On Set-Valued Functionals: Multivariate Risk Measures and Aumann Integrals

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Abstract

In this dissertation, multivariate risk measures for random vectors and Aumann integrals of set-valued functions are studied. Both are set-valued functionals with values in a complete lattice of subsets of $\mathbb{R}^m$.

Multivariate risk measures are considered in a general d-asset financial market with trading opportunities in discrete time. Specifically, the following features of the market are incorporated in the evaluation of multivariate risk: convex transaction costs modeled by solvency regions, intermediate trading constraints modeled by convex random sets, and the requirement of liquidation into the first $m \leq d$ of the assets. It is assumed that the investor has a "pure" multivariate risk measure R on the space of m-dimensional random vectors which represents her risk attitude towards the assets but does not take into account the frictions of the market. Then, the investor with a d-dimensional position minimizes the set-valued functional R over all m-dimensional positions that she can reach by trading in the market subject to the frictions described above. The resulting functional R^{mar} on the space of d-dimensional random vectors is another multivariate risk measure, called the market-extension of R. A dual representation for R^{mar} that decomposes the effects of R and the frictions of the market is proved.

Next, multivariate risk measures are studied in a utility-based framework. It is assumed that the investor has a complete risk preference towards each individual asset, which can be represented by a von Neumann-Morgenstern utility function. Then, an incomplete preference is considered for multivariate positions which is represented by the vector of the individual utility functions. Under this structure, multivariate shortfall and divergence risk measures are defined as the optimal values of set minimization problems. The dual relationship between the two classes of multivariate risk measures is constructed via a recent Lagrange duality for set optimization. In particular, it is shown that a shortfall risk measure can be written as an intersection over a family of divergence risk measures indexed by a scalarization parameter. Examples include the multivariate versions of the entropic risk measure and the average value at risk.

In the second part, Aumann integrals of set-valued functions on a measurable space are viewed as set-valued functionals and a Daniell-Stone type characterization theorem is proved for such functionals. More precisely, it is shown that a functional that maps measurable set-valued functions into a certain complete lattice of subsets of $\mathbb{R}^m$ can be written as the Aumann integral with respect to a measure if and only if the functional is (1) additive and (2) positively homogeneous, (3) it preserves decreasing limits, (4) it maps halfspace-valued functions to halfspaces, and (5) it maps shifted cone-valued functions to shifted cones. While the first three properties already exist in the classical Daniell-Stone theorem for the Lebesgue integral, the last two properties are peculiar to the set-valued framework and they suffice to complement the first three properties to identify a set-valued functional as the Aumann integral with respect to a measure.

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Contents

Chapter 1

Introduction

This dissertation is about two types of set-valued functionals: multivariate risk measures of financial mathematics and Aumann integrals of integration theory.

Set relations have recently been of interest in set-valued variational analysis. In [22, 23], a reflexive and transitive relation is defined on the power set of $\mathbb{R}^m$, $m \in \{1, 2, \ldots\}$, (more generally, a real linear space) with respect to a fixed convex cone $C \subseteq \mathbb{R}^m$. This relation induces an indifference relation whose set of all equivalence classes is given by

$$\mathcal{P}(C) := \mathcal{P}(\mathbb{R}^m, C) := \{D \subseteq \mathbb{R}^m \mid D = D + C\}.$$

An element of $\mathcal{P}(C)$ is called an *upper set* (with respect to C) of $\mathbb{R}^m$. It has been showed that upper sets have useful algebraic and order-theoretic properties. On the algebraic side, an addition operation and a multiplication with positive scalars can be defined on $\mathcal{P}(C)$. In other words, $\mathcal{P}(C)$ has a very similar structure to that of a linear space; however, $\mathcal{P}(C)$ is closed under multiplication with positive scalars only. Such structure is sometimes referred to as a *conlinear space*. On the order-theoretic side, the infimum and supremum of an arbitrary subset of $\mathcal{P}(C)$ are well-defined upper sets when $\mathcal{P}(C)$ is equipped with the usual superset relation $\supseteq$. In other words, $\mathcal{P}(C)$ is a *complete lattice* under $\supseteq$.

As properties of well-posedness, one usually works with the subfamilies of closed and closed convex elements of $\mathcal{P}(C)$, and the above properties of $\mathcal{P}(C)$ are inherited by these subfamilies.

The above algebraic and order-theoretic properties of upper sets and their consequences make it possible to generalize many concepts and results of variational analysis to functions on a topological linear space that take values in $\mathcal{P}(C)$. These include linearity (called *conlinearity* as only positive scalars are multiplied by upper sets), convexity, Legendre-Fenchel conjugation, the Fenchel-Moreau biconjugation theorem, infimal convolution; see [26] for a survey on set-valued convex analysis based on the so-called *complete lattice approach*. It turns out that many problems in vector and set optimization can be rewritten as problems for functions taking values in $\mathcal{P}(C)$, and the complete lattice approach provides new points of view for such problems, see [37, 26]. In particular, a Fenchel-Rockafellar duality and a Lagrange duality for set optimization are constructed in [24] and [28], respectively. As a prerequisite for the main chapters of the dissertation, a formal introduction to set-valued variational analysis based on the complete lattice approach is presented in Chapter 2.

In this dissertation, set-valued functionals on spaces of measurable vector- and set-valued functions are considered. More precisely, the following classes of functionals with values in $\mathcal{P}(C)$ are of interest.

1. Convex functionals on a linear space of random vectors: multivariate risk measures,

2. Conlinear functionals on a conlinear space of random sets: Aumann integrals.

Section 1.1 and Section 1.2 below provide detailed motivation, summary of the results obtained, and the concepts of set-valued variational analysis involved in the study of these functionals.

1.1 Multivariate risk measures

The modern study of risk measures starts with the seminal paper [3]. In the framework of [3], a financial position whose future worth is uncertain is modeled as a univariate random variable and a *(monetary) risk measure* is defined as a real-valued functional on a suitable linear space of financial positions. The value of the risk measure evaluated at a position can be interpreted as the minimal capital requirement that can compensate for the risk of the position when added to it.

In [3], a *coherent risk measure* is defined by a list of axioms that a sensible measure of risk should satisfy. These axioms capture the inadequacies of some conventional ways of measuring risk such as standard deviation and value at risk (quantile): under a coherent risk measure, a position yielding larger gains for all possible scenarios is less risky (*monotonicity*) and risk is reduced by diversification (*convexity*). In addition to monotonicity and convexity, a coherent risk measure is assumed to be scaled proportionally to the size of the position (*positive homogeneity*), and a deterministic amount added to the position reduces the risk by the same amount (*translativity*) - a property that is justified by the capital requirement interpretation of risk. The more general *convex risk measures* are introduced later in [19, 21] by dropping the positive homogeneity axiom. Since then, primal and dual representation results, extensions, examples and applications of risk measures have been studied extensively in the literature and the topic has become an active area of research in financial mathematics. The book [20] includes a chapter devoted to the main developments in risk measure theory.

In multi-asset markets, it is typical that more than one of the assets can be used as reference assets for deposits. In the existence of frictions such as proportional transaction costs or nonlinear illiquidities, the reference assets cannot be exchanged into each other freely, hence one needs to keep track of the amount in physical units held in each reference asset, which can be done by modeling financial positions as random vectors. Based on this physical units approach, a treatment of financial markets with proportional transaction costs is introduced in [34], which is later generalized in [42] to nonlinear frictions modeled by convex solvency regions.

In terms of risk measurement in these multi-asset markets, unlike the univariate framework described above, one has an unbounded set of deterministic portfolio vectors that can compensate the risk of a financial position, and, in general, the set of nondominated such portfolios, say, with respect to the componentwise ordering of vectors, is also an unbounded set. To that end, *multivariate risk measures* are defined as set-valued functionals which assign to a multivariate financial position its set of all risk compensating portfolios, and the boundary of this set gives the nondominated such portfolios. This set-valued approach is introduced in [33], where *coherent multivariate risk measures* are defined by generalizing the axioms of the univariate case in [3].

As already point out in [33, Property 3.1], the axioms for a coherent multivariate risk measure R guarantee that its values are upper sets with respect to the cone $C = R(0)$. This important observation bridges multivariate risk measures to the recent developments in set-valued variational analysis described earlier in this chapter. In [25], the more general *convex multivariate risk measures* are introduced and a dual representation theorem is proved as an application of the Fenchel-Moreau biconjugation theorem in [23]. This framework is extended in [27] for financial markets with random transaction costs.

In Chapter 3, convex multivariate risk measures are studied in a general market model with random frictions in discrete time by adopting the following two-stage scheme. As the first stage, it is assumed that the investor has a multivariate risk measure that represents her risk attitude towards the assets but does not take into account the frictions of the market. Basic definitions and results about multivariate risk measures are recalled in Section 3.1. The second stage updates the value of the initial risk measure by considering the trading opportunities of the market. More precisely, the value of the initial risk measure is to be minimized over all final positions that the investor can reach by trading in the market. The result of the optimization problem gives a new multivariate risk measure, which is called the *market-extension* of the initial risk measure. This notion is introduced in Section 3.2. It turns out that the market-extension can be formulated as a set-valued *infimal convolution* of the initial risk measure and the set-valued *indicator functions* of the solvency regions of the market frictions, which is the key observation to obtain the general dual representation result Theorem 3.2.11.

Chapter 4 introduces two classes of convex multivariate risk measures in a utility-based framework. In the literature, few examples of coherent multivariate risk measures have been introduced such as the *average value at risk* in [30] and the set of superhedging portfolios in markets with proportional transaction costs in [27, 38]. In the convex (non-coherent) case, to the best of our knowledge, the only examples in the literature are the set of superhedging portfolios with certain trading constraints in markets with frictions, and the relaxed market-extension of the so-called worst case risk measure in [29].

The basic assumption in the utility-based framework of Chapter 4 is that the investor has a complete risk preference towards each individual asset which can be represented by a von Neumann-Morgenstern utility function as in the classical framework of [50]. However, her risk preference towards multivariate positions is incomplete, that is, she is indecisive about some pairs of random vectors. This incomplete risk preference is assumed to be represented by the vector of individual utility functions.

Based on the above assumptions on the risk preference, the *shortfall risk* of a financial position is defined as the set of portfolios which, when added to the position, make the expected utility vector "large" enough with respect to a fixed preorder on vectors. The value of a shortfall risk measure can be formulated as a constrained set optimization problem. Using the recent Lagrange duality in [28], *divergence risk measures* are obtained as the dual counterparts of shortfall risk measures. It turns out that the divergence risk of a position is the value of a *partially scalarized* unconstrained set optimization problem. It has a decision-theoretic interpretation as it models the tradeoff between the consumption at initial time and the utility received at terminal time. Divergence risk measures also form the multi-objective versions of the so-called *optimized certainty equivalents* introduced in [8, 9]. One of the main results in Chapter 4 shows that a shortfall risk measure can be written as an intersection, that is, a set-valued supremum, over a family of divergence risk measures indexed by a scalarization parameter.

Set-valued versions of the well-known *entropic risk measure* (see [20] for the scalar case) are obtained as examples of shortfall risk measures. On the other hand, non-coherent versions of the set-valued average value at risk of [30] are obtained as examples of divergence risk measures.

The main contribution of Chapter 3 and Chapter 4 in the theory of multivariate risk measures can be summarized as follows.

- A method to incorporate various kinds of market frictions (proportional transaction costs, nonlinear illiquidities, liquidation to fewer assets, trading constraints) in the computation of multivariate risk measures is presented.

- A dual representation theorem for the resulting multivariate risk measure is proved using the concept of set-valued infimal convolution.

- A vector-valued utility function to adopt incomplete risk preferences in the computation of multivariate risk measures is proposed.

- Utility-based multivariate risk measures are defined, forming the first examples of non-coherent convex multivariate risk measures in the literature (besides the superhedging example in [29]).

- The recent solution concepts and the Lagrange duality for set optimization are employed in the context of financial mathematics.

- Set-valued generalizations of the well-known entropic risk measure are presented.

The results of Chapter 3 and Chapter 4 are presented in the article [1].

1.2 Aumann integrals

The second part of the dissertation is about the Aumann integration of functions with values in $\mathcal{P}(C)$ defined on a measurable space. The resulting integral with respect to a measure is regarded as a set-valued functional and a Daniell-Stone type characterization theorem is proved in Chapter 5.

Integration of set-valued functions on a measurable space dates back to the paper [6] by Aumann in 1965. Given a σ-finite measure, the Aumann integral of a measurable function whose values are subsets of $\mathbb{R}^m$ is defined as the set of all (vector) Lebesgue integrals of its integrable selections. The Aumann integral can be seen as a functional that maps a measurable set-valued function to a subset of $\mathbb{R}^m$. Conversely, one can consider a functional that maps from a certain class of measurable set-valued functions into a collection of subsets of $\mathbb{R}^m$ and look for conditions under which this functional can be written as the Aumann integral with respect to a measure.

This type of result is well known for the classical Lebesgue integral, see [49, 15], and is also referred to as the Daniell-Stone theorem. One version of the Daniell-Stone theorem states that a functional that maps into $[0, +\infty]$ from the set of all positive measurable functions on a measurable space can be written as the integral with respect to a measure if and only if it is conlinear and it preserves decreasing limits (monotone convergence property); see [15, Theorem I.4.21] for this version of the result and [49] for the original work of Stone. To the best of our knowledge, there seems to be no such characterizations of the Aumann integral in the literature so far.

The aim of Chapter 5 is to prove an analogue of the Daniell-Stone theorem for the Aumann integrals of measurable functions whose values are closed and convex upper subsets of $\mathbb{R}^m$. The Aumann integrals of such functions are again closed convex upper sets as shown in Proposition 5.2.2. This property makes it possible to use the same algebraic and order-theoretic rules for the values of the functions and their integrals.

The main result in this part of the dissertation is Theorem 5.3.1. It shows that a functional that maps measurable set-valued functions into closed convex upper sets can be written as the Aumann integral with respect to a measure if and only if the functional is conlinear, it preserves decreasing limits and it satisfies three additional properties that are peculiar to the set-valued framework. One of these properties ensures that the value of the functional at a given set-valued function can be computed in terms of the corresponding values at the supporting halfspaces of the function. This is already a well-known property of the Aumann integral as proved in [31]. The other two properties ensure that the functional maps halfspace-valued functions to halfspaces and "point+cone"-valued functions to "point+cone"s. It turns out that these three geometric properties, which are redundant in the scalar case, suffice to complement the already existing properties of the scalar theory (conlinearity, monotone convergence property and a technical condition that guarantees the σ-finiteness of the measure) in order to obtain a Daniell-Stone type characterization for the Aumann integral.

The results of Chapter 5 are published in [2].

Chapter 2

The complete lattice framework

This chapter is devoted to the standard framework of complete lattices for subsets of topological linear spaces. No proofs are presented. The reader is referred to [26] for a more detailed survey of the concepts and results of set-valued variational analysis based on complete lattices.

Let $\mathcal{Z}$ be a locally convex topological real linear space with topological dual $\mathcal{Z}^*$. In the chapters to follow, it will either be the case that $\mathcal{Z}$ is a finite-dimensional Euclidean space or else a space of (equivalence classes of) random vectors on a probability space.

The set of all subsets of $\mathcal{Z}$ is denoted by $2^{\mathcal{Z}}$. For $A, B \in 2^{\mathcal{Z}}$, $\lambda \in \mathbb{R}$, define

$$A + B := \{a + b \mid a \in A, b \in B\},$$
$$\lambda A := \{\lambda a \mid a \in A\},$$

as the *Minkowski sum* of A and B, and the *Minkowski multiplication* of A with the scalar λ. These operations are defined with the conventions $A + \emptyset = \emptyset + A = \emptyset$ and $0\emptyset = \{0\} \in 2^{\mathcal{Z}}$, and used with the shorthand notations $A - B = A + (-1)B$ and $z + A = \{z\} + A$ for $z \in \mathcal{Z}$.

Let $C \in 2^{\mathcal{Z}}$ be a fixed closed convex cone with $C \neq \mathcal{Z}$. The collection $\mathcal{P}(C)$ of subsets of $\mathcal{Z}$ that are invariant under the translation by the cone C, and its subcollection $\mathcal{G}(C)$ of closed and convex such sets are central objects in this dissertation. The motivation for the definition of $\mathcal{P}(C)$ is discussed in Section 2.1, the collection $\mathcal{G}(C)$ is introduced in Section 2.2, and the algebraic properties of $\mathcal{G}(C)$ are summarized in Section 2.3. Finally, Section 2.4 summarizes the common notation used for the special case $\mathcal{Z} = \mathbb{R}^m$.

2.1 The complete lattice of upper sets

On $\mathcal{Z}$, define the reflexive and transitive relation $\leq_C$ by

$$z_1 \leq_C z_2 \quad \Leftrightarrow \quad z_2 \in z_1 + C$$

for $z_1, z_2 \in \mathcal{Z}$. (If C is *pointed*, that is, if $C \cap -C = \{0\}$, then $\leq_C$ is a partial order on $\mathcal{Z}$.) This relation can be extended to the relation $\preccurlyeq_C$ on $2^{\mathcal{Z}}$ by letting

$$A \preccurlyeq_C B \quad \Leftrightarrow \quad B \subseteq A + C$$

for $A, B \in 2^{\mathcal{Z}}$. It is easy to check that $\preccurlyeq_C$ is also reflexive and transitive, and it holds

$$A \preccurlyeq_C B \quad \Leftrightarrow \quad B + C \subseteq A + C.$$

Hence, the indifference relation $\sim_C$ induced by $\preccurlyeq_C$, that is, the symmetric part of $\preccurlyeq_C$, satisfies

$$A \sim_C B \quad \Leftrightarrow \quad A + C = B + C.$$

Note that $\sim_C$ is an equivalence relation and the set of all equivalence classes of it can be identified by the collection

$$\mathcal{P}(C) := \mathcal{P}(\mathcal{Z}, C) := \{A \in 2^{\mathcal{Z}} \mid A = A + C\}. \tag{2.1}$$

An element of $\mathcal{P}(C)$ is called an *upper set* with respect to C. Note that $\preccurlyeq_C$ coincides with the usual superset relation $\supseteq$ on $\mathcal{P}(C)$. In particular, $\preccurlyeq_C$ is a partial order on $\mathcal{P}(C)$. A more nontrivial observation is that $(\mathcal{P}(C), \supseteq)$ is indeed a *complete lattice*, that is, every collection $\mathcal{A} \subseteq \mathcal{P}(C)$ has its infimum $\inf_{\mathcal{P}(C)} \mathcal{A}$ and supremum $\sup_{\mathcal{P}(C)} \mathcal{A}$ in $\mathcal{P}(C)$, and they are given by

$$\inf_{\mathcal{P}(C)} \mathcal{A} = \bigcup_{A \in \mathcal{A}} A, \qquad \sup_{\mathcal{P}(C)} \mathcal{A} = \bigcap_{A \in \mathcal{A}} A, \tag{2.2}$$

with the conventions $\inf_{\mathcal{P}(C)} \emptyset = \emptyset$ and $\sup_{\mathcal{P}(C)} \emptyset = \mathcal{Z}$.

2.2 Closed convex upper sets

As well-posedness assumptions in the study of set optimization problems, one usually works with functions whose values are closed convex sets. To that end, let $\mathcal{G}(C)$ be the collection of all closed convex upper sets with respect to C. It can be checked that

$$\mathcal{G}(C) = \left\{ A \in 2^{\mathcal{Z}} \mid A = \operatorname{cl}\operatorname{co}(A + C) \right\}, \tag{2.3}$$

where $\operatorname{cl}, \operatorname{co}$ denote the closure, convex hull operators, respectively.

An immediate consequence of the Hahn-Banach theorem is that every closed convex set can be written as the intersection of its supporting halfspaces. On the other hand, it can be checked easily that a set in $\mathcal{G}(C)$ may only be supported in the directions taken from the positive dual cone C^+ of C, which is defined by

$$C^+ = \{z^* \in \mathcal{Z}^* \mid \forall z \in C: \; z^*(z) \geq 0\};$$

see [51, Section 1.1, p. 7] for instance. Consequently, every $A \in \mathcal{G}(C)$ can be represented as

$$A = \bigcap_{z^* \in C^+ \setminus \{0\}} \{z \in \mathcal{Z} \mid z^*(z) \geq \sigma_A(z^*)\}, \tag{2.4}$$

where $\sigma_A \colon \mathcal{Z}^* \to [-\infty, +\infty]$ is the *support function* of A defined by

$$\sigma_A(z^*) = \inf_{a \in A} z^*(a) \tag{2.5}$$

for $z^* \in \mathcal{Z}^*$. (Notice the slight difference that the function $z^* \mapsto -\sigma_A(-z^*) = \sup_{a \in A} z^*(z)$ is usually referred to as the support function of A in the literature, see [44, 51].)

Similar to $\mathcal{P}(C)$, the set $\mathcal{G}(C)$ is also a complete lattice with respect to the relation $\supseteq$. In this case, while the supremum formula is the same as in (2.2), the infimum formula has additional structure to ensure that the infimum is a closed convex set: for an arbitrary collection $\mathcal{A} \subseteq \mathcal{G}(C)$, the infimum $\inf_{\mathcal{G}(C)} \mathcal{A}$ and supremum $\sup_{\mathcal{G}(C)} \mathcal{A}$ are given by

$$\inf_{\mathcal{G}(C)} \mathcal{A} = \operatorname{cl} \operatorname{co} \bigcup_{A \in \mathcal{A}} A, \qquad \sup_{\mathcal{G}(C)} \mathcal{A} = \bigcap_{A \in \mathcal{A}} A, \tag{2.6}$$

with the same conventions $\inf_{\mathcal{G}(C)} \emptyset = \emptyset$ and $\sup_{\mathcal{G}(C)} \emptyset = \mathcal{Z}$.

2.3 The algebraic structure of $\mathcal{G}(C)$

The Minkowski sum of two sets in $\mathcal{G}(C)$ is a convex upper set but may fail to be closed. Instead, $\mathcal{G}(C)$ is equipped with the *closed Minkowski addition* $\oplus \colon \mathcal{G}(C) \times \mathcal{G}(C) \to \mathcal{G}(C)$ defined by

$$A \oplus B := \operatorname{cl}(A + B)$$

for $A, B \in \mathcal{G}(C)$.

Note that $\mathcal{G}(C)$ is not closed under Minkowski multiplication with scalars. Indeed, for $A \in \mathcal{G}(C)$ and $\lambda > 0$, one has $\lambda A \in \mathcal{G}(C)$, $-\lambda A \in \mathcal{G}(-C)$, and $0A = \{0\} \notin \mathcal{G}(C) \cup \mathcal{G}(-C)$ unless $C = \{0\}$. For this reason, Minkowski multiplication with zero is modified by the convention

$$0A \coloneqq C.$$

Therefore, $\mathcal{G}(C)$ is closed under Minkowski multiplication with scalars in $[0, +\infty)$.

With the closed Minkowski addition $\oplus$ and the modified Minkowski multiplication with positive scalars, $\mathcal{G}(C)$ has a very similar structure to that of a linear space. Indeed, $\mathcal{G}(C)$ satisfies all defining properties of a linear space except that strictly negative scalars are missing in the multiplication operation and the addition operation does not have the inverse element property. This structure of $\mathcal{G}(C)$ is sometimes called a *conlinear space with convex elements.*

2.4 The special case $\mathcal{Z} = \mathbb{R}^m$

Let $\mathcal{Z} = \mathbb{R}^m$ with $m \in \{1, 2, \ldots\}$. For $z \in \mathbb{R}^m$, the notations $z = (z_1, \ldots, z_m)$ and $z = [z_i]_{i=1}^m$ are used.

The Euclidean space $\mathbb{R}^m$ is considered with its usual topology. It is usually considered with some arbitrary fixed norm $|\cdot|$. Besides, the usual inner and Hadamard (componentwise) products on $\mathbb{R}^m$ are defined by

$$\langle z, w \rangle \coloneqq \sum_{i=1}^{m} z_i w_i \in \mathbb{R},$$

$$z \circ w \coloneqq [z_i w_i]_{i=1}^m \in \mathbb{R}^m,$$

for $z, w \in \mathbb{R}^m$, respectively.

The cone $\mathbb{R}^m_+$ ($\mathbb{R}^m_{++}$) denotes the set of all $z \in \mathbb{R}^m$ with $z_i \geq 0$ ($z_i > 0$) for every $i \in \{1, \ldots, m\}$. If $m = 1$, it is written as $\mathbb{R}_+$ ($\mathbb{R}_{++}$). The *componentwise ordering* on

$\mathbb{R}^m$ is defined as $\leq := \leq_{\mathbb{R}^m_+}$. In other words, for $z, w \in \mathbb{R}^m$, one has $z \leq w$ if and only if $z_i \leq w_i$ for every $i \in \{1, \ldots, m\}$.

The topological dual of $\mathbb{R}^m$ is identified by $\mathcal{Z}^* = \mathbb{R}^m$. Hence, the support function of a closed convex set A takes the form

$$\sigma_A(w) = \inf_{a \in A} \langle a, w \rangle$$

for $w \in \mathbb{R}^m$.

Chapter 3

Multivariate risk measures

This chapter serves as a formal introduction to multivariate risk measures and studies their market-extensions in a general convex market model.

In Section 3.1, basic definitions and results about multivariate risk measures are presented. In Section 3.2, the market model is introduced first. The nonlinear illiquidities of the market are modeled by the convex solvency regions of [42]. Additional structure is assumed to model the trading constraints and the requirement of liquidation into fewer reference assets. Then, the market-extension of a multivariate risk measure is defined as another multivariate risk measure. Finally, the main result Theorem 3.2.11 provides a dual representation which decomposes the contributions of the original risk measure and the imposed market structure to the penalty function of the market-extension. The proof of Theorem 3.2.11 relies on the reformulation of the market-extension as a set-valued infimal convolution. This technical observation and the detailed proof are presented in Section 3.3.

3.1 Basic properties of multivariate risk measures

In this section, multivariate risk measures for d-dimensional random vectors are introduced, where $d \in \{1, 2, \ldots\}$.

Throughout, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. Let $L_d^0 := L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ be the linear space of all random variables taking values in $\mathbb{R}^d$, where two elements are considered identical if they are equal $\mathbb{P}$-almost surely. (In)equalities between random variables are understood in the $\mathbb{P}$-almost sure sense, unless otherwise stated. Besides, define

$$
\begin{aligned}
L_d^p &:= \{X \in L_d^0 \mid \mathbb{E}\left[|X|^p\right] < +\infty\}, \quad p \in [1, +\infty), \\
L_d^\infty &:= \{X \in L_d^0 \mid \operatorname{ess\,sup} |X| < +\infty\}, \\
L_{d,+}^p &:= \{X \in L_d^p \mid \mathbb{P}\{X \in \mathbb{R}_+^d\} = 1\}, \quad p \in [1, +\infty].
\end{aligned}
$$

For $p \in [1, +\infty)$, the space L_d^p is considered with its usual norm topology. The space L_d^∞ is considered with the weak* topology $\sigma(L_d^\infty, L_d^1)$.

In a multi-asset financial market, $d \in \{1, 2, \ldots\}$ denotes the number of assets of interest. A financial position is modeled as an element $X \in L_d^0$, where $X_i(\omega)$ represents the number of (physical) units in the ith asset for $i \in \{1, \ldots, d\}$ when the state of the world $\omega \in \Omega$ occurs.

In these markets, it is typically the case that only a small subset of all assets can be used for depositing purposes. Without loss of generality, suppose that the first $m \leq d$ assets are these reference assets. The space $\mathbb{R}^m$ is called space of *eligible portfolios*, that is, every $z \in \mathbb{R}^m$ is a potential deposit to be used at initial time in order to compensate for the risk of a financial position.

As it will be necessary to deal with vectors in $\mathbb{R}^d$ and $\mathbb{R}^m$ at the same time, let us introduce the linear operator $B : \mathbb{R}^m \to \mathbb{R}^d$ defined by

$$
Bz = (z_1, \ldots, z_m, 0, \ldots, 0) \tag{3.1}
$$

for $z \in \mathbb{R}^m$. The composition of B with random variables in L_m^0 will also be used. Given $Z \in L_m^0$, the random variable $BZ \in L_d^0$ is defined by $(BZ)(\omega) = B(Z(\omega))$ for

$\omega \in \Omega$. The adjoint $B^* : \mathbb{R}^d \to \mathbb{R}^m$ of B is given by

$$B^* x = (x_1, \ldots, x_m) \tag{3.2}$$

for $x \in \mathbb{R}^d$. Similarly, B^* can be composed with random variables in L^0_d.

Let $p \in [1, +\infty]$. The following is a formal definition of a multivariate risk measure as a set-valued functional on L^p_d.

Definition 3.1.1. *([27, Definition 2.3]) A functional $R\colon L^p_d \to 2^{\mathbb{R}^m}$ is said to be a multivariate risk measure if it satisfies the following properties.*

(i) Finiteness at zero: $\emptyset \neq R(0) \neq \mathbb{R}^m$.

(ii) Monotonicity: $X \leq Y$ implies $R(Y) \supseteq R(X)$ for every $X, Y \in L^p_d$.

(iii) Translativity: $R(X + Bz) = R(X) - z$ for every $X \in L^p_d$, $z \in \mathbb{R}^m$.

Using monotonicity and translativity, one can easily check that a multivariate risk measure R indeed takes values in the complete lattice $\mathcal{P}(\mathbb{R}^m_+)$ introduced in (2.1).

Definition 3.1.2. *([27, Definition 2.12]) Let $R\colon L^p_d \to \mathcal{P}(\mathbb{R}^m_+)$ be a functional. The set*

$$\operatorname{graph} R := \{(X, z) \in L^p_d \times \mathbb{R}^m \mid z \in R(X)\},$$

is called its graph. R is said to be convex if graph R *is a convex set, and closed if* graph R *is a closed set with respect to the product topology on $L^p_d \times \mathbb{R}^m$.*

It can be checked that a functional $R\colon L^p_d \to \mathcal{P}(\mathbb{R}^m_+)$ is convex if and only if

$$R(\lambda X + (1 - \lambda) Y) \supseteq \lambda R(X) + (1 - \lambda) R(Y),$$

for every $X, Y \in L^p_d$, $\lambda \in (0, 1)$. Besides, if R is closed and convex, then it takes values in the complete lattice $\mathcal{G}(\mathbb{R}^m_+)$ of closed convex upper sets; see (2.3). The converse does not hold in general as the closedness/convexity of the graph of a set-valued function is a much stronger property than the closedness/convexity of its values.

In particular, a closed convex multivariate risk measure takes values in $\mathcal{G}(\mathbb{R}^m_+)$.

Definition 3.1.3. *([27, Definition 2.1]) A set $A \in \mathcal{P}(L^p_{d,+})$ is called an acceptance set if it holds $\emptyset \neq \{z \in \mathbb{R}^m \mid Bz \in A\} \neq \mathbb{R}^m$.*

For a functional $R\colon L^p_d \to \mathcal{P}(\mathbb{R}^m_+)$, define the set

$$A_R := \{Y \in L^p_d \mid 0 \in R(Y)\}, \tag{3.3}$$

and for a set $A \in \mathcal{P}(L^p_{d,+})$, define the functional $R_A\colon L^p_d \to \mathcal{P}(\mathbb{R}^m_+)$ by

$$R_A(Y) := \{z \in \mathbb{R}^m \mid Y + Bz \in A\} \tag{3.4}$$

for $Y \in L^p_d$. The next proposition establishes a one-to-one correspondence between risk measures and acceptance sets, which is analogous to the well-known one for scalar risk measures (as in [20], for instance).

Proposition 3.1.4. *([27, Propositions 2.4, 2.5, 2.13]) Let $A \in \mathcal{P}(L^p_{d,+})$ be an (-, closed, convex) acceptance set. Then R_A is a (-, closed, convex) multivariate risk measure and $A = A_{R_A}$. Conversely, let R be a (-, closed, convex) multivariate risk measure. Then A_R is an (-, closed, convex) acceptance set and $R = R_{A_R}$.*

Finally, the dual representation of a closed convex multivariate risk measure is stated in Proposition 3.1.6 below. To that end, let $\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_d)$ be a d-dimensional *vector probability measure* in the sense that $\mathbb{Q}_i$ is a probability measure on $(\Omega, \mathcal{F})$ for each $i \in \{1, \ldots, d\}$. Define $\mathbb{E}^{\mathbb{Q}}[Y] = (\mathbb{E}^{\mathbb{Q}_1}[Y_1], \ldots, \mathbb{E}^{\mathbb{Q}_d}[Y_d])$ for every $Y \in L^0_d$ such that the components exist in $\mathbb{R}$. Denote by $\mathcal{M}_d(\mathbb{P})$ the set of all d-dimensional vector probability measures on $(\Omega, \mathcal{F})$ whose components are absolutely continuous with respect to $\mathbb{P}$. For $\mathbb{Q} \in \mathcal{M}_d(\mathbb{P})$, set

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \left(\frac{d\mathbb{Q}_1}{d\mathbb{P}}, \ldots, \frac{d\mathbb{Q}_d}{d\mathbb{P}} \right),$$

where, for each $i \in \{1, \ldots, d\}$, $\frac{d\mathbb{Q}_i}{d\mathbb{P}}$ denotes the Radon-Nikodym derivative of $\mathbb{Q}_i$ with respect to $\mathbb{P}$. For $w \in \mathbb{R}^d_+ \setminus \{0\}$, define the halfspace

$$H(w) \coloneqq \left\{ y \in \mathbb{R}^d \mid \langle y, w \rangle \geq 0 \right\}.$$

The set of dual variables to be used in Proposition 3.1.6 is given by

$$\mathcal{W}_{m,d} \coloneqq \left\{ (\mathbb{Q}, w) \in \mathcal{M}_d(\mathbb{P}) \times \mathbb{R}^d_+ \mid B^* w \neq 0,\ w \circ \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q_d \right\}, \tag{3.5}$$

where $q \in [1, +\infty]$ such that $p^{-1} + q^{-1} = 1$.

Definition 3.1.5. *([27, Definition 4.1]) A function $-\alpha\colon \mathcal{W}_{m,d} \to \mathcal{G}(\mathbb{R}^m_+)$ is called a penalty function if it satisfies the following properties.*

(i) $\bigcap_{(\mathbb{Q},w) \in \mathcal{W}_{m,d}} -\alpha(\mathbb{Q}, w) \neq \emptyset$, *and there is* $(\mathbb{Q}, w) \in \mathcal{W}_{m,d}$ *such that* $-\alpha(\mathbb{Q}, w) \neq \mathbb{R}^m$.

(ii) $-\alpha(\mathbb{Q}, w) = B^* \left[(B(-\alpha(\mathbb{Q}, w)) \oplus H(w)) \cap B\mathbb{R}^m \right]$ *for every* $(\mathbb{Q}, w) \in \mathcal{W}_{m,d}$.

Proposition 3.1.6. *([27, Theorem 4.2]) A function $R\colon L^p_d \to \mathcal{G}(\mathbb{R}^m_+)$ is a closed convex multivariate risk measure if and only if there exists a penalty function $-\alpha_R\colon \mathcal{W}_{m,d} \to \mathcal{G}(\mathbb{R}^m_+)$ such that*

$$R(Y) = \bigcap_{(\mathbb{Q},w) \in \mathcal{W}_{m,d}} \left[-\alpha_R(\mathbb{Q}, w) + B^* \left[\left(\mathbb{E}^{\mathbb{Q}}[-Y] + H(w) \right) \cap B\mathbb{R}^m \right] \right] \tag{3.6}$$

for $Y \in L_d^p$. In this case, (3.6) *holds when $-\alpha_R$ is replaced with the minimal penalty function $-\alpha_R^{\min}\colon \mathcal{W}_{m,d} \to \mathcal{G}(\mathbb{R}_+^m)$ of R defined by*

$$-\alpha_R^{\min}(\mathbb{Q}, w) \coloneqq \operatorname{cl} \bigcup_{Y \in L_d^p} \left[R(Y) + B^* \left[\left(\mathbb{E}^{\mathbb{Q}}[Y] + H(w) \right) \cap B\mathbb{R}^m \right] \right] \tag{3.7}$$

for $(\mathbb{Q}, w) \in \mathcal{W}_{m,d}$. Besides, for every penalty function $-\alpha_R$ such that (3.6) *holds, $-\alpha_R^{\min}(\mathbb{Q}, w) \subseteq -\alpha_R(\mathbb{Q}, w)$ for each $(\mathbb{Q}, w) \in \mathcal{W}_{m,d}$.*

3.2 Market-extensions of multivariate risk measures

The purpose of this section is to propose a method to incorporate the frictions of the market into the quantification of risk. As the first step of the method, it is assumed that there is a "pure" multivariate risk measure R that represents the attitude of the investor towards the assets of the market. However, this multivariate risk measure does not take into account the frictions of the market. Hence, the second step is to minimize (in the sense of set optimization) the value of R over the set of financial positions that can be reached with the given position by trading in the so-called convex market model. The result of the risk minimization, as a function of the given position, is called the market-extension of R.

In the literature, market-extensions are introduced in [27] and [30] for the special case of a conical market model. Here, this notion is considered for an arbitrary risk measure with the more general convex market model of [42] and the possibility of trading constraints and liquidation into fewer assets. The main dual representation result, Theorem 3.2.11 is also part of the contribution of this dissertation.

Consider a financial market with $d \in \{1, 2, \ldots\}$ assets. It is assumed that the market has convex transaction costs or nonlinear illiquidities in finite discrete time.

Following [42], convex solvency regions are used to model such frictions. To that end, let $T \in \{1, 2, \dots\}$ and $(\mathcal{F}_t)_{t=0}^T$ a filtration of $(\Omega, \mathcal{F}, \mathbb{P})$ augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. The number T denotes the time horizon, and $(\mathcal{F}_t)_{t=0}^T$ represents the evolution of information over time. Suppose that there is no information at time 0, that is, every $\mathcal{F}_0$-measurable function is deterministic $\mathbb{P}$-almost surely; and there is full information at time T, that is, $\mathcal{F}_T = \mathcal{F}$. For $p \in \{0\} \cup [1, +\infty]$, denote by $L_d^p(\mathcal{F}_t)$ the linear subspace of all $\mathcal{F}_t$-measurable random variables in L_d^p.

By the $\mathcal{F}_t$-measurability of a set-valued function $D \colon \Omega \to 2^{\mathbb{R}^d}$, it is meant that the graph $\{(\omega, y) \in \Omega \times \mathbb{R}^d \mid y \in D(\omega)\}$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-algebra on $\mathbb{R}^d$. For such function D, define the set $L_d^p(\mathcal{F}_t, D) := \{Y \in L_d^p(\mathcal{F}_t) \mid \mathbb{P}\{\omega \in \Omega \mid Y(\omega) \in D(\omega)\} = 1\}$ for $p \in \{0\} \cup [1, \infty]$.

The convex market model of [42] is introduced next. For each $t \in \{0, \dots, T\}$, let $\mathcal{C}_t \colon \Omega \to \mathcal{G}(\mathbb{R}_+^d)$ be an $\mathcal{F}_t$-measurable function such that $\mathbb{R}_+^d \subseteq \mathcal{C}_t(\omega)$ and $-\mathbb{R}_+^d \cap \mathcal{C}_t(\omega) = \{0\}$ for each $t \in \{0, \dots, T\}$ and $\omega \in \Omega$. The set $\mathcal{C}_t$ is called the *(random) solvency region* at time t, see [4, 42]. It models the bid and ask prices as a function of the magnitude of a trade, for instance, as in [13, 14, 46]; and thus, directly relates to the shape of the order book. More precisely, $\mathcal{C}_t(\omega)$ is the set of all portfolios which can be exchanged into ones with nonnegative components at time t when the outcome is ω. Convex solvency regions allow for the modeling of temporary illiquidity effects in the sense that they cover nonlinear illiquidities; however, they assume that agents have no market power and thus, their trades do not affect the costs of subsequent trades.

Example 3.2.1. An important special case is the conical market model introduced in [34]. Suppose that $\mathcal{C}_t(\omega)$ is a (closed convex) cone for each $t \in \{0, \dots, T\}$ and $\omega \in \Omega$. In this case, the transaction costs are proportional to the size of the orders.

From a financial point of view, it is possible to have additional constraints on the trading opportunities at intermediate times. For instance, trading may be allowed only up to a (possibly state- and time-dependent) threshold level for the assets (Example 3.2.2), or it may be the case that a certain linear combination of the trading units should not exceed a threshold level (Example 3.2.3). Such constraints are modeled via convex random sets. Given $t \in \{0, \ldots, T-1\}$, let $\mathcal{D}_t : \Omega \to 2^{\mathbb{R}^d}$ be an $\mathcal{F}_t$-measurable function such that $\mathcal{D}_t(\omega)$ is a closed convex set and $0 \in \mathcal{C}_t(\omega) \cap \mathcal{D}_t(\omega)$ for every $\omega \in \Omega$. Note that $\mathcal{D}_t$ does not necessarily map into $\mathcal{G}(\mathbb{R}^d_+)$, and this is why we prefer to work with $\mathcal{C}_t \cap \mathcal{D}_t$ instead of replacing $\mathcal{C}_t$ by $\mathcal{C}_t \cap \mathcal{D}_t$. For convenience, let us also set $\mathcal{D}_T \equiv \mathbb{R}^d$.

Example 3.2.2. For each $t \in \{0, \ldots, T-1\}$, suppose that

$$\mathcal{D}_t = \bar{Y}_t - \mathbb{R}^d_+,$$

for some $\bar{Y}_t \in L^0_d(\mathcal{F}_t, \mathbb{R}^d_+)$. In this case, trading in asset $i \in \{1, \ldots, d\}$ at time $t \in \{0, \ldots, T-1\}$ may not exceed the level $(\bar{Y}_t)_i$.

Example 3.2.3. For each $t \in \{0, \ldots, T-1\}$, suppose that

$$\mathcal{D}_t = \{y \in \mathbb{R}^d \mid \langle y, A_t \rangle \leq B_t\},$$

for some $A_t \in L^0_d(\mathcal{F}_t, \mathbb{R}^d_+ \backslash \{0\})$ and $B_t \in L^0_1(\mathcal{F}_t, \mathbb{R}_+)$. In this case, trading in each asset is unlimited but the linear combination of the trading units with the weight vector A_t cannot exceed the level B_t.

Let $p \in [1, +\infty]$ be fixed and define

$$\mathcal{K} := -\sum_{t=0}^{T} L^p_d(\mathcal{F}_t, \mathcal{C}_t \cap \mathcal{D}_t), \tag{3.8}$$

which is the set of all financial positions that can be obtained by trading in the market starting with the zero position. Hence, an investor with a financial position $Y \in L^p_d$

can ideally reach any element of the set $Y+\mathcal{K}$ by trading in the market. However, it may be the case that the risk of the resulting position is evaluated only through a (small) selection of the d assets, in other words, trading has to be done in such a way that the only possibly nonzero components of the resulting position can be in some selected subset of the d assets. Without loss of generality, suppose that liquidation is made into the first $m\leq d$ of the assets. The idea of liquidation is made precise by the notion of liquidation function introduced in Definition 3.2.4 below.

Definition 3.2.4. *The function $\Lambda_m\colon L_d^p\to\mathcal{P}(L_{m,+}^p)$ defined by*

$$\Lambda_m(Y)=\{X\in L_m^p \mid BX\in Y+\mathcal{K}\},$$

for $Y\in L_d^p$, is called the liquidation function associated with $\mathcal{K}$.

Hence, given $Y\in L_d^p$, the set $\Lambda_m(Y)$ consists of all possible resulting positions in $Y+\mathcal{K}$ that are already liquidated into the first m assets.

Finally, let us consider a multivariate risk measure $R\colon L_m^p\to\mathcal{P}(\mathbb{R}_+^m)$ which is used for risk evaluation after liquidating the resulting positions into the first m assets. As all the positions in $\Lambda_m(Y)$ are accessible to the investor with position $Y\in L_d^p$, the value of R is to be minimized over the set $\Lambda_m(Y)$ as the following definition suggests.

Definition 3.2.5. *The function $R^{\mathrm{mar}}: L_d^p\to\mathcal{P}(\mathbb{R}_+^m)$ defined by*

$$R^{\mathrm{mar}}(Y)\coloneqq\inf_{(\mathcal{P}(\mathbb{R}_+^m),\supseteq)}\{R(X)\mid X\in\Lambda_m(Y)\}=\bigcup_{X\in\Lambda_m(Y)}R(X),$$

for $Y\in L_d^p$, is called the market-extension of R.

Remark 3.2.6. In the case of the conical market model described in Example 3.2.1, when $\mathcal{D}_t\equiv\mathbb{R}^d$ for each $t\in\{0,\dots,T\}$ and no liquidation at $t=T$ is considered ($m=d$), Definition 3.2.5 recovers the notion of market-extension given in [30, Definition 2.8, Remark 2.9].

The next proposition shows that the market-extension is also a multivariate risk measure except for a finiteness condition.

Proposition 3.2.7. *The functional R^{mar} is a multivariate risk measure except possibly that $R^{\mathrm{mar}}(0) = \mathbb{R}^m$.*

Proof. Clearly, $R^{\mathrm{mar}}(0) \neq \emptyset$ since $0 \in \Lambda_m(0)$ and $R(0) \neq \emptyset$. To prove monotonicity, let $Y^1, Y^2 \in L^p_d$ with $Y^1 \leq Y^2$. Let $X \in \Lambda_m(Y^1)$. With $\tilde{Y} := Y^2 - Y^1 \in L^p_{d,+}$, it holds

$$
\begin{aligned}
BX &\in Y^1 + \mathcal{K} \\
&= Y^2 - (Y^2 - Y^1) + \mathcal{K} \\
&= Y^2 - \sum_{t=0}^{T-1} L^p_d(\mathcal{F}_t, \mathcal{C}_t \cap \mathcal{D}_t) - \left(\tilde{Y} + L^p_d(\mathcal{F}_T, \mathcal{C}_T)\right) \\
&\subseteq Y^2 - \sum_{t=0}^{T-1} L^p_d(\mathcal{F}_t, \mathcal{C}_t \cap \mathcal{D}_t) - \left(L^p_{d,+} + L^p_d(\mathcal{F}_T, \mathcal{C}_T)\right) \\
&\subseteq Y^2 + \mathcal{K},
\end{aligned}
$$

where the last inclusion holds since

$$
L^p_{d,+} + L^p_d(\mathcal{F}_T, \mathcal{C}_T) = L^p_d(\mathcal{F}_T, \mathbb{R}^d_+) + L^p_d(\mathcal{F}_T, \mathcal{C}_T) = L^p_d(\mathcal{F}_T, \mathcal{C}_T)
$$

due to $\mathcal{C}_T(\omega) \in \mathcal{G}(\mathbb{R}^d_+)$ for every $\omega \in \Omega$. Hence, $X \in \Lambda_m(Y^2)$. Therefore, $\Lambda_m(Y^1) \subseteq \Lambda_m(Y^2)$ which implies $R(Y^1) \subseteq R(Y^2)$.

To prove translativity, let $Y \in L^p_d$, $z \in \mathbb{R}^m$. For every $X \in L^p_m$, it holds

$$
\begin{aligned}
X \in \Lambda_m(Y + Bz) \quad &\Leftrightarrow \quad BX \in Y + Bz + \mathcal{K} \\
&\Leftrightarrow \quad B(X - z) \in Y + \mathcal{K} \\
&\Leftrightarrow \quad X - z \in \Lambda_m(Y).
\end{aligned}
$$

Hence,

$$\begin{aligned}
R^{\text{mar}}(Y+Bz) &= \bigcup_{X\in\Lambda_m(Y+Bz)} R(X)\\
&= \bigcup_{X-z\in\Lambda_m(Y)} R(X)\\
&= \bigcup_{\tilde{X}\in\Lambda_m(Y)} R(\tilde{X}+z)\\
&= \bigcup_{\tilde{X}\in\Lambda_m(Y)} R(\tilde{X})-z\\
&= R^{\text{mar}}(Y)-z,
\end{aligned}$$

from which translativity follows. □

Let $A_{R^{\text{mar}}} \in \mathcal{P}(L^p_{d,+})$ be the acceptance set of R^{mar} defined by (3.3). Proposition 3.2.8 below shows that $A_{R^{\text{mar}}}$ can be written in terms of the acceptance set $A_R \in \mathcal{P}(L^p_{m,+})$ of R and the set $\mathcal{K}$ of freely available portfolios.

Proposition 3.2.8. *It holds* $A_{R^{\text{mar}}} = BA_R - \mathcal{K} = \{BX - K \mid X \in A_R,\ K \in \mathcal{K}\}$.

Proof. For every $Y \in L^p_d$, it holds

$$\begin{aligned}
R^m_{BA_R-\mathcal{K}}(Y) &= \{z\in\mathbb{R}^m \mid Y+Bz \in BA_R-\mathcal{K}\}\\
&= \bigcup_{K\in\mathcal{K}} \{z\in\mathbb{R}^m \mid Y+K+Bz\in BA_R\}\\
&= \bigcup_{\{X\in L^p_m \mid BX\in Y+\mathcal{K}\}} \{z\in\mathbb{R}^m \mid BX+Bz\in BA_R\}\\
&= \bigcup_{X\in\Lambda_m(Y)} \{z\in\mathbb{R}^m \mid X+z\in A_R\}\\
&= \bigcup_{X\in\Lambda_m(Y)} R(X)\\
&= R^{\text{mar}}(Y).
\end{aligned}$$

By Proposition 3.1.4, the result follows. □

By Proposition 3.1.4 and Proposition 3.2.8, it follows that R^{mar} is convex if R is convex. On the other hand, R^{mar} may fail to be closed even if R is so. To recover the closedness property, which is crucial in obtaining dual representation results, closed versions of market-extensions are defined next.

Definition 3.2.9. *For a functional $F\colon L^p_d \to \mathcal{P}(\mathbb{R}^m_+)$, the functional $\operatorname{cl} F := R_{\operatorname{cl} A_F}$ is called the closed hull of F; see* (3.3), (3.4) *for notation. The closed hull of the market-extension R^{mar} of R is called the closed market-extension of R.*

Remark 3.2.10. In analogy with the scalar case, the closed hull of $F : L^p_d \to \mathcal{P}(\mathbb{R}^m_+)$ is the pointwise greatest closed function minorizing it, that is, if $F'\colon L^p_d \to \mathcal{P}(\mathbb{R}^m_+)$ is a closed function such that $F(Y) \subseteq F'(Y)$ for every $Y \in L^p_d$, then it holds $(\operatorname{cl} F)(Y) \subseteq F'(Y)$ for every $Y \in L^p_d$. This is by [23, Corollary 1(ii)].

By Proposition 3.1.4, the closed hull of a function $F\colon L^p_d \to \mathcal{P}(\mathbb{R}^m_+)$ is always closed. Translativity, monotonicity and convexity are preserved under taking the closed hull. In particular, in view of Proposition 3.2.7, the closed market-extension of a convex multivariate risk measure $R\colon L^p_m \to \mathcal{P}(\mathbb{R}^m_+)$ is a closed convex multivariate risk measure provided that $R^{\text{mar}}(0) \neq \mathbb{R}^m$.

The main result of this chapter, Theorem 3.2.11 below gives a dual representation of the closed market-extension in terms of the minimal penalty function of the "pure" multivariate risk measure R under the assumptions of convexity, closedness, and finiteness at zero. The special case of no trading constraints in a convex (conical) market model is given in Corollary 3.2.12 (Corollary 3.2.13).

Recall from (3.5) that the set of dual variables is given by

$$\mathcal{W}_{m,d} := \left\{ (\mathbb{Q}, w) \in \mathcal{M}_d(\mathbb{P}) \times \mathbb{R}^d_+ \mid B^* w \neq 0,\ w \circ \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q_d \right\},$$

where $q \in [1, +\infty]$ such that $p^{-1} + q^{-1} = 1$. In addition, with a slight abuse of notation, the linear operator $B^*\colon \mathbb{R}^d \to \mathbb{R}^m$ is also used in the context of vector probability measures. Given $\mathbb{Q} \in \mathcal{M}_d(\mathbb{P})$, define $B^*\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_m) \in \mathcal{M}_m(\mathbb{P})$. Besides, recall also the homogeneous halfspace $H(w) = \{y \in \mathbb{R}^d \mid \langle y, w\rangle \geq 0\}$ for $w \in \mathbb{R}^d_+ \backslash \{0\}$.

Theorem 3.2.11. *Suppose that $R : L^p_m \to \mathcal{G}(\mathbb{R}^m_+)$ is a closed convex multivariate risk measure with minimal penalty function $-\alpha^{\min}_R \colon \mathcal{W}_{m,m} \to \mathcal{G}(\mathbb{R}^m_+)$. Assume that* $\operatorname{cl} R^{\mathrm{mar}}(0) \neq \mathbb{R}^m$*. Then the closed market-extension* $\operatorname{cl} R^{\mathrm{mar}} \colon L^p_d \to \mathcal{G}(\mathbb{R}^m_+)$ *is a closed convex multivariate risk measure, and it has the following dual representation. For every $Y \in L^p_d$,*

$$(\operatorname{cl} R^{\mathrm{mar}})(Y) = \bigcap_{(\mathbb{Q},w) \in \mathcal{W}_{m,d}} \left[-\alpha^{\min}_{\operatorname{cl} R^{\mathrm{mar}}}(\mathbb{Q}, w) + B^*\left(\left(\mathbb{E}^{\mathbb{Q}}[-Y] + H(w)\right) \cap B\mathbb{R}^m\right)\right],$$

where $-\alpha^{\min}_{\operatorname{cl} R^{\mathrm{mar}}} \colon \mathcal{W}_{m,d} \to \mathcal{G}(\mathbb{R}^m_+)$ is defined by

$$\begin{aligned} -\alpha^{\min}_{\operatorname{cl} R^{\mathrm{mar}}}(\mathbb{Q}, w) = {} & -\alpha^{\min}_R(B^*\mathbb{Q}, B^*w) \\ & + \sum_{t=0}^{T} \operatorname{cl} \bigcup_{U \in L^p_d(\mathcal{F}_t, \mathcal{C}_t \cap \mathcal{D}_t)} B^*\left(\left(\mathbb{E}^{\mathbb{Q}}[U] + H(w)\right) \cap B\mathbb{R}^m\right) \end{aligned} \tag{3.9}$$

for $(\mathbb{Q}, w) \in \mathcal{W}_{m,d}$.

Recall that the *recession cone* of a nonempty convex set $C \subseteq \mathbb{R}^d$ is the convex cone $0^+C := \{y \in \mathbb{R}^d \mid y + C \subseteq C\}$; see [44, Section 8, p. 61] for instance.

Corollary 3.2.12. *Under the assumptions of Theorem 3.2.11, suppose that $\mathcal{D}_t \equiv \mathbb{R}^d$ for each $t \in \{0, \ldots, T\}$. Then $-\alpha^{\min}_{\operatorname{cl} R^{\mathrm{mar}}}$ given by* (3.9) *is concentrated on the set*

$$\mathcal{W}^{\mathrm{convex}}_{m,d} := \left\{(\mathbb{Q}, w) \in \mathcal{W}_{m,d} \mid \forall t \in \{0, \ldots, T\} \colon w \circ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right] \in L^q_d(\mathcal{F}_t, (0^+\mathcal{C}_t)^+)\right\},$$

where, for each $t \in \{0, \ldots, T\}$, $(0^+\mathcal{C}_t)^+ : \Omega \to \mathcal{G}(\mathbb{R}^d_+)$ is the measurable function defined by $(0^+\mathcal{C}_t)^+(\omega) := (0^+\mathcal{C}_t(\omega))^+$.

In other words, $-\alpha^{\min}_{\mathrm{cl}\, R^{\mathrm{mar}}}(\mathbb{Q}, w) = \mathbb{R}^m$ *for* $(\mathbb{Q}, w) \in \mathcal{W}_{m,d} \backslash \mathcal{W}^{\mathrm{convex}}_{m,d}$ *within the setting of the previous result.*

Proof. Let $(\mathbb{Q}, w) \in \mathcal{W}_{m,d} \backslash \mathcal{W}^{\mathrm{convex}}_{m,d}$. So there exist $t \in \{0, \ldots, T\}$ and $A \in \mathcal{F}_t$ such that $\mathbb{P}(A) > 0$ and $w \circ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,|\, \mathcal{F}_t\right](\omega) \notin (0^+\mathcal{C}_t(\omega))^+$ for each $\omega \in A$. Using the fact that the effective domain of the support function of a nonempty closed convex set in $\mathbb{R}^d$ is a subset of its recession cone, which is an easy consequence of [44, Corollary 14.2.1] for instance, it follows that

$$\inf_{y \in \mathcal{C}_t(\omega)} \left\langle y, w \circ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right](\omega) \right\rangle = -\infty$$

for each $\omega \in A$. As a result,

$$\begin{aligned}
&\mathrm{cl} \bigcup_{U \in L^p_d(\mathcal{F}_t, \mathcal{C}_t)} B^*\left(\left(\mathbb{E}^{\mathbb{Q}}[U] + H(w)\right) \cap B\mathbb{R}^m\right) \\
&= \left\{ z \in \mathbb{R}^m \mid \langle Bz, w \rangle \geq \inf_{U \in L^p_d(\mathcal{F}_t, \mathcal{C}_t)} \left\langle \mathbb{E}^{\mathbb{Q}}[U], w \right\rangle \right\} \\
&= \left\{ z \in \mathbb{R}^m \mid \langle z, B^* w \rangle \geq \inf_{U \in L^p_d(\mathcal{F}_t, \mathcal{C}_t)} \mathbb{E}\left[\left\langle U, w \circ \frac{d\mathbb{Q}}{d\mathbb{P}} \right\rangle\right] \right\} \\
&= \left\{ z \in \mathbb{R}^m \mid \langle z, B^* w \rangle \geq \inf_{U \in L^p_d(\mathcal{F}_t, \mathcal{C}_t)} \mathbb{E}\left[\left\langle U, w \circ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right] \right\rangle\right] \right\} \\
&= \left\{ z \in \mathbb{R}^m \mid \langle z, B^* w \rangle \geq \mathbb{E}\left[\inf_{y \in \mathcal{C}_t} \left\langle y, w \circ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right] \right\rangle\right] \right\} \\
&= \mathbb{R}^m,
\end{aligned}$$

where the fourth equality is by [45, Theorem 14.60]. Note that the passage to conditional expectations in the third line is necessary for the application of this theorem. By (3.9), it follows that $-\alpha^{\min}_{\mathrm{cl}\, R^{\mathrm{mar}}}(\mathbb{Q}, w) = \mathbb{R}^m$. □

Corollary 3.2.13. *Under the assumptions of Theorem 3.2.11 suppose that* $\mathcal{D}_t \equiv \mathbb{R}^d$ *for each* $t \in \{0, \ldots, T\}$ *and that the market model is conical as in Example 3.2.1.*

Consider the set

$$\mathcal{W}^{\text{cone}}_{m,d} := \left\{ (\mathbb{Q}, w) \in \mathcal{W}_{m,d} \mid \forall t \in \{0, \dots, T\} : w \circ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t \right] \in L^q_d(\mathcal{F}_t, \mathcal{C}^+_t) \right\},$$

where, for each $t \in \{0, \dots, T\}$, $\mathcal{C}^+_t : \Omega \to \mathcal{G}(\mathbb{R}^d_+)$ *is the* $\mathcal{F}_t$*-measurable function defined by* $\mathcal{C}^+_t(\omega) := (\mathcal{C}_t(\omega))^+$ *for* $\omega \in \Omega$*. Then,* (3.9) *reduces to*

$$-\alpha^{\min}_{\text{cl}\, R^{\text{mar}}}(\mathbb{Q}, w) = \begin{cases} -\alpha^{\min}_R(B^*\mathbb{Q}, B^*w) & \textit{if } (\mathbb{Q}, w) \in \mathcal{W}^{\text{cone}}_{m,d} \\ \mathbb{R}^m & \textit{else} \end{cases} \tag{3.10}$$

for $(\mathbb{Q}, w) \in \mathcal{W}_{m,d}$*. Hence,*

$$(\text{cl}\, R^{\text{mar}})(Y) = \bigcap_{(\mathbb{Q},w) \in \mathcal{W}^{\text{cone}}_{m,d}} \left[-\alpha^{\min}_R(B^*\mathbb{Q}, B^*w) + B^* \left(\left(\mathbb{E}^{\mathbb{Q}}[-Y] + H(w) \right) \cap B\mathbb{R}^m \right) \right]$$

for $Y \in L^p_d$*.*

Proof. Let $t \in \{0, \dots, T\}$. For each $\omega \in \Omega$,

$$\inf_{y \in \mathcal{C}_t(\omega)} \left\langle y, w \circ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t \right](\omega) \right\rangle = \begin{cases} 0 & \text{if } w \circ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t \right](\omega) \in (\mathcal{C}_t(\omega))^+ \\ -\infty & \text{else} \end{cases}$$

since $\mathcal{C}_t(\omega)$ is a nonempty closed convex cone. Similar to the calculation in the proof of Corollary 3.2.12, it holds

$$\begin{aligned} &\text{cl} \bigcup_{U \in L^p_d(\mathcal{F}_t, \mathcal{C}_t)} B^* \left(\left(\mathbb{E}^{\mathbb{Q}}[U] + H(w) \right) \cap B\mathbb{R}^m \right) \\ &\qquad = \left\{ z \in \mathbb{R}^m \mid \langle z, B^*w \rangle \geq \mathbb{E}\left[\inf_{y \in \mathcal{C}_t} \left\langle y, w \circ \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t \right] \right\rangle \right] \right\}, \end{aligned}$$

from which the result follows immediately. □

3.3 Proof of Theorem 3.2.11

The aim of this section is to establish the link between the notions of market-extension and set-valued *infimal convolution*. This relationship forms the main argument in the proof of Theorem 3.2.11.

Two key concepts from complete lattice-based set-valued variational analysis are introduced first; the reader is referred to [23] for details.

Definition 3.3.1. *([23, Example 1]) Let $\mathcal{Y} \subseteq L^p_d$. The function $\mathcal{I}^m_{\mathcal{Y}}\colon L^p_d \to \mathcal{G}(\mathbb{R}^m_+)$ defined by*

$$\mathcal{I}^m_{\mathcal{Y}}(Y) = \begin{cases} \mathbb{R}^m_+ & \text{if } Y \in \mathcal{Y} \\ \emptyset & \text{else} \end{cases},$$

for $Y \in L^p_d$, is called the indicator function of $\mathcal{Y}$.

Definition 3.3.2. *([23, Section 4.4(C)]) Let $N \in \{1, 2, \ldots\}$. For each $n \in \{1, \ldots, N\}$, let $F^n\colon L^p_d \to \mathcal{P}(\mathbb{R}^m_+)$ be a functional. The functional $\square_{n=1}^N F^n\colon L^p_d \to \mathcal{G}(\mathbb{R}^m_+)$ defined by*

$$(\square_{n=1}^N F^n)(Y) = \operatorname{cl}\operatorname{co} \bigcup_{Y^1,\ldots,Y^N \in L^p_d} \left(\sum_{n=1}^N F^n(Y^n) \mid Y^1 + \ldots + Y^N = Y \right),$$

for $Y \in L^p_d$, is called the infimal convolution of $F^1, \ldots, F^N$.

The next proposition basically shows that the market-extension of a multivariate risk measure R is the infimal convolution of R and the indicator function of the negative of the set $\mathcal{K}$ of freely available portfolios defined by (3.8).

Proposition 3.3.3. *Let $R\colon L^p_m \to \mathcal{G}(\mathbb{R}^m_+)$ be a convex multivariate risk measure and let $\tilde{R}\colon L^p_d \to \mathcal{G}(\mathbb{R}^m_+)$ be defined by*

$$\tilde{R}(Y) = \begin{cases} R(B^*Y) & \text{if } Y \in BL^p_m \\ \emptyset & \text{else} \end{cases}$$

for $Y \in L_d^p$. *Then,*

$$\begin{aligned}\operatorname{cl}(R^{\mathrm{mar}}(Y)) &= (\tilde{R} \,\square\, \mathcal{I}^m_{-\mathcal{K}})(Y) \\ &= (\tilde{R} \,\square\, \mathcal{I}^m_{L_d^p(\mathcal{F}_0, \mathcal{C}_0 \cap \mathcal{D}_0)} \,\square\, \ldots \,\square\, \mathcal{I}^m_{L_d^p(\mathcal{F}_T, \mathcal{C}_T \cap \mathcal{D}_T)})(Y)\end{aligned} \tag{3.11}$$

for each $Y \in L_d^p$.

Remark 3.3.4. It should be noted that, in (3.11) above, we are dealing with the the closure of the set $R^{\mathrm{mar}}(Y)$ for a given $Y \in L_d^p$, which is not equal to the value $(\operatorname{cl} R^{\mathrm{mar}})(Y)$ of the closed market-extension $\operatorname{cl} R^{\mathrm{mar}}$ at Y, in general.

Proof of Proposition 3.3.3. For each $Y \in L_d^p$,

$$\begin{aligned}R^{\mathrm{mar}}(Y) &= \bigcup_{\{X \in L_m^p \mid BX \in Y + \mathcal{K}\}} R(X) \\ &= \bigcup_{U \in Y + \mathcal{K}} \tilde{R}(U) \\ &= \bigcup_{U, U' \in L_d^p} \{\tilde{R}(U) + \mathcal{I}^m_{-\mathcal{K}}(U') \mid U + U' = Y\} \\ &= \bigcup_{U, U^0, \ldots, U^T \in L_d^p} \left\{ \tilde{R}(U) + \sum_{t=0}^{T} \mathcal{I}^m_{L_d^p(\mathcal{F}_t, \mathcal{C}_t \cap \mathcal{D}_t)}(U^t) \mid U + U^0 + \ldots + U^T = Y \right\}.\end{aligned}$$

Since each of the functions in the infimal convolution is convex, the convex hull operator in Definition 3.3.2 can be omitted; and the result follows. □

By Proposition 3.3.3, the market-extension can be formulated as an infimal convolution up to taking closures. As in the scalar theory, one is able to write the Legendre-Fenchel conjugate of the infimal convolution of finitely many convex functions as the sum of the Legendre-Fenchel conjugates of these convex functions, which is provided by [23, Lemma 2]. The application of this result is the main step of the proof of Theorem 3.2.11 below. For completeness, the definition of conjugate for set-valued functions is recalled next.

Definition 3.3.5. *([23, Definition 5]) Let $F : L_d^p \to \mathcal{P}(\mathbb{R}_+^m)$ be a functional. Let $q \in [1, +\infty]$ with $p^{-1} + q^{-1} = 1$. The functional $-F^* : L_d^q \times (\mathbb{R}_+^m \backslash \{0\})$ defined by*

$$-F^*(V, v) = \mathrm{cl} \bigcup_{Y \in L_d^p} \left(F(Y) + \{ z \in \mathbb{R}^m \mid \langle z, v \rangle \geq \mathbb{E}\left[\langle Y, -V \rangle \right] \} \right)$$

is called the (Fenchel) conjugate of F.

Proof of Theorem 3.2.11. Since $\mathrm{cl}\, R^{\mathrm{mar}}$ has closed values, Proposition 3.3.3 implies

$$\begin{aligned} R^{\mathrm{mar}}(Y) &\subseteq (\tilde{R} \,\square\, \mathcal{I}^m_{L_d^p(\mathcal{F}_0, \mathcal{C}_0 \cap \mathcal{D}_0)} \,\square\, \ldots \,\square\, \mathcal{I}^m_{L_d^p(\mathcal{F}_T, \mathcal{C}_T \cap \mathcal{D}_T)})(Y) \\ &= \mathrm{cl}(R^{\mathrm{mar}}(Y)) \\ &\subseteq (\mathrm{cl}\, R^{\mathrm{mar}})(Y), \end{aligned}$$

for each $Y \in L_d^p$. By [23, Remark 6 and Lemma 2], the three functions R^{mar}, $\mathrm{cl}(R^{\mathrm{mar}}(\cdot))$, $\mathrm{cl}\, R^{\mathrm{mar}}$ have the same conjugate (on $L_d^q \times (\mathbb{R}_+^m \backslash \{0\})$) which is given by

$$-\left(\tilde{R} \,\square\, \mathcal{I}^m_{L_d^p(\mathcal{F}_0, \mathcal{C}_0 \cap \mathcal{D}_0)} \,\square\, \ldots \,\square\, \mathcal{I}^m_{L_d^p(\mathcal{F}_T, \mathcal{C}_T \cap \mathcal{D}_T)} \right)^* = -\tilde{R}^* + \sum_{t=0}^{T} -(\mathcal{I}^m_{L_d^p(\mathcal{F}_t, \mathcal{C}_t \cap \mathcal{D}_t)})^*;$$

note that this is the set-valued version of the rule "the conjugate of the infimal convolution of finitely many convex functions is the sum of their conjugates." Let $(V, v) \in L_d^q \times (\mathbb{R}_+^m \backslash \{0\})$. As in Proposition 3.2.7, one can easily check that $\mathrm{cl}(R^{\mathrm{mar}}(\cdot))$ is a convex risk measure. Hence, by [27, Proposition 6.7] on the conjugate of a risk measure, for every $(V, v) \in L_d^q \times (\mathbb{R}_+^m \backslash \{0\})$, it holds $-(\mathrm{cl}(R^{\mathrm{mar}}(\cdot)))^*(V, v) = \mathbb{R}^m$ unless $V \in -L_{d,+}^q$ and $v = \mathbb{E}\left[-B^* V \right]$.

The next step is to pass from $L_d^q \times (\mathbb{R}_+^m \backslash \{0\})$ to $\mathcal{W}_{m,d}$ using the "change of variables formula" provided by [27, Lemma 3.4]. An application of this lemma yields that for every $V \in -L_{d,+}^q$ with $v = \mathbb{E}\left[-B^* V \right]$, there exists $(\mathbb{Q}, w) \in \mathcal{W}_{m,d}$ such that for every

$Y \in L_d^p$,

$$\{z \in \mathbb{R}^m \mid \langle z, v\rangle \geq \mathbb{E}\left[\langle Y, -V\rangle\right]\} = B^*\left(\left(\mathbb{E}^{\mathbb{Q}}\left[Y\right] + H(w)\right) \cap B\mathbb{R}^m\right), \tag{3.12}$$

and conversely, every $(\mathbb{Q}, w) \in \mathcal{W}_{m,d}$ can be obtained by some $V \in -(L_d^q)_+$ with $v = \mathbb{E}\left[-B^*V\right]$ such that (3.12) holds for every $Y \in L_d^p$. Note that $B^*\mathbb{R}^d = \mathbb{R}^m \times \{0\}^{d-m}$. For such corresponding pairs (V, v) and $(\mathbb{Q}, w)$, using (3.12), we first observe that

$$\begin{aligned}
-\tilde{R}^*(V, v) &= \operatorname{cl} \bigcup_{Y \in L_d^p} \left(\tilde{R}(Y) + \{z \in \mathbb{R}^m \mid \langle z, v\rangle \geq \mathbb{E}\left[\langle Y, -V\rangle\right]\}\right) \\
&= \operatorname{cl} \bigcup_{Y \in L_d^p} \left(\tilde{R}(Y) + B^*\left(\left(\mathbb{E}^{\mathbb{Q}}[Y] + H(w)\right) \cap B\mathbb{R}^m\right)\right) \\
&= \operatorname{cl} \bigcup_{Y \in BL_m^p} \left(R(B^*Y) + \mathbb{E}^{B^*\mathbb{Q}}[B^*Y] + H(B^*w)\right) \\
&= \operatorname{cl} \bigcup_{X \in L_m^p} \left(R(X) + \mathbb{E}^{B^*\mathbb{Q}}[X] + H(B^*w)\right) \\
&= -\alpha_R^{\min}(B^*\mathbb{Q}, B^*w).
\end{aligned}$$

Next, let $t \in \{0, \ldots, T\}$. For the same such pairs (V, v) and $(\mathbb{Q}, w)$, by Definition 3.3.1 and Definition 3.3.5, we have

$$\begin{aligned}
-\left(\mathcal{I}^m_{L_d^p(\mathcal{F}_t, \mathcal{C}_t \cap \mathcal{D}_t)}\right)^*(V, v) &= \operatorname{cl} \bigcup_{U \in L_d^p(\mathcal{F}_t, \mathcal{C}_t \cap \mathcal{D}_t)} \{z \in \mathbb{R}^m \mid \langle z, v\rangle \geq \mathbb{E}\left[\langle U, -V\rangle\right]\} \\
&= \operatorname{cl} \bigcup_{U \in L_d^p(\mathcal{F}_t, \mathcal{C}_t \cap \mathcal{D}_t)} B^*\left(\left(\mathbb{E}^{\mathbb{Q}}[U] + H(w)\right) \cap B\mathbb{R}^m\right).
\end{aligned}$$

Finally, note that $\operatorname{cl} R^{\mathrm{mar}}$ is a closed convex function that is finite at zero by assumption. Hence, by [23, Theorem 2], which is a biconjugation theorem for set-valued functions, it holds

$$(\operatorname{cl} R^{\mathrm{mar}})(Y) = \bigcap_{V \in -L^q_{d,+},\ v = \mathbb{E}[-B^*V]} \left[-(\operatorname{cl} R^{\mathrm{mar}})^*(V, v) + \{z \in \mathbb{R}^m \mid \langle z, v\rangle \geq \mathbb{E}\left[\langle Y, V\rangle\right]\}\right],$$

for every $Y \in L_d^p$, and the above calculations allow for a passage to vector probability measures: for every $Y \in L_d^p$,

$$(\operatorname{cl} R^{\text{mar}})(Y) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_{m,d}} \left[-\alpha^{\min}_{\operatorname{cl} R^{\text{mar}}}(\mathbb{Q},w) + B^* \left(\left(\mathbb{E}^{\mathbb{Q}}[-Y] + H(w)\right) \cap B\mathbb{R}^m\right)\right],$$

where

$$\begin{aligned}-\alpha^{\min}_{\operatorname{cl} R^{\text{mar}}}(\mathbb{Q},w) = &- \alpha^{\min}_{R}(B^*\mathbb{Q}, B^*w)\\ &+ \sum_{t=0}^{T} \operatorname{cl} \bigcup_{U\in L_d^p(\mathcal{F}_t,\mathcal{C}_t\cap\mathcal{D}_t)} B^* \left(\left(\mathbb{E}^{\mathbb{Q}}[U] + H(w)\right) \cap B\mathbb{R}^m\right)\end{aligned}$$

for $(\mathbb{Q},w) \in \mathcal{W}_{m,d}$. □

Chapter 4

Utility-based risk measures

The goal of this chapter is to study multivariate risk measures in a utility-based framework. In particular, two classes of convex multivariate risk measures are introduced in this framework: shortfall and divergence risk measures.

In Section 4.1, a precise definition of a *loss function* (the same as a *utility function* up to sign changes) is given, which represents a complete risk preference for univariate financial positions. As a preparation for the multivariate case, shortfall and divergence risk measures are presented in this section first in the univariate case. The material here is mostly due to [8, 9, 48]. However, the assumptions on the loss function are slightly more general in this dissertation. A discussion on the technical properties of the loss function is given in Section 4.2.

Section 4.3 is the main section of this chapter, where multivariate shortfall and divergence risk measures are studied. It is assumed that there is a complete risk preference towards the outcomes from each asset, which is represented by a loss function. Hence, the risk preference for the multivariate positions is incomplete and represented by a *vector loss function*. Then, multivariate shortfall and divergence risk measures are defined with respect to this vector-valued function. Theorem 4.3.8, one of the main results of this chapter, shows that a shortfall risk measure can be written as an intersection, that is, a set-valued supremum, over a family of divergence risk measures indexed by a scalarization parameter. To make the similarities and differences with the scalar theory more apparent, Section 4.1 and Section 4.3 are organized in a parallel way. As examples, multivariate *entropic risk measures* and multivariate *average value at risks* are discussed in Section 4.4.

It should be noted that the utility-based multivariate risk measures presented in this chapter form examples of the "pure" risk measures in Chapter 3, in other words, they do not take into account the frictions of the market. A remark on the *finiteness at zero* property of the market-extensions of shortfall and divergence risk measures (see Definition 3.1.1 and Proposition 3.2.7) is given in Section 4.5.

Throughout this chapter, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and continue using the notation from Chapter 3.

4.1 Scalar shortfall and divergence risk measures

In this section, we summarize the theory of (real-valued) shortfall and divergence risk measures on $L^\infty = L^\infty_1(\Omega, \mathcal{F}, \mathbb{P})$. Shortfall risk measures are introduced in [19]. Divergence risk measures are introduced in [8] and analyzed further in [9] with the name *optimized certainty equivalent* for their negatives. The dual relationship between these risk measures is pointed out in [48] and [9]. In terms of the assumptions on the loss function to be used, we slightly generalize the results of these papers; see Section 4.2 for a comparison. Most of the proofs in this section inherit the convex duality arguments in [9] rather than the analytic arguments in [19].

For a function $f \colon \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$, let us define its *effective domain* by

$$\operatorname{dom} f := \{x \in \mathbb{R} \mid f(x) < +\infty\}.$$

Definition 4.1.1. *A convex, lower semicontinuous function* $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *is said to be a loss function if it satisfies the following properties:*

(i) f *is nondecreasing.*

(ii) $0 \in \operatorname{dom} f$.

(iii) f *is not identically constant on* $\operatorname{dom} f$.

Throughout this section, let $\ell\colon \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ be a loss function. Definition 4.1.1 above guarantees that $\operatorname{int} \ell(\mathbb{R}) \neq \emptyset$, where int denotes the interior operator. Assume $x^0 \in \operatorname{int} \ell(\mathbb{R})$.

Definition 4.1.2. *The function* $\rho_\ell\colon L^\infty \to \mathbb{R} \cup \{\pm\infty\}$ *defined by*

$$\rho_\ell(X) = \inf\{s \in \mathbb{R} \mid \mathbb{E}\left[\ell(-X - s)\right] \leq x^0\},$$

for $X \in L^\infty$*, is called the* ℓ*-shortfall risk measure (on* L^∞ *with threshold level* x^0*).*

Proposition 4.1.3. *The function* ρ_ℓ *is a weak*-lower semicontinuous convex risk measure in the sense of [20, Definitions 4.1, 4.4]. In particular,* ρ_ℓ *takes values in* $\mathbb{R}$*.*

Proof. Monotonicity, translativity and convexity are trivial. Let $X \in L^\infty$. For every $s \in \mathbb{R}$,

$$\ell(-\operatorname{ess\,sup} X - s) \leq \mathbb{E}\left[\ell(-X - s)\right] \leq \ell(-\operatorname{ess\,inf} X - s).$$

Note that ℓ is strictly increasing on

$$\ell^{-1}(\operatorname{int} \ell(\mathbb{R})) \coloneqq \{x \in \mathbb{R} \mid \ell(x) \in \operatorname{int} \ell(\mathbb{R})\} = (a, b),$$

where

$$a \coloneqq \inf\left\{x \in \mathbb{R} \mid \ell(x) > \inf_{y\in\mathbb{R}} \ell(y)\right\} \in \mathbb{R} \cup \{-\infty\},$$

$$b \coloneqq \sup\{x \in \mathbb{R} \mid \ell(x) < +\infty\} \in \mathbb{R} \cup \{+\infty\}.$$

Hence, the inverse ℓ^{-1} is well-defined as a function from $\operatorname{int}\ell(\mathbb{R})$ to (a, b). We have $\mathbb{E}[\ell(-X-s)] \leq x^0$ for each $s \geq -\operatorname{ess\,inf} X - \ell^{-1}(x^0)$, and $\mathbb{E}[\ell(-X-s)] > x^0$ for each $s < -\operatorname{ess\,sup} X - \ell^{-1}(x^0)$. So $\rho_\ell(X) \in \mathbb{R}$. Besides, $\mathbb{E}[\ell(-X-\rho_\ell(X))] \leq x^0$ since the restriction of ℓ on $\operatorname{dom}\ell$ is a continuous function.

To show weak*-lower semicontinuity, let $(X^n)_{n\in\mathbb{N}}$ be a bounded sequence in L^∞ converging to some $X^\infty \in L^\infty$ $\mathbb{P}$-almost surely. Then, using Fatou's lemma together with the fact that the restriction of ℓ on $\operatorname{dom}\ell$ is nondecreasing and continuous, we have

$$\begin{aligned}
\mathbb{E}\left[\ell\left(-X^\infty - \liminf_{n\to\infty} \rho_\ell(X^n)\right)\right] &= \mathbb{E}\left[\ell\left(\liminf_{n\to\infty} \left(-X^n - \rho_\ell(X^n)\right)\right)\right] \\
&\leq \liminf_{n\to\infty} \mathbb{E}\left[\ell(-X^n - \rho_\ell(X^n))\right] \\
&\leq x^0.
\end{aligned}$$

This implies the so-called *Fatou property* of ρ_ℓ, namely, that

$$\rho_\ell(X^\infty) \leq \liminf_{n\to\infty} \rho_\ell(X^n).$$

By [20, Theorem 4.33], this is equivalent to the weak*-lower semicontinuity of ρ_ℓ. □

According to Definition 4.1.2, given $X \in L^\infty$, $\rho_\ell(X)$ can be seen as the value of a convex minimization problem. The next proposition computes $\rho_\ell(X)$ as the value of the corresponding Lagrangian dual problem. Its proof is an easy application of strong duality.

Proposition 4.1.4. *For every $X \in L^\infty$,*

$$\rho_\ell(X) = \sup_{\lambda>0}\left(\inf_{s\in\mathbb{R}}\left(s + \lambda\mathbb{E}\left[\ell(-X-s)\right]\right) - \lambda x^0\right). \tag{4.1}$$

Proof. Note that $m \mapsto \mathbb{E}[\ell(-X-m)]$ is a proper convex function on $\mathbb{R}$. Hence, by Definition 4.1.2, $\rho_\ell(X)$ is the value of a convex minimization problem. The corresponding Lagrangian dual objective function k is given by

$$k(\lambda) := \inf_{s\in\mathbb{R}} \left(s + \lambda \left(\mathbb{E}\left[\ell(-X-s)\right] - x^0\right)\right)$$

for $\lambda \in \mathbb{R}_+$. Since $k(0) = -\infty$, the value of the dual problem equals the right hand side of (4.1). Finally, the two sides of (4.1) are equal since the usual *Slater's condition* holds: There exists $\bar{s} \in \mathbb{R}$ such that $\mathbb{E}\left[\ell(-X-\bar{s})\right] < x^0$. This is because we have $\mathbb{E}\left[\ell(-X-s)\right] < x^0$ for each $s > -\operatorname{ess\,inf} X - \ell^{-1}(x^0)$, where ℓ^{-1} is the inverse function on $\operatorname{int}\ell(\mathbb{R})$ as in the proof of Proposition 4.1.3. □

Note that $X \mapsto \inf_{s\in\mathbb{R}} \left(s + \lambda\mathbb{E}\left[\ell(-X-s)\right]\right)$ on L^∞ is a monotone and translative function for each $\lambda \in \mathbb{R}_{++}$. Our aim is to determine the values of λ for which this function is a weak*-lower semicontinuous convex risk measure. To that end, we are interested in the properties of the Legendre-Fenchel conjugate $g\colon \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ of the loss function ℓ given by

$$g(y) := \ell^*(y) = \sup_{x\in\mathbb{R}}(xy - \ell(x)).$$

In the following, we will adopt the convention $(+\infty) \cdot 0 = 0$ as usual in convex analysis, see [45]. Besides, $1/+\infty = 0$ as well as $1/0 = +\infty$. We will also use the measure-theoretic indicator function 1_A and the convex-analytic indicator function I_A for $A \subseteq \mathbb{R}$: $1_A(x) = 1$, $I_A(x) = 0$ for $x \in A$, and $1_A(x) = 0$, $I_A(x) = +\infty$ for $x \in \mathbb{R}\backslash A$.

Definition 4.1.5. *A proper, convex, lower semicontinuous function* $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *is said to be a divergence function if it satisfies the following properties:*

(i) $\operatorname{dom}\varphi \subseteq \mathbb{R}_+$.

(ii) φ *attains its infimum over* $\mathbb{R}$.

(iii) φ *is not of the form* $y \mapsto +\infty \cdot 1_{\{y<0\}} + (ay+b) \cdot 1_{\{y \geq 0\}}$ *with* $a \in \mathbb{R}_+ \cup \{+\infty\}$ *and* $b \in \mathbb{R}$.

Proposition 4.1.6. *Legendre-Fenchel conjugation furnishes a bijection between loss and divergence functions.*

Proof. Let f be a loss function and $f^* : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ its conjugate function. Note that $\operatorname{dom} f^* \subseteq \mathbb{R}_+$ since, for each $y < 0$,

$$f^*(y) \geq \sup_{n \in \mathbb{N}} (-ny - f(-n)) \geq \sup_{n \in \mathbb{N}} (-ny) - f(0) = +\infty,$$

where we use the monotonicity of f for the second inequality. Clearly, $f^*(y) \geq -f(0)$ for each $y \in \mathbb{R}$. Besides, by [44, Theorem 23.3], we have $\partial f(0) \neq \emptyset$ and, by [44, Theorem 23.5], we have $f^*(y) = -f(0)$ for every $y \in \partial f(0)$. ($\partial f(x)$ denotes the subdifferential of f at $x \in \mathbb{R}$.) Hence, f^* attains its infimum. Finally, f^* is not of the form $y \mapsto +\infty \cdot 1_{\{y<0\}} + (ay+b) \cdot 1_{\{y \geq 0\}}$ for some $a \in \mathbb{R}_+ \cup \{+\infty\}$ and $b \in \mathbb{R}$ as otherwise we would get $f(x) = (f^*)^*(x) = +\infty \cdot 1_{\{x>a\}} - b \cdot 1_{\{x \leq a\}}$, $x \in \mathbb{R}$, so that ℓ would be identically constant on $\operatorname{dom} f$. Hence, f^* is a divergence function.

Conversely, let φ be a divergence function and $\varphi^* : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ its conjugate function. Let $x^1, x^2 \in \mathbb{R}$ with $x^1 \geq x^2$. Since $\operatorname{dom} \varphi \subseteq \mathbb{R}_+$, we have $x^1 y - \varphi(y) \geq x^2 y - \varphi(y)$ for each $y \in \operatorname{dom} \varphi$ so that $\varphi^*(x^1) \geq \varphi^*(x^2)$. Hence, φ^* is nondecreasing. Clearly, $\varphi^*(0) = -\inf_{y \in \mathbb{R}} \varphi(y) \in \mathbb{R}$ so that $0 \in \operatorname{dom} \varphi^*$. Finally, φ^* is not identically constant on $\operatorname{dom} \varphi^*$ as otherwise $\varphi = (\varphi^*)^*$ would fail to satisfy property (iii) in Definition 4.1.5. Hence, φ^* is a loss function. □

Remark 4.1.7. Let $\lambda > 0$. If f is a loss function, then λf is also a loss function. If φ is a divergence function, then the function $y \mapsto \varphi_\lambda(y) := \lambda \varphi(\frac{y}{\lambda})$ on $\mathbb{R}$ is also a divergence function. The functions f and φ are conjugates of each other if and only if λf and φ_λ are.

Definition 4.1.8. *Let φ be a divergence function, $\lambda > 0$ and $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$. The quantity*

$$\mathcal{I}_{\varphi,\lambda}(\mathbb{Q}\,|\,\mathbb{P}) := \mathbb{E}\left[\varphi_\lambda\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] = \lambda\mathbb{E}\left[\varphi\left(\frac{1}{\lambda}\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]$$

is called the (φ,λ)-divergence of $\mathbb{Q}$ with respect to $\mathbb{P}$.

Note that $g = \ell^*$ is a divergence function, and $\operatorname{dom} g$ is an interval (possibly a singleton) with some endpoints $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}_+ \cup \{+\infty\}$ with $\alpha \leq \beta$. We also have that $\operatorname{dom} g \neq \{0\}$ since otherwise g would be of the form $y \mapsto +\infty \cdot 1_{\{y<0\}} + (ay+b) \cdot 1_{\{y\geq 0\}}$ for $a = +\infty$ and some $b \in \mathbb{R}$. Finally, for each $\lambda > 0$, $y \mapsto g_\lambda(y) := \lambda g(\frac{y}{\lambda})$ on $\mathbb{R}$ is a divergence function by Remark 4.1.7, and the corresponding (g,λ)-divergences are defined according to Definition 4.1.8.

Theorem 4.1.9. *For every $\lambda > 0$ and $X \in L^\infty$,*

$$\inf_{s\in\mathbb{R}} \left(s + \lambda\mathbb{E}\left[\ell(-X-s)\right]\right) = \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})} \left(\mathbb{E}^{\mathbb{Q}}\left[-X\right] - \mathcal{I}_{g,\lambda}(\mathbb{Q}\,|\,\mathbb{P})\right). \tag{4.2}$$

Moreover, the quantity in (4.2) is finite if

$$\lambda \in 1/\operatorname{dom} g := \{1/y \mid 0 \neq y \in \operatorname{dom} g\} = [1/\beta, 1/\alpha] \cap \mathbb{R}_{++}, \tag{4.3}$$

and it is equal to $-\infty$ if $\lambda \notin 1/\operatorname{dom} g$.

Proof. Note that the right hand side of (4.2) can be rewritten as a maximization problem on $L^1 := L^1_1(\Omega, \mathcal{F}, \mathbb{P})$:

$$\sup_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})} \left(\mathbb{E}^{\mathbb{Q}}\left[-X\right] - \mathcal{I}_{g,\lambda}(\mathbb{Q}\,|\,\mathbb{P})\right) = \sup_{V\in L^1_+} \left\{\mathbb{E}\left[-XV\right] - \lambda\mathbb{E}\left[g\left(\frac{1}{\lambda}V\right)\right] \mid \mathbb{E}\left[V\right] = 1\right\},$$

where $L^1_+ = L^1_{1,+}(\Omega, \mathcal{F}, \mathbb{P})$. The value of the corresponding Lagrangian dual problem is computed as

$$\begin{aligned} q &:= \inf_{s\in\mathbb{R}} \sup_{V\in L^1_+} \left(\mathbb{E}\left[-XV\right] - \lambda\mathbb{E}\left[g\left(\frac{1}{\lambda}V\right)\right] + s(1-\mathbb{E}\left[V\right])\right) \\ &= \inf_{s\in\mathbb{R}} \left(s + \sup_{V\in L^1_+} \mathbb{E}\left[(-X-s)V - \lambda g\left(\frac{1}{\lambda}V\right)\right]\right) \\ &= \inf_{s\in\mathbb{R}} \left(s + \mathbb{E}\left[\sup_{z\in\mathbb{R}_+}\left((-X-s)z - \lambda g\left(\frac{1}{\lambda}z\right)\right)\right]\right) \\ &= \inf_{s\in\mathbb{R}} (s + \mathbb{E}\left[g^*_\lambda(-X-s)\right]), \end{aligned}$$

where the third equality is due to [45, Theorem 14.60], and g^*_λ is the conjugate of the divergence function g_λ; see Remark 4.1.7. Hence, $g^*_\lambda = \lambda\ell$ and q equals the left hand side of (4.2). Finally, to conclude (4.2), we consider the following cases:

(i) Suppose that $1 \in \operatorname{int}\operatorname{dom} g_\lambda$, that is, $\lambda\alpha < 1 < \lambda\beta$. (Recall that α and β are the endpoints of $\operatorname{dom} g$, see Definition 4.1.8 *et seq.*) Then the following constraint qualification holds, for instance, with $\bar{V} \equiv 1$:

$$\exists \bar{V} \in L^1_+ : \ \mathbb{E}\left[\bar{V}\right] = 1, \ \bar{V} \in \operatorname{int}\operatorname{dom} g_\lambda \ \ \mathbb{P}\text{-almost surely.} \tag{4.4}$$

By [10, Corollary 4.8], (4.4) suffices to conclude (4.2). Note that we have

$$\begin{aligned} -\mathbb{E}\left[X\right] - \lambda g\left(\frac{1}{\lambda}\right) &\le \sup_{V\in L^1_+} \left\{\mathbb{E}\left[-XV\right] - \lambda\mathbb{E}\left[g\left(\frac{1}{\lambda}V\right)\right] \mid \mathbb{E}\left[V\right] = 1\right\} \\ &\le -\operatorname{ess\,inf} X - \lambda \inf_{x\in\mathbb{R}} g(x), \end{aligned}$$

so that both sides of (4.2) are in $\mathbb{R}$.

(ii) Suppose that $\lambda\alpha = 1$ or $\lambda\beta = 1$. In this case, the only $V \in L^1_+$ with $\mathbb{E}\left[V\right] = 1$ and $V \in \operatorname{dom} g_\lambda$ $\mathbb{P}$-almost surely is $V \equiv 1$, and hence, the right hand side of (4.2) gives $-\mathbb{E}\left[X\right] - \lambda g(\frac{1}{\lambda}) \in \mathbb{R}$. Note that (4.4) fails to hold here. If $\alpha = \beta$, that is, if $\operatorname{dom} g = \{\alpha\}$, then $\ell(x) = \frac{x}{\lambda} - g(\frac{1}{\lambda}), x \in \mathbb{R}$, and the left hand side of (4.2) gives

$-\mathbb{E}[X] - \lambda g(\frac{1}{\lambda})$, showing (4.2). Suppose $\alpha \neq \beta$ so that $\operatorname{int}\operatorname{dom} g_\lambda \neq \emptyset$. If $\lambda\beta = 1$, then, using (4.2) for the previous case, we have

$$\begin{aligned}\inf_{s\in\mathbb{R}}(s + \lambda\mathbb{E}[\ell(-X-s)]) &= \lim_{\varepsilon\downarrow 0}\inf_{s\in\mathbb{R}}(s + (\lambda+\varepsilon)\mathbb{E}[\ell(-X-s)]) \\ &= \lim_{\varepsilon\downarrow 0}\sup_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})}\left(\mathbb{E}^{\mathbb{Q}}[-X] - (\lambda+\varepsilon)\mathbb{E}\left[g\left(\frac{1}{\lambda+\varepsilon}\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right) \\ &= \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})}\left(\mathbb{E}^{\mathbb{Q}}[-X] - \lambda\mathbb{E}\left[g\left(\frac{1}{\lambda}\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right).\end{aligned}$$

Here, the first and the last equalities follow by the fact that the proper, concave, upper semicontinuous function

$$\mathbb{R} \ni \gamma \longmapsto \inf_{s\in\mathbb{R}}(s + \gamma\mathbb{E}\ell(-X-s)) \in \mathbb{R}\cup\{-\infty\} \tag{4.5}$$

is right-continuous at $\gamma = \lambda = \beta^{-1}$. This argument applies for $\lambda\alpha = 1$ analogously if we switch the direction of the limit from $\varepsilon \downarrow 0$ to $\varepsilon \uparrow 0$ and use the fact that the function in (4.5) is left-continuous at $\gamma = \lambda = \alpha^{-1}$. Hence, we obtain (4.2).

(iii) Suppose $\lambda\alpha > 1$ or $\lambda\beta < 1$. In this case, there is no $Y \in L^1_+$ with $\mathbb{E}[Y] = 1$ and $Y \in \operatorname{dom} g_\lambda$ $\mathbb{P}$-almost surely. Hence, the right hand side of (4.2) gives $-\infty$. On the other hand, we have

$$\begin{aligned}\inf_{s\in\mathbb{R}}(s + \lambda\mathbb{E}[\ell(-X-s)]) &\leq \inf_{s\in\mathbb{R}}(s + \lambda\ell(-\operatorname{ess\,inf} X - s)) \\ &= -\operatorname{ess\,inf} X - \sup_{s\in\mathbb{R}}(s - \lambda\ell(s)) \\ &= -\operatorname{ess\,inf} X - \lambda g\left(\frac{1}{\lambda}\right) \\ &= -\infty.\end{aligned} \tag{4.6}$$

Hence, (4.2) holds. □

For every $\lambda > 0$, define a function $\delta_{g,\lambda}\colon L^\infty \to \mathbb{R} \cup \{-\infty\}$ by

$$\delta_{g,\lambda}(X) := \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})} (\mathbb{E}^{\mathbb{Q}}[-X] - \mathcal{I}_{g,\lambda}(\mathbb{Q}\,|\,\mathbb{P})) - \lambda x^0. \tag{4.7}$$

Corollary 4.1.10. *For every $X \in L^\infty$,*

$$\rho_\ell(X) = \sup_{\lambda\in 1/\operatorname{dom} g} \delta_{g,\lambda}(X).$$

Proof. By Theorem 4.1.9, $\delta_{g,\lambda}$ maps into $\mathbb{R}$ for $\lambda \in 1/\operatorname{dom} g$, and $\delta_{g,\lambda} \equiv -\infty$ for $\lambda \notin 1/\operatorname{dom} g$. The result is an immediate consequence of Proposition 4.1.4. □

Definition 4.1.11. *Let $\lambda \in 1/\operatorname{dom} g$. The function $\delta_{g,\lambda} : L^\infty \to \mathbb{R}$ defined by* (4.7) *is called the (g,λ)-divergence risk measure on L^∞ with threshold level x^0.*

By [20, Theorem 4.33], a weak*-lower semicontinuous convex risk measure $\rho\colon L^\infty \to \mathbb{R}$ has the dual representation

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})} \left(\mathbb{E}^{\mathbb{Q}}[-X] - \alpha_\rho^{\min}(\mathbb{Q})\right) \tag{4.8}$$

for $X \in L^\infty$, where $\alpha_\rho^{\min}\colon \mathcal{M}_1(\mathbb{P}) \to \mathbb{R} \cup \{+\infty\}$ is the *minimal penalty function* of ρ defined by

$$\alpha_\rho^{\min}(\mathbb{Q}) = \sup_{X\in L^\infty} \left(\mathbb{E}^{\mathbb{Q}}[-X] - \rho(X)\right)$$

for $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$. In Propositions 4.1.12 and 4.1.13 below, we compute the minimal penalty functions of ρ_ℓ and $\delta_{g,\lambda}$ for $\lambda \in 1/\operatorname{dom} g$.

Proposition 4.1.12. *Let $\lambda \in 1/\operatorname{dom} g$. The function $\delta_{g,\lambda}\colon L^\infty \to \mathbb{R}$ is a convex weak*-lower semicontinuous risk measure with minimal penalty function $\alpha_{\delta_{g,\lambda}}^{\min}\colon \mathcal{M}_1(\mathbb{P}) \to \mathbb{R} \cup \{+\infty\}$ given by*

$$\alpha_{\delta_{g,\lambda}}^{\min}(\mathbb{Q}) = \mathcal{I}_{g,\lambda}(\mathbb{Q}\,|\,\mathbb{P}) + \lambda x^0.$$

Proof. The definition of $\delta_{g,\lambda}$ given by (4.7) guarantees monotonicity, translativity, convexity and weak*-lower semicontinuity directly. Note that the function $\mathbb{Q} \mapsto \mathcal{I}_{g,\lambda}(\mathbb{Q} \,|\, \mathbb{P}) + \lambda x^0$ on $\mathcal{M}_1(\mathbb{P})$ is a penalty function in the sense that

$$\inf_{\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})} (\lambda x^0 + \mathcal{I}_{g,\lambda}(\mathbb{Q} \mid \mathbb{P})) = -\delta_{g,\lambda}(0) \in \mathbb{R}.$$

Finally, this function is indeed the minimal penalty function of $\delta_{g,\lambda}$ since, using the definition of minimal penalty function, we have

$$\begin{aligned}
\alpha^{\min}_{\delta_{g,\lambda}}(\mathbb{Q}) &= \sup_{X \in L^\infty} \left(\mathbb{E}^{\mathbb{Q}}[-X] - \delta_{g,\lambda}(X)\right) \\
&= \lambda x^0 + \sup_{X \in L^\infty} \left(\mathbb{E}^{\mathbb{Q}}[-X] + \sup_{s \in \mathbb{R}} (-s - \lambda \mathbb{E}[\ell(-X-s)])\right) \\
&= \lambda x^0 + \sup_{s \in \mathbb{R}} \left(-s + \sup_{X \in L^\infty} \mathbb{E}\left[-\frac{d\mathbb{Q}}{d\mathbb{P}} X - \lambda \ell(-X-s)\right]\right) \\
&= \lambda x^0 + \sup_{s \in \mathbb{R}} \left(-s + \mathbb{E}\left[\sup_{x \in \mathbb{R}} \left(-\frac{d\mathbb{Q}}{d\mathbb{P}} x - \lambda \ell(-x-s)\right)\right]\right) \\
&= \lambda x^0 + \sup_{s \in \mathbb{R}} \left(-s + \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} s + g_\lambda\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]\right) \\
&= \lambda x^0 + \mathcal{I}_{g,\lambda}(\mathbb{Q} \,|\, \mathbb{P}),
\end{aligned}$$

for each $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$. Note that, in this computation, the fourth equality follows by [45, Theorem 14.60] and the fifth equality follows from Remark 4.1.7. □

Recall that Corollary 4.1.10 summarizes the duality between the ℓ-shortfall risk measure ρ_ℓ and (g, λ)-divergence risk measures $\delta_{g,\lambda}$ with $\lambda \in 1/\operatorname{dom} g$. Proposition 4.1.13 tells that there is an analogous duality between the corresponding minimal penalty functions.

Proposition 4.1.13. *The minimal penalty function* $\alpha^{\min}_{\rho_\ell} \colon \mathcal{M}_1(\mathbb{P}) \to \mathbb{R} \cup \{+\infty\}$ *of* ρ_ℓ *is given by*

$$\alpha^{\min}_{\rho_\ell}(\mathbb{Q}) = \inf_{\lambda > 0} (\lambda x^0 + \mathcal{I}_{g,\lambda}(\mathbb{Q} \,|\, \mathbb{P})) = \inf_{\lambda \in 1/\operatorname{dom} g} \alpha^{\min}_{\delta_{g,\lambda}}(\mathbb{Q})$$

for $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$.

Proof. Let $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$. By the definition of minimal penalty function and Definition 4.1.2, we have

$$\begin{aligned}
\alpha_{\rho_\ell}^{\min}(\mathbb{Q}) &= \sup_{X \in L^\infty} \left(\mathbb{E}^{\mathbb{Q}}[-X] - \rho_\ell(X)\right) \\
&= \sup_{X \in L^\infty} \left(\mathbb{E}^{\mathbb{Q}}[-X] - \inf_{s \in \mathbb{R}} \left(s + I_{(-\infty,x^0]}\left(\mathbb{E}\left[\ell(-X-s)\right]\right)\right)\right) \\
&= \sup_{s \in \mathbb{R}} \left(-s - \inf_{X \in L^\infty} \left(\mathbb{E}^{\mathbb{Q}}[X] + I_{(-\infty,x^0]}\left(\mathbb{E}\left[\ell(-X-s)\right]\right)\right)\right) \\
&= \sup_{s \in \mathbb{R}} \sup_{X \in L^\infty} \left(\mathbb{E}^{\mathbb{Q}}[X] - I_{(-\infty,x^0]}(\mathbb{E}\left[\ell(X)\right])\right) \\
&= \sup_{X \in L^\infty} \left\{\mathbb{E}^{\mathbb{Q}}[X] \mid \mathbb{E}\left[\ell(X)\right] \leq x^0\right\}.
\end{aligned}$$

The value of the corresponding Lagrangian dual problem is

$$\begin{aligned}
q(\mathbb{Q}) &:= \inf_{\lambda > 0} \sup_{X \in L^\infty} \left(\mathbb{E}^{\mathbb{Q}}[X] + \lambda\left(x^0 - \mathbb{E}\left[\ell(X)\right]\right)\right) \\
&= \inf_{\lambda > 0} \left(\lambda x^0 + \sup_{X \in L^\infty} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} X - \lambda \ell(X)\right]\right) \\
&= \inf_{\lambda > 0} \left(\lambda x^0 + \mathbb{E}\left[\sup_{x \in \mathbb{R}} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} x - \lambda \ell(x)\right)\right]\right) \\
&= \inf_{\lambda > 0} \left(\lambda x^0 + \mathcal{I}_{g,\lambda}\left(\mathbb{Q} \,|\, \mathbb{P}\right)\right),
\end{aligned}$$

where, we use [45, Theorem 14.60] for the third equality and Remark 4.1.7 for the fourth equality again. Note that Slater's condition holds, that is, there exists $\bar{X} \in L^\infty$ such that $\mathbb{E}\left[\ell(\bar{X})\right] < x^0$; take, for example, $\bar{X} \equiv \ell^{-1}(x^0) - 1$, where ℓ^{-1} is the inverse function on $\operatorname{int} \ell(\mathbb{R})$ as in the proof of Proposition 4.1.3. Therefore, $\alpha_{\rho_\ell}^{\min}(\mathbb{Q}) = q(\mathbb{Q})$. Note that $\mathcal{I}_{g,\lambda}(\mathbb{Q} \,|\, \mathbb{P}) = +\infty$ for every $\lambda \notin 1/\operatorname{dom} g$; see case (iii) in the proof of Theorem 4.1.9. Hence, we also have $q(\mathbb{Q}) = \inf_{\lambda \in 1/\operatorname{dom} g} \alpha_{\delta_{g,\lambda}}^{\min}(\mathbb{Q})$. □

4.2 A remark about scalar loss functions

In [19] and [20], Proposition 4.1.13 ([19, Theorem 10], [20, Theorem 4.115]) is proved with the additional assumption that ℓ maps into $\mathbb{R}$. This assumption implies that the ℓ-shortfall risk measure is continuous from below and the supremum in (4.8) is attained ([20, Proposition 4.113]). Besides, the same assumption implies the so-called *superlinear growth condition* on g, namely, that $\lim_{y\to\infty}\frac{g(y)}{y}=+\infty$ ([19, Lemma 11]). The analytic proof for Proposition 4.1.13 in [19] makes use of this property instead of the dual relationship with divergence risk measures. Using this proposition and assuming that $1 \in \operatorname{dom} g$, Theorem 4.1.9 is proved for $\lambda = 1$ ([20, Theorem 4.122]), in which case $\delta_{g,1}$ is guaranteed to be a risk measure (has finite values). In our treatment, while 1 may not be in $\operatorname{dom} g$, there exists some $\bar{\lambda} \in 1/\operatorname{dom} g$ such that $1 \in \operatorname{dom} g_{\bar{\lambda}}$ and hence $\delta_{g,\bar{\lambda}}$ is a risk measure.

In [9], on the other hand, the divergence function g is of central importance: in addition to the assumptions in Definition 4.1.5, it is assumed in [9] that g attains its infimum at 1 with value 0, which is equivalent to assuming that $\ell(0) = 0$ and $1 \in \partial\ell(0)$. These assumptions make g a natural divergence function in the sense that the function $\mathbb{Q} \mapsto \mathbb{E}[g(\frac{d\mathbb{Q}}{d\mathbb{P}})]$ on $\mathcal{M}_1(\mathbb{P})$ has positive values and takes the value 0 if $\mathbb{Q} = \mathbb{P}$; $\mathbb{E}[g(\frac{d\mathbb{Q}}{d\mathbb{P}})]$ can be interpreted as the distance between some "subjective" measure $\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})$ and the physical measure $\mathbb{P}$. On the other hand, the additional assumptions on the loss function ℓ may be considered as restrictive. Here, we take ℓ as the central object by dropping these assumptions and use the convex duality approach as in [9]. Note that Theorem 4.1.9 ([9, Theorem 4.2]) and Proposition 4.1.12 ([9, Theorem 4.4]) are proved in [9] for the case $\lambda = 1$. Here, we generalize this proof, basically, by considering the cases where the constraint qualification (4.4), which is also used in the proof of [9, Theorem 4.2], fails to hold.

4.3 Multivariate shortfall and divergence risk measures

In this section, multivariate shortfall and divergence risk measures, the central objects of this chapter, are introduced.

Throughout this section, let $m \in \{1, 2, \ldots\}$ denote the number of eligible assets (see Section 3.1). Corresponding to each such asset $i \in \{1, \ldots, m\}$, it is assumed that there is a (scalar) loss function $\ell_i \colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ in the sense of Definition 4.1.1; g_i denotes its conjugate function, which is a divergence function in the sense of Definition 4.1.5. Let us define the *vector loss function* $\ell \colon \mathbb{R}^m \to \mathbb{R}^m \cup \{+\infty\}$ by

$$\ell(x) = \begin{cases} (\ell_1(x_1), \ldots, \ell_m(x_m)) & \text{if } x \in \bigtimes_{i=1}^m \operatorname{dom} \ell_i \\ +\infty & \text{else} \end{cases},$$

and similarly the *vector divergence function* $g \colon \mathbb{R}^m \to \mathbb{R}^m \cup \{+\infty\}$ by

$$g(x) = \begin{cases} (g_1(x_1), \ldots, g_m(x_m)) & \text{if } x \in \bigtimes_{i=1}^m \operatorname{dom} g_i \\ +\infty & \text{else} \end{cases}$$

for $x \in \mathbb{R}^m$. Here, the element $+\infty$ is added to $\mathbb{R}^m$ as a top element with respect to the partial order generated by $\mathbb{R}^m_+$, that is,

$$\forall z \in \mathbb{R}^m \cup \{+\infty\} : z \leq_{\mathbb{R}^m_+} +\infty.$$

The addition is extended from $\mathbb{R}^m$ to $\mathbb{R}^m \cup \{+\infty\}$ by $z + (+\infty) = (+\infty) + z = +\infty$ for every $z \in \mathbb{R}^m \cup \{+\infty\}$.

Remark 4.3.1. Of course, one could consider more general loss functions ℓ_i that depend on the vector $x \in \mathbb{R}^m$, and not only on the component x_i, $i \in \{1, \ldots, m\}$, or even vector loss functions $\ell\colon \mathbb{R}^d \to \mathbb{R}^m$ with $d > m$. However, it should be noted that (1) the interconnectedness of the components of a portfolio $x = X(\omega)$ at time t is already modeled by the prevailing exchange rates $\mathcal{C}_t(\omega)$ and trading constraints $\mathcal{D}_t(\omega)$, and thus, it will be included in the market-extension of a risk measure (see Section 3.2), (2) the dimension reduction, which is motivated by allowing only m of the d assets to be used as eligible assets for risk compensation, is modeled by forcing liquidation into L^∞_m in Definition 3.2.5 of the market-extension. This includes the case where a large number of assets d are denoted in a few ($m \ll d$) currencies, the currencies are used as eligible assets, and the loss functions are just defined for each of the m currencies (but not for each asset individually). Finally, (3) from a risk preference point of view, we assume that there is a complete risk preference for each of the m eligible assets which has a von Neumann-Morgenstern representation: this might be disputable, but it is already (much) more general than the assumption that there is a complete risk preference for multivariate positions (as in [11]) which even has a von Neumann-Morgenstern representation generated by a real-valued loss function defined on $\mathbb{R}^m$ (or $\mathbb{R}^d$) as in [12].

Using $\operatorname{dom}\ell \coloneqq \{x \in \mathbb{R}^m \mid \ell(x) \in \mathbb{R}^m\} = \bigtimes_{i=1}^m \operatorname{dom}\ell_i \subseteq \mathbb{R}^m$, we note

$$\operatorname{int}\ell(\operatorname{dom}\ell) = \bigtimes_{i=1}^{m} \operatorname{int}\ell_i(\mathbb{R}).$$

The expected loss operator is extended by $\mathbb{E}\left[\ell(X)\right] = +\infty$ whenever $\mathbb{P}\left\{X \in \operatorname{dom}\ell\right\} < 1$.

Let $x^0 = (x^0_1, \ldots, x^0_m) \in \operatorname{int}\ell(\operatorname{dom}\ell)$. Let $C \in \mathcal{G}(\mathbb{R}^m_+)$ be such that $0 \in \mathbb{R}^m$ is a boundary point of C.

Definition 4.3.2. *The function $R_\ell\colon L^\infty_m \to \mathcal{P}(\mathbb{R}^m_+)$ defined by*

$$R_\ell(X) = \left\{z \in \mathbb{R}^m \mid \mathbb{E}\left[\ell(-X - z)\right] \in x^0 - C\right\},$$

for $X \in L^\infty_m$, is called the multivariate ℓ-shortfall risk measure (on L^∞_m with threshold level x^0 and threshold set C).

The set C determines a rule with respect to which expected loss vectors are compared to the threshold level: the relation $\leq_C$ defined by $x \leq_C y :\Leftrightarrow y - x \in C$ is reflexive (since $0 \in C$), transitive if $C + C \subseteq C$ and antisymmetric if $C \cap (-C) = \{0\}$ (C is "pointed."). In particular, if C is a pointed convex cone, then $\leq_C$ is a partial order which is compatible with the linear structure on $\mathbb{R}^m$ as discussed in Section 2.1. The case $C = \mathbb{R}^m_+$ corresponds to the componentwise ordering of the expected loss vectors.

Proposition 4.3.3. *The function $R_\ell \colon L^\infty_m \to \mathcal{P}(\mathbb{R}^m_+)$ is a weak*-closed convex multivariate risk measure. In particular, it maps into $\mathcal{G}(\mathbb{R}^m_+)$.*

Proof. Monotonicity, translativity and convexity are trivial. Let $X \in L^\infty_m$. Using the proof of Proposition 4.1.3, we can find $z^1 \in \mathbb{R}^m$ with $\mathbb{E}\left[\ell(-X - z^1)\right] \in x^0 - \mathbb{R}^m_+$ and $z^2 \in \mathbb{R}^m$ with $\mathbb{E}\left[\ell(-X - z^2)\right] \in x^0 + \mathbb{R}^m_{++}$. So $R_\ell(X) \notin \{\emptyset, \mathbb{R}^m\}$.

To show weak*-closedness, let $(X^n)_{n\in\mathbb{N}}$ be a bounded sequence in L^∞_m converging to some $X \in L^\infty_m$ $\mathbb{P}$-almost surely. Let $z \in \mathbb{R}^m$ and suppose that there exists $z^n \in R_\ell(X^n)$, for each $n \in \mathbb{N}$, such that $(z^n)_{n\in\mathbb{N}}$ converges to z. Using the dominated convergence theorem, the closedness of $x_0 - C$, and the fact that the restriction of ℓ on $\operatorname{dom}\ell$ is continuous, we have

$$\mathbb{E}\left[\ell(-X - z)\right] = \mathbb{E}\left[\ell\Big(\lim_{n\to\infty}(-X^n - z^n)\Big)\right] = \lim_{n\to\infty}\mathbb{E}\left[\ell(-X^n - z^n)\right] \in x^0 - C,$$

that is, $z \in R_\ell(X)$. This shows the so-called *Fatou property* of R_ℓ, namely, that

$$\liminf_{n\to\infty} R_\ell(X^n) := \left\{z \in \mathbb{R}^m \mid \forall n \in \mathbb{N}\ \exists z^n \in R_\ell(X^n) \colon \lim_{n\to\infty} z^n = z\right\} \subseteq R_\ell(X).$$

By [25, Theorem 6.2], this is equivalent to the weak*-closedness of R_ℓ. $\square$

Proposition 4.3.3 above implies that R_ℓ has the dual representation given by Proposition 3.1.6. In Proposition 4.3.12 at the end of this section, we will compute its minimal penalty function $-\alpha^{\min}_{R_\ell}$; this will be the set-valued analogue of Proposition 4.1.13.

If $C = \mathbb{R}^m_+$, then R_ℓ becomes a trivial generalization of the scalar shortfall risk measures $\rho_{\ell_1}, \ldots, \rho_{\ell_m}$ in the sense that

$$R_\ell(X) = (\rho_{\ell_1}(X_1), \ldots, \rho_{\ell_m}(X_m)) + \mathbb{R}^m_+,$$

for every $X \in L^\infty_m$. In general, such an explicit representation of R_ℓ may not exist. However, given $X \in L^\infty_m$, one may write

$$R_\ell(X) = \inf_{(\mathcal{G}(\mathbb{R}^m_+), \supseteq)} \left\{ z + \mathbb{R}^m_+ \mid 0 \in \mathbb{E}\left[\ell(-X - z)\right] - x^0 + C,\ z \in \mathbb{R}^m \right\}, \tag{4.9}$$

that is, $R_\ell(X)$ is the (optimal) value of the set minimization problem

$$\text{minimize} \quad \Phi(z) \quad \text{subject to} \quad 0 \in \Psi(z),\ z \in \mathbb{R}^m, \tag{4.10}$$

where $\Phi\colon \mathbb{R}^m \to \mathcal{G}(\mathbb{R}^m_+)$ and $\Psi\colon \mathbb{R}^m \to \mathcal{G}(\mathbb{R}^m_+)$ are the (set-valued) objective function and constraint function, respectively, defined by

$$\Phi(z) = z + \mathbb{R}^m_+ \quad \text{and} \quad \Psi(z) = \mathbb{E}\left[\ell(-X - z)\right] - x^0 + C$$

for $z \in \mathbb{R}^m$. Here, it is understood that $\Psi(z) = \emptyset$ whenever $\mathbb{E}\left[\ell(-X - z)\right] = +\infty$. A Lagrange duality theory for problems of the form (4.10) is developed in the recent work [28]. Using this theory, we will compute the Lagrangian dual problem for $R_\ell(X)$ in Proposition 4.3.4 below. It turns out that, after a change of variables provided by

Proposition 4.3.7, the dual objective function has a simple form, which will give rise to multivariate divergence risk measures.

Proposition 4.3.4. *For every* $X \in L^\infty_m$,

$$R_\ell(X) = \bigcap_{\lambda,\, v \in \mathbb{R}^m_+ \setminus \{0\}} \Big\{ \eta \in \mathbb{R}^m \mid \langle \eta, v \rangle \geq \inf_{z \in \mathbb{R}^m} \left(\langle z, v \rangle + \langle \mathbb{E}\left[\ell(-X - z)\right], \lambda \rangle \right) + \inf_{x \in -x^0 + C} \langle x, \lambda \rangle \Big\}, \tag{4.11}$$

where $\langle +\infty, \lambda \rangle := +\infty$ *whenever* $\lambda \in \mathbb{R}^m_+ \setminus \{0\}$.

Proof. Consider the minimization problem (4.10). The halfspace-valued functions $S_{(\lambda,v)} \colon \mathbb{R}^m \to \mathcal{G}(\mathbb{R}^m_+)$ for $v \in \mathbb{R}^m_+ \setminus \{0\}$, $\lambda \in \mathbb{R}^m$, defined by

$$S_{(\lambda,v)}(x) = \left\{ z \in \mathbb{R}^m \mid \langle z, v \rangle \geq \langle x, \lambda \rangle \right\},$$

for $x \in \mathbb{R}^m$, will be used as set-valued substitutes for the (continuous) linear functionals of the scalar duality theory as in [23], [28]. Here, we have two types of dual variables: The variable $\lambda \in \mathbb{R}^m$ is the usual vector of Lagrange multipliers which will be used to scalarize the values of the constraint function Ψ whereas the variable $v \in \mathbb{R}^m_+$ is the weight vector which will be used to scalarize the values of the objective function Φ. Fix $X \in L^\infty_m$. The set-valued Lagrangian $L \colon \mathbb{R}^m \times \mathbb{R}^m \times (\mathbb{R}^m_+ \setminus \{0\}) \to \mathcal{G}(\mathbb{R}^m_+)$ and the objective function $K \colon \mathbb{R}^m \times (\mathbb{R}^m_+ \setminus \{0\}) \to \mathcal{G}(\mathbb{R}^m_+)$ of the dual problem for (4.10) are defined by

$$L(z, \lambda, v) = \operatorname{cl} \left(\Phi(z) + \inf_{(\mathcal{G}(\mathbb{R}^m_+), \supseteq)} \left\{ S_{(\lambda,v)}(x) \mid x \in \Psi(z) \right\} \right),$$

$$K(\lambda, v) = \inf_{(\mathcal{G}(\mathbb{R}^m_+), \supseteq)} \{ L(z, \lambda, v) \mid z \in \mathbb{R}^m \},$$

for $z, \lambda \in \mathbb{R}^m, v \in \mathbb{R}^m_+ \setminus \{0\}$; see [28, Section 2 and Section 6.1]. Using the definition of Ψ and the formula (2.6) for infimum in $\mathcal{G}(\mathbb{R}^m_+)$, for $(z, \lambda, v) \in \mathbb{R}^m \times \mathbb{R}^m \times (\mathbb{R}^m_+ \setminus \{0\})$, we obtain

$$
\begin{aligned}
L(z,\lambda,v) &= \operatorname{cl}\left(z+\mathbb{R}^m_+ + \operatorname{cl}\bigcup_{x\in(\mathbb{E}[\ell(-X-z)]-x^0+C)\cap\mathbb{R}^m} S_{(\lambda,v)}(x)\right)\\
&= \operatorname{cl}\left(z+\mathbb{R}^m_+ + \left\{\eta\in\mathbb{R}^m \mid \langle \eta, v\rangle \geq \inf_{x\in(\mathbb{E}[\ell(-X-z)]-x^0+C)\cap\mathbb{R}^m} \langle x,\lambda\rangle\right\}\right)\\
&= \Big\{\eta\in\mathbb{R}^m \mid \mathbb{E}\left[\ell(-X-z)\right]\in\mathbb{R}^m,\\
&\qquad\qquad \langle \eta, v\rangle \geq \langle z, v\rangle + \langle \mathbb{E}\left[\ell(-X-z)\right],\lambda\rangle + \inf_{x\in -x^0+C}\langle x,\lambda\rangle\Big\}.
\end{aligned}
$$

This gives, for $(\lambda, v)\in\mathbb{R}^m\times(\mathbb{R}^m_+\backslash\{0\})$,

$$
\begin{aligned}
K(\lambda,v) &= \inf_{(\mathcal{G}(\mathbb{R}^m_+),\supseteq)}\{L(z,\lambda,v)\mid z\in\mathbb{R}^m\}\\
&= \operatorname{cl}\bigcup_{z\in\mathbb{R}^m} L(z,\lambda,v)\\
&= \Big\{\eta\in\mathbb{R}^m \mid \mathbb{E}\left[\ell(-X-z)\right]\in\mathbb{R}^m,\\
&\qquad \langle \eta, v\rangle \geq \inf_{z\in\mathbb{R}^m}\left(\langle z, v\rangle + \langle \mathbb{E}\left[\ell(-X-z)\right],\lambda\rangle\right) + \inf_{x\in -x^0+C}\langle x,\lambda\rangle\Big\}.
\end{aligned}
\tag{4.12}
$$

Note that $L(z,0,v)=\emptyset$ whenever $\mathbb{E}\left[\ell(-X-z)\right]\notin\mathbb{R}^m$, but this cannot happen for all $z\in\mathbb{R}^m$ since $X\in L^\infty_m$. Hence $K(0,v)=\mathbb{R}^m$ for every $v\in\mathbb{R}^m_+\backslash\{0\}$. Suppose $\lambda\notin\mathbb{R}^m_+$. Since $C+\mathbb{R}^m_+\subseteq C$, there exists $\bar{x}\in C$ such that, for every $n\in\mathbb{N}$, we have $n\bar{x}\in C$ and $\langle\bar{x},\lambda\rangle<0$. Hence $\inf_{x\in C}\langle x,\lambda\rangle=-\infty$ and $K(\lambda,v)=\mathbb{R}^m$ for every $v\in\mathbb{R}^m_+\backslash\{0\}$. The dual optimal value is the supremum over the dual variables: since the supremum in $\mathcal{G}(\mathbb{R}^m_+)$ is the intersection, we have to take it over $K(\lambda,v)$ with λ, v running only through $\mathbb{R}^m_+\backslash\ \{0\}$, that is, the dual optimal value is

$$
d := \sup_{(\mathcal{G}(\mathbb{R}^m_+),\supseteq)}\{K(\lambda,v)\mid \lambda,v\in\mathbb{R}^m_+\backslash\{0\}\} = \bigcap_{\lambda,v\in\mathbb{R}^m_+\backslash\{0\}} K(\lambda,v). \tag{4.13}
$$

Noting that, for $\lambda, v \in \mathbb{R}^m_+ \setminus \{0\}$,

$$K(\lambda, v) = \left\{ \eta \in \mathbb{R}^m \mid \langle \eta, v \rangle \geq \inf_{z \in \mathbb{R}^m} \left(\langle z, v \rangle + \langle \mathbb{E}\left[\ell(-X - z)\right], \lambda \rangle \right) + \inf_{x \in -x^0 + C} \langle x, \lambda \rangle \right\}$$

as $\langle +\infty, \lambda \rangle = +\infty$ by convention, we see that d equals the right hand side of (4.11). Finally, the two sides of (4.11) are equal since the set-valued version of *Slater's condition* holds ([28, Theorem 6.6]): There exists $\bar{z} \in \mathbb{R}^m$ such that

$$\left(\mathbb{E}\left[\ell(-X - \bar{z})\right] - x^0 + C \right) \cap -\mathbb{R}^m_{++} \neq \emptyset.$$

This follows as for the scalar version (see proof of Proposition 4.1.4). □

Let us introduce the vector-valued versions of some notions used in Section 4.1.

Definition 4.3.5. *Let $r \in \mathbb{R}^m_+ \setminus \{0\}$ and $\mathbb{Q} \in \mathcal{M}_m(\mathbb{P})$. For each $i \in \{1, \ldots, m\}$, let*

$$\mathcal{I}_{g_i, r_i}(\mathbb{Q}_i \,|\, \mathbb{P}) := \begin{cases} r_i \mathbb{E}\left[g_i \left(\frac{1}{r_i} \frac{d\mathbb{Q}_i}{d\mathbb{P}} \right) \right] & \text{if } r_i > 0 \\ +\infty & \text{else} \end{cases}.$$

The element $\mathcal{I}_{g,r}(\mathbb{Q} \,|\, \mathbb{P}) \in \mathbb{R}^m \cup \{+\infty\}$ defined by

$$\mathcal{I}_{g,r}(\mathbb{Q} \,|\, \mathbb{P}) := \left(\mathcal{I}_{g_1, r_1}(\mathbb{Q}_1 \,|\, \mathbb{P}), \ldots, \mathcal{I}_{g_m, r_m}(\mathbb{Q}_m \,|\, \mathbb{P}) \right)$$

whenever the right hand side is in $\mathbb{R}^m$ and $+\infty$ otherwise is called the vector (g, r)-divergence of $\mathbb{Q}$ with respect to $\mathbb{P}$.

If $r_i > 0$, then $\mathcal{I}_{g_i, r_i}(\mathbb{Q}_i \,|\, \mathbb{P})$ is the (scalar) (g_i, r_i)-divergence of $\mathbb{Q}_i$ with respect to $\mathbb{P}$; see Definition 4.1.8. Recall that $\mathbb{E}\left[g_i \left(\frac{1}{r_i} \frac{d\mathbb{Q}_i}{d\mathbb{P}} \right) \right] = +\infty$ whenever $\mathbb{P}\left\{ \frac{1}{r_i} \frac{d\mathbb{Q}_i}{d\mathbb{P}} \in \operatorname{dom} g_i \right\} < 1$.

Definition 4.3.6. *For every $r \in \mathbb{R}^m_+ \backslash \{0\}$, define a function $\delta_{g,r} \colon L^\infty_m \to \mathbb{R}^m \cup \{-\infty\}$ by*

$$\delta_{g,r}(X) = (\delta_{g_1,r_1}(X_1), \ldots, \delta_{g_m,r_m}(X_m))$$

whenever the right hand side is in $\mathbb{R}^m$ and $\delta_{g,r}(X) = -\infty$ otherwise.

Of course, if $r_i > 0$, then δ_{g_i,r_i} given by

$$\begin{aligned}\delta_{g_i,r_i}(X_i) &\coloneqq \sup_{\mathbb{Q}_i \in \mathcal{M}_1(\mathbb{P})} \left(\mathbb{E}^{\mathbb{Q}_i}[-X_i] - \mathcal{I}_{g_i,r_i}(\mathbb{Q}_i \,|\, \mathbb{P})\right) - r_i x_i^0 \\ &= \inf_{z_i \in \mathbb{R}} \left(z_i + r_i \mathbb{E}\left[\ell_i(-X_i - z_i)\right]\right) - r_i x_i^0\end{aligned} \tag{4.14}$$

is the (scalar) (g_i, r_i)-divergence risk measure according to Definition 4.1.11, and we have $\delta_{g_i,0} \equiv -\infty$. The next proposition provides a more useful version of the representation in Proposition 4.3.4 in terms of the vector-valued functions $(\delta_{g,r})_{r \in \mathbb{R}^m_+}$ as a result of a "change of variables."

Recall from Section 2.4 that $x \circ y \coloneqq (x_1 y_1, \ldots, x_m y_m)$ for $x, y \in \mathbb{R}^m$.

Proposition 4.3.7. *For every $X \in L^\infty_m$,*

$$R_\ell(X) = \bigcap_{r,w \in \mathbb{R}^m_+ \backslash \{0\}} \left\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \langle \delta_{g,r}(X), w \rangle + \inf_{x \in C} \langle r \circ x, w \rangle \right\},$$

where $\langle -\infty, w \rangle = -\infty$ whenever $w \in \mathbb{R}^m_+ \backslash \{0\}$.

Proof. With (4.12) in view, for $r, w \in \mathbb{R}^m_+ \backslash \{0\}$, we define

$$M(r, w) = \left\{ \eta \in \mathbb{R}^m \mid \langle \eta, w \rangle \geq \langle \delta_{g,r}(X), w \rangle + \inf_{x \in C} \langle r \circ x, w \rangle \right\}, \tag{4.15}$$

and we will show

$$\bigcap_{\lambda, v \in \mathbb{R}^m_+ \backslash \{0\}} K(\lambda, v) = \bigcap \left\{ M(r, w) \mid r, w \in \mathbb{R}^m_+ \backslash \{0\},\ \langle r, w \rangle > 0 \right\}. \tag{4.16}$$

First, if $r, w \in \mathbb{R}^m_+ \setminus \{0\}$ and $\langle r, w \rangle > 0$, then we define $\lambda_i = r_i w_i$ and $v_i = w_i$ for $i \in \{1, \dots, m\}$. Then $\lambda, v \in \mathbb{R}^m_+ \setminus \{0\}$ as well as $K(\lambda, v) = M(r, w)$; see Definition 4.3.6 and (4.14). This means that the intersection on the left hand side runs over at least as many sets as the one on the right hand side, hence "$\subseteq$" holds true.

Conversely, if $\lambda, v \in \mathbb{R}^m_+ \setminus \{0\}$, then we define, for $n \in \mathbb{N}$ and $i \in \{1, \dots, m\}$,

$$\begin{aligned} r_i^n &= \frac{\lambda_i}{v_i} \quad \text{and} \quad w_i^n = v_i \quad \text{whenever} \quad v_i > 0, \\ r_i^n &= 1 \quad \text{and} \quad w_i^n = \lambda_i \quad \text{whenever} \quad v_i = 0,\ \lambda_i = 0, \\ r_i^n &= n\lambda_i \quad \text{and} \quad w_i^n = \frac{1}{n} \quad \text{whenever} \quad v_i = 0,\ \lambda_i > 0. \end{aligned}$$

Then $r^n, w^n \in \mathbb{R}^m_+ \setminus \{0\}$ and $\lambda_i = r_i^n w_i^n$. Moreover, $\langle r^n, w^n \rangle > 0$ since $\lambda \neq 0$. Assume

$$\eta \in \bigcap \left\{ M(r, w) \mid r, w \in \mathbb{R}^m_+ \setminus \{0\},\ \langle r, w \rangle > 0 \right\}.$$

If there is no $i \in \{1, \dots, m\}$ satisfying $v_i = 0$ and $\lambda_i > 0$, then $v = w^n$ and $K(\lambda, v) = M(r^n, w^n)$ for every $n \in \mathbb{N}$; hence $\eta \in K(\lambda, v)$. Next, assume there is at least one $j \in \{1, \dots, m\}$ with $v_j = 0$, $\lambda_j > 0$ as well as $g_j(0) = +\infty$. Then

$$\inf_{z_j \in \mathbb{R}} \mathbb{E}\left[\ell_j(-X_j - z_j)\right] = \inf_{y \in \mathbb{R}} \ell_j(y) = -g_j(0) = -\infty. \tag{4.17}$$

since ℓ_j is nondecreasing and $X_j \in L^\infty$. This implies

$$\sum_{i=1}^m \inf_{z_i \in \mathbb{R}} \left(v_i z_i + \lambda_i \mathbb{E}\left[\ell_i(-X_i - z_i)\right] \right) + \inf_{x \in -x^0 + C} \langle x, \lambda \rangle = -\infty$$

since $v_j = 0$ and none of the terms in the sum is $+\infty$. Hence $\eta \in K(\lambda, v)$. Finally, assume that there is at least one $j \in \{1, \dots, m\}$ with $v_j = 0$, $\lambda_j > 0$ as well as

$g_j(0) \in \mathbb{R}$. Since $\eta \in M(r^n, w^n)$ for every $n \in \mathbb{N}$, it follows

$$\begin{aligned} \sum_{\substack{i\colon v_i>0 \\ i\colon v_i=\lambda_i=0}} v_i\eta_i + \sum_{i\colon v_i=0,\,\lambda_i>0} \frac{\eta_i}{n} \geq &\sum_{\substack{i\colon v_i>0 \\ i\colon v_i=\lambda_i=0}} \inf_{z_i\in\mathbb{R}} \left(v_i z_i + \lambda_i \mathbb{E}\left[\ell_i(-X_i - z_i)\right]\right) \\ &+ \sum_{i\colon v_i=0,\,\lambda_i>0} \frac{1}{n} \inf_{z_i\in\mathbb{R}} \left(z_i + n\lambda_i \mathbb{E}\left[\ell_i(-X_i - z_i)\right]\right) \\ &+ \inf_{x\in -x^0+C} \langle x, \lambda\rangle. \end{aligned} \tag{4.18}$$

Whenever $\lambda_j > 0$, we obtain

$$\begin{aligned} -\operatorname{ess\,sup} X_j - n\lambda_j g_j\left(\frac{1}{n\lambda_j}\right) &\leq \inf_{z_j\in\mathbb{R}} \left(z_j + n\lambda_j \mathbb{E}\left[\ell_j(-X_j - z_j)\right]\right) \\ &\leq -\operatorname{ess\,inf} X_j - n\lambda_j g_j\left(\frac{1}{n\lambda_j}\right) \end{aligned}$$

for each $n \in \mathbb{N}$. This can be checked with a similar calculation as in (4.6). Since g_j is convex and lower semicontinuous, the restriction of g_j to $\operatorname{cl}\operatorname{dom} g_j$ is a continuous function (see [51, Proposition 2.1.6]), we obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \inf_{z_j\in\mathbb{R}} \left(z_j + n\lambda_j \mathbb{E}\left[\ell_j(-X_j - z_j)\right]\right) &= -\lim_{n\to\infty} \lambda_j g_j\left(\frac{1}{n\lambda_j}\right) \\ &= -\lambda_j g_j(0) \\ &= \lambda_j \inf_{y\in\mathbb{R}} \ell_j(y) \end{aligned}$$

whenever $v_j = 0$, $\lambda_j > 0$ and $g_j(0) \in \mathbb{R}$. Using the first two equalities in (4.17) and taking the limit in (4.18) as $n \to \infty$, we finally obtain

$$\begin{aligned}\langle \eta, v\rangle \geq & \sum_{\substack{i\colon v_i>0 \\ i\colon v_i=\lambda_i=0}} \inf_{z_i\in\mathbb{R}} \left(v_i z_i + \lambda_i \mathbb{E}\left[\ell_i(-X_i - z_i)\right]\right) \\ & + \sum_{i\colon v_i=0,\, \lambda_i>0} \lambda_i \inf_{y\in\mathbb{R}} \ell_i(y) + \inf_{x\in -x^0+C} \langle x, \lambda\rangle \\ = & \sum_{\substack{i\colon v_i>0 \\ i\colon v_i=\lambda_i=0}} \inf_{z_i\in\mathbb{R}} \left(v_i z_i + \lambda_i \mathbb{E}\left[\ell_i(-X_i - z_i)\right]\right) \\ & + \sum_{i\colon v_i=0,\, \lambda_i>0} \inf_{z_i\in\mathbb{R}} \left(v_i z_i + \lambda_i \mathbb{E}\left[\ell_i(-X_i - z_i)\right]\right) + \inf_{x\in -x^0+C} \langle x, \lambda\rangle \\ = & \sum_{i=1}^{m} \inf_{z_i\in\mathbb{R}} \left(v_i z_i + \lambda_i \mathbb{E}\left[\ell_i(-X_i - z_i)\right]\right) + \inf_{x\in -x^0+C} \langle x, \lambda\rangle ,\end{aligned}$$

that is, $\eta \in K(\lambda, v)$. Hence, (4.16) holds. Finally, note that $M(r, w) = \mathbb{R}^m$ whenever $\langle r, w\rangle = 0$; thus, we also have

$$\bigcap_{\lambda, v\in\mathbb{R}^m_+\setminus\{0\}} K(\lambda, v) = \bigcap_{r, w\in\mathbb{R}^m_+\setminus\{0\}} M(r, w).$$

By Proposition 4.3.4, the result follows. □

Given $r \in \mathbb{R}^m_+\setminus\{0\}$, define a set-valued function $D_{g,r}\colon L^\infty_m \to \mathcal{G}(\mathbb{R}^m_+)$ by

$$D_{g,r}(X) := \bigcap_{w\in\mathbb{R}^m_+\setminus\{0\}} \left\{ z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \langle \delta_{g,r}(X), w\rangle + \inf_{x\in C} \langle r \circ x, w\rangle \right\} \tag{4.19}$$

for $X \in L^\infty_m$.

Theorem 4.3.8. *For every $X \in L^\infty_m$,*

$$R_\ell(X) = \bigcap_{r\in 1/\operatorname{dom} g} D_{g,r}(X), \tag{4.20}$$

where

$$1/\operatorname{dom} g := \bigtimes_{i=1}^{m} 1/\operatorname{dom} g_i = \bigtimes_{i=1}^{m} \left\{ \frac{1}{z_i} \mid 0 \neq z_i \in \operatorname{dom} g_i \right\}. \tag{4.21}$$

Proof. By Theorem 4.1.9, we have $\delta_{g,r}(X) \in \mathbb{R}^m$ for all $X \in L^\infty_m$ if and only if $r \in 1/\operatorname{dom} g$. So $r \notin 1/\operatorname{dom} g$ implies $D_{g,r} \equiv \mathbb{R}^m$. The result directly follows from Proposition 4.3.7. □

Definition 4.3.9. *For $r \in 1/\operatorname{dom} g$, $D_{g,r}$ defined by* (4.19) *is called the multivariate* (g,r)*-divergence risk measure on L^∞_m with threshold level x^0 and threshold set C.*

Theorem 4.3.8 states that the multivariate ℓ-shortfall risk measure is the supremum of all multivariate (g,r)-divergence risk measures with $r \in 1/\operatorname{dom} g$; recall (2.2) for the infimum and supremum formulae for the complete lattice $(\mathcal{G}(\mathbb{R}^m_+), \supseteq)$. In general, there is no single $r \in 1/\operatorname{dom} g$ which yields this supremum, that is, the supremum is not attained in a single argument. Instead, one could look for a set $\Gamma \subseteq 1/\operatorname{dom} g$ such that (4.20) holds with $1/\operatorname{dom} g$ replaced by Γ and each $D_{g,r}(X)$ for $r \in \Gamma$ is a maximal element of the set $\{D_{g,r}(X) \mid r \in 1/\operatorname{dom} g\}$ with respect to $\supseteq$. This corresponds to the solution concept for set optimization problems introduced in [32] (see also [28, Definition 3.3]) and will be discussed for the entropic risk measure in Section 4.4.1.

Since Proposition 4.3.7 provides a multivariate divergence risk measure $D_{g,r}$ in terms of the vector $\delta_{g,r}$ of scalar divergence risk measures, in Proposition 4.3.10 below, we are also able to give a formula that relates the corresponding set-valued and scalar minimal penalty functions.

Note that the set of dual variables defined by (3.5) reduces to

$$\mathcal{W}_{m,m} = \mathcal{M}_m(\mathbb{P}) \times \left(\mathbb{R}^m_+ \backslash \{0\}\right),$$

as we work with the case $d = m$ and $p = \infty$.

Proposition 4.3.10. *For every $r \in 1/\operatorname{dom} g$ and $X \in L_m^\infty$,*

$$\begin{aligned} D_{g,r}(X) &= \delta_{g,r}(X) + r \circ C \\ &= \inf_{(\mathcal{G}(\mathbb{R}^m_+),\supseteq)} \left\{ -z + r \circ (\mathbb{E}\left[\ell(-X+z)\right] - x^0 + C) \mid z \in \mathbb{R}^m \right\}, \end{aligned}$$

where $r \circ A := \{r \circ x \mid x \in A\}$ for a set $A \subseteq \mathbb{R}^m$. Moreover, $D_{g,r}$ is a weak-closed convex multivariate risk measure with minimal penalty function $-\alpha^{\min}_{D_{g,r}}$ given by*

$$-\alpha^{\min}_{D_{g,r}}(\mathbb{Q}, w) = \left\{ z \in \mathbb{R}^m \mid \langle z, w\rangle \geq -\langle \mathcal{I}_{g,r}(\mathbb{Q} \,|\, \mathbb{P}), w\rangle + \inf_{x \in -x^0 + C} \langle r \circ x, w\rangle \right\} \quad (4.22)$$

for each $(\mathbb{Q}, w) \in \mathcal{M}_m(\mathbb{P}) \times (\mathbb{R}^m_+ \backslash \{0\})$.

Proof. Let $X \in L_m^\infty$. We have $\delta_{g,r}(X) \in \mathbb{R}^m$ so that

$$\begin{aligned} D_{g,r}(X) &= \delta_{g,r}(X) + \bigcap_{w \in \mathbb{R}^m_+ \backslash \{0\}} \left\{ z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \inf_{x \in C} \langle r \circ x, w\rangle \right\} \\ &= \delta_{g,r}(X) + \bigcap_{w \in \mathbb{R}^m_+ \backslash \{0\}} \left\{ z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \inf_{x \in r \circ C} \langle x, w\rangle \right\} \\ &= \delta_{g,r}(X) + r \circ C, \end{aligned}$$

where the last equality is due to $C \in \mathcal{G}(\mathbb{R}^m_+)$ and the fact that a closed convex set is the intersection of all of its supporting halfspaces; see [28, (5.2)]. This representation also guarantees that $D_{g,r}(X) \notin \{\emptyset, \mathbb{R}^m\}$. The second representation of $D_{g,r}(X)$ as a set-valued infimum is obvious from the definition of $\delta_{g,r}(X)$ given by Definition 4.3.6.

Let $w \in \mathbb{R}^m_+ \backslash \{0\}$ and $M(r,w)$ as in (4.15). For the moment, let us denote by $-\alpha$ the function defined by the right hand side of (4.22). By (4.14), we have

$$\begin{aligned}
M(r,w) &= \Big\{ z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \sum_{i=1}^{m} \sup_{\mathbb{Q}_i \in \mathcal{M}_1(\mathbb{P})} \big(w_i \mathbb{E}^{\mathbb{Q}_i}[-X_i] - w_i \mathcal{I}_{g_i, r_i}(\mathbb{Q}_i \,|\, \mathbb{P})\big) \\
&\qquad\qquad + \inf_{x \in -x^0 + C} \langle r \circ x, w\rangle \Big\} \\
&= \bigcap_{\mathbb{Q} \in \mathcal{M}_m(\mathbb{P})} \Big\{ z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \big\langle \mathbb{E}^{\mathbb{Q}}[-X] - \mathcal{I}_{g,r}(\mathbb{Q} \,|\, \mathbb{P}), w \big\rangle \\
&\qquad\qquad + \inf_{x \in -x^0 + C} \langle r \circ x, w\rangle \Big\} \\
&= \bigcap_{\mathbb{Q} \in \mathcal{M}_m(\mathbb{P})} \big(-\alpha(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}}[-X] \big).
\end{aligned}$$

Hence,

$$D_{g,r}(X) = \bigcap_{w \in \mathbb{R}^m_+ \backslash \{0\}} M(r,w) = \bigcap_{(\mathbb{Q}, w) \in \mathcal{M}_m(\mathbb{P}) \times (\mathbb{R}^m_+ \backslash \{0\})} \big(-\alpha(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}}[-X] \big).$$

This representation of $D_{g,r}$ guarantees monotonicity, translativity, convexity and weak*-closedness. Note that $-\alpha$ is a penalty function in the sense of Definition 3.1.5 because

$$\bigcap_{(\mathbb{Q}, w) \in \mathcal{M}_m(\mathbb{P}) \times (\mathbb{R}^m_+ \backslash \{0\})} -\alpha(\mathbb{Q}, w) = D_{g,r}(0) \notin \{\emptyset, \mathbb{R}^m\}.$$

Finally, we show that $-\alpha$ is indeed the minimal penalty function of $D_{g,r}$. By the definition of minimal penalty function, we obtain, for $\mathbb{Q} \in \mathcal{M}_m(\mathbb{P})$, $w \in \mathbb{R}^m_+ \backslash \{0\}$,

$$
\begin{aligned}
&-\alpha^{\min}_{D_{g,r}}(\mathbb{Q}, w) \\
&= \operatorname{cl} \bigcup_{X \in L^\infty_m} \left(\mathbb{E}^{\mathbb{Q}}[X] + H(w) + D_{g,r}(X)\right) \\
&= \operatorname{cl} \bigcup_{X \in L^\infty_m} \left(\mathbb{E}^{\mathbb{Q}}[X] + \delta_{g,r}(X) + r \circ C + H(w)\right) \\
&= \operatorname{cl} \bigcup_{X \in L^\infty_m} \left(\mathbb{E}^{\mathbb{Q}}[X] + \delta_{g,r}(X) + \left\{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \inf_{x \in C} \langle r \circ x, w\rangle\right\}\right) \\
&= \left\{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \inf_{X \in L^\infty_m} \left\langle \mathbb{E}^{\mathbb{Q}}[X] + \delta_{g,r}(X), w\right\rangle + \inf_{x \in C} \langle r \circ x, w\rangle\right\} \\
&= \left\{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \sum_{i=1}^m w_i \inf_{X_i \in L^\infty} \left(\mathbb{E}^{\mathbb{Q}_i}[X_i] + \delta_{g_i,r_i}(X_i)\right) + \inf_{x \in C} \langle r \circ x, w\rangle\right\} \\
&= \left\{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \sum_{i=1}^m w_i(-\mathcal{I}_{g_i,r_i}(\mathbb{Q}_i \,|\, \mathbb{P}) - r_i x^0_i) + \inf_{x \in C} \langle r \circ x, w\rangle\right\} \\
&= -\alpha(\mathbb{Q}, w),
\end{aligned}
$$

where the sixth equality follows from the analogous scalar result established in Proposition 4.1.12. Hence, $-\alpha^{\min}_{D_{g,r}} = -\alpha$ and (4.22) holds. □

Remark 4.3.11. Let us give a financial interpretation for $D_{g,r}(X)$ using the representation given in Proposition 4.3.10 for fixed $r \in 1/\operatorname{dom} g$ and $X \in L^\infty_m$. Suppose that there is an investor with the random portfolio X who wants to choose a deterministic portfolio $z \in \mathbb{R}^m$ to be received at initial time. Hence, she will hold $X - z$ at terminal time which corresponds to the expected loss vector $\mathbb{E}[\ell(-X + z)]$. The deterministic portfolios are compared with respect to the componentwise ordering cone $\mathbb{R}^m_+$ and the expected loss vectors are compared with respect to the set $D = -x^0 + C$ as discussed after Definition 4.3.2. With these comparison rules, suppose that the investor wants to maximize the immediate consumption z and minimize the expected loss $\mathbb{E}[\ell(-X + z)]$ at the same time, or equivalently, minimize both $-z$ and $\mathbb{E}[\ell(-X + z)]$ over $z \in \mathbb{R}^m$.

This can be formulated as the following set minimization problem where the objective function maps into $\mathcal{G}(\mathbb{R}^{2m}_+)$:

$$\text{minimize} \quad \begin{pmatrix} -z + \mathbb{R}^m_+ \\ \mathbb{E}\left[\ell(-X+z)\right] - x^0 + C \end{pmatrix} \quad \text{subject to} \quad z \in \mathbb{R}^m.$$

Considering r_i as the "relative weight" of $\mathbb{E}\left[\ell(-X_i+z_i)\right]$ with respect to $-z_i$ for each $i \in \{1,\dots,m\}$, we construct the "partially scalarized" problem

$$\text{minimize} \quad -z + \mathbb{R}^m_+ + r \circ \left(\mathbb{E}\left[\ell(-X+z)\right] - x^0 + C\right) \quad \text{subject to} \quad z \in \mathbb{R}^m.$$

Proposition 4.3.10 shows that the value of this (still set-valued) problem is $D_{g,r}(X)$.

We end this section by formulating the relationship between the minimal penalty functions of multivariate shortfall and divergence risk measures.

Proposition 4.3.12. *If $\mathbb{Q} \in \mathcal{M}_m(\mathbb{P})$ and $w \in \mathbb{R}^m_{++}$, then*

$$\begin{aligned} &-\alpha^{\min}_{R_\ell}(\mathbb{Q},w) \\ &= \left\{ z \in \mathbb{R}^m \mid \langle z,w\rangle \geq \sup_{r \in \mathbb{R}^m_+ \setminus \{0\}} \left(-\langle \mathcal{I}_{g,r}(\mathbb{Q} \,|\, \mathbb{P}), w\rangle + \inf_{x \in -x^0 + C} \langle r \circ x, w\rangle \right) \right\} \\ &= \bigcap_{r \in 1/\operatorname{dom} g} -\alpha^{\min}_{D_{g,r}}(\mathbb{Q},w). \end{aligned} \tag{4.23}$$

Proof. By the definition of minimal penalty function, we obtain, for $\mathbb{Q} \in \mathcal{M}_m(\mathbb{P})$, $w \in \mathbb{R}^m_{++}$,

$$
\begin{aligned}
-\alpha^{\min}_{R_\ell}(\mathbb{Q}, w) &= \operatorname{cl} \bigcup_{X \in L^\infty_m} \left(\mathbb{E}^{\mathbb{Q}}[X] + H(w) + R_\ell(X) \right) \\
&= \operatorname{cl} \bigcup_{X \in L^\infty_m} \Bigg(\mathbb{E}^{\mathbb{Q}}[X] + H(w) \\
&\qquad\qquad + \operatorname{cl} \bigcup_{z \in \mathbb{R}^m} \left\{ z + \mathbb{R}^m_+ \mid 0 \in \mathbb{E}\left[\ell(-X - z)\right] - x^0 + C \right\} \Bigg) \\
&= \operatorname{cl} \bigcup_{z \in \mathbb{R}^m} \bigcup_{X \in L^\infty_m} \left\{ z + \mathbb{E}^{\mathbb{Q}}[X] + H(w) \mid 0 \in \mathbb{E}\left[\ell(-X - z)\right] - x^0 + C \right\} \\
&= \operatorname{cl} \bigcup_{X \in L^\infty_m} \left\{ \mathbb{E}^{\mathbb{Q}}[-X] + H(w) \mid 0 \in \mathbb{E}\left[\ell(X)\right] - x^0 + C \right\} \\
&= \inf_{(\mathcal{G}(\mathbb{R}^m_+), \supseteq)} \left\{ \mathbb{E}^{\mathbb{Q}}[-X] + H(w) \mid 0 \in \mathbb{E}\left[\ell(X)\right] - x^0 + C,\, X \in L^\infty_m \right\}.
\end{aligned}
$$

Next, we compute the value of the Lagrangian dual problem for this convex set-valued minimization problem using the approach of [28]. Denote by $L\colon L^\infty_m \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \backslash \{0\} \to \mathcal{G}(\mathbb{R}^m_+)$ the Lagrangian and by $K\colon L^\infty_m \times \mathbb{R}^m_+ \backslash \{0\} \to \mathcal{G}(\mathbb{R}^m_+)$ the Lagrangian dual objective function. For $X \in L^\infty_m$, $\lambda \in \mathbb{R}^m_+$, $v \in \mathbb{R}^m_+ \backslash \{0\}$, we have $L(X, \lambda, v) = \mathbb{R}^m$ if $v \notin \{sw \mid s > 0\}$ and

$$
\begin{aligned}
L(X, \lambda, v) &= \mathbb{E}^{\mathbb{Q}}[-X] + H(sw) + \left\{ z \in \mathbb{R}^m \mid \langle z, sw \rangle \geq \langle \mathbb{E}\left[\ell(X)\right], \lambda \rangle + \inf_{x \in -x^0 + C} \langle x, \lambda \rangle \right\} \\
&= \left\{ z \in \mathbb{R}^m \mid \langle z, sw \rangle \geq \left\langle \mathbb{E}^{\mathbb{Q}}[-X], sw \right\rangle + \langle \mathbb{E}\left[\ell(X)\right], \lambda \rangle + \inf_{x \in -x^0 + C} \langle x, \lambda \rangle \right\}
\end{aligned}
$$

whenever $v = sw$ for some $s > 0$. Observe $H(sw) = H(w)$ for every $s > 0$.

Hence, for $\lambda \in \mathbb{R}^m_+$ and $s > 0$,

$$K(\lambda, sw) = \Big\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \inf_{X \in L^\infty_m} \left(\left\langle \mathbb{E}^{\mathbb{Q}}\left[-X\right], w \right\rangle + \left\langle \mathbb{E}\left[\ell(X)\right], s^{-1}\lambda \right\rangle \right) + \inf_{x \in -x^0 + C} \left\langle x, s^{-1}\lambda \right\rangle \Big\}$$

$$= K\left(s^{-1}\lambda, w\right).$$

The dual problem is

$$\sup_{(\mathcal{G}(\mathbb{R}^m_+), \supseteq)} \left\{ K\left(s^{-1}\lambda, w\right) \mid s > 0,\ \lambda \in \mathbb{R}^m_+ \right\} = \sup_{(\mathcal{G}(\mathbb{R}^m_+), \supseteq)} \left\{ K\left(\lambda, w\right) \mid \lambda \in \mathbb{R}^m_+ \right\}.$$

Since $w_i > 0$ for every $i \in \{1, \ldots, m\}$ by assumption, we have

$$K\left(\lambda, w\right) = M(r, w),$$

where $r_i = \frac{\lambda_i}{w_i}$, $i \in \{1, \ldots, m\}$, and

$$M(r, w) := \Big\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \inf_{X \in L^\infty_m} \left(\left\langle \mathbb{E}^{\mathbb{Q}}\left[-X\right], w \right\rangle + \langle r \circ \mathbb{E}\left[\ell(X)\right], w \rangle \right) + \inf_{x \in -x^0 + C} \langle r \circ x, w \rangle \Big\}.$$

Note that

$$\begin{aligned}
&\inf_{X \in L^\infty_m} \left(\left\langle \mathbb{E}^{\mathbb{Q}}\left[-X\right], w \right\rangle + \langle r \circ \mathbb{E}\left[\ell(X)\right], w \rangle \right) \\
&= \sum_{i=1}^m w_i \inf_{X_i \in L^\infty} \left(\mathbb{E}\left[-\frac{d\mathbb{Q}_i}{d\mathbb{P}} X_i \right] + r_i \mathbb{E}\left[\ell_i(X_i)\right] \right) \\
&= \sum_{i=1}^m w_i \mathbb{E}\left[\inf_{x_i \in \mathbb{R}} \left(-\frac{d\mathbb{Q}_i}{d\mathbb{P}} x_i + r_i \ell_i(x_i) \right) \right] \\
&= \langle \mathcal{I}_{g,r}(\mathbb{Q} \,|\, \mathbb{P}), w \rangle.
\end{aligned}$$

Therefore, the value of the dual problem equals the middle term in (4.23). Note that the set-valued version of Slater's condition holds, that is, there exists $\bar{X} \in L_m^\infty$ such that $(\mathbb{E}\left[\ell(\bar{X})\right] - x^0 + C) \cap -\mathbb{R}^m_{++} \neq \emptyset$. This is immediate from the scalar version as in the proof of Proposition 4.1.13. Hence, the first equality in (4.23) holds by the set-valued strong duality result [28, Theorem 6.6]. Since $\mathcal{I}_{g,r}(\mathbb{Q}\,|\,\mathbb{P}) \notin \mathbb{R}^m$ for $r \notin 1/\operatorname{dom} g$, we also have the second equality in (4.23). □

4.4 Examples

4.4.1 Multivariate entropic risk measures

In this section, we assume that the vector loss function ℓ of Section 4.3 is the *vector exponential loss function* with constant risk aversion vector $\beta \in \mathbb{R}^m_{++}$, that is, for each $i \in \{1, \dots, m\}$, we assume

$$\forall x \in \mathbb{R}\colon\ \ell_i(x) = \frac{e^{\beta_i x} - 1}{\beta_i},$$

which satisfies the conditions in Definition 4.1.1. The corresponding vector divergence function g is given by

$$\forall y \in \mathbb{R}\colon\ g_i(y) = \frac{y}{\beta_i}\log y - \frac{y}{\beta_i} + \frac{1}{\beta_i},$$

for each $i \in \{1, \dots, m\}$. Here and elsewhere, we make the convention $\log y = -\infty$ for every $y \leq 0$.

For convenience, let us define $x^{-1} \coloneqq (x_1^{-1}, \dots, x_m^{-1})$ and $\log x \coloneqq (\log x_1, \dots, \log x_m)$ for $x \in \mathbb{R}^m_{++}$, and $\log[A] \coloneqq \{\log x \mid x \in A\}$ for $A \subseteq \mathbb{R}^m_{++}$. We will also use $1 \coloneqq (1, \dots, 1)$ as an element of $\mathbb{R}^m$.

Let $x^0 \in \operatorname{int}\ell(\operatorname{dom}\ell) = -\beta^{-1} + \mathbb{R}^m_{++}$ and $C \in \mathcal{G}(\mathbb{R}^m_+)$ with $0 \in \mathbb{R}^m$ being a boundary point of C. We call the corresponding multivariate ℓ-shortfall risk measure $R^{\text{ent}} := R_\ell$ the *multivariate entropic risk measure* (with threshold level x^0 and threshold set C). The next proposition shows that R^{ent} has the simple form of "a vector-valued function plus a fixed set," which is, in general, not the case for an arbitrary loss function. Note that the functional ρ^{ent} in Proposition 4.4.1 is the vector of scalar entropic risk measures.

Proposition 4.4.1. *For every $X \in L_m^\infty$,*

$$R^{\text{ent}}(X) = \rho^{\text{ent}}(X) + C^{\text{ent}},$$

where

$$\begin{aligned} \rho^{\text{ent}}(X) &:= \left[\frac{1}{\beta_i}\log \mathbb{E}\left[e^{-\beta_i X_i}\right]\right]_{i=1}^m, \\ C^{\text{ent}} &:= -\beta^{-1}\circ \log\left[(1+\beta\circ(x^0 - C))\cap \mathbb{R}^m_{++}\right]. \end{aligned} \tag{4.24}$$

Proof. Using the definitions, we have

$$\begin{aligned} R^{\text{ent}}(X) &= \left\{z\in\mathbb{R}^m \mid \mathbb{E}\left[\ell(-X-z)\right] \in x^0 - C\right\} \\ &= \left\{z\in\mathbb{R}^m \mid \exists c\in C\ \forall i\in\{1,\dots,m\}:\ \frac{\mathbb{E}\left[e^{\beta_i(-X_i-z_i)}\right]-1}{\beta_i} = x_i^0 - c_i\right\} \\ &= \left\{z\in\mathbb{R}^m \mid \exists c\in C\ \forall i\in\{1,\dots,m\}:\ e^{-\beta_i z_i}\mathbb{E}\left[e^{-\beta_i X_i}\right] = 1+\beta_i(x_i^0 - c_i)\right\} \\ &= \Bigg\{z\in\mathbb{R}^m \mid \exists c\in C\ \forall i\in\{1,\dots,m\}:\ z_i = \frac{1}{\beta_i}\log\frac{\mathbb{E}\left[e^{-\beta_i X_i}\right]}{1+\beta_i(x_i^0-c_i)}, \\ &\qquad\qquad 1+\beta_i(x_i^0-c_i)\in\mathbb{R}_{++}\Bigg\} \\ &= \rho^{\text{ent}}(X) + C^{\text{ent}}. \end{aligned}$$

□

Note that the set $1/\operatorname{dom} g$ defined in (4.21) becomes $\mathbb{R}^m_{++}$. For $r \in \mathbb{R}^m_{++}$, let $D^{\text{ent}}_r := D_{g,r}$ be the multivariate (g,r)-divergence risk measure defined by Definition 4.3.9.

Proposition 4.4.2. *For every $r \in \mathbb{R}^m_{++}$ and $X \in L^\infty_m$,*

$$D^{\text{ent}}_r(X) = \rho^{\text{ent}}(X) + \bigcap_{w \in \mathbb{R}^m_+ \setminus \{0\}} \Big\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \big\langle \beta^{-1} \circ (1 - r + \log r), w \big\rangle + \inf_{x \in -x^0 + C} \langle r \circ x, w \rangle \Big\},$$

where $\rho^{\text{ent}}(X)$ is defined by (4.24)*.*

Proof. Recalling (4.19), it holds

$$D^{\text{ent}}_r(X) = \bigcap_{w \in \mathbb{R}^m_+ \setminus \{0\}} \Big\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \langle \delta_{g,r}(X), w \rangle + \inf_{x \in C} \langle r \circ x, w \rangle \Big\},$$

where, for each $i \in \{1, \ldots, m\}$,

$$\begin{aligned} \delta_{g_i, r_i}(X_i) &= \inf_{z_i \in \mathbb{R}} \left(z_i + r_i \mathbb{E}\left[\ell_i(-X_i - z_i) \right] \right) - r_i x^0_i \\ &= \frac{1}{\beta_i} \log \mathbb{E}\left[e^{-\beta_i X_i} \right] + \frac{1}{\beta_i}(1 - r_i + \log r_i) - r_i x^0_i \in \mathbb{R}. \end{aligned}$$

The result follows. □

Recall from (4.20) that $R^{\text{ent}}(\cdot)$ is the supremum of all $D^{\text{ent}}_r(\cdot)$ with $r \in \mathbb{R}^m_{++}$, that is, for $X \in L^\infty_m$,

$$R^{\text{ent}}(X) = \sup_{(\mathcal{G}(\mathbb{R}^m_+), \supseteq)} \left\{ D^{\text{ent}}_r(X) \mid r \in \mathbb{R}^m_{++} \right\} = \bigcap_{r \in \mathbb{R}^m_{++}} D^{\text{ent}}_r(X). \tag{4.25}$$

If $m = 1$, then the only choice for C is $\mathbb{R}_+$. In this case, one can check that, for $X \in L^\infty$,

$$R^{\text{ent}}(X) = D^{\text{ent}}_1(X) = \rho^{\text{ent}}(X) - x^0 + \mathbb{R}_+.$$

In other words, the supremum in (4.25) is *attained* at $r = 1$. In general, when $m \geq 2$, we may not be able to find some $\bar{r} \in \mathbb{R}_{++}$ for which $R^{\text{ent}}(X) = D^{\text{ent}}_{\bar{r}}(X)$. Instead, we will compute a *solution* to this set maximization problem in the sense of [28, Definition 3.3], that is, we will find a set $\Gamma \subseteq \mathbb{R}^m_{++}$ such that

(i) $R^{\text{ent}}(X) = \bigcap_{r\in\Gamma} D^{\text{ent}}_r(X)$,

(ii) for each $\bar{r} \in \Gamma$, $D^{\text{ent}}_{\bar{r}}(X)$ is a *maximal element* of the collection $\{D^{\text{ent}}_r(X) \mid r \in \mathbb{R}^m_{++}\}$ in the following sense:

$$\forall r \in \mathbb{R}^m_{++}: \quad D^{\text{ent}}_r(X) \subseteq D^{\text{ent}}_{\bar{r}}(X) \;\Rightarrow\; r = \bar{r}.$$

Moreover, the set Γ will be independent of the choice of X. Indeed, if $r \in \mathbb{R}^m_{++}$ and $X \in L^\infty_m$, we can rewrite $D^{\text{ent}}_r(X)$ as

$$D^{\text{ent}}_r(X) = \rho^{\text{ent}}(X) + \bigcap_{w\in\mathbb{R}^m_+\backslash\{0\}} \{z \in \mathbb{R}^m \mid w^\mathsf{T} z \geq -(f_w(r) + h_w(r))\},$$

where, for $w \in \mathbb{R}^m_+\backslash\{0\}$, $r \in \mathbb{R}^m_{++}$,

$$f_w(r) := \left\langle -\beta^{-1} \circ (1 - r + \log r) + r \circ x^0, w \right\rangle$$
$$h_w(r) := -\inf_{x\in C} \langle x, r \circ w\rangle = \sup_{x\in -C} \langle x, r \circ w\rangle.$$

Lemma 4.4.3. *Let $w \in \mathbb{R}^m_+\backslash\{0\}$. The function $f_w + h_w$ on $\mathbb{R}^m_{++}$ is either identically $+\infty$ or else it attains its infimum at a unique point $r^w \in \mathbb{R}^m_{++}$ which is determined by the following property: r^w is the only vector $r \in \mathbb{R}^m_{++}$ for which $-x^0 + C$ is supported at the point*

$$\beta^{-1} \circ \left(1 - r^{-1}\right)$$

by the hyperplane with normal direction $r \circ w$.

Proof. First, we extend f_w and h_w from $\mathbb{R}^m_{++}$ to $\mathbb{R}^m$ with their original definitions so that we have

$$\inf_{r\in\mathbb{R}^m_{++}} \left(f_w(r) + h_w(r)\right) = \inf_{r\in\mathbb{R}^m} \left(f_w(r) + h_w(r)\right).$$

Note that f_w is a proper, strictly convex, continuous function and has a unique minimum point. Hence, by [44, Theorem 27.1(d)], f_w has no directions of recession, that is, the recession function f_w0^+ of f_w always takes strictly positive values; see [44, p. 66 and p. 69] for definitions. Besides, h_w is a proper, convex, lower semicontinuous function. If $h_w \equiv +\infty$, then the infimum of $f_w + h_w$ is $+\infty$. Suppose that h_w is a proper function. Since $0 \in \mathbb{R}^m$ is a boundary point of $-C$, the function h_w always takes nonnegative values. Hence, the infimum of h_w is finite. By [44, Theorem 27.1(a), (i)], this implies that the recession function h_w0^+ of h_w always takes nonnegative values. Therefore, $f_w + h_w$ has no directions of recession since $(f_w + h_w)0^+ = f_w0^+ + h_w0^+$ by [44, Theorem 9.3]. Hence, by [44, Theorem 27.1(b), (d)] and the strict convexity of $f_w + h_w$, this function has a unique minimum point $r^w \in \mathbb{R}^m_{++}$ which is determined by the first order condition

$$\begin{aligned} 0 \in \partial(f_w + h_w)(r^w) = & w \circ \left(\beta^{-1} \circ (1 - (r^w)^{-1}) + x^0\right) \\ & + \left\{ w \circ \eta \mid \eta \in -C,\ \sup_{x\in -C} \langle x, r^w \circ w\rangle = \langle \eta, r^w \circ w\rangle \right\}, \end{aligned}$$

that is,

$$\begin{aligned} &\beta^{-1} \circ (1 - (r^w)^{-1}) \in -x^0 + C, \\ &\inf_{x\in -x^0+C} \langle x, r^w \circ w\rangle = \left\langle \beta^{-1} \circ (1 - (r^w)^{-1}), r^w \circ w \right\rangle, \end{aligned}$$

which form the claimed property of r^w. □

Proposition 4.4.4. *Using the notation in Lemma 4.4.3, the set*

$$\Gamma := \left\{ r^w \mid w \in \mathbb{R}^m_{+}\backslash\{0\},\ f_w + h_w \text{ is proper} \right\}$$

is a solution to the maximization problem in (4.25) *for every* $X \in L_m^\infty$.

Proof. By Lemma 4.4.3 , it is clear that, for each $w \in \mathbb{R}_+^m \backslash \{0\}$, we have

$$\inf_{r\in\mathbb{R}_{++}^m} (f_w(r) + h_w(r)) = \inf_{r\in\Gamma} (f_w(r) + h_w(r)) = f_w(r^w) + h_w(r^w).$$

Hence,

$$\begin{aligned}
R^{\text{ent}}(X) &= \bigcap_{r\in\mathbb{R}_{++}^m} D_r^{\text{ent}}(X) \\
&= \rho^{\text{ent}}(X) + \bigcap_{w\in\mathbb{R}_+^m\backslash\{0\},\, r\in\mathbb{R}_{++}^m} \{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq -(f_w(r) + h_w(r))\} \\
&= \rho^{\text{ent}}(X) + \bigcap_{w\in\mathbb{R}_+^m\backslash\{0\}} \left\{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq -\inf_{r\in\mathbb{R}_{++}^m} (f_w(r) + h_w(r))\right\} \\
&= \rho^{\text{ent}}(X) + \bigcap_{w\in\mathbb{R}_+^m\backslash\{0\}} \left\{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq -\inf_{r\in\Gamma} (f_w(r) + h_w(r))\right\} \\
&= \bigcap_{r\in\Gamma} D_r^{\text{ent}}(X).
\end{aligned}$$

Let $w \in \mathbb{R}_+^m\backslash\{0\}$ such that $f_w + h_w$ is proper and let $r \in \mathbb{R}_{++}^m$. Suppose that $D_r^{\text{ent}}(X) \subseteq D_{r^w}^{\text{ent}}(X)$. Then,

$$-(f_w(r) + h_w(r)) = \inf_{z\in D_r^{\text{ent}}(X)} \langle z, w\rangle \geq \inf_{z\in D_{r^w}^{\text{ent}}(X)} \langle z, w\rangle = -(f_w(r^w) + h_w(r^w)),$$

that is, $f_w(r) + h_w(r) \leq f_w(r^w) + h_w(r^w)$. By Lemma 4.4.3, this implies that $r = r^w$. □

Finally, we compute the minimal penalty function of R^{ent} in terms of the *vector relative entropies*

$$\mathcal{H}(\mathbb{Q}\,|\,\mathbb{P}) := \left[\mathbb{E}^{\mathbb{Q}_i}\left[\log\frac{d\mathbb{Q}_i}{d\mathbb{P}}\right]\right]_{i=1}^m$$

of vector probability measures $\mathbb{Q} \in \mathcal{M}_m(\mathbb{P})$.

Proposition 4.4.5. *For every* $(\mathbb{Q}, w) \in \mathcal{M}_m(\mathbb{P}) \times (\mathbb{R}^m_+ \backslash \{0\})$, *we have* $-\alpha^{\min}_{R^{\text{ent}}}(\mathbb{Q}, w) = \mathbb{R}^m$ *if* $h_w \equiv +\infty$ *and*

$$-\alpha^{\min}_{R^{\text{ent}}}(\mathbb{Q}, w) = -\beta^{-1} \circ \left(\mathcal{H}(\mathbb{Q} \,|\, \mathbb{P}) + \log\left[(1 + \beta \circ (x^0 - C)) \cap \mathbb{R}^m_{++}\right]\right) + H(w)$$

if h_w *is a proper function.*

Proof. Proposition 4.3.12 and Lemma 4.4.3 give

$$\begin{aligned}
& -\alpha^{\min}_{R_\ell}(\mathbb{Q}, w) \\
&= \bigcap_{r \in 1/\operatorname{dom} g} -\alpha^{\min}_{D_{g,r}}(\mathbb{Q}, w) \\
&= \bigcap_{r \in 1/\operatorname{dom} g} \left\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq -\langle \mathcal{I}_{g,r}(\mathbb{Q} \,|\, \mathbb{P}), w \rangle + \inf_{x \in -x^0 + C} \langle r \circ x, w \rangle \right\} \\
&= -\beta^{-1} \circ \mathcal{H}(\mathbb{Q} \,|\, \mathbb{P}) \\
&\quad + \bigcap_{r \in \mathbb{R}^m_{++}} \left\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \sum_{i=1}^m \frac{w_i}{\beta_i}(1 - r_i + \log r_i) + \inf_{x \in -x^0 + C} \langle r \circ x, w \rangle \right\} \\
&= -\beta^{-1} \circ \mathcal{H}(\mathbb{Q} \,|\, \mathbb{P}) + \bigcap_{r \in \mathbb{R}^m_{++}} \{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq -(f_w(r) + h_w(r)) \} \\
&= -\beta^{-1} \circ \mathcal{H}(\mathbb{Q} \,|\, \mathbb{P}) + \{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq -(f_w(r^w) + h_w(r^w)) \} \\
&= -\beta^{-1} \circ \mathcal{H}(\mathbb{Q} \,|\, \mathbb{P}) \\
&\quad + \left\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \sum_{i=1}^m \frac{w_i}{\beta_i}(1 - r^w_i + \log r^w_i) + \inf_{x \in -x^0 + C} \langle r \circ x, w \rangle \right\},
\end{aligned}$$

assuming that h_w is not identically $+\infty$ (otherwise $-\alpha^{\min}_{R_\ell}(\mathbb{Q}, w) = \mathbb{R}^m$). □

Thus, the minimal penalty function for the multivariate entropic risk measure is of the form "- vector relative entropy + a nonhomogeneous halfspace" (except for the trivial case).

4.4.2 Multivariate average value at risks

In this section, we assume that the vector loss function ℓ of Section 4.3 is the *(vector) scaled positive part function* with scaling vector $\alpha \in (0,1]^m$, that is, for each $i \in \{1,\dots,m\}$, we assume

$$\forall x \in \mathbb{R}\colon\ \ell_i(x) = \frac{x^+}{\alpha_i},$$

which satisfies the conditions in Definition 4.1.1. The corresponding vector divergence function g is given by

$$\forall y \in \mathbb{R}\colon\ g_i(y) = I_{\left[0,\frac{1}{\alpha_i}\right]}(y) = \begin{cases} 0 & \text{if } y \in \left[0,\frac{1}{\alpha_i}\right] \\ +\infty & \text{else} \end{cases}$$

for each $i \in \{1,\dots,m\}$.

Let $x^0 \in \operatorname{int} \ell(\operatorname{dom} \ell) = \mathbb{R}^m_{++}$ and $C \in \mathcal{G}(\mathbb{R}^m, \mathbb{R}^m_+)$ with $0 \in \mathbb{R}^m$ being a boundary point of C. The corresponding multivariate ℓ-shortfall risk measure is given by

$$R_\ell(X) = \left\{ z \in \mathbb{R}^m \mid \mathbb{E}\left[(z-X)^+\right] \in \alpha \circ (x^0 - C) \right\},$$

where the positive part function is applied component-wise.

Note that the set $1/\operatorname{dom} g$ defined in (4.21) becomes $\bigtimes_{i=1}^m [\alpha_i, +\infty)$. For $r \in \mathbb{R}^m_{++}$, the multivariate (g,r)-divergence risk measure is given by

$$D_{g,r}(X) = \delta_{g,r}(X) + r \circ C$$

for $X \in L^\infty_m$, where, for each $i \in \{1,\dots,m\}$,

$$\delta_{g_i,r_i}(X_i) = \inf_{z_i \in \mathbb{R}} \left(z_i + \frac{r_i}{\alpha_i} \mathbb{E}\left[(z_i - X_i)^+\right] \right) - r_i x_i^0.$$

When $r=(1,\dots,1)$ and $C=\mathbb{R}^m_+$, we obtain the set-valued average value at risk in the sense of [30, Definition 2.1 for $M=\mathbb{R}^m$], which is given by

$$\mathrm{AV@R}_\alpha(X) \coloneqq D_{g,1}(X)+x^0=\left[\inf_{z_i\in\mathbb{R}}\left(z_i+\frac{1}{\alpha_i}\mathbb{E}\left[(z_i-X_i)^+\right]\right)\right]_{i=1}^m+\mathbb{R}^m_+,\quad X\in L^\infty_m.$$

Hence, our framework offers the following generalization of multivariate average value at risk to a convex multivariate risk measure:

$$\mathrm{AV@R}_{\alpha,r}(X) \coloneqq D_{g,r}(X)+r\circ x^0=\left[\inf_{z_i\in\mathbb{R}}\left(z_i+\frac{r_i}{\alpha_i}\mathbb{E}\left[(z_i-X_i)^+\right]\right)\right]_{i=1}^m+r\circ C,\quad .$$

As in the scalar case, this definition even works for $X\in L^1_m$.

4.5 A remark about market-extensions

From Proposition 3.2.7, recall that the closed market-extension $\operatorname{cl} R^{\mathrm{mar}}\colon L^\infty_d\to\mathcal{G}(\mathbb{R}^m_+)$ $(m\le d)$ of a convex multivariate risk measure $R\colon L^\infty_m\to\mathcal{P}(\mathbb{R}^m_+)$ satisfies the definition of a weak*-closed convex multivariate risk measure except that one might have $\operatorname{cl} R^{\mathrm{mar}}(0)=\mathbb{R}^m$. In this section, we present sufficient conditions that guarantee this finite-valuedness property for the multivariate shortfall and divergence risk measures. Once this property is established, the dual representation for these closed market-extensions is provided by Proposition 3.2.13. For simplicity, we assume that the market model is conical in the sense of Example 3.2.1.

Assumption 4.5.1. *Suppose that the solvency cones of the market model share a common supporting halfspace in the sense that there exists $\bar{w}\in\mathbb{R}^d_+\backslash\{0\}$ such that for $\mathbb{P}$-almost every $\omega\in\Omega$ and every $t\in\{0,\dots,T\}$,*

$$\bar{w}\in(\mathcal{C}_t(\omega))^+,$$

or equivalently,

$$\inf_{y \in \mathcal{C}_t(\omega)} \langle y, \bar{w} \rangle > -\infty.$$

Remark 4.5.2. Assumption 4.5.1 states the existence of a halfspace

$$H(\bar{w}) = \{z \in \mathbb{R}^d \mid \langle z, \bar{w} \rangle \geq 0\}$$

for some $\bar{w} \in \mathbb{R}^d_+$ which satisfies $H(\bar{w}) \supseteq \mathcal{C}_t(\omega)$ for $\mathbb{P}$-almost every $\omega \in \Omega$ and all $t \in \{0, \ldots, T\}$. In particular, when the solvency cones are constructed from bid-ask prices (see [34]), this is equivalent to the ask prices having a uniform (in time and outcome) lower bound, or equivalently the bid prices having a uniform (in time and outcome) upper bound. That is, $\bar{w}_j \leq \pi_{ij}(\omega, t)\bar{w}_i$ for every $i, j \in \{1, \ldots, d\}$, $t \in \{0, \ldots, T\}$ and $\mathbb{P}$-almost every $\omega \in \Omega$, where $\pi_{ij}(\omega, t)$ is the number of units of asset i for which an agent can buy one unit of asset j at time t and state ω and thus, denotes the ask price of asset j in terms of asset i.

Proposition 4.5.3. *Suppose that Assumption 4.5.1 holds. Let* $r \in 1/\operatorname{dom} g$. *If*

$$\inf_{x \in C} \langle r \circ x, B^* \bar{w} \rangle > -\infty, \tag{4.26}$$

then the closed market-extension $\operatorname{cl} D^{\mathrm{mar}}_{g,r}$ *of* $D_{g,r}$ *is finite at zero. In particular, it is a weak*-closed convex multivariate risk measure with a dual representation provided by Corollary 3.2.13.*

Proof. Let $i \in \{1, \ldots, m\}$. From Remark 4.1.7, recall that, for every $s \in \mathbb{R}$,

$$r_i \ell_i(s) = \sup_{y \in \mathbb{R}} \left(sy - r_i g_i \left(\frac{y}{r_i} \right) \right) \geq s - r_i g_i \left(\frac{1}{r_i} \right).$$

Hence, given $X \in L_m^\infty$,

$$\begin{aligned}\delta_{g_i,r_i}(X_i) &= \inf_{y\in\mathbb{R}} \left(y + r_i \mathbb{E}\left[\ell_i(-X_i - y)\right]\right) - r_i x_i^0 \\ &\geq -\mathbb{E}\left[X_i\right] - r_i g_i\left(\frac{1}{r_i}\right) - r_i x_i^0\end{aligned} \tag{4.27}$$

for every $i \in \{1, \ldots, m\}$. Then,

$$\begin{aligned}&\inf_{z\in D_{g,r}^{\mathrm{mar}}(0)} \langle z, B^*\bar{w}\rangle \\ &= \inf_{X\in\Lambda_m(0)} \inf_{z\in D_{g,r}(X)} \langle z, B^*\bar{w}\rangle \\ &= \inf_{X\in\Lambda_m(0)} \inf\{\langle z, B^*\bar{w}\rangle \mid z \in \delta_{g,r}(X) + r\circ C\} \\ &= \inf_{X\in\Lambda_m(0)} \langle \delta_{g,r}(X), B^*\bar{w}\rangle + \inf_{x\in C} \langle r\circ x, B^*\bar{w}\rangle \\ &\geq \inf_{X\in\Lambda_m(0)} \langle \mathbb{E}\left[-X\right], B^*\bar{w}\rangle - \sum_{i=1}^m \bar{w}_i r_i g_i\left(\frac{1}{r_i}\right) + \inf_{x\in -x^0+C} \langle r\circ x, B^*\bar{w}\rangle \\ &= \inf_{X\in\Lambda_m(0)} \langle \mathbb{E}\left[-BX\right], \bar{w}\rangle - \sum_{i=1}^m \bar{w}_i r_i g_i\left(\frac{1}{r_i}\right) + \inf_{x\in -x^0+C} \langle r\circ x, B^*\bar{w}\rangle \\ &\geq \inf_{Y\in\mathcal{K}} \langle \mathbb{E}\left[-Y\right], \bar{w}\rangle - \sum_{i=1}^m \bar{w}_i r_i g_i\left(\frac{1}{r_i}\right) + \inf_{x\in -x^0+C} \langle r\circ x, B^*\bar{w}\rangle \\ &= \sum_{t=0}^T \inf_{U\in L_d^\infty(\mathcal{F}_t, \mathcal{C}_t\cap\mathcal{D}_t)} \mathbb{E}\left[\langle U, \bar{w}\rangle\right] - \sum_{i=1}^m \bar{w}_i r_i g_i\left(\frac{1}{r_i}\right) + \inf_{x\in -x^0+C} \langle r\circ x, B^*\bar{w}\rangle \\ &\geq \sum_{t=0}^T \inf_{U\in L_d^\infty(\mathcal{F}_t, \mathcal{C}_t)} \mathbb{E}\left[\langle U, \bar{w}\rangle\right] - \sum_{i=1}^m \bar{w}_i r_i g_i\left(\frac{1}{r_i}\right) + \inf_{x\in -x^0+C} \langle r\circ x, B^*\bar{w}\rangle \\ &=: a,\end{aligned}$$

where the first inequality follows from (4.27), the second inequality follows since $\Lambda_m(0) = \{X \in L_m^\infty \mid BX \in \mathcal{K}\}$, and the last inequality follows since $L_d^\infty(\mathcal{F}_t, \mathcal{C}_t\cap\mathcal{D}_t) \subseteq L_d^\infty(\mathcal{F}_t, \mathcal{C}_t)$ for each $t \in \{0, \ldots, T\}$. By the same arguments as in the proofs of

Corollary 3.2.12 and Corollary 3.2.13, the hypotheses guarantee that $\alpha > -\infty$. Hence,

$$D^{\mathrm{mar}}_{g,r}(0) \subseteq \left\{ \eta \in \mathbb{R}^m \mid \langle \eta, B^* \bar{w} \rangle \geq \inf_{z \in D^{\mathrm{mar}}_{g,r}(0)} \langle z, B^* \bar{w} \rangle \right\}$$
$$\subseteq \{ \eta \in \mathbb{R}^m \mid \langle \eta, B^* \bar{w} \rangle \geq a \} \neq \mathbb{R}^m.$$

Note that $L^\infty_m \ni X \mapsto \{ \eta \in \mathbb{R}^m \mid \langle \eta, B^* \bar{w} \rangle \geq a \} \in \mathcal{G}(\mathbb{R}^m_+)$ is a weak*-closed convex function. Hence, by Remark 3.2.10,

$$(\mathrm{cl}\, D^{\mathrm{mar}}_{g,r})(0) \subseteq \{ \eta \in \mathbb{R}^m \mid \langle \eta, B^* \bar{w} \rangle \geq a \} \neq \mathbb{R}^m,$$

which shows the desired finiteness condition. □

Proposition 4.5.4. *Suppose that Assumption 4.5.1 holds. If there exists* $r \in 1/\operatorname{dom} g$ *such that* (4.26) *holds, then the closed market-extension* $\mathrm{cl}\, R^{\mathrm{mar}}_\ell$ *of* R_ℓ *is finite at zero. In particular, it is a weak*-closed convex multivariate risk measure with a dual representation provided by Corollary 3.2.13.*

Proof. From (4.20), for every $X \in L^\infty_m$, we know that $R_\ell(X) \subseteq D_{g,r}(X)$; hence, by Remark 3.2.10, we have $(\mathrm{cl}\, R^{\mathrm{mar}}_\ell)(Y) \subseteq (\mathrm{cl}\, D^{\mathrm{mar}}_{g,r})(Y)$ for every $Y \in L^\infty_d$. The result follows directly from Proposition 4.5.3. □

Chapter 5

A characterization theorem for Aumann integrals

In this chapter, a Daniell-Stone type characterization theorem is proved for Aumann integrals of set-valued functions on a measurable space.

Basic concepts related to measurable set-valued functions are recalled in Section 5.1. The definition of the Aumann integral due to Aumann's 1965 paper [6] is recalled and basic properties of the integral are presented in Section 5.2. Some of these properties appear in the main theorem as part of the characterizing properties of the integral.

The main result Theorem 5.3.1 is stated in Section 5.3. Its proof follows in several steps and is presented in Section 5.4.

The use of Aumann integrals in the study of dynamic multivariate risk measures in continuous time is the subject of future research. Some remarks about this connection are discussed in Section 5.5.

Throughout this chapter, let $C \subseteq \mathbb{R}^m$ be a fixed closed convex cone such that $C \neq \mathbb{R}^m$. It is also assumed that C has nonempty interior, that is, $\operatorname{int} C \neq \emptyset$. Recall the complete lattice of closed convex upper subsets of $\mathbb{R}^m$ given by (2.3):

$$\mathcal{G}(C) = \{A \subseteq \mathbb{R}^m \mid A = \mathrm{co}(A \oplus C)\},$$

where $A \oplus C \coloneqq \mathrm{cl}(A + C)$.

5.1 Measurable set-valued functions

Let $(\Omega, \mathcal{E})$ be a measurable space. Given a function $F \colon \Omega \to \mathcal{G}(C) \backslash \{\emptyset\}$, the *preimage* of a set $A \subseteq \mathbb{R}^m$ under F is defined as

$$F^{-1}(A) \coloneqq \{\omega \in \Omega \mid F(\omega) \cap A \neq \emptyset\}.$$

Definition 5.1.1. *A function $F \colon \Omega \to \mathcal{G}(C) \backslash \{\emptyset\}$ is said to be measurable if $F^{-1}(A) \in \mathcal{E}$ for every closed set $A \subseteq \mathbb{R}^m$. The set of all such functions is denoted by $\mathscr{F}(C)$.*

This is the usual notion of measurability for set-valued functions as in the books [45, 41, 36]. In Theorem 2.3 of [41, Chapter 1], functions whose values are closed subsets of a Polish space are considered. Assuming the existence of a σ-finite measure ν under which $(\Omega, \mathcal{E}, \nu)$ is complete, it is shown that such a set-valued function is measurable in the sense of Definition 5.1.1 if and only if the preimage of every Borel set, or equivalently every open set, is measurable. Since we work with functions with values in $\mathcal{G}(C)$, it is possible to provide another characterization of measurability in terms of simple sets of the form "point minus cone" as shown in the following proposition.

Proposition 5.1.2. *Let $F \in \mathscr{F}(C)$. The following are equivalent:*

(i) F is measurable.

(ii) $F^{-1}(A) \in \mathcal{E}$ *for every* $A \in \mathcal{G}(-C)$.

(iii) $F^{-1}(y - C) \in \mathcal{E}$ *for every* $y \in \mathbb{R}^m$.

Proof. It is obvious that $(i) \Rightarrow (ii) \Rightarrow (iii)$. Note that the elements of $\mathcal{G}(C)\setminus\{\emptyset\}$ are regular closed sets in the sense that $A = \operatorname{cl}\operatorname{int} A$ for every $A \in \mathcal{G}(C)$. Indeed, for $A \in \mathcal{G}(C)\setminus\{\emptyset\}$, note that $\operatorname{int} A \neq \emptyset$ because for every $z \in A$ it holds $\emptyset \neq z + \operatorname{int} C = \operatorname{int}(z + C) \subseteq \operatorname{int}(A + C) = \operatorname{int} A$. Then, $A = \operatorname{cl}\operatorname{int} A$ follows from [45, Theorem 2.33]. Hence, by [45, Example 14.7], F is measurable if and only if $\{\omega \in \Omega \mid y \in F(\omega)\} \in \mathcal{E}$ for every $y \in \mathbb{R}^m$. Now fix $y \in \mathbb{R}^m$. We claim that

$$F^{-1}(y - C) = \{\omega \in \Omega \mid y \in F(\omega)\}.$$

Indeed, the $\supseteq$ part is clear since $0 \in C$. For the $\subseteq$ part, let $\omega \in \Omega$ such that $F(\omega) \cap (y - C) \neq \emptyset$. Assume that $y \notin F(\omega)$. There exists $z \in C$ such that $y - z \in F(\omega)$. Thus, $y \in F(\omega) + z \subseteq F(\omega) + C = F(\omega)$, a contradiction. Thus, $y \in F(\omega)$. Hence, $(i) \Leftrightarrow (iii)$ follows. □

For functions $F, G \colon \Omega \to \mathcal{G}(C)\setminus\{\emptyset\}$ and $\lambda \in \mathbb{R}_+$, the functions $F + G$, $F \oplus G$, λF are defined in the pointwise sense with the conventions of Section 2.2. These set-valued functions are measurable when F, G are measurable; see [45, Proposition 14.11].

Denote by $\mathcal{L}^0(\mathbb{R}^m)$ the linear space of all Borel measurable functions $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m$. For subsets of $\mathcal{L}^0(\mathbb{R}^m)$, Minkowski sum and multiplication with positive scalars are defined analogously as in the case of the subsets of $\mathbb{R}^m$.

Definition 5.1.3. *Let* $F \in \mathscr{F}(C)$. *A function* $f \in \mathcal{L}^0(\mathbb{R}^m)$ *is said to be a measurable selection of* F *if* $f(\omega) \in F(\omega)$ *for every* $\omega \in \Omega$. *The set of all measurable selections of* F *is denoted by* $\mathcal{L}^0(F)$.

Proposition 5.1.4. *[45, Corollary 14.6] Every $F \in \mathscr{F}(C)$ has a measurable selection, that is, $\mathcal{L}^0(F) \neq \emptyset$.*

5.2 Basic properties of Aumann integrals

Let μ be a σ-finite measure on $(\Omega, \mathcal{E})$ with $\mu(\Omega) > 0$. The set of all functions $f \in \mathcal{L}^0(\mathbb{R}^m)$ with $\int |f_i|\, d\mu < +\infty$ for every $i \in \{1, \ldots, m\}$ is denoted by $\mathcal{L}^\mu(\mathbb{R}^m)$. For $f \in \mathcal{L}^\mu(\mathbb{R}^m)$, define

$$\int f d\mu := \left(\int f_1 d\mu, \ldots, \int f_m d\mu \right).$$

Definition 5.2.1. *Let $F \in \mathscr{F}(C)$. Denote by $\mathcal{L}^\mu(F)$ the set of all μ-integrable measurable selections of F, that is, $\mathcal{L}^\mu(F) := \mathcal{L}^\mu(\mathbb{R}^m) \cap \mathcal{L}^0(F)$. The Aumann integral of F with respect to μ is defined as*

$$\int F d\mu := \operatorname{cl} \left\{ \int f d\mu \mid f \in \mathcal{L}^\mu(F) \right\}.$$

F is said to be μ-integrable if $\mathcal{L}^\mu(F) \neq \emptyset$.

Proposition 5.2.2. *The Aumann integral of a function $F \in \mathscr{F}(C)$ is a closed convex upper set, that is, $\int F d\mu \in \mathcal{G}(C)$.*

Proof. First, we show

$$\left\{ \int f d\mu \mid f \in \mathcal{L}^\mu(F) \right\} = \left\{ \int f d\mu \mid f \in \mathcal{L}^\mu(F) \right\} + C. \tag{5.1}$$

The $\subseteq$ part is obvious since $0 \in C$. For the $\supseteq$ part, let $f \in \mathcal{L}^\mu(F)$ and $z \in C$. Define $g \in \mathcal{L}^0(\mathbb{R}^m)$ by $g(\omega) = f(\omega) + (\mu(E))^{-1} 1_E(\omega) z$ for $\omega \in \Omega$, where $E \in \mathcal{E}$ is some nonempty set with $0 < \mu(E) < +\infty$, and $1_E(\omega) = 1$ for $\omega \in E$ and $1_E(\omega) = 0$ for $\omega \in E^c := \Omega \backslash E$. Then, $g(\omega) \in F(\omega) + C = F(\omega)$ for every $\omega \in \Omega$ since C is a cone with $0 \in C$. Besides, $g \in \mathcal{L}^\mu(F)$ and $\int g d\mu = \int f d\mu + z$.

Note that taking closures in (5.1) gives

$$\int F d\mu = \left\{ \int f d\mu \mid f \in \mathcal{L}^\mu(F) \right\} \oplus C = \int F d\mu \oplus C.$$

Next, we show that $\left\{ \int f d\mu \mid f \in \mathcal{L}^\mu(F) \right\}$ is a convex set. Indeed, let $f, g \in \mathcal{L}^\mu(F)$ and $\lambda \in (0,1)$. Clearly, $\lambda f + (1-\lambda) g \in \mathcal{L}^\mu(F)$ since the values of F are convex sets. Since $\lambda \int f d\mu + (1-\lambda) \int g d\mu = \int (\lambda f + (1-\lambda) g) d\mu$, convexity follows. Hence, $\int F d\mu$ is also convex. So

$$\int F d\mu = \operatorname{co}\left(\int F d\mu \oplus C \right) = \operatorname{cl}\operatorname{co}\left(\int F d\mu + C \right),$$

that is, $\int F d\mu \in \mathcal{G}$. □

Proposition 5.2.3. *[31, Theorem 5.4] Let $F, G \in \mathscr{F}(C)$ be μ-integrable. Then,*

$$\int (F \oplus G)\, d\mu = \int F d\mu \oplus \int G d\mu.$$

Proposition 5.2.4. *For every $F \in \mathscr{F}(C)$ and $\lambda \in \mathbb{R}_+$, it holds $\int \lambda F d\mu = \lambda \int F d\mu$. In particular, if $F \equiv C$, then $\int F d\mu = C$.*

Proof. The result is obvious when $\lambda > 0$. The case $\lambda = 0$ follows from the convention $0A = C$ for $A \in \mathcal{G}(C)$ once we show that $\int C d\mu = C$. Indeed, it is clear that $0 \in \mathcal{L}^\mu(C)$, hence $C = \{0\} \oplus C \subseteq \int C d\mu \oplus C = \int C d\mu$ by Proposition 5.2.2. Conversely, if $f \in \mathcal{L}^\mu(C)$, then, for every $w \in C^+ \backslash \{0\}$, we have $\langle f(\omega), w \rangle \geq 0$ for every $\omega \in \Omega$ and hence $\left\langle \int f d\mu, w \right\rangle \geq 0$. Therefore, $\int f d\mu \in C$. This proves $\int C d\mu = C$. □

Given $E \in \mathcal{E}$, define the function $\mathbf{1}_E F \colon \Omega \to \mathcal{G}(C)$ for $\omega \in \Omega$ by

$$(\mathbf{1}_E F)(\omega) = \begin{cases} F(\omega) & \text{if } \omega \in E \\ C & \text{else} \end{cases}. \tag{5.2}$$

Note that $\mathbf{1}_E F$ is not the same as the function $1_E F$ given by

$$(1_E F)(\omega) = 1_E(\omega)F(\omega) = \begin{cases} F(\omega) & \text{if } \omega \in E \\ \{0\} & \text{else} \end{cases},$$

for $\omega \in \Omega$. Indeed, we have $\mathbf{1}_E F = 1_E F + 1_{E^c} C$. Since we work with functions whose values are closed convex upper sets, it is often more advantageous in our framework to work with $\mathbf{1}_E F$ rather than $1_E F$.

Definition 5.2.5. *Let $F \in \mathscr{F}(C)$ and $E \in \mathcal{E}$. The integral of F over the set E with respect to μ is defined as*

$$\int_E F d\mu = \int \mathbf{1}_E F d\mu.$$

Proposition 5.2.6. *For every $F \in \mathscr{F}(C)$ and $E \in \mathcal{E}$, it holds*

$$\int_E F d\mu = \left\{ \int_E f d\mu \mid f \in \mathcal{L}^\mu(F) \right\} \oplus C.$$

Proof. By the definition of the integral,

$$\begin{aligned} \int_E F d\mu &= \mathrm{cl} \left\{ \int g d\mu \mid g \in \mathcal{L}^\mu(\mathbf{1}_E F) \right\} \\ &= \mathrm{cl} \left(\left\{ \int_E g^1 d\mu \mid g^1 \in \mathcal{L}^\mu(\mathbf{1}_E F) \right\} + \left\{ \int_{E^c} g^2 d\mu \mid g^2 \in \mathcal{L}^\mu(\mathbf{1}_E F) \right\} \right). \end{aligned}$$

Here, the $\subseteq$ part of the second equality is obvious and the $\supseteq$ part follows from the observation that if $g^1, g^2 \in \mathcal{L}^\mu(\mathbf{1}_E F)$, then $g = 1_E g^1 + 1_{E^c} g^2 \in \mathcal{L}^\mu(\mathbf{1}_E F)$ by triangle inequality and $\int g d\mu = \int_E g^1 d\mu + \int_{E^c} g^2 d\mu$.

We claim that

$$\left\{ \int_E g^1 d\mu \mid g^1 \in \mathcal{L}^\mu(\mathbf{1}_E F) \right\} = \left\{ \int_E f d\mu \mid f \in \mathcal{L}^\mu(F) \right\}. \tag{5.3}$$

To show $\subseteq$, let $g^1 \in \mathcal{L}^\mu(\mathbf{1}_E F)$. For some arbitrarily fixed $g^0 \in \mathcal{L}^\mu(F)$, we have $f = 1_E g^1 + 1_{E^c} g^0 \in \mathcal{L}^\mu(F)$ by triangle inequality and $\int_E g^1 d\mu = \int_E f d\mu$ as well. The $\supseteq$ part is trivial since $f \in \mathcal{L}^\mu(F)$ implies $1_E f \in \mathcal{L}^\mu(\mathbf{1}_E F)$ (as $0 \in C$) and we have $\int_E 1_E f d\mu = \int_E f d\mu$. Thus, (5.3) holds.

If $\mu(E^c) = 0$, then the result follows immediately since

$$\begin{aligned}\int_E F d\mu &= \operatorname{cl}\left(\left\{\int_E f d\mu \mid f \in \mathcal{L}^\mu(F)\right\} + \{0\}\right)\\ &= \operatorname{cl}\left\{\int_E f d\mu \mid f \in \mathcal{L}^\mu(F)\right\}\\ &= \left\{\int_E f d\mu \mid f \in \mathcal{L}^\mu(F)\right\} \oplus C,\end{aligned}$$

where the first equality uses (5.3) and the last equality is due to the fact that $\int_E F d\mu \in \mathcal{G}(C)$; see Proposition 5.2.2 and Definition 5.2.5.

Finally, suppose that $\mu(E^c) > 0$. To finish the proof, we claim that

$$\left\{\int_{E^c} g^2 d\mu \mid g^2 \in \mathcal{L}^\mu(\mathbf{1}_E F)\right\} = C. \tag{5.4}$$

To show $\subseteq$, let $g^2 \in \mathcal{L}^\mu(\mathbf{1}_E F)$. For every $\omega \in E^c$, we have $g^2(\omega) \in C$. Since C is a closed convex cone, this is equivalent to the following: for every $\omega \in E^c$ and $w \in C^+ \backslash \{0\}$, it holds $\langle g^2(\omega), w\rangle \geq 0$. Thus,

$$\left\langle \int_{E^c} g^2 d\mu, w \right\rangle = \int_{E^c} \langle g^2(\omega), w\rangle \, \mu(d\omega) \geq \int_{E^c} 0 d\mu = 0,$$

for every $w \in C^+ \backslash \{0\}$, which shows that $\int_{E^c} g^2 d\mu \in C$. To show $\supseteq$ in (5.4), let $z \in C$. Since μ is σ-finite and $\mu(E^c) > 0$, there exists $E_0 \in \mathcal{E}$ such that $0 < \mu(E^c \cap E_0) \leq \mu(E_0) < +\infty$. Note that for every $f \in \mathcal{L}^\mu(F)$, we have $g^2 = 1_E f + 1_{E^c \cap E_0} z \in \mathcal{L}^\mu(\mathbf{1}_E F)$ since

$$\int \left|g^2\right| d\mu \leq \int_E |f| \, d\mu + \mu(E^c \cap E_0) \, |z| < +\infty,$$

by triangle inequality, where $|\cdot|$ is some arbitrary fixed norm on $\mathbb{R}^m$. Besides, for such g^2,

$$\int_{E^c} g^2 d\mu = \mu(E^c \cap E_0)z.$$

Hence,

$$\left\{\int_{E^c} g^2 d\mu \mid g^2 \in \mathcal{L}^\mu(\mathbf{1}_E F)\right\} \supseteq \{\mu(E^c \cap E_0)z \mid z \in C\} = \mu(E^c \cap E_0)C = C$$

since C is a cone and $\mu(E^c \cap E_0) > 0$. Thus, (5.4) holds. □

Remark 5.2.7. When $\mu(E) > 0$ in Proposition 5.2.6, one has $\int_E F d\mu = \operatorname{cl}\left\{\int_E f d\mu \mid f \in \mathcal{L}^\mu(F)\right\}$, which coincides with the usual definition of (the closure of) the Aumann integral of F over the set E as in [31]. When $\mu(E) = 0$, Proposition 5.2.6 gives $\int_E F d\mu = C$ which deviates from the classical definition $\operatorname{cl}\left\{\int_E f d\mu \mid f \in \mathcal{L}^\mu(F)\right\} = \{0\}$. This is in total analogy to Definition 5.2.1 and aligns with the algebraic structure on $\mathcal{G}(C)$.

Since closed convex sets are identified by their support functions, it is natural to consider the relationship between the Aumann integral and the (random) support function of $F \in \mathscr{F}(C)$. Proposition 5.2.8 below shows that Aumann integration and computing support functions commute. It is a well-known result in Aumann integration and valid in a more general framework; see [31, Theorem 5.4]. In the context of this dissertation, it leads to the representation (5.5) of the Aumann integral as given below.

Proposition 5.2.8. *Let $F \in \mathscr{F}(C)$ be a μ-integrable set-valued function. For every $w \in C^+ \backslash \{0\}$, it holds*

$$\inf_{f \in \mathcal{L}^\mu(F)} \int \langle f, w \rangle \, d\mu = \int \inf_{y \in F(\omega)} \langle y, w \rangle \, \mu(d\omega).$$

In particular,

$$\int F d\mu = \bigcap_{w \in C^+ \setminus \{0\}} \left\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \int \inf_{y \in F(\omega)} \langle y, w \rangle \, \mu(d\omega) \right\}, \tag{5.5}$$

Proof. The first part of the result is provided by [31, Theorem 5.4]. The second part follows directly from (2.4) and the observation that

$$\inf_{z \in \int F d\mu} \langle z, w \rangle = \inf_{f \in \mathcal{L}^\mu(F)} \int \langle f, w \rangle \, d\mu$$

for $w \in C^+ \setminus \{0\}$. □

Using Proposition 5.2.8, we compute the integrals of two particular functions next. The results of these examples will be used in the proof of Theorem 5.3.1, the main result of this chapter. Recall that the halfspace with normal vector $w \in C^+ \setminus \{0\}$ is denoted by $H(w) = \{z \in \mathbb{R}^m \mid \langle z, w \rangle \geq 0\}$.

Example 5.2.9. Let $w \in C^+ \setminus \{0\}$ and $\xi \colon \Omega \to \mathbb{R} \cup \{-\infty\}$ be a Borel measurable function with $\int \xi^+ d\mu < +\infty$, where $\xi^+ := \max\{\xi, 0\}$. Let $F(\omega) = \{z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \xi(\omega)\}$ for $\omega \in \Omega$. Hence, $F \in \mathscr{F}(C)$ and, for every $\bar{w} \in C^+ \setminus \{0\}$ and $\omega \in \Omega$,

$$\inf_{y \in F(\omega)} \langle y, \bar{w} \rangle = \begin{cases} k\xi(\omega) & \text{if } \bar{w} = kw \text{ for some } k \in (0, +\infty) \\ -\infty & \text{else} \end{cases}.$$

Since $\int \xi^+ d\mu < +\infty$, it is easy to check that $f := \frac{\xi^+}{\langle c, w \rangle} c \in \mathcal{L}^\mu(F)$, where $c \in \operatorname{int} C$ is some fixed point. Hence, F is μ-integrable. Note that, for every $\bar{w} \in C^+ \setminus \{0\}$,

$$\int \inf_{y \in F(\omega)} \langle y, \bar{w} \rangle \, \mu(d\omega) = \begin{cases} k \int \xi d\mu & \text{if } \bar{w} = kw \text{ for some } k \in (0, +\infty) \\ -\infty & \text{else} \end{cases}.$$

Hence, by Proposition 5.2.8,

$$\int F d\mu = \bigcap_{k\in(0,+\infty)} \left\{ z \in \mathbb{R}^m \mid \langle z, kw\rangle \geq k \int \xi d\mu \right\} = \left\{ z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \int \xi d\mu \right\}.$$

In particular, if $F \equiv H(w)$, that is, if $\xi \equiv 0$, then $\int F d\mu = H(w)$.

Example 5.2.10. Let $f \in \mathcal{L}^\mu(\mathbb{R}^m)$ and $F(\omega) = f(\omega) + C$ for $\omega \in \Omega$. Clearly, $F \in \mathscr{F}(C)$ is μ-integrable. Note that, for every $w \in C^+\backslash\{0\}$,

$$\begin{aligned}
\int \inf_{y\in F(\omega)} \langle y, w\rangle \, \mu(d\omega) &= \int \left(\langle f(\omega), w\rangle + \inf_{y\in C} \langle y, w\rangle \right) \mu(d\omega) \\
&= \int \langle f, w\rangle \, d\mu \\
&= \left\langle \int f d\mu, w \right\rangle.
\end{aligned}$$

Hence, by Proposition 5.2.8,

$$\begin{aligned}
\int F d\mu &= \bigcap_{w\in C^+\backslash\{0\}} \left\{ z \in \mathbb{R}^m \mid \langle z, w\rangle \geq \left\langle \int f d\mu, w \right\rangle \right\} \\
&= \bigcap_{w\in C^+\backslash\{0\}} \left(\int f d\mu + H(w) \right) \\
&= \int f d\mu + C.
\end{aligned}$$

In Lebesgue integration, one version of the monotone convergence theorem states that the integral of the decreasing limit of measurable functions (dominated by some integrable function) is given as the decreasing limit of the corresponding integrals. In Proposition 5.2.11 below, a similar convergence result is proved for the Aumann integrals.

Proposition 5.2.11. *Let $(F_n)_{n\in\mathbb{N}}$ and F be in $\mathscr{F}(C)$. Suppose that $(F_n)_{n\in\mathbb{N}}$ decreases to F in the sense that, for μ-a.e. $\omega \in \Omega$, $F_n(\omega) \subseteq F_{n+1}(\omega)$, $n \in \mathbb{N}$, and*

$\operatorname{cl}\bigcup_{n\in\mathbb{N}} F_n(\omega) = F(\omega)$. *Suppose that F_1 is μ-integrable. Then, for every $n \in \mathbb{N}$,*

$$\int F_n d\mu \subseteq \int F_{n+1} d\mu,$$

and

$$\operatorname{cl}\bigcup_{n\in\mathbb{N}} \int F_n d\mu = \int F d\mu.$$

Proof. When μ is a probability measure, the result is provided by Theorem 1.44 of [41, Chapter 2]. The proof of that case relies on the classical monotone convergence theorem and works in the general σ-finite case as well. □

5.3 The characterization theorem

The aim of this section is to present a Daniell-Stone type characterization theorem for Aumann integrals of set-valued functions in $\mathscr{F}(C)$, which is the set of all measurable set-valued functions $F\colon \Omega \to \mathcal{G}(C)\setminus\{\emptyset\}$. Recall that $C \neq \mathbb{R}^m$ is a closed convex cone with $\operatorname{int} C \neq \emptyset$. Let $c \in \operatorname{int} C$ be fixed.

Theorem 5.3.1. *Let $\Phi\colon \mathscr{F}(C) \to \mathcal{G}(C)$ be a set-valued functional. Consider the following properties:*

- **(A) Additivity:** *For every $F, G \in \mathscr{F}(C)$ with $\Phi(F) \neq \emptyset$ and $\Phi(G) \neq \emptyset$, it holds $\Phi(F \oplus G) = \Phi(F) \oplus \Phi(G)$.*

- **(P) Positive homogeneity:** *For every $F \in \mathscr{F}(C)$ and $\lambda \in \mathbb{R}_+$, it holds $\Phi(\lambda F) = \lambda\Phi(F)$.*

- **(C) Continuity from above:** *For every $(F_n)_{n\in\mathbb{N}}$, F in $\mathscr{F}(C)$ with $\Phi(F_1) \neq \emptyset$, $F_n(\omega) \subseteq F_{n+1}(\omega)$ for every $n \in \mathbb{N}$ and $\omega \in \Omega$, and $\operatorname{cl}\bigcup_{n\in\mathbb{N}} F_n(\omega) = F(\omega)$ for every $\omega \in \Omega$, it holds $\Phi(F_n) \subseteq \Phi(F_{n+1})$ for every $n \in \mathbb{N}$, and $\operatorname{cl}\bigcup_{n\in\mathbb{N}} \Phi(F_n) = \Phi(F)$.*

(N) Nullity: *If* $F \equiv H(w)$ *for some* $w \in C^+ \setminus \{0\}$*, then* $\Phi(F) = H(w)$*.*

(I) Indicator property: *For every* $\xi \in \mathcal{L}^0(\mathbb{R}_+)$*, either* $\Phi(\xi c + C) = \emptyset$*, or else there exists* $k \in \mathbb{R}_+$ *such that* $\Phi(\xi c + C) = kc + C$*. In addition, there exists at least one* $\xi \in \mathcal{L}^0(\mathbb{R}_{++})$ *such that* $\Phi(\xi c + C) \notin \{\emptyset, C\}$*.*

(S) Interchangeability with supporting halfspaces: *Let* $F \in \mathscr{F}(C)$ *with* $\Phi(F) \neq \emptyset$*. For* $\omega \in \Omega$*,* $w \in C^+ \setminus \{0\}$*, define the supporting halfspace of* $F(\omega)$ *with normal* $w \in C^+ \setminus \{0\}$ *by*

$$F^w(\omega) = F(\omega) \oplus H(w) = \left\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq \inf_{y \in F(\omega)} \langle y, w \rangle \right\}. \qquad (5.6)$$

Then, it holds $\Phi(F) = \bigcap_{w \in C^+ \setminus \{0\}} \Phi(F^w)$*.*

Φ *satisfies the above properties if and only if there exists a unique nonzero* σ*-finite measure* μ *on* $(\Omega, \mathcal{E})$ *such that, for every* μ*-integrable* $F \in \mathscr{F}(C)$*,*

$$\Phi(F) = \int F d\mu.$$

The proof of Theorem 5.3.1 is given in Section 5.4.

Theorem 5.3.1 is a set-valued generalization of the following Daniell-Stone characterization of the Lebesgue integral: A functional on the set of positive measurable functions that maps into $[0, +\infty]$ is the Lebesgue integral with respect to some measure if and only if it is additive and positively homoegeneous, and it satisfies the monotone convergence property; see [15, Theorem I.4.21] for this version, and [49] for the original work. Indeed, these three properties are precisely equivalent to properties (A), (P), (C), respectively, in the scalar case where $m = 1$ and $C = \mathbb{R}_+$. Properties (A) and (P) together are sometimes called "conlinearity" meaning that the functional is linear except that negative scalars are not considered in multiplication.

Therefore, properties (N), (I) and (S) are the additional properties of the set-valued framework. Properties (N) and (I) put regularity conditions on how the functional preserves the geometric properties of some special set-valued functions. In particular, Property (N) assumes that a constant homogeneous halspace is mapped to itself under Φ. Together with the aforementioned properties, it ensures that halfspace-valued functions are mapped to halfspaces under Φ. Property (I) provides a similar regularity condition when the functional is applied to functions having a special "point plus cone" structure, which is already verified for Aumann integrals in Example 5.2.10. In particular, Property (I) describes the behavior of the functions of the form $\xi = 1_E$ where $E \in \mathcal{E}$. Recalling (5.2), note that $\xi c + C = 1_E c + C = \mathbf{1}_E F$, where $F \equiv c + C$. The function $1_E c + C$ can be thought as the set-valued indicator function of the measurable set E. The second part of Property (I) characterizes σ-finiteness (and the nontriviality) of the measure and also exists as an optional additional property in the scalar case; see [15, Exercise I.4.32]. Since the construction of the Aumann integral in Proposition 5.2.6 and Property (S) in its relation to Proposition 5.2.8 already assume σ-finiteness of the underlying measure, this condition is needed for the characterization theorem in the set-valued case.

Property (S) assumes that computing supporting halfspaces of functions on Ω commutes with the functional, which is verified for Aumann integrals in Proposition 5.2.8. While Proposition 5.2.8 is stated in terms of the Lebesgue integrals of (scalar) support functions, this is not the case for Property (S). Instead, we consider the supporting halfspace of a function as another function in $\mathscr{F}(C)$ and state Property (S) in terms of its Aumann integral. In the scalar case where $m = 1$ and $C = \mathbb{R}_+$, properties (N), (I) (the first part), and (S) become trivial and thus can be omitted.

5.4 Proof of Theorem 5.3.1

Before we prove Theorem 5.3.1, we begin with a technical result to be used in the proof.

Lemma 5.4.1. *For every* $f \in \mathcal{L}^\mu(\mathbb{R}^m)$, *there exists some* $\xi \in \mathcal{L}^\mu(\mathbb{R}_+)$ *such that* $\xi c - f \in \mathcal{L}^\mu(C)$.

Proof. Let

$$\xi \coloneqq \left(\sup_{w \in C^+ \setminus \{0\}} \frac{\langle f, w\rangle}{\langle c, w\rangle} \right)^+ .$$

Recall that $\langle c, w\rangle > 0$ for every $w \in C^+ \setminus \{0\}$. Hence,

$$\langle \xi c - f, w\rangle = \xi \langle c, w\rangle - \langle f, w\rangle \geq 0$$

for every $w \in C^+ \setminus \{0\}$, that is, $\xi c - f \in \mathcal{L}^0(C)$. It remains to show that $\xi \in \mathcal{L}^\mu(\mathbb{R}_+)$. Noting that $\frac{\langle f,w\rangle}{\langle c,w\rangle} = \frac{\langle f,kw\rangle}{\langle c,kw\rangle}$ for every $k > 0$ and $w \in C^+ \setminus \{0\}$, it follows that

$$\xi = \left(\sup_{w \in D(c)} \langle f, w\rangle \right)^+ ,$$

where $D(c) \coloneqq \{w \in C^+ \colon \langle c, w\rangle = 1\}$. Since $\operatorname{int} C \neq \emptyset$, the dual cone C^+ is pointed, that is, $C^+ \cap -C^+ = \{0\}$. Hence, an elementary exercise in convex analysis yields that $D(c)$ is a bounded set. Therefore, letting

$$a \coloneqq \sup_{w \in D(c)} \max_{i \in \{1,\dots,m\}} |w_i| < +\infty,$$

it holds

$$\xi \leq \sup_{w \in D(c)} \sum_{i=1}^m |f_i| \, |w_i| \leq a \sum_{i=1}^m |f_i| \, .$$

Since $f \in \mathcal{L}^\mu(\mathbb{R}^m)$, it follows that $\xi \in \mathcal{L}^\mu(\mathbb{R}_+)$. □

Proof of Theorem 5.3.1. If there exists a unique nonzero σ-finite measure μ on $(\Omega, \mathcal{E})$ such that $\Phi(F) = \int F d\mu$ for every μ-integrable $F \in \mathscr{F}(C)$, then all properties are already satisfied as shown in Proposition 5.2.3 (Property (A)), Proposition 5.2.4 (Property (P)), Proposition 5.2.11 (Property (C)), Example 5.2.9 (Property (N)), Example 5.2.10 (Property (I)), Proposition 5.2.8 (Property (S)).

Suppose that Φ satisfies the properties listed in the theorem.

Let $\xi \in \mathcal{L}^0(\mathbb{R}_+)$. If $\Phi(\xi c + C) = \emptyset$, then set $\varphi(\xi) = +\infty$. Otherwise, there exists $k \in \mathbb{R}_+$ with $\Phi(\xi c + C) = kc + C$ and set $\varphi(\xi) = k$. Here, such k is unique because for $\bar{k} \in \mathbb{R}$,

$$kc + C = \bar{k}c + C \quad \Longrightarrow \quad k = \bar{k}. \tag{5.7}$$

This is an easy exercise which follows from the fact that $c \in \operatorname{int} C$. Hence $\varphi(\xi)$ is well-defined. Next, we show that the function $\varphi\colon \mathcal{L}^0(\mathbb{R}_+) \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies the assumptions of the classical Daniell-Stone theorem:

(i) Monotonicity: Let $\xi, \zeta \in \mathcal{L}^0(\mathbb{R}_+)$ with $\xi(\omega) \leq \zeta(\omega)$ for every $\omega \in \Omega$. We claim $\varphi(\xi) \leq \varphi(\zeta)$. If $\varphi(\zeta) = +\infty$, the claim holds trivially. Assume that $\varphi(\zeta) < +\infty$, that is, $\Phi(\zeta c + C) \neq \emptyset$. Then, $F^1 := \zeta c + C$, $F := F^n := \xi c + C$ for $n \in \{2, 3, \ldots\}$ satisfy the assumptions of Property (C). Hence, $\emptyset \neq \Phi(\zeta c + C) \subseteq \Phi(\xi c + C)$. By Property (I), $\emptyset \neq \varphi(\zeta)c + C \subseteq \varphi(\xi)c + C$. By (5.7), it follows that $\varphi(\xi) \leq \varphi(\zeta)$.

(ii) Additivity: Let $\xi, \zeta \in \mathcal{L}^0(\mathbb{R}_+)$. We claim $\varphi(\xi + \zeta) = \varphi(\xi) + \varphi(\zeta)$. By the monotonicity of φ, it holds $\max\{\varphi(\xi), \varphi(\zeta)\} \leq \varphi(\xi + \zeta)$. Hence, if $\varphi(\xi) = +\infty$ or $\varphi(\zeta) = +\infty$, then the claim holds trivially. Assume that $\varphi(\xi) < +\infty$ and

$\varphi(\zeta) < +\infty$, that is, $\Phi(\xi c + C) \neq \emptyset$ and $\Phi(\zeta c + C) \neq \emptyset$. Hence, by Property (A),

$$\begin{aligned}
\Phi((\xi+\zeta)c + C) &= \Phi\left((\xi c + C) \oplus (\zeta c + C)\right) \\
&= \Phi(\xi c + C) \oplus \Phi(\zeta c + C) \\
&= (\varphi(\xi) + \varphi(\zeta))c + C \\
&\neq \emptyset.
\end{aligned}$$

By Property (I) and (5.7), it follows that $\varphi(\xi+\zeta) = \varphi(\xi) + \varphi(\zeta)$.

(iii) Positive homogeneity: Let $\xi \in \mathcal{L}^0(\mathbb{R}_+)$ and $\lambda \in \mathbb{R}_+$. We claim $\varphi(\lambda\xi) = \lambda\varphi(\xi)$. If $\lambda = 0$, then

$$\varphi(0)c + C = \Phi(C) = \Phi(0C) = 0\Phi(C) = C$$

by Property (P). Hence, the claim holds with $\varphi(\lambda\xi) = \varphi(0) = 0$. Assume that $\lambda > 0$. By Property (P) and the fact that C is a cone, it holds

$$\Phi(\lambda\xi c + C) = \Phi(\lambda(\xi c + C)) = \lambda\Phi(\xi c + C).$$

If $\varphi(\xi) = +\infty$, that is, $\Phi(\xi c + C) = \emptyset$, then $\Phi(\lambda\xi c + C) = \emptyset$, that is, $\varphi(\lambda\xi) = +\infty$. If $\varphi(\xi) < +\infty$, then

$$\varphi(\lambda\xi)c + C = \lambda(\varphi(\xi)c + C) = \lambda\varphi(\xi)c + C.$$

By (5.7), it follows that $\varphi(\lambda\xi) = \lambda\varphi(\xi)$.

(iv) Continuity from above: Let $(\xi^n)_{n\in\mathbb{N}}, \xi^\infty$ be in $\mathcal{L}^0(\mathbb{R}_+)$ such that $\varphi(\xi^1) < +\infty$, $\xi^n(\omega) \geq \xi^{n+1}(\omega)$ for every $n \in \mathbb{N}$ and $\omega \in \Omega$, and $\lim_{n\to\infty} \xi^n(\omega) = \xi^\infty(\omega)$ for every $\omega \in \Omega$. We claim $\varphi(\xi^n) \geq \varphi(\xi^{n+1})$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varphi(\xi^n) = \varphi(\xi^\infty)$. The first part of the claim follows by the monotonicity of φ. For the second part, let $F_n := \xi^n c + C$ for every $n \in \mathbb{N} \cup \{+\infty\}$. Then, $\Phi(F_1) \neq \emptyset$ and hence,

$$\Phi(F_\infty) = \text{cl} \bigcup_{n \in \mathbb{N}} \Phi(F_n) = \text{cl} \bigcup_{n \in \mathbb{N}} (\varphi(\xi^n)c + C) = \left(\lim_{n \to \infty} \varphi(\xi^n) \right) c + C \neq \emptyset$$

by Property (C). By (5.7), it follows that $\varphi(\xi^\infty) = \lim_{n \to \infty} \varphi(\xi^n)$.

Hence, by the classical Daniell-Stone theorem, there exists a unique measure μ such that

$$\varphi(\xi) = \int \xi d\mu \tag{5.8}$$

for every $\xi \in \mathcal{L}^0(\mathbb{R}_+)$. The second part of Property (I) implies that μ is nonzero and σ-finite; see [15, Exercise I.4.32].

Finally, we prove that $\Phi(F) = \int F d\mu$ for every μ-integrable function $F \in \mathscr{F}(C)$. We proceed in three main steps.

Step 1: Let $F \in \mathscr{F}(C)$ be a *negative set-valued function* in the sense that $0 \in F(\omega)$, that is, $C \subseteq F(\omega)$, for every $\omega \in \Omega$. Clearly, F is μ-integrable as $0 \in \mathcal{L}^\mu(F)$. We prove $\Phi(F) = \int F d\mu$. We first prove this claim for the case of halfspace-valued functions and then approximate F by its supporting halfspaces.

(1a) If $F \equiv H(w)$ for some $w \in C^+ \setminus \{0\}$, then $\Phi(F) = H(w) = \int F d\mu$ by Property (N) and Example 5.2.9.

(1b) Suppose that $F = \{z \in \mathbb{R}^m \mid \langle z, w \rangle \geq -\xi\}$ for some Borel function $\xi \in \mathcal{L}^0(\mathbb{R}_+)$ with $\int \xi d\mu < +\infty$ and some $w \in C^+ \setminus \{0\}$. Then,

$$F = \left\{ z \in \mathbb{R}^m \mid \langle z, w \rangle \geq -\frac{\xi}{\langle c, w \rangle} \langle c, w \rangle \right\} = \left(-\frac{\xi}{\langle c, w \rangle} c + C \right) \oplus H(w). \tag{5.9}$$

Since $\int \xi d\mu < +\infty$, the first part of Property (A) implies that

$$\begin{aligned} C &= \Phi(C) \\ &= \Phi\left(-\frac{\xi}{\langle c, w\rangle}c + C\right) \oplus \Phi\left(\frac{\xi}{\langle c, w\rangle}c + C\right) \\ &= \Phi\left(-\frac{\xi}{\langle c, w\rangle}c + C\right) \oplus \left(\varphi\left(\frac{\xi}{\langle c, w\rangle}\right)c + C\right) \\ &= \varphi\left(\frac{\xi}{\langle c, w\rangle}\right)c + \Phi\left(-\frac{\xi}{\langle c, w\rangle}c + C\right) \end{aligned}$$

so that

$$\Phi\left(-\frac{\xi}{\langle c, w\rangle}c + C\right) = -\varphi\left(\frac{\xi}{\langle c, w\rangle}\right)c + C = \left(-\int \frac{\xi}{\langle c, w\rangle}d\mu\right)c + C.$$

Hence, by (5.9), the first part of Property (A), and Example 5.2.9, it follows that

$$\begin{aligned} \Phi(F) &= \left(\left(-\int \frac{\xi}{\langle c, w\rangle}d\mu\right)c + C\right) \oplus H(w) \\ &= \left(-\int \frac{\xi}{\langle c, w\rangle}d\mu\right)c + H(w) \\ &= \left\{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq -\int \xi d\mu\right\} \\ &= \int F d\mu. \end{aligned}$$

(1c) Suppose that $F = \{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq -\xi\}$ for some Borel function $\xi\colon \Omega \to \mathbb{R}_+ \cup \{+\infty\}$ and $w \in C^+ \setminus \{0\}$. Then, there exists a sequence $(\xi^n)_{n\in\mathbb{N}}$ in $\mathcal{L}^0(\mathbb{R}_+)$ with $\xi^n \leq \xi^{n+1}$ and $\int \xi^n d\mu < +\infty$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} \xi^n(\omega) = \xi(\omega)$ for every $\omega \in \Omega$. Define $F_n := \{z \in \mathbb{R}^m \mid \langle z, w\rangle \geq -\xi^n\}$ for $n \in \mathbb{N}$. By the previous case, Property (C), monotone convergence theorem for Lebesgue integrals, and Example 5.2.9, it holds

$$\begin{aligned}\Phi(F) &= \operatorname{cl} \bigcup_{n\in\mathbb{N}} \Phi(F_n) \\ &= \operatorname{cl} \bigcup_{n\in\mathbb{N}} \left\{ z\in\mathbb{R}^m \mid \langle z,w\rangle \geq -\int \xi^n d\mu \right\} \\ &= \left\{ z\in\mathbb{R}^m \mid \langle z,w\rangle \geq -\lim_{n\to\infty}\int \xi^n d\mu \right\} \\ &= \left\{ z\in\mathbb{R}^m \mid \langle z,w\rangle \geq -\int \xi d\mu \right\} \\ &= \int F d\mu.\end{aligned}$$

(1d) Let $F \in \mathscr{F}(C)$ be an arbitrary negative set-valued function. For $w \in C^+\backslash\{0\}$, define the supporting halfspace F^w by (5.6). Using Property (S), Step (1c), Proposition 5.2.8 for the respective equalities, we have

$$\Phi(F) = \bigcap_{w\in C^+\backslash\{0\}} \Phi(F^w) = \bigcap_{w\in C^+\backslash\{0\}} \int F^w d\mu = \int F d\mu.$$

Step 2: Let $F = f + C$ for some $f \in \mathcal{L}^\mu(\mathbb{R}^m)$. We show that $\Phi(F) = \int F d\mu = \int f d\mu + C$.

(2a) If $f \in -\mathcal{L}^\mu(C)$, then Step 1 and Example 5.2.10 imply that $\Phi(f + C) = \int (f + C)\, d\mu = \int f d\mu + C$.

(2b) If $f = \xi c$ for some $\xi \in \mathcal{L}^\mu(\mathbb{R}_+)$, then $\varphi(\xi) = \int \xi d\mu < +\infty$ by (5.8). Hence, Property (I) implies that $\Phi(\xi c + C) = \left(\int \xi d\mu\right) c + C$. By Example 5.2.10, $\Phi(f + C) = \int (f + C)\, d\mu$.

(2c) If $f \in \mathcal{L}^\mu(\mathbb{R}^m)$, we may write $f = \xi c - g$ for some $\xi \in \mathcal{L}^\mu(\mathbb{R}_+)$ and $g \in \mathcal{L}^\mu(C)$ by Lemma 5.4.1. The previous two cases and the first part of Property (A) yield that

$$\begin{aligned}\Phi(f+C) &= \Phi\left((\xi c + C)\oplus(-g+C)\right)\\ &= \left(\int(\xi c)d\mu + C\right)\oplus\left(\int(-g)d\mu + C\right)\\ &= \int f d\mu + C.\end{aligned}$$

By Example 5.2.10, $\Phi(f+C) = \int (f+C)\, d\mu$.

Step 3: Let $F \in \mathscr{F}(C)$ be μ-integrable. Fix $f \in \mathcal{L}^\mu(F)$. Then $G := -f + F$ is a negative set-valued function as defined in Step 1. By the previous two steps and the first part of Property (A),

$$\begin{aligned}-\int f d\mu + \Phi(F) &= \left(-\int f d\mu + C\right)\oplus\Phi(F)\\ &= \Phi(-f+C)\oplus\Phi(F)\\ &= \Phi(G)\\ &= \int G d\mu\\ &= \int(-f+C)d\mu \oplus \int F d\mu\\ &= -\int f d\mu + \int F d\mu.\end{aligned}$$

It follows that $\Phi(F) = \int F d\mu$.

The uniqueness of μ follows from its definition and (5.7). □

Remark 5.4.2. In the proof of Theorem 5.3.1, the measure μ can alternatively be constructed without reference to the classical Daniell-Stone theorem. Indeed, for each $E \in \mathcal{E}$, one can take $\xi = 1_E$ in Property (I) and define $\mu(E) := \varphi(\xi)$. Hence, $\Phi(1_E c + C) = \mu(E)c + C$ if $\mu(E) < +\infty$ and $\Phi(1_E c + C) = \emptyset$ otherwise. Then, analogous to the properties (i)-(iv) of φ in the proof of Theorem 5.3.1, the following properties of the set function μ can be checked:

1. μ is monotone, that is, $E_1 \subseteq E_2$ implies $\mu(E_1) \leq \mu(E_2)$ for every $E_1, E_2 \in \mathcal{E}$.

2. μ is additive, that is, $\mu(E_1 + E_2) = \mu(E_1) + \mu(E_2)$ for every disjoint $E_1, E_2 \in \mathcal{E}$.

3. $\mu(\emptyset) = 0$.

4. μ is continuous from above, that is, if $(E_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}$ with $\mu(E_1) < +\infty$ and $E_n \supseteq E_{n+1}$ for every $n \in \mathbb{N}$, then $\mu(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu(E_n)$.

Hence, μ is a measure.

5.5 Connection to multivariate risk measures

In optimal control theory, Aumann integration has been a central object in the study of set-valued differential equations and differential inclusions. [5, 16] are among the monographs on these subjects in the deterministic setting. More recently, the stochastic counterparts of these equations and inclusions have also gained interest in the literature. In the stochastic setting, the integrals of set-valued functions are considered with respect to time, with respect to outcome (that is, expectations and conditional expectations of random sets), and with respect to a Wiener process (that is, stochastic integrals). Aumann's approach based on integrable selections have been extended for conditional expectations and stochastic integrals, which make it possible to study *set-valued stochastic differential equations* (SV-SDEs) and *stochastic differential inclusions* (SDIs); see [36, 39, 40], for instance.

On the other hand, in financial mathematics, dynamic evaluation of risk has been of interest in the last decade. While a (static) risk measure assigns a nonrandom number to a financial position as the minimal capital requirement, a *dynamic risk measure* assigns a stochastic process $(\rho_t(X))_{t\leq T}$ to a univariate position X at time T, where $\rho_t(X)$ can be interpreted as the minimal (random) capital requirement that makes X acceptable at time t; see [43] for instance. Dynamic risk measures in continuous time are closely related with *backward stochastic differential equations* (BSDEs). In [7, 47], it has been shown that a large class of (time-consistent) dynamic risk measures can be constructed as solutions of BSDEs.

In the multidimensional case, dynamic multivariate risk measures have been studied in [17, 18] in discrete and continuous time while the focus of the results related to time-consistency is on discrete time. In particular, the availability of a mechanism to construct time-consistent dynamic multivariate risk measures in continuous time, similar to the BSDEs in the scalar case, is an open problem.

The goal for future research in this subject is to investigate the relationship between dynamic multivariate risk measures and *set-valued backward stochastic differential equations* (SV-BSDEs) / *backward stochastic differential inclusions* (BSDIs), which are the counterparts of BSDEs in the set-valued setting.

As the values of multivariate risk measures are upper sets, the corresponding SV-BSDEs and BSDIs also need to work with functions whose values are (unbounded) upper sets. On the other hand, the current literature on these equations and inclusions focus on functions whose values are compact sets. In particular, the existence of solutions is proved under this assumption and use the *Hausdorff metric* on the space of compact convex sets; see [35, 39, 40] for instance.

As a result, the future work aims to establish a theory of SV-BSDEs and BSDIs with upper sets which provides existence, uniqueness and comparison theorems for the solutions. With these results, the next step is to construct the relationship between these equations/inclusions and dynamic multivariate risk measures, which includes necessary and sufficient conditions under which the solution of a SV-BSDE/BSDI gives rise to a dynamic multivariate risk measure, and a dual representation of this multivariate risk measure in terms of the structure of the SV-BSDE/BSDI.

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