NONCOMMUTATIVE PHASE AND THE UNITARIZATION OF THE QUANTUM GROUP $GL_{p,q}(2)$

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## ABSTRACT

## NONCOMMUTATIVE PHASE AND THE UNITARIZATION OF THE QUANTUM GROUP $GL_{p,q}(2)$

In this thesis, a new $*$–operation (or unitarized form) is defined for the two–parameter quantum group $GL_{p,q}(2)$ in the case that $pq$ is real, and the new group is denoted by $U_{p,q}(2)$. The most interesting aspect of our construction is the appearance of the noncommutative phase described by the unitary operator $u$. The operator $u$ with a central hermitian operator $s$ allow us to extend the algebra of the quantum group $GL_{p,q}(2)$ in order to obtain not only the $*$–operation but also the $*$–relations throughout the new algebra of $U_{p,q}(2)$. It is shown how certain $*$–representations of the quantum group $SU_q(2)$ can be extended in order to give $*$–representations of $U_{p,q}(2)$. This not only allows us to verify the algebraic relations in a representation, it also gives a hint of possible physical interpretations of the algebraic generators as operators.

# ÖZET

## DEĞİŞMELİ OLMAYAN FAZ OPERATÖRÜ ve $GL_{p,q}(2)$ KUANTUM GRUBUNUN ÜNİTERİZASYONU


Bu tezde *pq* reel olmak üzere iki parametre ile deforme edilmiş $GL_{p,q}(2)$ kuantum grubu üzerinde bir $*$ işlemcisi tanımlanmıştır. Bu $*$ işlemcisi kullanılarak $U_{p,q}(2)$ olarak gösterilen yeni bir kuantum grubu elde edilmiştir. Yaptıklarımızın en ilginç yanı, bu yapı inşaa edilirken, değişmeli olmayan bir üniter faz işlemci tanımlamamızdır. $u$ olarak adlandırdığımız bu işlemci, $s$ olarak adlandırdığımız merkezi ve hermisyen bir işlemci ile birlikte yeni cebir içerisinde $*$ işlemcisini tanımlamamızı sağlamaktadır. Böylece $U_{p,q}(2)$ olarak adlandırdığımız yeni cebir içerisinde tüm $*$ bağıntıları elde edilebilmektedir. $SU_q(2)$ kuantum grubunun bilinen temsillerinin, $U_{p,q}(2)$ kuantum grubunun temsillerine nasıl genişletilebileceğide bu tezde gösterilmektedir. Bu hem cebir bağıntılarının temsil içerisinde gerçeklemesini sağlamakta, hem de cebir üreteçlerinin işlemciler olarak fiziksel yorumları ile ilgili ipuçları vermektedir.

# TABLE OF CONTENTS


ACKNOWLEDGEMENTS . . . . . . . . . . . . . . . . . . . . . . . . . . . . iii

ABSTRACT . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . iv

ÖZET . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . v

LIST OF FIGURES . . . . . . . . . . . . . . . . . . . . . . . . . . . . vii

LIST OF SYMBOLS/ABBREVIATIONS . . . . . . . . . . . . . . . . . . . . viii

1. INTRODUCTION . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1

   1.1. Hopf Algebra and Quantum Groups . . . . . . . . . . . . . . . . . 2

   1.2. The Quantum Groups $GL_q(2)$ and $GL_{p,q}(2)$ . . . . . . . . . . . . . 10

   1.3. The Quantum Groups $SU_q(2)$ and $U_{\bar{q},q}(2)$ . . . . . . . . . . . . . 19

2. THE UNITARIZATION AND THE REPRESENTATION OF THE QUANTUM GROUP $GL_{p,q}(2)$ . . . . . . . . . . . . . . . . . . . . . . . . 21

   2.1. The Unitarization of the Quantum Group $GL_{p,q}(2)$ with $p \neq \bar{q}$ . . . . . 21

   2.2. The Representation of the Quantum Group $U_{p,q}(2)$ . . . . . . . . . . . 24

3. CONCLUSIONS . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 32

REFERENCES . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 33

# LIST OF FIGURES


Figure 1.1. Four morphism . . . . . 3

Figure 1.2. Associativity in an algebra $\mathcal{A}$ . . . . . 3

Figure 1.3. Unity in an algebra $\mathcal{A}$ . . . . . 3

Figure 1.4. Coassociativity in a coalgebra $\mathcal{C}$ . . . . . 4

Figure 1.5. Counity in a coalgebra $\mathcal{C}$ . . . . . 4

Figure 1.6. Connecting axiom in a bialgebra $\mathscr{A}$ . . . . . 5

Figure 1.7. Coalgebra morphism of $\mu$ . . . . . 6

Figure 1.8. Coalgebra morphism of $\eta$ . . . . . 6

Figure 1.9. Algebra morphism of $\Delta$ . . . . . 6

Figure 1.10. Algebra morphism of $\varepsilon$ . . . . . 7

Figure 1.11. Antipode of a bialgebra . . . . . 7

Figure 1.12. Commutativity in an algebra $\mathcal{A}$ . . . . . 9

Figure 1.13. Cocommutativity in a coalgebra $\mathcal{C}$ . . . . . 9

# LIST OF SYMBOLS/ABBREVIATIONS

| | |
|---|---|
| $\mathcal{A}$ | Algebra |
| $\mathscr{A}$ | Bialgebra |
| $\mathcal{C}$ | Coalgebra |
| $\mathbb{C}$ | Complex numbers |
| $\mathscr{D}$ | Quantum determinant of the quantum matrix $T$ |
| $\mathscr{H}$ | Hopf algebra |
| id | Identity map |
| $k$ | Compex field |
| $p, q$ | Deformation parameters |
| $R_q$, $R$ | Quantum $R$ matrix |
| $S$ | Antipode or coinverse map |
| $T$ | Quantum, representation or fundamental matrix of the quantum group |
| $\Delta$ | Coproduct map |
| $\epsilon(\sigma)$ | The quantum signature of $\sigma$ |
| $\varepsilon$ | Counit map |
| $\eta$ | Unit map |
| $\mu$ | Usual multiplication map |
| $\tau$ | Flip map |
| $\forall$ | For all |
| $\in$ | Element(s) of |
| $\otimes$ | Tensor multiplication |
| $\dot{\otimes}$ | Matrix multiplication with tensor product |
| $*$ | Star, hermitian conjugation operator |
| $\dagger$ | Hermitian conjugation operator |
| Fun $(G)$ | Linear space of $\mathbb{C}$–valued functions on a complex Lie group $G$ |

| | |
|---|---|
| $G$ | Complex Lie group |
| $GL_q(2)$ | One–parameter deformed quantum group formed by two by two matrices whose entries are noncommutative |
| $GL_{p,q}(2)$ | Two–parameter deformed quantum group formed by two by two matrices whose entries are noncommutative |
| $\mathfrak{M}_q(2)$ | One–parameter matrix quantum group formed by two by two matrices whose entries are noncommutative |
| $SU_q(2)$ | Special unitary one–parameter deformed quantum group |
| $U_{\bar{q},q}(2)$ | Unitary two–parameter deformed quantum group |

# 1. INTRODUCTION

The concept of quantum groups generalize the concept of symmetries to the realm of noncommutative geometry, which can be viewed as the quantization of the classical vector space in order to obtain the quantum vector space rather than the quantization of the classical physics as in quantum mechanics. This can be realized via appropriate deformation of the coordinate plane leading to attainment of the related quantum plane and noncommutative comultiplication in coalgebra structure achieved from a given algebra. More formally, the mathematical construction of a quantum group $G_q$ pertaining to a given Lie group $G$ is simply a deformation of a commutative Poisson-Hopf algebra defined over $G$ [1, 2]. This result in generalizing the classical groups in the sense of Hopf algebra.

Although the applications of quantum groups mainly concentrate on the studies of quantum integrable models using the quantum inverse scattering method and non–commutative geometry, there have been many phenomenological applications of quantum algebras in nuclear physics, condensed matter physics, molecular physics, quantum optics and elementary particle physics. The most important and remarkable application arose from the q–deformation of the known quantum mechanical harmonic oscillator algebra. The algebraic approaches to the oscillator algebras involve the known creation, annihilation and number operators. It is worth emphasizing the importance of the algebra possessing hermitian operators, giving rise to the ability of representing the physical observables. An algebra, therefore, needs to have a $*$ structure to be interpreted as an algebra of observables. The simplest matrix quantum group with such a structure is $SU_q(2)$ [3].

Another interesting quantum group which is under consideration in this thesis is the two parameter deformed quantum group $GL_{p,q}(2)$. This quantum group can be obtained through the quantization of both the coordinate and the exterior plane [4, 5]. The quantized coordinate and exterior plane are called $p$–plane and $1/q$–exterior plane respectively. The distinct feature of $GL_{p,q}(2)$ is that the quantum determinant of the fundamental matrix $T$ is not central if $p \neq q$ while the quantum determinant is central in one parameter deformed quantum group, e.g. $GL_q(2)$. It is important to note that for $p = q$ the quantum group $GL_{p,q}(2)$ becomes the quantum group $GL_q(2)$.

In this thesis, an algebra obtained by imposing $*$ relations on the operators $a, b, c$ and $d$ which are the matrix elements of the quantum group $GL_{p,q}(2)$ will be considered [6]. We are able to do this for $pq$ real. In the limit $p = \bar{q}$, our algebra coincides with $U_{q,\bar{q}}(2)$ [7]. We thus name this algebra $U_{p,q}(2)$. Representation of this algebra is constructed and the relationship of these representations to q–oscillators and to two–parameter coherent states are discussed. Let us review the definition of quantum groups via the notion of Hopf algebra, the formulation of two parameter deformed quantum group $GL_{p,q}(2)$ and its unitary form $U_{q,\bar{q}}(2)$ in order.

### 1.1. Hopf Algebra and Quantum Groups

The notion of quantum groups in physics is widely known to be the generalization of the symmetry properties of both classical Lie groups and Lie algebras, where two different mathematical blocks, namely deformation and co–multiplication, are simultaneously imposed either on the related Lie group or on the related Lie algebra.

A quantum group is defined algebraically as a quasi–triangular Hopf algebra. It can be either non-commutative or commutative. It is fundamentally a bi–algebra with an antipode so as to consist of either the q–deformed universal enveloping algebra of the classical Lie algebra or its dual, called the matrix quantum group, which can be understood as the q–analog of a classical matrix group [1]. Since a Hopf algebra is essentially a bialgebra with an antipode, one needs to define a bialgebra and an antipode.

The conventional way of defining a bialgebra is based on the possession of both algebra and coalgebra structures of a given algebra [2, 8, 9]. Assume that $\mathcal{A}$ is an associative algebra with unit 1, over a field $k$ which can be taken to be the set of complex numbers $\mathbb{C}$. A bialgebra $\mathscr{A}$ on $\mathcal{A}$ is, then, defined by four morphisms shown in Figure 1.1.

$$\mathscr{A} \otimes \mathscr{A} \xrightarrow{\mu} \mathscr{A} \xrightarrow{\Delta} \mathscr{A} \otimes \mathscr{A}$$

$$k \xrightarrow{\eta} \mathscr{A} \xrightarrow{\varepsilon} k$$

Figure 1.1. Four morphism

Algebra and coalgebra structures can be treated as the axioms which the four morphism above should satisfy and these axioms can be written as commutative diagrams.

An algebra is given by a triple $(\mathcal{A}, \mu, \eta)$ where $\mathcal{A}$ is a vector space and $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\eta : k \to \mathcal{A}$ are linear maps satisfying associativity and unity axioms as shown in Figure 1.2.

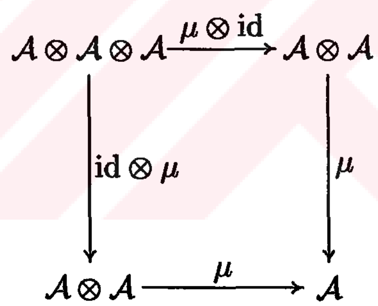


Figure 1.2. Associativity in an algebra $\mathcal{A}$

that is, $\mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu)$. The operation $\mu$ is the usual product in $\mathcal{A}$: $\mu(a \otimes b) = ab$ for $a, b \in \mathcal{A}$.

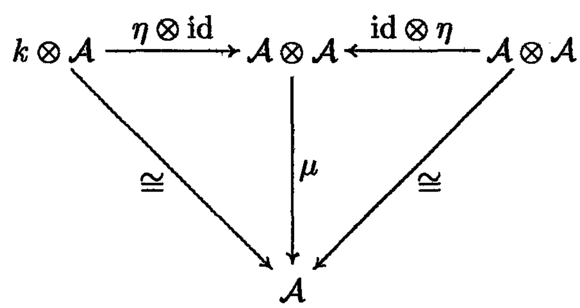


Figure 1.3. Unity in an algebra $\mathcal{A}$

that is, $\mu(a \otimes 1) = \mu(1 \otimes a) = a$ for all $a \in \mathcal{A}$. The operation $\eta$ is defined by $\eta(c) = c1$ for all $c \in k$.

On the other hand the notion of a coalgebra is similar to the concept of an algebra as regards its way of definition. A coalgebra is a triple $(\mathcal{C}, \Delta, \varepsilon)$ where $\mathcal{C}$ is a vector space and $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and $\varepsilon : \mathcal{C} \to k$ are linear maps satisfying coassociativity and counity axioms. It therefore can be said that the idea of a coalgebra is dual to the one of an algebra in this sense.

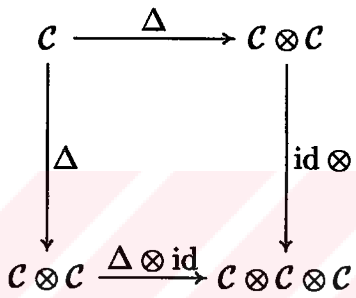


Figure 1.4. Coassociativity in a coalgebra $\mathcal{C}$

that is, $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. Coproduct $\Delta$ is a homomorphism of $\mathcal{C}$.

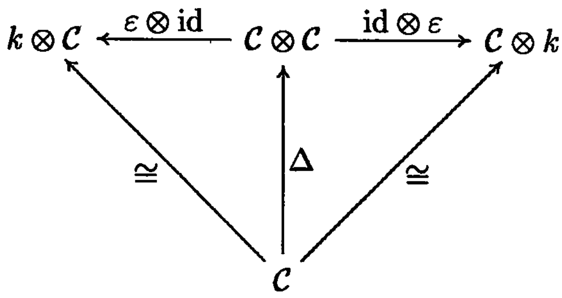


Figure 1.5. Counity in a coalgebra $\mathcal{C}$

that is, $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$, and $\varepsilon$ is also a homomorphism: $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ for all $a, b \in \mathcal{C}$.

There is an additional structure, called the connecting axiom [9], which is needed to link the algebra to its dual one. One can achieve a bialgebra $\mathscr{A}$ with the assistance of the connecting axiom shown in Figure 1.6.

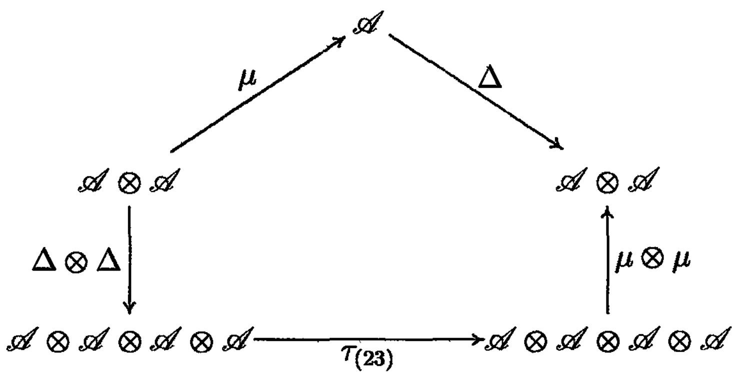


Figure 1.6. Connecting axiom in a bialgebra $\mathscr{A}$

where $\tau_{(23)}$ is the morphism exchanging the second and third places in the tensor product.

A bialgebra is, thus, defined by a vector space $\mathscr{A}$ equipped simultaneously with an algebra structure $(\mathcal{A}, \mu, \eta)$ and a coalgebra structure $(\mathcal{C}, \Delta, \varepsilon)$ with a connecting axiom. It is worth emphasizing that a morphism of bialgebras is a morphism for the underlying algebra and coalgebra structures. Since a bialgebra is basically a quintuple $(\mathscr{A}, \mu, \eta, \Delta, \varepsilon)$, the substructures should satisfy the equivalent conditions verified by a bialgebra itself. It is crucial that this definition be valid provided that the equivalency of the following two statements are fulfilled [8].

(i) The maps $\mu$ and $\eta$ are coalgebra morphisms.
(ii) The maps $\Delta$ and $\varepsilon$ are algebra morphisms.

The connecting axiom actually includes these statements expressed in Figure 1.6. The former in the first statement can be expressed by the commutativity of the following diagrams for $\mu$, giving rise to a coalgebra morphism of $\mu$,

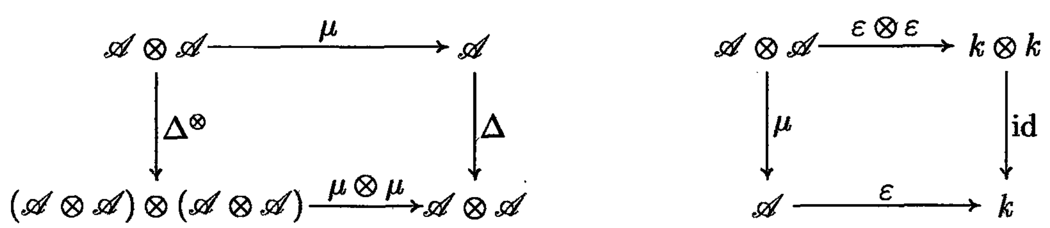


Figure 1.7. Coalgebra morphism of $\mu$

where $\Delta^{\otimes} \stackrel{\text{def}}{=} (\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta\otimes\Delta) : \mathscr{A}\otimes\mathscr{A} \longrightarrow (\mathscr{A}\otimes\mathscr{A})\otimes(\mathscr{A}\otimes\mathscr{A})$ and $\tau$ is a linear map and is flip switching the factors $\tau_{i,i+1}(a_1\otimes\ldots\otimes a_n) = a_1\otimes\ldots\otimes a_{i+1}\otimes a_i\otimes\ldots\otimes a_n$ in which $\forall a \in \mathscr{A}$ whereas the latter in the first statement can be expressed by the commutativity of the diagrams below for $\eta$, giving rise to a coalgebra morphism of $\eta$,

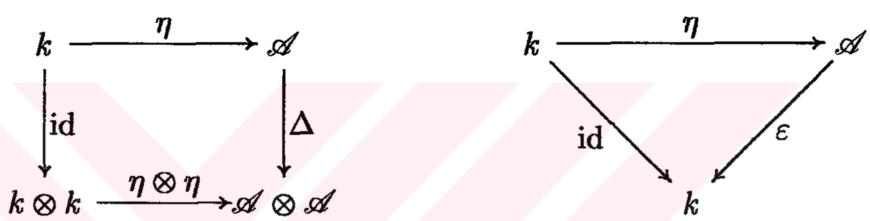


Figure 1.8. Coalgebra morphism of $\eta$

Similarly, the algebra morphism of $\Delta$ in the second statement is equivalent to the commutativity of the two diagrams below

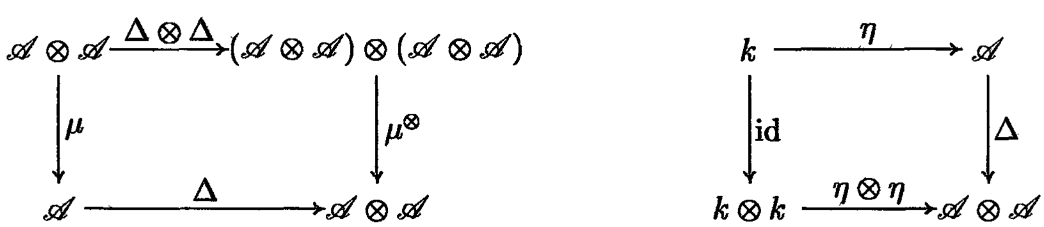


Figure 1.9. Algebra morphism of $\Delta$

where $\mu^{\otimes} \stackrel{\text{def}}{=} (\mu\otimes\mu)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id}) : (\mathscr{A}\otimes\mathscr{A})\otimes(\mathscr{A}\otimes\mathscr{A}) \longrightarrow \mathscr{A}\otimes\mathscr{A}$ whereas the algebra morphism of $\varepsilon$ in the second statement is equivalent to the commutativity of the following two diagrams

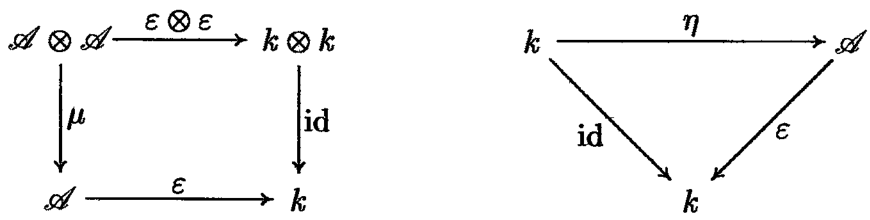


Figure 1.10. Algebra morphism of $\varepsilon$

Observation of the equivalency of the first four diagrams in Figures 1.7, 1.8 and the second four ones in Figures 1.9, 1.10 implies that the equivalency of the two statements. In this sense, the concept of bialgebra is selfdual.

An antipode of a bialgebra $(\mathscr{A}, \mu, \eta, \Delta, \varepsilon)$ is a linear map $S : A \to A$ such that the following diagram is commutative:

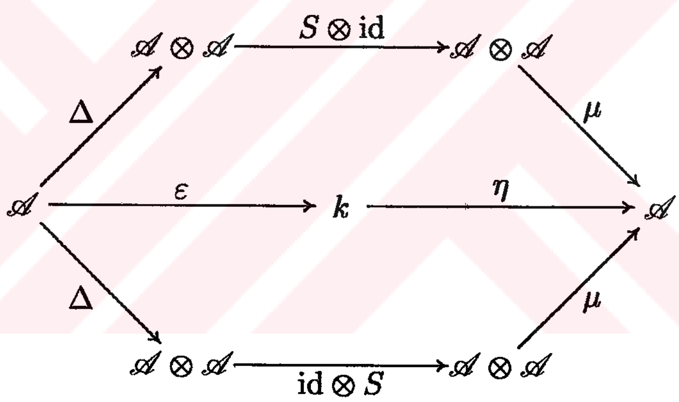


Figure 1.11. Antipode of a bialgebra

that is, $\mu(\mathrm{id} \otimes S)\Delta(a) = \mu(S \otimes \mathrm{id})\Delta(a) = \varepsilon(a)1$, where $a \in \mathscr{A}$. The antipode is an antihomomorphism: $S(ab) = S(b)S(a)$. The antipode $S$ reverses multiplication and coproduct, that is, defines a bialgebra morphism on the bialgebra $(\mathscr{A}, \mu, \eta, \Delta, \varepsilon)$.

A bialgebra $(\mathscr{A}, \mu, \eta, \Delta, \varepsilon)$ together with property $S$ is, thus, called a Hopf algebra with an antipode $S$ and denoted by $(\mathscr{H}, \mu, \eta, \Delta, \varepsilon, S)$. A morphism of Hopf algebra is a morphism between the underlying bialgebras commuting with the antipodes.

A $*$–algebra is an associative algebra $\mathcal{A}$ with unit $I$ equipped with a $*$-operation [10] with the properties

$$\begin{aligned}
(\alpha a+\beta b)^* &= \bar{\alpha}a^*+\bar{\beta}b^* \quad \text{(anti–linearity)}, && (1.1)\\
(a^*)^* &= a \quad \text{(involutivity)}, && (1.2)\\
(ab)^* &= b^*a^* \quad \text{(anti–multiplicativity)}, && (1.3)\\
I^* &= I\,. && (1.4)
\end{aligned}$$

A Hopf algebra is said to be a $*$–Hopf algebra [10] if and only if $\mathscr{H}$ is equipped with the $*$–operation with the properties (1.1)–(1.3) and such that

$$S\left(\left(S\left(a^*\right)\right)^*\right)=a\,, \quad a\in\mathscr{H}\,, \quad \text{that is} \quad S\circ *\circ S\circ * = \mathrm{id}\,, \tag{1.5}$$

and if $\Delta$ and $\varepsilon$ are $*$–homomorphism, that is

$$\varepsilon\left(a^*\right)=\overline{\varepsilon\left(a\right)}\,, \quad a\in\mathscr{H}\,, \tag{1.6}$$

and

$$\Delta\left(a^*\right)=\sum_i b_i^*\otimes c_i^*\,, \tag{1.7}$$

where $\Delta\left(a\right)=\sum_i b_i\otimes c_i$. The equation (1.7) is equivalent to the form

$$\Delta\circ * = (*\otimes *)\circ\Delta \tag{1.8}$$

It is important to note that $S$ and $*$ can be noncommutative.

So far, commutativity and noncommutativity of a bialgebra have not been mentioned yet so these of a Hopf algebra have not been either. An algebra $(\mathcal{A},\mu,\eta)$ is commutative if and only if $\mu=\mu\circ\tau$ which is equivalent to the commutativity of the diagram below

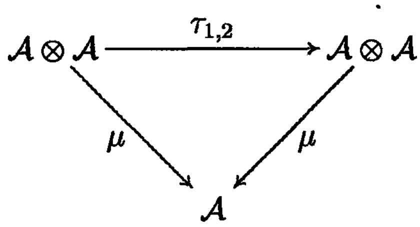


Figure 1.12. Commutativity in an algebra $\mathcal{A}$

Similarly, a coalgebra $(\mathcal{C}, \Delta, \varepsilon)$ is cocommutative if and only if $\Delta = \tau \circ \Delta$ which is equivalent to the commutativity of the following diagram

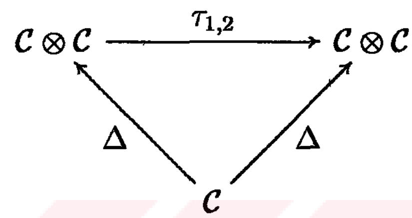


Figure 1.13. Cocommutativity in a coalgebra $\mathcal{C}$

It is apparent that commutativity and cocommutativity can be achieved provided that the Lie group $G$, in which $\mathscr{H} = \mathrm{Fun}(G)$, is abelian. For instance, if cocommutativity of a bialgebra is under consideration, then

$$\begin{aligned}
(\Delta f)(g_1, g_2) &= (f_1 \otimes f_2)(g_1, g_2) \\
&= f_1(g_1) f_2(g_2) \\
&= f(g_1 g_2) = f(g_2 g_1) \\
&= (\tau \circ \Delta f)(g_1, g_2) \qquad (1.9)
\end{aligned}$$

is satisfied if and only if $g_1 g_2 = g_2 g_1$ for a given function $f \in \mathrm{Fun}(G)$ and for elements $g_1, g_2 \in G$.

If the underlying Lie group $G$ is a noncommutative group, then the Hopf algebra $\mathscr{H} = \mathrm{Fun}(G)$ is actually both a noncommutative and a noncocommutative Hopf algebra. This implies that antipode of the Hopf algebra is also noncommuting. Moreover, the Hopf algebra under consideration has a $*$ structure, i.e., it is a $*$–Hopf algebra, then $*$ is noncommuting as well.

In quantum groups, Hopf algebras are encountered without the condition of commutativity. Quantum groups are a generalization of the concept of groups in the Hopf algebra sense with an appropriate deformation parameter $q$ as mentioned before. They can be defined by quantum matrices with noncommuting entries, which coincide with the general linear matrices if $q = 1$. These matrices form a Hopf algebra which is both noncommutative and noncocommutative rather than forming a group under matrix multiplication.

## 1.2. The Quantum Groups $GL_q(2)$ and $GL_{p,q}(2)$

The quantum group $GL_q(2)$ can be thought as a noncommutative Hopf algebra freely generated by the elements $a$, $b$, $c$ and $d$ of the two–by–two matrix called quantum, fundamental or representation matrix of the quantum group

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{1.10}$$

The entries of the matrix (1.10) satisfies the commutation relations

$$\begin{aligned} &ab = qba \quad ac = qca \\ &bd = qdb \quad cd = qdc \\ &bc = cb \quad\ \ ad - da = (q - q^{-1})\, bc\,. \end{aligned} \tag{1.11}$$

The commutation relations (1.11) can be achieved through the quantum plane relation and the quantum $R_q$ matrix, or shortly $R$. The former is acquired by the deformation of two dimensional plane which gives us to the quantum plane [8, 9]. The coordinates of the quantum plane do not commute

$$x^1 x^2 = q x^2 x^1 \,, \tag{1.12}$$

where $q \in \mathbb{C}\backslash\{0\}$ while the exterior plane, whose coordinates are defined by $\xi^1 = \mathrm{d}x^1$, $\xi^2 = \mathrm{d}x^2$, commutation relations are given by

$$\xi^1\xi^2 = -q^{-1}\xi^2\xi^1 \quad (\xi^1)^2 = (\xi^2)^2 = 0\,. \tag{1.13}$$

The quantum matrix $T$, the representation matrix of the quantum group $GL_q(2)$, can be treated as the matrix which transforms the quantum plane into another one

$$\begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \tag{1.14}$$

$$\begin{pmatrix} \xi'^1 \\ \xi'^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}. \tag{1.15}$$

The relation (1.14) leads to

$$ac = qca \quad bd = qdb \quad ad - da = -q^{-1}bc + qcb\,, \tag{1.16}$$

whereas the relation (1.15) gives rise to

$$bc = cb \quad ad - da = -q^{-1}cb + qbc\,. \tag{1.17}$$

Exchanging $b$ and $c$ in (1.16) results in

$$ab = qba \quad cd = qdc\,. \tag{1.18}$$

Thus all the commutation relations between the entries of the representation matrix of the one parameter deformed quantum group $GL_q(2)$ given by (1.11) have been achieved.

The latter is the way of using the quantum $R_q$ matrix, shortly $R$ which accounts for the noncommutativity of the quantum matrix $T$. The quantum $R$ matrix will be mentioned in the $GL_{p,q}(2)$ case in detail. The $R$ matrix for the quantum group $GL_q(2)$

is given by

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} . \tag{1.19}$$

The noncommuting matrix entries of $T^a{}_b$ satisfy the relation

$$R^{ab}{}_{ef} T^e{}_c T^f{}_d = T^b{}_f T^a{}_e R^{ef}{}_{cd} , \tag{1.20}$$

in which the repeated indices should be summed over and they receive their values from one to two. The commutation relations (1.11) can be obtained by the relation (1.20).

The quantum determinant $\mathscr{D}$ of an $n$–dimensional matrix quantum group $\mathfrak{M}_q(n)$ [2, 9] is defined by

$$\mathscr{D} = \det_q(T) \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} \epsilon(\sigma) T^1{}_{\sigma(1)} T^2{}_{\sigma(2)} \cdots T^n{}_{\sigma(n)} , \tag{1.21}$$

where the sum is over all permutations $\sigma$ of the symmetric group $S_n$, and the quantum signature $\epsilon(\sigma)$ is given for each element of $S_n$ by

$$\epsilon(\sigma) = \prod_{\substack{j<k \\ \sigma(j)<\sigma(k)}} (-q) = (-q)^{l(\sigma)} , \tag{1.22}$$

in which $l(\sigma)$ is the length of $\sigma$, that is, the minimal number of inversions in the permutation $\sigma$. The $q$–determinant is central for the one–parameter quantum groups, that is, it commutes with $T^a{}_b$ for each $a, b = 1, \ldots, n$. The quantum determinant $\mathscr{D}$ of the quantum group $GL_q(2)$ is therefore given by

$$\mathscr{D} = ad - qbc . \tag{1.23}$$

A Hopf algebra $\mathscr{H}$ is an algebra $\mathscr{H}$ which is endowed with the homomorphism $\Delta : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ and $\varepsilon : \mathscr{H} \to \mathbb{C}$ and the antihomomorphism $S : \mathscr{H} \to \mathscr{H}$ as mentioned in Section (1.1). The coproduct $\Delta$ of the quantum group $GL_q(2)$ is given by

$$\begin{aligned} \Delta(T) &= T \dot{\otimes} T \\ &= T^a_{\ b} \otimes T^b_{\ c} \\ &= \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix} . \end{aligned} \tag{1.24}$$

The counit $\varepsilon$ is given by

$$\varepsilon(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{1.25}$$

The antipode $S$ is given by

$$S(T) = \mathscr{D}^{-1} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} . \tag{1.26}$$

The quantum group $GL_{p,q}(2)$ is obtained by the deformation of two dimensional differential calculus on the two dimensional coordinate plane. This results in obtaining the quantum plane whose coordinates do not commute and also accounts for the noncocommutativity of the comultiplication [4, 5]. The quantum plane is spanned by the coordinates $x^1, x^2$ whose commutation relation is given by

$$x^1 x^2 = p^2 x^2 x^1 , \tag{1.27}$$

where $p \in \mathbb{C}\backslash\{0\}$ whereas the exterior plane, which is Grassmannian, is spanned by the coordinates $\xi^1 = \mathrm{d}x^1$, $\xi^2 = \mathrm{d}x^2$ whose commutation relation is given by

$$\xi^1 \xi^2 = -q^{-2} \xi^2 \xi^1 \quad (\xi^1)^2 = (\xi^2)^2 = 0 , \tag{1.28}$$

where $q \in \mathbb{C}\backslash\{0\}$. The quantum matrix $T$, the representation matrix of the quantum group $GL_{p,q}(2)$, is given by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.29}$$

The quantum matrix $T \in \mathscr{H}$ can be viewed as transformations of a quantum vector space, that is,

$$\begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \tag{1.30}$$

$$\begin{pmatrix} \xi'^1 \\ \xi'^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}. \tag{1.31}$$

The relation (1.30) leads to

$$ac = p^2 ca \quad bd = p^2 db \quad ad - da = -p^{-2} bc + p^2 cb\,, \tag{1.32}$$

whereas the relation (1.31) gives rise to

$$bc = p^2 q^{-2} cb \quad ad - da = -q^{-2} cb + q^2 bc\,. \tag{1.33}$$

Exchanging $b$ and $c$ and replacing $p^2$ by $q^2$ simutaneously in (1.32) results in

$$ab = q^2 ba \quad cd = q^2 dc\,. \tag{1.34}$$

Thus all the commutation relations between the entries of the representation matrix of the two parameter deformed quantum group $GL_{p,q}(2)$ are given by

$$\begin{aligned} &ab = q^2 ba && ac = p^2 ca \\ &bd = p^2 db && cd = q^2 dc \\ &bc = p^2 q^{-2} cb && ad - da = (q^2 - p^{-2})\, bc\,. \end{aligned} \tag{1.35}$$

One can achieve the same commutation relations by means of the quantum $R_q$ matrix, shortly $R$, which is the solution of the quantum Yang–Baxter equation

$$R^{a_1b_1}{}_{a_2b_2}R^{a_2c_1}{}_{a_3c_2}R^{b_2c_2}{}_{b_3c_3} = R^{b_1c_1}{}_{b_2c_2}R^{a_1c_2}{}_{a_2c_3}R^{a_2b_2}{}_{a_3b_2}\,, \tag{1.36}$$

in which the repeated indices should be summed over and they receive their values from one to two. The noncommutativity of $T^a_{\ b}$ is controlled by the $R$ matrix and the matrix $T^a_{\ b}$ satisfy the relation

$$R^{ab}{}_{ef}T^e{}_cT^f{}_d = T^b{}_fT^a{}_eR^{ef}{}_{cd}\,, \tag{1.37}$$

in which the repeated indices should, again, be summed over and they receive their values from one to two.

In general, the matrices $T$ and $R$ are given by

$$T = \left(T^i_{\ j}\right) \quad i,j = 1,\ldots,n \qquad R \in \mathbb{C}^{n^2\times n^2}\,. \tag{1.38}$$

Quantum groups are obtained through noncommutative continuous deformations of the Hopf algebra $\mathscr{H} = \mathrm{Fun}(G)$ as mentioned before. Thereby, for one-parameter deformed quantum groups $q \to 1$ corresponds to the classical limit which means that

$$R^{ab}{}_{cd} \to \delta^a{}_c\delta^b{}_d\,, \tag{1.39}$$

such that the noncommutative matrix entries of $T^a_{\ b}$ becomes commutative. This can be easily seen from the Hecke condition [1, 2] which the $\hat{R}^{ab}{}_{cd}$ matrix satisfies

$$\hat{R}^2 = (q-q^{-1})\,\hat{R} + I\,, \quad \text{for } A_{n-1} \quad (\text{Hecke condition})\,, \tag{1.40}$$

where $\hat{R}^{ab}{}_{cd} = R^{ba}{}_{cd}$.

For the quantum group $GL_{p,q}(2)$, the $R$ matrix can be given by

$$R = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & p-q^{-1} & pq^{-1} & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{1.41}$$

The commutation relations in (1.35) can be obtained through both the $R$ matrix above and (1.37).

The Hopf algebra structure of the quantum group $GL_{p,q}(2)$ is given by the coproduct $\Delta$, the counit $\varepsilon$, and the antipode( matrix inverse) $S$ of the matrix $T$, whose bialgebra is generated by the matrix elements $a$, $b$, $c$ and $d$. The coproduct $\Delta$ is given by

$$\begin{aligned} \Delta(T) &= T \dot{\otimes} T \\ &= T^a_{\ b} \otimes T^b_{\ c} \\ &= \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{pmatrix} . \end{aligned} \tag{1.42}$$

The counit $\varepsilon$ is given by

$$\varepsilon(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{1.43}$$

The anipode $S$ is given by

$$S(T) = \mathscr{D}^{-1} \begin{pmatrix} d & -p^{-2}b \\ -p^2 c & a \end{pmatrix} = \begin{pmatrix} d & -q^{-2}b \\ -q^2 c & a \end{pmatrix} \mathscr{D}^{-1} , \tag{1.44}$$

where the quantum determinant of $T$ is defined by

$$\mathscr{D} \equiv det_{p,q} T = ad - q^2 bc = ad - p^2 cb , \tag{1.45}$$

The quantum determinant $\mathscr{D}$ can also be achieved by the Borel decomposition of the matrix $T$ as follows

$$\begin{aligned} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \det \left[ \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - cd^{-1}b & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} \right] \\ &= \left(a - cd^{-1}b\right) d \\ &= ad - p^2 cb \\ &= ad - q^2 bc\,. \end{aligned} \tag{1.46}$$

The commutation relations between $\mathscr{D}$, which is not central unless $p = q$, and $a$, $b$, $c$, $d$ are given by

$$\begin{aligned} a\mathscr{D} &= \mathscr{D}a & b\mathscr{D} &= p^2q^{-2}\mathscr{D}b \\ c\mathscr{D} &= p^{-2}q^2\mathscr{D}c & d\mathscr{D} &= \mathscr{D}d\,, \end{aligned} \tag{1.47}$$

whereas $\mathscr{D}^{-1}$ obeys the following commutation relations

$$\begin{aligned} a\mathscr{D}^{-1} &= \mathscr{D}^{-1}a & b\mathscr{D}^{-1} &= p^{-2}q^2\mathscr{D}^{-1}b \\ c\mathscr{D}^{-1} &= p^2q^{-2}\mathscr{D}^{-1}c & d\mathscr{D}^{-1} &= \mathscr{D}^{-1}d\,, \end{aligned} \tag{1.48}$$

The matrix with the entries $S(a)$, $S(b)$, $S(c)$ and $S(d)$ is both left and right inverse for the matrix $T$, that is,

$$\begin{aligned} \begin{pmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} \\ &= \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{1.49}$$

The coproduct and the antipode of the quantum determinant are given by

$$\Delta(\mathscr{D}) = \mathscr{D}\otimes\mathscr{D}, \tag{1.50}$$

$$S(\mathscr{D}) = \mathscr{D}^{-1}. \tag{1.51}$$

The coproduct of the inverse of the quantum determinant $\mathscr{D}^{-1}$ is given by

$$\Delta\left(\mathscr{D}^{-1}\right) = \mathscr{D}^{-1}\otimes\mathscr{D}^{-1}, \tag{1.52}$$

which is consistent with $\mathscr{D}\mathscr{D}^{-1} = 1 = \mathscr{D}^{-1}\mathscr{D}$, that is,

$$\begin{aligned} \Delta\left(\mathscr{D}\mathscr{D}^{-1}\right) &= \Delta(\mathscr{D})\,\Delta\left(\mathscr{D}^{-1}\right) \\ &= (\mathscr{D}\otimes\mathscr{D})\left(\mathscr{D}^{-1}\otimes\mathscr{D}^{-1}\right) \\ &= \left(\mathscr{D}\mathscr{D}^{-1}\right)\otimes\left(\mathscr{D}\mathscr{D}^{-1}\right) \\ &= 1\otimes 1 \\ &= \Delta(1). \end{aligned} \tag{1.53}$$

The antipode of the inverse of the quantum determinant $\mathscr{D}^{-1}$ is given by

$$S\left(\mathscr{D}^{-1}\right) = \mathscr{D}, \tag{1.54}$$

which is again consistent with $\mathscr{D}\mathscr{D}^{-1} = 1 = \mathscr{D}^{-1}\mathscr{D}$. However, it is important to note that $S^2 \equiv S\circ S \neq \mathrm{id}$ generally if the algebra under consideration is noncommutative [2], for instance, in the $GL_{p,q}(2)$ algebra

$$\begin{array}{ll} S^2(a) = a & S^2(b) = p^{-2}q^{-2}b \\ S^2(c) = p^2q^2c & S^2(d) = d\,. \end{array} \tag{1.55}$$

## 1.3. The Quantum Groups $SU_q(2)$ and $U_{\bar{q},q}(2)$

The quantum group $SU_q(2)$ is defined as the algebra $\mathrm{Fun}_q(SL(2,\mathbb{C}))$ which admits an anti–involution defined by $T^\dagger = T^{-1}$ for $q \in \mathbb{R}^+$ [3, 9], i.e., the unitarity condition imposed on the quantum group. $T^\dagger$ is given by

$$T^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}. \tag{1.56}$$

In order to obtain the quantum group $SL_q(2)$, the quantum determinant of the quantum group $GL_q(2)$ should be equalized to one. Afterwards the unitarity condition, $T^\dagger = T^{-1}$, is imposed on this quantum group so as to achieve the quantum group $SU_q(2)$

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}. \tag{1.57}$$

The quantum matrix of the resultant quantum group $SU_q(2)$ read from the equation (1.57) is given by

$$T = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}. \tag{1.58}$$

The new commutation relations of the unitary quantum group are given by

$$\begin{aligned} &ac = qca \quad ac^* = qc^*a \\ &cc^* = c^*c \quad aa^* - a^*a = (q^2-1)\,cc^*, \end{aligned} \tag{1.59}$$

together with the conjugate relations

$$c^*a^* = qa^*c^* \quad ca^* = qa^*c. \tag{1.60}$$

The quantum determinant $\mathscr{D}$ of the quantum group $SU_q(2)$ is defined as

$$\mathscr{D} = aa^* + q^2cc^* = 1\,. \tag{1.61}$$

Lastly, the quantum matrix $T \in SU_q(2)$ satisfies the relation

$$T^\dagger T = TT^\dagger = 1\,. \tag{1.62}$$

A unitarized form of $GL_{p,q}(2)$, named $U_{\bar{q},q}(2)$, can be found in Jagannathan and Van Der Jeugt [7]. It is important to notice that our notation is different from the usual one as regards the usage of the deformation parameters $p$ and $q$. The deformation parameters $p$ and $q$ should be replaced by $p^{1/2}$ and $q^{1/2}$ to obtain the usual convention in [7]. The fundamental $T$–matrix of the quantum group is given in [7] by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -\bar{q}\mathscr{D}c^* \\ c & \mathscr{D}a^* \end{pmatrix} = \begin{pmatrix} a & -qc^*\mathscr{D} \\ c & a^*\mathscr{D} \end{pmatrix}, \tag{1.63}$$

where the matrix elements satisfy

$$\begin{array}{ll} ac = \bar{q}ca & ac^* = qc^*a \\ a\mathscr{D} = \mathscr{D}a & \mathscr{D}c^* = e^{2i\theta}c^*\mathscr{D} \\ cc^* = c^*c & \mathscr{D}^*\mathscr{D} = \mathscr{D}\mathscr{D}^* = 1 \\ a^*a + c^*c = 1 & aa^* + |q|^2\,c^*c = 1\,. \end{array} \tag{1.64}$$

Here $q = |q|\,e^{i\theta}$, $p = \bar{q}$ and $\theta$ is a phase. The case $\mathscr{D} = 1$ which also implies $\theta = 0$ corresponds to $SU_q(2)$, (1.59)–(1.60).

# 2. THE UNITARIZATION AND THE REPRESENTATION OF THE QUANTUM GROUP $GL_{p,q}(2)$

## 2.1. The Unitarization of the Quantum Group $GL_{p,q}(2)$ with $p \neq \bar{q}$

In order to obtain $SU_q(2)$, elements of the fundamental matrix $T$, $T \in GL_q(2, \mathbb{C})$, are chosen in such a way that $T^\dagger = T^{-1}$ and $\det_q(T) = 1$. This choice brings about a restriction on the elements of the matrix such that $b = -qc^*$ and $d = a^*$.

The procedure applied to the quantum group $GL_{p,q}(2, \mathbb{C})$ to carry out the unitarization [6] is similar to the one applied to the one–parameter deformed quantum group in order to transform $GL_q(2, \mathbb{C})$ into $U_q(2)$ but it is not completely the same. The most important point of the procedure we have studied is that the matrix of the quantum group should be factorized into a product which consists of the square root of the quantum determinant and a new matrix whose determinant is unity.

$$\mathscr{D} = \delta^2, \tag{2.1}$$

$$T = \delta T_{right}. \tag{2.2}$$

The coproduct and the antipode of $\delta$ are given by

$$\Delta(\delta) = \delta \otimes \delta, \tag{2.3}$$

$$S(\delta) = \delta^{-1}, \tag{2.4}$$

which are consistent with the equations (1.50) and (1.51). The commutation relations between $\delta$ and the entries of the matrix $T$ are given by

$$\begin{array}{ll} a\delta = \delta a & d\delta = \delta d \\ b\delta = pq^{-1}\delta b & c\delta = p^{-1}q\delta d\,. \end{array} \tag{2.5}$$

The next step is to impose the unitarity condition on the new matrix, resulting in finding the relation between the elements of this matrix as in the $U_q(2)$ case. Therefore, the elements of the matrix $T_{right}$ become the elements of $SU_r(2)$ with $r \in \mathbb{R}$

$$r = pq = \bar{p}\bar{q}\,. \tag{2.6}$$

The commutation relations between $\delta$ and $T_{right}$, obtained through (2.2) and (2.5) under the unitarity condition, i.e., using the commutation relations in (1.59) and (1.60), are given by

$$\begin{array}{ll} a_R\delta = \delta a_R & a_R^*\delta = \delta a_R^* \\ c_R^*\delta = pq^{-1}\delta c_R^* & c_R\delta = p^{-1}q\delta c_R\,, \end{array} \tag{2.7}$$

in which $a_R$, $a_R^*$, $c_R$ and $c_R^* \in SU_r(2)$.

Lastly, the relations between the original matrix elements can be achieved through the relations between the new ones obtained after the unitarization of the matrix with the condition

$$\delta\delta^* = s^2\delta^*\delta\,, \tag{2.8}$$

where $s$ is a central element of the resultant unitarized algebra of $GL_{p,q}(2)$. It commutes with all elements in the algebra and is also hermitian $s = s^*$. The coproduct, counit and the antipode of the central element $s$ are given by

$$\Delta(s) = s \otimes s\,, \tag{2.9}$$

$$\varepsilon(s) = 1\,, \tag{2.10}$$

$$S(s) = s^{-1}\,. \tag{2.11}$$

The equation (2.8) allows us to determine the commutation relations between the entries of $T$ and $\delta^*$, which are

$$\begin{array}{ll} s^2\delta^* a = a\delta^* & s^2\delta^* d = d\delta^* \\ s^2\delta^* b = \bar{p}^{-1}\bar{q} b\delta^* & s^2\delta^* c = \bar{p}\bar{q}^{-1} c\delta^* \,. \end{array} \tag{2.12}$$

These lead to the matrix elements $b$ and $d$ of the matrix $T$ being respectively replaced by a combination of $c^*$ and $a^*$ multiplied by inverse of $s$ and a unitary operator $u$. The new fundamental matrix $T$ of the unitary quantum group is given by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -\bar{q}^2 s^{-1} u c^* \\ c & s^{-1} u a^* \end{pmatrix} . \tag{2.13}$$

It can be easily checked that with these relations $a$, $c$ and $b$, $d$ defined by (2.13) satisfy the commutation relations (1.35) of $GL_{p,q}(2)$. The coproduct, the counit and the antipode of the unitary operator $u$ are given by

$$\Delta(u) = u \otimes u \,, \tag{2.14}$$

$$\varepsilon(u) = 1 \,, \tag{2.15}$$

$$S(u) = u^* \,. \tag{2.16}$$

The whole algebra of the unitarized two–parameter quantum group, which the matrix elements obey, is given by

$$\begin{array}{ll} ac = p^2 ca & a^* c^* = \bar{p}^{-2} c^* a^* \\ ac^* = q^2 s^2 c^* a & a^* c = \bar{q}^{-2} s^{-2} c a^* \\ cc^* = \frac{q\bar{q}}{p\bar{p}} s^2 c^* c & aa^* - s^2 a^* a = (1 - r^2)\, cc^* \\ ua = s^2 au & u^* a^* = s^{-2} a^* u^* \\ ua^* = s^2 a^* u & u^* a = s^{-2} a u^* \\ uc = \frac{p\bar{q}}{\bar{p}q} s^2 cu & u^* c^* = \frac{p\bar{q}}{\bar{p}q} s^{-2} c^* u^* \\ uc^* = \frac{\bar{p}q}{p\bar{q}} s^2 c^* u & u^* c = \frac{\bar{p}q}{p\bar{q}} s^{-2} c u^* \\ uu^* = u^* u = 1 \,. & \end{array} \tag{2.17}$$

It can be shown that the commutation relations (2.17) satisfy the co–product algebra homomorphism and the antipode algebra anti–homomorphism. For $p = \bar{q}$, the algebra introduced in (2.17) coincides with the $U_{\bar{q},q}(2)$ algebra in (1.64) since $s = 1$ in this limit.

### 2.2. The Representation of the Quantum Group $U_{p,q}(2)$

The operators constituting an $SU_q(2)$ matrix which corresponds to $\mathscr{D} = 1$, $q$ real in (1.63) can be represented by their action on states $|n, m\rangle$ where $n$ is non–negative integer corresponding to the particle number associated with the creation operator $a^*$ and $m$ is a positive or negative integer associated with the Fourier transform of $c$ [11]. This serves two purposes. One is that it proves the algebra presented in the previous section is consistent. The second is that it gives physical insight on the oscillator properties of the operators. The action of the operators $a$, $a^*$ and $c$, $c^*$ of $SU_q(2)$ on the states $|n, m\rangle$ is given by

$$a\,|n,m\rangle = \sqrt{1-q^{2n}}\,|n-1,m\rangle \,, \tag{2.18}$$

$$a^*\,|n,m\rangle = \sqrt{1-q^{2n+2}}\,|n+1,m\rangle \,, \tag{2.19}$$

$$c\,|n,m\rangle = q^n\,|n,m-1\rangle \,, \tag{2.20}$$

$$c^*\,|n,m\rangle = q^n\,|n,m+1\rangle \,. \tag{2.21}$$

Here $m$ is an integer and $n$ is a nonnegative integer. Motivated by this, we look for a representation of the algebra (2.17) on such states. Firstly, it is appropriate to investigate the representation of the $SU_r(2)$ through the actions of the operators $c_R$ and $a_R$ on such states $|\gamma\rangle$ that

$$c_R|\gamma\rangle = \gamma|\gamma\rangle\,, \tag{2.22}$$

$$c_R^*|\gamma\rangle = \bar{\gamma}|\gamma\rangle\,, \tag{2.23}$$

where $\gamma \in \mathbb{C}$. If the commutation relations given in (1.59)–(1.60) with $r$ instead of $q$, the equations below can be achieved,

$$c_R\left(a_R|\gamma\rangle\right) = r^{-1}\gamma\left(a_R|\gamma\rangle\right) , \tag{2.24}$$

which gives

$$a_R|\gamma\rangle = \alpha\left(\gamma\right)|r^{-1}\gamma\rangle . \tag{2.25}$$

Similar holds for the operator $a_R^*$,

$$c_R\left(a_R^*|\gamma\rangle\right) = r\gamma\left(a_R^*|\gamma\rangle\right) , \tag{2.26}$$

which also gives

$$a_R^*|\gamma\rangle = \tilde{\alpha}\left(\gamma\right)|r\gamma\rangle . \tag{2.27}$$

If the equations below are treated in the same way,

$$a_R a_R^* + r^2 c_R^* c_R = 1 , \tag{2.28}$$

$$a_R^* a_R + c_R^* c_R = 1 , \tag{2.29}$$

the following equations can easily be found,

$$\alpha\left(\gamma\right)\tilde{\alpha}\left(r^{-1}\gamma\right) + \|\gamma\|^2 = 1 , \tag{2.30}$$

$$\tilde{\alpha}\left(\gamma\right)\alpha\left(r\gamma\right) + r^2\|\gamma\|^2 = 1 . \tag{2.31}$$

In addition, the multiplication of the state $a_R|\gamma\rangle$ by the state $\langle r^{-1}\gamma|$ from left gives

$$\langle r^{-1}\gamma|a_R|\gamma\rangle = \alpha\left(\gamma\right) . \tag{2.32}$$

Taking the hermitian conjugate of the both sides of (2.32) brings about

$$\langle\gamma|a_R^*|r^{-1}\gamma\rangle = \overline{\alpha(\gamma)} \quad \Rightarrow \quad a_R^*|\gamma\rangle = \overline{\alpha(r\gamma)}|r\gamma\rangle\,. \tag{2.33}$$

Comparing this result with (2.27), one can easily find

$$\tilde{\alpha}(\gamma) = \overline{\alpha(r\gamma)}\,. \tag{2.34}$$

This result with the equations (2.30)–(2.31) leads to

$$\|\alpha(\gamma)\|^2 = 1 - \|\gamma\|^2\,. \tag{2.35}$$

The number operator can therefore be given by

$$a_R^* a_R|\gamma\rangle = \left(1 - \|\gamma\|^2\right)|\gamma\rangle\,. \tag{2.36}$$

The ground state or vacuum is defined by

$$a_R|\gamma_{top}\rangle = 0\,, \tag{2.37}$$

$$a_R^* a_R|\gamma_{top}\rangle = 0\,, \quad \exists\gamma_{top} \quad \|\gamma_{top}\|^2 = 1\,, \quad \text{for } q < 1\,, \tag{2.38}$$

in order to obtain the harmonic oscillator properties of the operators.

One can construct the Fock space which is the full Hilbert space of states through defining the state $|n\rangle$ with the creation operators,

$$c_R\left(a_R^*|\gamma_{top}\rangle\right) = r\gamma_{top}\left(a_R^*|\gamma_{top}\rangle\right)\,, \tag{2.39}$$

$$c_R|n\rangle \cong c_R\left[\left(a_R^*\right)^n|\gamma_{top}\rangle\right]\,, \tag{2.40}$$

$$= r^n\gamma_{top}|n\rangle\,, \tag{2.41}$$

in which $\gamma_{top}$ has to be

$$\gamma_{top} = e^{i\alpha}\,, \quad \text{with} \quad \alpha \in \mathbb{R}/2\pi\,. \tag{2.42}$$

The action of $c_R$ and $c_R^*$ on the states $|n,\alpha\rangle$ are given by

$$c_R|n,\alpha\rangle = r^n e^{i\alpha}|n,\alpha\rangle\,, \tag{2.43}$$

$$c_R^*|n,\alpha\rangle = r^n e^{-i\alpha}|n,\alpha\rangle\,, \tag{2.44}$$

with the normalization $\langle n,\alpha|n',\alpha'\rangle = \delta_{nn'}\delta(\alpha-\alpha')$. Therefore, the state $|0\rangle$ is made to relate $||\gamma|| = 1$ and so forth.

$$\begin{array}{llllll} ||\gamma|| & : & 1 & r & r^2 & \dots \\ |n\rangle & : & |0\rangle & |1\rangle \propto a_R^*|0\rangle & |2\rangle \propto a_R^*|1\rangle & \dots \\ a_R^*a_R|n\rangle & : & 0 & (1-r^2)\,|1\rangle & (1-r^4)\,|2\rangle & \dots \end{array} \tag{2.45}$$

The action of the operators $a_R$ and $a_R^*$ on the states $|n,\alpha\rangle$are given by

$$a_R|n,\alpha\rangle = \sqrt{1-r^{2n}}|n-1,\alpha\rangle\,, \tag{2.46}$$

$$a_R^*|n,\alpha\rangle = \sqrt{1-r^{2n+2}}|n+1,\alpha\rangle\,. \tag{2.47}$$

If the states $|n,\alpha\rangle$ are expanded in Fourier Series as,

$$|n,\alpha\rangle = \frac{1}{\sqrt{2\pi}}\sum_m e^{im\alpha}|n,m\rangle\,, \tag{2.48}$$

one can find the action of the operator $c_R$ under these transformations

$$c_R|n,\alpha\rangle = r^n e^{i\alpha}|n,\alpha\rangle\,, \tag{2.49}$$

$$= \frac{r^n}{\sqrt{2\pi}}\sum_m e^{i(m+1)\alpha}|n,m\rangle\,, \tag{2.50}$$

$$= \frac{r^n}{\sqrt{2\pi}}\sum_m e^{im\alpha}|n,m-1\rangle\,. \tag{2.51}$$

If the states on the left–hand side in the last step is also expanded in Fourier Series without the fulfillment of the action of the operator $c_R$ on the states, the following relations can be acquired,

$$\frac{1}{\sqrt{2\pi}}\sum_m e^{im\alpha}c_R|n,m\rangle = \frac{r^n}{\sqrt{2\pi}}\sum_m e^{im\alpha}|n,m-1\rangle\,. \tag{2.52}$$

The action of the operator $c_R$ on the states $|n,m\rangle$ is thus defined by,

$$c_R|n,m\rangle = r^n|n,m-1\rangle\,. \tag{2.53}$$

If the same calculation is done for the operator $a_R$, it can be found that

$$\frac{1}{\sqrt{2\pi}}\sum_m e^{im\alpha}a_R|n,m\rangle = \frac{\sqrt{1-r^{2n}}}{\sqrt{2\pi}}\sum_m e^{im\alpha}|n-1,m\rangle\,, \tag{2.54}$$

$$\therefore \quad a_R|n,m\rangle = \sqrt{1-r^{2n}}|n-1,m\rangle\,. \tag{2.55}$$

Likewise, similiar equations can be found for the other two operators $c_R^*$ and $a_R^*$. The complete set of equations for the action of the operators $c_R$, $c_R^*$, $a_R$ and $a_R^*$ for the quantum group $SU_r(2)$ is given by

$$c_R|n,m\rangle = r^n|n,m-1\rangle\,, \tag{2.56}$$

$$c_R^*|n,m\rangle = r^n|n,m+1\rangle\,, \tag{2.57}$$

$$a_R|n,m\rangle = \sqrt{1-r^{2n}}|n-1,m\rangle\,, \tag{2.58}$$

$$a_R^*|n,m\rangle = \sqrt{1-r^{2n+2}}|n+1,m\rangle\,, \tag{2.59}$$

where $m$ is an integer and $n$ is a nonnegative integer as mentioned before.

In the second place the representation of the $\delta$ should be determined in order to find the representation of the elements of the quantum matrix $T$. It is a good idea to give an ansatz for the action of the operator $\delta$ on the states $|n,m\rangle$

$$\delta|n,m\rangle = \sum_{n'm'} C\left(n,m,n',m'\right)|n,m\rangle\,. \tag{2.60}$$

It is possible to choose the function $C\left(n,m,n',m'\right)$ as

$$C\left(n,m,n',m'\right) = \delta\left(n,n'\right) D\left(m,m'\right) \tag{2.61}$$

Through the commutation relations (2.7) and the representations of the quantum group $SU_r(2)$ in (2.56)–(2.59), the relation which the function $D\left(m,m'\right)$ satisfies can be found

$$\begin{aligned}
\delta c_R|n,m\rangle &= \frac{p}{q} c_R \delta|n,m\rangle\,, && (2.62)\\
r^n \sum_{m'} D\left(m-1,m'\right)|n,m'\rangle &= r^n \frac{p}{q} \sum_{m'} D\left(m,m'\right)|n,m'-1\rangle\,, && (2.63)\\
\therefore \quad D\left(m,m'+1\right) &= \frac{q}{p} D\left(m-1,m'\right)\,. && (2.64)
\end{aligned}$$

Same result can be achieved via the commutation relation which includes the operators $\delta$ and $c_R^*$.

Let us choose the function $D(m,m')$ as

$$D(m,m') = F(m-m',m+m')\,, \tag{2.65}$$

which gives

$$\begin{aligned}
F\left(m-m',m+m'+2\right) &= \frac{q}{p} F\left(m-m',m+m'\right)\,, && (2.66)\\
\therefore \quad F(x,y) &= \left(\frac{q}{p}\right)^{y/2} F(x)\,. && (2.67)
\end{aligned}$$

Both the function $D\left(m,m'\right)$ and the action of the operator $\delta$ on the states $|n,m\rangle$ can be rewritten in terms of the new function $F(m, m')$ as

$$D(m,m') = F(m-m')\left(\frac{q}{p}\right)^{\frac{m+m'}{2}}, \qquad (2.68)$$

$$\delta|n,m\rangle = \sum_{m'} F(m-m')\left(\frac{q}{p}\right)^{\frac{m+m'}{2}} |n,m'\rangle . \qquad (2.69)$$

If the function $F(m-m')$ is chosen to be

$$F(m-m') = \delta_{m,m'+k} B(k), \qquad (2.70)$$

the representation of the operator $\delta$ with the states $|n,m\rangle$ can be found as

$$\delta|n,m\rangle = \left(\frac{q}{p}\right)^{m} |n,m-k\rangle , \qquad (2.71)$$

with the function

$$B(k) = \left(\frac{q}{p}\right)^{k/2} . \qquad (2.72)$$

This kind of choice for the function $B(k)$ is for simplification and also does not make any changes in the commutation relations (2.17). The action of the operator $\delta^*$ on the states $|n,m\rangle$ is also given by

$$\delta^*|n,m\rangle = \left(\frac{\bar{q}}{\bar{p}}\right)^{m+k} |n,m+k\rangle . \qquad (2.73)$$

The deformation parameters $p$ and $q$ are reparametrized in order to achieve a convenient form for the representation

$$p = \sqrt{\frac{r}{t}} e^{i\theta/2} \quad q = \sqrt{rt} e^{-i\theta/2} , \qquad (2.74)$$

where $r$, $t$ and $\theta$ are real independent parameters. The parameters $r$ and $t$ are positive by definition and $\theta$ is a phase angle. The representation also depends on a real integer parameter $k$ associated with the eigenvalue of the central element $s$. The special case where $t = 1$, $\theta = 0$, $k = 0$, and therefore $p = q$, corresponds to the $SU_q(2)$ algebra for which it is necessary that $q \in (0,1)$ whereas the case $k = 0$, $t = 1$ corresponds to $U_{\bar{q},q}(2)$ discussed in Section (2.2). The operators $c$, $c^*$, $a$, $a^*$, $u$, $u^*$ and $s$ act on states as

$$c\,|n,m\rangle = r^n \left(te^{-i\theta}\right)^{m-1} |n, m-(k+1)\rangle\ , \tag{2.75}$$

$$c^*\,|n,m\rangle = r^n \left(te^{i\theta}\right)^{m+k} |n, m+(k+1)\rangle\ , \tag{2.76}$$

$$a\,|n,m\rangle = \sqrt{1-r^{2n}} \left(te^{-i\theta}\right)^{m} |n-1, m-k\rangle\ , \tag{2.77}$$

$$a^*\,|n,m\rangle = \sqrt{1-r^{2n+2}} \left(te^{i\theta}\right)^{m+k} |n+1, m+k\rangle\ , \tag{2.78}$$

$$u\,|n,m\rangle = e^{i(k-2m)\theta}\,|n, m-2k\rangle\ , \tag{2.79}$$

$$u^*\,|n,m\rangle = e^{i(2m+3k)\theta}\,|n, m+2k\rangle\ , \tag{2.80}$$

$$s|n,m\rangle = t^k|n,m\rangle\,, \tag{2.81}$$

which explicitly shows that $u$ is actually a non–commutative unitary phase operator. It can be easily seen that setting $t = 1$, $\theta = 0$, $k = 0$ leads to $p = q$. The representation above then reduces to the representation (2.18–2.21) by the replacement of $p$ and $q$ by $p^{1/2}$ and $q^{1/2}$.

# 3. CONCLUSIONS

The usual method of unitarizing a quantum group is merely imposing the (unitarity) condition $TT^{\dagger} = T^{\dagger}T = 1$ on the quantum matrix $T$. The quantum group $U_q(2)$ can, thus, be obtained from $GL_q(2)$ in this manner. During this process, consistency also requires that the complex parameter $q$ of the quantum group $GL_q(2)$ is constrained to be real. Hence $U_q(2)$ exists only for real $q$. If one applies the same method to $GL_{p,q}(2)$, it is found that consistency requires $p = \bar{q}$ and the resulting group is the quantum group $U_{\bar{q},q}(2)$. The quantum group $GL_{p,q}(2)$ is made to reduce to the quantum group $SU_q(2)$ for the special case in which $p = q$. The physical importance of unitarization is that it imposes star relations onto the algebra formed by the noncommutative entries of the quantum matrix $T$. The star relations achieved in this manner allow a physical interpretation of a subset of these operators which are interpreted as deformed creation and annihilation operators.

In our approach we are able to impose such relations and thus generalize $U_{\bar{q},q}(2)$ to $U_{p,q}(2)$ with $pq$ real. It is achieved by extending the oscillator algebra related to the quantum group $U_{\bar{q},q}(2)$ by adding a noncommutative unitary phase operator $u$ and a hermitian central operator $s$. For the special case $p = \bar{q}$, s becomes unit operator and $u$ becomes a commutative phase, which results in that our algebra coincides with $U_{\bar{q},q}$ algebra whereas our representation reduces to the representation of $SU_q(2)$ for the special case in which $p = q$.

Taking everything mentioned in this thesis into account, the obvious conclusion to be drawn is that a rigorous foundation for a noncommutative unitary phase operator lies in the two–parameter deformed quantum group. Whether applications such as the quantum phase operator for a quantized boson can be incorporated into this formalism will be the subject of further research.

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