THEORY OF GENERATING FUNCTIONS AND THEIR APPLICATIONS

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ABSTRACT

The generating functions are important instruments for solving enumerative problems in combinatorial analysis and in number theory. Enumerative problems arise when we need to be explicit about the number of ways of choosing particular elements from a finite set. The application of generating functions in this situation consists of establishing a correspondence between the elements of the set and the terms of the products of some series; the solution of enumerative problem is reduced, in fact, to finding a suitable method for the multiplication of these series.

The method of generating functions can be effectively applied to enumerative problems of graph theory, that is, problems arising when counting graphs with specific properties. In number theory, the generating functions can be used to prove some identities.

In this thesis, we try to understand the benefits of the generating functions and discuss many identities that come from 'Partitions of Integers', and 'Stirling Numbers'. We shall show how we can easily prove these identities by using generating functions.

KISA ÖZET

'Doğuran fonksiyonlar' *Kombinatorik Analiz* ve *Sayılar Teorisinde* karşılaşılan sayma problemlerinin çözümünde etkili bir rol oynarlar. Sayma problemleri sonlu bir kümeden belirli özellikleri taşıyan elemanları kaç farklı şekilde seçebileceğimizi hesaplamamız gerektiğinde karşımıza çıkar. Bu durumda, doğuran fonksiyonların uygulamaları kümenin elemanlarıyla bir serinin terimlerinin çarpımı arasında bir ilişki bulmaktan ibarettir. Bu yolla sayma problemlerinin çözümü bu serilerin çarpımına ilişkin uygun bir metod bulmaya indirgenir.

Ayrıca doğuran fonksiyonlar *Graf Teorisinde* belirli özelliklere sahip grafların sayılmasında etkili olarak kullanılır. *Sayılar Teorisinde* ise bazı özdeşlikler doğuran fonksiyonlar yardımıyla ispatlanır.

Bu tezde doğuran fonksiyonların yukarıda belirtilen konulardaki kulanımlarını anlamaya çalıştık. Bu amaçla *Tamsayıların Gruplanması* ve *Stirling Sayıları* ile ilgili bir çok özdeşliği sunduk.

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1. INTRODUCTION

This thesis is about generating functions and some of their uses in discrete mathematics. Generating functions are a bridge between discrete mathematics, on the one hand, and continuous analysis on the other. It is possible to study them solely as tools for solving discrete problems. As such there is much that is powerful and magical in the way generating functions give unified methods for handling such problems.

Here are some of the things that we will often be able to do with generating functions:

1. **Find an exact formula for the members of a sequence.** Not always in a pleasant way, if the sequence is complicated. But at least we will have a good shot at finding such a formula.

2. **Find a recurrence formula.** Most often generating functions arise from recurrence formulas. Sometimes, however, from the generating functions we will find a new recurrence formula, not the one we started with, that gives new insights into the nature of the sequence.

3. **Generating function is used for combinatorial identities.** Many identities are known, in combinatorics and elsewhere in mathematics. The identities that we refer to are those in which a certain formula is asserted to be equal to another formula for stated values of the free variable(s).

For example, it is well known that

$$\sum_j \binom{n}{j}^2 = \binom{2n}{n}, \qquad n = 0, 1, 2, \dots$$

One way to prove such identities is to consider the generating function whose coefficients are the sequence shown on the left side of the claimed identity, and to consider the generating function formed from the sequence on the right side of the claimed identity, and to show that these are the same function. Snake Oil method that we present in this thesis explores some of these ideas.

2. GENERATING FUNTIONS

2.1. Ideas and Examples

In this section we will try to understand how the generating function method works for recurrence relations and will use them to find general expressions for some recurrence relations.

In a recurence relation n^{th} element a_n is computed from the knowledge of the previous elements, $a_0, a_1, ..., a_{n-1}$. Generating functions method tries to find a function $f(x)$ which gives a_n directly. Here are some simple examples.

Example 2.1. We aim to find the closed form of the recurrence relation that satisfies the condition

$$a_{n+1} = 3a_n + 2 , \qquad n \geqslant 0, \quad a_0 = 0.$$

We collect the unknown a_n's in a power series: $F(x) = \sum a_n x^n$. To find $F(x)$, we multiply the both sides of the recurrence relation by x^n and sum over the values of n :

$$\sum_{n\geqslant 0} a_{n+1}x^n = 3\sum_{n\geqslant 0} a_n x^n + \sum_{n\geqslant 0} 2x^n.$$

But,

$$\sum_{n\geqslant 0} a_{n+1}x^n = \frac{F(x) - a_0}{x} = \frac{F(x)}{x}$$

$$3\sum_{n\geqslant 0} a_n x^n = 3F(x) \text{ and } \sum_{n\geqslant 0} 2x^n = \frac{2}{1-x}.$$

Thus,

$$\frac{F(x)}{x} = 3F(x) + \frac{2}{1-x}$$

and,

$$F(x) = \frac{2x}{(1-3x)(1-x)} = \frac{1}{1-3x} - \frac{1}{1-x}.$$

We now expand $F(x)$ in a power series:

$$F(x) = \sum_{n\geqslant 0} 3^n x^n - \sum_{n\geqslant 0} x^n = \sum_{n\geqslant 0}(3^n - 1)x^n,$$

that is, we have

$$a_n = 3^n - 1.$$

Example 2.2. Let's look at the Fibonacci Sequence

$$F_{n+1} = F_n + F_{n-1}, \quad n \geqslant 1, \quad F_0 = F_1 = 1.$$

We will solve for the generating function

$$F(x) = \sum_n F_n x^n.$$

As in the Example above we multiply the both sides of the recurrence relation by x^n and sum over $n \geqslant 1$:

$$\sum_{n\geqslant 1} F_{n+1}x^n = \sum_{n\geqslant 1} F_n x^n + \sum_{n\geqslant 1} F_{n-1}x^n$$

$$\frac{F(x) - 1 - x}{x} = F(x) - 1 + xF(x).$$

So,

$$F(x) = \frac{1}{1 - x - x^2} = \frac{1}{r_+ - r_-}\left(\frac{r_+}{1 - xr_+} - \frac{r_-}{1 - xr_-}\right), \quad r_\pm = \frac{1 \pm \sqrt{5}}{2}.$$

Hence

$$F_n = \frac{1}{\sqrt{5}}(r_+^{n+1} - r_-^{n+1}), \quad n = 0, 1, 2, \ldots$$

Since $r_+ > 1$ and $|r_-| < 1$, the second term in F_n, for large n, will be very small compared to the first, so an extremely good approximation to F_n will be

$$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1}.$$

The above two examples are very typical for the method of generating functions. That is, we multiply both sides of the given recurrence relation by x^n and sum over all values of n. We then solve the resulting equation for the unknown generating function $F(x)$. In particular, if we can express $F(x)$ as a rational function, then success will result from expanding it in partial fractions and then handling each of the resulting term seperately. We would like to mention that we do not worry about the convergence of the power series involved; that is, we do everything formally [1].

2.2. The Calculus of Generating Functions

Operation on formal series involve corresponding operations on their coefficients. If the series actually converge and represent functions, then operations on those functions correspond to certain operations on the power series coefficients of the expansions of those functions. We will explore some of these relationships. They are of great importance in helping to spot which kind of generating function is appropriate for which kind of recurrence relation or other combinatorial situation.

Definition 2.1. *The symbol* $f \xrightarrow{ops} \{a_n\}$ *will mean that the series* f *is the ordinary power series generating function for the sequence* $\{a_n\}$*. That is,* $f = \sum a_n x^n$*.*

We have the following rules for ordinary power series generating functions to solve many enumerative problems easier:

Rule1- If $f \xrightarrow{ops} \{a_n\}$, then for any integer $h > 0$

$$\frac{f - a_o - a_1 x - \ldots - a_{h-1}x^{h-1}}{x^h} \xrightarrow{ops} \{a_{n+h}\}.$$

Rule2- If $f \xrightarrow{ops} \{a_n\}$, and P is a polynomial, then

$$P(xD)f \xrightarrow{ops} \{P(n)a_n\}.$$

Example 2.3. We will find a closed formula for the sum of the squares of the first n positive integers. Let

$$f = \sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}$$

and choose $P(x) = x^2$. Then Rule 2 gives:

$$P(xD)f \xrightarrow{ops} \{P(n)\}.$$

Specializing to $x = 1$, we can conclude that

$$\sum_{k=0}^{n} k^2 = (xD)^2 \left\{ \frac{x^{n+1}-1}{x-1} \right\}_{x=1} = \frac{n(n+1)(2n+1)}{6}.$$

The following third rule comes from the properties of multiplication of ordinary power series.

Rule3- If $f \overset{ops}{\to} \{a_n\}$ and $g \overset{ops}{\to} \{b_n\}$, then

$$f.g \overset{ops}{\to} \left\{ \sum_{r=0}^{n} a_r b_{n-r} \right\}.$$

If we multiply f by k times, then we can get the following rule,

Rule4- If $f \overset{ops}{\to} \{a_n\}$ and k be positive integer, then

$$f^k \overset{ops}{\to} \left\{ \sum_{n_1+n_2+...+n_k=n} a_{n_1} a_{n_2} ... a_{n_k} \right\}.$$

For example, since $\frac{1}{1-x} \overset{ops}{\to} \{1\}$, $\frac{1}{(1-x)^k}$ is the generating function of a_{nk} which denote the number of ways that the nonnegative integer n can be written as an ordered sum of k nonnegative integers.

Next is a special rule that helps us to find the partial sum of coefficients of power series that comes from the generating function.

Rule5- If $f \overset{ops}{\to} \{a_n\}$, then

$$\frac{f}{1-x} \overset{ops}{\to} \left\{ \sum_{i=0}^{n} a_i \right\}.$$

Now we will study the following two examples to understand usefulness of these rules.

Example 2.4. We will prove that the Fibonacci numbers satisfy

$$F_0 + F_1 + ... + F_n = F_{n+2} - 1, \qquad n \geq 0.$$

By Rule 5, the ordinary power series generating function of $F_0 + F_1 + ... + F_n$ is $\frac{F}{1-x}$ where $F = \frac{1}{1-x-x^2}$ is the generating function of the Fibonacci numbers, that is

$$\frac{F-1-x}{x^2} \overset{ops}{\to} \{F_{n+2}\}.$$

So,

$$\frac{F-1-x}{x^2}-\frac{1}{1-x}\overset{ops}{\rightarrow}\{F_{n+2}-1\},$$

$$\frac{F-1-x}{x^2}-\frac{1}{1-x}=\frac{F}{1-x},$$

thus proving the claim.

Example 2.5. (Fountain of Coins) By a fountain of coins we mean an arrangement of n coins in rows such that the coins in the first row form a single contiguous block, and that in all higher rows each coin touches exactly two coins from the row beneath it. If the first row contains k coins, we will speak of an $(n,k)-fountain$ of coins. A fountain of coins which every row consists of just a single contiguous block of coins is called block fountains. We shall find how many block fountains have a first row that consists of exactly k coins.

Let $F\overset{ops}{\rightarrow}\{f(k)\}$, where $f(k)$ is the number of block fountains having k coins in its first row. Above the row of k coins, we will place a block fountain whose first row has j coins. For $j=0$ there is just one way to do that. Otherwise there are $k-j$ ways to do it, $0\leq j\leq k-1$. So $f(0)=1$, and

$$f(k)=\sum_{j=1}^{k}(k-j)f(j)+1, \qquad k=1,2,...$$

By using Rule3, we see

$$F(x)=\frac{x}{(1-x)^2}F(x)+\frac{x}{1-x}\text{ and }F(x)=\frac{x(1-x)}{1-3x+x^2},$$

$$F(x)=-1+\frac{1}{r_+-r_-}(\frac{2-r_-}{1-r_-x}+\frac{r_+-2}{1-r_+x}), \quad r_\pm=\frac{3\pm\sqrt{5}}{2}.$$

Thus,

$$\left[x^k\right]F(x)=f(k)=-1+\frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^k+\frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^k.$$

The sequence $\{f(k)\}$ begins with $1,1,2,5,13,34,89,...$[1].

We shall now investigate the rules for **exponential power series** generating functions.

Definition 2.2. *The symbol* $f \overset{egf}{\to} \{a_n\}$ *means that the series* f *is the exponential generating function of the sequence* $\{a_n\}$, *i.e.* $f = \sum a_n \frac{x^n}{n!}$.

We have the following rules:

Rule 1- If $f \overset{egf}{\to} \{a_n\}$, then for any integer $h \geqslant 0$

$$D^h f \overset{egf}{\to} \{a_{n+h}\}.$$

Rule 2- If $f \overset{egf}{\to} \{a_n\}$ and P is a given polynomial, then

$$P(xD) f \overset{egf}{\to} \{P(n)a_n\}.$$

Example 2.6. (Fibonacci Numbers) Let us find the exponential generating function of the Fibonacci numbers. Now, with just a glance at the recurrence

$$F_{n+2} = F_{n+1} + F_n, \qquad (n \geqslant 0)$$

we see from Rule1 that the exponential generating function satisfies the differential equation

$$f'' = f' + f.$$

We solve the differential equation, getting

$$f(x) = c_1 e^{r_+ x} + c_2 e^{r_- x}, \qquad r_\pm = \frac{-1 \pm \sqrt{5}}{2}$$

and c_1 and c_2 are to be determined by the initial conditions $f(0) = f'(0) = 1$. After applying these two conditions, we find that $c_1 = \frac{r_+}{\sqrt{5}}$ and $c_2 = \frac{r_-}{\sqrt{5}}$, from which the exponential generating function of the Fibonacci sequence is

$$f = \frac{1}{\sqrt{5}}(r_+ e^{r_+ x} - r_- e^{r_- x}).$$

Rule 3- If $f \overset{egf}{\to} \{a_n\}$ and $g \overset{egf}{\to} \{b_n\}$, then $f.g$ generates the sequence

$$f.g \overset{egf}{\to} \left\{ \sum_{r=0}^{n} \binom{n}{r} a_r b_{n-r} \right\}.$$

Now, we will study the following examples to understand usefulness of these rules.

Example 2.7. (Bell Numbers) Let us look at the Bell numbers, which we may write in the form

$$b(n+1) = \sum_k \binom{n}{k} b(k), \qquad n \geqslant 0; b(0) = 1.$$

Let B be the required exponential generating function. By using Rule1 and Rule3, we have

$$B' = e^x B$$

as the equation that we must solve in order to find the unknown B, for which the solution is $B = c\exp(e^x)$, and since $B(0) = 1$, we must have $c = e^{-1}$. Thus

$$B(x) = \exp(e^x - 1).$$

Example 2.8. (Derangement of Letters) By a derangement of n letters we mean a permutation of them that has no fixed points. Let D_n denotes the number of derangements of n letters, and let $D(x)$ be an exponential generating function of D_n. We will find a recurrence for sequence, then $D(x)$, an explicit formula for the members of the sequence. The number of permutations of n letters that have a particular set of $k \leq n$ letters as their set of fixed points is clearly D_{n-k}. There are $\binom{n}{k}$ ways to choose the set of k fixed points, and so there are exactly $\binom{n}{k} D_{n-k}$ permutations of n letters that have exactly k fixed points. Since every permutation has some set of fixed points, it must be that

$$n! = \sum_k \binom{n}{k} D_{n-k}, \qquad n \geqslant 0.$$

By Rule3

$$\frac{1}{1-x} = e^x D(x).$$

Thus,

$$D(x) = \frac{e^{-x}}{1-x}.$$

By using Rule5 and identifying the coefficients of the power series

$$\frac{D_n}{n!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!}\,.$$

2.3. Theories of Generating Funtions

The problem arises of trying to explain combinatorially why certain types of generating functions such as

$$\sum a_n x^n \text{ and } \sum a_n \frac{x^n}{n!}$$

often arise, while other types like

$$\sum a_n \frac{x^n}{1+n^2} \text{ or} \sum a_n \frac{x^n}{1^1 2^2 ... n^n}$$

never seem to occur.

Two abstract theories of generating functions have been formulated to try to solve this problem, the Doubilet-Rota-Stanley theory of reduced incidence algebras, and the Bender-Goldman theory of prefabs.

2.3.1. Prefabs

The classical problems of enumerating unlabeled graphs motivate our introduction of a general structure for generating functions. We can think of an unlabeled graph as being decomposed into its connected components. Conversely, given any two graphs we can construct a new graph, their composition, whose components are the components of each of the given graphs. The connected graphs play the role of basic building blocks or primes [2].

Definition 2.3. *A prefab $(S, \circ, f)$ is a set S together with a multivalued binary operation $\circ$, $(a, b \in S$ implies $a \circ b \subseteq S)$ and a real valued function f satisfying properties below.*
The composition $\circ$

$a_1)$ *is associative*

$a_2)$ *is commutative*

$a_3)$ *has an identity i, $a \circ i = i \circ a = a$ for all a.*

We extend $\circ$ to subsets of S by

$$A \circ B = \{c : c \in a \circ b \text{ for some } a \in A, b \in B\}.$$

Definition 2.4. *We call $p \in S$ a prime if $p \in a \circ b$ implies $a = i$ or $b = i$. Then we have:*

b_1) Unique factorization; every $a \in S$ factors uniquely into primes in the sense that

$$a \in p_1^{i_1} \circ p_2^{i_2} \circ \ldots \circ p_n^{i_n},$$

where the $p_i^{'}$ s are distinct primes and they are unique up to order.

b_2) Very unique factorization; if $c \in \prod_i p_i^{\alpha_i} \circ \prod_i q_i^{\beta_i}$ where $p's$ and $q's$ are distinct primes, then there exist unique elements $a \in \prod_i p_i^{\alpha_i}$ and $b \in \prod_i q_i^{\beta_i}$ such that $c \in a \circ b$.

Then function f satisfies,

c_1) if $c_1, c_2 \in a \circ b$ then $f(c_1) = f(c_2)$

c_2) Let a and b have no common factors other then i, where $|A|$ denotes the size of A,

$$|a \circ b| = \frac{f(a \circ b)}{f(a).f(b)}.$$

Definition 2.5. *Let $(S, \circ, f)$ be a prefab. Then a multiplicative function w on S taking values in an integral domain is called a weight. By multiplicative we mean $w(c) = w(a).w(b)$ whenever $c \in a \circ b$. Then the four tuple $(S, \circ, f, w)$ is called a weighted prefab.*

Definition 2.6. *If $(S, \circ, f, w)$ is a weighted prefab and $A \subseteq S$, then the generating function or enumerator $g(A)$ of A, is given by the formal sum,*

$$g(A) = \sum_{a \in A} \frac{w(a)}{f(a)}.$$

Theorem 2.1 (Product Theorem). *If $(S, \circ, f, w)$ is weighted prefab and A, B are subsets of S such that no element of A has any prime factor in common with any element of B, then*

$$g(A \circ B) = g(A)g(B).$$

Proof. Let $c \in A \circ B$ then $c \in \prod_i p_i^{\alpha_i} \circ \prod_i q_i^{\beta_i}$ where $\prod_i p_i^{\alpha_i} \in A$ and $\prod_i q_i^{\beta_i} \in B$ so there exist unique $a \in \prod_i p_i^{\alpha_i}$ and $b \in \prod_i q_i^{\beta_i}$ such that $c \in a \circ b$. Hence,

$$g(A \circ B) = \sum_{c \in A \circ B} \frac{w(c)}{f(c)} = \sum_{a \in A} \sum_{b \in B} \sum_{c \in a \circ b} \frac{w(c)}{f(c)}.$$

But we have known that,

$$\sum_{c \in a \circ b} \frac{w(c)}{f(c)} = |a \circ b| \frac{w(a \circ b)}{f(a \circ b)} = \frac{w(a)w(b)}{f(a)f(b)}.$$

Therefore,

$$g(A \circ B) = \sum_{a \in A} \sum_{b \in B} \frac{w(a)w(b)}{f(a)f(b)} = g(A)g(B). \blacksquare$$

Corollary 2.2. *If $(S, \circ, f, w)$ is a weighted prefab, then*

$$g(S) = \prod_p \sum_k \frac{w(p)^k}{f(p^k)} |p^k|$$

where $p^0 = \{i\}$ and p ranges over all primes in S.

Proof. If $p_1, p_2, ...$ are the primes of S and P_i is the union of all powers of p_i then

$$g(P_i) = \sum_k \sum_{c \in p_i^k} \frac{w(c)}{f(c)} = \sum_k \frac{w(p)^k}{f(p)^k} |p^k|,$$

and,

$$g(S) = \prod_i g(P_i). \blacksquare$$

Corollary 2.3. *If composition is unique, then we may take $f \equiv 1$ and we have*

$$g(S) = \prod_p \frac{1}{1 - w(p)}.$$

Corollary 2.4. *If $(S, \circ, f \equiv 1, w)$ is a weighted prefab with unique composition, then we have,*

$$g(S) = \exp\left\{\sum_n \frac{g_n}{n}\right\}, \qquad \text{where} \qquad g_n = \sum_p w(p^n) = \sum_p w(p)^n.$$

Proof. Let $(S, \circ, f \equiv 1, w)$ be a weighted prefab with unique composition. Then,

$$g(S) = \prod_p \frac{1}{1 - w(p)},$$

$$\begin{aligned} \log g(S) &= \log \prod_p \frac{1}{1 - w(p)} = -\sum_p \log(1 - w(p)) \\ &= \sum_p \sum_n \frac{w(p)^n}{n} = \sum_n \frac{1}{n} \sum_p w(p)^n, \end{aligned}$$

$$g(S) = \exp\left\{\sum_p \sum_n \frac{w(p)^n}{n}\right\}. \blacksquare$$

Theorem 2.5. *If $(S, \circ, f, w)$ is a weighted prefab such that* $|p^k| = \frac{f(p^k)}{f(p)^k k!}$ *or equivalently*

$$g(p^k) = \frac{1}{k!}\left(\frac{w(p)}{f(p)}\right)^k$$

for all primes p and integers k, then, $g(S) = e^{g(P)}$ *where P, the set of primes of S, has enumerator*

$$g(P) = \sum_{p \in P} \frac{w(p)}{f(p)}.$$

Furthermore, under our assumptions $\frac{g(p)^n}{n!}$ *is the generating function of the set of elements which are products of n primes (not necessarily distinct).*

Proof. Take

$$|p^k| = \frac{f(p^k)}{f(p)^k k!}$$

and put it into

$$g(S) = \prod_p \sum_k \frac{w(p)^k}{f(p^k)} |p^k|.$$

Then,

$$\begin{aligned} g(S) &= \prod_p \sum_k \frac{w(p)^k f(p^k)}{f(p^k) f(p)^k k!} = \prod_p \exp\left\{\frac{w(p)}{f(p)}\right\} \\ &= \exp \sum_p \frac{w(p)}{f(p)} = \exp g(P). \end{aligned}$$

Now define a new weight $w^*(a) = w(a) y^n$ when a is the product of n primes. Then,

$$g^*(P) = \sum_p \frac{w^*(p)}{f(p)} = \sum_p \frac{w(p) y}{f(p)}$$

$$g^*(S) = \exp\{g^*(P)\} = \exp\{y g(P)\} = \sum_n \frac{g^n(P)}{n!} y^n. \blacksquare$$

Example 2.9. (Labeled Graphs) Let the set S consists of all labeled graphs (a graph with n vertices uses the labels $1, 2, ..., n$ for the vertices). Composition $a \circ b$ of two graphs a and b with m and n vertices respectively is defined as follows: With each partition of $\{1, 2, ... m+n\}$ into a set of size m and one of size n, say $\{v_1, ... v_m\}$ and $\{u_1, ..., u_n\}$ associate the graph whose components are those of a and b labelled by replacing i in a by v_i and i in b by u_i. Run through all partitions of $\{1, 2, ..., m+n\}$ to get $a \circ b$. If a graph arises in more than one way by this construction we keep only one copy of it in $a \circ b$. The primes are the connected graphs. If a has m vertices and b has n vertices and a and b have no common factors then,

$$|a \circ b| = \binom{m+n}{n} = \frac{(m+n)!}{m!n!}.$$

By choosing $f(a) = m!$ and $w(a) = x^m$

$$f(p^k) = (mk)! \text{ and } |p^k| = \frac{(mk)!}{k!(m!)^k}.$$

Thus,

$$\sum \frac{w(p)^k}{f(p^k)} |p^k| = \sum \frac{w(p)^k}{(m!)^k k!} = \exp\left\{\frac{w(P)}{f(P)}\right\},$$

$$g(S) = \prod_p \exp\left\{\frac{w(p)}{f(p)}\right\} = \exp\{g(C)\},$$

where $g(C)$ is the enumerator of connected graphs [2].

2.3.2. Incidence Algebra

A very general approach to the construction of generating function was presented in [3]. Some ideas related to this approach will be given in this section.

Definition 2.7. *A poset (or partially ordered set) P is locally finite if every interval*

$$[x, y] = \{z \in P : x \leqslant z \leqslant y\}$$

of P is finite.

Definition 2.8. *A binomial poset is a partially ordered set P satisfying the following three conditions:*

i) P is locally finite and contains arbitrarily long finite chains. (A chain is a totally ordered subset of P.)

ii) For every interval $[x, y] \in P$, all maximal chains between x and y have same length, which we denote by $n(x, y)$. If $n(x, y) = n$, then we call $[x, y]$ an n-interval, (the length of a chain is one less then its number of elements).

iii) For all $n \in N$ any two n-intervals contains the same number $B(n)$ of maximal chains.

Example 2.10.1) If $P = N$ with the usual order, then $B(n) = 1$ for all $n \in N$.

2) If P is the lattice of all finite subsets of N, ordered by imclusion, then $B(n) = n!$.

To see the connection between binomial posets and generating functions of the form

$$\sum a_n \frac{x^n}{B(n)},$$

it is necessary to consider incidence algebra.

Definition 2.9. *If P is any locally finite poset, the incidence algebra $I(P)$ of P is the vector space of all functions $f : S(P) \to C$, where $S(P)$ is the set of all nonvoid intervals $[x, y]$ of P, endowed with the multiplication (convolution). The multiplication $*$ of these functions is defined as follows: $f * g = h$ means*

$$h(x, y) = \sum_{x \leqslant z \leqslant y} f(x, z) g(z, y).$$

Note that the above sum is finite since P is locally finite. Multiplication as defined above is associative and distributive with respect to addition. The neutral element of $I(P)$ with respect to addition is Kronecker symbol.

Let $R(P)$ be the subspace of $I(P)$ consisting of functions f constant on n-intervals, i.e. $f(x, y) = f(z, w)$ whenever $[x, y]$ and $[z, w]$ have the same length. If $f \in R(P)$, we write $f(n)$ for $f(x, y)$, where $[x, y]$ is an $n-$interval.

A fundemental property of binomial poset is that $R(P)$ is a sub algebra of $I(P)$, i.e. $R(P)$ is closed under convolution. Note also that $\delta \in R(P)$. We can see that

$$h(n) = \sum_i \left\langle \frac{n}{i} \right\rangle f(i) g(n - i),$$

where $\langle \frac{n}{i} \rangle$ means the number of distinct elements z from a segment $[x, y]$ of type n such that $[x, z]$ is a segment of type i and $[z, y]$ of type $n - i$. This symbol is called the incidence coefficient.

Since $B(i)B(n - i)$ maximal chains of $[x, y]$ pass through a given such z, we have

$$\left\langle \frac{n}{i} \right\rangle = \frac{B(n)}{B(i)B(n - i)}.$$

This is the P-analogue of the formula

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

Indeed, an element from $R(X)$ is uniquely determined by the sequence $[a_n]$ of real numbers if we put $f(i,j) = a_{j-i}$ where $i \leqslant j$. Multiplication of elements is defined by the equality

$$h(i,j) = \sum_{i \leqslant k \leqslant j} f(i,k)g(k,j) = \sum_{i \leqslant k \leqslant j} a_{k-i}b_{j-k}.$$

Setting $r = k - i$ and $n = j - i$, we can obtain

$$h(i,j) = \sum_{r=0}^{n} a_r b_{n-r} = c_n.$$

Hence it follows that the mapping of the set of power series into $R(X)$ defined as

$$F(t) = \sum_n a_n x^n \longrightarrow f(i,j) = a_{j-i}, \qquad j \geqslant i,$$

is an isomorphism. We can conclude:

Theorem 2.6. *Let P be a binomial poset. Then $R(P)$ is isomorphic to the ring of formal power series over the complex numbers C, where*

$$f \longrightarrow F_f(X) = \sum_n f(n)\frac{x^n}{B(n)}.$$

Let us consider some applications. In the following example, P is assumed to be a binomial poset.

Example 2.11. Define $f \in R(P)$ by $f(n) = 1$ for all $n \in N$. Then for an n-interval $[x, y]$

$$f^2(n) = f^2(x,y) = \sum_{x \leqslant z \leqslant y} f(x,z)f(z,y) = \sum_{x \leqslant z \leqslant y} 1 = |[x,y]|\,.$$

Hence by the theorem, the cardinality of n-interval is given by

$$\sum_n a_n \frac{x^n}{B(n)} = \left(\sum_n \frac{x^n}{B(n)}\right)^2.$$

But since $P = N$, $B(n) = 1$. And we have that the cardinality a_n of a chain of length n satisfies

$$\sum_n a_n x^n = \left(\sum_n x^n\right)^2 = \frac{1}{(1-x)^2} = \sum_n (n+1)x^n.$$

So, $a_n = n + 1$.

Similarly, the number of subsets of n-element set satisfies, by putting $B(n) = n!$

$$\sum_n a_n \frac{x^n}{n!} = \left(\sum_n \frac{x^n}{n!}\right)^2 = e^{2x} = \sum_n 2^n \frac{x^n}{n!}.$$

Thus we conclude that $a_n = 2^n$ [3].

2.4. The Main Counting Theorems

In this section we will discuss a particularly rich vein of applications of the theory of generating functions to counting problems. The exponential formula, which is our main goal here, is a cornerstone of the art of counting. It deals with the question of counting structures that are built out of connected pieces. The structures themselves need not be connected, but their pieces always are. The question is, if we know how many pieces of each size there are, how many structures of each size can we build out of those pieces?

We are going to develop the theory in a context of playing cards and hands. In order to do that, we will see a number of definitions of the basic terminology[1].

Definition 2.10. *Let P be the collection of all $n-$tuples whose entries are the integers $1, 2, ..., n$. An element of P is called a picture.*

Definition 2.11. *A card $C(S,p)$ is a pair consisting of a finite set S (the label set) of positive integers, and a picture $p \in P$. The weight of C is $n = |S|$.*

For instance, one card of weight 3 might be

$$C(S,p) = (\{5,7,9\}, 2-1-3),$$

which would correspond to the connected labelled graph $7-5-9$.

Definition 2.12. *A card of weight n is called standard if its label set is $[n]$, where $[n] = \{1, 2, ..., n\}$.*

Definition 2.13. *A hand H is a set of cards whose label sets form a partition of $[n]$ for some n. The weight of hand is the sum of the weights of the cards in the hand.*

Definition 2.14. *A deck D is a finite set of standard cards whose weights are all the same and whose pictures are all different. The weight of the deck is the common weight of all of the cards in the deck.*

Definition 2.15. *An exponential family F is a collection of decks $D_1, D_2, ...$ where for each $n = 1, 2, 3, ...$ the deck D_n is of weight n.*

If F is an exponential family, we will write d_n for the number of cards in deck D_n, and we will call $D(x)$, the exponential generating function of the sequence d_n, the deck enumerator of the family. Let $h(n,k)$ denotes the number of hands H of weight n that consists of k cards and

$$H(x,y) = \sum_{n,k} h(n,k)\frac{x^n.y^k}{n!},$$

which is called the hand enumerator of the family.

Let F' and F'' be two exponential families whose picture sets P', $P''$are disjoint. We form a third family F which is called their merger, and write $F = F' \oplus F''$ as follows: Fix $n \geqslant 1$, from F' we take all of the d'_n cards of deck D'_n and put them in a new pile. Then from F'' we take all d''_n of its cards from deck D''_n and add these d''_n cards to the pile, which now contains $d_n = d'_n + d''_n$ different cards. Repeat this for each $n \geqslant 1$.

Lemma 2.7. *Let F' and F'' be two exponential families, and let $F = F' \oplus F''$ be their merger. If $H'(x,y)$, $H''(x,y)$ and $H(x,y)$ be the respective two variable hand enumerator of these families then,*

$$H(x,y) = H'(x,y)H''(x,y).$$

Proof. Consider a hand H in the merged family F. Some of its cards came from F' and some came from F''. The collection of cards that came from F' forms a sub-hand H' of weight, say n', and having k' cards, that has been relabeled, in an order preserving way, with a certain label set $S \subset [n]$. All hands H in the merged family are uniquely determined by a particular hand H' from F', the choice of new labels S with which that hand is to be relabeled, and the remaining sub-hand H'' from F'', which must be relabeled, again preserving the order of the labels, with $[n] - S$.

Consequently the number of hands in the merged family that have weight n and have exactly k cards is

$$\begin{aligned} h(n,k) &= \sum_{n',k'} \binom{n}{n'} .h'(n',k').h''(n-n',k-k') \\ &= \left[\frac{x^n.y^k}{n!}\right] .H'(x,y).H''(x,y). \end{aligned}$$

Thus,

$$H(x,y) = H'(x,y)H''(x,y). \blacksquare$$

Theorem 2.8 (The Exponential Formula). *Let F be an exponential family whose deck and hand enumerators are $D(x)$ and $H(x,y)$ respectively. Then*

$$H(x,y) = e^{yD(x)}$$

$$i.e., \quad h(n,k) = \left[\frac{x^n}{n!}\right] \left\{\frac{D(x)^k}{k!}\right\}.$$

Proof.

Step 1: Fix a positive integer r. Let $d_r = 1$ and $d_j = 0$, $\quad j \neq r$. The deck enumerator is

$$D(x) = \sum_n d_n \frac{x^n}{n!} = \frac{x^r}{r!} .$$

A hand consists of some number, say s, of copies of one card that exists. The weight of H is rs. Therefore the number of hands of k cards and of weight n is $h(n,k) = 0$ unless $n = kr$. If $n = kr$ then

$$h(kr,r) = \frac{n!}{k!r!^k}.$$

The hand enumerator of this elementary family is

$$H(x,y) = \sum_{n,k} h(n,k)\frac{x^n y^k}{n!} = \sum_k \frac{x^{kr}y^k}{k!r!^k} = \exp\left\{\frac{yx^r}{r!}\right\}.$$

Step 2: Let $d_r = n$ and $d_j = 0$ if $j \neq r$. We will show that

$$H(x,y) = \exp\left\{\frac{ynx^r}{r!}\right\}.$$

The proof is by induction on d_r. The equality holds when $d_r = 1$ by Step 1. Suppose the equality holds for $d_r = 1, 2, ..., n-1$. Let the family F have n cards in its r'th deck. Then F is the result of merging a family with $n-1$ cards in the r'th deck and a family with 1 card in that deck. By the Fundamental Lemma

$$\exp\left\{\frac{y(n-1)x^r}{r!}\right\}\exp\left\{\frac{yx^r}{r!}\right\} = \exp\left\{\frac{ynx^r}{r!}\right\},$$

and the claim is proved.

Step 3: Now let $D_1, D_2, ...$ is a full sequence of non-empty decks that is the merger of the special families $F_r(r = 1, 2, ...)$, each of which has just a single non-empty deck D_r. By the Fundamental Lemma, the hand enumerator of the general family is the product of the hand enumerators of the special families.

$$H(x,y) = \prod_{r \geqslant 1} H_r(x,y) = \exp\left\{y\sum_r d_r\frac{x^r}{r!}\right\} = \exp\{yD(x)\}.$$

In detail

$$\begin{aligned} H(x,y) &= \sum_{n,k} h(n,k)\frac{x^n y^k}{n!} = \sum_k \frac{y^k D(x)^k}{k!}, \\ h(n,k) &= \left[\frac{x^n}{n!}\right]\left\{\frac{D(x)^k}{k!}\right\}. \blacksquare \end{aligned}$$

Theorem 2.9. *The counting sequences $\{d_n\}$ and $\{h_n\}$, of decks and hands in an exponential family satisfy the recurrence*

$$nh_n = \sum_k \binom{n}{k} kd_k h_{n-k}, \qquad n \geqslant 1, \quad h_0 = 1.$$

Proof. By the Exponential Formula

$$\begin{aligned} H(x) &= \exp\{D(x)\} \\ &\Rightarrow \sum_n h_n\frac{x^n}{n!} = \exp\left\{\sum_n d_n\frac{x^n}{n!}\right\} \\ &\Rightarrow \log\sum_n h_n\frac{x^n}{n!} = \sum_n d_n\frac{x^n}{n!} \\ &\Rightarrow \frac{\sum_n nh_n\frac{x^{n-1}}{n!}}{\sum_n h_n\frac{x^n}{n!}} = \sum_n nd_n\frac{x^{n-1}}{n!} \\ &\Rightarrow \sum_n nh_n\frac{x^n}{n!} = \sum_n h_n\frac{x^n}{n!}\sum_n nd_n\frac{x^n}{n!} \\ &\Rightarrow \sum_n nh_n\frac{x^n}{n!} = \sum_n\left(\sum_k\binom{n}{k}kd_k h_{n-k}\right)\frac{x^n}{n!}. \end{aligned}$$

Therefore,

$$nh_n = \sum_k \binom{n}{k} kd_k h_{n-k}. \blacksquare$$

Example 2.12. (Set Partition) By a partition of a set S we will mean a collection of non-empty pairwise disjoint sets whose union is S. Let for each $n \geqslant 1$, in the deck D_n there is just one card of weight n. On that card there is a picture, and there is the label set $[n]$. There is a hand H corresponding to every partition of the set $[n]$. So $h(n,k)$ denotes the number of hands of weight n that have k cards which is equal to the number of partitions of the set $[n]$ into k classes. We called those numbers $s(n,k)$, the Stirling numbers of the second kind.So,

$$D(x) = \sum_{n \geqslant 1} d_n \frac{x^n}{n!} = \sum_{n \geqslant 1} \frac{x^n}{n!} = e^x - 1.$$

Now by the exponential formula the enumerator of hands is

$$H(x,y) = \exp\left\{y(e^x - 1)\right\},$$

and in particular

$$s(n,k) = \left[\frac{x^n}{n!}\right]\left\{\frac{(e^x-1)^k}{k!}\right\}.$$

We have therefore shown:

Theorem 2.10. *The exponential generating function of the Bell numbers is*

$$\exp(e^x - 1).$$

i.e. the coefficient of $\frac{x^n}{n!}$ *in the power series expansion of* $\exp(e^x - 1)$ *is the number of partitions of a set of* n *elements* [1].

An alternative approach to the proof of the previous theorem is following [4]. Consider an auxiliary finite set U having u elements. We shall examine the structure of the set U^S of functions with domain S, a set with n elements, and range a subset of U. The basic fact is that there are u^n distinct such functions, as is evidenced by the most elementary of counting arguments. We shall now examine this set of functions in greater detail.

To every functions $f : S \mapsto U$ there is naturally associated a partition π of the set S, called the kernel of f, defined as follows. Two elements a and b of S are to belong to the same subsets of π, if and only if $f(a) = f(b)$.

Let $N(\pi)$ denote the number of the distinct subsets of the partition π. A function having kernel π must take distinct values on distinct subsets of π. Thus, such a function takes altogether $N(\pi)$ distinct values, and the number of distinct such functions equals to the number of one-to-one functions from a set of $N(\pi)$ elements to the set U. It is well known that such a number is

$$u(u-1)(u-2)...(u-N(\pi)+1) = (u)_{N(\pi)}$$

and this expression is called the factorial power of the number u, with exponent $N(\pi)$.

Now, every function has a unique kernel. We have

$$\sum_{\pi}(u)_{N(\pi)} = u^n.$$

We now come to the main idea. Let V be the vector space over the reals consisting of all polynomials in the single variable u. Any sequence of polynomials of degrees $0, 1, 2, ...$ is a basis for this vector space, in particular, the sequence $(u)_n$.

Take $(u)_0 = 1, (u)_1, (u)_2, ...$ is a basis and a linear functional L on V which is defined as

$$L((u)_k) = 1 \text{ for } k = 0, 1, 2, ...$$

Therefore,

$$L(\sum_{\pi} N(\pi)) = L(u^n).$$

But, by the definition of L, the left side simplifies to a sum of as many ones as there are partitions of the set S. In other words,

$$B_n = L(u^n)$$

where B_n is the number of the partition of $[n]$. We claim that

$$B_{n+1} = \sum_k \binom{n}{k} B_k.$$

Now, since

$$u(u-1)_n = (u)_{n+1}$$

we have

$$L(u(u-1)_n) = L((u)_n),$$
$$L(up(u-1)) = L(p(u))$$

for every polynomial p. In particular, for $p(u) = (u+1)^n$ we obtain

$$L(u^{n+1}) = L((u+1)^n).$$

Thus the claim is proved. Now,

$$\sum_n B_n \frac{x^n}{n!} = \sum_n L(u^n)\frac{x^n}{n!} = L(\sum_n u^n \frac{x^n}{n!}) = L(e^{xu}).$$

Set $e^x = 1 + v$. So

$$\sum_n B_n \frac{x^n}{n!} = L((1+v)^u) = L(\sum_n (u)_n \frac{v^n}{n!})$$
$$= \sum_n L((u)_n)\frac{v^n}{n!} = e^v = \exp\{e^x - 1\}.$$

2.4.1. Unlabeled Cards and Hands

A card $C = C(n,p)$ now has only its weight n and its picture p. For each $n = 1, 2, 3, ...$ there is a deck D_n that contains d_n cards, all of weight n. A hand is a multiset of cards. That is, we may reach into one of the decks D_r and pull out of it some number of copies of a single card $C(r, p')$, then a number of copies of $C(r, p'')$, and so forth, then from another deck we can take more cards, etc.

No significance attaches to the sequence of cards in the hand. What matters is which cards have been sellected and with which multiplicities. The weight of a hand is the sum of the weights of the cards in the hand. And instead of exponential families we will call prefabs P. So P consists of a sequence of decks $D_1, D_2, ...$

Now, for the unlabeled cards and hands

$$H(x,y) = \sum_{n,k} h(n,k)x^n y^k$$

$$D(x) = \sum_n d(n)x^n.$$

If P' and P'' are prefabs whose picture sets are disjoint, then by their merger

$$P = P' \oplus P''$$

we mean the prefab whose deck D_n, for each n, is the union of the corresponding decks of P' and P''. If there were d'_n, d''_n cards, respectively, then there are $d_n = d'_n + d''_n$ cards in D_n.

Lemma 2.11. *Let $H'(x,y)$, $H''(x,y)$ and $H(x,y)$ be the hand enumerators of prefabs P', P'' and $P = P' \oplus P''$, respectively. Then*

$$H(x,y) = H'(x,y).H''(x,y).$$

Proof. Consider a hand $H \in P$, of weight n, and containing exactly k cards. Some k' of those cards come from P', and their total weight is, say, n', while the remaining $k - k'$ cards come from P'', and their total weight must be $n - n'$. Thus,

$$h(n,k) = \sum_{n',k'} h'(n',k').h''(n-n',k-k').$$

But, by a strange coincidence, that is exactly the relationship which holds between the coefficients of the power series H, H' and H''. So,

$$H(x,y) = H'(x,y)H''(x,y). \blacksquare$$

Theorem 2.12. *In a prefab P whose hand enumerator is $H(x,y)$, we have*

$$H(x,y) = \prod_n \frac{1}{(1-yx^n)^{d_n}}$$

where d_n is the number of cards in the n'th deck.

Let W be fixed set of non-negative integers, containing 0. For each n and k, we let $h(n,k,W)$ be the number of hands of weight n that have exactly k cards, each appearing with a multiplicity that belongs to W.

Let

$$H(x,y,W) = \sum_{n,k} h(n,k,W)x^n y^k \qquad \text{and} \qquad W(t) = \sum_{k \in W} t^k.$$

Theorem 2.13. *Let the prefab P contains decks of sizes $d_1, d_2, \ldots$ and let W be a set of non-negative integers, $0 \in W$. If $h(n,k,W)$ is the number of hands of k cards of weight n, such that each card appears with a multiplicity that belongs to W, then*

$$H(x,y,W) = \sum_{n,k} h(n,k,W)x^n y^k = \prod_{r\geqslant 1} W(yx^r)^{d_r}.$$

Proof. Consider a prefab with just 1 card of weight r, and no other decks. Then $h(n,k,W) = 1$ if $k \in W$, $n = kr$, and 0 otherwise. So

$$H(x,y,W) = \sum_{k\in W} x^{kr} y^r = W(yx^r).$$

If there are d_r cards in the $r'th$ deck, and no other cards. Then

$$H(x,y,W) = W(yx^r)^{d_r}. \blacksquare$$

Example 2.13. (Partitions of Integers) A partition of a positive integer n is representation

$$n = r_1 + r_2 + \ldots + r_k, \qquad (r_1 \geqslant r_2 \geqslant \ldots \geqslant r_k \geqslant 1).$$

The number of $r_1, r_2, \ldots, r_k$ are the parts of the partition. The number of partitions of n is denoted by $p(n)$ and $p(n,k)$ is the number of partitions of n into k parts. Then

$$\sum_{n,k} p(n,k)x^n y^k = \prod_{i\geqslant 1}(1-x^i)^{-1}, \qquad p(o,k) = \delta_{0,k}.$$

With, $y = 1$, we find

$$\sum_{n\geqslant 0} p(n)x^n = \prod_{n\geqslant 1}(1-x^n)^{-1}$$

as the generating function for $\{p(n)\}$ itself [1].

3. APPLICATIONS OF GENERATING FUNCTIONS

3.1. Partitions of Integers and Partition Identities

The concept of partition of integers belongs to number theory as well as to combinatorial analysis. This theory was established at the end of the 18-th century by Euler. In this section we will show some partition identities by using generating functions.

A partition of a positive integer n is defined as a way of writing n as the sum of positive integers. We shall denote by $p(n)$ the number partitions of n. Thus, for example, since 5 can be expressed as the sum of positive integers by

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1 \text{ and } 1+1+1+1+1$$

we have $p(5) = 7$, see [5][6].

Theorem 3.1. *The generating function for $p(n)$ is given by*

$$f(x) = \prod_{n \geqslant 1} (1 - x^n)^{-1}, \qquad |x| < 1.$$

Proof. Since

$$\sum_{n \geqslant 0} x^n = \frac{1}{1-x},$$

the function

$$(1 - x^n)^{-1} = \sum_{j \geqslant 0} x^{jn}$$

and taking $n = 1, 2, 3, ..., m$ we can find

$$\begin{aligned}
\prod_{n=1}^{m} (1 - x^n)^{-1} &= \prod_{n=1}^{m} \sum_{j} x^{jn} \\
&= (1 + x^{1.1} + x^{2.1} + ...)(1 + x^{1.2} + x^{2.2} + ...)...(1 + x^{1.m} + x^{2.m} + ...) \\
&= \sum_{j_1} \sum_{j_2} ... \sum_{j_m} x^{j_1.1 + j_2.2 + ... + j_m.m} = \sum_{j=0}^{m} c_j x^j
\end{aligned}$$

where c_j is the number of solution of $j_1.1 + j_2.2 + ... + j_m.m = j$ where $j_j \geqslant 0$ for all j. That is $c_j = P_m(j)$.

Let $x \in [0,1)$. We introduce the functions

$$f_m(x) = \prod_{n=1}^{m}(1-x^n)^{-1} \text{ and } f(x) = \prod_{n\geqslant 1}(1-x^n)^{-1} = \lim f_m(x).$$

The product defining $f(x)$ converges for $|x| < 1$ because $\sum_k x^k$ do. For fixed $x \in [0,1)$, the series $f_m(x)$ grows monotonically. Therefore

$$f_m(x) \leqslant f(x) \text{ for fixed } x \in [0,1).$$

Since $f_m(x)$ is product of a finite number of absolutely convergent series, $f_m(x)$ is absolutely convergent series and can be written as

$$f_m(x) = \sum_{n\geqslant 0} P_m(n)x^n$$

where $P_m(n)$ denotes the number of partition.

For $m \geqslant n$, $P_m(n) = P(n)$. Since $P_m(n) \leqslant P(n)$, $\lim P_m(n) = P(n)$. Now,

$$\begin{aligned} f_m(x) &= \sum_{n=0}^{m} P_m(n)x^n + \sum_{n=m+1} P_m(n)x^n \\ &= \sum_{n=0}^{m} P(n)x^n + \sum_{n=m+1} P_m(n)x^n. \end{aligned}$$

Because $x \geqslant 0$

$$\sum_{n=0}^{m} P(n)x^n \leqslant f_m(x) \leqslant f(x).$$

Consequently $\sum_n P(n)x^n$ converges, because $P_m(n) \leqslant P(n)$,

$$\sum_n P_m(n)x^n \leqslant \sum_n P(n)x^n \leqslant f(x).$$

Consequently, the series $\sum_{n\geqslant 0} P_m(n)x^n$ converges uniformly for all m and

$$\begin{aligned} f(x) &= \lim f_m(x) = \lim \sum_{n\geqslant 0} P_m(n)x^n \\ &= \sum_{n\geqslant 0} \lim P_m(n)x^n = \sum_{n\geqslant 0} P(n)x^n. \end{aligned}$$

So, $f(x) = \prod_{n\geqslant 1}(1-x^n)^{-1} = \sum_{n\geqslant 0} P(n)x^n$. ∎

Theorem 3.2. *We have*

$$\begin{aligned} f(x) &= 1+\sum_{n\geqslant 1} P(n)x^n = \prod_{i\geqslant 1}(1-x^i)^{-1} \\ &= 1+\sum_{m\geqslant 1}\frac{x^m}{(1-x)(1-x^2)...(1-x^m)} \end{aligned}$$

$$\begin{aligned} f(x,u) &= 1+\sum_{n\geqslant 1} P(n,m)x^n u^m = \prod_{i\geqslant 1}(1-ux^i)^{-1} \\ &= 1+\sum_{m\geqslant 1}\frac{x^m u^m}{(1-x)(1-x^2)...(1-x^m)}, \end{aligned}$$

where $P(n,m)$ is the number of partitions of n into m summands and $1\leqslant m\leqslant n$.

Proof. Let

$$f(x,u)=\sum_{m\geqslant 0} C_m u^m,$$

where $C_m = C_m(x)$. Since

$$f(x,u)=\prod_{i\geqslant 1}(1-ux^i)^{-1} \text{ and } f(x,xu)=(1-xu)f(x,u)$$

so,

$$\sum_{n\geqslant 0} C_n x^n u^n = (1-xu)\sum_{n\geqslant 0} C_n u^n = \sum_{n\geqslant 0}(C_n - xC_{n-1})u^n.$$

Hence

$$x^n C_n = C_n - xC_{n-1} and C_n = \frac{x}{1-x^n}C_{n-1}.$$

Therefore

$$\begin{aligned} C_n &= \frac{x^n}{(1-x)(1-x^2)...(1-x^n)}, \\ f(x,u) &= 1+\sum_{n\geqslant 1}\frac{x^n u^n}{(1-x)(1-x^2)...(1-x^n)} \end{aligned}$$

By putting $u=1$

$$f(x)=1+\sum_{n\geqslant 1}\frac{x^n}{(1-x)(1-x^2)...(1-x^n)}.$$

∎

Theorem 3.3 (Durfee Square Identity).

$$\sum_{n\geqslant 0} P(n)x^n = 1 + \sum_{n\geqslant 1} \frac{x^{n^2}}{(1-x)^2.(1-x^2)^2....(1-x^n)^2}.$$

Proof. Define

$$f(x,u) = \prod_{i\geqslant 1}(1-x^i u)^{-1} \text{ and } f_m(x,u) = \prod_{k=1}^{m}(1-x^k u)^{-1}.$$

Let

$$f(x,u) = \sum_{m\geqslant 1} C_m(x)u^m f_m(x,u), \text{ where } C_0(x) = 1.$$

Then

$$f(x,xu) = (1-xu)f(x,u),$$

and

$$f_m(x,xu) = (1-xu)\left\{f_m(x,u) + x^{m+1}uf_{m+1}(x,u)\right\}.$$

So,

$$f(x,xu) = \sum_{m\geqslant 1} C_m(x)x^m u^m (1-xu)\left\{f_m(x,u) + x^{m+1}uf_{m+1}(x,u)\right\}$$

$$f(x,u) = \sum_{m\geqslant 1} C_m(x)x^m u^m \left\{f_m(x,u) + x^{m+1}uf_{m+1}(x,u)\right\}.$$

By identifying the coefficients of u^m, we can obtain

$$C_m(x) = \frac{C_{m-1}(x)x^{2m-1}}{1-x^m}, \text{ where } C_0(x) = 1.$$

Thus,

$$C_n(x) = \frac{x^{n^2}}{(1-x).(1-x^2)...(1-x^n)}.$$

Consider the coefficients of u^n for the first equation, that is, $C_n(x)F_n(x)$ and by putting $u = 1$ we find

$$\sum_{n\geqslant 0} P(n)x^n = 1 + \sum_{n\geqslant 0} \frac{x^{n^2}}{(1-x)^2.(1-x^2)^2...(1-x^n)^2}.$$

∎

Theorem 3.4. *Let $P(n)$ be the number of partition of n, then*

$$nP(n) = \sum_{lk \leq n} l.P(n - lk).$$

Proof. Suppose $|x| < 1$

$$f(x) = \prod_{i \geqslant 1}(1 - x^i)^{-1} = 1 + \sum_{l \geqslant 1} P(l)x^l$$

$$\log f(x) = \log \prod_{i \geqslant 1}(1 - x^i)^{-1} = \sum_{i \geqslant 1} \log(1 - x^i)^{-1}$$

$$\frac{f'(x)}{f(x)} = \sum_{i \geqslant 1} \frac{ix^{i-1}}{1 - x^i} = \frac{1}{x} \sum_{l \geqslant 1} \frac{lx^l}{1 - x^l}.$$

Differentiating $f(x)$ from the series expansion

$$\begin{aligned} \sum_{n \geqslant 1} nP(n)x^n &= xf'(x) = f(x) \sum_{l \geqslant 1} \sum_{k \geqslant 1} lx^{lk} \\ &= \left\{1 + \sum_{m \geqslant 1} P(m)x^m\right\} \sum_{l \geqslant 1} \sum_{k \geqslant 1} lx^{lk}. \end{aligned}$$

By comparison of coefficients we find

$$nP(n) = \sum_{lk \leq n} lP(n - lk). \blacksquare$$

Theorem 3.5. *The number of partitions of n into exactly m parts (m is a given positive integer) is equal to the number of partitions of n into parts the largest of which is m.*

Proof. A partition of n can be represented graphically. Let $n = a_1 + a_2 + ... + a_m$. We may presume that $a_1 \geqslant a_2 \geqslant ... \geqslant a_m$. Then the graph of this partition is the array of points having a_1 points in the top row a_2 in the next row, and so on down to a_m in the bottom row. If we read the graph vertically instead of horizontally, we obtain a possibly different partition. Given a partition $n = a_1 + a_2 + ... + a_m$ consisting of m parts with the largest part a_1, the conjugate partition of n has a_1 parts with largest part m. Since this correspondence reversible, the theorem is proved. $\blacksquare$

Theorem 3.6 (Euler). *The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

Proof. Let $f(x) = \prod_{n\geqslant 1}(1+x^n)$ be the generating function of n into distinct parts, and $g(x) = \prod_{n\geqslant 0}(1-x^{2n+1})^{-1}$ be the generating function of the partition of n into odd parts. Then,

$$\begin{aligned} f(x) &= \prod_{n\geqslant 1}(1+x^n) = (1+x)(1+x^2)(1+x^3)... \\ &= \frac{1-x^2}{1-x}.\frac{1-x^4}{1-x^2}.\frac{1-x^6}{1-x^3}.... = \prod_{n\geqslant 0}(1-x^{2n+1})^{-1} = g(x). \end{aligned}$$

∎

Theorem 3.7. *The number of partitions of n into m unequal summands equals the number of partitions of n* $- \binom{m+1}{2}$ *into at most m summands* [7].

Proof. Let

$$f(x,u) = 1 + \sum_{1\leq m\leq n} Q(n,m)x^n u^m = \prod_{i\geqslant 1}(1+ux^i).$$

So,

$$f(x,u) = (1+xu)f(x,xu).$$

Define

$$f(x,u) = \sum_{n\geqslant 0} C_n u^n = (1+xu)\sum_{n\geqslant 0} C_n x^n u^n$$

$$\sum_{n\geqslant 0} C_n u^n = \sum_{n\geqslant 0} C_n x^n u^n + \sum_{n\geqslant 0} C_n x^{n+1} u^{n+1}.$$

By the comparison of the coefficients of u^n, we can find,

$$C_n = C_n x^n + C_{n-1} x^{n-1}$$

and

$$C_n = \frac{x^{n-1}}{1-x^n} C_{n-1} = \frac{x^{n-1}.x^{n-2}....}{(1-x^n).(1-x^{n-1})....}$$

$$f(x,u) = 1 + \sum_{1\leq m\leq n} Q(n,m)x^n u^m = 1 + \sum_{m\geqslant 1} \frac{u^m x^{\binom{m+1}{2}}}{(1-x)(1-x^2)...(1-x^m)}.$$

So $Q(n,m)$ equals to the coefficient of $x^{n-\binom{m+1}{2}}$, thus

$$Q(n,m) = P(n - \binom{m+1}{2}, m).$$ ∎

If $Q^e(n)$ is the number of partitions of n into an even number of distinct parts and $Q^o(n)$ is the number of partitions of n into an odd number of distinct parts. Then we have the following theorem [7].

Theorem 3.8. *If* $n \geq 0$, *then*

$$\begin{aligned} Q^e(n) - Q^o(n) &= (-1)^j \text{ if } n = (3j^2 \pm j)/2 \text{ for some } j = 0, 1, 2, \ldots \\ &= 0 \text{ otherwise.} \end{aligned}$$

Proof. For $n = 0$ we have $Q^e(n) - Q^o(n) = 1$. Now suppose $n \geqslant 1$ and consider a partition $n = a_1 + a_2 + \ldots + a_r$ into distinct parts. In the graph of this partition we let A_1denote the point farthest to the right in the first row. Since the parts are distinct, there will be no point directly below A_1. If $a_2 = a_1 - 1$, then there will be a point A_2 directly below the point that is immediately to the left of A_1. If $a_2 < a_1 - 1$, then there will be no such point A_2. If $a_3 = a_1 - 2$, then $a_2 = a_1 - 1$ and there will be a point A_3 directly below the point that is immediately to the left of A_2. If $a_2 = a_1 - 1$ and $a_3 < a_2 - 1$, there will be no point A_3. We continue this process as far as possible, thus obtaining a set of points $A_1, A_2, \ldots A_s$, $s \geq 1$ lying on a line through A_1 with slope 1. We also label the points of the bottom row $B_1, B_2, \ldots B_t$, $\quad t = a_r$. Notice that B_t and A_s may be the same point.

Now we wish to change the graph into the graph of another partition of n into distinct parts. First, we try taking the points $B_1, B_2, \ldots, B_t$ and placing them to the right of $A_1, A_2, \ldots, A_t$; B_1 to the right of A_1, B_2 to the right of A_2 and so on. It is obvious that we cannot do this $t > s$, or if $t = s$ and $B_t = A_s$. However we can do it if $t < s$ or if $t = s$ and $B_t \neq A_s$, and we obtain a graph of a partition into distinct parts. Second, we try the reverse process, putting $A_1, A_2, \ldots, A_s$ underneath $B_1, B_2, \ldots, B_s$. This will give a proper graph if and only if $s < t - 1$ or $s = t - 1$ and $B_t \neq A_s$.

The transformation just described acts in one of three different ways on a partition π of the fixed integer n, say $n = \varepsilon_1 + a_2 + \ldots + a_r$. If the partition π has $t < s$, or $t = s$ with distinct points A_s and B_t, the transformation removes the entire bottom row of the graph, $B_1, B_2, \ldots, B_t$, and extends the first t rows of the graph by one point each.

If the partition π has $s < t-1$, or $s = t-1$, again with $A_s \neq B_t$, the transformation moves the points $A_1, A_2, ..., A_s$ to form an additional bottom row in the graph. The third type of partition π has $A_s = B_t$ with $s = t$ or $s = t-1$; for partitions π of this type, the transformation leaves π unchanged. The three types account for all possible partitions of n with distinct parts.

With the first and second types of partition P, the partition P' also has distinct parts, but has one fewer or one more part than P. Thus, apart from partitions of the third type, we have paired off partitions with an odd number of parts and those with an even parts.

Now consider the exceptional partitions of the third type, with $A_s = B_t$ and $s = t$ or $s = t-1$. Since A_s and B_t are identical points, it follows that $s = r$ and $a_1, a_2, ..., a_r$ are consecutive integers, with a_1 largest. Since $t = a_r$ in all cases, the partition has the form

$$n = (t+s-1) + (t+s-2) + ... + (t+1) + t$$

If $s = t$ we have $n = (3s^2 - s)/2$, whereas if $s = t-1$ then $n = (3s^2 + s)/2$. Hence if $n = (3s^2 \pm s)/2$ for some natural number s, that is, if n is one of the numbers $1, 2, 5, 7, 12, 15, 22, 26, ...$, then $Q^e(n)$ exceeds $Q^o(n)$ by 1 if s is even, but $Q^o(n)$ is larger by 1 if s is odd. For all other values of n, there are no partitions of the third type, and $Q^o(n) = Q^e(n)$. ∎

We can multiply out $\prod_{n\geqslant 1}(1-x^n)$ formally to get

$$\prod_{n\geqslant 1}(1-x^n) = \sum_{n\geqslant 0}(Q^e(n) - Q^o(n))x^n.$$

Then this identity implies,

$$\prod_{n\geqslant 1}(1-x^n) = 1 + \sum_{j\geqslant 1}(-1)^j(x^{(3j^2+j)/2} + x^{(3j^2-j)/2}).$$

This is known as Euler's formula.

Theorem 3.9 (Euler's Identity). *For any positive integer n,*

$$\begin{aligned} P(n) &= P(n-1) + P(n-2) - P(n-5) - P(n-7) + P(n-12) + P(n-15) - ... \\ &= \sum_{j\geqslant 0}(-1)^{j+1}P(n-(3j^2+j)/2) + \sum_{j\geqslant 0}(-1)^{j+1}P(n-(3j^2-j)/2). \end{aligned}$$

Proof. From the Euler's formula and the fact that $\prod_{n\geqslant 1}(1-x^n)^{-1}$ is the generating function for $P(n)$, we can write

$$\left[1+\sum_{j\geqslant 1}(-1)^j\left\{x^{(3j^2+j)/2}+x^{(3j^2-j)/2}\right\}\right]\sum_{k\geqslant 0}P(k)x^k=1.$$

Equating coefficients of x^n on the two sides, we get

$$P(n)-P(n-1)-P(n-2)+P(n-5)+P(n-7)-P(n-12)-...=0,$$

thus the theorem is established. ∎

Theorem 3.10 (Jacobi Identity).

$$\prod_{j\geqslant 0}\{(1-x^{2i+2})(1+x^{2i+1}u)(1+x^{2i+1}u^{-1})\}=\sum_{n\in Z}x^{n^2}u^n.$$

Proof. Let,

$$\prod_{i\geqslant 0}(1+x^iu)=\sum_{n\geqslant 0}\frac{x^{\binom{n}{2}}u^n}{(1-x)...(1-x^n)}$$

and

$$\prod_{i\geqslant 0}(1+x^iu)^{-1}=\sum_{n\geqslant 0}\frac{(-1)^nu^n}{(1-x)...(1-x^n)}.$$

Then,

$$\begin{aligned}\prod_{i\geqslant 0}(1+x^{2i+1}u) &= \sum_{n\geqslant 0}\frac{x^{n^2}u^n}{(1-x^2)...(1-x^{2n})}\\ &= \sum_{n\geqslant 0}x^{n^2}u^n\frac{\prod_{j\geqslant 0}(1-x^{2n+2+2j})}{\prod_{j\geqslant 0}(1-x^{2j+2})}\\ &= \frac{1}{\prod_{j\geqslant 0}(1-x^{2j+2})}\sum_{n}x^{n^2}u^n\prod_{j\geqslant 0}(1-x^{2n+2+2j})\\ &= \frac{1}{\prod_{j\geqslant 0}(1-x^{2j+2})}\sum_{n\in Z}x^{n^2}u^n\sum_{m\geqslant 0}\frac{(-1)^mx^{m^2+m+2mn}}{(1-x^2)...(1-x^{2m})}\\ &= \frac{1}{\prod_{j\geqslant 0}(1-x^{2j+2})}\sum_{m\geqslant 0}\frac{(-1)^m(xu^{-1})^m}{(1-x^2)...(1-x^{2m})}\sum_{n\in Z}x^{(m+n)^2}u^{m+n}\\ &= \frac{1}{\prod_{j\geqslant 0}(1-x^{2j+2})}\left\{\prod_{j\geqslant 0}(1+x^{2j+1}u^{-1})\right\}^{-1}\sum_{n\in Z}x^{n^2}u^n.\end{aligned}$$

∎

If we replace x by x^k and u by $\pm x^l$ in the Jacobi Identity where k and l are integers > 0, then

$$\prod_{j\geqslant 0}(1+x^{2kj+k-l})(1+x^{2kj+k+l})(1-x^{2kj+2k}) = \sum_{n\in Z} x^{kn^2+nl},$$

$$\prod_{j\geqslant 0}(1-x^{2kj+k-l})(1-x^{2kj+k+l})(1-x^{2kj+2k}) = \sum_{n\in Z}(-1)^n x^{kn^2+nl}.$$

By putting $k=\frac{3}{2}$ and $l=\frac{1}{2}$ we can obtain Euler Identity

$$\prod_{j\geqslant 0}(1-x^{3j+1})(1-x^{3j+2})(1-x^{3j+3}) = \prod_{j\geqslant 0}(1-x^j) = \sum_{n\in Z}(-1)^n x^{(3n^2+n)/2}.$$

Theorem 3.11 (Rogers-Ramanajuan). *The number of partitions of n into parts differing by at least 2 is equal to the number of partitions of n into parts which are congruent to 1 or 4 modulo 5* [5].

Proof. Firstly, we need to derive the generating function for the number of partitions of n into parts differing by at least 2. Since parts must differ by at least 2, each line must have at least 2 more dots than the one below. Thus, if the partition has exactly m parts, the graph must have at least $1+3+5+7+...+(2m-1) = m^2$ dots. Consequently a partition of n into m parts differing by at least 2 can be graphically represented by a triangle with m^2 dots and a partition of $n-m^2$ into at most m parts. But we know that the generating function for the number of partitions of n into parts not exceeding m is

$$\prod_{i=1}^{m}(1-x^i)^{-1}.$$

So the coefficient of x^n in $\frac{x^{m^2}}{(1-x)(1-x^2)...(1-x^m)}$ equals the number of partitions of $n-m^2$ into parts not exceeding m. To find the number of all partitions of n into parts differing by at least 2, we need to sum the coefficients of x^n for $m=1,2,3,...$

$$\sum_{m\geqslant 1}\frac{x^{m^2}}{(1-x)(1-x^2)...(1-x^m)}.$$

Now, we need to prove

$$\sum_{m\geqslant 1}\frac{x^{m^2}}{(1-x)(1-x^2)...(1-x^m)} = \prod_{i\geqslant 1}\frac{1}{(1-x^{5i+1})(1-x^{5i+4})}.$$

You can find the proof in [8]. ∎

This theorem implies another Rogers-Ramanajuan Identitiy.

Theorem 3.12 (Rogers-Ramanajuan). *The number of partitions of n into parts differing by at least 2, each part being greater than or equal to 2, is equal to the number of partitions of n into parts which are congruent to 2 or 3 modulo 5.*

This theorem is expressed as an identity of generating functions as follows

$$\sum_{m\geqslant 0}\frac{x^{m^2+m}}{(1-x)(1-x^2)...(1-x^m)}=\prod_{n\geqslant 0}\frac{1}{(1-x^{5n+2})(1-x^{5n+3})}.$$

3.2. Stirling Numbers

3.2.1. Stirling Numbers of the Second Kind S(n,k)

Definition 3.1. *The number $S(n,k)$ of k-partitions is called Stirling number of the second kind. In other words, $S(n,k)$ is the number of equivalence relations with k classes on N. It is also the number of distributions of n distinct balls into k distinguishable boxes (the order of the boxes does not count) such that no box is empty.*

For example, there are seven ways to split a four-element set into two parts:

$$\{1,2,3\}\cup\{4\},\quad \{1,2,4\}\cup\{3\},\quad \{1,3,4\}\cup\{2\},\quad \{2,3,4\}\cup\{1\}$$
$$\{1,2\}\cup\{3,4\},\quad \{1,3\}\cup\{2,4\},\quad \{1,4\}\cup\{2,3\}.$$

thus $S(4,2)=7$ [6][9].

Theorem 3.13. *The Stirling number of the second kind satisfies*

$$S(n,k)=\frac{1}{k!}\sum_{0\leqslant j\leqslant k}(-1)^j\binom{k}{j}(k-j)^n,$$

while $S(n,k)=0,\quad$ *if* $\;1\leqslant n\leqslant k.$

Proof. Let E be the set of maps of N into $[k] = \{1, 2, ..., k\}$ and let F be the subset of E consisting of the surjective maps. Then $|E| = k^n$ and any $f \in F$ corresponds to precisely one partition of N, namely the partition consisting of the k pre-images $f^{-1}(i)$, $i \in [k]$, $|F| = k!S(n,k)$.

Now let B_i be the set of $f \in E$ such that $\forall x \in N$, $f(x) \neq i$. Evidently $F = \bigcap_{i=1}^{k} B_i^c$. Thus

$$\begin{aligned} k!S(n,k) &= |F| = \left|\bigcap_{i=1}^{k} B_i^c\right| \\ &= |E| - \binom{k}{1}|B_1| + \binom{k}{2}\left|B_1 \bigcap B_2\right| - \binom{k}{3}\left|B_1 \bigcap B_2 \bigcap B_3\right| + \ldots \\ &= \sum_{1 \leqslant j \leqslant k} (-1)^j \binom{k}{j} (k-j)^n. \ \blacksquare \end{aligned}$$

Theorem 3.14. *The Stirling numbers of the second kind $S(n,k)$ satisfies*

$$f_k(x) = \sum_{n \geqslant k} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k, \qquad k \geq 0,$$

and

$$f(x,u) = \sum_{n,k \geqslant 0} S(n,k)\frac{x^n}{n!}u^k = 1 + \sum_{n \geqslant 1} \frac{x^n}{n!} \sum_{1 \leqslant k \leqslant n} S(n,k)u^k = \exp\{u(e^x - 1)\}.$$

Proof.

$$\begin{aligned} f_k(x) &= \sum_{n \geqslant 0} S(n,k)\frac{x^n}{x!} = \frac{1}{k!}\sum_{n \geqslant 0}(-1)^j \binom{k}{j}(k-j)^n \frac{x^n}{n!}, \qquad 0 \leqslant j \leqslant k \\ &= \frac{1}{k!}\sum_{0 \leqslant j \leqslant k}(-1)^j\binom{k}{j}\sum_{n \geqslant 0}(k-j)^n\frac{x^n}{n!} \\ &= \frac{1}{k!}\sum_{0 \leqslant j \leqslant k}\binom{k}{j}(-1)^j e^{x(k-j)} = \frac{1}{k!}(e^x - 1)^k. \end{aligned}$$

Similarly

$$f(x,u) = \sum_{k \geqslant 0} u^k \sum_{n \geqslant k} S(n,k)\frac{x^n}{n!} = \sum_{k \geqslant 0}\frac{1}{k!}u^k(e^x - 1)^k = \exp\{u(e^x - 1)\}. \ \blacksquare$$

Theorem 3.15.

$$x^n = \sum_{0 \leqslant k \leqslant n} S(n,k)(x)_k,$$

where $(x)_n = x(x-1)(x-2)...(x-k+1)$, $(x)_0 = 1$.

Proof. By identifying the coefficients of $\frac{u^n}{n!}$

$$\begin{aligned}\sum_{n\geqslant 0} x^n \frac{u^n}{n!} &= e^{ux} = \{1+(e^u-1)\}^x = \sum_{k\geqslant 0}\binom{x}{k}(e^u-1)^k \\ &= \sum_{k\geqslant 0}(x)_k\frac{(e^u-1)^k}{k!} = \sum_{0\leqslant k\leqslant n}(x)_k S(n,k)\frac{u^n}{n!}.\end{aligned}$$

∎

Theorem 3.16. *We have the following rational generating function*

$$f_k = \sum_{n\geqslant k} S(n,k)u^n = \frac{u^k}{(1-u)(1-2u)...(1-ku)}, \qquad k \geq 1$$

Proof.

$$\begin{aligned}f_k &= \frac{u^k}{(1-u)(1-2u)...(1-ku)} = \sum_{0\leqslant j\leqslant k}\frac{(-1)^j}{k!}\binom{k}{j}\frac{1}{1-(k-j)u} \\ &= \sum_{0\leqslant j\leqslant k}\frac{(-1)^j}{k!}\binom{k}{j}\sum_{n\geqslant 0}u^n(k-j)^n = \sum_{n\geqslant 0}u^n\frac{1}{k!}\sum_{0\leqslant j\leqslant k}(-1)^j\binom{k}{j}(k-j)^n \\ &= \sum_{n\geqslant 0}S(n,k)u^n. \ ∎\end{aligned}$$

Theorem 3.17.

$$S(n,k) = \sum_{c_1+c_2+...+c_k=n-k} 1^{c_1}.2^{c_2}...k^{c_k}.$$

Proof.

$$\begin{aligned}\frac{f_k}{u^k} &= \prod_{1\leqslant j\leqslant k}\frac{1}{1-ju} = \prod_{1\leqslant j\leqslant k}\prod_{c_j\geqslant 0} j^{c_j}u^{c_j} \\ &= \sum_{c_1,c_2,...c_k\geqslant 0}(1^{c_1}.2^{c_2}....k^{c_k})u^{c_1+c_2+...+c_k} \\ &= \sum_{n\geqslant 0}S(n,k)u^{n-k}. \ ∎\end{aligned}$$

Theorem 3.18. *The Stirling numbers of the second kind $S(n,k)$ satisfy the recurrence relation*

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \qquad n,k \geqslant 1$$

$$S(n,0) = S(0,k) = 0, \qquad S(0,0) = 1.$$

Proof. By equating the coefficients of $(x)_n$

$$\begin{aligned}\sum_{k\geqslant 0} S(n,k)(x)_k &= x^n = x\sum_{h\geqslant 0} S(n-1,h)(x)_h \\ &= \sum_{h\geqslant 0} S(n-1,h)\left\{(x)_{h+1} + h(x)_h\right\}\end{aligned}$$

because

$$(x)_{h+1} + h(x)_h = (x)_h(x-h+h) = x(x)_h. \blacksquare$$

Theorem 3.19. *The $S(n,k)$ satisfy the vertical recurrence relations*

$$S(n,k) = \sum_{k-1\leqslant j\leqslant n-1} \binom{n-1}{j} S(j,k-1),$$

$$S(n,k) = \sum_{k\leqslant j\leqslant n} S(j-1,k-1)k^{n-j}.$$

Proof.

$$g_k(x) = \sum_{n\geqslant k} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k$$

$$\begin{aligned}\frac{dg_k}{dx} &= \sum_{n\geqslant 0} S(n,k)\frac{x^{n-1}}{(n-1)!} = \frac{1}{(k-1)!}(e^x-1)^{k-1}e^x = e^x g_{k-1} \\ &= \sum_{l,m\geqslant 0} S(l,k-1)\frac{x^{l+m}}{l!m!} = \sum_{l,n\geqslant 0} \frac{S(l,k-1)(n-1)!}{l!(n-1-l)!}\frac{x^{n-1}}{(n-1)!}.\end{aligned}$$

By identifying the coefficients of $\frac{x^{n-1}}{(n-1)!}$, and using the rational generating function theorem

$$f_k = \sum_{n\geqslant k} S(n,k)u^n = \frac{u}{1-ku} f_{k-1} = \sum_{l,m\geqslant 0} S(l-1,k-1)k^m u^{l+m}. \blacksquare$$

Theorem 3.20 (Bernoulli and Stirling Numbers).

$$B_n = \sum_{k=0}^{n} (-1)^k k! \frac{S(n,k)}{k+1}.$$

Proof. By the definition of the Bernoulli numbers

$$\sum_{n\geqslant 0} B_n \frac{t^n}{n!} = \frac{t}{e^t-1}$$

$$\frac{t}{e^t-1} = \frac{\log(1+(e^t-1))}{e^t-1} = \frac{\sum_{n\geqslant 0}(-1)^n\frac{(e^t-1)^{n+1}}{n+1}}{e^t-1}$$
$$= \sum_{n\geqslant 0}(-1)^n\frac{(e^t-1)^n}{n+1} = \sum_{n\geqslant 0}\frac{(-1)^n.n!}{n+1}\sum_{m\geqslant n}S(m,n)\frac{t^m}{m!}$$

So by equating the coefficients of $\frac{t^m}{m!}$ we get

$$B_m = \sum_{0\leqslant n\leqslant m}\frac{(-1)^n.n!}{n+1}S(m,n). \blacksquare$$

3.2.2. Stirling Numbers of the First Kind s(n,k)

Definition 3.2. *The Stirling numbers of the first kind $s(n,k)$ counts the number of ways to arrange n objects into k cycles instead of k subsets* [9].

For example there are eleven different ways to make two cycles from four elements.

[123] [4], [124] [3], [134] [2], [234] [1],
[132] [4], [142] [3], [143] [2], [243] [1],
[12] [34], [13] [24], [14] [23].

Hence $s(4,2) = 11$.

In general $(n-1)!$ different n cycles can be made from any n elements set whenever $n > 0$. Therefore $s(n,1) = (n-1)!$. For all non-negative integers k, $s(0,k) = \delta_{ok}$.

The $s(n,k)$ have double generating function

$$f(x,u) = \sum_{n,k\geqslant 0}s(n,k)\frac{x^n}{n!}u^k = 1 + \sum_{n\geqslant 1}\frac{x^n}{n!}\sum_{1\leqslant k\leqslant n}s(n,k)u^k = (1+x)^u.$$

We will prove that the infinite lower triangular matrix of the $s(n,k)$ is the inverse of the matrix of the $S(n,k)$. The $s(n,k)$ are not all positive, their sign is given by $|s(n,k)| = (-1)^{k+n}s(n,k)$

For the generating function of the $|s(n,k)|$, take $-x$ and $-u$ instead of x and u. Then we can get

$$\sum_{n,k\geqslant 0} |s(n,k)| \frac{x^n}{n!} u^k = (1-x)^{-u}.$$

Theorem 3.21. *The* $s(n,k)$ *have*

$$(x)_n = \sum_{0\leqslant k\leqslant n} s(n,k) x^k,$$

$$\langle x\rangle_n = \sum_{0\leqslant k\leqslant n} |s(n,k)|\, x^k,$$

where $(x)_n = x.(x-1)...(x-n+1)$ *and* $\langle x\rangle_n = x.(x+1)...(x+n-1)$, $(x)_0 = \langle x\rangle_0 = 1$.

Proof. By the definition of the $s(n,k)$ and $|s(n,k)|$

$$\sum_{n,k\geqslant 0} s(n,k) \frac{x^n}{n!} u^k = (1+x)^u = \sum_{n\geqslant 0} \binom{u}{n} x^n = \sum_{n\geqslant 0} (u)_n \frac{x^n}{n!}.$$

Similarly,

$$\sum_{n,k\geqslant 0} |s(n,k)| \frac{x^n}{n!} u^k = (1-x)^{-u} = \sum_{n\geqslant 0} \binom{-u}{n} (-1)^n x^n = \sum_{n\geqslant 0} \langle u\rangle_n \frac{x^n}{n!}.$$

∎

Now consider

$$y^n = (x+1)^n = \sum_{k\geqslant 0} \binom{n}{k} x^k.$$

By the inverse relation

$$x^n = (y-1)^n = \sum_{k\geqslant 0} (-1)^{n+k} \binom{n}{k} y^k.$$

By substitution of either into the other and equating coefficients of powers of the variable x and y, we have the following equation

$$\delta_{nm} = \sum_{k\geqslant 0} (-1)^{k+m} \binom{n}{k} \binom{k}{m}.$$

The Stirling numbers of the first kind and second kind are defined by the inverse relation that we proved

$$(x)_n = \sum_{k\geqslant 0} s(n,k) x^k \text{ and } x^n = \sum_{k\geqslant 0} S(n,k) (x)_k.$$

Hence,

$$\delta_{nm} = \sum_k s(n,k)S(k,m) = \sum_k S(n,k)s(k,m).$$

Thus, the infinite lower triangular matrix of the $s(n,k)$ is the inverse of the matrix of the $S(n,k)$.

Theorem 3.22. *Bell Polynomials and Stirling numbers of the first kind satisfies*

$$B_{n,k}(0!, 1!, 2!, ...) = |s(n,k)|,$$

where $B_{n,k}$ is the Bell polynomial.

Proof. By the definition of $B_{n,k}$

$$f(t,u) = \exp(u \sum_{m\geqslant 1} x_m \frac{t^m}{m!}) = \sum_{n,k} B_{n,k} \frac{t^n}{n!} u^k.$$

Put $x_m = (m-1)!$

$$f(t,u) = \exp(u \sum_{m\geqslant 1} \frac{t^m}{m!}) = \exp(-u \log(1-t)) = (1-t)^{-u},$$

which is the generating function of the $|s(n,k)|$.

∎

Theorem 3.23.

$$f_n(x) = \sum_{0\leqslant k\leqslant n} s(n,k)x^{n-k} = (1-x).(1-2x)...(1-x(n-1)),$$

$$f_n(-x) = \sum_{0\leqslant k\leqslant n} |s(n,k)|\, x^{n-k} = (1+x).(1+2x)...(1+x(n-1)).$$

Proof. Since

$$(x)_n = \sum_{0\leqslant k\leqslant n} s(n,k)x^k,$$

replace x by x^{-1}

$$(x^{-1})_n = \sum_{0\leqslant k\leqslant n} s(n,k)x^{-k} = \frac{(1-x).(1-2x)...(1-x(n-1))}{x^n}.$$

Similarly put x by x^{-1} in the

$$\langle x \rangle_n = \sum_{0\leqslant k\leqslant n} |s(n,k)|\, x^k.$$

Consequently the second identity holds. ∎

Theorem 3.24. *The Stirling numbers of the first kind $s(n,k)$ satisfy the recurrence relation*

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \qquad n,k \geqslant 1,$$

$$|s(n,k)| = |s(n-1,k-1)| + (n-1)\,|s(n-1,k)|\,,$$

$$s(n,0) = \delta_{n,0}.$$

Proof.

$$\begin{aligned}\sum_{k\geqslant 0} s(n,k)x^k &= (x)_n = \{x-(n-1)\}\,(x)_{n-1} \\ &= \{x-(n-1)\}\sum_{j\geqslant 0} s(n-1,j)x^j,\end{aligned}$$

and,

$$\begin{aligned}\sum_{k\geqslant 0} |s(n,k)|\,x^k &= \langle x\rangle_n = \{x+(n-1)\}\,\langle x\rangle_{n-1} \\ &= \{x+(n-1)\}\sum_{j\geqslant 0} |s(n-1,j)|\,x^j\end{aligned}$$

∎

By the recurrence relation of $|s(n,k)|$, let us try to find a closed form for $|s(n,2)|$, the number of permutations of n objects that have exactly two cycles.

$$|s(n+1,2)| = n\,|s(n,2)| + |s(n,1)| = n\,|s(n,2)| + (n-1)!$$

$$\frac{1}{n!}\,|s(n+1,2)| = \frac{1}{(n-1)!}\,|s(n,2)| + \frac{1}{n}.$$

Then we can conclude that

$$\frac{1}{n!}\,|s(n+1,2)| = H_n$$

and

$$|s(n+1,2)| = n!H_n,$$

where H_n is the first order harmonic numbers.

In order to find $|s(n,3)|$ and $|s(n,4)|$, we have the following theorem [6].

Theorem 3.25.

$$|s(n+1,k+1)| = \frac{n!}{k!} Y_k(\xi_n(1), -1!\xi_n(2), 2!\xi_n(3), ...),$$

where Y_k stands for the Bell polynomials and

$$\xi_n(s) = \sum_{j=1}^{n} j^{-s}.$$

Proof. Since

$$\sum_{0 \leqslant k \leqslant n} |s(n+1,k+1)|\, x^k = (x+1).(x+2)...(x+n)$$

$$\begin{aligned}
\sum_{k\geqslant 0} |s(n+1,k+1)|\, x^k &= n!(1+x)(1+\frac{x}{2})(1+\frac{x}{3})...(1+\frac{x}{n}) \\
&= n!\exp(\sum_{j=1}^{n} \log(1+xj^{-1})) \\
&= n!\exp(\sum_{j=1}^{n} \sum_{s\geqslant 1} (-1)^{s-1} x^s s^{-1} j^{-s}) \\
&= n!\exp(\sum_{s\geqslant 1} (-1)^{s-1} x^s s^{-1} \xi_n(s)) \\
&= n! \sum_{k\geqslant 0} Y_k(\xi_n(1), -1!\xi_n(2), 2!\xi_n(3), ...) \frac{x^k}{k!}.
\end{aligned}$$

∎

Then we have,

$$\sum_{k\geqslant 0} |s(n+1,k+1)|\, x^k = n! \sum_{k\geqslant 0} (\sum_{s\geqslant 1} (-1)^{s-1} x^s s^{-1} \xi_n(s))^k . \frac{1}{k!}.$$

Put $k = 1$ and consider the coefficient of x on the right hand side of the equation

$$|s(n+1,2)| = n!\xi_n(1) = n!H_n.$$

Similarly put $k = 2$ and $k = 3$ to obtain

$$|s(n+1,3)| = \frac{n!}{2!}(H_n^2 - H_n^{(2)})$$

and

$$|s(n+1,4)| = \frac{n!}{3!}(H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}).$$

3.3. Connected and Colored Graphs

3.3.1. Connected Graphs

We consider enumerative problems of graph theory, that is, problems arising when counting graphs with specific properties. The method of generating functions can be effectively applied to these problems. As a result, we obtain either explicit formulae or some expressions for generating functions which allow us to find relationships between colored connected graphs and colored graphs [10].

Definition 3.3. *A graph* $\Gamma = \Gamma(X, W)$ *consists of a set* X *containing* $n \geqslant 1$ *elements called vertices and of a set* W *of unordered pairs of vertices called edges. Usually a graph is geometrically represented on the plane by points corresponding to the vertices, and lines corresponding to the edges which join pairs of vertices from W. The intersections of the lines at points that differ from the vertices are not taken into account.*

Definition 3.4. *An alternating sequence* $x_1, w_1, x_2, w_2, ..., w_{n-1}, x_n$, *of vertices and edges where* $x_i \in X$, *and* $w_i \in W$, $w_i \neq w_j$ *when* $i \neq j$ *and* w_i *is incident to* x_i *and* x_{i+1}, *is called a path connecting the vertices* x_1 *and* x_n. *A path is called a simple path if* $x_i \neq x_j$ *when* $i \neq j$. *A closed path with* $x_1 = x_n$ *is called a cycle.*

Definition 3.5. *A graph is connected if any pair of its vertices is connected by a simple path.*

Definition 3.6. *A graph* Γ' *is a subgraph of a graph* Γ' *if all vertices and all edges of* Γ' *belong to* Γ. *A maximal connected subgraph of a graph* Γ *is called a connected component of the graph* Γ.

Two graphs with labelled vertices are counted as the same if and only if for all i and j the same number of lines go from the vertex i to the vertex labelled j in both graphs. Two topological equivalent graphs may be counted as distinct labelled graphs if they are labelled differently.

Let P be any property of connected graphs. Let $T_{n,m}$ be the total number of graphs having n labelled vertices m unlabeled lines and such that every connected component of the graph has the property P. Suppose the numbers $T_{n,m}$ are known and consider the number $C_{n,m}$ of connected graphs of n labelled vertices and m unlabeled lines which have the property P. To get a recurrence equation for $C_{n,m}$ note that in a graph J with $n+1$ vertices and m lines the vertex labelled $n+1$ belongs to a connected component K having some number n' of other vertices and some number m' of lines. The remaining part $J-K$ of J has $n-n'$ vertices and $m-m'$ lines.

Hence we conclude,

$$\begin{aligned} T_{n+1,m} &= \sum_{n',m'\geqslant 0} \binom{n}{n'} C_{n'+1,m'} T_{n-n',m-m'}, \\ T_{0,m} &= \delta_{0m} \text{ by convention.} \end{aligned}$$

Now, define generating functions such that

$$C_n(y) = \sum_{m\geqslant 0} C_{n,m} y^m$$

$$T_n(y) = \sum_{m\geqslant 0} T_{n,m} y^m.$$

Then,

$$T_{n+1}(y) = \sum_{n'\geqslant 0} \binom{n}{n'} C_{n'+1}(y) T_{n-n'}(y).$$

We can find the number of connected graphs by using these generating functions.

Theorem 3.26. *The generating function for the number of connected graphs having n labelled vertices, m lines and having property P is $(n-1)!$ times the coefficient of x^{n+1} in the power series for the quotient*

$$\sum_{n\geqslant 0} T_{n+1}(y)\frac{x^n}{n!} \Big/ \sum_{n\geqslant 0} T_n(y)\frac{x^n}{n!}.$$

In these series $T_0(y) = 1$ by convention.

Proof. In order to prove the theorem, we use a special pair of parentheses $\{\}$. The expressions inside such parentheses will be analytic functions depending on x and two letters C and T. The term $T^m C^n x^i$ is replaced by $T_m(y)C_n(y)x^i$ for all m, n, i. Then, generating functions of C and T become

$$C(x,y) = \{\exp Cx\},$$

$$T_{n+1}(y) = \{C(T+C)^n\}.$$

Multiplying both sides of $T_{n+1}(y)$ by $\frac{x^n}{n!}$ and summing on n, one derives

$$\{T \exp Tx\} = \{C \exp(T + C)x\} = \{C \exp Cx\} \{\exp Tx\} .$$

Integrating both sides with respect to x from 0 to x one also has

$$\{\exp Cx\} - \{C^0\} = \log \{\exp Tx\} .$$

By convention we put $\{C^0\} = C_0(y) = 0$, then

$$C(x, y) = \{\exp Cx\} = \log \{\exp Tx\} .$$

Thus $C_n(y)$ is also $n!$ times the coefficient of x^n in the power series for

$$\log \sum_{n \geqslant 0} T_n(y) \frac{x^n}{n!}.$$

∎

There are

$$\binom{\frac{1}{2}n(n-1)}{m}$$

graphs with m lines and n labelled vertices and no lines in parallel. Hence by putting

$$T_n(y) = (1 + y)^{\frac{1}{2}n(n-1)}$$

in the theorem, we count connected graphs with n labelled vertices and m lines, none in parallel. Setting $y = 1$, we count these graphs by vertices, allowing any number of lines. For $n = 1, 2, 3, 4, 5$ we find $1, 1, 4, 38,$ and 728 connected graphs [10].

Let P be the property that every cycle of the graph must be of even length. We use L for the number of labelled lines. The vertices of even graph separate into two classes A and B with the property that every line has one end point in A and one in B. Let a denote the number of vertices in A and b the number in B. The number of graphs with these values of a and b is just $S(L, a)S(L, b)$ where $S(L, k)$ is the number of ways of putting L different objects (the end points) into k groups (the vertices) so that no group is empty.

We know that,

$$\sum_{l,k \geqslant 0} S(L, k) y^k \frac{x^L}{L!} = \exp(y \exp(x - 1)).$$

Hence, the generating function $T_n(y)$ for the number of graphs with m vertices is

$$T_n(y) = \sum_{a,m\geqslant 0} S(n,a)S(n,m-a)y^m = (\sum_{k\geqslant 0} S(L,k)y^k)^2.$$

3.3.2. Colored Graphs

By a coloring of a graph Γ in k colors, we shall mean a mapping of the vertex of the graph onto a set of k colors $c_1, c_2, ..., c_k$ such that no two vertex which are joined by one edge are mapped onto the same color. A graph so colored in exactly k colors will be called a k-colored graph [11].

Suppose we are given a set of n vertex $a_1, a_2, ..., a_n$, a set of positive non-zero integers $n_1, n_2, ..., n_k$ such that $n_1 + n_2 + ... + n_k = n$ and a set of integers $e_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ..., k)$. We shall count the number of k-colored graphs on these n vertex which are such that n_α vertex allocated the color c_α and $e_{\alpha\beta}$ edges join vertex allocated the color c_α to vertex allocated the color c_β .

Clearly,

$$E = \sum_{\alpha<\beta} e_{\alpha\beta}$$

is the total number of edges.

Firstly, allocate the colors to the various vertex. This is possible in,

$$\frac{n!}{n_1!.n_2!....n_k!}$$

different ways. Next, consider the number of ways of choosing $e_{\alpha\beta}$ edges joining vertex colored in c_α and c_β. There are $n_\alpha.n_\beta$ possible edges so the choice can be made in,

$$\binom{n_\alpha.n_\beta}{e_{\alpha\beta}}$$

different ways. Thus the total number of graphs is,

$$\frac{n!}{n_1!.n_2!....n_k!}\prod_{\alpha<\beta}\binom{n_\alpha.n_\beta}{e_{\alpha\beta}}.$$

This is the coefficient of $t^{e_{\alpha\beta}}$ in

$$F(t) = \frac{n!}{n_1!.n_2!...n_k!}\prod_{\alpha<\beta}(1+t)^{n_\alpha.n_\beta} = \frac{n!}{n_1!.n_2!...n_k!}(1+t)^{\sum n_\alpha.n_\beta},$$

and is therefore

$$\frac{n!}{n_1!.n_2!...n_k!}\binom{\sum n_\alpha.n_\beta}{E}.$$

But,

$$\sum_{\alpha<\beta} n_\alpha.n_\beta = \frac{1}{2}(\sum_\alpha n_\alpha)^2 - \frac{1}{2}\sum_\alpha n_\alpha^2.$$

This coefficient is equal to

$$\frac{n!}{n_1!.n_2!...n_k!}\binom{\frac{1}{2}n^2 - \frac{1}{2}\sum n_\alpha^2}{E}.$$

When there are n colors and n vertex with different colors, i.e. $n_1 = n_2 = ... = n_n = 1$, this number reduces to

$$n!\binom{\frac{1}{2}(n-1)n}{E}.$$

For finding the number of k-colored graphs whatever the number of edges, sum the coefficient of t^E for all possible values of E, and we see that this is equal to

$$\frac{n!}{n_1!.n_2!...n_k!}2^{\frac{1}{2}n^2 - \frac{1}{2}\sum n_\alpha^2}.$$

Finally, we can change the number of colors with $n = n_1 + n_2 + ... + n_k$. So to find the total number of k-colored graphs we need to sum

$$F_n(k) = \sum_{(n)} \frac{n!}{n_1!.n_2!....n_k!}2^{\frac{1}{2}n^2 - \frac{1}{2}\sum n_\alpha^2} = n!2^{\frac{1}{2}n^2}\left[\sum_{s\geqslant 1}\frac{2^{-\frac{1}{2}s^2}}{s!}x^s\right]^k.$$

Hence,

$$\sum_{n\geqslant 1} 2^{-\frac{1}{2}n^2}F_n(k)\frac{x^n}{n!} = \left(\sum_{s\geqslant 1} 2^{-\frac{1}{2}s^2}\frac{x^s}{s!}\right)^k.$$

3.4. The Snake Oil Method for Easier Combinatorial Identities

Combinatorial mathematics is full of dazzling identities. It is a fine skill for a working discrete mathematician to have if he or she is able to evaluate or simplify complicated looking sums that involve combinatorial numbers, because they have a way of turning up in connection with problems in graphs, algorithms, enumerations, etc. In this section we will study the Snake Oil method that uses generating functions to deal with the evaluation of combinatorial sums. The method works beautifully, within its limitations, on sums involving other combinatorial numbers.

This method works step by step. Identify the free variable, say n, that the sum depends on and give a name to the sum that you are working on; call it $f(n)$. Let $F(x)$ be the ordinary power series generating function whose $[x^n]$ is $f(n)$. Multiply the sum by x^n, and sum on n. Your generating function is now expressed as a double sum over n. Interchange the order of the two summations that you are now looking at, and perform the inner one in simple closed form. Finally try to identify the coefficients of the generating function of the answer, because those coefficients are what we want to find [1].

Example 3.1. Consider the sum

$$\sum_{k\geqslant 0}\binom{k}{n-k}, \qquad (n=0,1,2,\ldots).$$

Let

$$f(n)=\sum_{k\geqslant 0}\binom{k}{n-k}.$$

Multiply both sides by x^n and sum over n. Then,

$$\begin{aligned} F(x) &= \sum_{n\geqslant 0}x^n\sum_{k\geqslant 0}\binom{k}{n-k}=\sum_{k\geqslant 0}\sum_{n\geqslant 0}\binom{k}{n-k}x^n \\ &= \sum_{k\geqslant 0}x^k\sum_{n\geqslant 0}\binom{k}{n-k}x^{n-k}=\sum_{k\geqslant 0}x^k\sum_{r\geqslant 0}\binom{k}{r}x^r \\ &= \sum_{k\geqslant 0}x^k(1+x)^k=\sum_{k\geqslant 0}(x+x^2)^k=\frac{1}{1-x-x^2}. \end{aligned}$$

This is the generating function of Fibonacci numbers as we have seen before. Hence, $f(n) = F_n$, i.e.

$$\sum_{k\geqslant 0} \binom{k}{n-k} = F_n, \qquad (n = 0, 1, 2, \ldots).$$

Example 3.2. Evaluate the sum

$$f_n = \sum_{k\geqslant 0} \binom{n+k}{2k} 2^{n-k}.$$

Let F be the ordinary power series generating function of f_n.

$$\begin{aligned} F(x) &= \sum_{k\geqslant 0} 2^{-k} \sum_{n\geqslant 0} \binom{n+k}{2k} 2^n x^n = \sum_{k\geqslant 0} 2^k (2x)^{-k} \sum_{n\geqslant 0} \binom{n+k}{2k} (2x)^{n+k} \\ &= \sum_{k\geqslant 0} 2^{-k} (2x)^{-k} \frac{(2x)^k}{(1-2x)^{2k+1}} = \frac{1}{1-2x} \sum_{k\geqslant 0} \left\{ \frac{x}{(1-2x)^2} \right\}^k \\ &= \frac{1}{1-2x} \frac{1}{1-\frac{x}{(1-2x)^2}} = \frac{1-2x}{(1-4x)(1-x)} = \frac{2}{3(1-4x)} + \frac{1}{3(1-x)}. \end{aligned}$$

Hence the coefficient of x^n is

$$f_n = \frac{2^{2n+1}+1}{3}.$$

Example 3.3. Suppose we have two complicated sums and we want to show that they are the same. Then the generating function method, if it works, should be very easy to carry out. Let us prove that

$$\sum_{k\geqslant 0} \binom{m}{k} \binom{n+k}{m} = \sum_{k\geqslant 0} \binom{m}{k} \binom{n}{k} 2^k.$$

Multiply on the left by x^n, sum on n

$$\begin{aligned} \sum_{k\geqslant 0} \binom{m}{k} x^{-k} \sum_{n\geqslant 0} \binom{n+k}{m} x^{n+k} &= \sum_{k\geqslant 0} \binom{m}{k} x^{-k} \frac{x^m}{(1-x)^{m+1}} \\ &= \frac{x^m}{(1-x)^{m+1}} \left(1 + \frac{1}{x}\right)^m = \frac{(1+x)^m}{(1-x)^{m+1}}, \end{aligned}$$

and,

$$\begin{aligned} \sum_{k\geqslant 0} \binom{m}{k} 2^k \sum_{n\geqslant 0} \binom{n}{k} x^n &= \frac{1}{1-x} \sum_{k\geqslant 0} \binom{m}{k} \left(\frac{2x}{1-x}\right)^k \\ &= \frac{1}{1-x} \left(1 + \frac{2x}{1-x}\right)^m = \frac{(1+x)^m}{(1-x)^{m+1}}. \end{aligned}$$

Hence the two sums are equal, even if we don't know what they are.

Example 3.4. We will show that

$$\sum_{m\geqslant 0}\binom{r}{m}\binom{s}{t-m}=\binom{r+s}{t}.$$

Let

$$f_t=\sum_{m\geqslant 0}\binom{r}{m}\binom{s}{t-m}$$

and F be the generating function of f_t

$$\begin{aligned}
F(x) &= \sum_{t\geqslant 0}x^t\sum_{m\geqslant 0}\binom{r}{m}\binom{s}{t-m}=\sum_{m\geqslant 0}\binom{r}{m}x^m\sum_{t\geqslant 0}\binom{s}{t-m}x^{t-m}\\
&= \sum_{m\geqslant 0}\binom{r}{m}x^m(1+x)^s=(1+x)^s(1+x)^r=(1+x)^{s+r}\\
&= \sum_{t\geqslant 0}\binom{r+s}{t}x^t i.e. f_t=\binom{r+s}{t}.
\end{aligned}$$

If we take $r=s$ and $t=2m$, then we can show that

$$\sum_{m\geqslant 0}\binom{r}{m}^2=\binom{2r}{2m}.$$

4. CONCLUSION

One of the fundamental concepts in combinatorial theory is that of enumeration, and one of the basic techniques for dealing with problems of enumeration is that of generating functions. In this thesis we tried to analyze the solutions of some enumerative problems by using generating functions. For these problems, we have used exponential generating functions or ordinary power series generating functions, but a problem arises when we have to decide which kind of generating function can be used. In order to solve this problem we studied the Theory of Generating Functions which is based on R. Stanley's Theory of Incidence Algebra and Bender-Goldman Theory of Prefabs. The exponential generating functions can be used for solving enumerative problems that are related with permutations and the ordinary power series generating functions can be used when we have a problem that is related with usual order.

Generating functions can be used for proving some identities that come from Partitions of Integers and Stirling Numbers. I studied the proofs of the Euler's and Jacobi's identities and tried to understand the methods of generating functions. The most famous identities studied in this thesis are the Rogers-Ramanujan identities.

The problem of evaluating $p(n)$ where $p(n)$ is the number of partitions of integers has a long history. There is no simple formula in general for $p(n)$, but there are remarkable and quite sophisticated methods to compute $p(n)$. In this thesis, we have learned how we can compute $p(n)$ by using generating function of the partition of integers.

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