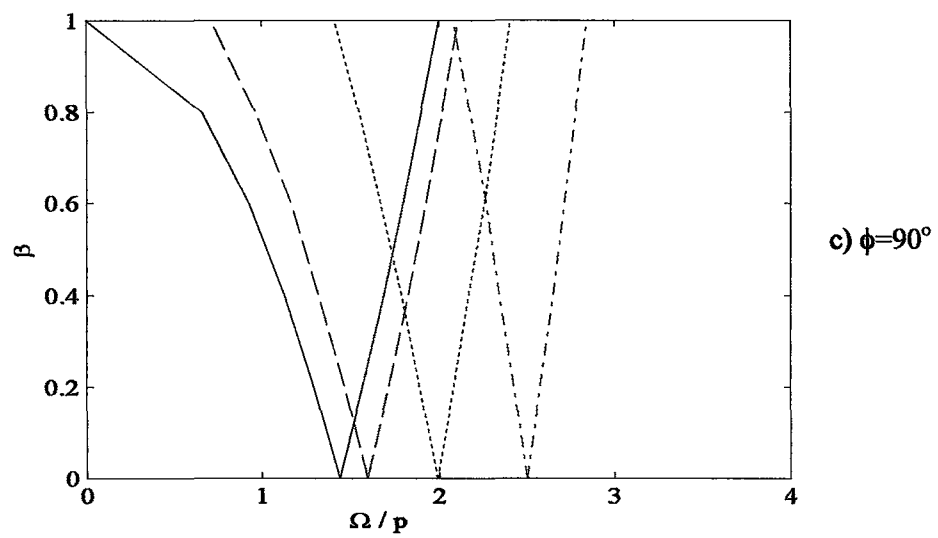
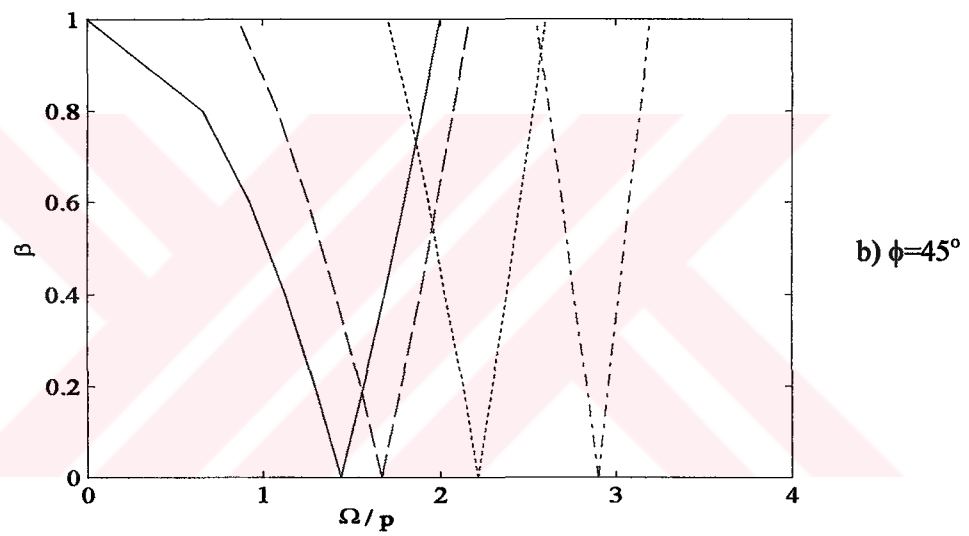
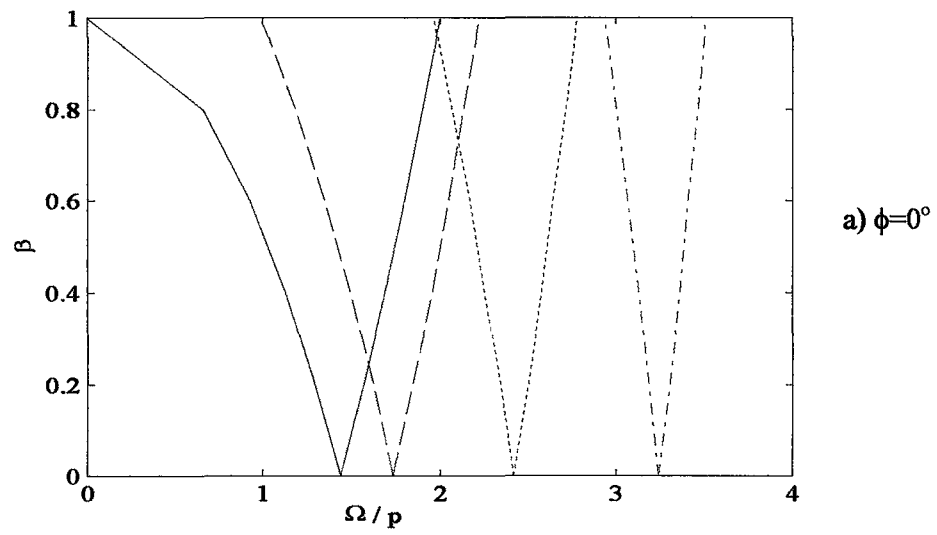


Figure 3.13 Effect of stagger angle on the regions of the dynamic instability of an aerofoil blade
 ($\alpha=0,5$ — $\omega=0$, — — — $\omega=2000$, ····· $\omega=4000$, — · — $\omega=6000$ rpm)

Figure 3.14 shows the rotational speed effect on the regions of dynamic instability of an aerofoil cross section straight blade. With an increase in rotational speed, the instability regions shift to the right and the widths of the regions are decreased. Obviously, increase in the rotational speed increases the stiffness of the rotating blade and makes the blade more stable.

Figure 3.15 shows the effect of disc radius on the dynamic instability region of an aerofoil cross section straight blade. As the disc radius increases the regions are shifted towards the right and the widths of these regions are decreased. As a result, an increase in the disc radius increases the potential energy of the beam. Thus it makes the blade more stable.

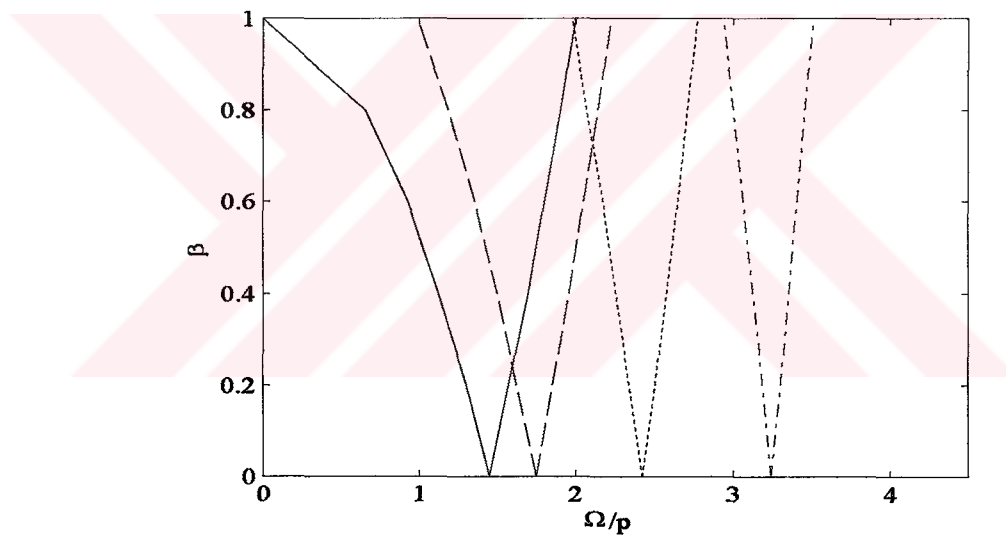


Figure 3.14 Effect of rotation on the regions of dynamic instability of an aerofoil blade
($\alpha=0,5$ $\phi=0^\circ$, — $\omega=0$, ---- $\omega=2000$, ... $\omega=4000$, -.-.- $\omega=6000$ rpm)

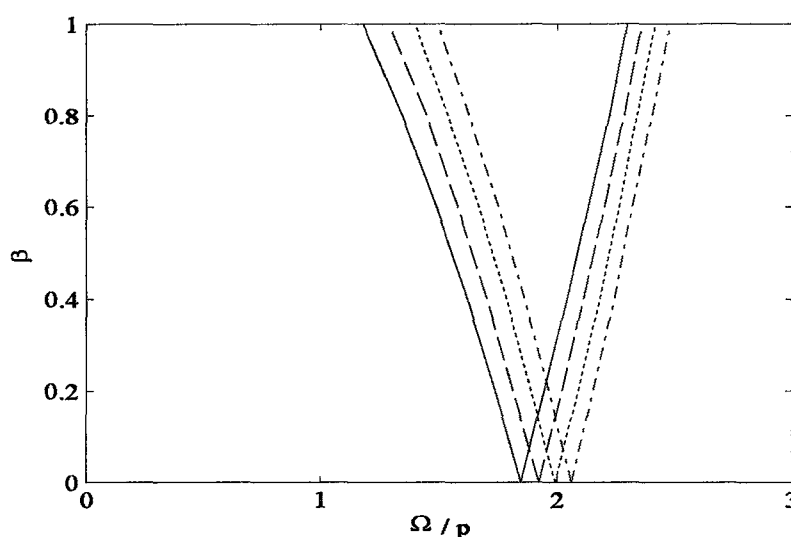


Figure 3.15 Effect of disc radius on the regions of dynamic instability of an aerofoil blade
 ($\alpha=0.5$, $\phi=0^\circ$, — $r=0$, - - $r=0.0254$, ... $r=0.0508$, - · - $r=0.0762$ m)

3.5.2 For pretwisted blades

In this section static and dynamic stability and free vibration analysis of asymmetric cross section rotating and non-rotating pretwisted blades were investigated. As a first step, convergence characteristic study of the developed element was carried out.

Figure 3.16 and figure 3.17 show the convergence characteristic of frequencies and buckling load parameters as the element number increases. It can be seen from the figures that the convergence is very rapid for all cases. Even though four elements seem to be sufficient to get accurate results, throughout this investigation eight elements were employed.

In general the modes of vibration of pretwisted asymmetric aerofoil cross section blades are of coupled bending-bending-torsion types. For the particular cross section beam used in this study the coupling between bending and torsion is very negligible in the first and second modes and vibrations bending-bending types. However, the third, fourth and fifth modes are coupled bending-bending-torsion types as

mentioned by Carnegie & Dawson, (1971) and Thomas & Sabuncu, (1979). These particular modes are also considerably affected by the distance of center of flexure from the centroid.

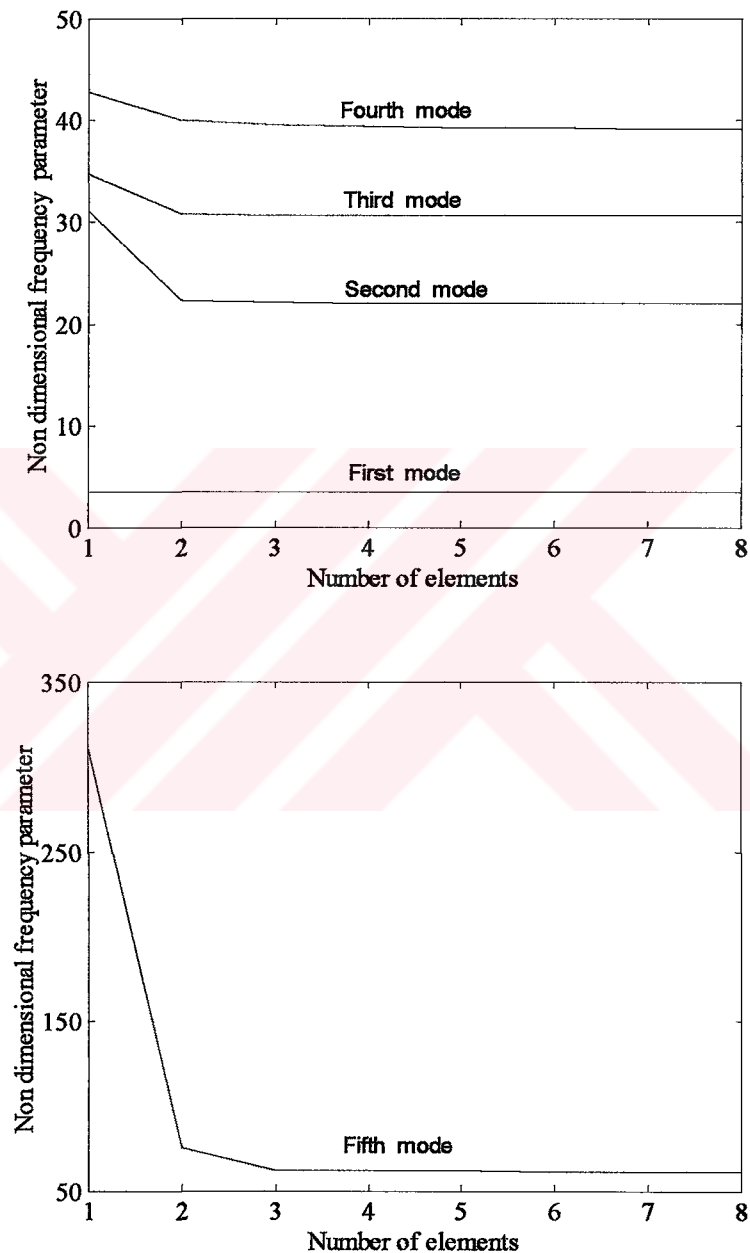


Figure 3.16 Converges characteristics of non-dimensional frequency parameter of a straight blade for the first five modes. ($d_x=0,193e-3$ m, $d_y=1,1938e-3$ m, $\alpha_p=0^\circ$, $\phi=0^\circ$, $\omega=0$ rpm)

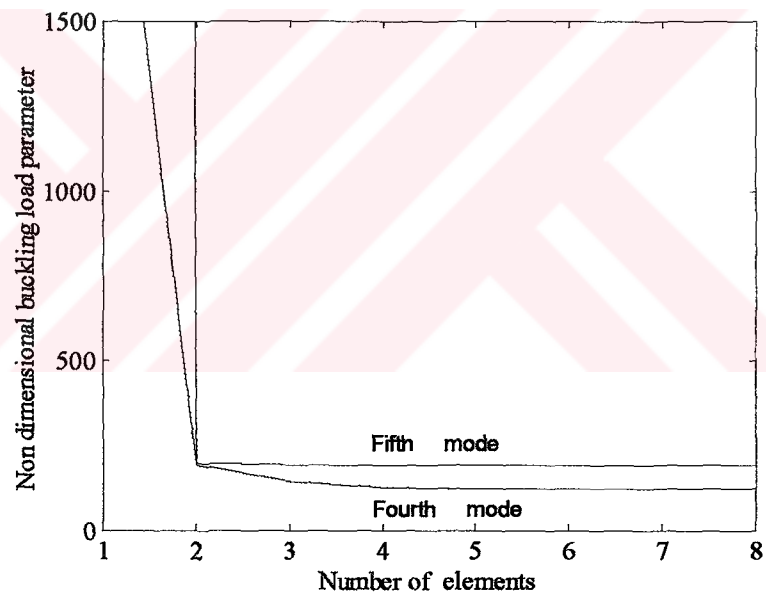
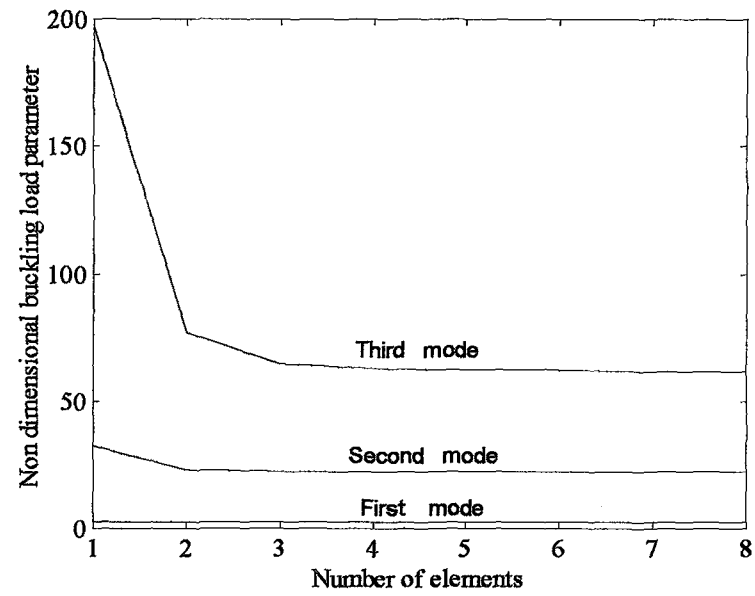


Figure 3.17 Converges characteristics of non-dimensional buckling parameter of a straight blade for the first five modes. ($d_x=0,193e-3$ m, $d_y=1,1938e-3$ m, $\alpha_p=0^\circ$, $\phi=0^\circ$, $\omega=0$ rpm)

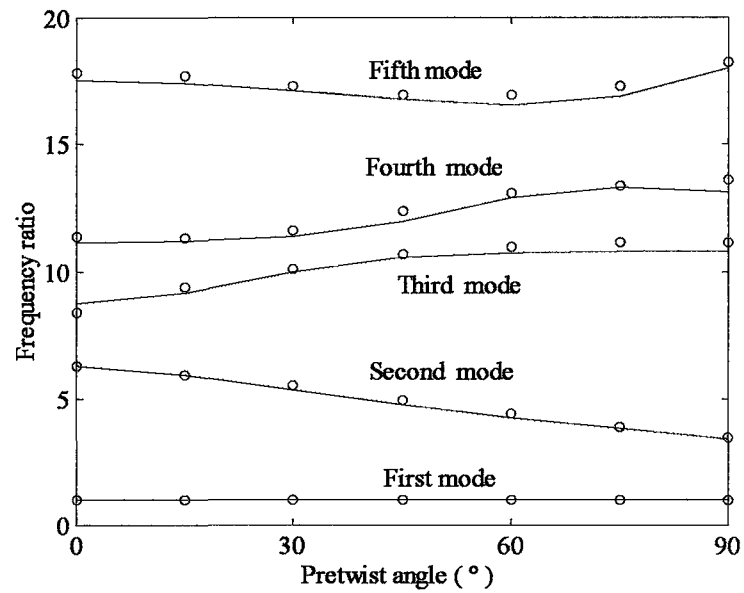


Figure 3.18 Frequency ratio p/p_0 against different pretwist angles. (— Present, (o) Carnegie & Dawson (1971), $\phi=0^\circ$, $\omega=0$ rpm)

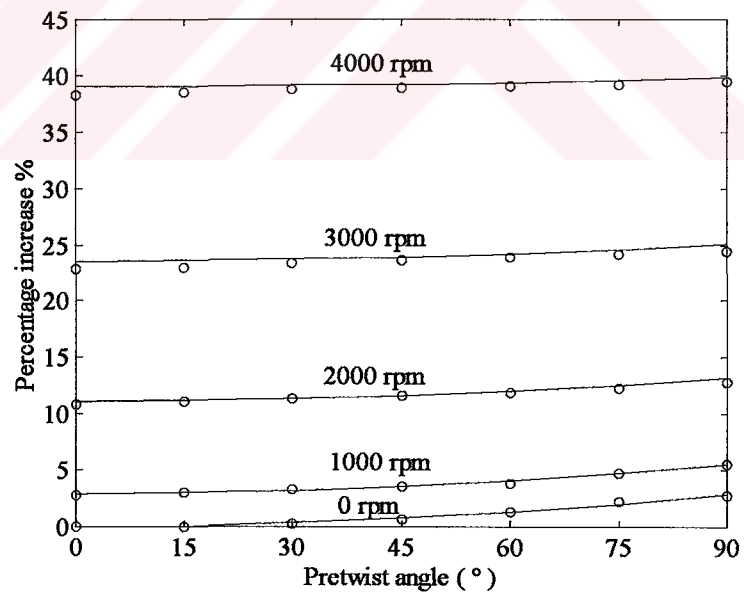


Figure 3.19 Percentage frequency rise against pretwist angle for various disc rotational speeds. (— Present, (o) Thomas & Sabuncu (1979), $\phi=0^\circ$)

Table 3.6 Buckling load parameters (λ_b) against pretwist angle for various d_x and d_y distances

$d_x = 0, d_y = 0$				
Mode	$\alpha_p=0^\circ$	$\alpha_p=30^\circ$	$\alpha_p=60^\circ$	$\alpha_p=90^\circ$
1	2,467	2,492	2,569	2,697
2	22,209	20,366	16,518	12,893
3	61,76	60,705	57,954	54,085
4	121,448	119,45	115,92	112,05
5	197,113	199,4	199,4	198,66
$d_x = 0,193e-3 \text{ m}, d_y = 1,1938e-3 \text{ m}$				
1	2,467	2,492	2,568	2,696
2	22,209	20,359	16,51	12,884
3	61,756	60,663	57,845	53,998
4	121,433	119,306	115,57	111,68
5	195,18	198,94	198,53	197,76
$d_x = 0,193e-3 \text{ m}, d_y = 2,54e-3 \text{ m}$				
1	2,467	2,492	2,567	2,695
2	22,209	20,332	16,478	12,849
3	61,756	60,478	57,462	53,709
4	121,435	118,67	114,292	110,393
5	188,67	197,035	195,138	194,661

Figure 3.18 shows a comparison between the present results and those of (Carnegie & Dawson, 1971) for aerofoil blades with various pretwist angles. Agreement between the two theoretical results is good. The figure also illustrates the frequency ratios (p/p_0) of vibration resulting from the coupling of deflections along the two principal axes and the torsional deflections as pretwist angle increases.

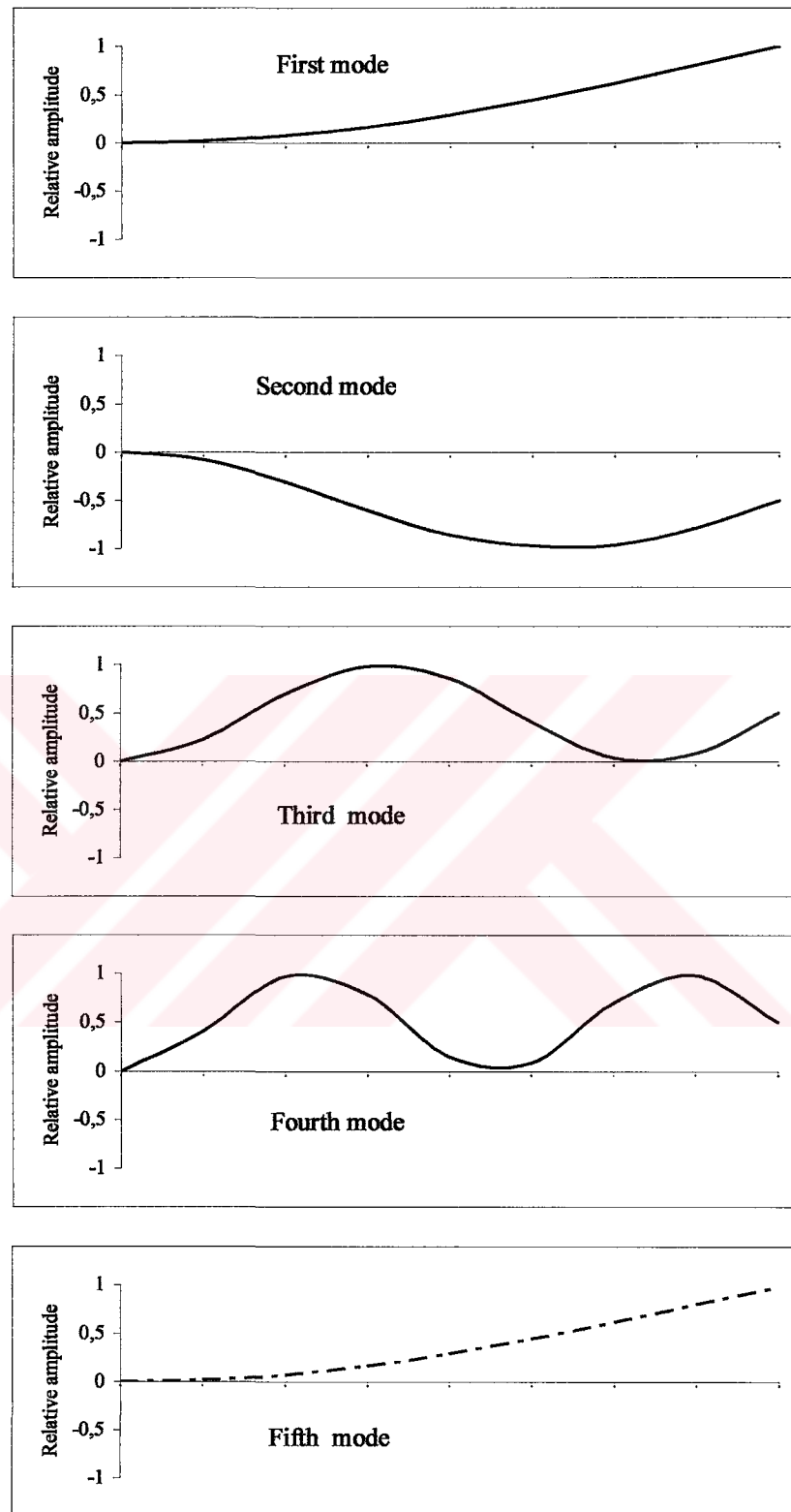


Figure 3.20 Buckling mode shapes for an aerofoil blade ($d_x=0$, $d_y=0$, $\alpha_p=0^\circ$, $\phi=0^\circ$, $\omega=0$ rpm uncoupled mode, — YY, - - - - XX)

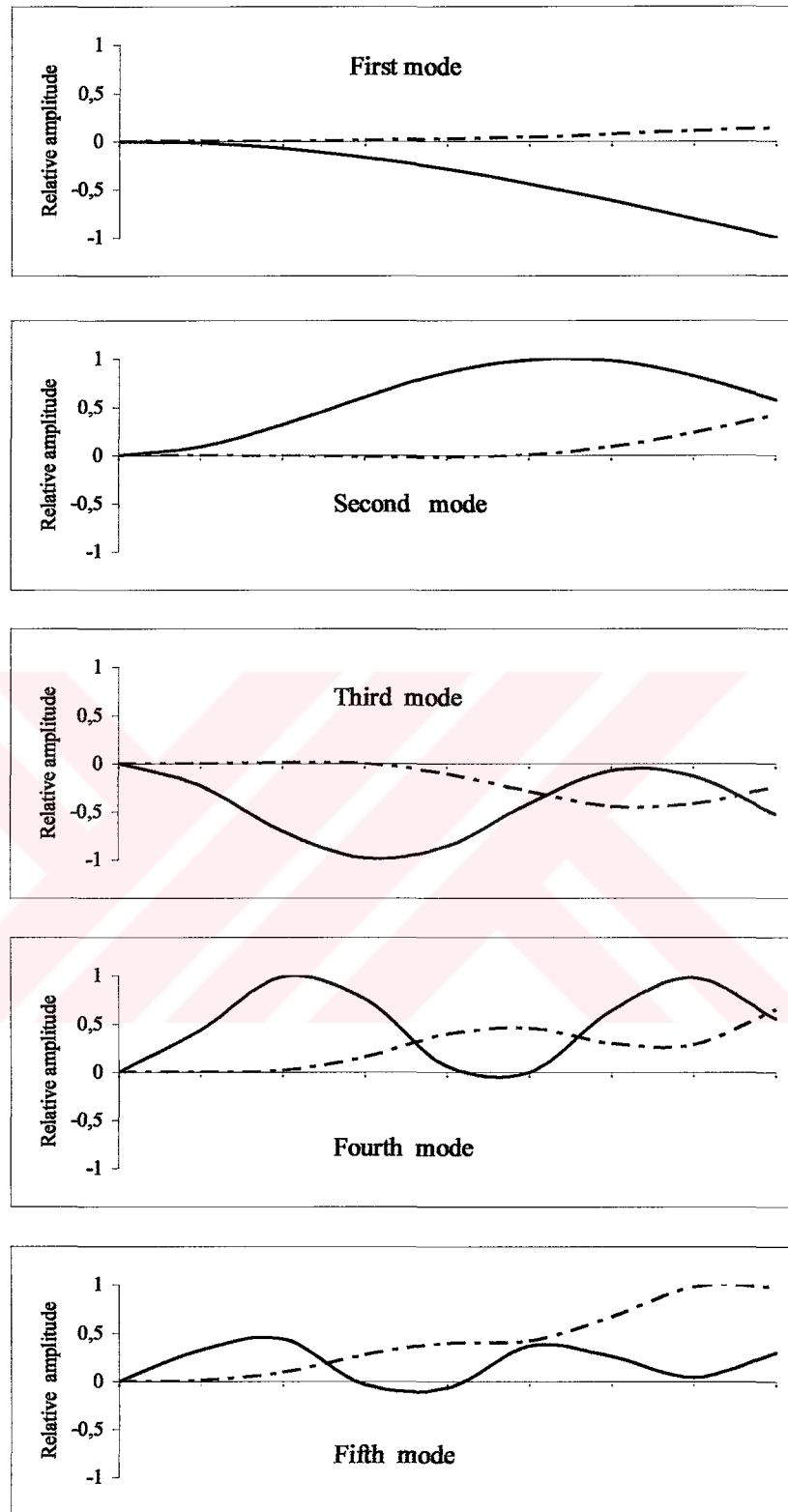


Figure 3.21 Buckling mode shapes for an aerofoil pretwisted blade ($d_x=0$, $d_y=0$, $\alpha_p=30^\circ$, $\phi=0^\circ$, $\omega=0$ rpm, — YY, ---- XX)

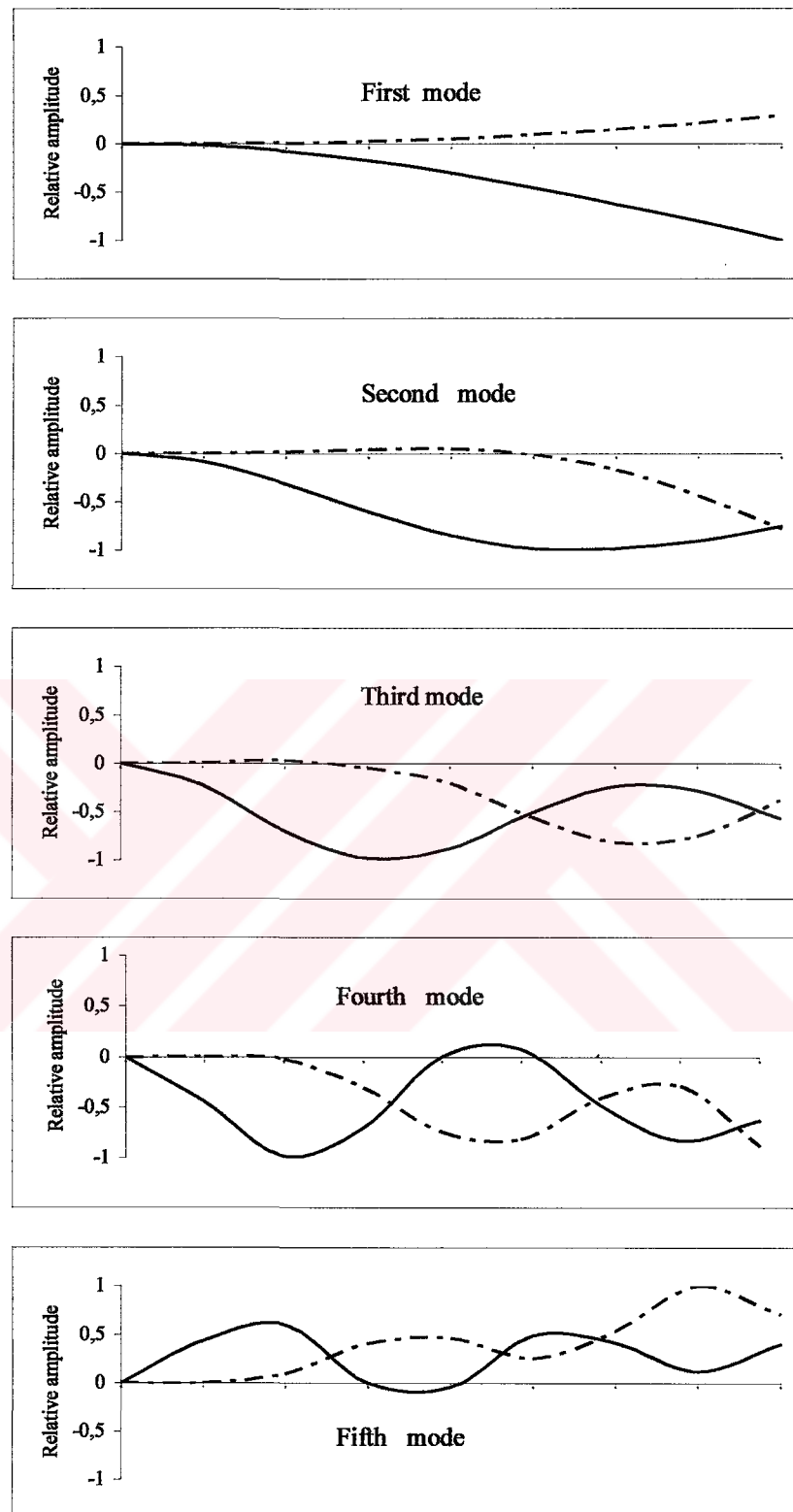


Figure 3.22 Buckling mode shapes for an aerofoil pretwisted blade ($d_x=0$, $d_y=0$, $\alpha_p=60^\circ$, $\phi=0^\circ$, $\omega=0$ rpm, — YY, - - - XX)

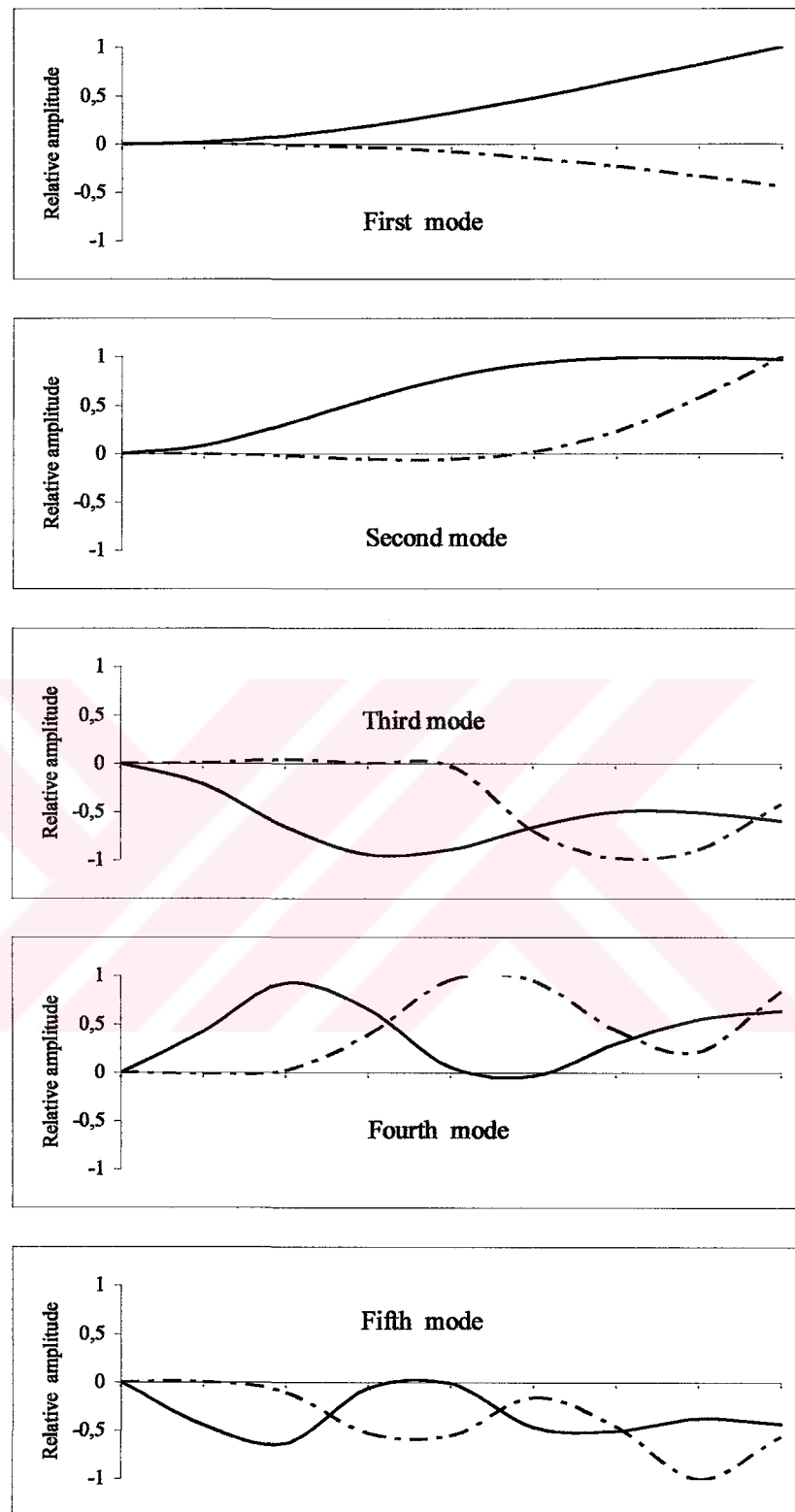


Figure 3.23 Buckling mode shapes for an aerofoil pretwisted blade ($d_X=0$, $d_Y=0$, $\alpha_p=90^\circ$, $\phi=0^\circ$, $\omega=0$ rpm, — YY, - - - XX)

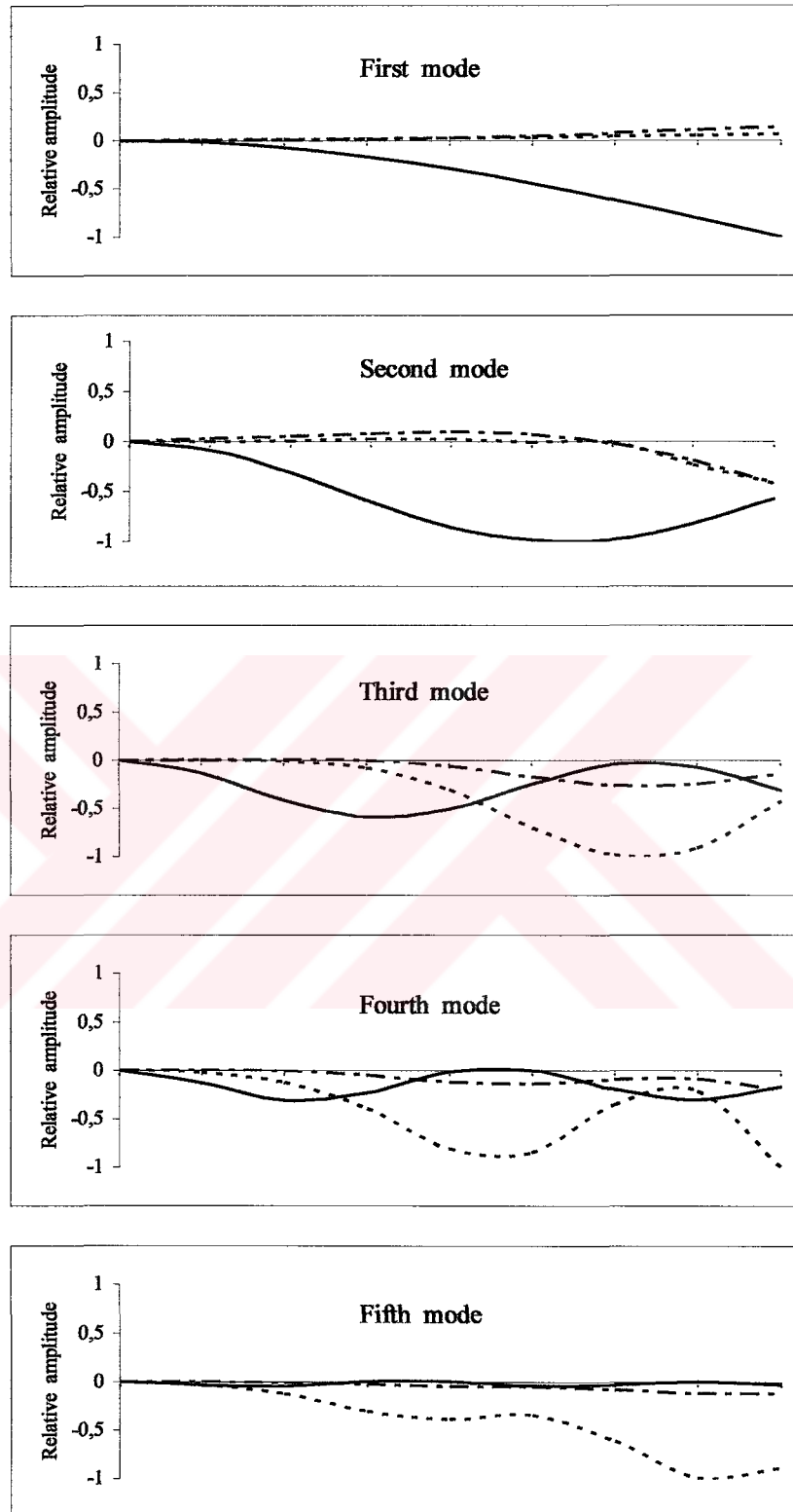


Figure 3.24 Buckling mode shapes for an aerofoil pretwisted blade ($d_x=0,193e-3$, $d_y=1,1938e-3$ m, $\alpha_p=30^\circ$, $\phi=0^\circ$, $\omega=0$ rpm, — YY, - - - XX, . . . 00)

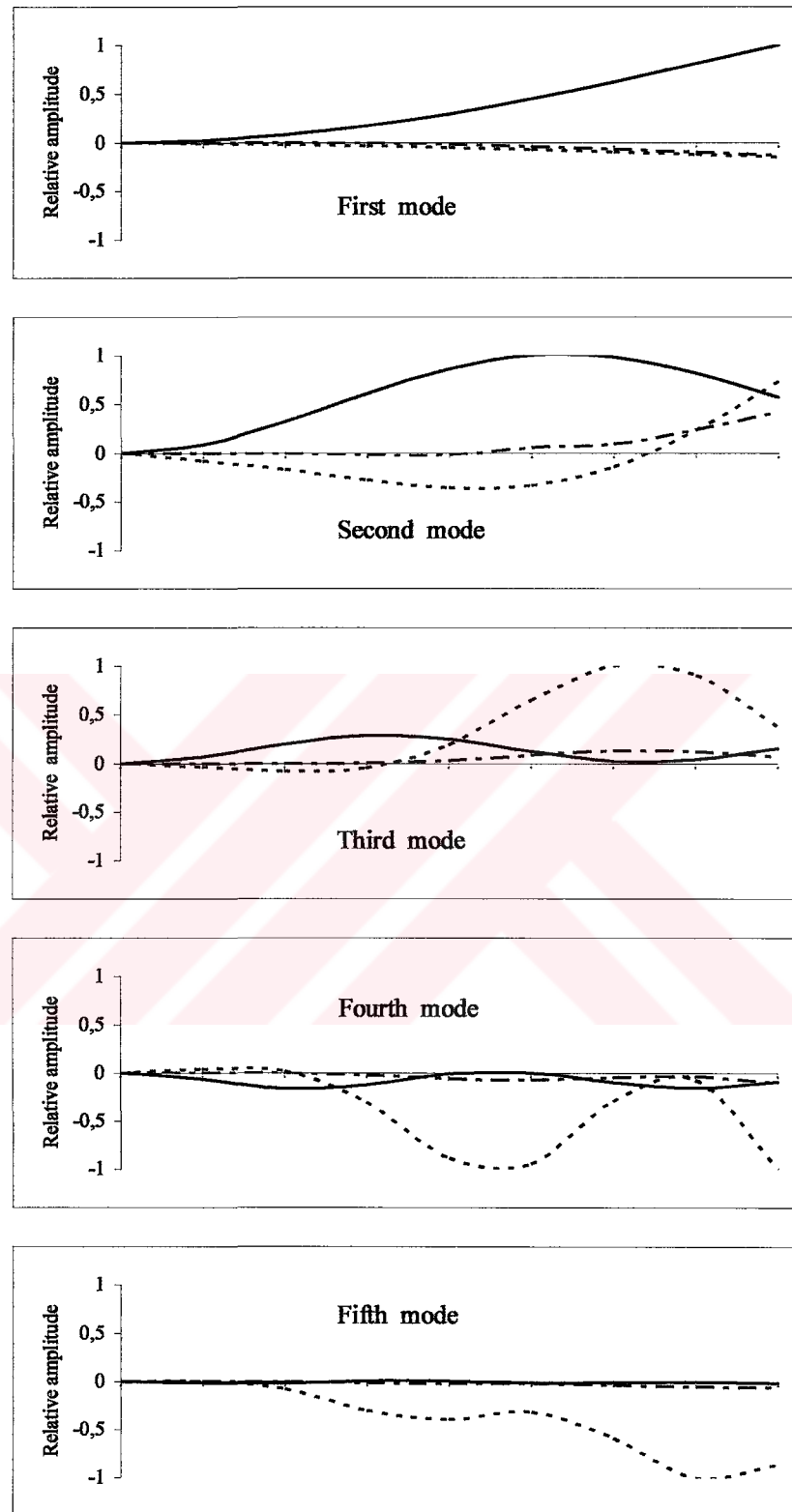
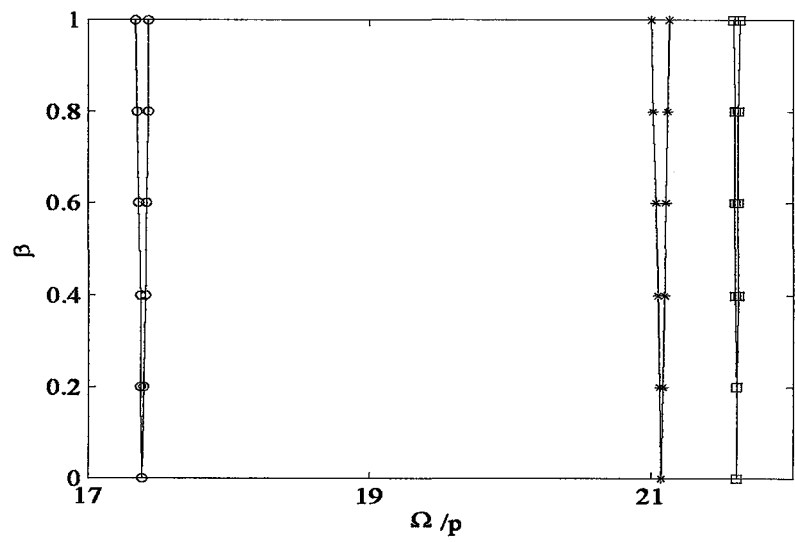
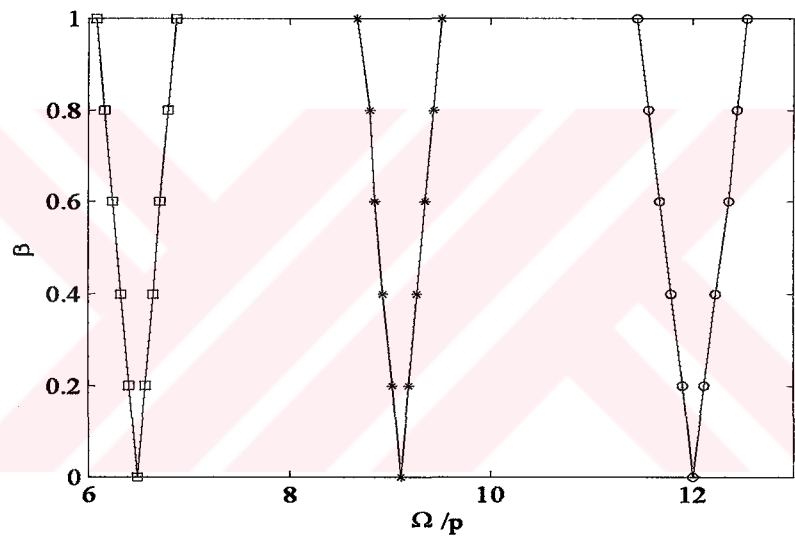
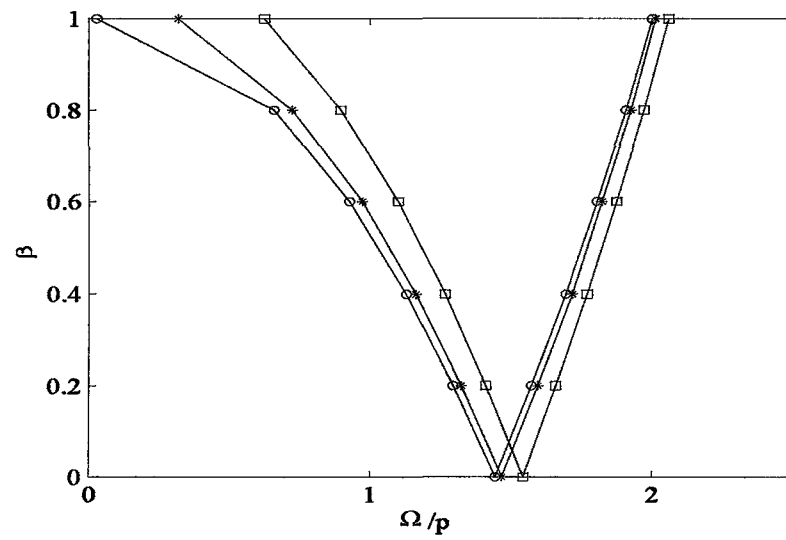


Figure 3.25 Buckling mode shapes for an aerofoil pretwisted blade ($d_x=0,193\text{e-}3$, $d_y=2,54\text{e-}3$ m, $\alpha_p=30^\circ$, $\phi=0^\circ$, $\omega=0$ rpm, — YY, - · - · XX, - - - $\theta\theta$)

Figure 3.19 shows the percentage increase of frequencies of the first mode vibration of blades with various pretwist angles as the speed of rotation increases. It is seen from the figure that for the first mode of vibration, the percentage increase is very little for all pretwist angles where as it increases as the speed of rotation increases (Thomas & Sabuncu, 1979). The finite element model is applied to the vibration analysis of rotating asymmetric blades in the first instance to compare the results obtained with the results of other investigators in order to establish the accuracy of the model developed as shown in figure 3.18 and figure 3.19.

Figure 3.20 illustrates the change in the mode shapes of buckling in the first five modes for uncoupled ($d_x=0$, $d_y=0$) motion. Figures 3.21, 3.22, 3.23 show the effect of pretwist on the buckling mode shapes for an aerofoil blade ($d_x=0$, $d_y=0$). It can be seen that when the blade pretwists the mode shapes give components of bending deformations in yy and xx planes. For first and second modes bending deformations in xx axes are small but for third, fourth and fifth modes they increase. As the pretwist angle increases the ratio of relative displacement along the xx principal axis increases for all modes. In figure 3.23 when the pretwist angle equals to 90° the relative displacement along the xx axis is the same as relative displacement along the yy axis for the first and second modes and for other modes the relative displacement along the xx axis more increases.

Figure 3.24 and 3.25 show the change in the mode shapes of buckling in the first five modes for a bending- bending torsional motion of a pretwisted blade. In these figures it can be seen that when an asymmetric blade pretwist the mode shapes give components of bending deformations in yy and xx planes and torsional deformations even for lower modes. For this particular blade (asymmetric and pretwisted) it is seen that torsional coupling plays an important role in the third, fourth and fifth modes of buckling.



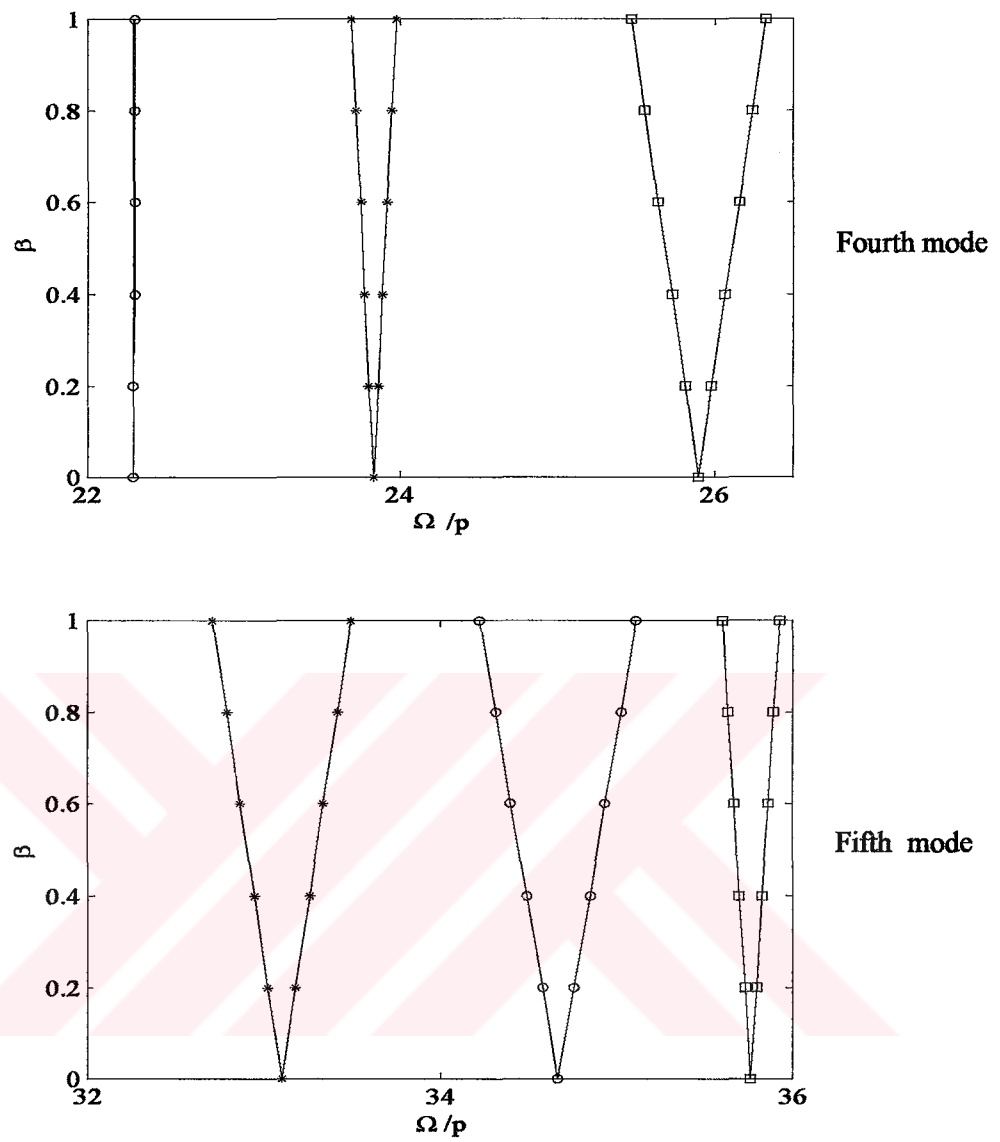


Figure 3.26 Effect of pretwist on the regions of dynamic instability of an aerofoil cross section pretwisted blade for five modes. ($\alpha=0,5$ $\phi=0^\circ$, $\omega=0$, (o) $\alpha_p=0^\circ$, (*) $\alpha_p=45^\circ$, ($\square$) $\alpha_p=90^\circ$)

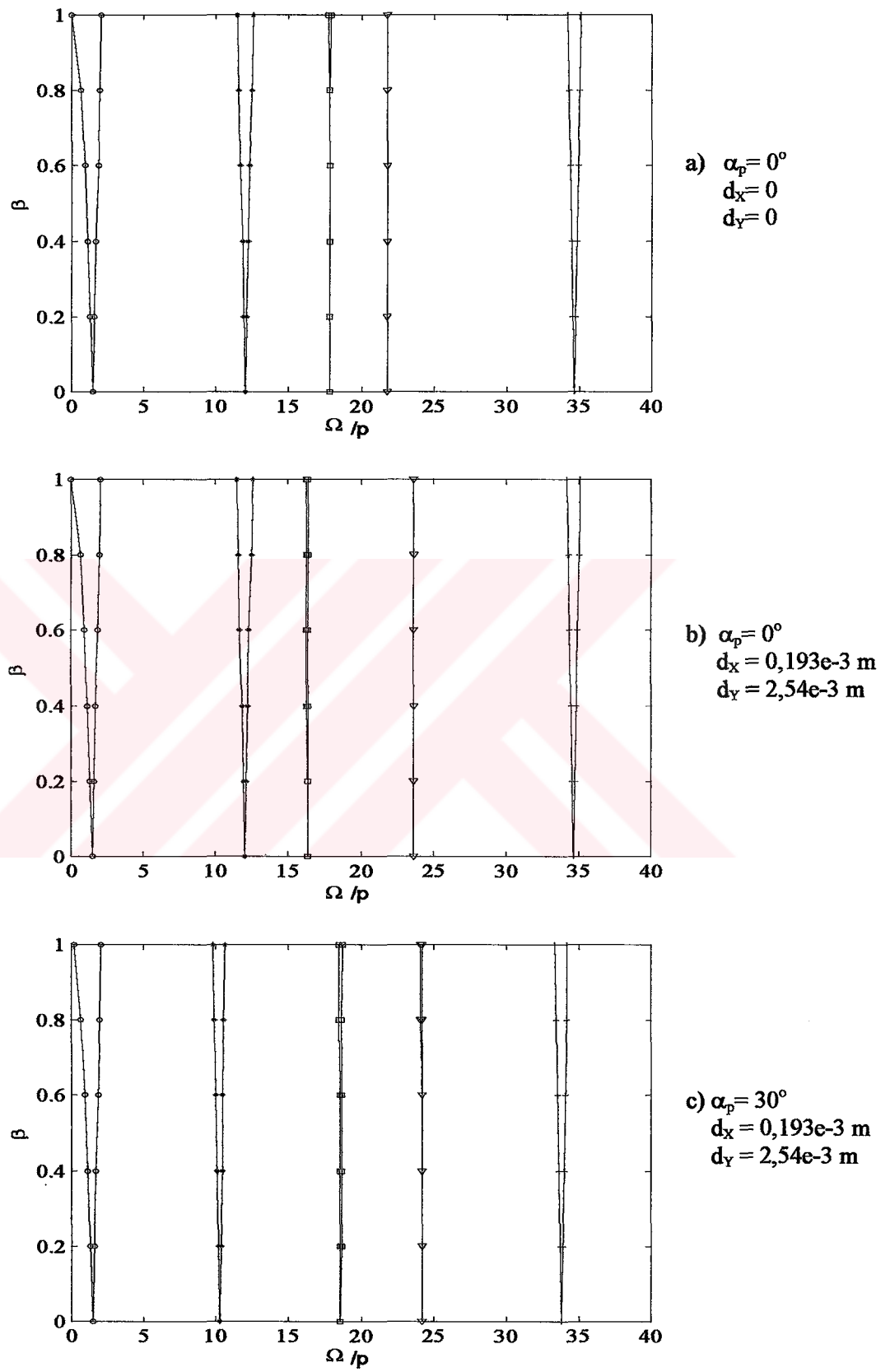


Figure 3.27 Effect of coupling on the regions of dynamic instability of an aerofoil cross section pretwisted blade ($\phi=0^\circ$, (o)first mode, (*)second mode, (□)third mode, (v)fourth mode, (+)fifth mode)

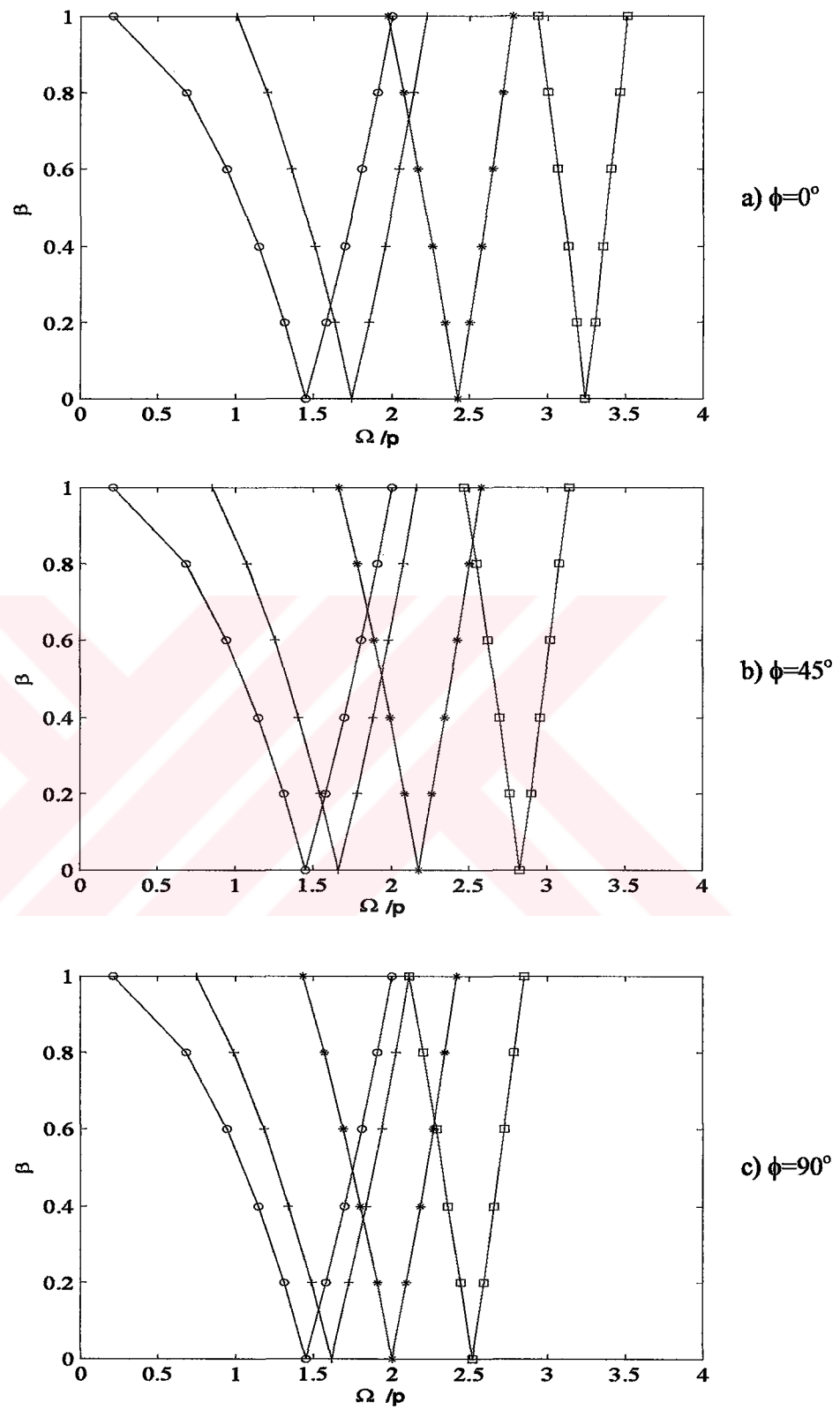


Figure 3.28 Effect of stagger angle on the regions of dynamic instability of an aerofoil cross section pretwisted blade ($\alpha=0,5$ $\alpha_p=30^\circ$) (o) $\omega=0$, (+) $\omega=2000$ rpm, (*) $\omega=4000$ rpm, ($\square$) $\omega=6000$ rpm)

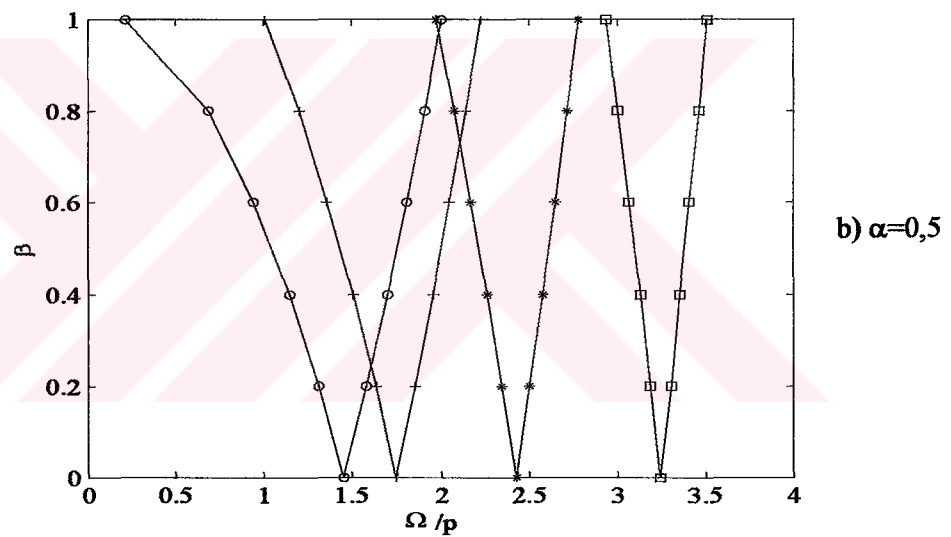
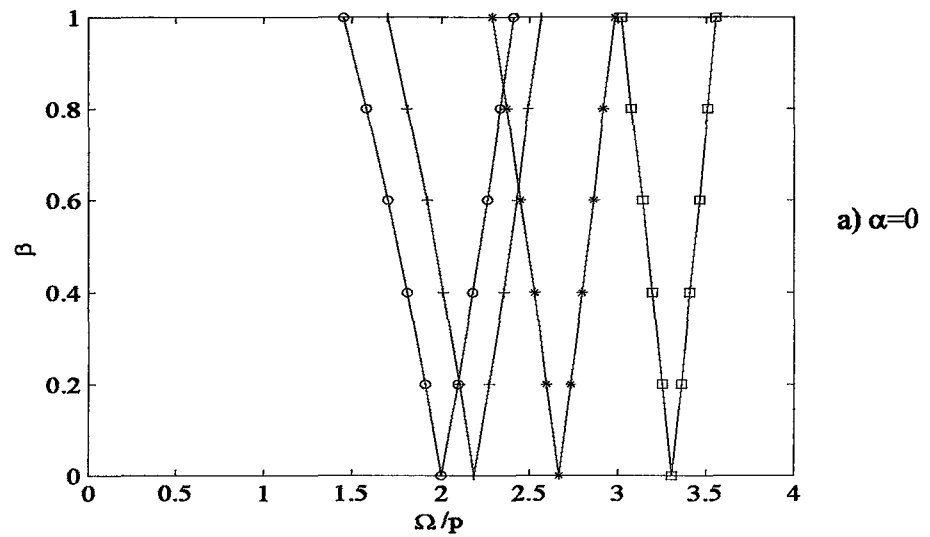


Figure 3.29 Effect of rotational speed and static load factor on the regions of dynamic instability of an aerofoil cross section pretwisted blade ($\phi=30^\circ$, $\alpha_p=30^\circ$) (o) $\omega=0$, (+) $\omega=2000$ rpm, (*) $\omega=4000$ rpm ($\square$) $\omega=6000$ rpm)

Table 3.6 shows the critical buckling load parameters for various d_x and d_y distances as pretwist angle increases. For the first mode as the pretwist angle increases critical buckling load parameter increases for all d_x and d_y distances, but for other modes as pretwist angle increases critical buckling load parameters decrease.

Table 3.6 also shows that the coupling effect on the critical buckling load parameter. The coupling effect is not significant in the first four yy bending static buckling load modes. The fifth critical buckling load parameter decreases when d_y/d_x ratio increases, because this mode (xx bending mode) is affected by d_y . When the blade pretwist angle increases the critical buckling load parameter doesn't change considerably for the first mode but decreases for the other modes.

Figure 3.26 shows the effect of pretwist angle on the regions of dynamic instability. For the fundamental mode as the pretwist angle increases the regions shift to the right and widths of the regions decrease and the blade becomes more stable. For the second mode as pretwist angle increases the instability regions shift to the left and the beam becomes unstable as the pretwist angle increases. For the third and fourth modes as pretwist angle increases the instability regions shift to the right and for the third mode change of width of regions seems to be very little but for the fourth mode the width of regions increases a lot. The pretwist plays a very different role on the fifth mode for 45° pretwist angle, the instability region shifts to the left and changes place with the unpretwisted beam instability region. A similar phenomenon takes place with the change of the fifth natural frequency as the pretwist angle changes as shown in figure 3.18. In the third, fourth and fifth modes torsional coupling plays an important role.

Figure 3.27 shows the effect of coupling caused by the distance of center of flexure from the centroid on the regions of dynamic instability. In figures 3.25 (a) and 3.25 (b) the first and second regions don't change for various center of flexure and centroid distances. The third region is affected by d_y and as the value of d_y increases the regions shift to the left towards static vertical axis. Thus the effect is one of the destabilization. The fourth region shifted to the right with an increase in d_y and it makes the blade more stable. When the blade pretwists the region shifts to the right and the width of the region decreases and the blade becomes more stable for the first mode in figure 3.27 (c). For the second mode the instability region shifts to the left and its width decreases. For the third and fourth modes the instability

regions shift to the right and the widths of the regions slightly increase. The instability region for the fifth mode shifts to the left.

Figure 3.28 shows the effect of stagger angle for various rotational speeds on the regions of dynamic instability for the first mode. As the stagger angle increases the blade becomes more flexible so the regions shift to the left and the width of these regions increases, thus this effect makes the blade less stable. Similar cases can be observed for the other modes.

Figure 3.29 (a) shows the rotational speed effect on the regions of dynamic instability of an aerofoil cross section pretwisted blade. With an increase in rotational speed, the instability regions shift to the right and the widths of the regions are decreased as in the case of symmetric cross section blades. Obviously an increase in the rotational speed increases the stiffness of the rotating blade and makes the blade more stable as expected. Figure 3.29 (b) shows the effect of static load factor on the dynamic instability of an asymmetric aerofoil blade. Increasing the static force component of the dynamic load affects the instability regions. As seen from the figure, the regions of dynamic instability are shifted to the left in parallel along the frequency ratio axis, indicating the unstable behavior of the aerofoil blade due to increasing values of static load factor.

CHAPTER FOUR

THEORETICAL CONSIDERATION OF CURVED BEAMS

The vibration problem of curved beams has been the subject of interest of several investigators due to its importance in many practical applications. Especially in turbine design, the blades are often joined together to reduce the resonant amplitude by a single shroud ring either at their tips or at some intermediate points.

It is well known that the shrouding significantly alters the natural frequencies of the system. The analysis of curved beam gains importance in the design of turbine blade packets.

In the flexural vibrations of a curved beam the central line of the undeformed beam is assumed to be in plane curve and its plane a principal plane of the beam at each point. These assumptions lead us to the uncoupling of the in-plane and the out-of-plane vibrations of curved beams.

In this section, shroud is represented by curved beam elements. Shroud is assumed to have a rectangular cross section. It is known that if the mass and shear centers of a symmetric cross section, i.e. square, rectangular etc. coincide there is no coupling between torsion and bending modes of vibration. This means that for a curved beam although torsional and out-of-plane vibrations are coupled, in-plane and out-of-plane vibrations are independent of each other and can be examined separately. For curved beams with small width/thickness ratio, thin beam theory is applied by neglecting shear deformation and rotary inertia effects. Hence in-plane and out-of-plane vibrations are dealt with separately and combined together to form the shroud element before the shroud blade assembly is made.

4.1 In-plane vibration

A curved beam finite element is developed, whose shape functions are derived from independent polynomial expressions for the general strains rather than displacements. Deflections functions used in the analysis are given by Sabir & Ashwell, (1971) as,

$$w = a_1 \cos \phi + a_2 \sin \phi + a_4 - a_6 \phi$$

$$v = -a_1 \sin \phi + a_2 \cos \phi + a_3 + a_5 \phi + \frac{1}{2} a_6 \phi^2, \quad \phi = y/R \quad (4.1)$$

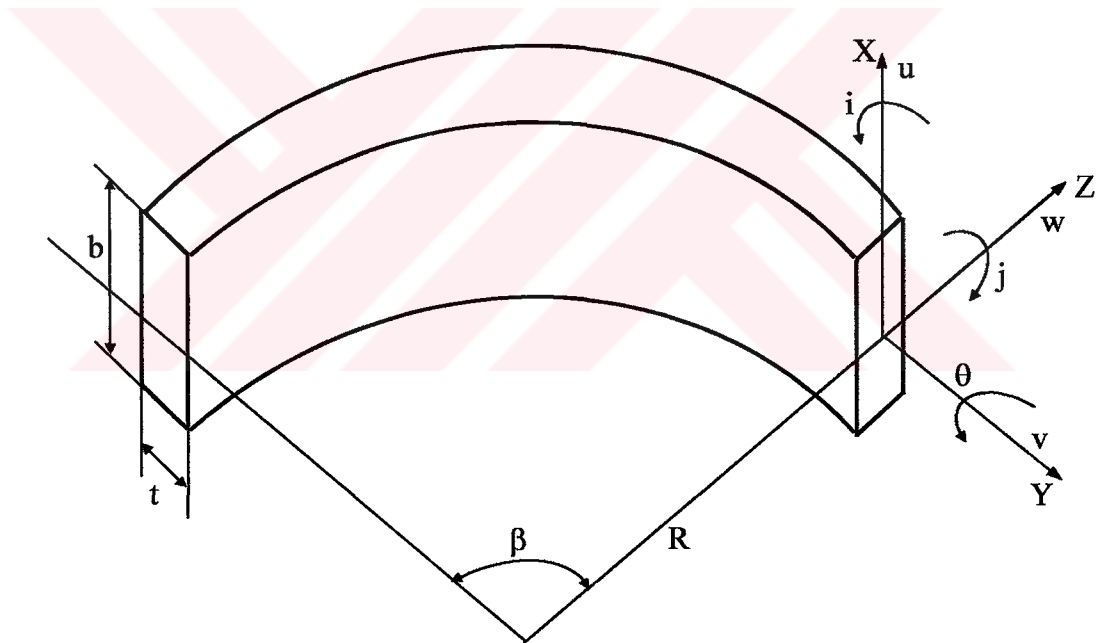


Fig 4.1 Coordinate system and displacements of the curved beam finite element

4.1.1 Strain energy of in-plane vibration of a curved beam

Strain energy of a curved beam element vibrating in its own plane is given by Belek, (1977) as,

$$V_I = \int_0^{l_{ab}} \left[E_s I_{xxs} \left(w'' - \frac{\dot{v}}{R} \right)^2 + E_s A_s \left(\frac{w}{R} + \dot{v} \right)^2 \right] dy \quad (4.2)$$

Equation (4.2) can be written as

$$V_I = \frac{1}{2} \{q_I\}^T [C_I]^{-T} [k_I] [C_I]^{-1} \{q_I\} \quad (4.3)$$

where

$$\{q_I\}^T = [v_i \quad w_i \quad \dot{v}_i \quad v_{i+1} \quad w_{i+1} \quad \dot{v}_{i+1}] \quad \dot{v}_i = \frac{\partial w}{\partial y} - \frac{v}{R} \quad (4.4)$$

$$[k_I] = \frac{E_s I_{xxs}}{R^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & Q\beta & Q\beta & 0 & \\ \text{Symmetric} & & \beta(1+Q) & \beta^2/2 & & \\ & & & \beta^3/3 & & \end{bmatrix} \quad (4.5)$$

$$Q = \frac{12 R^2}{t^2}$$

$$[C_I]^{-1} = \begin{bmatrix} -B_4 & B_1 & -R B_4 & -B_5 & -B_1 & -R B_5 \\ -B_3 & B_2 & -R B_3 & -B_1 & -B_2 & -R B_1 \\ B_1 & -B_2 & R B_3 & B_1 & B_2 & R B_1 \\ B_4 & -B_3 & R B_4 & B_5 & B_1 & R B_5 \\ B_6 & -B_7 & R B_8 & B_6 & B_7 & R B_9 \\ -B_1 & B_2 & -R B_1 & -B_1 & -B_2 & -R B_1 \end{bmatrix} \quad (4.6)$$

$$D = 2 \cos \beta - 2 + \beta \sin \beta$$

$$B_1 = (\cos \beta - 1) / D$$

$$B_2 = \sin \beta / D$$

$$B_3 = (1 - \beta \sin \beta - \cos \beta) / D$$

$$B_4 = (\sin \beta - \beta \cos \beta) / D$$

$$B_5 = (\beta - \sin \beta) / D$$

$$B_6 = B_1 \beta / 2$$

$$B_7 = B_2 \beta / 2$$

$$B_8 = B_6 + 1 / \beta$$

$$B_9 = B_6 - 1 / \beta$$

4.1.2 External work done of a curved beam under uniform compression forces in the plane of curvature

External work will be done when the shroud is subjected to uniform compression force P in the plane of curvature. The additional energy V_{gl} due to periodic force is (Papangelis & Trahair 1987)

$$V_{gl} = \int_0^{l_{sh}} PR \left(\frac{dw}{dy} - \frac{v}{R} \right)^2 dy \quad (4.7)$$

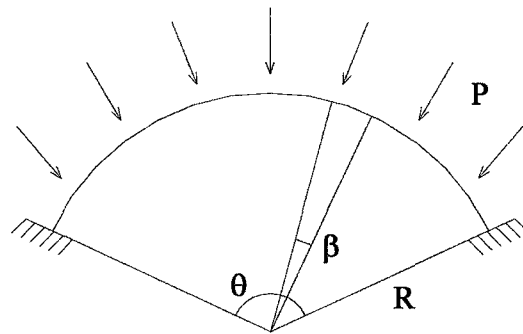


Figure 4.2 Curved beam subjected to uniform compression force

It may be written in the matrix form as

$$V_{gl} = \frac{1}{2} \{q_I\}^T [C_I]^T [k_{gl}] [C_I]^{-1} \{q_I\} \quad (4.8)$$

Where the corresponding degrees of freedom $\{q_I\}^T$ and transformation matrix $[C_I]^{-1}$ have been given in equations (4.4) and (4.6) respectively.

$$[k_{gl}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & \beta^2/2 & \beta + \beta^3/6 & & \\ 0 & 0 & 0 & 0 & & \\ \beta^3/3 & \beta^2/2 + \beta^4/8 & & & & \\ \beta + \beta^3/3 + \beta^5/20 & & & & & \end{bmatrix} \quad (4.9)$$

4.1.3 Increase of strain energy in the shroud due to the centrifugal force

A curved beam, in a centrifugal field, acting as a shroud attached at the tips of the rotating blades, increases its own strain energy as well as the strain energy of the whole system. The centrifugal force effect of the shroud mass on the blade as shown in figure 4.3 has been examined, and the derivation of the strain energy in the shroud due to the centrifugal force is given by Sabuncu, (1978).

The increase in strain energy in the shroud element is written as

$$V_c = -\frac{1}{2} \rho_s \omega^2 \int_0^{l_{sh}} \int_{A_{sh}} \left\{ \left(-zw + \frac{(R+z)v}{R} \right)^2 + 2w^2 \right\} dA dy \quad (4.10)$$

$$V_c = \frac{1}{2} \rho_s \omega^2 \{q_I\}^T [C_I^{-1}]^T [k_{sf}] [C_I^{-1}] \{q_I\} \quad (4.11)$$

Where

$$[k_{sf}] = \chi_1 [W_r] + \frac{\chi_2}{2} [V_r W_r] + \chi_3 [V_r] \quad (4.12)$$

$$\chi_1 = -\rho_s \omega^2 \left(\frac{bt^3}{12} + 2bt \right)$$

$$\chi_2 = \rho_s \omega^2 \frac{bt^3}{24R}$$

$$\chi_3 = -\rho_s \omega^2 \left(\frac{bt^3}{12R^2} + bt \right)$$

$$[W_r] = \begin{bmatrix} C_2 S & CS & 0 & C & 0 & -C_1 \\ & C_2 S & 0 & S & 0 & -S_1 \\ & & 0 & 0 & 0 & 0 \\ & & & \beta & 0 & -\beta^2/2 \\ & & & & 0 & 0 \\ & & & & & \beta^3/3 \end{bmatrix} \quad (4.13)$$

$$[V_r] = \begin{bmatrix} S_2 S & -CS & -S & 0 & -S_1 & -S_2/2 \\ & C_2 & C & 0 & C_1 & C_2/2 \\ & & \beta & 0 & \beta^2/2 & \beta^3/6 \\ & & & 0 & 0 & 0 \\ & & & & \beta^3/3 & \beta^4/8 \\ & & & & & \beta^5/20 \end{bmatrix} \quad (4.14)$$

$$[V_r W_r] = \begin{bmatrix} -2CS & C_2 S & C & -S & C_1 & C_2/2 + S_1 \\ & 2CS & S & C & S_1 & S_2/2 - C_1 \\ & & 0 & \beta & 0 & -\beta^2/2 \\ & & & 0 & \beta^2/2 & \beta^3/6 \\ & & & & 0 & -\beta^3/3 \\ & & & & & -\beta^4/4 \end{bmatrix} \quad (4.15)$$

$$C_2S = 0.5(\beta + 0.5 \sin 2\beta)$$

$$CS = 0.25(\cos 2\beta - 1)$$

$$C = \sin \beta$$

$$C_1 = \cos \beta - 1 + \beta \sin \beta$$

$$C_2 = 2\beta \cos \beta + (\beta^2 - 2) \sin \beta$$

$$S_2S = 0.5(\beta - 0.5 \sin 2\beta)$$

$$S = -(\cos \beta - 1)$$

$$S_1 = \sin \beta - \beta \cos \beta$$

$$S_2 = 2\beta \sin \beta + 2(\cos \beta + 1) - \beta^2 \cos \beta$$

In our analysis the centrifugal force effect is assumed to be in the plane of curvature of the curved beam, so the potential energy of the curved beam vibrating in its own plane can be written, by using equations (4.3) and (4.11) as

$$V = \frac{1}{2} \{q_I\}^T [C_1^{-1}]^T [k_{IC}] [C_1^{-1}] \{q_I\} \quad (4.16)$$

$$[k_{IC}] = [k_I] + \frac{R^3}{E_s I_{xxs}} [k_{sf}] \quad (4.17)$$

4.1.4 Kinetic energy of in-plane vibration of a curved beam

The kinetic energy expression of a curved beam element under combined bending-bending displacements is given by Belek, (1977) as,

$$T_I = \int_0^{l_h} [\rho_s A_s (\dot{v}^2 + \dot{w}^2)] dy \quad (4.18)$$

$$T_I = \frac{1}{2} \{\dot{q}_I\}^T [C_I]^{-T} [m_I] [C_I]^{-1} \{\dot{q}_I\} \quad (4.19)$$

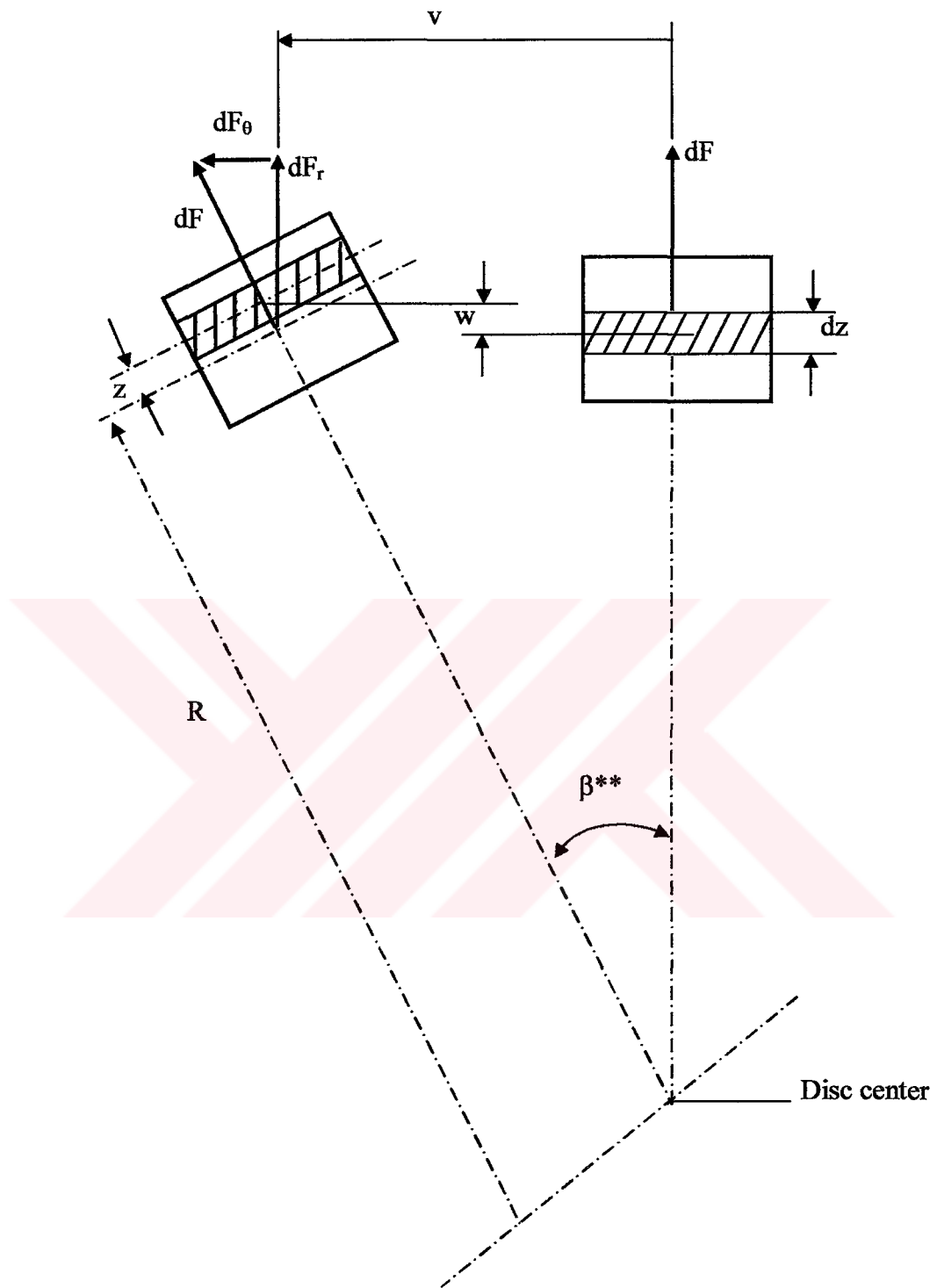


Figure 4.3 View of centrifugal force effect on a shroud

Where the corresponding degrees of freedom $\{q_i\}^T$ and the transformation matrix $[C_i]^{-1}$ have been given in equations (4.4) and (4.6) respectively.

The mass matrix $[m_i]$ is;

$$[m_i] = \rho_s A_s R \begin{bmatrix} \beta & 0 & \cos\beta - 1 & \sin\beta & \beta \cos\beta - \sin\beta & C_0 \\ & \beta & \sin\beta & 1 - \cos\beta & \cos\beta + \beta \sin\beta - 1 & C_1 \\ & & \beta & 0 & \beta^2/2 & \beta^3/6 \\ & & & \beta & 0 & -\beta^2/2 \\ & & & & \beta^3/3 & \beta^4/8 \\ & & & & & \beta^5/20 + \beta^3/3 \end{bmatrix} \quad (4.20)$$

$$C_0 = (\beta^2 - 4) \cos\beta/2 - 2\beta \sin\beta + 2$$

$$C_1 = (\beta^2 - 4) \sin\beta/2 + 2\beta \cos\beta$$

4.2 Out-of plane vibration

The out-of-plane curved beam element used in this analysis is given by Belek, (1977) and is based on strain functions. The strain displacement equations are integrated by making an assumption of uniform torsional strain and linear increase in curvature.

Deflection functions used in the analysis are given by Belek, (1977) as

$$u = a_1 R \cos\phi + a_2 R \sin\phi + a_3 R - a_4 R\phi - a_6 R\phi$$

$$\theta = a_1 \cos\phi + a_2 \sin\phi - a_5 - a_6\phi, \quad \phi = y/R \quad (4.21)$$

4.2.1 Strain energy of out-of-plane vibration of a curved beam

Strain energy of a curved beam element vibrating out-of plane direction is given by Belek, (1977) as,

$$V_o = \int_0^{l_{sh}} \left[E_s I_{yys} \left(u + \frac{\theta}{R} \right)^2 + G_s J_{xxs} \left(\dot{\theta} - \frac{\dot{u}}{R} \right)^2 \right] dy \quad (4.22)$$

$$V_o = \frac{1}{2} \{q_o\}^T [C_o]^T [k_o] [C_o]^{-1} \{q_o\} \quad (4.23)$$

where

$$\{q_o\}^T = [\theta_i \ u_i \ j_i \ \theta_{i+1} \ u_{i+1} \ j_{i+1}] \quad j = -\frac{\partial u}{\partial y} \quad (4.24)$$

$$[k_o] = \frac{E_s I_{yys}}{R} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ C_2 \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & \beta^2/2 & 0 \\ 0 & 0 & 0 & 0 & \beta^3/3 & 0 \end{bmatrix} \quad (4.25)$$

$$C_2 = \frac{G_s J_{xxs}}{E_s I_{yys}}$$

$$[C_o]^{-1} = \begin{bmatrix} 0 & -B_{o1}/R & B_{o2} & 0 & B_{o1}/R & B_{o3} \\ 0 & -B_{o5}/R & -B_{o4} & 0 & B_{o5}/R & -B_{o1} \\ 0 & B_{o4}/R & -B_{o2} & 0 & -B_{o1}/R & -B_{o3} \\ -1/\beta & 1/(R\beta) & 0 & 1/\beta & -1/(R\beta) & 0 \\ -1 & -B_{o1}/R & B_{o2} & 0 & B_{o1}/R & B_{o3} \\ 1/\beta & 2B_{o1}/(R\beta) & -B_{o1} & -1/\beta & -2B_{o1}/(R\beta) & -B_{o1} \end{bmatrix} \quad (4.26)$$

$$D_o = 2 - 2 \cos \beta - \beta \sin \beta$$

$$B_{o1} = (\cos \beta - 1) / D_o$$

$$B_{o2} = \sin \beta / D_o$$

$$B_{o3} = (\sin \beta - \beta) / D_o$$

$$B_{o4} = (1 - \cos \beta - \beta \sin \beta) / D_o$$

$$B_{o5} = \sin \beta / D_o$$

4.2.2 External work done in the out-of-plane direction of a curved beam due to uniform in-plane compression forces

External work will be done in the out of plane direction when the shroud is subjected to uniform compression force P. The additional energy V_{go} due to periodic force (Papangelis & Trahair, 1987)

$$V_{go} = \int_0^{l_{sh}} PR \left[\dot{u}^2 + r_x^2 \dot{\theta}^2 + r_y^2 \left(\dot{\theta} - \frac{\dot{u}}{R} \right)^2 \right] dy \quad (4.27)$$

$$r_x^2 = \frac{I_{xxs}}{A_s}, \quad r_y^2 = \frac{I_{yys}}{A_s} \quad (4.28)$$

$$V_{go} = \frac{1}{2} \{q_o\}^T [C_o]^T [k_{go}] [C_o]^{-1} \{q_o\} \quad (4.29)$$

$$g_1 = \begin{bmatrix} \sin^2 \theta & -\sin \theta \cos \theta & 0 & \sin \theta & 0 & \sin \theta \\ & \cos^2 \theta & 0 & -\cos \theta & 0 & -\cos \theta \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 1 \end{bmatrix} \quad (4.30.a)$$

$$g_2 = \begin{bmatrix} \sin^2 \theta & -\sin \theta \cos \theta & 0 & 0 & 0 & \sin \theta \\ & \cos^2 \theta & 0 & 0 & 0 & -\cos \theta \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 1 \end{bmatrix} \quad (4.30.b)$$

$$g_3 = \begin{bmatrix} -\sin^2 \theta & \sin \theta \cos \theta & 0 & -\sin \theta & 0 & -\sin \theta \\ \sin \theta \cos \theta & -\cos^2 \theta & 0 & \cos \theta & 0 & \cos \theta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & -1 & 0 & -1 \end{bmatrix} \quad (4.30.c)$$

$$g_4 = \begin{bmatrix} -\sin^2 \theta & \sin \theta \cos \theta & 0 & 0 & 0 & -\sin \theta \\ \sin \theta \cos \theta & -\cos^2 \theta & 0 & 0 & 0 & \cos \theta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 & 0 & -1 \end{bmatrix} \quad (4.30.d)$$

Using the above matrices the matrix $[k_{go}]$ is obtained

$$[KG] = [R^2 g_1 + r_x^2 g_2 + r_y^2 (g_1 + g_2 + g_3 + g_4)] \quad (4.31)$$

$$[k_{go}] = \int_0^\beta [KG] d\theta \quad (4.32)$$

4.2.3 Kinetic energy of out-of-plane vibration of a curved beam

The kinetic energy expression of a curved beam under combined bending-torsional displacements is given by Belek, (1977) as,

$$T_o = \int_0^{l_b} \left[\rho_s A_s (\dot{u})^2 + \rho_s I_p (\dot{\theta})^2 \right] dy \quad (4.33)$$

$$T_o = \frac{1}{2} \{\dot{q}_o\}^T [C_o]^T [m_o] [C_o]^{-1} \{\dot{q}_o\} \quad (4.34)$$

Where the corresponding degrees of freedom $\{q_o\}^T$ and the transformation matrix $[C_o]^{-1}$ have been given in equations (4.24) and (4.26) respectively and the mass matrix $[m_o]$ is

$$[m_o] = \rho_s A_s R^3 \begin{bmatrix} C_4 M_1 & C_4 M_2 & \sin \beta & M_3 & -C_3 \sin \beta & C_4 M_3 \\ & C_4 M_4 & M_5 & M_6 & -C_3 M_5 & C_4 M_6 \\ & & \beta & -\beta^2 / 2 & 0 & -\beta^2 / 2 \\ & & & \beta^3 / 3 & 0 & \beta^3 / 3 \\ & & & & C_3 \beta & C_3 \beta^2 / 2 \\ & & & & & C_4 \beta^3 / 3 \end{bmatrix} \quad (4.35)$$

$$M_1 = \beta / 2 + \sin 2\beta / 4$$

$$M_2 = \sin^2 \beta / 2$$

$$M_3 = 1 - \cos \beta - \beta \sin \beta$$

$$M_4 = \beta / 2 - \sin 2\beta / 4$$

$$M_5 = 1 - \cos \beta$$

$$C_3 = \frac{I_p}{(A_s R^2)}$$

$$C_4 = 1 + C_3$$

4.3 Results

In this section some of the results obtained for the frequencies and buckling loads of curved beams in the in-plane and out-of-plane directions are shown and compared with some other researchers' results in order to indicate the degree of accuracy and adaptability that can be expected in the application of the finite element method.

Table 4.2 shows the comparison of the in-plane frequencies obtained by the present analysis and finite element solutions of other investigators for a uniform fixed-fixed curved beam. As seen from the table, the results of uniform finite element models based on simple strain functions are in close agreement.

Table 4.3 shows the comparison of the out-of-plane frequencies obtained by the present analysis and the results of Ojalva & Newman (1964). The frequency factors are taken from the graphs and converted into the corresponding frequencies. There could be a slight error in reading the frequency factors from graphs which could cause slight discrepancies between actual results and results read from the graphs. In spite of that, the agreement between the results of Ojalvo & Newman (1964) and present results is good (Sabuncu, 1978).

Table 4.4 shows the comparison between the present results and analytical results obtained by Timoshenko & Gere (1961). It can be seen that obtained results are in agreement very closely with the analytical results.

Figure 4.4 shows the variation of subtended angle of an arch on the natural frequencies in the in-plane-direction. When the subtended angle increases the frequency decreases.

Figure 4.5 and 4.6 show the effect of variation of subtended angle and the radius on the in-plane buckling load of an arch. When the subtended angle and radius increase, the buckling load of curved beams decrease. This phenomenon can be

explained as follows; when the subtended angle and radius of an arch increase, as a result the length of curved beam increases, consequently beams become very flexible. The effect of variation of radius on the flexibility is more dominant than the effect of variation of subtended angle.

Figure 4.7 shows the effect of variation of thickness on the in-plane buckling load. When the t/b ratio increases consequently rigidity of arch increases as a result the buckling load increases.

Figure 4.8 shows the variation of subtended angle of an arch on the natural frequencies towards the out-of-plane direction. When the subtended angle increases the frequency decreases as the flexibility of the arch decreases.

Figure 4.9 shows the effect of variation of subtended angle on the out-of-plane buckling load of an arch. When the subtended angle increases, the buckling load of curved beam decreases. This phenomenon can also be explained as follows; when the subtended angle of an arch increases, the length of curved beams also increases, consequently beam becomes very flexible.

Figure 4.10 shows the effect of variation of t/b ratio on the buckling load of an arch both in the in-plane and out-of-plane directions. When the t/b ratio increases the buckling of an arch also increases. Direction of buckling mode changes from the in-plane of curvature to out-of-plane direction depending on the arch rigidity.

Figure 4.11 shows the change of buckling mode shapes of an arch in the plane of curvature for the first four modes. When t/b ratio equals to 0.2 the arch has buckled in the plane of curvature for the first eight modes. The out-of plane buckling takes place for the ninth mode. This is due to the higher in-plane buckling mode shapes are difficult to be formed than the out-of-plane buckling mode shapes.

Figure 4.12 shows the change of buckling mode shapes of an arch in the out-of-plane direction for the first four modes. When t/b ratio equals to 3 the arch has

buckled in the out-of-plane direction for the first nine modes. The in-plane buckling takes place for the tenth mode. This is due to the higher out-of-plane buckling mode shapes are difficult to be formed than the in-plane buckling mode shapes.

Table 4.1 The physical properties of arch used in the analysis

t (m)	b (m)	E (N/m ²)	ρ (kg/m ³)	R (m)
0,001587	0,02753	2,126e11	7833,4	0,3556

Table 4.2 Comparison of the frequencies (Hz) of in-plane vibration of a fixed-fixed curved beam
($t=0,0003302$ m, $b=0,0254$ m, $L_{sh}=0,1016$ m $R=0,0762$ m, $\rho=2770$ kg/m³, $E=6,89e10$ N/m²)

Mode	Natural frequency (Hz)	
	Present	(Petyt & Fleischer 1971)
1	451,7	448,69
2	716	713,82
3	1084,7	1082,9
4	14666,4	1463
5	2211,3	2216,8

Table 4.3 Comparison of out-of plane frequencies for a curved cantilever beam
($t=0,0127$ m, $b=0,0127$ m, $R=0,0254$ m, $\rho=2770$ kg/m³, $E=6,89e10$ N/m², $\theta=270^\circ$)

Mode	Natural frequency (Hz)	
	Present	(Ojalva & Newman 1964)
1	9,7	8,48
2	23,25	22,26
3	70,69	72,56
4	177	171,5

Table 4.4 Comparisons of in-plane buckling load parameters of a fixed-fixed curved beam (Legend for table 1)

Subtended angle	Buckling load parameter (k)	
	Present	(Timoshenko & Gere 1961)
60°	8,66	8,62
120°	4,42	4,37
180°	3	3

$$k = \sqrt{\frac{P_k R^3}{EI}} \quad \text{Non dimensional buckling load parameter (in-plane direction)}$$

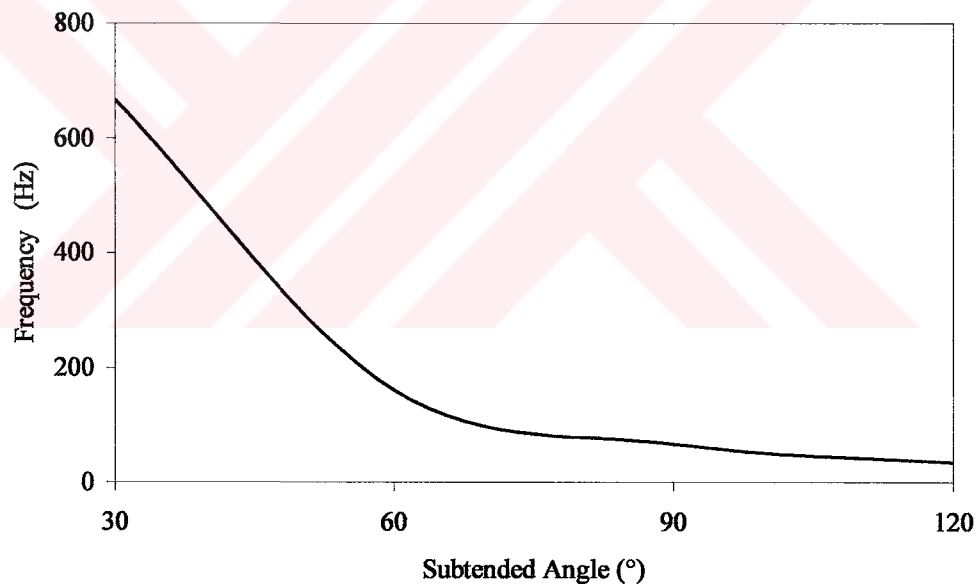


Figure 4.4 The effect of variation of subtended angle of an arch on the natural frequencies in the plane of curvature (Legend for table 1)

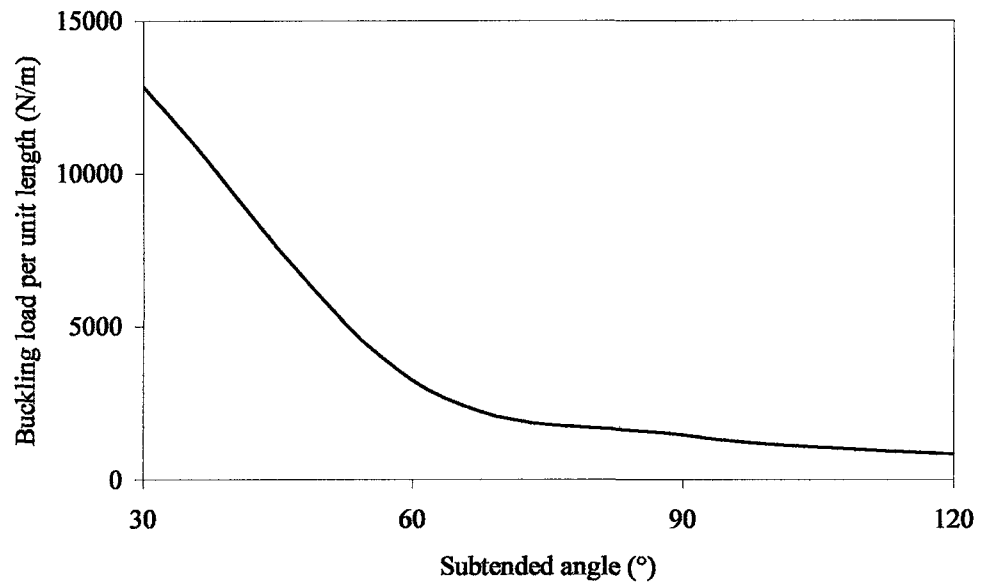


Figure 4.5 The effect of variation of subtended angle of an arch on the critical buckling load in the plane of curvature (Legend for table 1)

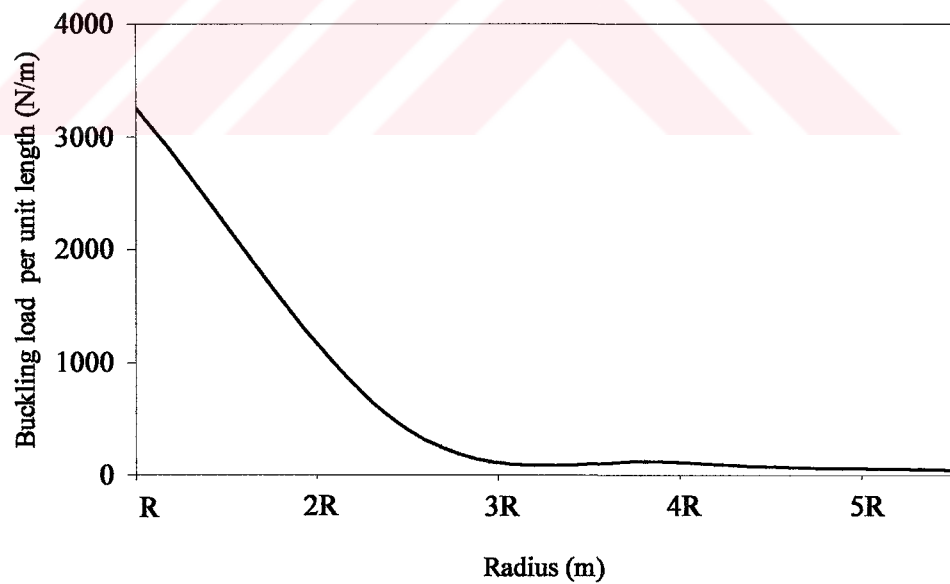


Figure 4.6 The effect of variation of radius of an arch on the critical buckling load in the plane of curvature (Legend for table 1)

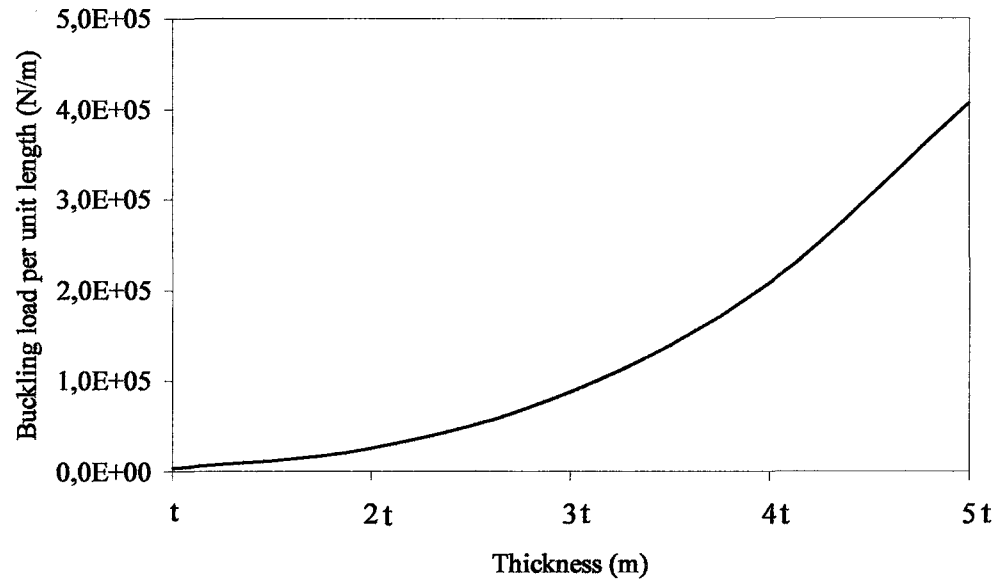


Figure 4.7 The effect of thickness variation of an arch on the critical buckling load in the plane of curvature (Legend for table 1)

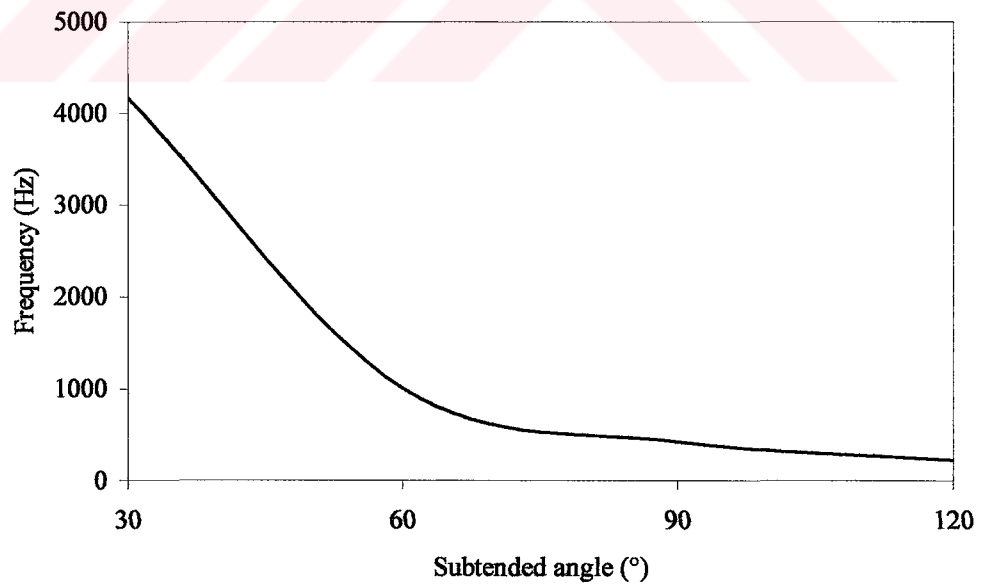


Figure 4.8 The effect of variation of subtended angle of an arch on the natural frequencies in the out-of-plane direction (Legend for table 1)

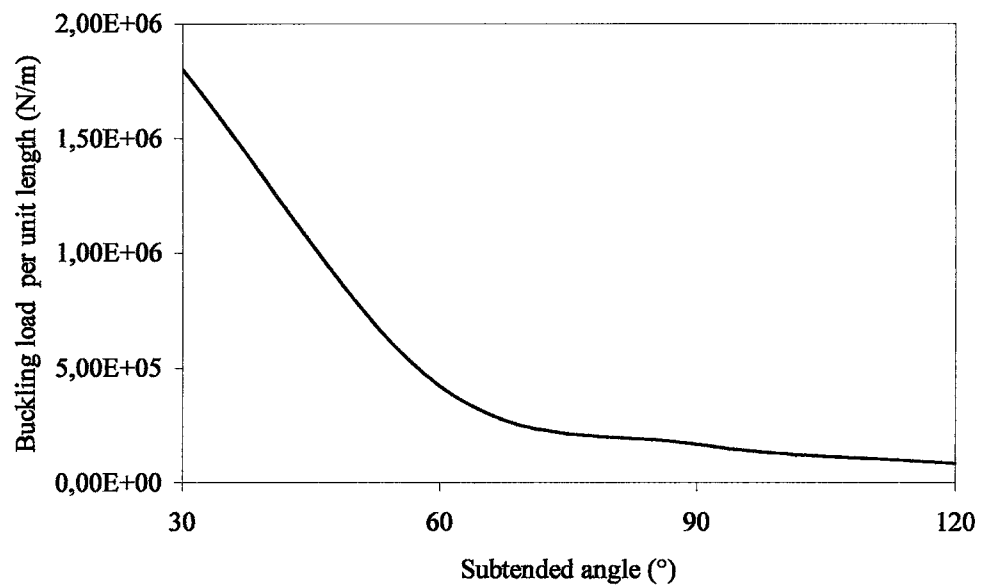


Figure 4.9 The effect of variation of subtended angle of an arch on the buckling load in the out-of-plane direction (Legend for table 1)

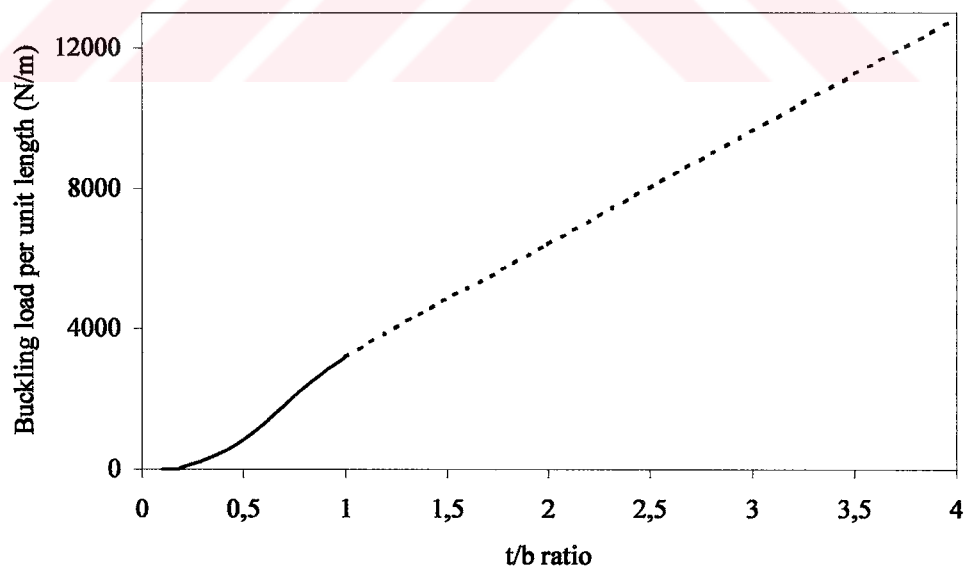


Figure 4.10 The effect of variation of t/b ratio of an arch on the buckling load for the first mode (Legend for table 1, — in-plane, - - - - - out-of plane)

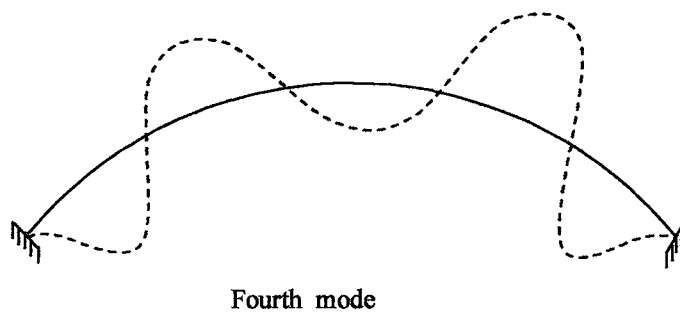
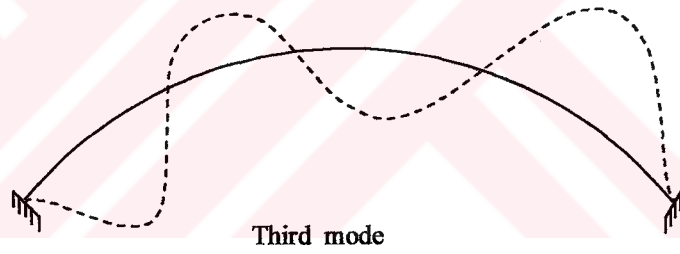
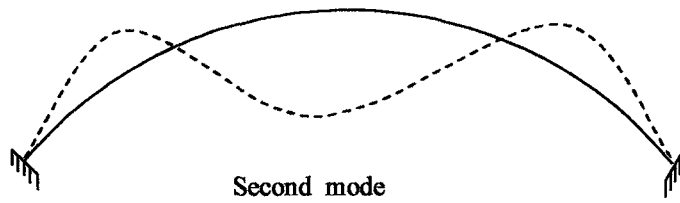
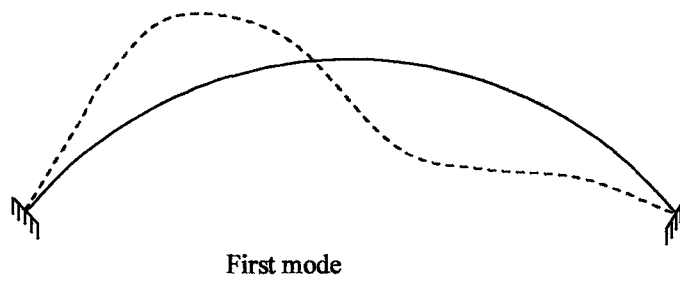
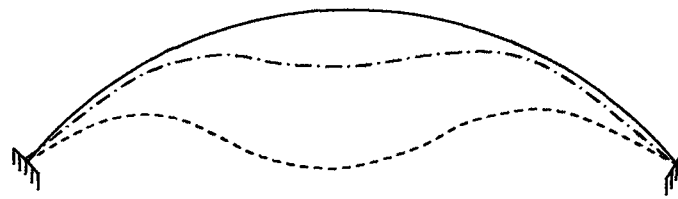
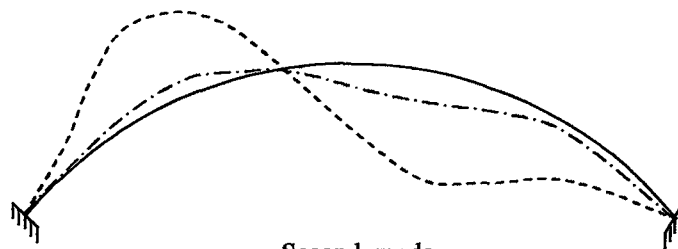


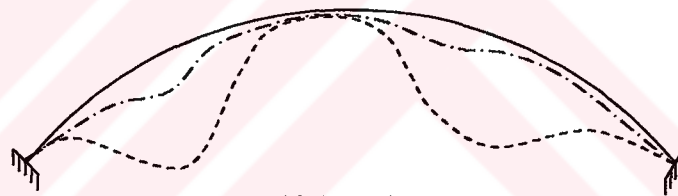
Figure 4.11 Buckling mode shapes for in-plane directions (----- Bending)



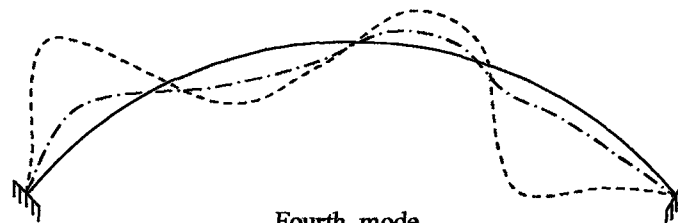
First mode



Second mode



Third mode



Fourth mode

Figure 4.12 Buckling mode shapes for out-of-plane directions (----- Torsion , -.-.-.- Bending)

CHAPTER FIVE

THEORETICAL CONSIDERATION OF BLADE PACKETS

Turbine blades are usually joined together, either at their tips or at some intermediate location, by steel bands. The band is called a shroud when it connects the blades at their free tips. The shroud may be continuous, joining all the blades in a row, or segmented so that only a group of adjacent blades are connected. In this case it is said that the blades are grouped in packets.

In practice, turbine blades are formed into packets by fitting an identical metal strip to the tip of each blade. These strips come into hard contact under the influence of the centrifugal force of rotation and form a continuous shrouding band.

Blades may also be grouped into packets by using lacing wires. The number of lacing wires and their positions in the packet varies to reduce resonance amplitude.

Turbine blades are usually straight or pretwisted along their length and also have asymmetric cross section. In this chapter the free vibration characteristics of aerofoil cross section straight and pretwisted blade packets are studied to show the accuracy of the model. Then the effects of pretwist angle, stagger angle, rotational speed, shroud length, shroud thickness, shroud width, and the distance of center of flexure from the centroid on the static buckling loads and dynamic instability regions of aerofoil cross section of blade packets are investigated using the finite element method.

The blades are joined together by a shroud, forming a packet, and rigidly fixed at the periphery of a disc rotating at a constant angular velocity. It is assumed that the shroud is subjected to a uniformly distributed radial periodic force. The scheme of the two bladed packet is shown in figure 5.1. The blades and shroud are divided into a number of elements and then stiffness matrix, mass matrix and stability matrix of the k^{th} element are developed.

5.1 Energy expression of a straight blade packet

Energy expressions of blade and shroud elements are derived under non-rotational and rotational conditions separately.

5.1.1 Energy expression of a blade element

The equations for strain and kinetic energies of a rotating straight and pretwisted uniform beam have been given in chapter 3. When the blade and shroud assembly is formed the blade and shroud elements must be analyzed by using dimensional form, otherwise the compatibility conditions can not be satisfied unless the blade and shroud have the same length. This is of course a special case and is not encountered in practice very often.

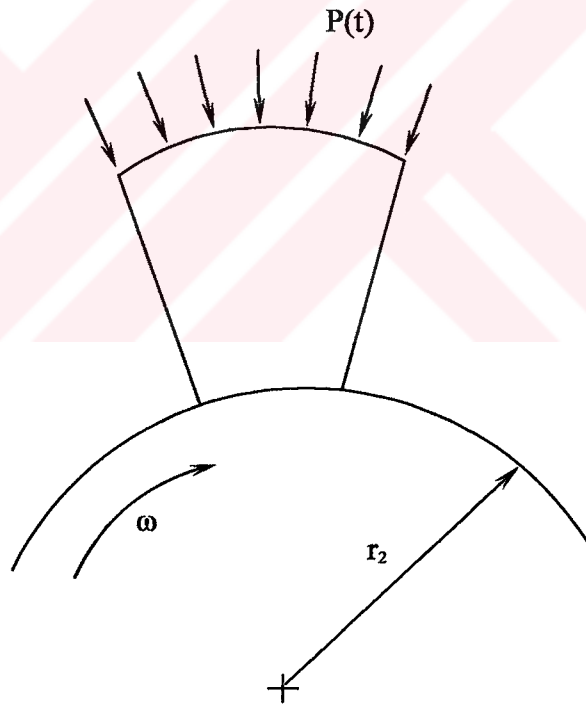


Figure 5.1 Two bladed curved packet

The energy expressions of a beam under non-rotating and rotating conditions are given in equations (3.2), (3.3) and additional energy expressions have to be derived due to the centrifugal force effect of the shroud on the blade.

5.1.1.1 Strain energy

Strain energies of a blade vibrating under non rotating and rotating conditions are derived for the blades joined together by a shroud, forming a packet, and rigidly fixed at the periphery of a disc rotating at constant angular velocity. As mentioned in chapter three there is an increase in potential energy due to rotation in a single blade, this potential energy increase is also increased by the centrifugal force of the rotating shroud mass. All the theoretical expressions of the energies are derived as follows;

5.1.1.1.1 Strain energy of a stationary blade

The strain energy of a stationary beam element has been given in equation (3.2).

5.1.1.1.2 Increase in strain energy of a blade element due to centrifugal force effect

A rotating shrouded blade mounted at the periphery of a rotating disc has its potential energy increased by the rotation. For a single blade the effect of rotation on the strain energy has been examined by Carnegie, (1959). When the shroud is fixed at the tip of the blades, the centrifugal force effect of the shroud mass increases the strain energy of the blade and the shroud.

Referring to figure (5.2) the centrifugal force, caused by the rotation of shroud mass along the δ direction, can be written as

$$F_{\delta} = F \sin \beta^1 \quad (5.1)$$

from the figure

$$\sin \beta^1 = \frac{\delta_{\max}}{R} \quad \text{where} \quad \delta_{\max} = \delta_n \quad (5.2)$$

If the centrifugal force is written as

$$F = \rho_s A_s L_{sh} R \omega^2 = \frac{W_s R \omega^2}{g} \quad (5.3)$$

Substitution of equations (5.2) and (5.3) into equation (5.1) gives

$$F_\delta = \frac{W_s}{g} \omega^2 \delta_{\max} \quad (5.4)$$

Force component F_z in the z direction is

$$F_z = F \cos \beta^1 \quad (5.5)$$

since β^1 is small $\cos \beta^1 \cong 1$

$$F_z = W_s R \omega^2 \frac{1}{g} \quad (5.6)$$

These forces F and F_z will be acting on each element of the blade. Due to differing transverse deflections the component F does different amount of work for each element of the blade as follows;

$$dU_\delta = -\frac{1}{2g} W_s \omega^2 \delta \delta_{\max} dz \quad (5.7)$$

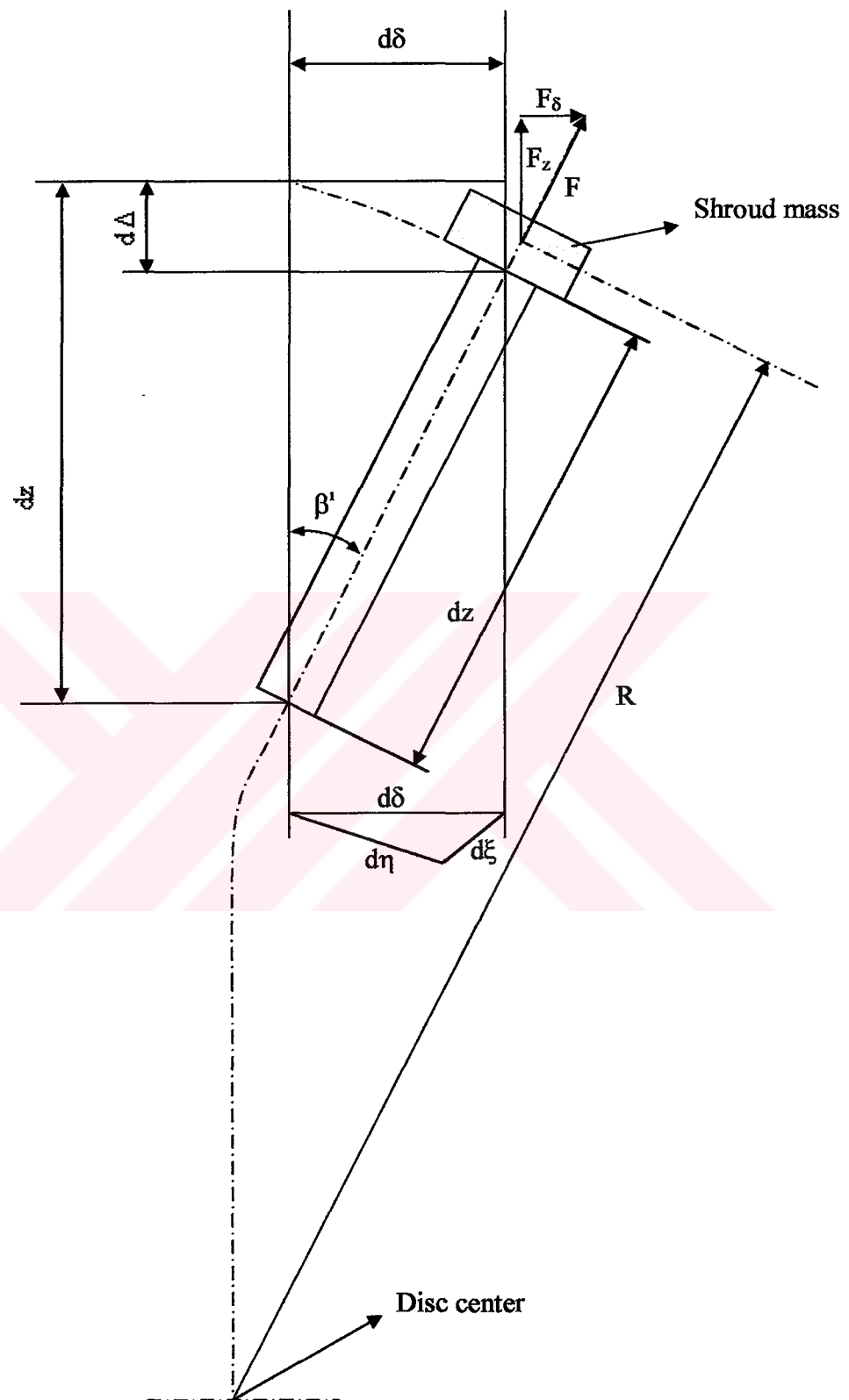


Figure 5.2 Centrifugal force effects on a blade due to shroud mass

Where δ is the deflection in the plane of disc for each element. Since deflection and force are in the same direction, the gain in potential energy becomes negative. Integrating over an element it becomes

$$U_{\delta} = -\frac{1}{2g} W_s \omega^2 \int_0^{\ell} \delta \delta_{\max} dz \quad (5.8)$$

This can also be written in the following form to make the global matrix symmetric

$$U_{\delta} = -\frac{1}{2g} W_s \omega^2 \left\{ \int_0^{\ell} \frac{1}{2} \delta^T \delta_{\max} dz + \int_0^{\ell} \frac{1}{2} \delta_{\max}^T \delta dz \right\} \quad (5.9)$$

The strain energy dU_z due to F_z force on the element is;

$$dU_z = F_z \Delta \quad (5.10)$$

Referring to figure (5.2) the total displacement, Δ of the element at z is;

$$\Delta = \frac{1}{2} \int_0^z \left\{ \left(\frac{\partial \xi}{\partial z} \right)^2 + \left(\frac{\partial \delta}{\partial z} \right)^2 \right\} dz \quad (5.11)$$

For small displacement Δ , the potential energy in a blade element can be written as

$$U_z = \frac{1}{2g} \int_0^{\ell} W_s R \omega^2 \int_0^z \left\{ \left(\frac{\partial \xi}{\partial z} \right)^2 + \left(\frac{\partial \delta}{\partial z} \right)^2 \right\} dz dz \quad (5.12)$$

Thus for the entire blade, the total strain energy increase caused by the centrifugal force can be obtained by the addition of the equations (3.3), (5.9) and (5.12)

$$U_f = \frac{W_b \omega^2}{2g} \int_0^L \left\{ \left(r_2 + z \right) + \frac{W_s R}{W_b} \right\} \int_0^z \left\{ \left(\frac{\partial \xi}{\partial z} \right)^2 + \left(\frac{\partial \delta}{\partial z} \right)^2 \right\} dz - \delta^2 - \frac{W_s}{W_b} \delta_{\max} \delta \right\} dz \quad (5.13)$$

Using the relationship

$$\begin{aligned}\xi &= v_1 \cos \phi + u_1 \sin \phi \\ \delta &= v_1 \sin \phi - u_1 \cos \phi\end{aligned}\quad (5.14)$$

The total strain energy of the k^{th} element along the blade can be written as,

$$\begin{aligned}V_b &= \int_0^\ell \left\{ \frac{EI_{yy}}{2} \left(\frac{\partial^2 u}{\partial z^2} \right)^2 + \frac{EI_{xx}}{2} \left(\frac{\partial^2 v}{\partial z^2} \right)^2 + \frac{C}{2} \left(\frac{\partial \theta}{\partial z} \right)^2 + \frac{EA}{2} \left(\frac{\partial w}{\partial z} \right)^2 \right\} dz + \\ &\frac{W_b \omega^2}{2g} \int_0^\ell [r_2 + (k-1)\ell + C_{sh1} + z_k] \left[\int_0^\ell \left(\frac{\partial u_1}{\partial z_1} \right)^2 dz_1 + \dots + \int_0^{z_k} \left(\frac{\partial u_k}{\partial z_k} \right)^2 dz_k + \int_0^\ell \left(\frac{\partial v_1}{\partial z_1} \right)^2 dz_1 + \dots + \right. \\ &\left. \int_0^{z_k} \left(\frac{\partial v_k}{\partial z_k} \right)^2 dz_k \right] dz - \frac{W_b \omega^2}{2g} \left[\int_0^\ell (u_1^2 \cos^2 \phi - 2u_1 v_1 \sin \phi \cos \phi + v_1^2 \sin^2 \phi) dz - \right. \\ &C_{sh2} \left\{ \int_0^\ell ((u_1^T u_{max} + u_{max}^T u_1) \cos^2 \phi - (v_1^T u_{max} + u_{max}^T v_1 + u_1^T v_{max} + v_{max}^T u_1) \sin \phi \cos \phi + \right. \\ &\left. (v_1^T v_{max} + v_{max}^T v_1) \sin^2 \phi) dz \right\} \end{aligned}\quad (5.15)$$

where

$$C_{sh1} = \frac{W_s R}{W_b}, \quad C_{sh2} = \frac{W_s}{2W_b} \quad (5.16)$$

Using deflection functions in cubic polynomial form for u , v and linear θ , w , being given by equation (3.10) and substituting into equation (5.15) and replacing a_r , b_r ($r=0,1,2,3$) and c_r , d_r ($r=0,1$) and after integration carried out from 0 to ℓ then strain energy expression of a blade with shroud attached at the tip becomes;

$$V = \frac{EI_{xx}}{2L^3} \{q\}^T [C^*]^{-T} [[S_s] + [A_s] + [E_s] + [E_l] + [E_T] + C_{sh2} [E_T]] [C^*]^{-1} \{q\} \quad (5.17)$$

$$[E_s] = \Gamma \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{A_2}{2} & \frac{A_3}{3} & \frac{A_4}{4} & 0 & 0 & 0 & 0 & 0 & \frac{d_x A_2}{2} & 0 & 0 \\ & & (\frac{4}{3})\frac{A_3}{3} & (\frac{6}{4})\frac{A_5}{5} & 0 & 0 & 0 & 0 & 0 & \frac{d_x A_3}{3} & 0 & 0 \\ & & & (\frac{9}{5})\frac{A_6}{6} & 0 & 0 & 0 & 0 & 0 & \frac{d_x A_4}{4} & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \frac{A_2}{2} & \frac{A_3}{3} & \frac{A_4}{4} & 0 & \frac{d_y A_2}{2} & 0 & 0 \\ & & & & & & (\frac{4}{3})\frac{A_4}{4} & (\frac{6}{4})\frac{A_5}{5} & 0 & \frac{d_y A_3}{3} & 0 & 0 \\ & & & & & & & (\frac{9}{5})\frac{A_6}{6} & 0 & \frac{d_y A_4}{4} & 0 & 0 \\ & & & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & & & \frac{(d_x^2 + d_y^2)A_2}{2} & 0 & 0 \\ & & & & & & & & & & 0 & 0 \\ & & & & & & & & & & & 0 \end{bmatrix}$$

Symmetric

$$[E_t] = H \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{A_3}{3} & \frac{A_4}{4} & \frac{A_5}{5} & 0 & 0 & 0 & 0 & 0 & \frac{d_x A_3}{3} & 0 & 0 \\ & & (\frac{4}{3})\frac{A_5}{5} & (\frac{6}{4})\frac{A_6}{6} & 0 & 0 & 0 & 0 & 0 & \frac{d_x A_4}{4} & 0 & 0 \\ & & & (\frac{9}{5})\frac{A_7}{7} & 0 & 0 & 0 & 0 & 0 & \frac{d_x A_5}{5} & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \frac{A_3}{3} & \frac{A_4}{4} & \frac{A_5}{5} & 0 & \frac{d_y A_3}{3} & 0 & 0 \\ & & & & & & (\frac{4}{3})\frac{A_5}{5} & (\frac{6}{4})\frac{A_6}{6} & 0 & \frac{d_y A_4}{4} & 0 & 0 \\ & & & & & & & (\frac{9}{5})\frac{A_7}{7} & 0 & \frac{d_y A_5}{5} & 0 & 0 \\ & & & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & & & \frac{(d_x^2 + d_y^2)A_3}{3} & 0 & 0 \\ & & & & & & & & & & 0 & 0 \\ & & & & & & & & & & & 0 \end{bmatrix}$$

Symmetric

	[E _T]=H											
Symmetric	SS A ₁	$\frac{SS A_2}{2}$	$\frac{SS A_3}{3}$	$\frac{SS A_4}{4}$	SC A ₁	$\frac{SC A_2}{2}$	$\frac{SC A_3}{3}$	$\frac{SC A_4}{4}$	SS d _x A ₁ +SC d _y A ₁	SS d _x $\frac{A_2}{2}$ +SC d _y $\frac{A_2}{2}$	0	0
	$\frac{SS A_3}{3}$	$\frac{SS A_4}{4}$	$\frac{SS A_5}{5}$	$\frac{SS A_6}{6}$	$\frac{SC A_2}{2}$	$\frac{SC A_3}{3}$	$\frac{SC A_4}{4}$	$\frac{SC A_5}{5}$	SS d _x $\frac{A_2}{2}$ +SC d _y $\frac{A_2}{2}$	SS d _x $\frac{A_3}{3}$ +SC d _y $\frac{A_3}{3}$	0	0
	$\frac{SS A_4}{4}$	$\frac{SS A_5}{5}$	$\frac{SS A_6}{6}$	$\frac{SS A_7}{7}$	$\frac{SC A_3}{3}$	$\frac{SC A_4}{4}$	$\frac{SC A_5}{5}$	$\frac{SC A_6}{6}$	SS d _x $\frac{A_3}{3}$ +SC d _y $\frac{A_3}{3}$	SS d _x $\frac{A_4}{4}$ +SC d _y $\frac{A_4}{4}$	0	0
	$\frac{SS A_5}{5}$	$\frac{SS A_6}{6}$	$\frac{SS A_7}{7}$		$\frac{SC A_4}{4}$	$\frac{SC A_5}{5}$	$\frac{SC A_6}{6}$	$\frac{SC A_7}{7}$	SS d _x $\frac{A_4}{4}$ +SC d _y $\frac{A_4}{4}$	SS d _x $\frac{A_5}{5}$ +SC d _y $\frac{A_5}{5}$	0	0
				CC A ₁	$\frac{CC A_2}{2}$	$\frac{CC A_3}{3}$	$\frac{CC A_4}{4}$	$\frac{CC A_5}{5}$	CC d _x A ₁ +SC d _x A ₁	CC d _y $\frac{A_2}{2}$ +SC d _x $\frac{A_2}{2}$	0	0
					$\frac{CC A_3}{3}$	$\frac{CC A_4}{4}$	$\frac{CC A_5}{5}$	$\frac{CC A_6}{6}$	CC d _x $\frac{A_3}{3}$ +SC d _x $\frac{A_3}{3}$	CC d _y $\frac{A_3}{3}$ +SC d _x $\frac{A_3}{3}$	0	0
						$\frac{CC A_4}{4}$	$\frac{CC A_5}{5}$	$\frac{CC A_6}{6}$	CC d _x $\frac{A_4}{4}$ +SC d _x $\frac{A_4}{4}$	CC d _y $\frac{A_4}{4}$ +SC d _x $\frac{A_4}{4}$	0	0
							$\frac{CC A_5}{5}$	$\frac{CC A_6}{6}$	CC d _x $\frac{A_5}{5}$ +SC d _x $\frac{A_5}{5}$	CC d _y $\frac{A_5}{5}$ +SC d _x $\frac{A_5}{5}$	0	0
								$\frac{CC A_6}{6}$	CC d _x $\frac{A_6}{6}$ +SC d _x $\frac{A_6}{6}$	CC d _y $\frac{A_6}{6}$ +SC d _x $\frac{A_6}{6}$	0	0
								$\frac{CC A_7}{7}$	CC d _x $\frac{A_7}{7}$ +SC d _x $\frac{A_7}{7}$	CC d _y $\frac{A_7}{7}$ +SC d _x $\frac{A_7}{7}$	0	0
									CC d _x $\frac{A_8}{8}$ +SC d _x $\frac{A_8}{8}$	CC d _y $\frac{A_8}{8}$ +SC d _x $\frac{A_8}{8}$	0	0
									CC d _x $\frac{A_9}{9}$ +SC d _x $\frac{A_9}{9}$	CC d _y $\frac{A_9}{9}$ +SC d _x $\frac{A_9}{9}$	0	0
									CC d _x $\frac{A_{10}}{10}$ +SC d _x $\frac{A_{10}}{10}$	CC d _y $\frac{A_{10}}{10}$ +SC d _x $\frac{A_{10}}{10}$	0	0

External work will be done when the blade is subjected to a periodic axial force $P(t)$. The additional energy V_{gb} due to the periodic force can be expressed by Chen & Chern, (1992) as,

$$V_{gb} = \frac{P(t)}{2} \int_0^L \left(\left(\frac{\partial u_1}{\partial z} \right)^2 + \left(\frac{\partial v_1}{\partial z} \right)^2 \right) dz \quad (5.18)$$

If the relationship between u_1 and u , v_1 and v being given by equation (3.8) is used and substituted by the cubic polynomial expansion for u and v and linear expression for θ given in equation (3.10) into equation (5.18) and replacing the coefficients by a_r, b_r ($r = 0, 1, 2, 3$) and c_r ($r = 0, 1$) and after integration carried out then equation (5.18) becomes

$$V_{gb} = \frac{P}{2} \{q\}^T [C^*]^T [K_{gb}] [C^*]^{-1} \{q\} \quad (5.19)$$

$$[K_{gb}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & A_1 & A_2 & A_3 & 0 & 0 & 0 & 0 & 0 & d_x A_1 & 0 & 0 \\ & & \frac{4}{3} A_3 & \frac{6}{4} A_4 & 0 & 0 & 0 & 0 & 0 & d_x A_2 & 0 & 0 \\ & & & \frac{9}{5} A_5 & 0 & 0 & 0 & 0 & 0 & d_x A_3 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & A_1 & A_2 & A_3 & 0 & d_y A_1 & 0 & 0 \\ & & & & & & \frac{4}{3} A_3 & \frac{6}{4} A_4 & 0 & d_y A_2 & 0 & 0 \\ & & & & & & & \frac{9}{5} A_5 & 0 & d_y A_3 & 0 & 0 \\ & & & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & & & (d_x^2 + d_y^2) A_1 & 0 & 0 \\ & & & & & & & & & & 0 & 0 \\ & & & & & & & & & & & 0 \end{bmatrix}$$

Symmetric

Where

$$\{\mathbf{q}\}^T = \begin{bmatrix} v_1 & \dot{v}_1 & u_1 & \dot{u}_1 & \theta_1 & w_1 & v_2 & \dot{v}_2 & u_2 & \dot{u}_2 & \theta_2 & w_2 \end{bmatrix}$$

$$\Lambda = H \left[\sum_{k=2}^n \{ (k-1)\ell + r_2 + 0.5 \} + C_{sh1} \right]$$

$$\Gamma = H[(k-1)\ell + r_2 + C_{sh1}]$$

$$H = \frac{\rho A \omega^2}{E I_{xx}}$$

$$A_1 = L^3 \ell, A_2 = L^3 \ell^2, A_3 = L^3 \ell^3, A_4 = L^3 \ell^4, A_5 = L^3 \ell^5, A_6 = L^3 \ell^6, A_7 = L^3 \ell^7$$

$$P = \frac{I_{yy}}{I_{xx}}$$

$$R = \frac{C}{E I_{xx}}$$

$$S = \frac{A}{I_{xx}}$$

$$CC = -\sin^2 \phi$$

$$SS = -\cos^2 \phi$$

$$CS = \sin \phi \cos \phi$$

5.1.1.2 Kinetic energy of a straight blade

The kinetic energy T of a blade vibrating in combined bending-bending- torsion mode is given as

$$T = \frac{\rho}{2} \int_0^L \left\{ I_{cg} \dot{\theta}^2 + A (\dot{u}_1^2 + \dot{v}_1^2 + \dot{w}^2) \right\} dz \quad (5.20)$$

Using equations (3.23) and (3.25), kinetic energy of a blade becomes

$$T = \frac{\rho}{2} \int_0^L \left\{ I_{cg} \dot{\theta}^2 + A (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + 2A\dot{u}\dot{\theta}dy + 2A\dot{v}\dot{\theta}dx \right\} dz \quad (5.21)$$

When the shear center distance from the centroid is constant along the blade and there is no pretwist, the equation (3.9) can be simplified to

$$d_x = d_x, \quad d_y = d_y \quad (5.22)$$

Substituting the cubic polynomial expansion for u and v and linear expression for θ and w given in equation (3.10) into equation (5.21) and replacing the coefficients a_r, b_r ($r = 0, 1, 2, 3$) and c_r, d_r ($r = 0, 1$) and after integration carried out then kinetic energy expression becomes

$$T = \frac{\rho AL}{2} \{q\}^T [C^*]^{-T} [m_{cs}] [C^*]^{-1} \{q\} \quad (5.23)$$

$[C^*]^{-1}$ is given in appendix 1.

$$[m_{cs}] = \begin{bmatrix} Kt_1 & Kt_2 & Kt_3 & Kt_4 & 0 & 0 & 0 & 0 & d_x Kt_1 & d_x Kt_2 & 0 & 0 \\ & Kt_3 & Kt_4 & Kt_5 & 0 & 0 & 0 & 0 & d_x Kt_2 & d_x Kt_3 & 0 & 0 \\ & & Kt_5 & Kt_6 & 0 & 0 & 0 & 0 & d_x Kt_3 & d_x Kt_4 & 0 & 0 \\ & & & Kt_7 & 0 & 0 & 0 & 0 & d_x Kt_4 & d_x Kt_5 & 0 & 0 \\ & & & & Kt_1 & Kt_2 & Kt_3 & Kt_4 & d_y Kt_1 & d_y Kt_2 & 0 & 0 \\ & & & & & Kt_3 & Kt_4 & Kt_5 & d_y Kt_2 & d_y Kt_3 & 0 & 0 \\ & & & & & & Kt_5 & Kt_6 & d_y Kt_3 & d_y Kt_4 & 0 & 0 \\ & & & & & & & Kt_7 & d_y Kt_4 & d_y Kt_5 & 0 & 0 \\ & & & & & & & & CT_1 & CT_2 & 0 & 0 \\ & & & & & & & & & CT_3 & 0 & 0 \\ & & & & & & & & & & Kt_1 & Kt_2 \\ & & & & & & & & & & & Kt_3 \end{bmatrix}$$

Symmetric

Where

$$Kt_1 = \frac{\ell}{L}$$

$$Kt_2 = \frac{\ell^2}{2L}$$

$$Kt_3 = \frac{\ell^3}{3L}$$

$$Kt_4 = \frac{\ell^4}{4L}$$

$$Kt_5 = \frac{\ell^5}{5L}$$

$$Kt_6 = \frac{\ell^6}{6L}$$

$$Kt_7 = \frac{\ell^7}{7L}$$

$$CT_1 = \frac{I_{cf} \ell}{LA}$$

$$CT_2 = \frac{I_{cf} \ell^2}{2LA}$$

$$CT_3 = \frac{I_{cf} \ell^3}{3LA}$$

5.2 Energy expression of a pretwisted blade packet

When the individual energy of each part of the packet is considered separately, it can be noticed that the energy of a curved beam is not affected by the blade pretwist in the packet, although the pretwist alters the characteristics of the system. Energy expressions of a curved beam, have been given above and is not repeated here. Considering only the effect of pretwisted on the blade energy, energy expressions of a blade are given as follows;

5.2.1 Strain energy of a shrouded-blade element in a centrifugal field

The strain energy equation of a rotating shrouded pretwisted blade element can be written in dimensional form for the k^{th} element as

$$\begin{aligned}
 V = & \int_0^\ell \left\{ \frac{EI_{yy}}{2} \left(\frac{\partial^2 u}{\partial z^2} \right)^2 + \frac{EI_{xx}}{2} \left(\frac{\partial^2 v}{\partial z^2} \right)^2 + EI_{xy} \left(\frac{\partial^2 u}{\partial z^2} \right) \left(\frac{\partial^2 v}{\partial z^2} \right) + \frac{C}{2} \left(\frac{\partial \theta}{\partial z} \right)^2 + \frac{EA}{2} \left(\frac{\partial w}{\partial z} \right)^2 \right\} dz + \\
 & \frac{W_b \omega^2}{2g} \int_0^\ell [r_2 + (k-1)\ell + C_{sh1} + z_k] \left[\int_0^\ell \left(\frac{\partial u_1}{\partial z_1} \right)^2 dz_1 + \dots + \int_0^{z_k} \left(\frac{\partial u_k}{\partial z_k} \right)^2 dz_k + \int_0^\ell \left(\frac{\partial v_1}{\partial z_1} \right)^2 dz_1 + \dots + \right. \\
 & \left. \int_0^{z_k} \left(\frac{\partial v_k}{\partial z_k} \right)^2 dz_k \right] dz - \frac{W_b \omega^2}{2g} \left[\int_0^\ell (u_1^2 \cos^2 \phi - 2u_1 v_1 \sin \phi \cos \phi + v_1^2 \sin^2 \phi) dz - \right. \\
 & C_{sh2} \left\{ \int_0^\ell ((u_1^T u_{\max} + u_{\max}^T u_1) \cos^2 \phi - (v_1^T u_{\max} + u_{\max}^T v_1 + u_1^T v_{\max} + v_{\max}^T u_1) \sin \phi \cos \phi + \right. \\
 & \left. \left. (v_1^T v_{\max} + v_{\max}^T v_1) \sin^2 \phi) dz \right\} \right]
 \end{aligned} \tag{5.24}$$

where

$$C_{sh1} = \frac{W_s R}{W_b}, \quad C_{sh2} = \frac{W_s}{2W_b} \tag{5.25}$$

The variation of pretwist along the blade length can be written in dimensional form as

$$\gamma = (k-1) \frac{\alpha_p}{L} \ell + \frac{\alpha_p}{L} z \quad (5.26)$$

Where k is the order number of elements.

The relationship between the moment of inertias,

$$\begin{aligned} I_{xx} &= I_{YY} \sin^2(\alpha_b \eta + \phi_i) + I_{XX} \cos^2(\alpha_b \eta + \phi_i) \\ I_{yy} &= I_{YY} \cos^2(\alpha_b \eta + \phi_i) + I_{XX} \sin^2(\alpha_b \eta + \phi_i) \\ I_{xy} &= \frac{(I_{YY} - I_{XX}) \sin 2(\alpha_b \eta + \phi_i)}{2} \end{aligned} \quad (5.27)$$

By using the relationship between u_1 and u , v_1 and v , from equation (3.8) and using the sign conversion, d_x and d_y become

$$\begin{aligned} d_x &= d_X \cos(\alpha_b \eta + \phi_i) - d_Y \sin(\alpha_b \eta + \phi_i) \\ d_y &= d_X \sin(\alpha_b \eta + \phi_i) + d_Y \cos(\alpha_b \eta + \phi_i) \end{aligned} \quad (5.28)$$

Using deflection functions in cubic polynomial form for u , v and linear θ , w being given by equation (3.10) and substituting into equation (5.24) and replacing a_r , b_r ($r=0,1,2,3$) and c_r , d_r ($r=0,1$) and integrating from 0 to ℓ , we have the stiffness matrices of non-rotating and rotating pretwisted blade elements $[k_s]$ and $[k_r]$ respectively, whose elements are given in dimensional form as follows;

$k_s = \text{zeros}(12,12)$

$k_s(3,3) = AA$

$k_s(3,4) = BB$

$k_s(3,7) = FF$

$k_s(3,8) = GG$

$k_s(4,4) = DD$

$k_s(4,7) = GG$

$k_s(4,8) = GG$

$k_s(7,7) = PP$

$k_s(7,8) = QQ$

$k_s(8,8) = RR$

$k_s(10,10) = R \ A1$

$k_s(12,12) = S \ A1$

$k_r(1,1) = SS \ A1$

$k_r(1,2) = SS \ A2$

$k_r(1,3) = SS \ A3$

$k_r(1,4) = SS \ A4$

$k_r(1,5) = CS \ A1$

$k_r(1,6) = CS \ A2$

$k_r(1,7) = CS \ A3$

$k_r(1,8) = CS \ A4$

$k_r(1,9) = TS \ R15 + RS \ I15$

$k_r(1,10) = TS \ R16 + RS \ I16$

$k_r(2,1) = k_r(1,2)$

$k_r(2,2) = SS \ A3 + \psi \ A1 + A23$

$k_r(2,3) = SS \ A4 + \psi \ A2 + A34$

$k_r(2,4) = SS \ A5 + \psi \ A3 + A45$

$k_r(2,5) = CS \ A2$

$k_r(2,6) = CS \ A3$

$k_r(2,7) = CS \ A4$

$$k_r(2,8)=CS\ A5$$

$$k_r(2,9)=TS\ R25+RS\ I25+H\ (\Lambda\ G25+\Gamma\ FI25+FS25)\ GC$$

$$k_r(2,10)=TS\ R26+RS\ I26+H\ (\Lambda\ G26+\Gamma\ FI26+FS26)\ GC$$

$$k_r(3,1)=k_r(1,3)$$

$$k_r(3,2)=k_r(2,3)$$

$$k_r(3,3)=(4/3)A45 +4\ \psi\ A3+SS\ A5$$

$$k_r(3,4)=1.5\ A56 +6\ \psi\ A4+SS\ A6$$

$$k_r(3,5)=CS\ A3$$

$$k_r(3,6)=CS\ A4$$

$$k_r(3,7)=CS\ A5$$

$$k_r(3,8)=CS\ A6$$

$$k_r(3,9)=TS\ R35+RS\ I35+H\ (\Lambda\ G35+\Gamma\ FI35+FS35)\ GC$$

$$k_r(3,10)=TS\ R36+RS\ I36+H\ (\Lambda\ G36+\Gamma\ FI36+FS36)\ GC$$

$$k_r(4,1)=k_r(1,4)$$

$$k_r(4,2)=k_r(2,4)$$

$$k_r(4,3)=k_r(3,4)$$

$$k_r(4,4)=(9/5)\ A67 +9\ \psi\ A5+SS\ A7$$

$$k_r(4,5)=CS\ A4$$

$$k_r(4,6)=CS\ A5$$

$$k_r(4,7)=CS\ A6$$

$$k_r(4,8)=CS\ A7$$

$$k_r(4,9)=TS\ R45+RS\ I45H +(\Lambda\ G45+\Gamma\ FI45+FS45)\ GC$$

$$k_r(4,10)=TS\ R46+RS\ I46H +(\Lambda\ G46+\Gamma\ FI46+FS46)\ GC$$

$$k_r(5,1)=k_r(1,5)$$

$$k_r(5,2)=k_r(2,5)$$

$$k_r(5,3)=k_r(3,5)$$

$$k_r(5,4)=k_r(4,5)$$

$$k_r(5,5)=CC\ A1$$

$$k_r(5,6)=CC\ A2$$

$$k_r(5,7)=CC\ A3$$

$$k_r(5,8)=CC\ A4$$

$$k_r(5,9)=TC\ I15+RS\ R15$$

$$k_r(5,10)=TC\ I16+RS\ R16$$

$$k_r(6,1)=k_r(1,6)$$

$$k_r(6,2)=k_r(2,6)$$

$$k_r(6,3)=k_r(3,6)$$

$$k_r(6,4)=k_r(4,6)$$

$$k_r(6,5)=k_r(5,6)$$

$$k_r(6,6)=CC\ A3+\psi\ A1+A23$$

$$k_r(6,7)=CC\ A4+2\ \psi\ A2+A34$$

$$k_r(6,8)=CC\ A5+3\ \psi\ A3+A45$$

$$k_r(6,9)=TC\ I25+RS\ R25+H\ (\Lambda\ Z25+\Gamma\ TI25+TS25)\ GC$$

$$k_r(6,10)=TC\ I26+RS\ R26+H\ (\Lambda\ Z26+\Gamma\ I26+TS26)\ GC$$

$$k_r(7,1)=k_r(1,7)$$

$$k_r(7,2)=k_r(2,7)$$

$$k_r(7,3)=k_r(3,7)$$

$$k_r(7,4)=k_r(4,7)$$

$$k_r(7,5)=k_r(5,7)$$

$$k_r(7,6)=k_r(6,7)$$

$$k_r(7,7)=(4/3)A45+4\psi\ A3+CC\ A5$$

$$k_r(7,8)=1.5\ A56+6\psi\ A4+CC\ A6$$

$$k_r(7,9)=TC\ I35+RS\ R35+H\ (\Lambda\ Z35+\Gamma\ TI35+TS35)\ G35$$

$$k_r(7,10)=TC\ I36+RS\ R36+H\ (\Lambda\ Z36+\Gamma\ TI36+TS36)\ G35$$

$$k_r(8,1)=k_r(1,8)$$

$$k_r(8,2)=k_r(2,8)$$

$$k_r(8,3)=k_r(3,8)$$

$$k_r(8,4)=k_r(4,8)$$

$$k_r(8,5)=k_r(5,8)$$

$$k_r(8,6)=k_r(6,8)$$

$$k_r(8,7)=k_r(7,8)$$

$$k_r(8,8)=(9/5) A67 +9 \psi A5+CC A7$$

$$k_r(8,9)= TC I45+RS R45+H (\Lambda Z45+\Gamma TI45+TS45) GC$$

$$k_r(8,10)= TC I46+RS R46+H (\Lambda Z46+\Gamma TI46+TS46) GC$$

$$k_r(9,1)=k_r(1,9)$$

$$k_r(9,2)=k_r(2,9)$$

$$k_r(9,3)=k_r(3,9)$$

$$k_r(9,4)=k_r(4,9)$$

$$k_r(9,5)=k_r(5,9)$$

$$k_r(9,6)=k_r(6,9)$$

$$k_r(9,7)=k_r(7,9)$$

$$k_r(9,8)=k_r(8,9)$$

$$k_r(9,9)=TC I55+TS R55+ 2 RS Y55+H (\Lambda (Z55+G55)+\Gamma (TI55+FI55)+TS55+FS55) GC$$

$$k_r(9,10)= TC I56+TS R56+2RS Y56+H (\Lambda (Z56+G56)+\Gamma (TI56+FI56)+TS56+FS56) GC$$

$$k_r(10,1)= k_r(1,10)$$

$$k_r(10,2)= k_r(2,10)$$

$$k_r(10,3)= k_r(3,10)$$

$$k_r(10,4)= k_r(4,10)$$

$$k_r(10,5)= k_r(5,10)$$

$$k_r(10,6)= k_r(6,10)$$

$$k_r(10,7)= k_r(7,10)$$

$$k_r(10,8)=k_r(8,10)$$

$$k_r(10,9)= k_r(9,10)$$

$$k_r(10,10)= TC I66+TS R66+2RS Y66 +H (\Lambda (Z66+G66)+\Gamma (TI66+FI66)+TS66+FS66) GC$$

$$H = \frac{\rho A \omega^2}{E I_{xx}}$$

$$\Lambda = \Lambda + (r_2 A_1 + 2 A_2 (k-1)) / L^3 + A_2 / L^3 + L^3 (t b R^2 \alpha_b \pi) / (180 A \rho A_1) \rho_s$$

$$\Gamma = (r_2 A_1 + 2 A_2 (k-1)) / L^3 + L^3 (t b R^2 \alpha_b \pi \rho_s) / (180 A \rho)$$

$$\psi = \Lambda H$$

$$A_1 = L^3 \ell, \quad A_2 = L^3 \ell^2 / 2, \quad A_3 = L^3 \ell^3 / 3, \quad A_4 = L^3 \ell^4 / 4, \quad A_5 = L^3 \ell^5 / 5, \quad A_6 = L^3 \ell^6 / 6, \\ A_7 = L^3 \ell^7 / 7$$

$$R = \frac{C}{E I_{xx}}$$

$$S = \frac{A}{I_{xx}}$$

$$CC = -H \sin^2 \phi$$

$$SS = -H \cos^2 \phi$$

$$CS = H \sin \phi \cos \phi$$

$$GC = L^3$$

$$TC = CC L^3$$

$$TS = SS L^3$$

$$RS = CS L^3$$

$$A_{23} = H(\Gamma A_2 + A_3)$$

$$A_{34} = H(\Gamma A_3 + A_4)$$

$$A_{45} = H(\Gamma A_4 + A_5)$$

$$A_{56} = H(\Gamma A_5 + A_6)$$

$$A_{67} = H(\Gamma A_6 + A_7)$$

External work will be done when the blade is subjected to a periodic axial force $P(t)$. The additional energy V_{gb} due to the periodic force can be expressed (Chen & Chern, 1992) as,

$$V_{gb} = \frac{P(t)}{2} \int_0^L \left(\left(\frac{\partial u_1}{\partial z} \right)^2 + \left(\frac{\partial v_1}{\partial z} \right)^2 \right) dz \quad (5.29)$$

If the relationships between u_1 and u , v_1 and v being given by equation (3.8) and equation (5.28) are used and substituted by the cubic polynomial expansion for u and v and linear expression for θ given in equation (3.10) into equation (5.29) and replacing the coefficients by a_r , b_r ($r = 0,1,2,3$) and c_r ($r = 0,1$) and after integration carried out from 0 to ℓ , then equation (5.29) becomes

$$V_{gb} = \frac{P}{2} \{q\}^T [C^*]^T [K_{gbp}] [C^*]^{-1} \{q\} \quad (5.30)$$

$$[K_{gbp}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \ell & \ell^2 & \ell^3 & 0 & 0 & 0 & 0 & G25 & G26 & 0 & 0 \\ & & \frac{4}{3}\ell^3 & \frac{6}{4}\ell^4 & 0 & 0 & 0 & 0 & G35 & G36 & 0 & 0 \\ & & & \frac{9}{5}\ell^5 & 0 & 0 & 0 & 0 & G45 & G46 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \ell & \ell^2 & \ell^3 & Z25 & Z26 & 0 & 0 \\ & & & & & & \frac{4}{3}\ell^3 & \frac{6}{4}\ell^4 & Z35 & Z36 & 0 & 0 \\ & & & & & & & \frac{9}{5}\ell^5 & Z45 & Z46 & 0 & 0 \\ & & & & & & & & G55 + Z55 & G56 + Z56 & 0 & 0 \\ & & & & & & & & & Z66 + G66 & 0 & 0 \\ & & & & & & & & & & 0 & 0 \\ & & & & & & & & & & & 0 \end{bmatrix}$$

Symmetric

The details used to calculate the variables are given in appendix 3.

5.2.2 Kinetic energy of a pretwisted blade

The kinetic energy expression of a pretwisted blade element under combined bending-bending-torsional vibration is given in dimensional form in equation (3.23). As the flexure center from the centroid is varying along the length of the blade due to pretwist, the expressions of these variations can be written in dimensional form, as in equation (5.28).

Using equation (5.28) and also for the representation of the displacements equation (3.8) and replacing the coefficients a_r , b_r ($r=0,1,2,3$) and c_r , d_r ($r=0,1$) and integrating from 0 to ℓ , the kinetic energy of a beam element becomes

$$T = \frac{\rho A L}{2} \{q\}^T [C^*]^T [m_{cp}] [C^*]^{-1} \{q\} \quad (5.31)$$

$[C^*]^{-1}$ is given in appendix 1.

$$[m_{cp}] = \begin{bmatrix} Kt_1 & Kt_2 & Kt_3 & Kt_4 & 0 & 0 & 0 & 0 & A3S/L & B3S/L & 0 & 0 \\ & Kt_3 & Kt_4 & Kt_5 & 0 & 0 & 0 & 0 & B3S/L & C3S/L & 0 & 0 \\ & & Kt_5 & Kt_6 & 0 & 0 & 0 & 0 & C3S/L & D3S/L & 0 & 0 \\ & & & Kt_7 & 0 & 0 & 0 & 0 & D3S/L & E3S/L & 0 & 0 \\ & & & & Kt_1 & Kt_2 & Kt_3 & Kt_4 & AS/L & BS/L & 0 & 0 \\ & & & & & Kt_3 & Kt_4 & Kt_5 & BS/L & CS/L & 0 & 0 \\ & & & & & & Kt_5 & Kt_6 & CS/L & DS/L & 0 & 0 \\ & & & & & & & Kt_7 & DS/L & ES/L & 0 & 0 \\ & & & & & & & & & CT_1 & CT_2 & 0 & 0 \\ & & & & & & & & & & CT_3 & 0 & 0 \\ & & & & & & & & & & & Kt_1 & Kt_2 \\ & & & & & & & & & & & & Kt_3 \end{bmatrix}$$

Symmetric

Where

$$\begin{aligned}
 Kt_1 &= \frac{\ell}{L} & CT_1 &= \frac{I_{cf}l}{LA} \\
 Kt_2 &= \frac{\ell^2}{2L} & CT_2 &= \frac{I_{cf}\ell^2}{2LA} \\
 Kt_3 &= \frac{\ell^3}{3L} & CT_3 &= \frac{I_{cf}\ell^3}{3LA} \\
 Kt_4 &= \frac{\ell^4}{4L} \\
 Kt_5 &= \frac{\ell^5}{5L} \\
 Kt_6 &= \frac{\ell^6}{6L} \\
 Kt_7 &= \frac{\ell^7}{7L}
 \end{aligned}$$

Where the details of other variables are given in appendix 3.

5.3 Assembly procedure and theoretical consideration

The finite element model of the curved packet of straight blades and pretwisted blades having aerofoil cross section is formulated as follows.

The shroud is represented by a curved beam element and the blades are modeled by beam elements with 12 degrees of freedom which represent the coupled bending-bending-torsion vibration. However, to ensure that the compatibility conditions satisfied at the shroud-blade connections, in-plane and out-of-plane curved beam element matrices are combined to give overall matrices which give the vibration characteristics of the curved beam in three directions.

Stiffness and mass matrices of a curved beam in the in-plane and out-of-plane directions are given by equations (4.3), (4.19), (4.23), (4.34) respectively.

To form the combination of the matrices, in-plane mass and stiffness matrices can be written as follows;

$$\rho_s A_s R [m_I] = \rho_s A_s \ell_{sh} \frac{1}{\beta} [m_I], \quad \beta = \frac{\ell_{sh}}{R} \quad (5.24)$$

$$\frac{E_s I_{xxs}}{R^3} [k_{IC}] = \frac{E_s I_{xxs}}{\ell_{sh}^3} \beta^3 [k_{IC}] \quad (5.32)$$

Similarly out-of-plane mass and stiffness matrices can also be written as follows;

$$\rho_s A_s R^3 [m_o] = \rho_s A_s \ell_{sh}^3 \frac{1}{\beta^3} [m_o] \quad (5.33)$$

$$\frac{E_s I_{yys}}{R} [k_o] = \frac{E_s I_{yys}}{\ell_{sh}} \beta [k_o] \quad (5.34)$$

Corresponding degrees of freedom of in-plane and out-of-plane elements are given in equations (4.4), (4.24) as;

$$\{q_I\}^T = [v_i \quad w_i \quad i_i \quad v_{i+1} \quad w_{i+1} \quad i_{i+1}] \quad (5.35)$$

$$\{q_o\}^T = [\theta_i \quad u_i \quad j_i \quad \theta_{i+1} \quad u_{i+1} \quad j_{i+1}] \quad (5.36)$$

respectively.

Finally, in-plane and out-of-plane stiffness and mass matrices are combined as follows;

$$[k_s] = \frac{E_s I_{xxs}}{\ell_{sh}^3} \begin{bmatrix} S_1[k_{ic}] & 0 \\ 0 & S_2[k_o] \end{bmatrix} \quad (5.37)$$

$$[m_s] = \rho_s A_s \ell_{sh} \begin{bmatrix} M_1[m_1] & 0 \\ 0 & M_2[m_o] \end{bmatrix} \quad (5.38)$$

As defined previously, the sequences of generalized coordinates are given by;

$$\{q_s\}^T = [\{q_i\}^T : \{q_o\}^T] \quad (5.39)$$

and to normalize the two matrices, the appropriate factors are written as;

$$S_1 = \beta^3, \quad S_2 = R^2 \beta^3 \frac{I_{yys}}{I_{xxs}} \quad (5.40)$$

$$M_1 = \frac{1}{\beta}, \quad M_2 = \frac{R^2}{\beta} \quad (5.41)$$

To make the blade shroud assembly easier, the sequence of generalized coordinates of the curved beam is listed by node as shown below;

$$\{q_s\}^T = [\{q_i\}^T : \{q_{i+1}\}^T] \quad (5.42)$$

or more explicitly;

$$\{q_s\}^T = [w_i \ v_i \ u_i \ \theta_i \ j_i \ i_i \ w_{i+1} \ v_{i+1} \ u_{i+1} \ \theta_{i+1} \ j_{i+1} \ i_{i+1}] \quad (5.43)$$

An explicit view of the blade-shroud connection is shown in figures (5.4) and (5.5) with shroud and blade local coordinates and the global coordinates of the system.

To normalize the shroud and blade matrices the following appropriate factors are given and shroud stiffness and mass matrices become

$$[k] = \frac{EI_{xx}}{L^3} c_s [k_s] \quad (5.44)$$

$$[m] = \rho AL c_m [m_s] \quad (5.45)$$

where

$$c_s = \frac{E_s I_{xx_s}}{\ell_{sh}^3} \frac{L^3}{EI_{xx}}, \quad c_m = \frac{\rho_s A_s \ell_{sh}}{\rho AL} \quad (5.46)$$

To ensure that the compatibility conditions are satisfied at the shroud-blade connections the displacements in the local coordinates have to be converted to those in the global system.

$$\{q_L\} = [\Delta] \{q\} \quad (5.47)$$

This is done by the use of transformation matrices which are given as follows

For the blade and shroud matrices,

$$[\Delta] = \begin{bmatrix} [\Delta_1] & 0 & 0 & 0 \\ 0 & [\Delta_1] & 0 & 0 \\ 0 & 0 & [\Delta_1] & 0 \\ 0 & 0 & 0 & [\Delta_1] \end{bmatrix} \quad (5.48)$$

Where

$$[\Delta_1] = \begin{bmatrix} [\Delta_{o\xi}] \\ [\Delta_{o\delta}] \\ [\Delta_{oz}] \end{bmatrix} \quad (5.49)$$

$$\begin{aligned} [\Delta_{o\xi}] &= [l_{ox} \quad m_{ox} \quad n_{ox}] \\ [\Delta_{o\delta}] &= [l_{oy} \quad m_{oy} \quad n_{oy}] \\ [\Delta_{oz}] &= [l_{oz} \quad m_{oz} \quad n_{oz}] \end{aligned} \quad (5.50)$$

They show the matrices of direction cosines for the $o\xi$, $o\delta$, oz directions measured in the global axis system. If ϕ is the stagger angle of the blade as shown in figure (5.3) then the direction cosines of the blade element can be written as follows,

$$\begin{aligned} l_{ox} &= 0^\circ & m_{ox} &= 90^\circ & n_{ox} &= 90^\circ \\ l_{oy} &= 90^\circ & m_{oy} &= \phi & n_{oy} &= 90^\circ + \phi \\ l_{oz} &= 90^\circ & m_{oz} &= 90^\circ - \phi & n_{oz} &= \phi \end{aligned} \quad (5.51)$$

Similarly the direction cosines for the shroud are written as;

$$\begin{aligned} l_{ox} &= 90^\circ & m_{ox} &= 0^\circ & n_{ox} &= 90^\circ \\ l_{oy} &= 0^\circ & m_{oy} &= 90^\circ & n_{oy} &= 90^\circ \\ l_{oz} &= 90^\circ & m_{oz} &= 90^\circ & n_{oz} &= 180^\circ \end{aligned} \quad (5.52)$$

Therefore, before assembling the blade and shroud matrices the transformation into the global axes system has to be made by using the well known transformation operation,

$$[k_G] = [\Delta]^T [k] [\Delta] \quad (5.53)$$

$$[m_G] = [\Delta]^T [m] [\Delta] \quad (5.54)$$

Suffix G denotes the transformed matrix in the global axis system. Having formulated the mass and stiffness matrices of the packet structure, eigenvalue evaluation procedures are applied to these matrices.

5.4 Results

5.4.1 For straight two bladed packets

In this section some of the results obtained for the frequencies of vibration of various straight aerofoil cross section two bladed packets are shown and compared with some other researchers' results in order to indicate the degree of accuracy and adaptability that can be expected in the application of the finite element method.

Table 5.3 (a) and 5.3 (b) show the comparison between the theoretical and experimental results of two various aerofoil cross section two bladed packets under non-rotating conditions. It can be seen from the tables obtained, the results are in agreement with the experimental results. The theoretical and experimental results for the same packet under various rotational speeds are shown in table 5.4. The agreement between the results is reasonably good.

Table 5.5 shows the effect of shroud thickness on the buckling load parameters of a two bladed packet. When the shroud thickness decreases the critical buckling load parameter decreases for all modes. In the higher modes this decrease is more noticeable.

Table 5.6 shows the effect of shroud length on the buckling load parameters of a two bladed packet. While the shroud length decreases the buckling load parameters

increase for all modes, because the system becomes stiffer as the shroud length decreases.

Table 5.7 shows the stagger angle effect on the buckling load parameters of a two bladed packet. The system becomes more flexible in the out-of-plane direction as the stagger angle increases. Whereas it becomes stiffer in the plane of the curvature.

Table 5.8 shows the rotational speed effect on the buckling load parameters of a two bladed packet. The critical buckling load parameters always increase with the increase of rotational speed. This increase becomes more pronounced at higher angular velocities.

Table 5.9 shows the effect of the distance between shear center and centroid on the critical buckling load parameters of a two bladed packet. When the distance between shear center and centroid increases the buckling load parameters are not much affected so the coupling effect has less effect on the buckling load parameters even at higher modes.

Figure 5.5 shows the effect of the distance between shear center and centroid on the dynamic stability regions for the first nine modes. In the figure when the distance between shear center and centroid increases, the instability regions do not change for the lower modes as the coupling between bending and torsional modes on the dynamic stability regions is negligible.

In the higher modes such as seventh, eighth and ninth modes, when the distance between shear center and centroid increases, the instability regions shifted to the left for these modes as shown in the figure 5.6.

Figure 5.7 shows the rotational speed effect on the regions of dynamic instability of an aerofoil cross section two bladed packet for the first two modes. With an increase in rotational speed, the instability regions shift to the right and the widths of

the regions are decreased. Obviously, increase in the rotational speed increases the stiffness of the rotating two bladed packet and makes the system more stable.

Figure 5.8 shows the effect of stagger angle on the regions of dynamic instability for the first four modes. As stagger angle increases the regions are shifted towards the static buckling axis, and the width of these regions increases, thus the effect of increase in stagger angle makes the packet more sensitive to periodic excitation in the out-of-plane direction.

The effect of the shroud length on the dynamic stability region is shown in the figure 5.9. As the shroud length increases the regions are shifted to the left and the widths of these regions are increased. As a result, an increase in the shroud length decreases the stiffness of the packet. Thus it makes the packet more unstable against periodic in-plane forces.

Figure 5.10 shows the shroud thickness on the regions of dynamic instability for the first four modes. When the shroud thickness decreases the instability regions shift to the left (low frequency ratios) and the width of the regions increases. This makes the packet less stable to periodic excitation.

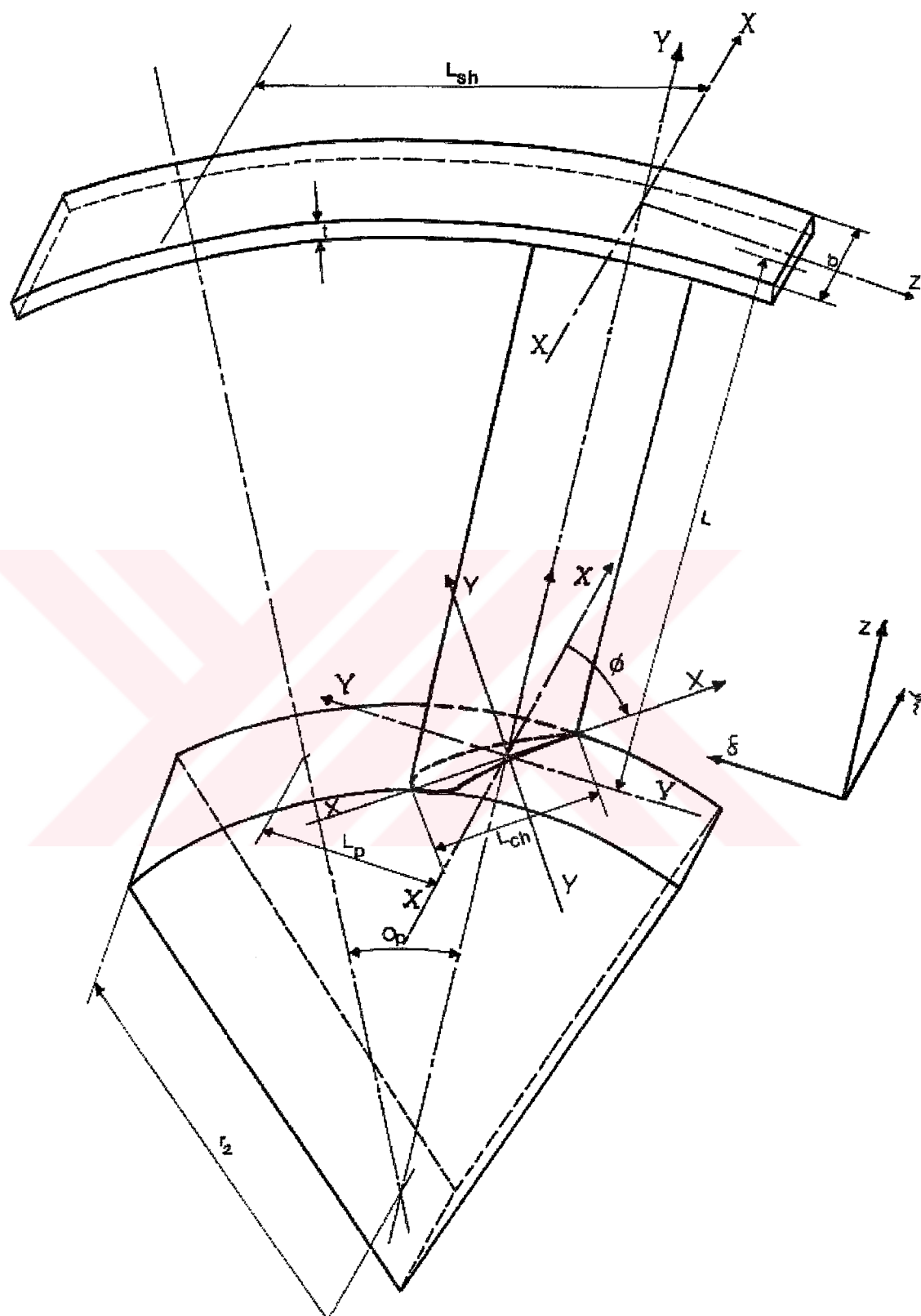


Figure 5.3 View of a curved packet

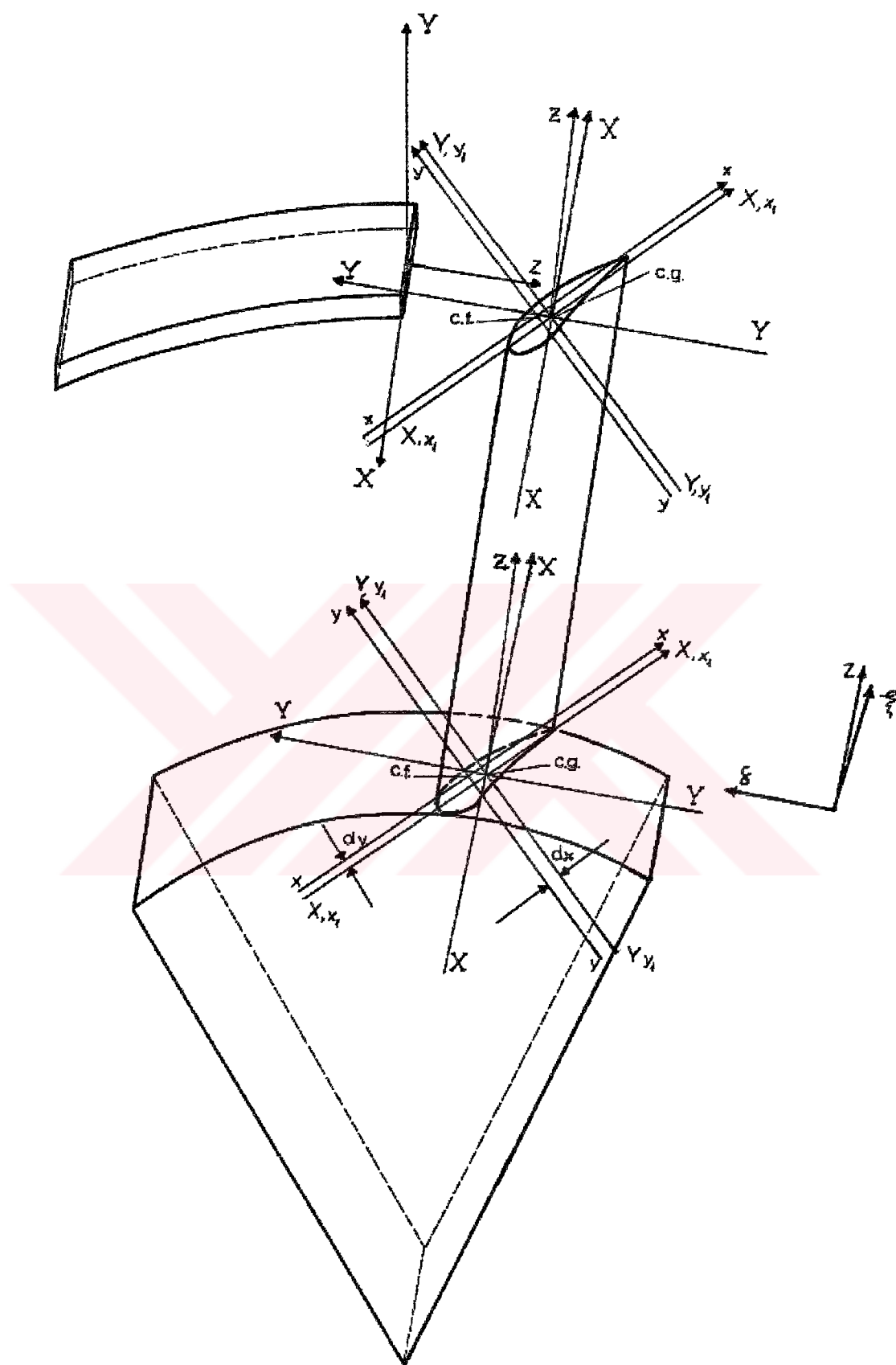


Figure 5.4 Local and global coordinates of a section of a curved packet

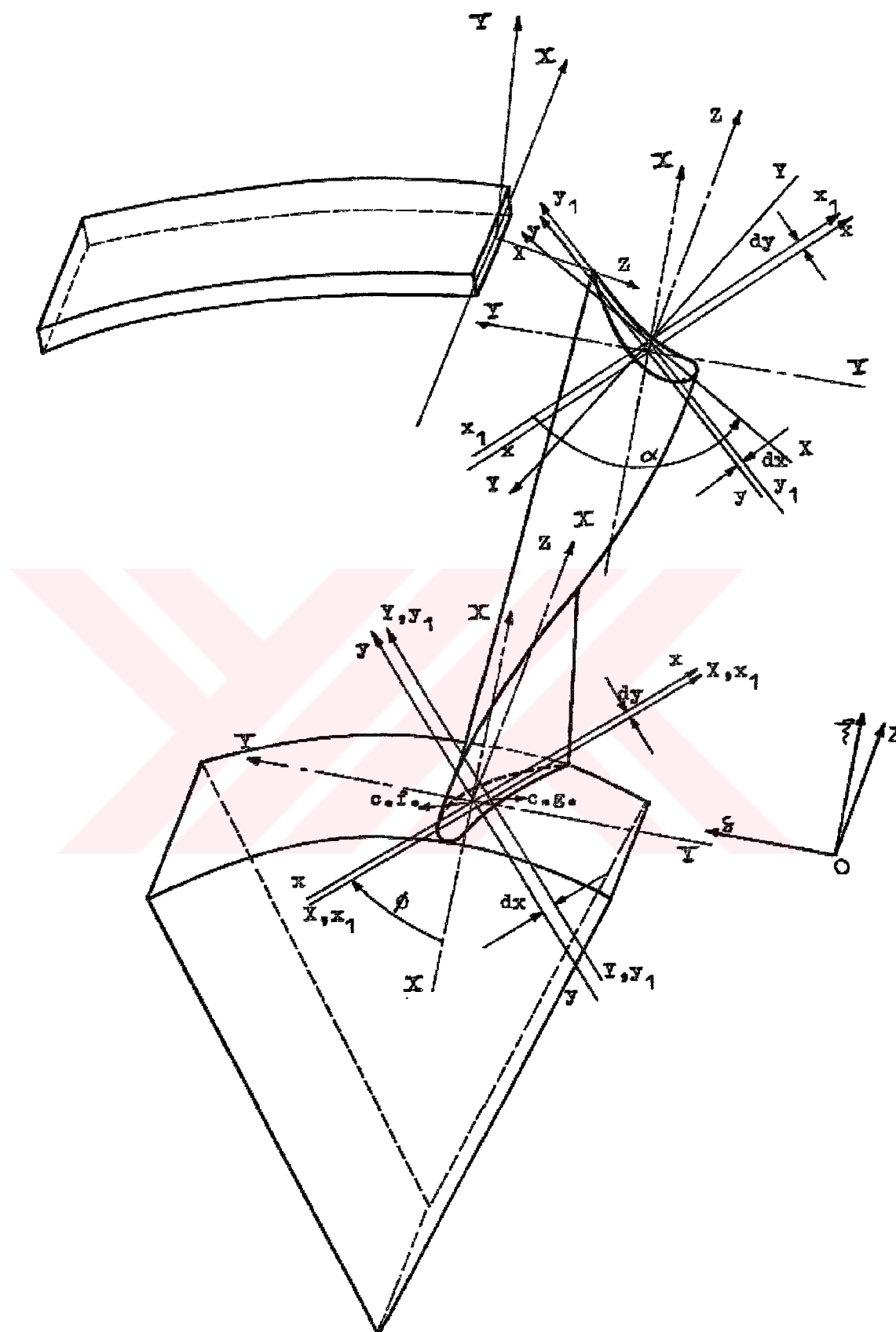


Figure 5.5 Local and global coordinates of a section of a pretwisted blade packet

Table 5.1 Details of the two bladed packets used in the analysis where blades have aerofoil cross section

Blades	Shroud
$A=58,96e-6 \text{ m}^2$	$t=0,001587 \text{ m}$
$L=0,1524 \text{ m}$	$b=0,02753 \text{ m}$
$E=2,126e11 \text{ N/m}^2$	$E_s=2,126e11 \text{ N/m}^2$
$\rho=7833,4 \text{ kg/m}^3$	$L_{sh}=0,1117 \text{ m}$
$I_{xx}=34,96e-12 \text{ m}^4$	$\rho_s=7833,4 \text{ kg/m}^3$
$I_{yy}=2,79e-9 \text{ m}^4$	$r_2=0,2032 \text{ m}$
$d_x=0,193e-3 \text{ m}$	
$d_y=1,1938e-3 \text{ m}$	
$C=9,14 \text{ Nm}^2/\text{rad}$	

Table 5.2 Details of the two bladed packets used in the analysis where blades have aerofoil cross section.

Blades	Shroud
$A=58,96e-6 \text{ m}^2$	$t=0,001587 \text{ m}$
$L=0,1524 \text{ m}$	$b=0,00635 \text{ m}$
$E=2,137e11 \text{ N/m}^2$	$E_s=2,137e11 \text{ N/m}^2$
$\rho=7473,5 \text{ kg/m}^3$	$L_{sh}=0,1058 \text{ m}$
$I_{xx}=34,96e-12 \text{ m}^4$	$\rho_s=7473,5 \text{ kg/m}^3$
$I_{yy}=2,79e-9 \text{ m}^4$	$r_2=0,2521 \text{ m}$
$d_x=0,193e-3 \text{ m}$	$\phi=30^\circ$
$d_y=1,1938e-3 \text{ m}$	
$C=9,14 \text{ Nm}^2/\text{rad}$	

Table 5.3 Comparison of experimental and theoretical values of frequencies two aerofoil cross section bladed packets. (Legend for table 5.1 and table 5.2 respectively)

Frequency (Hz)		
Mode No	Theoretical	Experimental (Sabuncu, 1978)
1	80	80
2	380	383
3	538,8	540
4	599,1	-
5	678,2	684
6	753	-
7	1376	-
8	1461	1454
9	1862	1849
10	2064	-

a)

Frequency (Hz)		
Mode No	Theoretical	Experimental (Sabuncu, 1978)
1	94,2	97
2	406,3	404
3	579,9	592
4	765,8	718
5	780,4	-
6	823,8	801
7	1289,8	1290
8	1493,6	1498
9	1615,5	1628
10	1884,8	-

b)

Table 5.4 Comparison of experimental and theoretical values of a two aerofoil cross section bladed packet with different rotational speeds.(Legend for table 5.2)

Mode No	Frequency (Hz)			
	Rotational speed (rpm)			
	1000		2000	
	Theoretical	Experimental (Sabuncu, 1978)	Theoretical	Experimental (Sabuncu, 1978)
1	100,6	102	117,6	114
2	410,7	402	423,3	409
3	587,5	602	609,8	609
4	765,8	714	766,5	721
5	781,2	-	783,8	-
6	826,2	-	833,3	-
7	1290,6	-	1292,7	-
8	1500	1520	1519,8	1551
9	1622,3	-	1462,4	-
10	1884,8	-	1885	-

Table 5.5 Buckling load parameters of a two bladed packet for different shroud thickness. (Legend for table 5.1)

Mode No	Buckling load parameter (λ_c)	
	Shroud thickness (m)	
	t	t/2
1	3,97	2,72
2	22,28	20,55
3	26,41	22,66
4	59,84	52,31
5	65,24	61,89
6	121,38	66

Table 5.6 Buckling load parameters of a two bladed packet for different shroud lengths. (Legend for table 5.1)

Buckling load parameter (λ_c)			
Mode No	Shroud length (m)		
	$0,2L_{sh}$	$0,5L_{sh}$	L_{sh}
1	5,31	4,37	3,66
2	26,34	23,1	21,8
3	32,42	28	25,6
4	63,59	60,98	58,14
5	72,43	66,95	63,97
6	126,45	122,64	118,83

Table 5.7 Buckling load parameters of a two bladed packet for different stagger angles. (Legend for table 5.1)

Buckling load parameter (λ_c)				
Mode No	Stagger angle (degree)			
	0°	30°	60°	90°
1	3,97	3,7	2,99	2,52
2	22,28	22,54	22,7	9,54
3	26,41	25,48	23,4	22,27
4	59,84	59,85	52,86	25,25
5	65,24	64,42	62,78	61,95
6	121,38	120,9	72,27	63,8

Table 5.8 Buckling load parameters of a two bladed packet for different rotational speeds. (Legend for table 5.1)

Buckling load parameter (λ_c)				
Mode No	Rotational speed (rpm)			
	0	1000	2000	4000
1	3,97	4,94	7,85	19,46
2	22,28	23,52	27,24	41,28
3	26,41	27,72	31,66	47,34
4	59,84	61,2	65,28	81,54
5	65,24	66,61	70,72	87,21
6	121,38	122,73	126,78	142,94

Table 5.9 Buckling load parameters of a two bladed packet for different d_x distances. (Legend for table 5.1)

Buckling load parameter (λ_c)					
Mode No	d_x distance (m)				
	0,000381	0,000762	0,0015	0,0023	0,00381
1	3,97	3,97	3,97	3,97	3,96
2	22,28	22,27	22,24	22,2	22,05
3	26,41	26,4	26,36	26,29	26,09
4	59,83	59,78	59,6	59,28	58,3
5	65,22	65,17	64,95	64,59	63,47
6	121,33	121,16	120,48	119,35	115,83
7	126,19	126	125,27	124,05	120,27
8	189,39	189,39	189,38	186,36	187,63
9	198,56	198,55	198,51	196,24	189,4

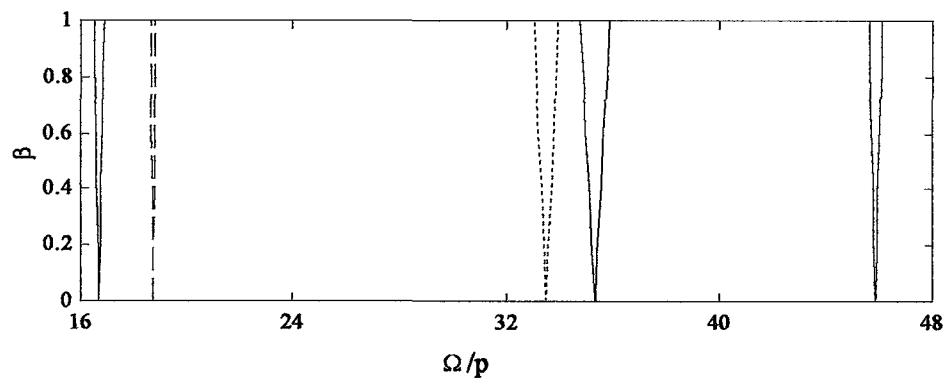
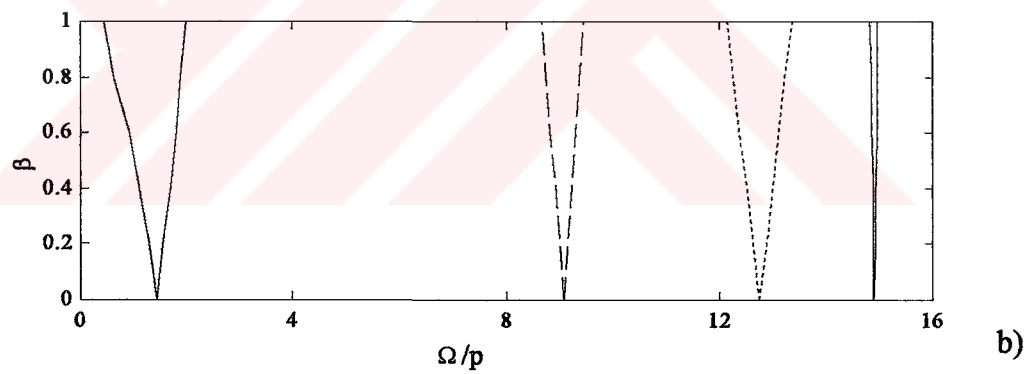
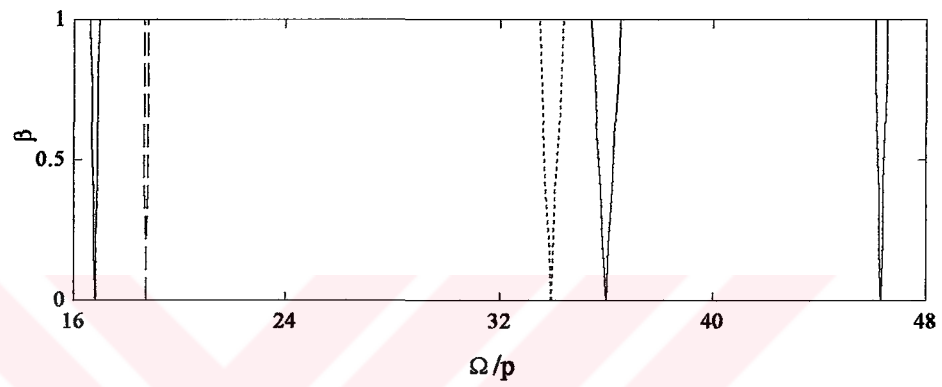
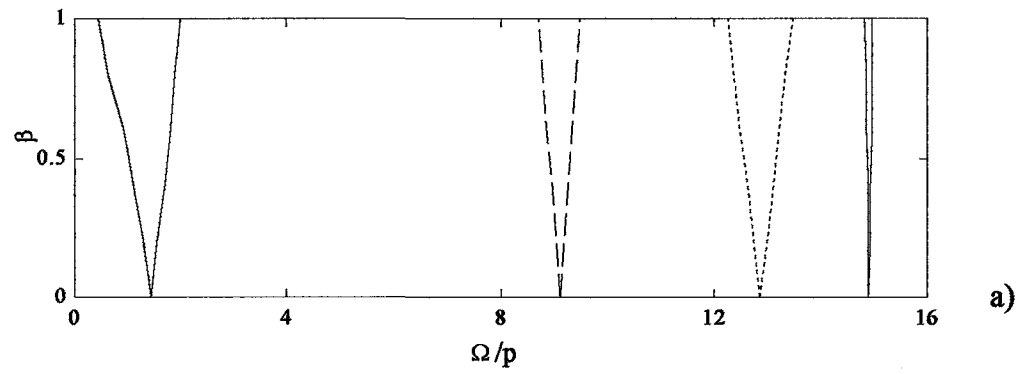


Figure 5.5 Effect of coupling on the regions of dynamic instability of a two bladed packet for first nine modes respectively. ($\alpha=0,5$ $\phi=0^\circ$, $\omega=0$ rpm, a) $d_x=0,000381$ m, b) $d_x=0,00381$ m)

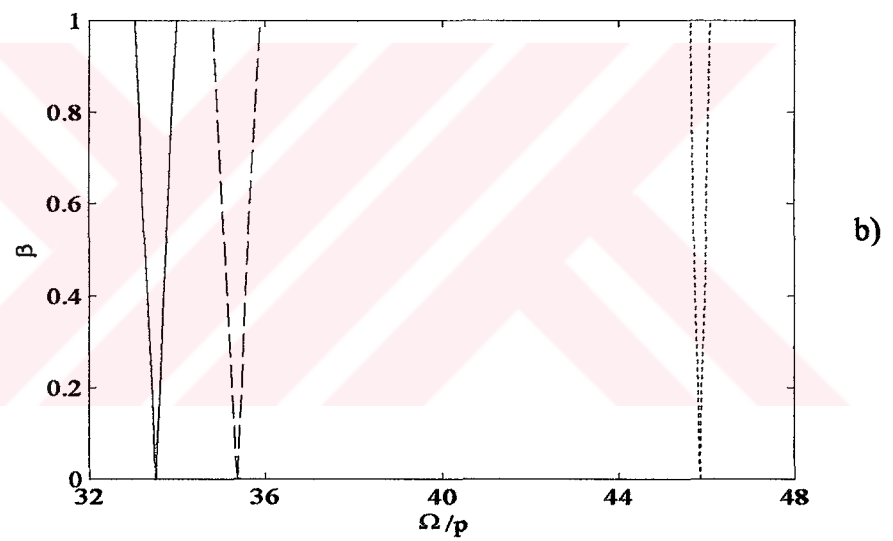
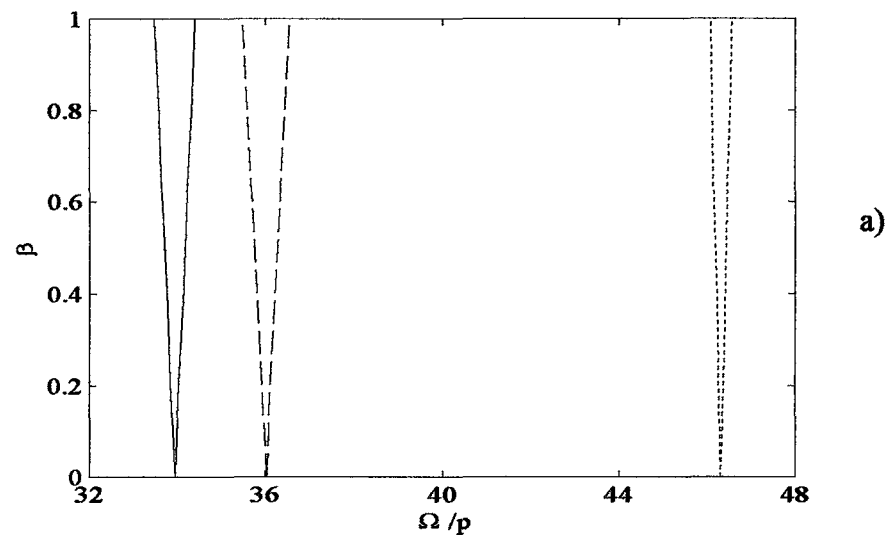
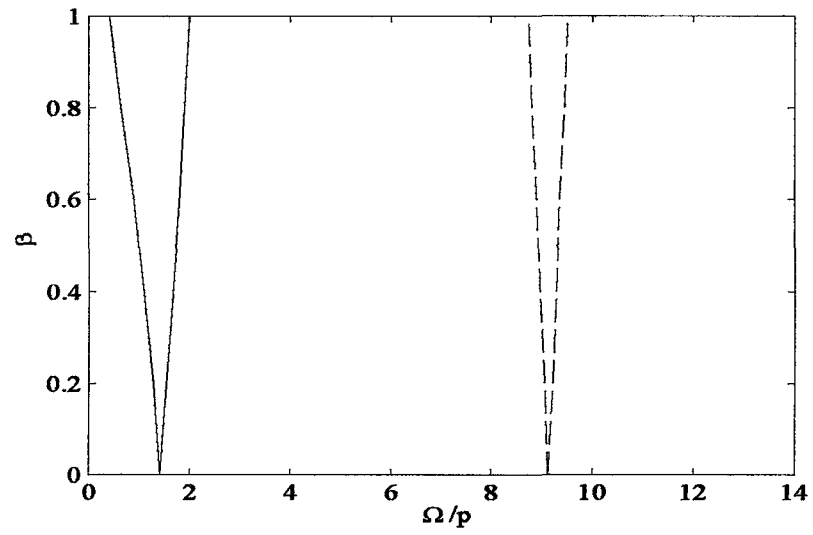
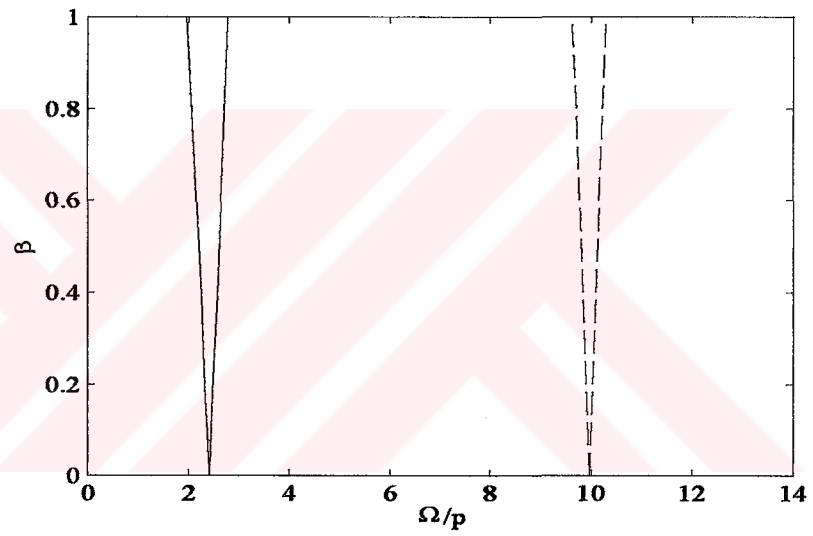


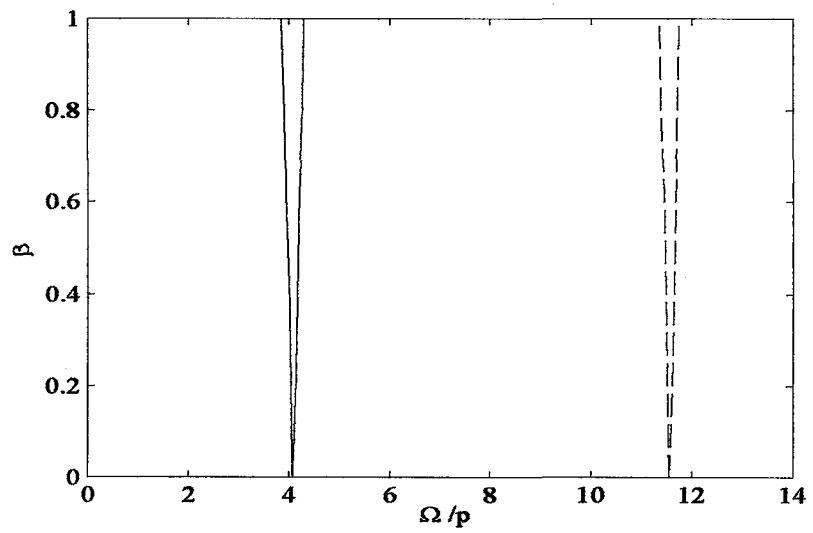
Figure 5.6 Effect of coupling on the regions of dynamic instability of a two bladed packet for seventh, eighth and ninth modes respectively. ($\alpha=0,5$ $\phi=0^\circ$, $\omega=0$ rpm, a) $d_x=0,000381$ m, b) $d_x=0,00381$ m)



a)



b)



c)

Figure 5.7 Effect of rotational speed on the regions of dynamic instability of a two bladed packet for first two modes. ($\alpha=0.5$, $\phi=0^\circ$, a) $\omega=0$, b) $\omega=2000$ rpm, c) $\omega=4000$ rpm)

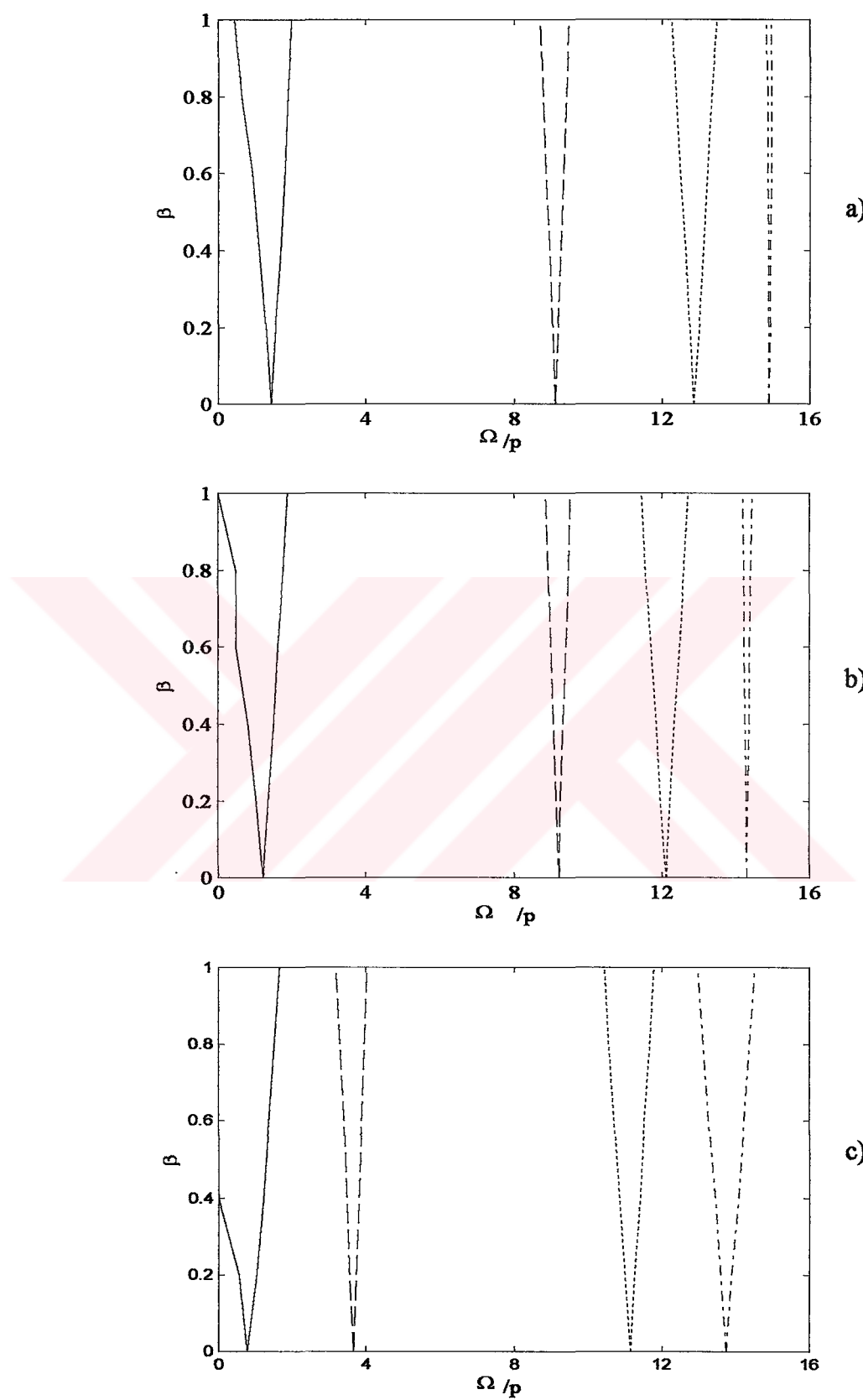


Fig.5.8 Effect of stagger angle on the regions of dynamic instability of a two bladed packet for first four modes. ($\alpha=0.5$, $\omega=0$ rpm, a) $\phi=0^\circ$ b) $\phi=45^\circ$ c) $\phi=90^\circ$)

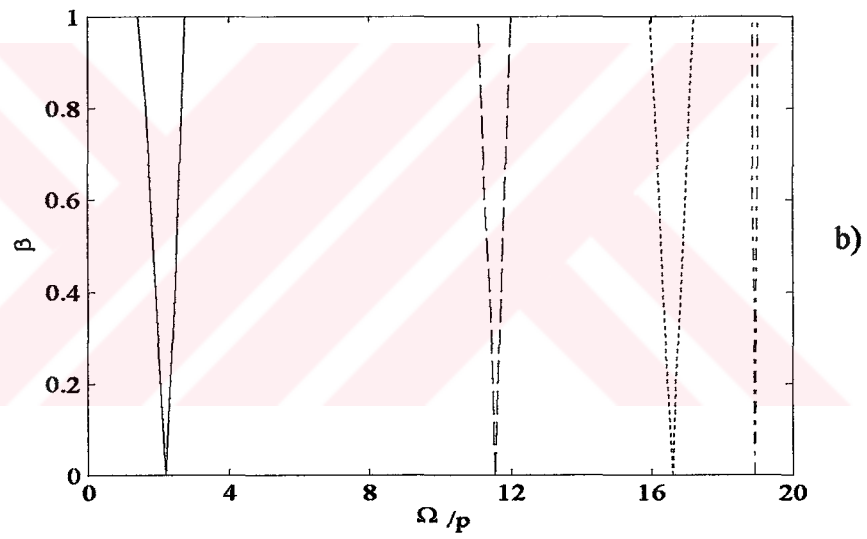
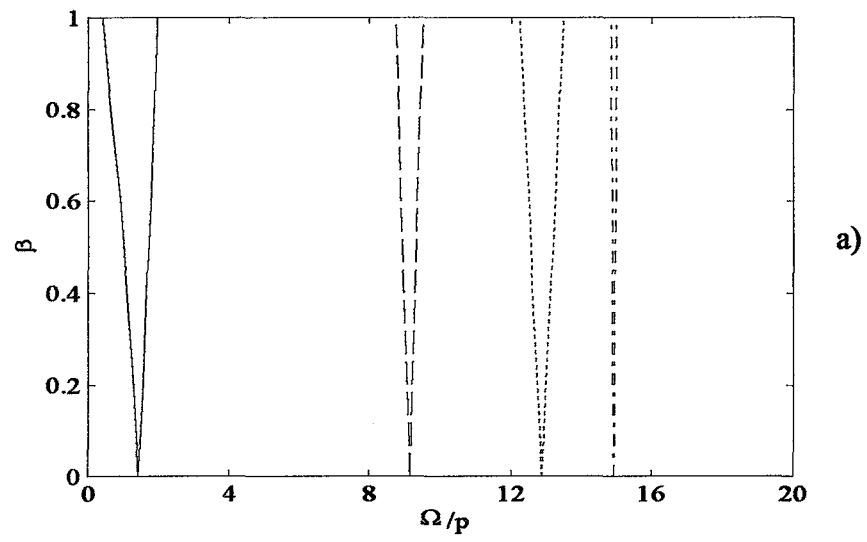


Figure 5.9 Effect of shroud length on the regions of dynamic instability of a two bladed packet for first four modes. ($\alpha=0,5$ $\phi=0^\circ$, $\omega=0$ rpm, a) L_{sh} b) $L_{sh}/4$)

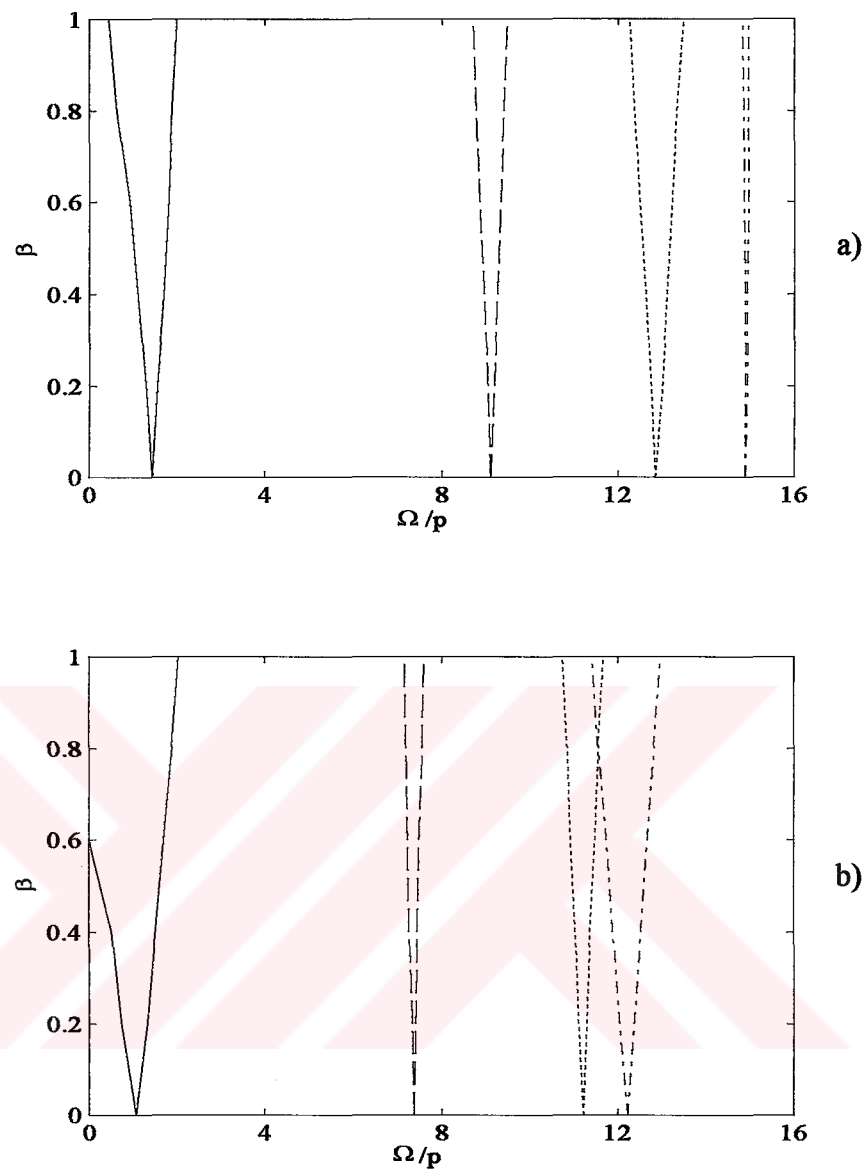


Figure 5.10 Effect of shroud thickness on the regions of dynamic instability of a two bladed packet for first two modes. ($\alpha=0,5$ $\phi=0^\circ$, $\omega=0$ rpm, a) $t=0,001587$ m b) $t=0,00079375$ m)

5.4.2 For pretwisted two bladed packets

In this section some of the results obtained for the frequencies of vibration of pretwisted aerofoil cross section two bladed packets are shown and compared with some other researchers' results in order to indicate the degree of accuracy and adaptability that can be expected in the application of the finite element method.

Table 5.11 shows the comparison between the theoretical and experimental results of a pretwisted aerofoil cross section blade packet under non-rotating condition. There are some discrepancies between calculated and measured frequencies of about 3.5% and in one mode 7%. Some error was probably caused by the uncertainty of Young's modulus, the density and residual stresses in the test piece. The residual stresses were caused by pretwisting the blades in a small operated torsion machine. It was concluded that the present results are sufficiently accurate for calculations, especially when the complex blade configurations are considered (Sabuncu & Thomas, 1992).

Figure 5.11 shows the effect of pretwist angle on the regions of dynamic instability of an aerofoil cross section pretwisted blade packet. For the first mode as the pretwist angle increases the instability regions are shifted to the right and the width of the regions are decreased slightly. An increase of pretwist angle makes the system stiffer for the first mode. For the second mode as the pretwist angle increases the instability regions are shifted to the left and the widths of the regions are increased. When the stability of a pretwisted blade was analyzed (Şakar & Sabuncu, 2004), it was noticed that the instability of second mode increased with increasing pretwist angle. It may be said that a similar phenomenon is happening here although there is a coupling trough shroud interaction. For the third mode as the pretwist angle increases the instability regions are shifted to the left and the widths of the regions are decreased. For the fourth mode as the pretwist angle increases the instability regions are shifted to the right and the widths of the regions are decreased. For the fifth and sixth modes as the pretwist angle increases the instability regions are shifted to the right and the widths of the regions are increased.

These variations of instability regions are related to the flexibility of the system in the relevant direction depending on the pretwist angle.

Figure 5.12 shows the effect rotational speed on the regions of dynamic instability of an aerofoil cross section pretwisted blade packet. With an increase in rotational speed the instability regions shift to the right and the widths of the regions are decreased as in the case of rotating pretwisted single blade. Obviously an increase in rotational speed increases the stiffness of the rotating two blade packet and makes the system more stable.

Figure 5.13 shows the effect of coupling on the dynamic instability regions for the first nine modes. In the figure when the distance between shear center and centroid increases the instability regions do not change for the lower modes. In the higher modes such as seventh, eighth and ninth modes, when the distance between shear center and centroid increases the instability regions shifted to the left. These modes are affected by bending and torsional modes between blades and shrouds. These are shown in Figure 5.14.

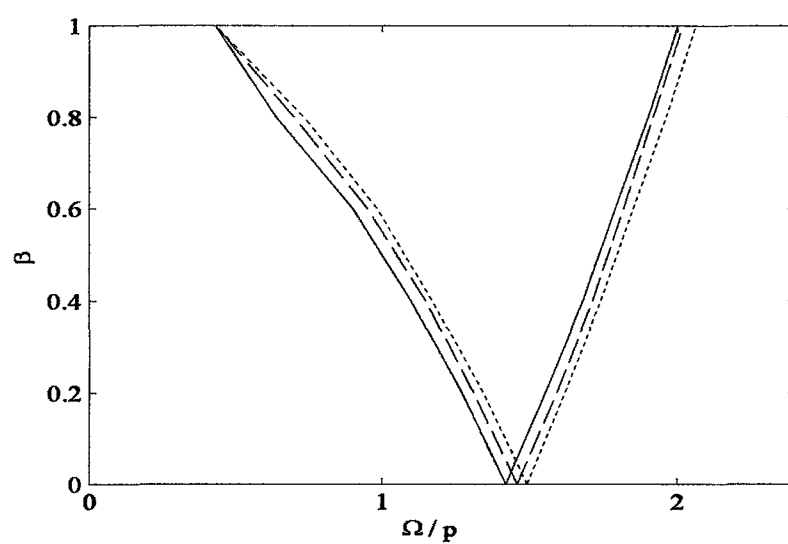
Figure 5.15 shows the effect of stagger angle on the regions of dynamic instability for the first four modes. As the stagger angle increases the three regions are shifted towards the static buckling axis, and the widths of these regions increase. For the fourth modes the instability regions shifted to right and the widths of the region decreased for the 45° pretwist angle, for the 90° pretwist angle the regions are shifted to left and the widths of the regions decreased. Thus the effect of increase in stagger angle makes the packet more sensitive to periodic excitation in the out-of-plane direction. The effect of the shroud length on the dynamic stability region is shown in the Figure 5.16 As the shroud length decreases the regions are shifted to the right and the widths of these regions are decreased for the first modes and slightly increased for the other modes. As a result, a decrease in the shroud length increases the stiffness of the packet. Thus it makes the packet more stable against periodic in-plane forces.

Table 5.10 Details of the two bladed packets used in the analysis where blades have aerofoil cross section.

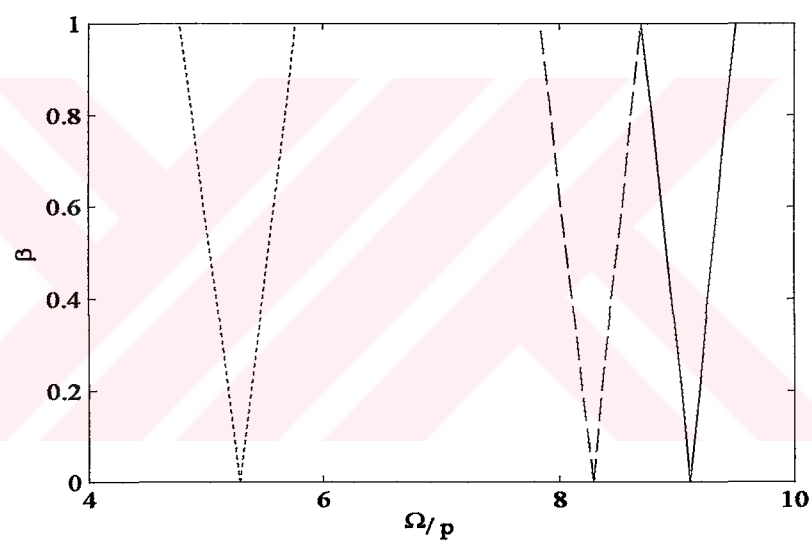
Blades	Shroud
$A=58,96e-6 \text{ m}^2$	$t=0,001587 \text{ m}$
$L=0,1524 \text{ m}$	$b=0,02753 \text{ m}$
$E=2,126e11 \text{ N/m}^2$	$E_s=2,126e11 \text{ N/m}^2$
$\rho=7833,4 \text{ kg/m}^3$	$L_{sh}=0,1117 \text{ m}$
$I_{xx}=34,96e-12 \text{ m}^4$	$\rho_s=7833,4 \text{ kg/m}^3$
$I_{yy}=2,79e-9 \text{ m}^4$	$r_2=0,2032 \text{ m}$
$d_x=0,193e-3 \text{ m}$	$\phi=0$
$d_y=1,1938e-3 \text{ m}$	$\alpha_p=45^\circ$
$C=9,14 \text{ Nm}^2/\text{rad}$	

Table 5.11 Comparison of experimental and theoretical values of frequencies pretwisted two aerofoil cross section bladed packets. (Legend for table 5.10)

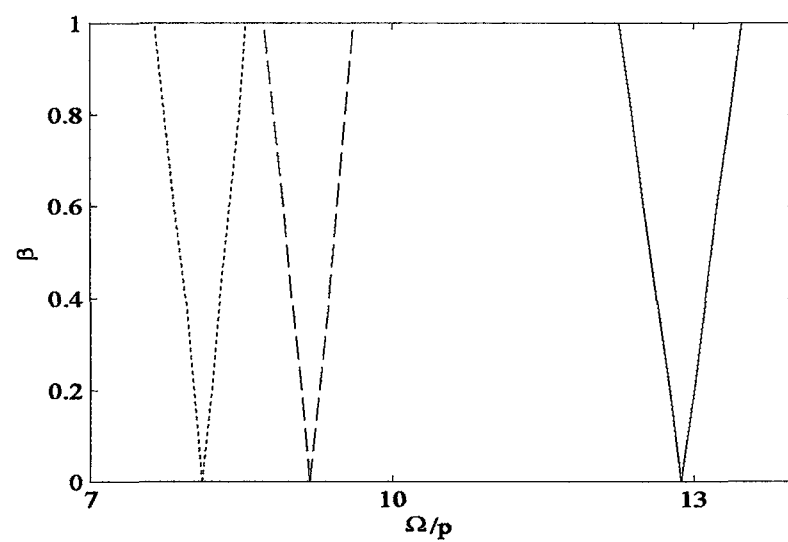
Frequency (Hz)		
Mode No	Theoretical	Experimental (Sabuncu, 1978)
1	88,5	92
2	389,9	380
3	395,6	405
4	642,5	656
5	911,9	868
6	976,2	-
7	1397,4	1305
8	1506,9	1483
9	1812,3	-
10	1899,7	-



First mode



Second mode



Third mode

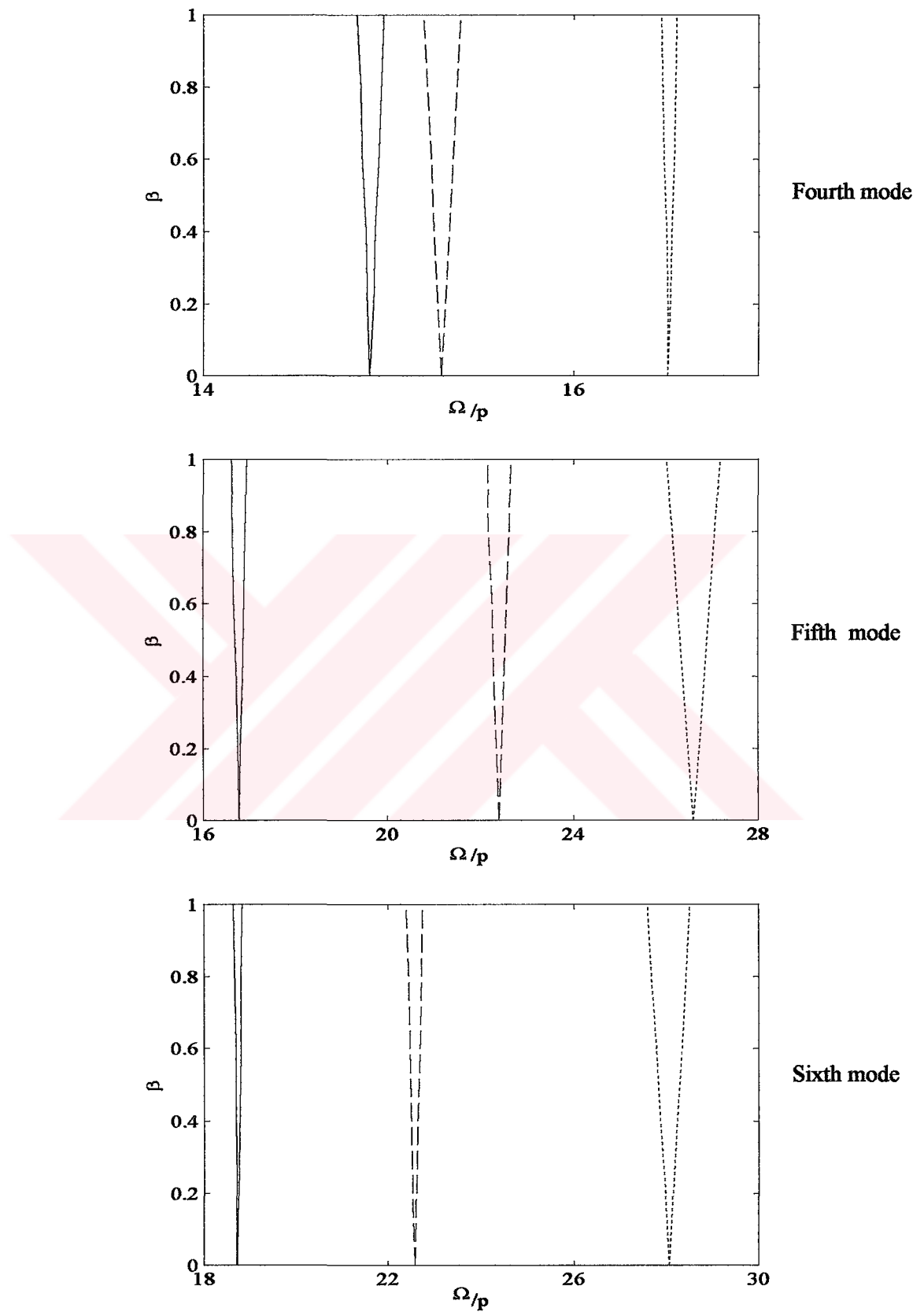


Figure 5.11 Effect of pretwist angle on the dynamic instability regions of a pretwisted blade packet for the first six modes. ($\alpha=0,5$ $\phi=0^\circ$, $\omega=0$ rpm, — $\alpha_p=0^\circ$, --- $\alpha_p=45^\circ$, ... $\alpha_p=90^\circ$)

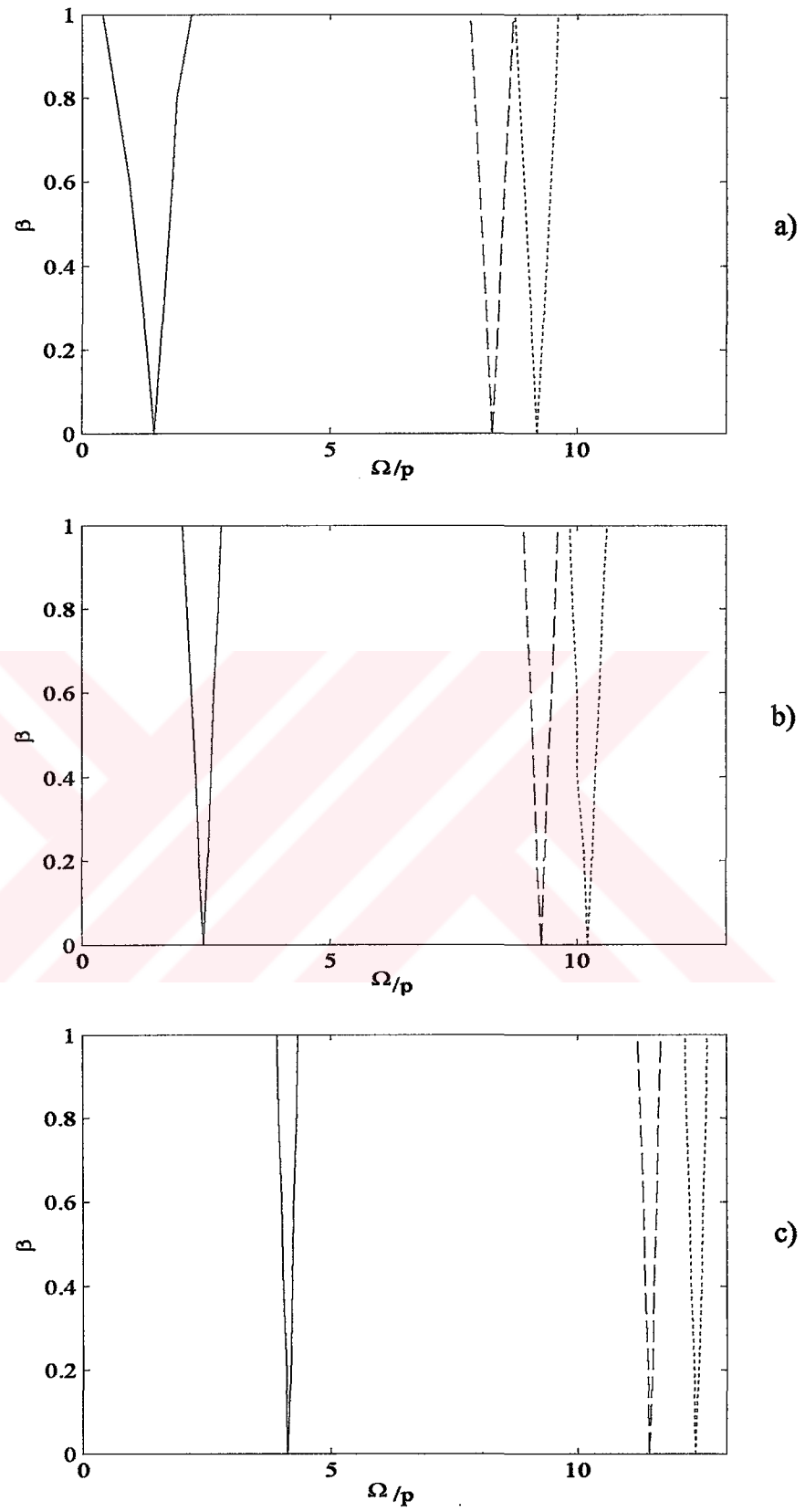


Figure 5.12 Effect of rotational speed on the dynamic instability regions of a pretwisted blade packet for the first three modes. ($\alpha=0,5$ $\phi=0^\circ$, $\alpha_p=45^\circ$, a) $\omega=0$ rpm, b) $\omega=2000$ rpm, c) $\omega=4000$ rpm)

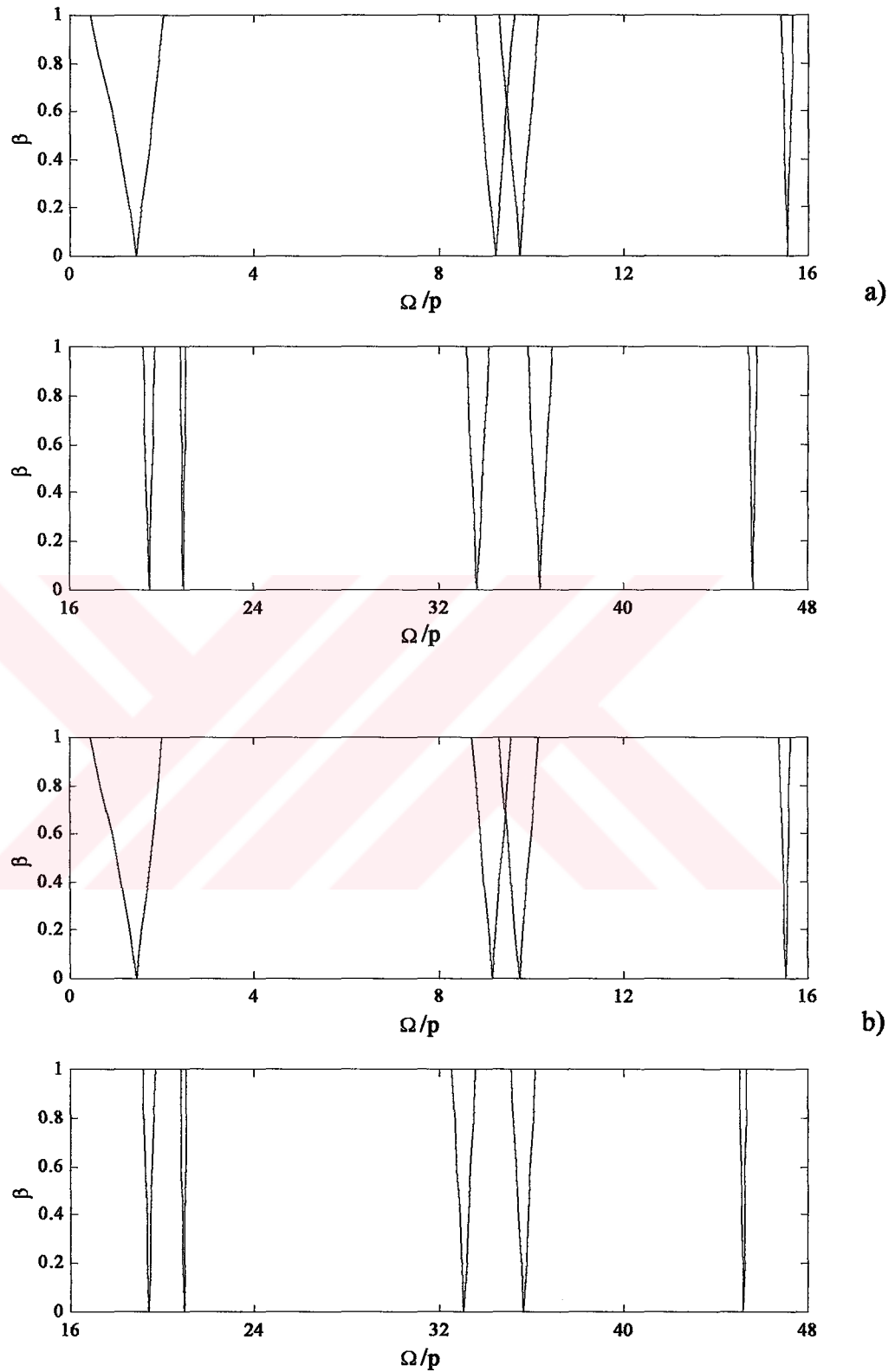


Figure 5.13 Effect of coupling on the dynamic instability regions of a pretwisted blade packet for the first nine modes respectively. ($\alpha=0.5$, $\phi=0^\circ$, $\omega=0$ rpm, $\alpha_p=30^\circ$, a) $d_x=0.000381$ m, b) $d_x=0.00381$ m)

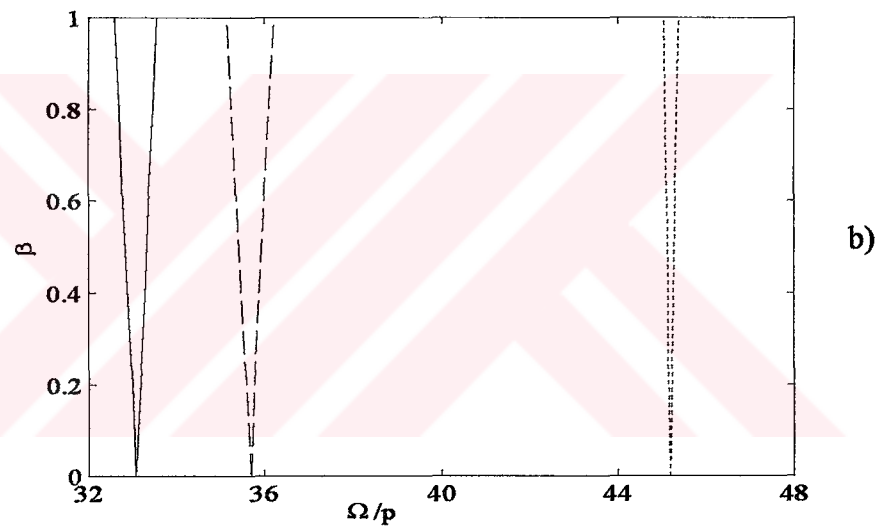
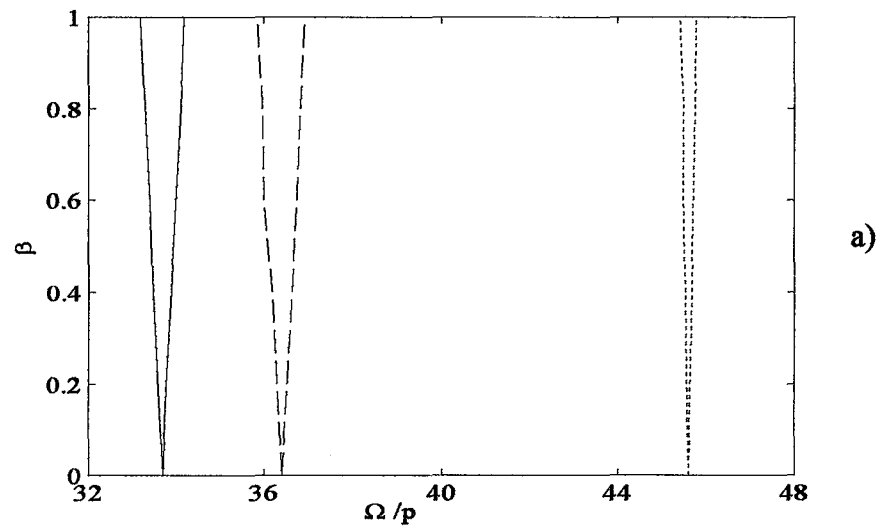


Figure 5.14 Effect of coupling on the dynamic instability regions of a pretwisted blade packet for seventh, eighth and ninth modes. ($\alpha=0,5$ $\phi=0^\circ$, $\omega=0$ rpm, $\alpha_p=30^\circ$, a) $d_x=0,000381$ m, b) $d_x=0,00381$ m)

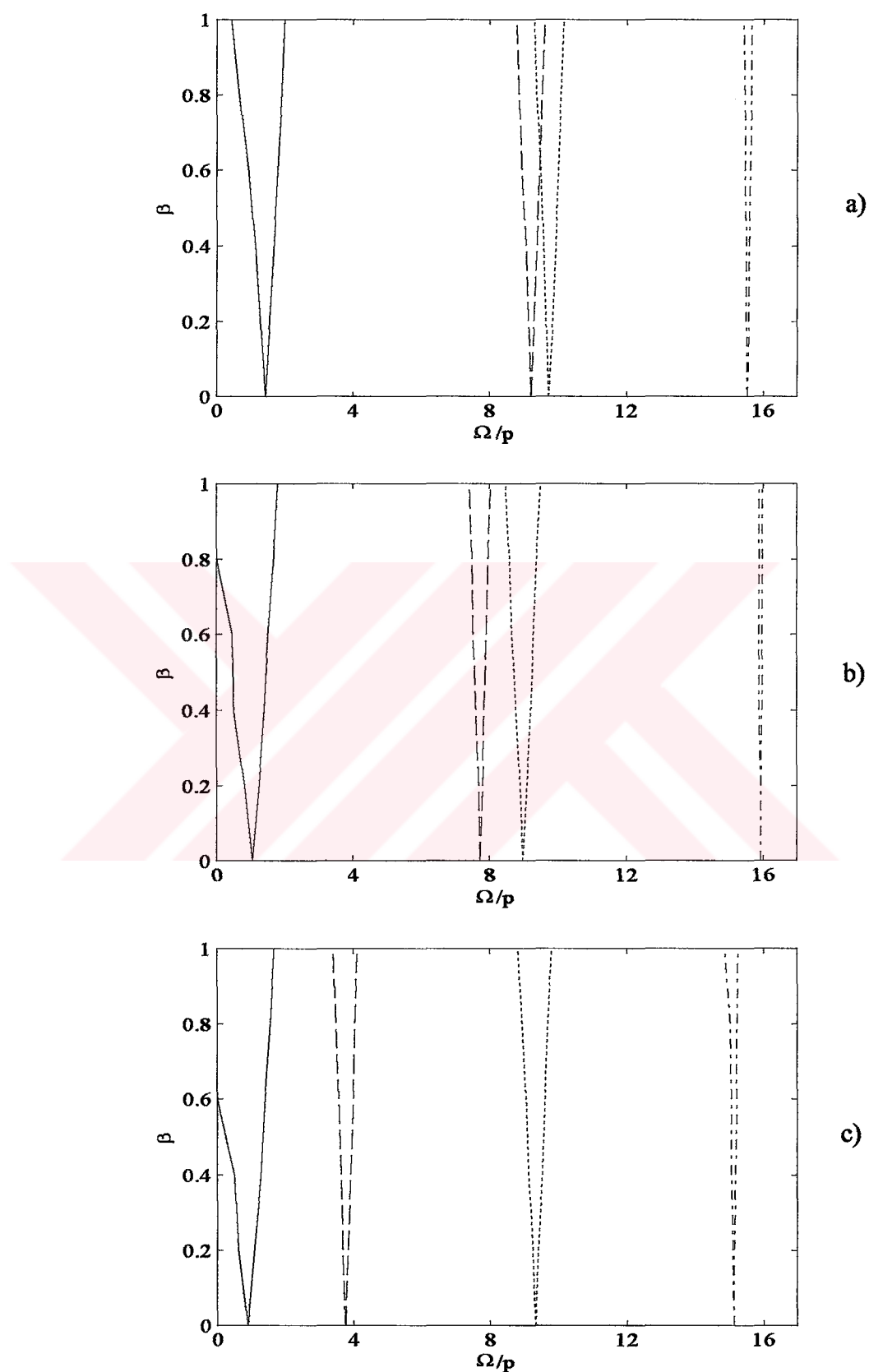


Figure 5.15 Effect of stagger angle on the dynamic instability regions of a pretwisted packet for the first four modes. ($\alpha=0,5$ $\omega=0$ rpm, $\alpha_p=30$, a) $\phi=0^\circ$, b) $\phi=45^\circ$, c) $\phi=90^\circ$)

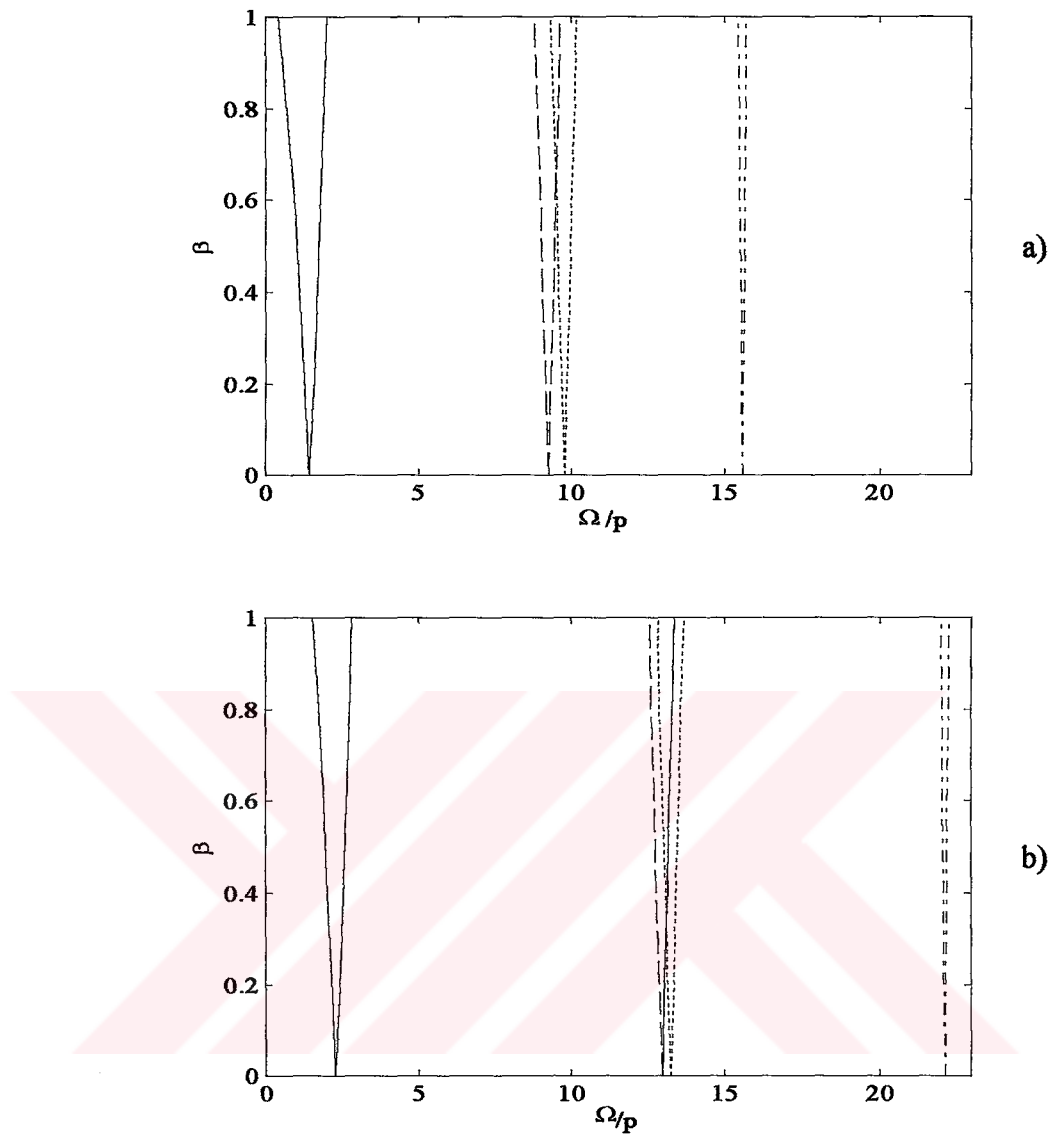


Figure 5.16 Effect of shroud length on the dynamic instability regions of a pretwisted blade packet for the first four modes. ($\alpha=0.5$, $\phi=0^\circ$, $\omega=0$ rpm, $\alpha_p=30$, a) L_{sh} b) $L_{sh}/4$)

CHAPTER SIX

CONCLUSIONS

In this investigation the application of the vibration, static and dynamic stability analysis of aerofoil cross section straight and pretwisted rotating blades and blade packets under in-plane periodic loading are carried out and the following conclusions are drawn.

The finite element method used in this investigation is ideal for the vibration, static and dynamic stability analysis of nonrotating and rotating straight and pretwisted aerofoil cross section blades. The results show very good agreement when compared with the theoretical and experimental results of other investigators.

The coupling effect is not significant in the first four bending modes for critical buckling loads. The critical buckling load of the fifth XX bending dominated coupled mode has been reduced considerably from the value of the uncoupled mode, because this mode is affected by the distance d_y that is much larger than d_x .

The coupling effect is very important in the third and fourth modes depending on d_y distance on the dynamic stability region.

Due to rotation the critical buckling load parameters increase for all modes for various d_x and d_y distances.

Critical buckling load parameters also increase for all stagger angles as the rotational speed increases.

The increase in stagger angle makes the beam more sensitive to periodic excitation. Similar cases can be seen for the other modes for straight rotating blades.

With an increase in rotational speed and disc radius, the instability regions are decreased, so the blade becomes more stable.

The vibration and static and dynamic stability analysis of pretwisted asymmetric aerofoil cross section blades are considerably affected by the distance of center of flexure from the centroid.

For the first mode as the pretwist angle increases critical buckling load parameter increases for all d_x and d_y distances, but for other modes as the pretwist angle increases the critical buckling load parameters decreases. The pretwist angle effect on the dynamic stability regions of an asymmetric cross section blade is very different from the straight blade.

The increase in stagger angle of a rotating blade makes the beam more sensitive to periodic excitation.

With an increase in rotational speed, the instability regions are decreased, so the blade becomes more stable and the increase in static load factor is one of the destabilization effect for the pretwisted blade.

The curved beam elements whose shape functions are based on simple strain displacement relations give fast convergence for the vibration and static stability analysis.

When the subtended angle and disc radius increase the buckling load of curved beams decrease in the in-plane and out-of-plane directions. When the t/b ratio increases the buckling modes of an arch changes from the in-plane of curvature to out-of-plane direction depending on the arch rigidity.

The coupling effect is not significant at the lower modes in critical buckling loads for the an aerofoil cross section straight blade packet. The coupling effect is very

important in the higher modes depending on center of flexure distance from the centroid on the dynamic stability regions.

Due to rotation the critical buckling load parameters increase for all modes whatever the d_x and d_y distances are. Critical buckling load parameters increase as the rotational speed increases. With an increase in rotational speed, the instability regions are decreased, so the two bladed packet becomes more stable.

The increase in stagger angle makes the two bladed packet more sensitive to periodic excitation.

An increase in the shroud length decreases the stiffness of the straight bladed packet. Thus it makes the packet more unstable against periodic in-plane forces.

When the shroud thickness decreases the packet becomes less stable to periodic excitation.

For the first mode an increase of pretwist angle makes an aerofoil cross section pretwisted blade packet stiffer. In the variations of instability regions of an aerofoil cross section pretwisted blade packet are related to the flexibility of the system in the relevant direction depending on the pretwist angle.

The coupling effect on the instability regions of an aerofoil cross section pretwisted blade packet is more effective in the higher modes. These modes are affected by bending and torsional modes between blades and shrouds.

The effect of increase in stagger angle makes the packet more sensitive to periodic excitation in the out-of plane direction.

A decrease in the shroud length increases the stiffness of the packet. Thus it makes the packet more stable against periodic in-plane forces.

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APPENDIX 1

The inverse matrix of a beam element which is represented by a cubic polynomial deflection function is given as;

$$[C_{v,u}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{\ell^2} & -\frac{2}{\ell} & \frac{3}{\ell^2} & -\frac{1}{\ell} \\ \frac{2}{\ell^3} & \frac{1}{\ell^2} & -\frac{2}{\ell^3} & \frac{1}{\ell^2} \end{bmatrix} \quad (A.1.1)$$

The inverse matrix, for the longitudinal and torsional displacements of an element are represented by a linear displacement function, is given as

$$[\xi]^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} \quad (A.1.2)$$

The inverse matrix of a beam element can be formed by using (A.1.1) and (A.1.2) as

$$[C^*]^{-1} = \begin{bmatrix} [C_{v,u}]^{-1} & 0 & 0 & 0 \\ 0 & [C_{v,u}]^{-1} & 0 & 0 \\ 0 & 0 & [\xi]^{-1} & 0 \\ 0 & 0 & 0 & [\xi]^{-1} \end{bmatrix} \quad (A.1.3)$$

The inverse matrix of a beam element which is represented by a cubic polynomial deflection function is given in non-dimensional form as

$$[C_1]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \quad (A.1.4)$$

The inverse matrix, for the longitudinal and torsional displacements of an element are represented by a linear displacement function is given in dimensional form as

$$[\xi^*]^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad (A.1.5)$$

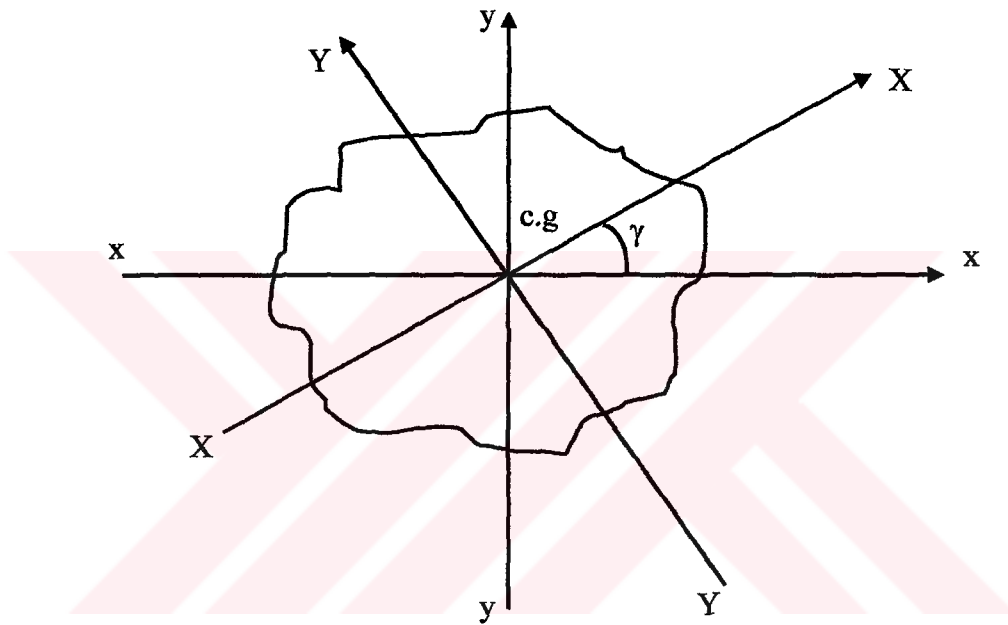
The inverse matrix of a beam element can be formed by using (A.1.4) and (A.1.5) as

$$[C_*]^{-1} = \begin{bmatrix} [C_1]^{-1} & 0 & 0 & 0 \\ 0 & [C_1]^{-1} & 0 & 0 \\ 0 & 0 & [\xi^*]^{-1} & 0 \\ 0 & 0 & 0 & [\xi^*]^{-1} \end{bmatrix} \quad (A.1.6)$$

APPENDIX 2

Relationship between I_{xx} , I_{yy} , I_{xy} , and I_{XX} , I_{YY} are as follows;

The following figure shows the cross section of a beam with its principal axis XX and YY and fixed axis xx and yy .



The relationship between these axes systems can be written as

$$y = X \sin \gamma + Y \cos \gamma$$

$$x = X \cos \gamma - Y \sin \gamma$$

where

$$\gamma = ((k-1) + \eta) \frac{\alpha}{n} \quad (\text{in non-dimensional form})$$

$$\gamma = (\ell(k-1) + z) \frac{\alpha}{L} \quad (\text{in dimensional form})$$

$$I_{xx} = I_{yy} \sin^2 \gamma + I_{xx} \cos^2 \gamma$$

$$I_{yy} = I_{yy} \cos^2 \gamma + I_{xx} \sin^2 \gamma$$

$$I_{xy} = (I_{yy} - I_{xx}) \frac{\sin 2\gamma}{2}$$

or they can be written as

$$I_{xx} = I_{xx} \left[\left(\frac{1}{2} \left(\frac{I_{yy}}{I_{xx}} + 1 \right) - \frac{1}{2} \left(\frac{I_{yy}}{I_{xx}} - 1 \right) \cos 2\gamma \right) \right]$$

$$I_{yy} = I_{xx} \left[\left(\frac{1}{2} \left(\frac{I_{yy}}{I_{xx}} + 1 \right) + \frac{1}{2} \left(\frac{I_{yy}}{I_{xx}} - 1 \right) \cos 2\gamma \right) \right]$$

$$I_{xy} = I_{xx} \frac{1}{2} \left(\frac{I_{yy}}{I_{xx}} - 1 \right) \sin 2\gamma$$

APPENDIX 3

The calculations of integrations terms used in chapters are carried out as follows

1. If η is replaced by 0 and 1 the integrations are carried out in non-dimensional form.
 2. If η is replaced by 0 and ℓ the integrations are carried out in dimensional form.
- a) Some of the symbols, which are used for the uniform straight aerofoil cross-section blade, have the following values.

a.1) In the strain energy matrix

$$AA=4$$

$$BB=6$$

$$DD=12$$

$$PP=4 \frac{I_{xx}}{I_{yy}}$$

$$QQ=6 \frac{I_{xx}}{I_{yy}}$$

$$RR=12 \frac{I_{xx}}{I_{yy}}$$

Product inertia term vanishes because $I_{xx} = I_{xx}$ and $I_{yy} = I_{yy}$ axes coincide consequently $GG=HH=FF=0$.

$$G26 = d_x$$

$$Z26 = d_y$$

$$G36 = d_x$$

$$Z36 = d_y$$

$$G46 = d_x$$

$$Z46 = d_y$$

$$G66 = d_x^2$$

$$Z66 = d_y^2$$

$$TI26 = d_Y / 2$$

$$TI36 = d_Y / 3$$

$$TI46 = d_Y / 4$$

$$TI66 = d_Y^2 / 2$$

$$FI26 = d_X / 2$$

$$FI36 = d_X / 3$$

$$FI46 = d_X / 4$$

$$FI66 = d_X^2 / 2$$

$$TS26 = d_Y / 3$$

$$TS36 = d_Y / 4$$

$$TS46 = d_Y / 5$$

$$TS66 = d_Y^2 / 3$$

$$FS26 = d_X / 3$$

$$FS36 = d_X / 4$$

$$FS46 = d_X / 5$$

$$FS66 = d_X^2 / 3$$

$$I15 = d_Y$$

$$I16 = d_Y / 2$$

$$I25 = d_Y / 2$$

$$I26 = d_Y / 3$$

$$I35 = d_Y / 3$$

$$I36 = d_Y / 4$$

$$I45 = d_Y / 4$$

$$I46 = d_Y / 5$$

$$I55 = d_Y^2$$

$$I56 = d_Y^2 / 2$$

$$I66 = d_Y^2 / 3$$

$$R15 = d_X$$

$$R16 = d_X / 2$$

$$R25 = d_X / 2$$

$$R26 = d_X / 3$$

$$R35 = d_X / 3$$

$$R36 = d_X / 4$$

$$R45 = d_X / 4$$

$$R46 = d_X / 5$$

$$R55 = d_X^2$$

$$R56 = d_X^2 / 2$$

$$R66 = d_X^2 / 3$$

$$Y55 = d_X d_Y$$

$$Y56 = d_X d_Y / 2$$

$$Y65 = d_X d_Y / 2$$

$$Y66 = d_X d_Y / 3$$

Rest of the symbols which appear in the pretwisted blade strain energy matrices are zero.

a.2) In the kinetic energy matrix

$$A3S = d_x$$

$$B3S = d_x / 2$$

$$C3S = d_x / 3$$

$$D3S = d_x / 4$$

$$E3S = d_x / 5$$

$$AS = d_y$$

$$BS = d_y / 2$$

$$CS = d_y / 3$$

$$DS = d_y / 4$$

$$ES = d_y / 5$$

b) Pretwisted aerofoil cross-section blade

$$B1 = 1/\alpha_b$$

$$B2 = 1/\alpha_b^2$$

$$B3 = 1/\alpha_b^3$$

$$B4 = 1/\alpha_b^4$$

$$B5 = 1/\alpha_b^5$$

$$SI1 = -B1 \cos \alpha_b \eta$$

$$SI2 = B2 \sin \alpha_b \eta - B1 \eta \cos \alpha_b \eta$$

$$SI3 = 2B2 \eta \sin \alpha_b \eta + (2B3 - B1 \eta^2) \cos \alpha_b \eta$$

$$CO1 = B1 \sin \alpha_b \eta$$

$$CO2 = B2 \cos \alpha_b \eta + B1 \eta \sin \alpha_b \eta$$

$$CO3 = 2B2 \eta \cos \alpha_b \eta + (B1 \eta^2 - 2B3) \sin \alpha_b \eta$$

$$SI4 = -B1 \eta^3 \cos \alpha_b \eta + 3 B1 CO3$$

$$CO4 = B1 \eta^3 \sin \alpha_b \eta - 3 B1 SI3$$

$$SI5 = -B1 \eta^4 \cos \alpha_b \eta + 4 B1 CO4$$

$$CO5 = B1 \eta^4 \sin \alpha_b \eta - 4 B1 SI4$$

$$SI21 = -0,5 B1 \cos 2\alpha_b \eta$$

$$SI22 = 0,25 B2 \sin 2\alpha_b \eta - 0,5 B1 \eta \cos 2\alpha_b \eta$$

$$SI23 = 0,5 B2 \eta \sin 2\alpha_b \eta + 0,25 B3 \cos 2\beta \eta - 0,5 B1 \eta^2 \cos 2\alpha_b \eta$$

$$CO21 = 0,5 B1 \sin 2\alpha_b \eta$$

$$CO22 = 0,25 B2 \cos 2\alpha_b \eta + 0,5 B1 \eta \sin 2\alpha_b \eta$$

$$CO23 = 0,5 B2 \eta \cos 2\alpha_b \eta + (0,5 B1 \eta^2 - 0,25 B3) \sin 2\alpha_b \eta$$

$$CO24 = 0,5 B1 \eta^3 \sin 2\alpha_b \eta - 1,5 B1 SI23$$

$$SI24 = 0,5 B1 \eta^3 \cos 2\alpha_b \eta + 1,5 B1 CO23$$

$$A = 0.5 \left(\frac{I_{yy}}{I_{xx}} + 1 \right)$$

$$B = 0.5 \left(\frac{I_{yy}}{I_{xx}} - 1 \right)$$

$$M1 = d_x \cos \phi_i - d_y \sin \phi_i$$

$$M2 = d_x \sin \phi_i + d_y \cos \phi_i$$

$$AO = \sin 2\phi_i CO21 + \cos 2\phi_i SI21$$

$$BO = \cos 2\phi_i CO21 - \sin 2\phi_i SI21$$

$$CO = \sin 2\phi_i CO22 + \cos 2\phi_i SI22$$

$$PO = \cos 2\phi_i CO22 - \sin 2\phi_i SI22$$

$$EO = \sin 2\phi_i CO23 + \cos 2\phi_i SI23$$

$$FO = \cos 2\phi_i CO23 - \sin 2\phi_i SI23$$

$$AA = 4(A\eta - BBO)$$

$$BB = 12(A\eta^2 / 2 - BPO)$$

$$DD = 36(A\eta^3 / 3 - BFO)$$

$$FF = 4BAO$$

$$GG = 12BCO$$

$$HH = 36BEO$$

$$PP = 4(A\eta + BBO)$$

$$QQ = 12(A\eta^2 / 2 + BPO)$$

$$RR = 36(A\eta^3 / 3 + BFO)$$

$$G25 = -\alpha_b (M1 SI1 + M2 CO1)$$

$$G26 = M1 CO1 - M2 SI1 - \alpha_b (M1 SI2 + M2 CO2)$$

$$G35 = -2 \alpha_b (M1 SI2 + M2 CO2)$$

$$G36 = 2 (M1 CO2 - M2 SI2) - 2 \alpha_b (M1 SI3 + M2 CO3)$$

$$G45 = -3 \alpha_b (M1 SI3 + M2 CO3)$$

$$G46 = 3 (M1 CO3 - M2 SI3) - 3 \alpha_b (M1 SI4 + M2 CO4)$$

$$G55 = \alpha_b^2 (0,5 M1^2 (\eta - CO21) + M1 M2 SI21 + 0,5 M2^2 (CO21 + \eta))$$

$$G56 = -\alpha_b (0,5 M1^2 SI21 + M1 M2 CO21 - 0,5 M2^2 SI21) + \alpha_b^2 (0,5 M1^2 (\eta^2 / 2 - CO22) + M1 M2 SI22 + 0,5 M2^2 (CO22 + \eta^2 / 2))$$

$$G66 = 0,5 M1^2 (CO21 + \eta) - M1 M2 SI21 + 0,5 M2^2 (\eta - CO21) - 2 \alpha_b (0,5 M1^2 SI22 + M1 M2 CO22 - 0,5 M2^2 SI22) + \alpha_b^2 (0,5 M1^2 (\eta^3 / 3 - CO23) + M1 M2 SI23 + 0,5 M2^2 (CO23 + \eta^3 / 3))$$

$$Z25 = \alpha_b (M1 CO1 - M2 SI1)$$

$$Z26 = M1 SI1 + M2 CO1 + \alpha_b (M1 CO2 - M2 SI2)$$

$$Z35 = 2 \alpha_b (M1 CO2 - M2 SI2)$$

$$Z36 = 2 (M1 SI2 + M2 CO2) + 2 \alpha_b (M1 CO3 - M2 SI3)$$

$$Z45 = 3 \alpha_b (M1 CO3 - M2 SI3)$$

$$Z46 = 3 (M1 SI3 + M2 CO3) + 3 \alpha_b (M1 CO4 - M2 SI4)$$

$$Z55 = \alpha_b^2 (0,5 M1^2 (\eta + CO21) - M1 M2 SI21 + 0,5 M2^2 (\eta - CO21))$$

$$Z56 = \alpha_b (0,5 M1^2 SI21 + M1 M2 CO21 - 0,5 M2^2 SI21) + \alpha_b^2 (0,5 M1^2 (\eta^2/2 + CO22) - M1 M2 SI22 + 0,5 M2^2 (\eta^2/2 - CO22))$$

$$Z66 = 0,5 M1^2 (\eta - CO21) + M1 M2 SI21 + 0,5 M2^2 (\eta + CO21) + 2 \alpha_b (0,5 M1^2 SI22 + M1 M2 CO22 - 0,5 M2^2 SI22) + \alpha_b^2 (0,5 M1^2 (\eta^3/3 + CO23) - M1 M2 SI23 + 0,5 M2^2 (\eta^3/3 - CO23))$$

$$FI25 = M1 (CO1 - \eta) - M2 SI1$$

$$FI26 = M1 CO2 - M2 SI2$$

$$FI35 = -2 M1 (B1 SI1 - CO2) - 2 M2 (SI2 + \alpha_b (CO1 - \eta))$$

$$FI36 = 2 M1 (-B2 (CO1 - \eta) - B1 SI2 + CO3) + 2 M2 (B2 SI1 - B1 CO2 - SI3)$$

$$FI45 = -3 M1 (2 B1 SI2 + 2 B2 (CO1 - \eta) - CO3) - 3 M2 (2 B1 CO2 + SI3 - 2 B2 SI1)$$

$$FI46 = 3 M1 (-4 B2 CO2 - 2 B1 SI3 + 4 B3 SI1 + CO4) + 3 M2 (4 B2 SI2 - SI4 + 4 B3 (CO1 - \eta) - 2 B1 CO3)$$

$$FI55 = \alpha_b^2 (0,5 M1^2 (\eta^2/2 - SI21 B1/2) - 0,5 B1 M1 M2 (CO21 - \eta) + 0,5 M2^2 (\eta^2/2 + 0,5 B1 SI21))$$

$$FI56 = 0,5 M1^2 (CO21 0,25 - 0,25 \eta + \alpha_b^2 \eta^3/6 - 0,5 \alpha_b SI22) + 0,5 M2^2 (-0,25 CO21 + 0,25 \eta + \alpha_b^2 \eta^3/6 + 0,5 \alpha_b SI22) + M1 M2 (-0,25 SI21 - 0,5 \alpha_b CO22)$$

$$FI66 = 0,5 M1^2 (0,5 \eta^2 + 0,25 B1 SI21 + 0,5 CO22 + \alpha_b^2 \eta^4/12 - 0,5 \beta SI23) + 0,5 M2^2 (0,5 \eta^2 - 0,25 B1 SI21 - 0,5 CO22 + \alpha_b^2 \eta^4/12 + 0,5 \beta SI23) + M1 M2 (-0,5 SI22 - 0,5 \alpha_b CO23 - 0,25 B1 + 0,25 B1 CO21)$$

$$TI25 = M2 (CO1 - \eta) + M1 SI1$$

$$TI26 = M2 CO2 + M1 SI2$$

$$TI35 = 2 M1 (B1 (CO1 - \eta) + SI2) - 2 M2 (B1 SI1 - CO2)$$

$$TI36 = 2 M1 (-B2 SI1 + SI3 + B1 CO2) + 2 M2 (-B2 (CO1 - \eta) - B1 SI2 + CO3)$$

$$TI45 = 3 M1 (2 B1 CO2 + SI3 - 2 B2 SI1) - 3 M2 (2 B1 SI2 + 2 B2 (CO1 - \eta) - CO3)$$

$$TI46 = 3 M1 (-4 B2 SI2 - 4 B3 (CO1 - \eta) + 2 B1 CO3 + SI4) + 3 M2 (4 B3 SI1 - 2 B1 SI3 - 4 B2 CO2 + CO4)$$

$$TI55=\alpha_b^2 (0,5 M1^2 (\eta^2/2+0,5 SI21 B1)+0,5 B1 M1 M2 (CO21-\eta)+0,5 M2^2 (\eta^2/2-0,5 B1 SI21))$$

$$TI56=0,5 M1^2 (-CO21 0,25+0,25 \eta+\alpha_b^2 \eta^3/6+0,5 \beta SI22)+0,5 M2^2 (0,25 CO21-0,25 \eta+\alpha_b^2 \eta^3/6-0,5 \alpha_b SI22)+M1 M2 (0,25 SI21+0,5 \alpha_b CO22)$$

$$TI66=0,5 M1^2 (0,5 \eta^2-0,25 B1 SI21-0,5 CO22+\alpha_b^2 \eta^4/12+0,5 \beta SI23)+0,5 M2^2 (0,5 \eta^2+0,25 B1 SI21+0,5 CO22+\alpha_b^2 \eta^4/12-0,5 \beta SI23)+M1 M2 (0,5 SI22-0,25 B1 (CO21-\eta)+0,5 \alpha_b CO23)$$

$$TS25=M1 SI2+M2 (CO2-0,5\eta^2)$$

$$TS26=M1 SI3+M2 CO3$$

$$TS35=2 M1 (B1 (CO2-0,5 \eta^2)+SI3)-2 M2 (B1 SI2-CO3)$$

$$TS36=2 M1 (-B2 SI2+SI4+B1 CO3)+2 M2 (-B2 (CO2-0,5 \eta^2)-B1 SI3+CO4)$$

$$TS45=3 M1 (2 B1 CO3+SI4-2 B2 SI2)-3 M2 (2 B1 SI3+2 B2 (CO2-0,5 \eta^2)-CO4)$$

$$TS46=3 M1 (-4 B2 SI3-4 B3 (CO2-0,5 \eta^2)+2 B1 CO4+SI5)+3 M2 (4 B3 SI2-2 B1 SI4-4 B2 CO3+CO5)$$

$$TS55=\alpha_b^2 (0,5 M1^2 (\eta^3/3+SI22 B1/2))+0,5 B1 M1 M2 (CO22-0,5 12)+0,5 M2^2 (\eta^3/3-0,5 B1 SI22)$$

$$TS56=0,5 M1^2 (-0,25 CO22+\eta^2/8+0,5 \alpha_b SI23+\alpha_b^2 \eta^4/8)+0,5 M1 M2 (0,5 SI22-\alpha_b CO23)+0,5 M2^2 (0,25 CO22-\eta^2/8-0,5 \beta SI23+\alpha_b^2 \eta^4/8)$$

$$TS66=0,5 M1^2 (\eta^3/3-0,25 B1 SI22+\alpha_b^2 \eta^5/15+0,5 CO23+0,5 \alpha_b SI24-SI23)+0,5 M2^2 (\eta^3/3+SI23+0,25 B1 SI22+\alpha_b^2 \eta^5/15-0,5 CO23-0,5 \alpha_b SI24)+M1 M2 (0,5 SI23-0,25 B1 (CO22-\eta^2/2)+0,5 \alpha_b CO24)$$

$$FS25=M1 (CO2-0,5 \eta^2)-M2 SI2$$

$$FS26=M1 CO3-M2 SI3$$

$$FS35=-2 M1 (B1 SI2-CO3)-2 M2 ((CO2-0,5 \eta^2) B1+SI3)$$

$$FS36=2 M1 (-B2 (CO2-0,5 \eta^2)-B1 SI3+CO4)+2 M2 (B2 SI2-B1 CO3-SI4)$$

$$FS45=-3 M1 (2 B1 SI3+2 B2 (CO2-0,5 \eta^2)-CO4)-3 M2 (2 B1 CO3+SI4-2 B2 SI2)$$

$$FS46=3 M1 (-4 B2 CO3-2 B1 SI4+4 B3 SI2+CO5)+3 M2 (4 B2 SI3-SI5+4 B3 (CO2-0,5 \eta^2)-2 B1 CO4)$$

$$FS55=\alpha_b^2 (0,5 M1^2 (\eta^3/3-SI22 B1/2)-0,5 B1 M1 M2 (CO22-0,5 \eta^2)+0,5 M2^2 (\eta^3/3+0,5 B1 SI22))$$

$$FS56=0,5 M1^2 (0,25 CO22-\eta^2/8-0,5 \beta SI23+\beta^2 \eta^4/8)+0,5 M1 M2 (-0,5 SI22-\alpha_b CO23)+0,5 M2^2 (-0,25 CO22+\eta^2/8+0,5 \alpha_b CO23+\alpha_b^2 \eta^4/8)$$

$$FS66=0,5 M1^2 (\eta^3/3+0,25 B1 SI22+\alpha_b^2 \eta^5/15+0,5 CO23-0,5 \alpha_b SI24)+0,5 M2^2 (\eta^3/3-0,25 B1 SI22+\alpha_b^2 \eta^5/15-0,5 CO23+0,5 \alpha_b SI24)+M1 M2 (-0,5 SI23+0,25 B1 (CO22-\eta^2/2)-0,5 \alpha_b CO24)$$

$$I15=M1 SI1+M2 CO1$$

$$I16=M1 SI2+M2 CO2$$

$$I25=I16$$

$$I26=M1 SI3+M2 CO3$$

$$I35=I26$$

$$I36=M1 SI4+M2 CO4$$

$$I45=I36$$

$$I46=M1 SI5+M2 CO5$$

$$I55=0,5 M1^2 (\eta-CO21)+0,5 M2^2 (\eta+CO21)+M1 M2 SI21$$

$$I56=0,5 M1^2 (0,5\eta^2-CO22)+0,5 M2^2 (0,5\eta^2+CO22)+M1 M2 SI22$$

$$I66=0,5 M1^2 (\eta^3/3-CO23)+0,5 M2^2 (\eta^3/3+CO23)+M1 M2 SI23$$

$$R15=M1 CO1-M2 SI1$$

$$R16=M1 CO2-M2 SI2$$

$$R25=R16$$

$$R26=M1 CO3-M2 SI3$$

$$R35=R26$$

$$R36=M1 CO4-M2 SI4$$

$$R45=R36$$

$$R46=M1 CO5-M2 SI5$$

$$R55=0,5 M1^2 (\eta+CO21)+0,5 M2^2 (\eta-CO21)-M1 M2 SI21$$

$$R56=0,5 M1^2 (0,5\eta^2+CO22)+0,5 M2^2 (0,5\eta^2-CO22)-M1 M2 SI22$$

$$R66=0,5 M1^2 (\eta^3/3+CO23)+0,5 M2^2 (\eta^3/3-CO23)-M1 M2 SI23$$

$$Y55=0,5 (M1^2-M2^2) SI21+M1 M2 CO21$$

$$Y56=0,5 (M1^2-M2^2) SI22+M1 M2 CO22$$

$$Y66=0,5 (M1^2-M2^2) SI23+M1 M2 CO23$$

$$A9=\sin\phi_i CO1+\cos\phi_i SI1$$

$$B9=\cos\phi_i CO1-\sin\phi_i SI1$$

$$C9=\sin\phi_i CO2+\cos\phi_i SI2$$

$$D9=\cos\phi_i CO2-\sin\phi_i SI2$$

$$E9=\sin\phi_i CO3+\cos\phi_i SI3$$

$$F9=\cos\phi_i CO3-\sin\phi_i SI3$$

$$G9=\sin\phi_i CO4+\cos\phi_i SI4$$

$$H9=\cos\phi_i CO4-\sin\phi_i SI4$$

$$I9=\sin\phi_i CO5+\cos\phi_i SI5$$

$$J9=\cos\phi_i CO5-\sin\phi_i SI5$$

$$AS=d_x A9+d_y B9$$

$$BS=d_x C9+d_y D9$$

$$CS=d_x E9+d_y F9$$

$$DS=d_x G9+d_y H9$$

$$ES=d_x I9+d_y J9$$

$$A3S=d_x B9-d_y A9$$

$$B3S=d_x D9-d_y C9$$

$$C3S=d_x F9-d_y E9$$

$$D3S=d_x H9-d_y G9$$

$$E3S=d_x J9-d_y I9$$

LIST OF SYMBOLS

A	Area of blade cross section
C	Torsional stiffness
c_f	Center of flexure
c_g	Center of blade cross section
E	Modulus of elasticity of blade
ρ	Density of blade material
I_{xx}	Second moment of area of cross section about XX
I_{yy}	Second moment of area of cross section about YY
I_{xx}	Second moment of area of cross section about x_1x_1
I_{yy}	Second moment of area of cross section about y_1y_1
I_{xy}	Product moment of area of cross section about x_1x_1 and y_1y_1
g	Gravitational acceleration
I_{cg}	Polar moment of inertia per unit length about centroid
I_{cf}	Polar moment of inertia per unit length about center of flexure
L	Blade length
r_2	Disc radius
XX, YY	Principal axes through centroid of blade cross- section
xx, yy	Coordinate axes through center of flexure of blade cross section
x_1x_1, y_1y_1	Coordinate axes through centroid of blade cross section
u	Displacement of center of flexure in x direction
u_1	Displacement of centroid in x direction
v	Displacement of center of flexure in y direction
v_1	Displacement of centroid in y direction
z	Coordinate distance measured along undeflected blade from root
θ	Torsional displacement
$\delta\delta$	Coordinate axis through centroid of blade cross section parallel to plane of disc
$\xi\xi$	Coordinate axis through centroid of blade cross section perpendicular to plane of the disc

ϕ	Angle between $\delta\delta$, $\xi\xi$ and axes x_1x_1 , y_1y_1 , stagger angle of a blade
ℓ	Length of beam element
q	Element coordinate vector
T	Kinetic energy
V	Strain energy
U_f	Strain energy due to the centrifugal force effects
d_x, d_y	Coordinates of center of flexure with respect to principal axis
d_x, d_y	Coordinates of center of flexure with respect to fixed axis system
R	Radius of curvature
ϕ_i	Pretwist angle of each beam element at their starting point
α	Fraction representing static component of load
β	Fraction representing time dependent component of load, element subtended angle
α_p	Angle between coordinates axes (x_1x_1 , y_1y_1) and principal axes (XX,YY) angle of pretwist
α_b	Pretwist angle along the beam element
p	Circular natural frequency
p_0	Fundamental natural frequency without pretwist
P^*	Fundamental static buckling load of a blade and two bladed packet
W_b	Weight of blade per unit length
W_s	Weight of shroud which resting on each blade
n	Number of elements
ℓ_{sh}	Curved beam element length
ω	Rotational speed
b	Axial thickness of the curved beam
q_i	Generalized displacements for an in-plane element
q_o	Generalized displacements for an out-of- plane element
I_{xxs}	Second moment of area of shroud cross- section about XX
I_{yys}	Second moment of area of shroud cross- section about YY
J_{xxs}	St. Venant torsion constant of shroud
i	Rotation of the tangent of a curved beam $i = \frac{\partial w}{\partial y} - \frac{v}{R}$

G_s	Shear modulus of shroud material
A_s	Area of shroud cross section (curved beam)
E_s	Modulus of elasticity of shroud material (curved beam)
ρ_s	Density of shroud
λ_b	Buckling load parameter of blade $\lambda_b = P * L^2 / EI_{xx}$
λ_c	Buckling load parameter of blade $\lambda_c = P * L^2 / EI_{xx}$
j	Slope of u deflection of a curved beam $j = -\frac{\partial u}{\partial y}$
L_{sh}	Total length of curved beam
$[]^T$	Denotes transpose of a matrix
$[]^{-1}$	Denotes inverse of a matrix
$\{ \}$	Denotes column matrix